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Water waves with compactly supported vorticity

A functional-analytic approach to bifurcation theory and the mathematical theory of traveling water waves

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Abstract

We study the mathematical theory of water waves. Local bifurcation theory is also discussed, including the Crandall–Rabinowitz theorem; an abstract theorem used to establish the presence of bifurcation points in the zero set of maps on Banach spaces. A functional-analytic approach is used to prove the existence of a family of localized traveling waves with one or more point vortices, by bifurcating from a trivial solution. This is done in the setting of the incompressible Euler equations with gravity and surface tension, on finite depth. Our result is an extension of a recent result by Shatah, Walsh and Zeng, where existence was shown for a single point vortex on infinite depth. The properties of the resulting waves are also examined: We find that the properties depend significantly on the position of the point vortices in the water column.

Sammendrag

Vi studerer den matematiske teorien bak vannbølger. Lokal bifurkasjonsteori blir også diskutert, deriblant Crandall–Rabinowitz' teorem; et abstrakt teorem brukt til å etablere tilstedeværelse av bifurkasjonspunkter i nullmengden til funksjoner mellom Banachrom. En funksjonalanalytisk tilnærming blir brukt til å bevise eksistens av en familie lokaliserte reisende bølger, med én eller flere punktvirvler, ved bifurkasjon fra triviell løsning. Dette blir gjort for de inkompressible Eulerlikningene med tyngdekraft og overflatespenning, på endelig dyp. Vårt resultat er en utvidelse av et nyere resultat ved Shatah, Walsh og Zeng, hvor eksistens ble vist for en enkel punktvirvel på uendelig dyp. De resulterende bølgenes egenskaper blir også undersøkt: Vi finner at egenskapene i høy grad avhenger av punktvirvlenes posisjon i vannsøylen.

Preface

This master's thesis was written in the spring of 2014, as the final part of the study program Industrial Mathematics within Applied Physics and Mathematics at the Norwegian University of Science and Technology (NTNU).

The topic of the thesis was chosen as a continuation of my specialization project (TMA4500) in the autumn of 2013, which is written in lieu of a longer master's thesis for students in engineering programs at NTNU. In the project, my main goal was to provide tools that could be useful for bifurcation theory. In this master's thesis, bifurcation theory is applied in order to establish the existence of traveling waves; albeit not the tools that were established in the project. In addition to the traveling waves, we also discuss local bifurcation theory in general Banach spaces.

To be more specific on the main results, we prove the existence of localized traveling water waves containing one or more point vortices, on finite depth. The idea for this topic came from several recent investigations (see Chapter 1), and in particular from [Shatah et al., 2013], which established the existence of traveling water waves with compactly supported vorticity, on infinite depth. One of the existence results proved in the investigation is for localized waves with a single point vortex, which this thesis extends: Aside from working on finite depth, we consider multiple point vortices, and also investigate the solutions in more detail. On the other hand, Shatah et al. additionally consider *vortex patches* and global bifurcation, which we will not pursue here. The plan is to continue work in this area for my doctoral dissertation.

A summary of the contents of each chapter in the thesis is provided below. We include some motivation and explanation for the chapters.

Chapter 1 In this introduction, the history of mathematical research on water waves is discussed, placing this thesis into context. We start with the early beginnings of research in the 18th century, and end with very recent developments in the area.

Chapter 2 A short collection of useful notation, definitions and conventions that are used throughout. These are gathered in one place for convenience.

Chapter 3 We gather some preliminary results and theory that will be used in later chapters. Fractional Sobolev spaces on $\mathbb{R}^d$ and on open sets are introduced, and analytic operators between Banach spaces are defined. Of special interest is Theorem 3.7, a result on analytic functions, which is used to establish Theorem 6.17.

Theorem 6.17 concerns the applicability of the existence theorem for waves with multiple point vortices.

Chapter 4 This is a largely self-contained chapter on local bifurcation theory. We prove the Lyapunov–Schmidt reduction, Theorem 4.12, which can be interpreted as an extension of the implicit function theorem, and apply this in order to prove the Crandall–Rabinowitz theorem, Theorem 4.13. This is an abstract theorem used for showing the existence of bifurcation points in a general setting. The books [Buffoni and Toland, 2003, Kielhöfer, 2012] were used as references for these theorems. While the proofs are my own, the approaches are standard.

Chapter 5 In this chapter, we introduce the incompressible Euler equations, the water-wave problem and a weak vorticity equation that must be satisfied at any point vortices that are present. The Zakharov–Craig–Sulem formulation of these equations is derived, reducing the problem to one at the water surface.

Chapter 6 We prove the existence of localized traveling waves with a single point vortex in Theorem 6.11, using the Zakharov–Craig–Sulem formulation from Chapter 5. The approach is influenced by the one employed in [Shatah et al., 2013], mentioned above. The book [Lannes, 2013] was helpful in providing general results for operators used. After the existence result, we give properties for the leading order surface term in Proposition 6.13 and Theorem 6.14. With Theorem 6.15 we extend existence to waves with any finite number of point vortices on the same vertical line. Finally, we prove results concerning when Theorem 6.15 is applicable.

Chapter 7 An explicit stream function that can be used for an existence theorem on periodic water waves is constructed. Although we do not pursue the actual existence theorem, we apply the result in order to provide simpler expressions for the leading order terms in the expansions for periodic waves with a single point vortex on infinite depth, improving upon the results in [Shatah et al., 2013]. The leading order wave velocity and velocity field are given in closed form, and a Fourier series is given for the leading order surface profile. These were given as series and as the solution of a differential equation, respectively, in the original article.

I would like to thank my advisor, Professor Mats Ehrnström, for his help during weekly meetings in the past year. His proofreading and advice has been invaluable in writing my project and this thesis.

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Contents

1 Introduction

Before we move to the current state of mathematical research on the theory of water waves, we begin with what has lead up to this point. The history of work on water waves is a long one, and wrought with detail. As such, we will only be able to skim the surface here. Hopefully, the history will serve as motivation for this thesis, and explain the direction of other recent research in the area. We will keep the mathematical minutiae to a minimum, and focus on the overarching themes.

1.1 Early history

The first real mathematical work on water waves started with Euler's publication of [Euler, 1757], where he derived the equations for the dynamics of inviscid fluid flow. These equations bear his name today; they are known as the Euler equations[1]. Several mathematicians worked on water waves in the latter half of the 18th century, among them Laplace and Lagrange. Laplace came near a full description of irrotational linear water waves driven by gravity in his memoir [de Laplace, 1776], while Lagrange considered small-amplitude linear waves on shallow water (those are waves for which the wavelength is large compared with the water depth) a few years later. In this context, linear water waves mean waves that satisfy linearized versions of the governing equations.

More serious accounts appeared several decades later, with [Cauchy, 1816, Poisson, 1818] (sharing many commonalities), where, among other things, Poisson covered what Laplace was not able to finish. These were later recognized as being important to the development of the mathematical theory of water waves, but were not well-received by their contemporaries. The works were, at the time, viewed as very inaccessible, and of little practical interest: An attest to this is Airy's opinion, which was that

> ... as regards their physical results these elaborate treatises are entirely uninteresting; although they rank among the leading works of the present century in regard to the improvement of pure mathematics.
>
> (Airy, 1841)

[1]Euler was so prolific that, as of the moment of writing, the online encyclopedia Wikipedia lists 14 different equations that include his name.

Airy was one of several authors publishing works on water waves, of varying novelty and impact, at the time (the mid 1800s); others being Green, Kelland and Earnshaw. An important contribution was [Airy, 1841], from which the above quote is taken. In this article, Airy covered many aspects of water wave theory in great depth. To this day, linear water wave theory is sometimes known as Airy wave theory, due to his contributions.

Another important contemporary figure was Russell. While Russell did not publish mathematics on water waves, he did produce several reports containing experimental data concerning waves. He researched waves in manifold locations and regimes, providing invaluable observations for those who would try to explain them. His last and most important work among these was [Russell, 1844], which contained a discovery that has shaped mathematical research ever since. In the oft-quoted first paragraph of his chapter concerning this discovery, Russell recalls his observation at the Union Canal in Scotland:

> I believe I shall best introduce this phænomenon [sic] by describing the circumstances of my own first acquaintance with it. I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped — not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phænomenon which I have called the Wave of Translation, a name which it now very generally bears; which I have since found to be an important element in almost every case of fluid resistance, and ascertained to be the type of that great moving elevation of the sea, which, with the regularity of a planet, ascends our rivers and rolls along our shores.
>
> (Russell, 1844)

The *Wave of Translation*, as Russell calls it, was the first observation of a solitary wave (also known as a *soliton*, although this more modern term, coined in [Zabusky and Kruskal, 1965], typically implies further particle-like properties of such waves). Solitary waves are waves that are localized and that travel at constant speed, while retaining their shape. A collection of Russell's original drawings of "his" wave can be seen in Figure 1.1, and appear in [Russell, 1844]. His observations of the wave of translation were met with skepticism from other researchers at the time, such as Airy. Even so, Green, Kelland and Earnshaw all tried to explain the solitary wave, but with little success. Such waves cannot exist in linear theory, which was dominating at the time, and a nonlinear theory is needed; solitary waves are the result of the careful balancing of nonlinear and dispersive effects (an equation is said to be dispersive if waves of different wavelengths travel at different speeds). It would not be before the advent of the Boussinesq equation, introduced in [Boussinesq, 1872], and the famous Korteweg–de Vries (KdV) equation, introduced in [Korteweg and De Vries, 1895], that Russell's solitary wave would be mathematically explained. Both the Boussinesq and KdV equations are simplified (from the Euler equations) nonlinear model equations for water waves on shallow water. More on the history of these equations can be found in e.g. [Bullough and Caudrey, 1995].

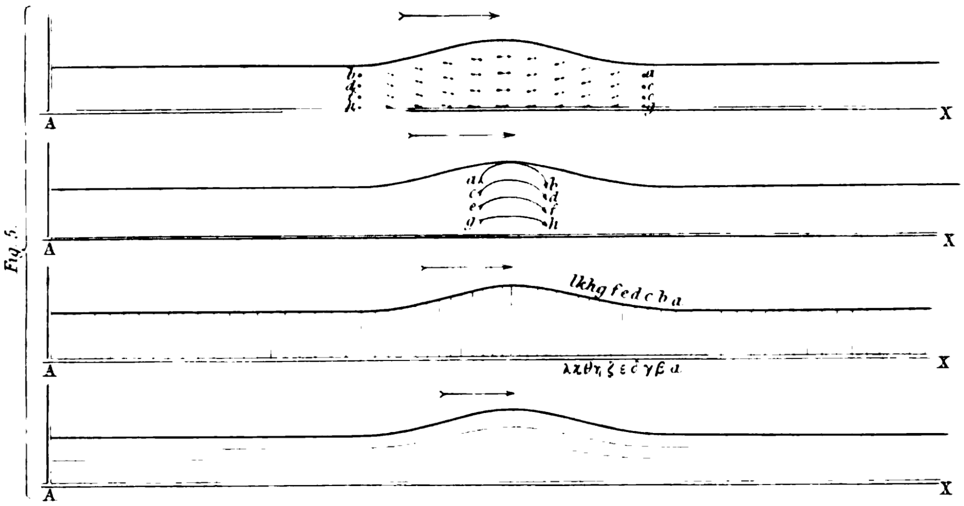


Figure 1.1: Some of Russell's drawings of his wave. Reproduced from a scan of the public domain report [Russell, 1844] on [Google Books]. A few of the lines in the drawing have disappeared, but the wave itself is clearly visible.

One of the biggest names in the early mathematical theory of water waves is Stokes, who in fluid dynamics is perhaps most well known for [Stokes, 1845], where he rederived[2] the equations for viscous flow that we know today as the Navier–Stokes equations. His most significant contribution to the theory of water waves was [Stokes, 1847], where he describes the nonlinear waves that are now known as Stokes waves. Stokes waves are symmetric, periodic, irrotational, traveling gravity-driven water waves that have exactly one trough and crest per minimal period, and which are monotone between adjacent crests and troughs. In the paper, Stokes gives an approximation for such waves with small amplitude, by the means of perturbation. His illustration of such a wave, including terms up to third order in the amplitude, can be seen in Figure 1.2.

In [Stokes, 1847], Stokes also deduced that there is a net forward mass transport in Stokes waves, called Stokes drift. This was unexpected, because the particle paths to the first order in linear waves are closed ellipses[3] (which had been known for some time). If this was true in general, the net mass transport would of course always be zero.

[2]Navier had derived the equations two decades prior, but Stokes did so in more generality.

[3]It would not be until [Constantin and Villari, 2008] that it would be shown that, in fact, they are not closed for the linear waves either! (Although it should be mentioned that the situation is not quite so clear-cut, see [Ehrnström and Villari, 2009] for an overview.)

Figure 1.2: Illustration by Stokes from [Stokes, 1847], from a scan on [The Internet Archive]. The illustration is accompanied by this explanation: "The following figure represents a vertical section of the waves propagated along the surface of deep water. The figure is drawn for the case in which $a = \frac{7\lambda}{80}$. The term of third order in (27) is retained, but it is almost insensible. The straight line represents a section of the plane of mean level." (Stokes, 1847)

Stokes also comments on Russell's solitary wave, expressing his doubts about such waves being able to propagate without change of form, which would not be quenched before Boussinesq's article [Boussinesq, 1872], mentioned above.

It would be years before Stokes would publish anything more significant on the subject of water waves; in fact, everything he wrote on fluid flow is limited to the time intervals 1842–1850 and 1880-1898 (between which he worked mainly on optics). In [Stokes, 1880–1905], Stokes' articles written in the period 1880–1905 are collected. Contained within is the introduction of the stream function; only the velocity potential had been used up to that point. If the velocity field is denoted by (u, v), then the stream function ψ and the velocity potential φ are scalar functions such that

$$\begin{aligned} u &= \varphi_x = -\psi_y \\ v &= \varphi_y = \psi_x. \end{aligned}$$

He uses the stream function and velocity potential together to establish a hodograph transform, essentially using the stream function and velocity potential as independent variables. This enabled him to give more terms for the expansion from [Stokes, 1847]. Also contained in the collection is an 11-page appendix to his 1847-article. There, he famously describes what is now known as the *Stokes conjecture*:

> This however leaves untouched the question whether the disturbance can actually be pushed to the extent of yielding crests with sharp edges, or whether on the other hand there exists a limit, for which the outline is still a smooth curve, beyond which no waves of the oscillatory irrotational kind can be propagated without change of form.
>
> After careful consideration I feel satisfied that there is no such earlier limit, but that we may actually approach as near as we please to the form in which the curvature at the vertex becomes infinite, and the vertex becomes a multiple point where the two branches with which alone we are concerned enclose an angle of 120°. But whether in the limiting form the inclination of the wave to the horizon continually increases from the trough to the summit, and is consequently limited to 30°, or whether on the other hand the points of inflexion which the profile presents in the general case remain at a finite

> distance from the summit when the limiting form is reached, so that on passing from the trough to the summit the inclination attains a maximum from which it begins to decrease before the summit is reached, is a question which I cannot certainly decide, though I feel little doubt that the former alternative represents the truth.
>
> (Stokes, 1880)

The Stokes conjecture concerns what Stokes calls the *wave of greatest height* (when the wavelength is fixed). In an elegant way, he argues that if such a wave exists, it should have sharp crests with an interior angles of 120° (Rankine had earlier argued that the angle was 90°), and that there is a stagnation point (meaning a point where a fluid particle is stationary with respect to the frame moving with the wave) at the crest. Then he conjectures, quoted above, that the wave should be convex between crests, and that it should be a limit of a family of smooth waves of lesser height. We will return to this conjecture below.

More detailed accounts on the fascinating history of research on water waves can be found in the article [Craik, 2004], and the article [Craik, 2005] for material on Stokes in particular. We now turn to more recent events.

1.2 Irrotational waves

A non-dimensionalized version of the KdV equation is

$$\partial_t \eta + \partial_x^3 \eta + 6\eta \partial_x \eta = 0, \tag{1.1}$$

where η represents the elevation of the water surface (in one spatial dimension). Equation (1.1) admits solutions of the form

$$\eta(x,t) = \frac{c}{2} \operatorname{sech}^2 \left(\frac{\sqrt{c}}{2}(x - ct - \alpha) \right),$$

where $c > 0$ is the wave speed, and $\alpha \in \mathbb{R}$ is a phase shift; a simple derivation of which can be found in [Evans, 2010, 4.2.1b]. These solutions look very much Russell's solitary wave in Figure 1.1, indeed, and are the prototypical examples of solitons. The KdV equation exhibits true soliton behavior, in that it in some respects behaves like a linear equation; solitons that collide stay unchanged, except for a change in phase. The equation is also remarkable because it is completely integrable[4], and the general initial value problem can be solved by the means of inverse scattering (pioneered in [Gardner et al., 1967, 1974]). For this reason, the equation (and similar equations, like the nonlinear Schrödinger equation) has attracted much attention over the years. The Stokes wave analog for the KdV equation are the *cnoidal waves*, so named because they can be expressed in terms

[4]The concept is technical, and there seems to be no single agreed-upon definition. For the KdV equation it manifests itself with the existence of a so-called Lax pair formulation, and infinitely many conservation laws (for instance momentum and energy), [Miura et al., 1968].

of the Jacobi elliptic function $\operatorname{cn}(\cdot, k)$. They approach the sech^2-soliton in the limit of infinite wavelength, see [Whitham, 1999, p. 470].

Although the Boussinesq and KdV equations exhibited solitary waves, like the one observed by Russell, this did not settle the question of existence of solitary wave solutions for the full Euler equations (on finite depth). The question for gravity waves was answered in the affirmative in [Friedrichs and Hyers, 1954], while existence was shown for gravity-capillary waves in the presence of large surface tension (strictly above a critical value) in [Amick and Kirchgässner, 1989, Sachs, 1991]. For surface tension strictly below the critical value, but sufficiently large, solitary waves do not exist, as was shown in [Sun, 1999]. The question for the remaining values of the surface tension is still open, although the result by Sun is thought to generalize. We mention that the solitary waves for the full Euler equation are not believed to exhibit the true soliton behavior of the KdV equation, see [Olver, 1982, 1983].

Much work has been done on Stokes waves since the time of Stokes, and his Stokes conjecture has spurred a fair share of this. A more "modern" way[5] of proving the existence of Stokes waves, or the properties of such waves, is through Nekrasov's integral equation, which was introduced in [Nekrasov, 1921]. It is possible to show that there is a correspondence between Stokes waves and odd continuous functions $\theta\colon [-\pi, \pi] \to [0, \pi/2)$ satisfying

$$\theta(t) = \frac{1}{3\pi} \int_0^\pi \log\left(\left| \frac{\sin((t+s)/2)}{\sin((t-s)/2)} \right| \right) \frac{\sin(\theta(s))}{\mu^{-1} + \int_0^s \sin(\theta(r))\, dr}\, ds, \quad t \in [-\pi, \pi] \tag{1.2}$$

for some constant $\mu > 0$ (note that such functions necessarily satisfy $\theta(0) = \theta(\pi) = 0$). The function θ will then represent the angle between the surface and the horizontal in one minimal period of a Stokes wave, the crest corresponding to $t = 0$, after a conformal map to the unit circle. We mention that there is no known (explicit) nontrivial solution of Equation (1.2).

From this point, K will denote a particular closed convex cone in the Banach space of continuous functions $[-\pi, \pi] \to \mathbb{R}$, whose exact definition shall not be important to us. In [Krasovskiĭ, 1961], abstract theory for operators on cones was used to establish a set of solutions (μ, θ) to Equation (1.2) in $(0, \infty) \times K$, which was such that it contained solutions with angles up to, but not including, $\pi/6$ ($30°$). This naturally led Krasovskiĭ to conjecture that Equation (1.2) admits no solutions taking values greater than $\pi/6$, but this was shown to be wrong in [McLeod, 1997]. A proper upper bound for solutions of Equation (1.2) is $31.14°$, proven in [Amick, 1987]. Global bifurcation theory due to [Rabinowitz, 1971, Dancer, 1973] was later used in [Keady and Norbury, 1978] to deduce that there is a connected set C with the same properties as the one found by Krasovskiĭ.

An important result came with[6] [Toland, 1978], where it was shown that there is a

[5]Another recent approach, which typically yields stronger results, uses a nonlinear equation involving the so-called periodic Hilbert transform, due to [Babenko, 1987].

[6]The 1979 preprint version of [McLeod, 1997], where the same results were independently obtained, should also be mentioned here.

sequence $(\mu_n, \theta_n)_{n\in\mathbb{N}} \subseteq C$ which is such that

$$\mu_n \to \infty, \quad \max_{t\in[0,\pi]} \theta_n(t) \to \frac{\pi}{6},$$

and such that $(\theta_n)_{n\in\mathbb{N}}$ has a pointwise limit to a function θ that is continuous except at the origin, satisfying Equation (1.2) with $\mu = \infty$. Along with some other technical details in the article, this came tantalizingly close to proving part of the Stokes conjecture, but it would not be before [Amick et al., 1982] (the "et al." including Toland) that it would finally be proved that

$$\lim_{t\downarrow 0} \theta(t) = \frac{\pi}{6}.$$

The only remaining part of the Stokes conjecture was then the convexity of the extreme surface, corresponding to θ being monotone on $(0, \pi]$. This was finally settled in the affirmative with [Plotnikov and Toland, 2004], 124 years after Stokes made his conjecture.

A great survey on Stokes waves is [Toland, 1996], where condensed proofs of many of the main results about Stokes waves and the Stokes conjecture up to that point can be found (the proof of convexity came later). Mild assumptions lead to Stokes waves; it is for instance not necessary to assume that the surface is the graph of a function, and the surface profile is automatically real analytic. Another source is [Buffoni and Toland, 2003, Part 4], which also uses Babenko's equation in addition to Equation (1.2). An interesting question connected to the Stokes conjecture that is still open is the uniqueness of the limiting function θ, the wave of greatest height. It is known that the set of such functions is connected in an appropriate space ([Buffoni and Toland, 2003, Remark 11.2.3]).

1.3 Rotational waves

Everything we have discussed up to this point has been on the topic of irrotational waves, meaning waves where the curl of the velocity field is zero. This assumption implies that the velocity field is the gradient of a harmonic function (the velocity potential). Together with the stream function, this enables the use of many powerful results on harmonic functions and complex analysis. Examples of such tools are the maximum principle and the Hopf boundary point lemma for harmonic functions, [Gilbarg and Trudinger, 2001, Lemma 3.4, Theorem 3.5]. Because of this, most mathematical research on water waves has been on the subject of irrotational waves until fairly recently, and as such the state of knowledge on rotational waves is still in its infancy when compared with that on irrotational waves.

There are several situations where rotational waves are appropriate. Due to Kelvin's circulation theorem, flows that are initially irrotational will remain so for all time, as long as they are only affected by conservative forces (e.g. gravity). However, effects such as wind, temperature or salinity gradients can all induce rotation. Irrotational waves are also poor analogs for rotational waves. For instance, in rotational waves it is possible to have internal stagnation points and critical layers enclosing areas of closed streamlines known as so-called cat's eye vortices, see Figure 1.3. The term cat's eye was used by

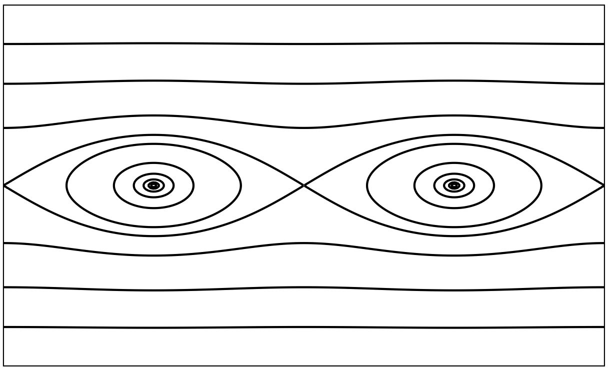


Figure 1.3: Cat's eye vortices with point vortices at the center. Kelvin commented on a similar diagram with "For this case the stream-lines are as represented in the annexed diagram, in which the region of translational velocity greater than wave-propagational velocity is separated from the region of translational velocity less than wave-propagational velocity by a cat's eye border pattern of elliptic whirls" (Kelvin, 1880)

Kelvin (at the time, his name was Thomson) in [Kelvin, 1880], where he was commenting on a stream function found by Rayleigh.

The first result on rotational waves came surprisingly early, already in the beginning of the 1800s with [Gerstner, 1809]. There, Gerstner gave[7] the first, and still the only known explicit (nontrivial) gravity-wave solutions to the Euler equations on infinite depth. The simplest description of the wave is a Lagrangian one (i.e., one that is focused on the individual particles), where the particles follow circles of radius exponentially decreasing with depth. This exponential decay is also enjoyed by the vorticity of the wave (it was Stokes that later realized that the wave was rotational). While significant because it is an exact solution, it is viewed as more of a mathematical curiosity, even today. This is partly because it is rotational, but there are also other reasons for this; see for instance [Constantin, 2011, Chapter 4.3]. In an appendix to [Stokes, 1847] in [Stokes, 1880–1905], Stokes expresses his objections stemming from the lack of net particle drift (which he had observed in Stokes waves): "...for deep water the absence of progressive motion is doubtless peculiar to the former case [of no progressive motion] ..." (Stokes, 1880). A good overview of Gerstner's wave can be found in [Henry, 2008].

Much later, with [Dubreil-Jacotin, 1934] (which was the author's doctoral dissertation), came the first existence result for small-amplitude waves with quite general vorticity distributions. A vorticity distribution is a function $\gamma\colon \mathbb{R} \to \mathbb{R}$ such that

$$\Delta\psi = \gamma \circ \psi, \tag{1.3}$$

where ψ is the relative stream function (which is still available for rotational waves, but is not harmonic). A sufficient, but not necessary, condition for such a vorticity distribution to exist is that the wave has no stagnation points. Several improvements have been

[7] The same solution was later independently rediscovered by Rankine in [Rankine, 1863]. Gerstner's original result was mostly overlooked.

made to the existence result of Dubreil-Jacotin, but it was not until the pioneering article [Constantin and Strauss, 2004] that large waves were constructed, using global bifurcation theory. This renewed interest in rotational waves. In both [Dubreil-Jacotin, 1934] and [Constantin and Strauss, 2004], a semi-hodograph transform was used to transfer the free boundary problem to one on a rectangle.

Proving the existence of rotational waves is typically done with a functional-analytic approach, using bifurcation theory. [Kozlov and Kuznetsov, 2011] provides flat-surface *streams* with general vorticity distributions, from which bifurcation can occur[8]. Different ways of transforming the Euler equations have been utilized, each with their own benefits, depending on what the goal is. Because of the transform used in the two previously mentioned existence results by Dubreil-Jacotin and Constantin, the resulting waves do not exhibit stagnation or critical layers.

The first existence result for waves with a critical layer was [Wahlén, 2009], where small-amplitude waves with constant vorticity were constructed, using a different transformation. This was followed by [Constantin and Varvaruca, 2011], which instead used a generalization of Babenko's equation (see Footnote 5) to waves with constant vorticity. In addition to being simpler, the newer approach has the advantage that it is amenable to global bifurcation theory, and allows for waves with overhanging surface profiles (there is numerical evidence for the existence of such waves, e.g. [Vanden-Broeck, 1996], but this is still an open problem). A natural generalization of the existence results for constant-vorticity waves with stagnation is of course for waves with an affine vorticity distribution. Existence for small-amplitude waves was established with the articles [Ehrnström et al., 2011, Ehrnström and Wahlén, 2013], also allowing for bi- and trimodal waves (meaning waves with surface including two or three Fourier modes in the leading order).

Recently, spurred by the above results, there has also been much interest in studying the properties of these waves below the surface. Linear waves with constant vorticity were considered in [Ehrnström and Villari, 2008], where it was shown that for certain values of the vorticity, cat's eye vortices appear. The particle paths were also considered. Later, the result was extended to the full Euler equations with [Wahlén, 2009], confirming the features found for the linear waves. In [Ehrnström et al., 2012], the interior dynamics of the waves constructed in [Ehrnström et al., 2011] (see above) was examined. The solutions were found to admit arbitrarily many critical layers.

Common for all the previously mentioned works is the feature that the vorticity is supported on the entire fluid domain, and that they are gravity waves. Recently, gravity-capillary waves with compactly supported vorticity were constructed in [Shatah et al., 2013], on infinite depth. This includes both periodic and solitary small-amplitude waves with either a point vortex (which is the simplest possible compactly supported vorticity) or a vortex patch, and global bifurcation theory is also used for the periodic waves with a point vortex. In this thesis, we aim to extend the existence result for small-amplitude solitary waves with a point vortex to finite depth.

[8]This also indicates that two similar surface profiles can have wildly different fluid behavior beneath them when rotation is allowed.

2 Notation and conventions

Given a normed space X, we will use the notation

$$\|\cdot\|_X$$

for the norm on X. The exception is the standard euclidean norm, which we denote by just $|\cdot|$. The inner product on an inner product space will be written as $\langle \cdot, \cdot \rangle_X$, and the metric on a metric space as $d_X(\cdot, \cdot)$. Similarly, we use the notation

$$|\cdot|_X$$

for a seminorm on a vector space X. An open ball of radius r centered at $x_0 \in X$ in a metric space will be denoted by $B_r(x_0, X)$, or just $B_r(x_0)$ if the underlying space is understood. The notation $B(X, Y)$ will be used for the space of bounded linear operators between normed spaces X and Y. For bounded multilinear operators (for instance higher order Fréchet derivatives) $X^k \to Y$, we use $B^k(X, Y)$.

If A and B are subsets of a topological space, we will write

$$A \Subset B,$$

read as "A compactly contained in B", when the closure of A is compact and contained in the interior of B. We will also use $\sqcup$ to indicate a disjoint union. Typically, we will use the blackboard bold letter $\mathbb{K}$ (from German *Körper* for field) to denote either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{K}$. By *domain* we mean an open, connected set, which is not necessarily bounded. The natural numbers $\mathbb{N}$ are the positive integers, and the notation $\mathbb{N}_0$ means $\mathbb{N} \cup \{0\}$.

Partial derivatives will be denoted by $\partial_x f$, f_x or $D^\alpha f$, depending on the context. The latter is the multi-index notation: If $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, then we define

$$|\alpha| \coloneqq \sum_{i=1}^{d} \alpha_i, \quad \alpha! \coloneqq \prod_{i=1}^{d} \alpha_i!, \quad x^\alpha \coloneqq \prod_{i=1}^{d} x_i^{\alpha_i},$$

and the differential operator

$$D^\alpha \coloneqq \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}}.$$

The differential operator $\nabla^\perp$ is defined by

$$\nabla^\perp f \coloneqq (-\partial_y f, \partial_x f)$$

for functions f defined on $\mathbb{R}^2$. We mention that some authors use the opposite sign convention. For Fréchet derivatives we simply use D, or D with subscripts indicating partial derivatives. If X, Y are Banach spaces and $U \subseteq X$ is open, we write $C^k(U, Y)$ for the set of functions $U \to Y$ whose (Fréchet) derivatives up to order k exist and are continuous. In addition, we define the subspace

$$C_c^k(U, Y) \coloneqq \{f \in C^k(U, Y) : \operatorname{supp} f \Subset U\}$$

of such functions with compact support, and the Banach space (see Lemma A.3 in Appendix A)

$$BC^k(U, Y) \coloneqq \left\{ f \in C^k(U, Y) : \|f\|_{BC^k(U,Y)} < \infty \right\},$$

where ($B^0(X, Y)$ is identified with Y)

$$\|f\|_{BC^k} \coloneqq \sum_{j=0}^{k} \sup_{x \in U} \|D^j f(x)\|_{B^j(X,Y)}. \tag{2.1}$$

The Lebesgue spaces $L^p(X)$ for $p \in [1, \infty]$ and a measure space (X, Σ, μ) are the spaces

$$L^p(X) \coloneqq \{f \colon X \to \mathbb{K} : f \text{ is measurable}, \|f\|_{L^p(X)} < \infty\},$$
$$\|f\|_{L^p(X)} \coloneqq \begin{cases} \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} & p \in [1, \infty) \\ \operatorname{ess\,sup} |f| & p = \infty \end{cases},$$

with μ-a.e. equal functions identified and $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, depending on the context. These L^p-spaces are Banach spaces, [Folland, 1999, Theorems 6.6, 6.8], and Hilbert spaces if $p = 2$. Local versions of the Lebesgue spaces are introduced through

$$L^p_{\mathrm{loc}}(X) \coloneqq \{f \colon X \to \mathbb{K} : f|_K \in L^p(K) \text{ for } K \Subset X\}$$

when X has a topology. These local spaces ignore behavior at infinity.

If X, Y are metric spaces, we denote the space of functions $f \colon X \to Y$ for which the Lipschitz seminorm defined by

$$|f|_{\mathrm{Lip}(X,Y)} \coloneqq \sup_{\substack{x,y \in X \\ x \neq y}} \frac{d_Y(f(x), f(y))}{d(x, y)}$$

is finite by $\mathrm{Lip}(X, Y)$. For the Fourier transform, we use the convention

$$(\mathscr{F} f)(\xi) \coloneqq (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} \, dx$$

for integrable functions defined on $\mathbb{R}^d$, later extended to distributions (Equation (3.2)).

3 Preliminaries

3.1 Sobolev spaces

We denote the space of Schwartz functions on $\mathbb{R}^d$, $d \geq 1$, by $S(\mathbb{R}^d)$. Those are functions $f \in C^\infty(\mathbb{R}^d, \mathbb{C})$ whose derivatives of all orders are of rapid decay, or more precisely

$$|f|_{S(\mathbb{R}^d),\alpha,\beta} \coloneqq \sup_{x\in\mathbb{R}^d} |x^\alpha (D^\beta f)(x)| < \infty \tag{3.1}$$

for all multi-indices $\alpha, \beta \in \mathbb{N}_0^d$. The countable family of seminorms defined in Equation (3.1) makes $S(\mathbb{R}^d)$ a Fréchet space, and the space $S'(\mathbb{R}^d)$ of continuous linear functionals on $S(\mathbb{R}^d)$ is known as the tempered distributions on $\mathbb{R}^d$. The topology on $S'(\mathbb{R}^d)$ is the weak* topology, i.e., pointwise convergence. If a distribution $T \in S'(\mathbb{R}^d)$ is of the form

$$\langle T, \varphi\rangle \coloneqq \int_{\mathbb{R}^d} f\varphi \, d\mu$$

for some $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, then T is said to be a regular distribution. It is conventional to use the notation $\langle T, \varphi\rangle$ instead of $T(\varphi)$, making equations such as Equation (3.2) more suggestive, and also to drop the distinction between a regular distribution T and the function f defining it. The distributions that are not regular are called singular distributions.

Many of the operations that can be performed on functions can be performed on tempered distributions, by defining the operations to extend identities that hold for sufficiently regular distributions. For instance, defining

$$\langle \mathscr{F} f, \varphi\rangle \coloneqq \langle f, \mathscr{F}\varphi\rangle, \quad f \in S'(\mathbb{R}^d), \varphi \in S(\mathbb{R}^d), \tag{3.2}$$

extends the Fourier transform to all tempered distributions. We shall assume some degree of familiarity with distributions.

The usual Sobolev spaces on $\mathbb{R}^d$ are the subspaces

$$H^s(\mathbb{R}^d) \coloneqq \{f \in S'(\mathbb{R}^d) : \langle\cdot\rangle^s \mathscr{F} f \in L^2(\mathbb{R}^d)\}, \quad s \in \mathbb{R}, \tag{3.3}$$

of the tempered distributions, and are Hilbert spaces in the norms

$$\|f\|_{H^s(\mathbb{R}^d)} \coloneqq \|\langle\cdot\rangle^s \mathscr{F} f\|_{L^2(\mathbb{R}^d)}.$$

Here $\langle\cdot\rangle \in C^\infty(\mathbb{R}^d, (0,\infty))$ is the *Japanese bracket* defined through $x \mapsto \sqrt{1+|x|^2}$. The bracket notation is useful in simplifying expressions, especially in the context of Sobolev spaces.

It may not be immediately clear from Equation (3.3) what distributions in $H^s(\mathbb{R}^d)$ "look like". Note that it is not hard to check that if $-\infty < s \le t < \infty$, then

$$H^t(\mathbb{R}^d) \hookrightarrow H^s(\mathbb{R}^d),$$

where $\hookrightarrow$ denotes a continuous embedding. The Sobolev space $H^0(\mathbb{R}^d)$ coincides with $L^2(\mathbb{R}^d)$ since the Fourier transform is an isometry on $L^2(\mathbb{R}^d)$ by Plancherel's theorem, and the functions in $H^s(\mathbb{R}^d)$ become more and more regular as s increases from 0. For instance, we have

$$H^s(\mathbb{R}^d) \hookrightarrow BC^{\lceil s-\frac{d}{2}\rceil-1}(\mathbb{R}^d) \tag{3.4}$$

for any $s > \frac{d}{2}$. This must of course be understood in the sense that every $f \in H^s(\mathbb{R}^d)$ can be represented by a function in $BC^{\lceil s-\frac{d}{2}\rceil-1}$. Hence, as an example, every $f \in H^{\frac{1}{2}+\varepsilon}(\mathbb{R})$ can be represented by a continuous and bounded function when $\varepsilon > 0$. That Equation (3.4) holds is Theorem A.5 (see Appendix A), which we have included a proof of because it is both instructive and quite simple to prove. This behavior of increasing regularity reflects the way that the Fourier transform converts regularity into integrability, and conversely.

For negative s, the situation is the opposite. In fact, one can prove that

$$H^s(\mathbb{R}^d) \subseteq L^1_{\mathrm{loc}}(\mathbb{R}^d) \iff s \ge 0,$$

see e.g. [Runst and Sickel, 1996, 2.2.4 Theorem 2]. In other words, Sobolev spaces of negative order always contain singular distributions, and they allow for "more singular" distributions as s decreases. One may check that $\delta \in H^s(\mathbb{R}^d)$ if $s < -\frac{d}{2}$, where δ denotes the Dirac delta distribution defined by

$$\langle \delta, \varphi\rangle \coloneqq \varphi(0), \quad \varphi \in S(\mathbb{R}^d).$$

These sharp, dimension-dependent barriers on the regularity parameter s are typical of Sobolev spaces.

It is not hard to see that every $f \in H^{s-1}(\mathbb{R})$ is *not* the derivative of some $\tilde{f} \in H^s(\mathbb{R})$. For instance, the function defined by

$$x \mapsto x\langle x\rangle^{-2}$$

has Fourier transform

$$\xi \mapsto -i\sqrt{\frac{\pi}{2}}\,\mathrm{sgn}(\xi)e^{-|\xi|},$$

and so is a member of $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$. However,

$$x\langle x\rangle^{-2} = (\log\langle x\rangle)',$$

and $x \mapsto \log\langle x\rangle$ is not in $L^2(\mathbb{R}) = H^0(\mathbb{R})$ (and neither is any other antiderivative), being unbounded. We therefore introduce the spaces

$$\partial_x H^s(\mathbb{R}) = \{f \in H^{s-1}(\mathbb{R}) : \text{There exists } \tilde{f} \in H^s(\mathbb{R}) \text{ such that } f = \tilde{f}'\},$$

with the norms

$$\|f\|_{\partial_x H^s(\mathbb{R})} \coloneqq \|\tilde{f}\|_{H^s(\mathbb{R})} \geq \|f\|_{H^{s-1}(\mathbb{R})}.$$

Note that this is well defined, because $\tilde{f}$ is certainly unique given f (constants have singular Fourier transforms). The reason for introducing a new norm is the following proposition.

Proposition 3.1. *The norm $\|\cdot\|_{\partial_x H^s(\mathbb{R})}$ makes $\partial_x H^s(\mathbb{R})$ a Hilbert space.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\partial_x H^s(\mathbb{R})$. Then it is immediate from the definition of the norm on $\partial_x H^s(\mathbb{R})$ that $(\tilde{f}_n)_{n\in\mathbb{N}}$ is Cauchy in $H^s(\mathbb{R})$. Hence there exists a function $\tilde{f} \in H^s(\mathbb{R})$ such that $\tilde{f}_n \to \tilde{f}$ in $H^s(\mathbb{R})$. But then $f_n \to \tilde{f}'$ in $\partial_x H^s(\mathbb{R})$. □

Remark. We emphasize that $\partial_x H^s(\mathbb{R})$ is in general not a closed subspace in the standard topology on $H^{s-1}(\mathbb{R})$. This can be seen by considering the sequence $(f_n)_{n\in\mathbb{N}}$ in $\partial_x H^1(\mathbb{R})$, defined by

$$f_n(x) = x\langle x\rangle^{-(5/2+1/n)}, \quad \text{with } \tilde{f}_n(x) = -(1/2+1/n)^{-1}\langle x\rangle^{-(1/2+1/n)}.$$

The sequence converges in $H^0(\mathbb{R}) = L^2(\mathbb{R})$ to the function f defined by $f(x) = x\langle x\rangle^{-5/2}$, but $f \notin \partial_x H^1(\mathbb{R})$ since $\tilde{f}(x) = -2\langle x\rangle^{-1/2}$ is not square integrable.

There are also other ways to deal with this complication of antiderivatives not lying in a Sobolev space. In addition to the standard Sobolev spaces, we will have use for the Beppo–Levi (or homogeneous Sobolev) spaces $\dot{H}^s(\mathbb{R}^d)$, which are defined by[1]

$$\dot{H}^s(\mathbb{R}^d) \coloneqq \{f \in L^2_{\mathrm{loc}}(\mathbb{R}^d)/\mathbb{R} : \nabla f \in H^{s-1}(\mathbb{R}^d)^d\}. \tag{3.5}$$

for $s \geq 1$. By $L^2_{\mathrm{loc}}/\mathbb{R}$ we mean that functions in L^2_{loc} that differ by constants are identified. The spaces defined in Equation (3.5) are also Hilbert spaces, when endowed with the norm

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} \coloneqq \|\nabla f\|_{H^{s-1}(\mathbb{R}^d)^d} = \left(\sum_{i=1}^d \|\partial_{x_i} f\|^2_{H^{s-1}(\mathbb{R}^d)}\right)^{\frac{1}{2}},$$

a proof of which can be found in [Lannes, 2013, Proposition 2.3]. There are situations where one is only interested in derivatives, and where the function itself may not necessarily be a member of H^s. We have already seen an example of this above. The function $f \in L^2_{\mathrm{loc}}(\mathbb{R})$ defined by $f(x) \coloneqq -2\langle x\rangle^{-1/2}$ is *not* in $L^2(\mathbb{R})$, whereas its derivative $x \mapsto x\langle x\rangle^{-5/2}$ is. Hence f is a member of $\dot{H}^1(\mathbb{R})$, but f is not in $H^1(\mathbb{R})$.

It can be useful to know that the Sobolev space $H^s(\mathbb{R}^d)$ coincides with the Sobolev space $W^s_2(\mathbb{R}^d)$, the Besov space $B^s_{2,2}(\mathbb{R}^d)$ and the Triebel-Lizorkin space $F^s_{2,2}(\mathbb{R}^d)$ ([Adams and Fournier, 2003, 7.62, 6.7]), because results are often proved in a more general setting in the literature. Some care should be taken, however, because there are many ways to define these spaces.

[1]These spaces are not standardized, and other authors may define them in other nonequivalent ways. Examples are the books [Grafakos, 2009, Bahouri et al., 2011], where they are defined in two other ways. Common for all the definitions is that integrability is sacrificed in some way, and that one typically consider only equivalence classes of distributions.

3.2 Sobolev spaces on open sets

There are several ways to define Sobolev spaces on open sets $\Omega \subseteq \mathbb{R}^d$. Of course, the Schwartz spaces are no longer available on such sets, and so there is no such thing as tempered distributions on them. Therefore, the space $D(\Omega) := C_c^\infty(\Omega, \mathbb{R})$ of smooth functions with compact support will act as the test functions instead. The topology that is usually put on $D(\Omega)$ is not as simple to describe (see [Rudin, 1991, Definition 6.3]) as that of $S(\mathbb{R}^d)$, but it turns out that for our purposes it is sufficient to know what the convergent sequences are. A sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges to a function φ in $D(\Omega)$ if and only if there exists a compact set $K \subseteq \Omega$ such that $\operatorname{supp}\varphi_n \subseteq K$ for all $n \in \mathbb{N}$ and $(D^\alpha \varphi_n)_{n\in\mathbb{N}}$ converges uniformly to $D^\alpha\varphi$ for every multi-index $\alpha \in \mathbb{N}_0^d$. A linear functional on $D(\Omega)$ is continuous if and only if it is sequentially continuous ([Rudin, 1991, Theorem 6.6]), and the space $D'(\Omega)$ of such functionals are known as the distributions on Ω. As with $S'(\mathbb{R}^d)$, one imposes the weak* topology on $D'(\Omega)$.

For nonnegative integers $k \in \mathbb{N}_0$ and any open set $\Omega \subseteq \mathbb{R}^d$, a transparent definition of the Sobolev space $H^k(\Omega)$ is

$$H^k(\Omega) := \{f \in D'(\Omega) : D^\alpha f \in L^2(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^d \text{ such that } |\alpha| \le k\}, \tag{3.6}$$

with the norm

$$\|f\|_{H^k(\Omega)} := \left(\sum_{i=0}^{k} |f|_{H^i(\Omega)}^2\right)^{\frac{1}{2}}, \tag{3.7}$$

where $|\cdot|_{H^i(\Omega)}$ are the Sobolev seminorms on $H^i(\Omega)$ defined by

$$|f|_{H^i(\Omega)}^2 = \sum_{\alpha\in\mathbb{N}_0^d : |\alpha|=i} \|D^\alpha f\|_{L^2(\Omega)}^2.$$

The spaces defined in Equation (3.6) are Hilbert spaces, a simple proof of which can be found in [Adams and Fournier, 2003, Theorem 3.3]. Furthermore, if $\Omega = \mathbb{R}^d$, then the spaces defined in Equation (3.6) are the same as those defined in Equation (3.3), and they have equivalent norms ([Adams and Fournier, 2003, 7.62]). This justifies using the same notation for them, which should cause no confusion.

The situation for Sobolev spaces with nonintegral exponents is more complicated; in [Adams and Fournier, 2003] they are defined as interpolation spaces, and there are also various other approaches. A particularly simple way to introduce them is through restrictions, which is the method that we will use. Specifically, we define

$$\tilde{H}^s(\Omega) := \{ f|_\Omega \in D'(\Omega) : f \in H^s(\mathbb{R}^d)\}, \quad s \in \mathbb{R}^d, \tag{3.8}$$

with the norm

$$\|f\|_{\tilde{H}^s(\Omega)} := \inf\{\|g\|_{H^s(\mathbb{R}^d)} : g \in H^s(\mathbb{R}^d),\, g|_\Omega = f\}.$$

By identifying the space $\tilde{H}^s(\Omega)$ with the quotient space

$$H^s(\mathbb{R}^d)/\{f \in H^s(\mathbb{R}^d) : \operatorname{supp} f \subseteq \mathbb{R}^d \setminus \Omega\}$$

in the natural way, $\tilde{H}^s(\Omega)$ is seen to be a Hilbert space by Proposition 4.3. Furthermore, we have the following proposition, which yields a condition under which $H^k(\Omega)$ and $\tilde{H}^k(\Omega)$ coincide.

Proposition 3.2 (Equivalence of Sobolev space definitions)**.** *If Ω is an open set, $k \in \mathbb{N}_0$, and there exist bounded linear extension operators*

$$S\colon H^k(\Omega) \to H^k(\mathbb{R}^d), \quad T\colon \tilde{H}^k(\Omega) \to H^k(\mathbb{R}^d),$$

then $H^k(\Omega) = \tilde{H}^k(\Omega)$, and their norms are equivalent.

Proof. Suppose first that $f \in H^k(\Omega)$. Then $Sf \in H^k(\mathbb{R}^d)$, so its restriction $(Sf)|_\Omega = f$ is a member of $\tilde{H}^k(\Omega)$ by definition. Also, one has

$$\|f\|_{\tilde{H}^k(\Omega)} \leq \|Sf\|_{H^k(\mathbb{R}^d)} \leq \|S\|_{B(H^k(\Omega),H^k(\mathbb{R}^d))} \|f\|_{H^k(\Omega)},$$

where the first inequality stems from Sf extending f, and the norm in $\tilde{H}^k(\Omega)$ being the infimum over all such extensions. Similarly, if $f \in \tilde{H}^k(\Omega)$, then $Tf \in H^k(\mathbb{R}^d)$, and again this means that the restriction $(Tf)|_\Omega = f$ is in $H^k(\Omega)$. Moreover,

$$\|f\|_{H^k(\Omega)} \leq \|Sf\|_{H^k(\mathbb{R}^d)} \leq \|S\|_{B(\tilde{H}^k(\Omega),H^k(\mathbb{R}^d))} \|f\|_{\tilde{H}^k(\Omega)}. \qquad \square$$

Proposition 3.2 tells us that we should look for some assumption on Ω that guarantees the existence of such extension operators. It turns out that this is intimately connected with the regularity of the boundary of Ω. In particular, it is sufficient to assume that Ω is a domain (open and connected) that satisfies a so-called strong local Lipschitz condition, Definition 3.3. Then the Stein extension theorem, see for instance [Adams and Fournier, 2003, Theorem 5.24], guarantees the existence of a bounded linear extension operator $H^k(\Omega) \to H^k(\mathbb{R}^d)$. A proof that the corresponding extension operator for $\tilde{H}^k(\Omega)$ exists in the same setting can be found in [Rychkov, 1999], where a much more general result is established in the context of Besov and Triebel-Lizorkin spaces.

Definition 3.3 (Strong local Lipschitz condition)**.** A set $\Omega \subseteq \mathbb{R}^d$ is said to satisfy a strong local Lipschitz condition if there exists some real number $\delta > 0$, a locally finite open cover $(U_n)_{n\in\mathbb{N}}$ of $\partial\Omega$ in $\mathbb{R}^d$ and a corresponding bounded sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathrm{Lip}(\mathbb{R}^{d-1}, \mathbb{R})$ such that:

(i) There is some $N \in \mathbb{N}$ such that any intersection of N distinct U_n is empty.

(ii) Up to a rotation and translation,

$$U_n \cap \Omega = \{(x_1, \ldots, x_d) \in U_n : x_d < \varphi_n(x_1, \ldots, x_{d-1})\}$$

for each $n \in \mathbb{N}$.

(iii) If $V_n \coloneqq \{x \in U_n : d(x, \partial U_i) > \delta\}$, $n \in \mathbb{N}$, then for any pair $x, y \in \Omega$ such that $\max\{|x - y|, d(x, \partial\Omega), d(y, \partial\Omega)\} < \delta$ one has $x, y \in V_m$ for some $m \in \mathbb{N}$.

For the case that we will be interested in, see Proposition 6.4, the conditions in Definition 3.3 will turn out to be almost trivial to verify. From now on we will therefore drop the tildes on $\tilde{H}^s(\Omega)$, and just write $H^s(\Omega)$. In a completely analogous fashion to $H^s(\Omega)$, one may also construct the so-called Beppo-Levi spaces $\dot{H}^s(\Omega)$ as restrictions of distributions in $\dot{H}^s(\mathbb{R}^d)$.

The question of traces (restrictions to a manifold of lower dimension) in Sobolev spaces is not a simple one, especially if the goal is to weaken the regularity assumptions as much as possible. If one wishes to make sense of pointwise values of functions in $H^s(\Omega)$, then one must demand that $H^s(\Omega) \subseteq BC(\Omega)$, i.e., that $s > d/2$. The "natural" space for boundary values of $H^s(\Omega)$-functions is $H^{s-1/2}(\partial\Omega)$ (one loses half a derivative), which can be defined on sufficiently regular domains by straightening out the boundary. See for instance [Marschall, 1987], where it is proved that the trace operator is a bounded linear operator $H^s(\Omega) \to H^{s-1/2}(\partial\Omega)$ when the boundary is $C^{\lceil s-\frac{1}{2}\rceil}$. This essentially means that the Lipschitz map in Definition 3.3 is switched out with $C^{\lceil s-\frac{1}{2}\rceil}$-diffeomorphisms. It is also possible to work with weaker assumptions on the boundary: For bounded Lipschitz domains there is a bounded trace operator $H^s(\Omega) \to H^{s-1/2}(\partial\Omega)$ when $1/2 < s < 3/2$ and $H^s(\Omega) \to H^1(\partial\Omega)$ when $s > 3/2$ ([Ding, 1996]).

3.3 Function spaces of periodic functions

In addition to the Sobolev spaces on open subsets of $\mathbb{R}^d$, one may introduce Sobolev spaces on the d-torus, $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, corresponding to Sobolev spaces of periodic functions. If $f \in L^1(\mathbb{T}^d)$, then f has a Fourier transform $\hat{\varphi}\colon \mathbb{Z}^d \to \mathbb{C}$, defined by

$$\hat{f}(k) \coloneqq \int_{\mathbb{T}^d} f(x) e^{-2\pi i k\cdot x} dx$$

for each $k \in \mathbb{Z}^d$. We can extend this to distributions $f \in D'(\mathbb{T}^d)$ in a natural way, by defining

$$\hat{f}(k) \coloneqq \langle f, e^{-2\pi i k\cdot x}\rangle, \quad k \in \mathbb{Z}^d,$$

coinciding with the above definition when f is regular. In the sense of distributions, one has the familiar Fourier series

$$f = \sum_{k\in\mathbb{Z}^d} \hat{f}(k) e^{2\pi i k\cdot x} \coloneqq \lim_{N\to\infty} \sum_{|k|\le N} \hat{f}(k) e^{2\pi i k\cdot x},$$

which can be seen by the identity

$$\left\langle \sum_{|k|\le N} \hat{f}(k) e^{2\pi i k\cdot x}, \varphi \right\rangle = \left\langle f, \sum_{|k|\le N} \hat{\varphi}(k) e^{2\pi i k\cdot x} \right\rangle,$$

and using the fact that the Fourier series for φ, and all its derivatives, converges uniformly. An example of such a Fourier series is

$$\delta = \sum_{k\in\mathbb{Z}^d} e^{2\pi i k\cdot x}.$$

The Sobolev spaces $H^s(\mathbb{T}^d)$ are now introduced in an analogous way to that of $H^s(\mathbb{R}^d)$, by

$$H^s(\mathbb{T}^d) \coloneqq \{f \in D'(\mathbb{T}^d) : \langle\cdot\rangle^s \hat{f} \in \ell^2(\mathbb{Z}^d)\},$$

where $\ell^2(\mathbb{Z}^d)$ is $L^2(\mathbb{Z}^d)$ using counting measure. By putting the norm

$$\|f\|_{H^s(\mathbb{T}^d)} \coloneqq \|\langle\cdot\rangle^s \hat{f}\|_{\ell^2(\mathbb{Z}^d)}$$

on $H^s(\mathbb{T}^d)$, it becomes a Hilbert space. Due to the definitions being so similar, many of the properties of $H^s(\mathbb{R}^d)$ carry over to $H^s(\mathbb{T}^d)$. Sobolev spaces of distributions with other periods can be introduced in the same manner.

3.4 Fourier multipliers

When working in Sobolev spaces on $\mathbb{R}^d$, one often encounters operators that are most naturally defined by how they act on the Fourier transform of a function (or distribution). This is where the concept of Fourier multipliers come in. For convenience, we define the differential operator

$$D \coloneqq \frac{1}{i}\nabla = -i\nabla,$$

where meaning of D should be clear from context[2]. Note that the motivation behind introducing this notation is that

$$\widehat{Df} = \xi\hat{f} \tag{3.9}$$

for tempered distributions, i.e. the D acts as multiplication by the smooth function $\xi \mapsto \xi$ on the Fourier transform.

Suppose that $m \in C^\infty(\mathbb{R}^d, \mathbb{C})$ is a smooth function that is slowly increasing; that is, m and all its derivatives are of at most polynomial growth. Then $S(\mathbb{R}^d)$ is closed under multiplication by m, and hence so is $S'(\mathbb{R}^d)$. If we have a tempered distribution $f \in S'(\mathbb{R}^d)$, we may therefore define a new distribution $m(D)f$ by

$$\widehat{m(D)f} \coloneqq m\hat{f},$$

inspired by Equation (3.9). We may also view $m(D)\colon S'(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ as a linear operator on $S'(\mathbb{R}^d)$, and such operators are known as Fourier multipliers. One may note that such multipliers also make sense with weaker assumptions on m, as long as their domain is restricted. In the context of Sobolev spaces, for instance, the Fourier transform is always regular, meaning that the product $m\hat{f}$ makes sense as a tempered distribution for a wide array of m.

As an example of Fourier multipliers, we may for instance write the norm on Sobolev spaces as

$$\|f\|_{H^s(\mathbb{R}^d)} = \|\langle D\rangle^s f\|_{L^2(\mathbb{R}^d)},$$

[2] This notation is perhaps somewhat unfortunate, since D is already an overloaded letter, but it is quite standard.

after recalling that the Fourier transform is an isometry on $L^2(\mathbb{R}^d)$. Observe also that if $m_1(D), m_2(D)$ are two Fourier multipliers, then

$$\mathscr{F}(m_1(D)m_2(D)f) = m_1 m_2 \hat{f} = m_2 m_1 \hat{f} = \mathscr{F}(m_2(D)m_1(D)f), \quad f \in S'(\mathbb{R}^d),$$

meaning that Fourier multipliers always commute with each other.

3.5 Analytic operators

It is possible to extend the concept of analytic functions on $\mathbb{R}$ or $\mathbb{C}$ to the more general setting of Banach spaces in a quite natural way. Many of the well known properties of analytic functions remains the same; derivatives of analytic operators will be analytic, for instance. One also has analytic versions of the inverse and implicit function theorems, see e.g. [Buffoni and Toland, 2003, Theorems 4.5.3 and 4.5.4].

In the following definition, we use the notation x^k for $x \in X, k \in \mathbb{N}$, to mean the point $(x, \ldots, x) \in X^k$. For $k = 0$ we identify $B^0(X, Y)$ with Y and set $x^0 = 1 \in \mathbb{K}$ when X and Y are Banach spaces over $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$). This simplifies the notation.

Definition 3.4 (Analytic operator)**.** Let X, Y be Banach spaces and U be an open set in X. A map $f: U \to Y$ is said to be analytic at $x_0 \in U$ if there exists some $r > 0$, and symmetric operators $P_k \in B^k(X, Y)$ for all $k \geq 0$, such that

$$\sup_{k \geq 0} r^k \|P_k\|_{B^k(X,Y)} < \infty$$

and

$$f(x) = \sum_{k=0}^{\infty} P_k (x - x_0)^k \tag{3.10}$$

in a neighborhood of x_0. We say that f is analytic in U, and write $f \in C^\omega(U, Y)$, if f is analytic at every $x_0 \in U$.

It is a standard result, see e.g. [Buffoni and Toland, 2003, Proposition 4.3.4], that if f is analytic at x_0, then the operators P_k are what we expect. Indeed, if f is defined by Equation (3.10) on some open neighborhood of x_0, then f is infinitely differentiable there, and

$$D^k f(x_0) = k! P_k$$

for every $k \geq 0$. Thus Equation (3.10) reads

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k f(x_0)(x - x_0)^k,$$

i.e., analytic operators have Taylor series that converge absolutely, and in fact uniformly, in a neighborhood of each point.

As an example of analytic operators, we prove the following result, which shows that inversion of bounded linear operators is an analytic operation. An application of Lemma 3.5 is the proof of regularity in the (ordinary) inverse and implicit function theorems, since the lemma in particular shows that inversion is smooth.

Lemma 3.5 (Analyticity of inversion)**.** *Let X, Y be Banach spaces and let $U \subseteq B(X, Y)$ be the set of all invertible bounded linear operators $X \to Y$. Then U is open, and the inversion map $f \colon U \to B(Y, X)$ defined by*

$$f(T) \coloneqq T^{-1}$$

is analytic. Its derivatives are given by

$$D^k f(S)(T_1, \ldots, T_n) = \sum_{\sigma \in S_k} \left(\prod_{j=1}^{k} S^{-1} T_{\sigma(j)} \right) S^{-1},$$

where S_k is the set of permutations of $1, \ldots, k$.

Proof. Fix $S \in U$. Then for any $T \in B(X, Y)$ we have

$$T = S - (S - T) = S(I_X - S^{-1}(S - T)),$$

whence for $T \in B_r(S)$, with $r \coloneqq \|S^{-1}\|_{B(Y,X)}^{-1}$, T is in U and

$$f(T) = \sum_{k=0}^{\infty} (S^{-1}(S - T))^k S^{-1}.$$

Proceed now to define maps $P_k \in B^k(B(X, Y), B(Y, X))$ by

$$P_k(T_1, \ldots, T_k) \coloneqq \frac{1}{k!} \sum_{\sigma \in S_k} \left(\prod_{j=1}^{k} S^{-1} T_{\sigma(j)} \right) S^{-1},$$

where the multilinearity and symmetry should be clear. We have

$$\begin{aligned} \|P\|_{B^k(B(X,Y),B(Y,X))} &= \sup_{\substack{\|T_1\|_{B(X,Y)}=1, \\ \cdots \\ \|T_k\|_{B(X,Y)}=1}} \|P_k(T_1, \ldots, T_k)\|_{B(Y,X)} \\ &\leq \frac{1}{k!} \sum_{\sigma \in S_k} \|S^{-1}\|_{B(Y,X)}^{k+1} = \|S^{-1}\|_{B(Y,X)}^{k+1}, \end{aligned}$$

where we used that $|S_k| = k!$, so that

$$\sup_{k \geq 0} r^k \|P_k\|_{B^k(B(X,Y),B(Y,X))} \leq \|S^{-1}\|_{B(Y,X)} < \infty.$$

Moreover,

$$P_k T^k = (S^{-1} T)^k S^{-1}$$

for any $k \in \mathbb{N}$ and $T \in B(X, Y)$. Hence, by the above,

$$f(T) = \sum_{k=0}^{\infty} P_k (S - T)^k$$

for any $T \in B_r(S)$. Thus f is analytic at S, and since S was arbitrary, f is analytic on U. ∎

As illustrated by Lemma 3.6, which extends the familiar result for functions on $\mathbb{R}$ or $\mathbb{C}$, analyticity of an operator is a strong property,

Lemma 3.6. *Suppose that X, Y are Banach spaces, and that $U \subseteq X$ is a connected, open set. If $f \in C^\omega(U, Y)$ vanishes on a nonempty open set, then $f = 0$.*

Proof. The set
$$E \coloneqq \bigcap_{k \geq 0} (D^k f)^{-1}(\{0\})$$
of points where f and all its derivatives vanish is closed, being an intersection of closed sets because $f \in C^\omega(U, Y) \subseteq C^\infty(U, Y)$. It is also open; suppose that $x_0 \in E$, then
$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k f(x_0)(x - x_0)^k = 0$$
in an open ball $B_r(x_0)$ around x_0. It follows that $B_r(x_0) \subseteq E$, meaning that E is open.

Finally, by assumption, the set E is nonempty. Thus, by the assumption that U is connected, we must have $E = U$. □

For operators defined on $\mathbb{R}^d$ or $\mathbb{C}^d$, we can say something more, namely Theorem 3.7 below, which is a result that we will need later for Theorem 6.17. Note that this is a much stronger result than the above, because there are sets of positive measure that have empty interior. An example in $\mathbb{R}$ is the Smith–Volterra–Cantor set (or "fat" Cantor set), described in [Folland, 1999, p. 39], which one can take a Cartesian product of with a ball in $\mathbb{R}^{d-1}$ to obtain an example in $\mathbb{R}^d$ for $d \geq 2$.

Theorem 3.7. *Suppose that Y is a Banach space over $\mathbb{K}$, and that $U \subseteq \mathbb{K}^d$ is open and connected. If $f \in C^\omega(U, Y)$ vanishes on a set of positive measure, then $f = 0$.*

Proof. We will denote the Lebesgue measure on $\mathbb{K}^d$ by μ. Let $E \coloneqq f^{-1}(\{0\})$, which by assumption has positive measure. The Lebesgue differentiation theorem, [Folland, 1999, Theorem 3.21], used on the characteristic function χ_E then says that almost every point in $x \in E$ is a density point, meaning a point where
$$\lim_{r \downarrow 0} \frac{\mu(E \cap B_r(x))}{\mu(B_r(x))} = 1,$$
a result which is sometimes known as Lebesgue's density theorem.

We will show that the derivative of f vanishes at every density point. To that end, let x_0 be a density point of E. By assumption of f being analytic, we may choose $R > 0$ and $C > 0$ such that
$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k f(x_0)(x - x_0)^k = Df(x_0)(x - x_0) + g(x),$$
$$\|g(x)\|_Y \leq C|x - x_0|^2,$$

for all $x \in B_R(x_0)$. Let now $\varepsilon > 0$, and define

$$\delta \coloneqq \min\left\{\frac{1}{\sqrt{2}}, \frac{\varepsilon}{2\sqrt{d}(1 + \|Df(x_0)\|_{B(\mathbb{K}^d,Y)})}\right\}.$$

As x_0 is a density point, there is some real number r satisfying

$$0 < r < \min\left\{R, \frac{\varepsilon}{2C\sqrt{d}}\right\},$$

such that

$$\mu(E \cap B_r(x_0)) > \left(1 - \left(\frac{\delta}{2}\right)^m\right)\mu(B_r(x_0)),$$

where m is d if $\mathbb{K} = \mathbb{R}$ and $2d$ if $\mathbb{K} = \mathbb{C}$.

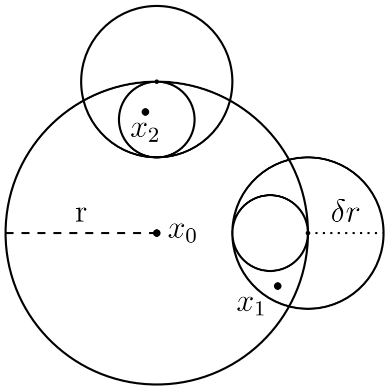


Figure 3.1: The balls used in the proof, illustrated in $\mathbb{R}^2$.

Denote the elements of the standard basis of $\mathbb{K}^d$ by e_i for $1 \leq i \leq d$. Observe now that the balls

$$B_{\delta r}(x_0 + re_1), \ldots, B_{\delta r}(x_0 + re_n),$$

see Figure 3.1, are disjoint (this is not required for the proof, but makes the idea clearer). Indeed, if $x \in B_{\delta r}(x_0 + re_i), y \in B_{\delta r}(x_0 + re_j)$ then

$$\begin{aligned}|x - y| &= |r(e_i - e_j) + [x - (x_0 + re_i)] - [y - (x_0 + re_j)]| \\ &\geq |r||e_i - e_j| - |[x - (x_0 + re_i)] - [y - (x_0 + re_j)]|| \\ &> \sqrt{2}r - 2\delta r \geq 0\end{aligned}$$

when $i \neq j$. Here we have used that $2\delta \geq \sqrt{2}$. Moreover, for each i, the ball $B_{\delta r/2}(x_0 + (1 - \delta/2)re_i)$ is contained in $B_r(x_0) \cap B_{\delta r}(x_0 + re_i)$; take any $x \in B_{\delta r/2}(x_0 + (1 - \delta/2)re_i)$, then it follows that

$$\begin{aligned}
|x - x_0| &= |x - (x_0 + (1 - 2^{-1}\delta)re_i) + (1 - 2^{-1}\delta)re_i| \\
&< \frac{1}{2}\delta r + \left(1 - \frac{\delta}{2}\right) r = r, \\
|x - (x_0 + re_i)| &= |x - (x_0 + (1 - 2^{-1}\delta)re_i) - 2^{-1}\delta r| \\
&< \frac{1}{2}\delta r + \frac{1}{2}\delta r = \delta r.
\end{aligned}$$

This implies that

$$\begin{aligned}
\mu(B_r(x_0) \setminus B_{\delta r}(x_0 + re_i)) &\leq \mu(B_r(x_0)) - \mu(B_{\delta r/2}(x_0 + (1 - \delta/2)re_i)) \\
&= \left(1 - \left(\frac{\delta}{2}\right)^m\right) \mu(B_r(x_0)),
\end{aligned}$$

which in turn implies that

$$\begin{aligned}
\mu(E \cap B_r(x_0) \cap B_{\delta r}(x_0 + re_i)) &= \mu(E \cap B_r(x_0)) - \mu(E \cap B_r(x_0) \setminus B_{\delta r}(x_0 + re_i)) \\
&\geq \mu(E \cap B_r(x_0)) - \mu(B_r(x_0) \setminus B_{\delta r}(x_0 + re_i)) \\
&> 0
\end{aligned}$$

by how we chose r.

Hence, these sets are nonempty, and we can select elements

$$x_i \in E \cap B_r(x_0) \cap B_{\delta r}(x_0 + re_i)$$

for each $1 \leq i \leq d$. Now, for any choice of $\alpha \in \mathbb{K}^d$, we have

$$\begin{aligned}
\|Df(x_0)\alpha\|_Y &= \left\| Df(x_0) \left(\sum_{i=1}^{d} \alpha_i e_i \right) \right\|_Y = \left\| \sum_{i=1}^{d} \alpha_i Df(x_0) e_i \right\|_Y \\
&= \left\| \sum_{i=1}^{d} \frac{\alpha_i}{r} [Df(x_0)(x_i - x_0) - Df(x_0)(x_i - (x_0 + re_i))] \right\|_Y \\
&\leq \frac{1}{r} \sum_{i=1}^{d} |\alpha_i| \left(\|Df(x_0)(x_i - x_0)\|_Y + \|Df(x_0)(x_i - (x_0 + re_i))\|_Y \right),
\end{aligned}$$

where, since $x_i \in E \cap B_r(x_0)$,

$$\|Df(x_0)(x_i - x_0)\|_Y = \|g(x_i)\|_Y \leq C|x_i - x_0|^2 \leq Cr^2,$$

and since $x_i \in B_{\delta_r}(x_0 + re_i)$,

$$\|Df(x_0)(x_i - (x_0 + re_i))\|_Y \leq \|Df(x_0)\|_{B(\mathbb{K}^d, Y)} \delta r.$$

Thus

$$\begin{aligned}\|Df(x_0)\alpha\|_Y &\leq (Cr + \delta\|Df(x_0)\|_{B(\mathbb{K}^d,Y)})\sum_{i=1}^{d}|\alpha_i| \\ &\leq \sqrt{d}(Cr + \delta\|Df(x_0)\|_{B(\mathbb{K}^d,Y)})|\alpha|,\end{aligned}$$

whence

$$\|Df(x_0)\|_{B(\mathbb{K}^d,Y)} \leq \sqrt{d}(Cr + \delta\|Df(x_0)\|_{B(\mathbb{K}^d,Y)}) < \varepsilon,$$

which shows that $Df(x_0) = 0$, as ε was arbitrary.

So far, we have proven that Df vanishes almost everywhere on E. Since Df is also analytic, we may repeat the argument to deduce that $D^2 f$ vanishes almost everywhere on E. By doing this for the derivatives of all orders, we conclude that the intersection

$$\bigcap_{k\geq 0} (D^k f)^{-1}(\{0\})$$

has positive measure (equal to that of E). In particular, it is nonempty, which means that there exists a point in U in which f and all its derivatives vanish. Since f is analytic, this implies that f vanishes in an open ball around this point. By Lemma 3.6, f vanishes identically, and the proof of Theorem 3.7 concludes. □

Remark. If $d = 1$, then an even stronger result holds; any zero of an analytic function (on a domain) that does not vanish identically must be isolated ([Markushevich, 1965a, Theorem 17.1]). The proof of Theorem 3.7 also shows that if $f \in C^k(U, Y)$ for some $k \geq 2$ and vanishes on a set of positive measure, then its derivatives up to order $k-1$ also vanish on that set, up to a subset of measure zero.

The behavior exhibited in Theorem 3.7 is very different from that of simply being C^∞. Indeed, it can be shown that the set on which a C^∞-function between Euclidean spaces vanishes can be any closed set, see for instance [Krantz and Parks, 2002, Proposition 3.3.6].

4 Local bifurcation theory

Suppose that we have Banach spaces X and Y over a field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$) and a map $f\colon X \times \mathbb{K} \to Y$ satisfying

$$f(x_0, \lambda_0) = 0$$

for some $x_0 \in X$ and $\lambda_0 \in \mathbb{K}$. One may think of $\mathbb{K}$ as a parameter space controlling the equation

$$f(x, \lambda) = 0, \tag{4.1}$$

where we look for solutions $x \in X$ for each $\lambda \in \mathbb{K}$. If f is at least C^1 on a neighborhood of (x_0, λ_0) and the partial derivative $D_x f(x_0, \lambda_0)$ is invertible, then the implicit function theorem tells us that there is a unique C^1 solution curve $\lambda \mapsto (x(\lambda), \lambda)$ in a neighborhood of (x_0, λ_0). In other words, there are no drastic local changes to the solution set of Equation (4.1) as we vary the parameter $\lambda \in \mathbb{K}$ near λ_0.

The focus of local bifurcation theory is to find conditions under which small changes of λ *do* cause radical changes locally to the solution set, in which case (x_0, λ_0) is called a bifurcation point. From what we have just seen, a necessary condition for this to happen when f is of sufficient regularity is that $D_x f(x_0, \lambda_0)$ is *not* invertible.

What is actually meant by the term bifurcation point may depend on the context, but in the setting of Equation (4.1) we formalize the meaning in Definition 4.1. Note that the solutions on the curve γ are sometimes viewed as the "trivial" solutions.

Definition 4.1 (Bifurcation point)**.** We say that (x_0, λ_0) is a bifurcation point for Equation (4.1) if there exists some C^1 solution curve $\gamma\colon I \to X \times I$, where $I \subseteq \mathbb{K}$ is an open neighborhood of λ_0, of the form

$$\begin{aligned} \gamma(\lambda) &= (x(\lambda), \lambda), \quad \lambda \in I, \\ x(\lambda_0) &= x_0, \end{aligned}$$

and the point (x_0, λ_0) is a limit point of the set $f^{-1}(\{0\}) \setminus \gamma(I)$ of solutions not lying on the solution curve γ.

We shall arrive at the Crandall–Rabinowitz theorem (Theorem 4.13), which establishes sufficient conditions for the existence of a transcritical or pitchfork bifurcation for Equation (4.1). A sketch of such a bifurcation can be found in Figure 4.1.

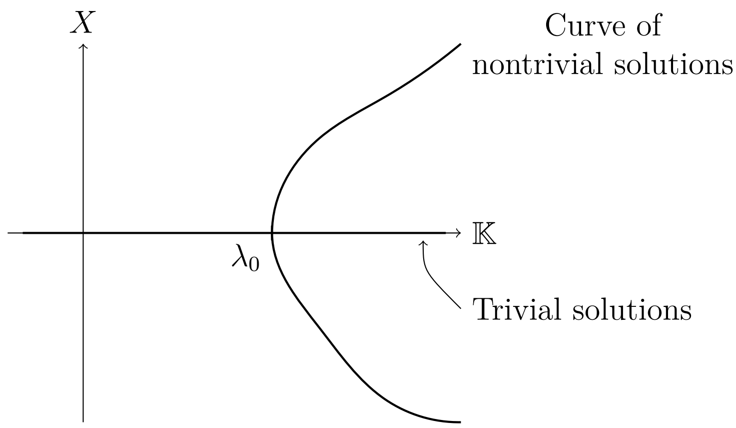


Figure 4.1: A pitchfork bifurcation for Equation (4.1). The point $(0, \lambda_0)$ is a bifurcation point in the sense of Definition 4.1.

4.1 Fredholm operators

Important for what follows is the concept of a Fredholm operator. Earlier, we discovered that non-invertibility of $D_x f(x_0, \lambda_0)$ is a prerequisite for bifurcations to occur in Equation (4.1), at least when f is C^1. However, a general element of $B(X, Y)$ which fails to be invertible is not easy to work with. To remedy this, we will typically require that $D_x f(x_0, \lambda_0)$ in addition be Fredholm; Fredholm operators being a class of more well-behaved operators.

Definition 4.2 (Fredholm operator)**.** Let X, Y be Banach spaces. An operator $T \in B(X, Y)$ is Fredholm if

$$\dim \ker T < \infty, \quad \operatorname{codim} \operatorname{im} T < \infty.$$

The quantity defined by

$$\operatorname{ind} T \coloneqq \dim \ker T - \operatorname{codim} \operatorname{im} T$$

is said to be its index.

In Definition 4.2, $\operatorname{codim} \operatorname{im} T$ means the dimension of the quotient space $Y / \operatorname{im} T$. Given a Banach space X and a closed subspace $E \subseteq X$, the quotient space X/E is a Banach space in the norm $\|\cdot\|_{X/E}$ defined by

$$\|x + E\|_{X/E} \coloneqq \inf_{y \in E} \|x - y\|_X = d_X(x, E) : \tag{4.2}$$

Proposition 4.3. *If X is a Banach space and E is a closed subspace of X, then X/E is a Banach space.*

Proof. The only aspect of Equation (4.2) defining a norm that requires elaboration is positive definiteness. If $x \in X$ is such that $\|x + E\|_{X/E} = d(x, E) = 0$, then we must necessarily have $x \in E$, because E is closed. Hence $x + E = 0$ in X/E.

Suppose now that $(x_n + E)_{n\in\mathbb{N}}$ is a Cauchy sequence in X/E. Then there is a strictly increasing sequence $(n_j)_{j\in\mathbb{N}} \subseteq \mathbb{N}$ such that

$$\|(x_{n_{j+1}} + E) - (x_{n_j} + E)\|_{X/E} = \inf_{e\in E} \|x_{n_{j+1}} - x_{n_j} + e\|_X < 2^{-j}, \quad j \in \mathbb{N},$$

so we may pick a sequence $(\tilde{e}_j)_{j\in\mathbb{N}} \subseteq E$ such that

$$\|x_{n_{j+1}} - x_{n_j} + \tilde{e}_j\|_X < 2^{-j} \tag{4.3}$$

holds for every $j \in \mathbb{N}$. If we now define a new sequence $(e_j)_{j\in\mathbb{N}} \subseteq E$ by $e_j = \sum_{i=1}^{j-1} \tilde{e}_j$, then $\tilde{e}_j = e_{j+1} - e_j$ for each $j \in \mathbb{N}$, so Equation (4.3) implies that the sequence $(x_{n_j} + e_j)_{j\in\mathbb{N}} \subseteq X$ is Cauchy, and hence converges to some $x \in X$. Finally,

$$\|(x_{n_j} + E) - (x + E)\|_{X/E} = \inf_{e\in E} \|x_{n_j} - x + e\|_X \leq \|(x_{n_j} + e_j) - x\|_X, \quad j \in \mathbb{N},$$

shows that $(x_{n_j} + E)_{j\in\mathbb{N}}$ is a convergent subsequence of $(x_n + E)_{n\in\mathbb{N}}$. □

We mention that, in addition to the assumptions in Definition 4.2, some authors (e.g. [Kato, 1995, p. 230]) demand that $\operatorname{im} T$ be closed for Fredholm operators. This condition is, in fact, redundant; a fact which we prove in Theorem 4.7. Having closed range is one of several consequences of being Fredholm that we shall have use for in the proof of Theorem 4.12. The following lemmas are also useful.

Lemma 4.4. *Suppose that X is a Banach space and that E is a finite-dimensional subspace of X. Then there exists a bounded projection onto E.*

Proof. Let $\{e_1, \ldots, e_d\}$ form a basis for E, and let $\{f_1, \ldots, f_d\}$ be its dual basis. By finite-dimensionality of E, the elements of the dual basis are all bounded. Hence we may extend them to elements of X' through the use of the Hahn–Banach theorem, while keeping the same notation. Now define $P \in B(X)$ by

$$Px = \sum_{i=1}^{d} f_i(x) e_i,$$

where the boundedness of P follows from that of the f_i. It is immediate that P is idempotent, so it is a bounded projection with $\operatorname{im} P = E$. □

In particular, Lemma 4.4 implies the existence of a closed subspace $F \coloneqq \ker P$ of X such that

$$X = E \oplus F,$$

called a topological complement of E.

Lemma 4.5. *Let X be a Banach space, and let E be a subspace of X. If E has finite codimension, then E has a finite-dimensional algebraic complement F. Moreover,*

$$\dim F = \operatorname{codim} E.$$

Proof. Choose a basis $\{x_1 + E, \ldots, x_d + E\}$ of X/E, and let $F = \operatorname{span}\{x_1, \ldots, x_d\}$. If we now take some $x \in X$, then

$$x + E = \sum_{i=1}^{d} \alpha_i (x_i + E) = \sum_{i=1}^{d} \alpha_i x_i + E,$$

where the α_i are uniquely determined by x. Hence

$$x - \sum_{i=1}^{d} \alpha_i x_i \in E,$$

which proves the result. □

Remark. Lemma 4.5 also shows that if E is closed and has finite codimension, then it has a finite-dimensional topological complement, as finite-dimensional subspaces are closed.

The next lemma is an almost immediate corollary of the open mapping theorem.

Lemma 4.6. *Let X be a Banach space and suppose that E, F are closed subspaces of X such that*

$$X = E \oplus F.$$

Then the projection onto E along F is bounded.

Proof. Note that by the assumption of E, F being closed, $E \oplus F$ is a Banach space in the norm

$$\|(e, f)\|_{E \oplus F} := \|e\|_X + \|f\|_X.$$

Denote the projection onto E along F by P, and define the linear map $T\colon E \oplus F \to X$ by $(e, f) \mapsto e + f$. Then T is bounded by the triangle inequality in X, and is clearly a bijection, with inverse $x \mapsto (Px, (I - P)x)$. By the open mapping theorem, the inverse of T is bounded, and so

$$\|Px\|_X \leq \|Px\|_X + \|(I - P)x\|_X = \|T^{-1}x\|_{E \oplus F} \leq \|T^{-1}\|_{B(X, E \oplus F)} \|x\|_X$$

shows that P is bounded. □

Finally, we prove that Fredholm operators have closed range.

Theorem 4.7. *Suppose that X, Y are Banach spaces over $\mathbb{K}$ and that $T \in B(X, Y)$ is Fredholm. Then $\operatorname{im} T$ is closed.*

Proof. Note that $\ker T$ is closed since T is bounded, and so $X/\ker T$ is a Banach space (Proposition 4.3). Now, by Lemma 4.5, $\operatorname{im} T$ has a finite dimensional algebraic complement E in Y. Let $\{e_1, \ldots, e_d\}$ be a basis for E, and proceed to define the linear operator $S\colon (X/\ker T) \times \mathbb{K}^d \to Y$ by

$$S(x + \ker T, \alpha) := Tx + \sum_{i=1}^{d} \alpha_i e_i. \tag{4.4}$$

For any $y \in \ker T$, we have

$$\|Tx\|_Y = \|T(x-y)\|_Y \le \|T\|_{B(X,Y)}\|x-y\|_X,$$

and so $\|Tx\|_Y \le \|T\|_{B(X,Y)}\|x+\ker T\|_{X/\ker T}$ by taking the infimum over y, meaning that

$$\|S(x+\ker T,\alpha)\|_Y \le \|T\|_{B(X,Y)}\|x+\ker T\|_{X/\ker T} + \sqrt{d}\max_{i=1,\dots,d}\|e_i\|_Y|\alpha|.$$

Hence $S \in B((X/\ker T)\times\mathbb{K}^d, Y)$ if we use any of the usual norms on the product space.

Observe now that since $Y = \operatorname{im} T \oplus E$ and $\{e_1,\dots,e_d\}$ is a basis for E, S is certainly surjective. It is also injective for the same reason, since we have modded out $\ker T$. Hence, by the open mapping theorem, S is a homeomorphism, and thus

$$S(\overbrace{(X/\ker T)\times\{0\}}^{\text{closed}}) = \operatorname{im} T$$

is closed. □

With that done, we need another definition before we move on. We use the notation $L(X,Y)$ to denote the set of linear operators $X \to Y$.

Definition 4.8 (Compact operator)**.** Let X, Y be Banach spaces and suppose that $T \in L(X,Y)$. We say that T is compact if, for every bounded set B, the set $T(B)$ is relatively compact.

By relatively compact, we mean having compact closure. It is almost immediate from the definition that every compact operator is also bounded. Indeed, we have the following.

Proposition 4.9. *Let X, Y be Banach spaces. Then every compact operator $T \in L(X,Y)$ is bounded.*

Proof. By definition of T being compact, the set $T(\overline{B}_1(0,X))$ is relatively compact, and hence bounded. Hence

$$\|T\|_{B(X,Y)} = \sup_{\|x\|_X\le 1}\|Tx\|_Y = \sup_{y\in T(\overline{B}_1(0,X))}\|y\|_Y < \infty,$$

which shows that T is bounded. □

The converse is certainly not true in general; the identity operator $I_X \in B(X)$ is compact if and only if X is finite-dimensional. This follows from the well-known characterization of finite-dimensional Banach spaces as being precisely those where the closed unit ball is compact, see e.g. [Kreyszig, 1989, p. 80].

For compact operators we have the so-called Fredholm alternative, Theorem 4.10. It states that compact perturbations of the identity are Fredholm. Together with Corollary 4.11 below, this theorem is a rich source of such Fredholm operators.

Theorem 4.10 [Brezis, 2011] (Fredholm alternative)**.** *Let X be a Banach space and $T \in B(X)$ be compact. Then $I_X - T$ is a Fredholm operator of index zero.*

Proof (partial). The simple part is finite-dimensionality of the kernel. Since $Tx = x$ for any $x \in \ker(I_X - T)$, we have

$$T(B_1(0, \ker(I_X - T))) = B_1(0, \ker(I_X - T)),$$

so by compactness of T, the closed unit ball in $\ker(I_X - T)$ is compact. Hence $\ker(I_X - T)$ is finite-dimensional. For the rest of the proof, see [Brezis, 2011, p. 160]. □

The origin for the name of Theorem 4.10 is that, given a compact operator $T \in B(X)$, there are precisely two alternatives (which can be elaborated on).

(i) One has $\dim\ker(I_X - T) = \operatorname{codim}\operatorname{im}(I_X - T) = 0$, in which case $I_X - T$ is invertible. In other words, the equation

$$Tx - x = y$$

has a unique solution $x \in X$ for any $y \in Y$.

(ii) Otherwise, $0 < \dim\ker(I_X - T) = \operatorname{codim}\operatorname{im}(I_X - T) < \infty$. Then the equation

$$Tx - x = 0$$

has a nonzero solution.

From the Fredholm alternative, we have the following corollary.

Corollary 4.11. *Suppose that X, Y are Banach spaces, and that $S, T \in B(X, Y)$ with S invertible and T compact. Then $S + T$ is Fredholm with index zero.*

Proof. If we let $L = S + T$, then $S^{-1}L = I_X + S^{-1}T$. Observe that since T is compact and S^{-1} is bounded, $S^{-1}T$ is compact. This follows from continuous functions mapping compact sets to compact sets. Hence $S^{-1}L$ is Fredholm of index zero by the Fredholm alternative, and thus so is L. □

4.2 The Lyapunov–Schmidt reduction

The Lyapunov–Schmidt reduction is a method to locally reduce Equation (4.1) to a finite-dimensional problem around a solution (x_0, λ_0), assuming that the derivative of f at that point is Fredholm. In some sense, it can be interpreted as a generalization of the implicit function theorem. The precise statement is as follows:

Theorem 4.12 (Lyapunov–Schmidt reduction)**.** *Let X, Y be Banach spaces. Suppose that $f \in C^k(U \times I, Y)$, where $k \geq 1$ and $U \times I \subseteq X \times \mathbb{K}$ is open. Assume further that $(x_0, \lambda_0) \in U \times I$ solves Equation* (4.1) *and that the partial derivative $L := D_x f(x_0, \lambda_0) \in B(X, Y)$ is Fredholm with $d := \operatorname{codim}\operatorname{im} T$. Then there exists an open neighborhood $\tilde{U} \times \tilde{I} \subseteq U \times I$ of (x_0, λ_0), an open neighborhood $V \subseteq \ker L$ of 0 and maps $\varphi \in C^k(V \times \tilde{I}, \tilde{U}), \psi \in$*

$C^k(V \times \tilde{I}, \mathbb{K}^d)$ *such that* $\varphi(0, \lambda_0) = x_0$, $\psi(0, \lambda_0) = 0$, *and such that for* $(x, \lambda) \in \tilde{U} \times \tilde{I}$ *we have*

$$f(x, \lambda) = 0$$

if and only if there exists $z \in V$ *such that*

$$\begin{aligned} \psi(z, \lambda) &= 0, \\ \varphi(z, \lambda) &= x. \end{aligned}$$

Proof. The theorem will follow almost as a corollary of the implicit function theorem, we need only find the correct map to apply it to. As L is Fredholm, $\ker L$ is finite-dimensional, and so, by Lemma 4.4, there is a closed subspace E of X such that

$$X = E \oplus \ker L.$$

Similarly, $\operatorname{im} L$ is closed by Theorem 4.7 and has finite codimension d. Hence, by Lemma 4.5, there is a d-dimensional subspace F of Y such that

$$Y = \operatorname{im} L \oplus F.$$

Let P be the projection onto F along $\operatorname{im} L$, which is bounded by Lemma 4.6, and let U_E and $U_{\ker L}$ be open neighborhoods of 0 in E and $\ker L$, respectively, such that $x_0 + U_E + U_{\ker L} \subseteq U$. We can now define the map $g\colon U_E \times U_{\ker L} \times I \to \operatorname{im} L$ by

$$g(y, z, \lambda) \coloneqq (I - P) f(x_0 + y + z, \lambda),$$

from which we observe that

$$\begin{aligned} g(0, 0, \lambda_0) &= 0, \\ D_y g(0, 0, \lambda_0) &= L, \end{aligned}$$

where L is understood as an operator $E \to \operatorname{im} L$. By construction, then, $D_y g(0, 0, \lambda_0)$ is invertible. Hence we may apply the implicit function theorem to deduce that there is an open neighborhood

$$\tilde{U}_E \times \tilde{U}_{\ker L} \times \tilde{I} \subseteq U_E \times U_{\ker L} \times I$$

of $(0, 0, \lambda_0)$ and a map $h \in C^k(\tilde{U}_{\ker L} \times \tilde{I}, \tilde{U}_E)$ such that for $(y, z, \lambda) \in \tilde{U}_E \times \tilde{U}_{\ker L} \times \tilde{I}$ we have $(I - P) f(x_0 + y + z, \lambda) = 0$ if and only if $y = h(z, \lambda)$. Note that this in particular implies that $h(0, \lambda) = 0$. Denote the set $x_0 + \tilde{U}_E + \tilde{U}_{\ker L}$ by $\tilde{U}$. If we now define $\varphi\colon \tilde{U}_{\ker L} \times \tilde{I} \to \tilde{U}$ and $\psi\colon \tilde{U}_{\ker L} \times \tilde{I} \to F$ by

$$\begin{aligned} \varphi(z, \lambda) &\coloneqq x_0 + h(z, \lambda) + z, \\ \psi(z, \lambda) &\coloneqq P f(\varphi(z, \lambda), \lambda), \end{aligned}$$

then for $(x,\lambda) = (x_0 + y + z, \lambda) \in \tilde{U} \times \tilde{I}$, we have

$$\begin{aligned} f(x,\lambda) = 0 &\iff [Pf(x_0 + y + z, \lambda) = 0 \text{ and } (I-P)f(x_0 + y + z, \lambda) = 0] \\ &\iff [Pf(x_0 + y + z, \lambda) = 0 \text{ and } y = h(z,\lambda)] \\ &\iff [\psi(z,\lambda) = 0 \text{ and } y = h(z,\lambda)] \\ &\iff [\psi(z,\lambda) = 0 \text{ and } x = \varphi(z,\lambda)]. \end{aligned}$$

The conclusion of the theorem now follows by composing ψ with an isomorphism between F and $\mathbb{K}^d$ and letting $V = \tilde{U}_{\ker L}$. $\square$

Observe that Theorem 4.12 tells us that solving the equation $f(x,\lambda) = 0$ in a neighborhood of (x_0, λ_0) is equivalent to solving the equation $\psi(z,\lambda) = 0$. The benefit, and why the theorem is useful, is that this new equation is completely finite-dimensional, and thus easier to work with.

Remark. Exchanging $\mathbb{K}$ with a general Banach space changes nothing in the proof of Theorem 4.12. For instance, we could use $\mathbb{K}^m$ to include dependence on m parameters in $\mathbb{K}$ and still obtain a finite-dimensional problem.

4.3 The Crandall–Rabinowitz theorem

Suppose that the map f from Equation (4.1) satisfies

$$f(0,\lambda) = 0$$

for all $\lambda \in \mathbb{K}$. Then the set $\{0\} \times \mathbb{K} \subseteq X \times \mathbb{K}$ is said to be a trivial line of solutions for Equation (4.1). A natural question to ask is whether or not there are other solutions, and if there are, how these interact with the trivial line. This is the question that the Crandall–Rabinowitz theorem is concerned with.

The result, or rather a slightly generalization thereof, was first introduced in the paper [Crandall and Rabinowitz, 1971]. The generalization stems from the fact that if we instead have a curve (of some regularity) of solutions $\mathbb{K} \to X \times \mathbb{K}$, given by $t \mapsto (\chi(t), l(t))$, then we can reduce to a trivial line by making the change of variables $(x,\lambda) \mapsto (x + \chi(\lambda), l(\lambda))$. This is usually done before applying the Crandall-Rabinowitz theorem as formulated in Theorem 4.13.

Theorem 4.13 (Crandall–Rabinowitz theorem)**.** *Suppose that X, Y are Banach spaces over $\mathbb{K}$, and that $f \in C^k(U \times I, Y)$, where $k \geq 2$ and $U \times I \subseteq X \times \mathbb{K}$ is an open neighborhood of $(0, \lambda_0)$. Furthermore, suppose that f vanishes on the trivial line of solutions $\{0\} \times I$, and that*

(i) the operator $L \coloneqq D_x f(0,\lambda_0)$ is Fredholm of index zero,

(ii) one has $\ker L = \operatorname{span}\{z_0\}$ for some $0 \neq z_0 \in \ker L$, and

(iii) the condition $D^2_{x\lambda} f(0,\lambda_0)(z_0, 1) \notin \operatorname{im} L$ holds. (Transversality)

Then $(0, \lambda_0)$ *is a bifurcation point in the sense of Definition 4.1. Moreover, there exists an open neighborhood* $\tilde{U} \times \tilde{I} \subseteq U \times I$ *of* $(0, \lambda_0)$*, a real number* $\varepsilon > 0$ *and maps* $\chi \in C^{k-1}(B_\varepsilon(0, \mathbb{K}), \tilde{U}), l \in C^{k-1}(B_\varepsilon(0, \mathbb{K}), \tilde{I})$ *such that* $(\chi(0), l(0)) = (0, \lambda_0)$*,* $\chi'(0) = z_0$*, and such that for* $(x, \lambda) \in \tilde{U} \times \tilde{I}$ *we have*

$$f(x, \lambda) = 0$$

if and only if (x, λ) *is on the trivial line of solutions or on the nontrivial solution curve* $t \mapsto (\chi(t), l(t))$*.*

Proof. Since L is Fredholm, we can apply Theorem 4.12. We adopt all the sets and maps as they are defined in the proof, without the identification of F and $\mathbb{K}^d$. Observe that because f vanishes on the trivial line of solutions, so do the maps g, h, φ and ψ.

The Lyapunov–Schmidt reduction reduces the problem to looking at the zero set of ψ. If we choose $\delta > 0$ such that $B_{\delta \|z_0\|_X}(0, \ker L) \subseteq \tilde{U}_{\ker L}$, we may write

$$\begin{aligned} \frac{1}{t}(\psi(tz_0, \lambda) - \psi(0, \lambda)) &= \frac{1}{t} \int_0^t [r \mapsto \psi(rz_0, \lambda)]'(s)\, ds \\ &= \int_0^1 D_z\psi(stz_0, \lambda) z_0 \, ds \end{aligned}$$

for $t \in B_\delta(0, \mathbb{K}) \setminus \{0\}$. Hence, if we define $\eta\colon B_\delta(0, \mathbb{K}) \times \tilde{I} \to F$ by

$$\eta(t, \lambda) := \int_0^1 D_z\psi(stz_0, \lambda) z_0 \, ds, \tag{4.5}$$

then

$$\eta(t, \lambda) = \begin{cases} t^{-1}\psi(tz_0, \lambda) & t \neq 0 \\ D_z\psi(0, \lambda) z_0 & t = 0 \end{cases}, \quad (t, \lambda) \in B_\delta(0, \mathbb{K}) \times \tilde{I}. \tag{4.6}$$

It should be evident from Equation (4.5) that we "lose" one derivative, so that $\eta \in C^{k-1}(B_\delta(0, \mathbb{K}) \times \tilde{I}, F)$.

We wish to apply the implicit function theorem to η, and since

$$\begin{aligned} \eta(0, \lambda_0) &= D_z\psi(0, \lambda_0) z_0, \\ D_\lambda \eta(0, \lambda_0) &= D^2_{z\lambda}\psi(0, \lambda_0)(z_0, 1), \end{aligned}$$

we have to calculate the derivatives of ψ. As $\psi(z, \lambda) = Pf(z + h(z, \lambda), \lambda)$, we obtain

$$D_z\psi(0, \lambda) z_0 = PD_x f(0, \lambda)(z_0 + D_z h(0, \lambda) z_0),$$

or in particular $D_z\psi(0, \lambda_0) z_0 = 0$, since $\ker P = \operatorname{im} L$. Taking another derivative yields

$$\begin{aligned} D^2_{z\lambda}\psi(0, \lambda_0)(z_0, 1) &= PD^2_{x\lambda} f(0, \lambda_0)(z_0 + D_z h(0, \lambda_0) z_0, 1) + PLD^2_{z\lambda} h(0, \lambda)(z_0, 1) \\ &= PD^2_{x\lambda} f(0, \lambda_0)(z_0, 1), \end{aligned} \tag{4.7}$$

where the last term in Equation (4.7) vanishes because $\ker P = \operatorname{im} L$, and

$$\begin{aligned} D_z h(0,\lambda_0)z_0 &= -D_y g(0,0,\lambda_0)^{-1} D_z g(0,0,\lambda_0) z_0 \\ &= -D_y g(0,0,\lambda_0)^{-1} L z_0 \\ &= 0, \end{aligned} \tag{4.8}$$

as $z_0 \in \ker L$.

We have now shown that $\eta(0,\lambda_0) = 0$ and that $D_\lambda \eta(0,\lambda_0) = D^2_{z\lambda}\psi(0,\lambda_0)(z_0,1) \neq 0$ (and is therefore invertible since $\dim F = 1$) by the assumption of transversality. Thus, by the implicit function theorem, there is an open neighborhood $B_\varepsilon(0,\mathbb{K}) \times \hat{I} \subseteq B_\delta(0,\mathbb{K}) \times \tilde{I}$ of $(0,\lambda_0)$ and a map $l \in C^{k-1}(B_\varepsilon(0,\mathbb{K}),\hat{I})$ with $l(0) = \lambda_0$, such that for $(t,\lambda) \in B_\varepsilon(0,\mathbb{K}) \times \hat{I}$ we have $\eta(t,\lambda) = 0$ if and only if $\lambda = l(t)$. From Equation (4.6), we deduce that this means that

$$\psi(tz_0,\lambda) = 0 \iff [t = 0 \text{ or } \lambda = l(t)]$$

for $(t,\lambda) \in B_\varepsilon(0,\mathbb{K}) \times \tilde{I}$.

Hence, if we denote the set $\tilde{U}_E + B_{\varepsilon\|z_0\|_X}(0,\ker L)$ by $\hat{U}$ and define $\chi\colon B_\varepsilon(0,\mathbb{K}) \to \hat{U}$ by $\chi(t) := \varphi(tz_0, l(t))$, then for $(x,\lambda) = (y + tz_0, \lambda) \in \hat{U} \times \hat{I}$ we have

$$\begin{aligned} f(x,\lambda) = 0 &\iff [\psi(tz_0,\lambda) = 0 \text{ and } x = \varphi(tz_0,\lambda)] \\ &\iff [(t = 0 \text{ or } \lambda = l(t)) \text{ and } x = \varphi(tz_0,\lambda)] \\ &\iff [x = 0 \text{ or } (x = \chi(t) \text{ and } \lambda = l(t))]. \end{aligned}$$

Finally, rename $\hat{U}, \hat{I}$ to $\tilde{U}, \tilde{I}$ and observe that

$$\begin{aligned} \chi'(0) &= D_z\varphi(0,\lambda_0)z_0 + D_\lambda\varphi(0,\lambda_0)l'(0) \\ &= z_0 + D_z h(0,\lambda_0)z_0 \\ &= z_0, \end{aligned}$$

since φ vanishes on the trivial line of solutions and $D_z h(0,\lambda_0)z_0 = 0$ by Equation (4.8). $\square$

Recall that the beginning of this chapter features Figure 4.1, which depicts this kind of bifurcation. The name *pitchfork bifurcation* for one type of bifurcation obtained from Theorem 4.13 stems from the way that the bifurcation typically looks (abstractly) when $l'(0) = 0$. The other type is a *transcritical bifurcation*, corresponding to $l'(0) \neq 0$. An example of a function where Crandall–Rabinowitz is applicable is the map $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x,\lambda) = x\sin(x + 1 - e^\lambda)$. Every point of the form $(0, \log(1 + n\pi))$ for $n \in \mathbb{N}_0$ is a bifurcation point by Theorem 4.13. One may check that if $z_0 = 1$, then the tangents of the (transcritical) bifurcating solution curves are given[1] by $x'(0) = 1, l'(0) = 1/(1 + n\pi)$.

As with the Lyapunov–Schmidt reduction, Theorem 4.12, the theorem of Crandall and Rabinowitz can be extended to multi-dimensional parameter spaces — given that the kernel of the derivative L is of the same dimension. The precise statement is not important for our purposes, but we mention that [Kielhöfer, 2012, Theorem I.19.6] is such a generalization.

[1] We give no general formula for $l'(0)$ in Theorem 4.13, but we mention that this derivative is given by $l'(0) = -f_{xx}(0,\log(1+n\pi))/(2f_{x\lambda}(0,\log(1+n\pi)))$ here. See [Kielhöfer, 2012, Chapter I.6] for general formulas for $l'(0)$, and even the second derivative $l''(0)$.

5 The water-wave problem

We are interested in studying water waves, and there are various equations that describe water waves in some form or another. The most general among these are the famous[1] Navier–Stokes equations. Under the assumption of inviscid (absence of viscosity) and incompressible (constant fluid density) flow, these reduce to the so-called incompressible Euler equations. For describing water waves on the open sea, these are realistic assumptions ([Johnson, 1997, Lighthill, 1978]), and are therefore standard. We will further assume two-dimensional flow under the influence of gravity, where the Cartesian coordinates (x, y) describe the horizontal and vertical direction, respectively. Then the incompressible Euler equations read

$$\begin{aligned} w_t + (w \cdot \nabla)w &= -\nabla p - ge_2, && \text{(Conservation of momentum)} \\ \nabla \cdot w &= 0. && \text{(Conservation of mass, incompressibility)} \end{aligned} \tag{5.1}$$

Here w is the velocity of the fluid, p is the pressure distribution and $-ge_2 = (0, -g)$ is the constant gravitational acceleration[2]. We mention that the density, which must be included in the full Euler equations, has been absorbed in the pressure. It is also useful to introduce the notation $w = (u, v)$ for the components of w, which we shall do. Furthermore, note that, for the moment, we assume that the various functions described are such that the equations make sense. The proper spaces that they live in will be specified later, and some of the equations must be understood in the sense of distributions.

We now proceed to describe our domain. Assume that we have an impermeable flat bottom at finite depth, which we for convenience place at $\{(x, y) \in \mathbb{R}^2 : y = -h\}$, and a free boundary at the surface, whose deviation is described by a function $\eta\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The first variable represents space, and the second time. For simplicity, we normalize the surface to be at $\{(x, y) \in \mathbb{R}^2 : y = \eta(x, t)\}$ at time t, where it is natural to assume that $\eta(\cdot, t)$ is bounded, continuous and strictly bounded below by $-h$. It should be emphasized that, due to the free boundary assumption, the function η is a priori unknown; determining it is part of the problem. Given a function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $\inf f > -h$,

[1] Showing global existence and regularity for this equation in three dimensions is one of the seven celebrated Clay Institute Millenium Problems.

[2] The constant g is approximately $9.8\,\mathrm{m/s^2}$, varying by less than $0.4\,\%$ on the Earth's suface (see [Hirt et al., 2013]).

we define the hypograph $\Omega(f)$ of f by

$$\Omega(f) \coloneqq \{(x, y) \in \mathbb{R}^2 : -h < y < f(x)\},$$

whence we see that our time-dependent domain will be

$$\Omega(\eta(\cdot, t))$$

at time t. The continuity assumption on $\eta(\cdot, t)$ ensures that this set is open, and the strict lower bound ensures that it is connected.

In addition to Equation (5.1), we require boundary conditions to match our domain. In order to model the bottom being impermeable, we will demand that

$$v\big|_{y=-h} = 0,$$

with which we mean that $v(x, -h, t) = 0$ for all x and t. Now, for the first boundary condition at the surface, we will assume that any fluid particle at the surface will stay at the surface. This is a natural condition to demand, considering that fluid particles are what makes up the surface, due to the free boundary. If we suppose that $(x(t), y(t))$ describes the position of a fluid particle at the surface at time t, then this is the assumption that

$$y(t) = \eta(x(t), t)$$

for all t. Differentiating this, we find that

$$u\eta_x + \eta_t = v \quad \text{(Kinematic boundary condition)}$$

holds on the surface. We take this equation as a boundary condition.

We also require a boundary condition for the pressure at the surface, which we will take to be

$$p = -\alpha^2 \kappa(\eta), \quad \text{(Dynamic boundary condition)} \tag{5.2}$$

where $\alpha^2 > 0$ is the surface tension and κ is the nonlinear differential operator

$$\kappa(\eta) \coloneqq \left(\frac{\eta'}{\langle \eta' \rangle}\right)' = \frac{\eta''}{\langle \eta' \rangle^3},$$

yielding the mean curvature of the surface. Equation (5.2) is known by physicists as the Young–Laplace equation, and states that the pressure difference across a fluid interface (in this case water/air) is proportional to its curvature.

As we assume the presence of gravity and surface tension, the waves described by these equations are known as *gravity-capillary waves* (see [Vanden-Broeck, 2010]). Note that in the lower limit $\alpha^2 = 0$, the dynamic boundary condition corresponds to the simplifying assumption of constant pressure on the surface, but we will require that α^2 be strictly positive. The proof of e.g. Theorem 6.11 relies upon the assumption that $\alpha^2 > 0$.

5.1 The strong and weak vorticity equation

We define the *vorticity* of the velocity field w by[3]

$$\omega \coloneqq \nabla \times w = v_x - u_y,$$

which can be thought of as the fluid's local tendency to rotate[4]. If one takes the curl in Equation (5.1), one obtains after some simple calculations that

$$\omega_t + \nabla \cdot (\omega w) = 0, \tag{5.3}$$

which is called the *vorticity transport equation*. The name stems from the fact that this equation implies that the vorticity ω is transported by the vector field w. Indeed, if $(x(t), y(t))$ describes the position of a particle at time t, then

$$\begin{aligned} \tfrac{d}{dt}[t \mapsto \omega(x(t), y(t), t)] &= \omega_t + u\omega_x + v\omega_y \\ &\overset{\nabla\cdot w=0}{=} \omega_t + \nabla \cdot (\omega w) = 0 \end{aligned}$$

by Equation (5.3). We will have need for a weaker form of Equation (5.3), because we will allow ω to be a combination of Dirac delta distributions supported in isolated points, corresponding to point vortices in those points. The following proposition, which is a standard result, will be useful.

Proposition 5.1 (Newtonian potential)**.** *The distribution $\Gamma \in L^2_{loc}(\mathbb{R}^2)$ defined by*

$$\Gamma(x, y) \coloneqq \frac{1}{4\pi} \log(x^2 + y^2)$$

satisfies

$$\nabla^\perp \Gamma(x, y) = (-\Gamma_y, \Gamma_x)(x, y) = \frac{1}{2\pi} \frac{(-y, x)}{x^2 + y^2}$$

and

$$\Delta \Gamma = \nabla \times \nabla^\perp \Gamma = \delta.$$

Proof. The fact that $\Gamma \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ follows from changing to polar coordinates and using that

$$\lim_{r \downarrow 0} r(\log r)^2 = 0,$$

as the origin is clearly the only obstacle. Observe that for $\varphi \in D(\mathbb{R}^2)$,

$$\begin{aligned} \langle \nabla^\perp \Gamma, \varphi \rangle &= -\langle \Gamma, \nabla^\perp \varphi \rangle \\ &= -\frac{1}{4\pi} \int_{\mathbb{R}^2} \log(x^2 + y^2) \nabla^\perp \varphi(x, y) \, d\mu \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(-y, x)}{x^2 + y^2} \varphi(x, y) \, d\mu, \end{aligned}$$

[3]The use of $\nabla\times$ here is a slight abuse of notation, but it should be clear what is meant.

[4]Informally, the vorticity describes the speed at which an infinitesimal paddle wheel placed in the fluid will rotate.

which follows from integration by parts and the compact support of φ. The second property is only slightly trickier, as we do not get a regular distribution. We find

$$\begin{aligned}\langle \Delta\Gamma, \varphi\rangle &= \langle \Gamma, \Delta\varphi\rangle \\ &= \frac{1}{4\pi}\left(\int_{B_\varepsilon(0)} + \int_{\mathbb{R}^2\setminus B_\varepsilon(0)}\right) \log(x^2+y^2)\Delta\varphi(x,y)\,d\mu\end{aligned}$$

for any $\varepsilon > 0$. Now, as $\Gamma \in L^2_{\text{loc}}(\mathbb{R}^2) \subseteq L^1_{\text{loc}}(\mathbb{R}^2)$, the first integral vanishes as $\varepsilon \downarrow 0$ by the dominated convergence theorem. For the second integral, we use one of Green's identities to deduce that

$$\frac{1}{4\pi}\int_{\mathbb{R}^2\setminus B_\varepsilon(0)} \log(x^2+y^2)\Delta\varphi(x,y)\,d\mu = \varepsilon\log(\varepsilon)\fint_{|(x,y)|=\varepsilon} \nabla\varphi\cdot n\,d\gamma + \fint_{|(x,y)|=\varepsilon} \varphi\,d\gamma,$$

from which the result follows. Here, the notation $\fint$ means the integral divided by the measure of the (finite) measure space; in this case $2\pi\varepsilon$. □

Remark. As $D(\mathbb{R}^d)$ is dense in $S(\mathbb{R}^d)$, Proposition 5.1 also holds when Γ is viewed as a tempered distribution.

By the vorticity transport equation, Equation (5.3), it is natural to expect that if the vorticity consists of a point vortex at some time, then it will remain a point vortex at all times, and be transported with the flow. It should be emphasized that, for now, this is not rigorously justified; the multiplication of δ with w is not well defined, as w will not be smooth at the point vortex. If ω is of the form

$$\omega(t) = \delta_{(x_0(t),y_0(t))},$$

then we deduce from Proposition 5.1 that w is of the form

$$w(x,y,t) = \frac{1}{2\pi}\frac{(y_0(t)-y, x-x_0(t))}{(x-x_0(t))^2+(y-y_0(t))^2} + \hat{w}(x,y,t),$$

where $\hat{w}$ satisfies $\nabla\cdot\hat{w} = 0$ and $\nabla\times\hat{w} = 0$. This will, in fact, imply that $\hat{w}$ is C^∞ in space (see the discussion before Equation (5.11)). As the first term, which we may think of as the part of w generated by the point vortex, is singular and odd around $(x_0(t), y_0(t))$, it is not unreasonable to think that the dynamics of the point vortex should depend only on $\hat{w}$. In other words, that the path $t \mapsto (x_0(t), y_0(t))$ on which the point vortex moves should satisfy

$$(\dot{x}_0, \dot{y}_0) = \hat{w}. \tag{5.4}$$

This can indeed be made rigorous, also in the case where ω consists of some linear combination of point vortices. In [Marchioro and Pulvirenti, 1994, Theorems 4.1 and 4.2] it is proved that if one considers initial data consisting of vortex patches (compactly supported vorticity) converging in the sense of distributions to point vortices, then the weak solutions of the vorticity transport equation converge to point vortices in an appropriate sense. Moreover, each point vortex in the ensemble moves with the velocity field obtained by removing the part that it itself generates. It should be mentioned that it is possible for these point vortices to collide with each other or hit the boundary in finite time, unless further assumptions are made. The dynamics of such point vortices without any boundaries is similar to that of the famous n-body problem in physics. For more on this, see [Marchioro and Pulvirenti, 1994, Chapter 4].

Using the weaker vorticity equation in Equation (5.4), then, will be part of making the concept of a solution of the Euler equation weaker. We allow for point vortices, as long as they are propagated in the fluid as in Equation (5.4), or its multiple vortex equivalent.

5.2 Traveling waves

Of particular interest to us are *traveling-wave* solutions to Equation (5.1). Those are waves that propagate horizontally at constant speed without changing shape. One can differentiate between two types of such waves:

- Localized, or solitary, waves. The functions describing the velocity and surface vanish at infinity.
- Periodic waves. The velocity and surface have finite (and identical) minimal period.

We will for the moment only focus on the localized waves. In either case, we assume that there are functions $\tilde{w}, \tilde{p}, \tilde{\eta}$, depending only on space, and a constant speed $c \in \mathbb{R}$ such that

$$\begin{aligned} w(x, y, t) &= \tilde{w}(x - ct, y), \\ p(x, y, t) &= \tilde{p}(x - ct, y), \\ \eta(x, t) &= \tilde{\eta}(x - ct) \end{aligned}$$

for all $(x, y) \in \Omega(\eta(\cdot, t))$ and all t. Positive and negative c then correspond to waves moving in the positive and negative x-directions, respectively. We will drop the tildes on these functions from now on, which should cause no confusion. In the new *steady* variables $(\tilde{x}, \tilde{y}) = (x - ct, y)$, which we will proceed to denote by x and y, our equations read

$$((w - ce_1) \cdot \nabla)w = -\nabla p - ge_2, \qquad \text{(Conservation of momentum)} \tag{5.5}$$

$$\nabla \cdot w = 0, \qquad \text{(Conservation of mass)} \tag{5.6}$$

with boundary conditions

$$v = 0, \qquad \text{at } y = -h, \tag{5.7}$$

$$(u - c)\eta' = v, \qquad \text{at } y = \eta(x), \qquad \text{(Kinematic)} \tag{5.8}$$

$$p = -\alpha^2 \kappa(\eta), \qquad \text{at } y = \eta(x), \qquad \text{(Dynamic)} \tag{5.9}$$

on the now time-independent domain

$$\Omega(\eta) = \{(x, y) \in \mathbb{R}^2 : -h < x < \eta(x)\}.$$

We call the problem of finding w, p and η such that these equations are satisfies the *steady water-wave problem*. An illustration of how $\Omega(\eta)$ may look is shown in Figure 5.1.

Note that the vorticity transport equation, Equation (5.3), reduces to

$$\nabla \cdot (\omega(w - ce_1)) = 0$$

for traveling waves. The weak version, given in Equation (5.4), reduces to

$$(c, 0) = \hat{w}(x_0, y_0) \tag{5.10}$$

for a single point vortex centered at $(x_0, y_0) \in \Omega(\eta)$. For the case of several point vortices, this equation must be satisfied at each point vortex.

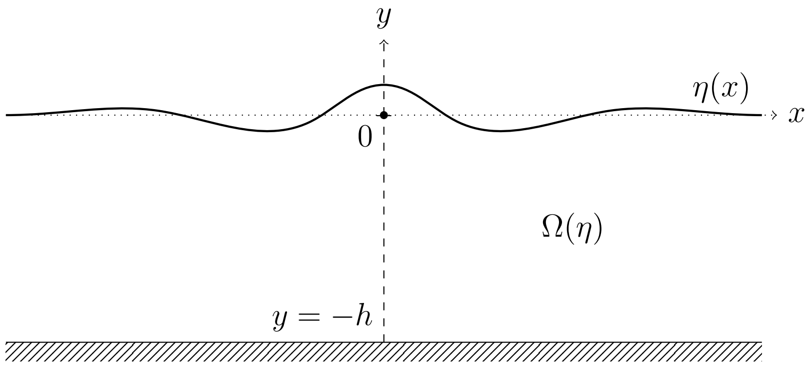


Figure 5.1: The domain, with a qualitative surface profile.

5.3 The Zakharov–Craig–Sulem formulation

It turns out that it is possible to reduce the steady[5] water-wave problem to an entirely one-dimensional one on the surface in a clever way. This is known as the Zakharov–Craig–Sulem formulation, and was first introduced by Zakharov in [Zakharov, 1968], and then put on a firmer mathematical basis in [Craig et al., 1992, Craig and Sulem, 1993]. In the last of the mentioned papers, it was used as the foundation for an algorithm for numerical simulation of gravity waves. The formulation relies on the fluid being irrotational, but it is in fact only necessary that this holds near the surface. This is where the compact support of the vorticity comes in. Our version of the formulation will be slightly different, because of the presence of vorticity.

Like in [Shatah et al., 2013], we use the stream function, and not the velocity potential, for the irrotational part. Most of the literature in this area uses the velocity potential, since this is the only approach that generalizes to flow in three dimensions (and higher for that matter, but this is of course not physically relevant). The book [Lannes, 2013] offers

[5]The method also works for the full equation in time, but we have no need for this. For three-dimensional waves, the reduction is to a two-dimensional problem on the surface.

a wealth of results in this direction. Appropriate versions of the theorems proven there should still hold for the stream function; the only real difference is between a Dirichlet and Neumann boundary condition in the Laplace equation defining them. In our case, there is actually a real reason as to why we cannot use the velocity potential, which we will comment on later.

Suppose that we have solved the steady water wave problem for some w, p, η. It is then convenient to split the velocity w as

$$w = \hat{w} + W,$$

where $\hat{w} = (\hat{u}, \hat{v})$ is irrotational, i.e., $\nabla \times \hat{w} = 0$ and $\nabla \times W = \omega$. We also assume that we have $\nabla \cdot \hat{w} = \nabla \cdot W = 0$. Here one thinks of $W = (U, V)$ as in some sense "known", or at least something we can control, and in that sense this splitting is unique.

Although we will allow for ω to be a non-regular (or *singular*) distribution, in particular that it is some combination of shifted Dirac delta functions, the vector field $\hat{w}$ will be assumed to be at least in $H^1(\Omega(\eta))^2$, and W to be at least $L^1_{\text{loc}}(\Omega(\eta))$ (regular). By the assumption of $\nabla \cdot \hat{w} = 0$, the differential

$$\hat{v}\, dx - \hat{u}\, dy$$

on $\Omega(\eta)$ is closed. Hence, as $\Omega(\eta)$ is simply connected and open in $\mathbb{R}^2$, the differential is exact by a generalization of the Poincaré lemma; see [Mardare, 2008, Theorem 3.1]. Thus, there is a function $\hat{\psi} \in \dot{H}^2(\Omega(\eta))$, determined uniquely modulo constants by $\hat{w}$, such that

$$\hat{w} = \nabla^\perp \hat{\psi} := (-\hat{\psi}_y, \hat{\psi}_x).$$

Moreover, by assumption of irrotationality of $\hat{w}$ ($\nabla \times \hat{w} = 0$), the function $\hat{\psi}$ is harmonic in the sense of distributions, i.e.,

$$\Delta \hat{\psi} = 0.$$

The function $\hat{\psi}$ is said to be a *stream function* for $\hat{w}$, and as it is harmonic in the sense of distributions, it is harmonic in the classical sense by Weyl's lemma, Lemma A.2 in Appendix A. In particular, it is C^∞ on $\Omega(\eta)$. Note that, while the gradient is orthogonal to the level curves, $\nabla^\perp \hat{\psi}$ is tangent to the level curves of $\hat{\psi}$, pointing such that $\hat{\psi}$ increases from the left hand side to the right hand side.

Furthermore, as $\nabla \cdot W = 0$, also the differential

$$V\, dx - U\, dy$$

is closed. Hence, by [Mardare, 2008, Theorem 2.1], there exists a distribution Ψ, determined uniquely modulo constants by W, such that

$$W = \nabla^\perp \Psi := (-\Psi_y, \Psi_x), \quad \text{where then } \Delta \Psi = \omega,$$

for some Ψ. This distribution is necessarily also regular (L^1_{loc}) by [Deny and Lions, 1953–54, Corollaire 2.1]. Further assumptions on W would imply more regularity on Ψ, but for now observe that Ψ is necessarily C^∞ outside $\operatorname{supp} \omega$, by the same reasoning as for $\hat{\psi}$.

By the above, we thus have that

$$w = \nabla^{\perp}(\hat{\psi} + \Psi) \tag{5.11}$$

holds for the velocity field w. Note that the boundary condition Equation (5.7) at the bottom translates to

$$v = \hat{\psi}_x + \Psi_x = 0 \quad \text{at } y = -h \tag{5.12}$$

for the stream function. Suppose now that W is chosen such that $\Psi = 0$ at the bottom (recall the uniqueness up to constants). Then the boundary condition in Equation (5.12) translates to

$$\hat{\psi}_x\Big|_{y=-h} = 0,$$

meaning that also $\hat{\psi}$ is constant along the bottom. Since $\hat{\psi}$ is unique modulo constants, we may as well take this to be

$$\hat{\psi}\Big|_{y=-h} = 0$$

instead.

We will now apply the assumption that that $\operatorname{supp} \omega \Subset \Omega(\eta)$. Because of the compact support of ω, there is some $\varepsilon > 0$ such that the (simply connected) set

$$O \coloneqq \Omega(\eta) \setminus \overline{\Omega(\eta - \varepsilon)}$$

does not intersect $\operatorname{supp} \omega$. Hence, as w is irrotational on this set, the differential

$$u\, dx + v\, dy$$

on O is closed. Thus, we deduce the existence of a function $\varphi \colon O \to \mathbb{R}$, known as the velocity potential, determined uniquely up to constants by $w|_O$, such that

$$w|_O = \nabla \varphi, \tag{5.13}$$

and which is harmonic by the assumption of incompressibility.

We use Equation (5.13) to deduce a version of the Bernoulli equation for solutions of the steady water-wave problem. Indeed, observe that by Equation (5.5) one has[6]

$$\begin{aligned} 0 &= -c\partial_x \nabla\varphi + (\nabla\varphi \cdot \nabla)\nabla\varphi + \nabla p + g e_2 \\ &= \nabla\left(-c\varphi_x + \frac{1}{2}|\nabla\varphi|^2 + p + gy\right), \end{aligned}$$

and so the expression inside the parentheses is constant in O. Assuming then that $\nabla\varphi$ and p has a well-defined trace to $\partial\Omega(\eta)$, we obtain

$$c(\hat{\psi}_y + \Psi_y) + \frac{1}{2}|\nabla\hat{\psi} + \nabla\Psi|^2 - \alpha^2\kappa(\eta) + g\eta = C, \quad \text{at } y = \eta(x), \tag{5.14}$$

for some $C \in \mathbb{R}$. Here we have inserted the boundary condition for the pressure at the surface, Equation (5.9). For the case of localized waves, we can let $|x| \to \infty$ to immediately deduce that $C = 0$. We now need the following formal definitions to proceed, which will be specified later on.

[6]Here y means the map $(x, y) \mapsto y$. We have also used the identity $(A \cdot \nabla)A = \frac{1}{2}\nabla(|A|^2) - A \times (\nabla \times A)$ for vector fields on $\mathbb{R}^3$, by adding a zero-component to $\nabla\varphi$.

Definition 5.2 (Harmonic extension operator)**.** Given η, we define the harmonic extension operator $H(\eta)$ as the operator mapping each function $\zeta\colon \mathbb{R} \to \mathbb{R}$ to the harmonic function $\hat{\psi}\colon \Omega(\eta) \to \mathbb{R}$ satisfying

$$\begin{aligned} \hat{\psi}(\cdot, \eta(\cdot)) &= \zeta, \\ \hat{\psi}(\cdot, -h) &= 0. \end{aligned}$$

Remark. Some comments are in order: First, Definition 5.2 requires that traces are well defined from $\Omega(\eta)$ to $\partial\Omega(\eta)$ in the function spaces considered, or at least that some meaning is put to boundary values. Second, one must have the property that every ζ of interest actually can be extended to a harmonic function on $\Omega(\eta)$. Lastly, one has the question of uniqueness. (It is well known that one must employ some kind of limitation on the growth of a solution of the Laplace equation on unbounded domains in $\mathbb{R}^2$ in order to obtain uniqueness. See the Phragmén-Lindelöf principle, [Markushevich, 1965b, Section 34].)

Definition 5.3 (Dirichlet-to-Neumann operator)**.** Given η, we define the Dirichlet-to-Neumann operator $N(\eta)$ as the operator mapping Dirichlet data to Neumann data; that is, the operator defined by

$$N(\eta)\zeta \coloneqq n \cdot \nabla[H(\eta)\zeta]\big|_{y=\eta}, \quad \text{where } n \coloneqq \frac{(-\eta', 1)}{\langle \eta' \rangle} \text{ is the unit surface normal,}$$

for functions $\zeta\colon \mathbb{R} \to \mathbb{R}$.

The Dirichlet-to-Neumann operator is an example of a more general class of operators known as Poincaré–Steklov operators, mapping one kind of boundary data for solutions of elliptic partial differential equations to another kind. Because of the somewhat inconvenient factor of $\langle \eta' \rangle^{-1}$ in the definition of $N(\eta)$, we will typically use the non-normalized Dirichlet-to-Neumann operator $G(\eta)$ defined by

$$G(\eta)\zeta \coloneqq (-\eta', 1) \cdot \nabla[H(\eta)\zeta]$$

instead.

With Definition 5.2 in mind, define $\zeta\colon \mathbb{R} \to \mathbb{R}$ by

$$\zeta \coloneqq \hat{\psi}(\cdot, \eta(\cdot)), \tag{5.15}$$

that is, the trace of $\hat{\psi}$ on the surface. By our assumptions, then, we have

$$\hat{\psi} = H(\eta)\zeta,$$

and we will use this to reformulate Equation (5.14) in a way that only involves ζ and Ψ. Note that

$$\begin{aligned} \zeta' &= \hat{\psi}_x + \eta' \hat{\psi}_y, \\ G(\eta)\psi &= -\eta' \hat{\psi}_x + \hat{\psi}_y, \end{aligned}$$

where the right hand side is evaluated at $y = \eta(x)$. By inverting these relations, we find that

$$\hat{\psi}_x = \frac{\zeta' - \eta' G(\eta)\zeta}{\langle \eta' \rangle^2}, \quad \hat{\psi}_y = \frac{\eta'\zeta' + G(\eta)\zeta}{\langle \eta' \rangle^2}, \quad \text{at } y = \eta(x), \tag{5.16}$$

which can be inserted into Equation (5.14) to yield[7]

$$c\left[\frac{\eta'\zeta' + G(\eta)\zeta}{\langle \eta' \rangle^2} + \Psi_y\right] + \frac{(\zeta' + (1,\eta')\cdot\nabla\Psi)^2 + (G(\eta)\zeta + (-\eta',1)\cdot\nabla\Psi)^2}{2\langle \eta' \rangle^2} + g\eta - \alpha^2\kappa(\eta) = 0 \tag{5.17}$$

in the localized case, after some simple calculations.

In a similar fashion, one obtains

$$c\eta' + \zeta' + (1,\eta')\cdot\nabla\Psi = 0 \tag{5.18}$$

from the kinematic boundary condition in Equation (5.8). We emphasize that Ψ is evaluated at $y = \eta(x)$ in Equations (5.17) and (5.18), which we suppress for readability. Equations (5.17) and (5.18) form the Zakharov–Craig–Sulem formulation, which we will combine with a suitable vorticity equation. One may note that the pressure, p, has been eliminated from the formulation entirely. We will cover its recovery in Section 6.4.

5.4 The operators $H(0)$ and $G(0)$ on the strip

It is possible to write down explicit expressions for the harmonic extension operator and Dirichlet-to-Neumann operator, introduced in Definitions 5.2 and 5.3, in terms of Fourier multipliers when $\eta = 0$, i.e. when the fluid domain $\Omega(\eta)$ is the strip $\mathbb{R} \times (-h, 0)$. The calculation is simple, but may be instructive to go through: Since we will bifurcate from a trivial solution in Theorem 6.11, the operators for the strip will play a role. We will assume that all functions involved are such that our manipulations are justified.

The equation of interest is

$$\begin{aligned} \Delta\psi = 0, &\quad \text{in } \mathbb{R} \times (-h,0), \\ \psi|_{y=0} = \zeta, &\quad \psi|_{y=-h} = 0, \end{aligned}$$

which, if we use hats on ψ for the Fourier transform in the first variable, is equivalent to the equation

$$\begin{aligned} -\xi^2\hat{\psi} + \hat{\psi}_{yy} = 0, &\text{in } \mathbb{R} \times (-h,0), \\ \hat{\psi}\Big|_{y=0} = \hat{\zeta}, &\quad \hat{\psi}\Big|_{y=-h} = 0 \end{aligned}$$

[7] We mention that this is a slightly different equation than the one used in [Shatah et al., 2013], but that they are equivalent after inserting the kinematic boundary condition, Equation (5.18).

for the Fourier transforms. For fixed ξ, this is an ordinary differential equation in y, which has a general solution that may be written in the form

$$\hat{\psi}(\xi, y) = A(\xi)\cosh((y+h)|\xi|) + B(\xi)\sinh((y+h)|\xi|), \tag{5.19}$$

where the coefficients $A(\xi)$ and $B(\xi)$ need to be determined.

In order to satisfy the boundary conditions, we must demand that

$$\begin{gathered} A(\xi)\cosh(h|\xi|) + B(\xi)\sinh(h|\xi|) = \hat{\zeta}(\xi), \\ A(\xi) = 0, \end{gathered}$$

or

$$A(\xi) = 0, \quad B(\xi) = \frac{1}{\sinh(h|\xi|)}\hat{\zeta}(\xi), \tag{5.20}$$

yielding that $\hat{\psi}$ is given by

$$\hat{\psi}(\xi, y) = \frac{\sinh((y+h)|\xi|)}{\sinh(h|\xi|)}\hat{\zeta}(\xi),$$

by inserting Equation (5.20) into Equation (5.19). Thus, the harmonic extension operator $H(0)$ may be written in terms of Fourier multipliers as

$$(H(0)\zeta)(\cdot, y) = \frac{\sinh((y+h)|D|)}{\sinh(h|D|)}\zeta,$$

and taking partial derivatives, we see that the Dirichlet-to-Neumann operator $G(0)$ is given by

$$G(0) = |D|\coth(h|D|).$$

One may observe that $G(0)$ is well defined[8] as an operator $H^s(\mathbb{R}) \to H^{s-1}(\mathbb{R})$, as

$$|\xi|\coth(h|\xi|) \sim \langle\xi\rangle,$$

and in fact that it is an isomorphism. It is for instance not hard to show that

$$\frac{1}{2}\min(1, h^{-1})\langle\xi\rangle \leq |\xi|\coth(h|\xi|) \leq \langle h^{-1}\rangle\langle\xi\rangle$$

for all $h > 0$ and $\xi \in \mathbb{R}$. This is not true for the velocity potential, as one will then find

$$\tilde{G}(0) = |D|\tanh(h|D|)$$

instead, [Lannes, 2013, p. 15]. Both the function defined by $\xi \mapsto |\xi|\tanh(h|\xi|)$ and its derivative vanish at the origin.

[8] We mention that this is in agreement with the more general Theorem 6.7 in Chapter 6.

6 Existence of traveling waves with compactly supported vorticity

We now focus on proving the existence of a family of small amplitude and small velocity traveling waves with vorticity consisting of a point vortex situated on the y-axis. In other words, solutions with vorticity of the form

$$\omega = \varepsilon\delta_\theta, \tag{6.1}$$

where $0 < |\varepsilon| \ll 1$, $\theta \in (0,1)$ and where we have defined

$$\delta_\theta \coloneqq \delta_{(0,-(1-\theta)h)}.$$

The constant θ then corresponds to the relative position of the point vortex above the bottom, and the parameter ε, describing the strength of the vortex, will be used as the bifurcation parameter. In order to do this, we will use the Zakharov–Craig–Sulem formulation of the problem, which we introduced in the previous chapter. The reason for adding the dependence on θ is, of course, that this will prove the existence of such traveling waves with a point vortex at any given depth within the fluid domain.

One can opt to scale away the water depth h, but we choose to keep it. This makes the dependence on the height easier to see at a glance, and adds little complexity.

We will from here on always assume that η is such that $\min\eta > -(1-\theta)h$, as these values of η are the only ones that make physical sense. Furthermore, we will assume that $\max\eta < (1-\theta)h$. The reason for this is purely technical (as we will see after Proposition 6.1), but note that this is not really a restriction, as we only consider waves of small amplitude[1]. For the purpose of accounting for these assumptions, define the set

$$\Gamma_\theta \coloneqq \{\eta \in BC(\mathbb{R}) : \inf\eta > -(1-\theta)h,\ \sup\eta < (1-\theta)h\}, \tag{6.2}$$

which is clearly open in $BC(\mathbb{R})$, and thus so is the intersection $\Gamma_\theta \cap H^s(\mathbb{R})$ in $H^s(\mathbb{R})$ for any $s > \frac{1}{2}$ by Theorem A.5. The only admissible surface profiles η will be those in Γ_θ. One may note that since Γ_θ is a ball in $BC(\mathbb{R})$ with radius $(1-\theta)h$, it shrinks to a point (the trivial surface profile $\eta = 0$) as $\theta \uparrow 1$.

[1]If one wishes to do global bifurcation theory, one may wish to modify the argument in order to avoid making the technical assumption. We shall not concern ourselves with this.

Observe that if we define $\Psi \in D'(\Omega(\eta))$ by

$$\Psi(x,y) := \frac{1}{4\pi}\log\left(\frac{x^2+(y+(1-\theta)h)^2}{x^2+(y+(1+\theta)h)^2}\right) \tag{6.3}$$

then

$$\nabla^\perp\Psi(x,y) = -\frac{1}{2\pi}\left(\frac{(y+(1-\theta)h,-x)}{x^2+(y+(1-\theta)h)^2} - \frac{(y+(1+\theta)h,-x)}{x^2+(y+(1+\theta)h)^2}\right), \tag{6.4}$$
$$\Delta\Psi = \delta_\theta, \quad \Psi|_{y=-h} = 0$$

by Proposition 5.1, as the point $(0,-(1+\theta)h)$ is outside of $\Omega(\eta)$. Up to the factor ε, the distribution Ψ satisfies what we require. The denominator in Equation (6.3) is added in order to satisfy the condition on Ψ at the bottom, a trick which is known as the method of images (see Figure 6.1(a)). It also has the effect of making Ψ decay at infinity. Indeed, while the logarithm of the numerator in Equation (6.3) clearly does not decay, one may observe that

$$\Psi(x,y) = \frac{1}{4\pi}\log\left(1-4\theta\frac{y+h}{x^2+(y+(1+\theta)h)^2}\right)$$

is $O(|x|^{-2})$ as $|(x,y)|\to\infty$ in $\Omega(\eta)$ (recall that $\Omega(\eta)$ is bounded in the y-direction). The same increased decay is also enjoyed by $\nabla^\perp\Psi$. While the first term in the parentheses in Equation (6.4) alone has norm $|(x,y+(1-\theta)h)|^{-1}$, one has

$$|\nabla^\perp\Psi(x,y)| = \frac{\theta h}{\pi}\frac{1}{|(x,y+(1-\theta)h)||(x,y+(1+\theta)h)|} = O(|(x,y)|^{-2})$$

as $|(x,y)|\to\infty$, because of cancellations. We will not have direct need for this decay, but note that one still has $\Psi \notin \dot{H}^1(\Omega(\eta))$, because $\nabla^\perp\Psi$ is *not* locally square integrable at $(0,-(1-\theta)h)$.

While Ψ has a particularly simple form, and could be used for the existence proof (Theorem 6.11), it is beneficial to introduce a different distribution, only differing from Ψ by a harmonic function, which in addition to vanishing at the bottom vanishes on the surface when $\eta = 0$. The distribution Φ, described in Proposition 6.1 below, has this property (see Figure 6.1(b)). It is slightly more cumbersome to work with, but will give us exact expressions that we have not been able to obtain with Ψ.

Proposition 6.1. *Let $\eta \in \Gamma_\theta$ and define $\Phi\colon \Omega(\eta)\to\mathbb{R}$ by*

$$\Phi(x,y) := \frac{1}{4\pi}\log\left(\frac{\cosh(\pi x/h)+\cos(\pi(y/h-\theta))}{\cosh(\pi x/h)+\cos(\pi(y/h+\theta))}\right). \tag{6.5}$$

Then Φ defines a regular distribution, and

$$\begin{aligned}\Delta\Phi &= \delta_\theta,\\ \Phi|_{y=0} &= 0,\\ \Phi|_{y=-h} &= 0.\end{aligned} \tag{6.6}$$

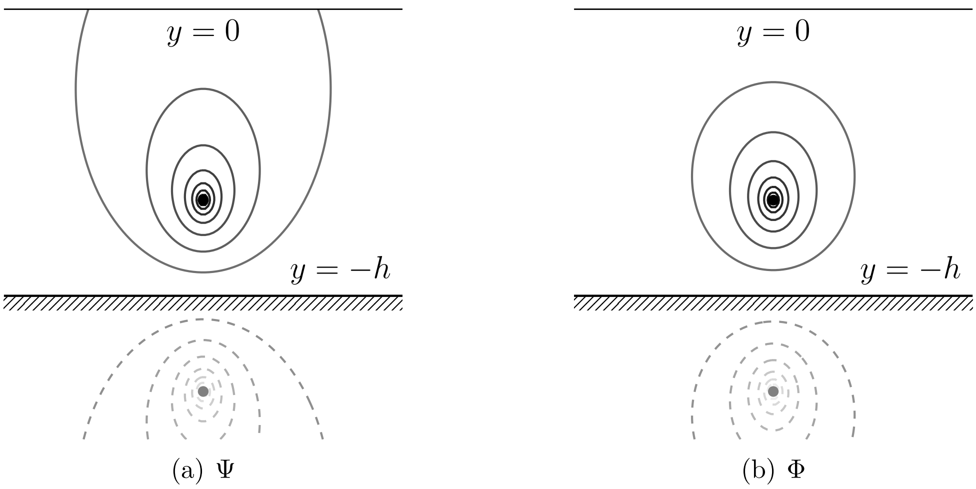


Figure 6.1: Placement of the point vortex and its "mirror vortex" when $\theta = 1/3$, with contours for Ψ and Φ.

Moreover (for completeness),

$$\begin{aligned}\nabla^\perp \Phi(x, y) &= \frac{1}{4h} \left(\frac{(\sin(\pi(y/h - \theta)), \sinh(\pi x/h))}{\cosh(\pi x/h) + \cos(\pi(y/h - \theta))} - \frac{(\sin(\pi(y/h + \theta)), \sinh(\pi x/h))}{\cosh(\pi x/h) + \cos(\pi(y/h + \theta))} \right) \\ &= -\frac{\sin(\pi\theta)}{2h} \frac{(\cos(\pi\theta) + \cos(\pi y/h)\cosh(\pi x/h), \sin(\pi y/h)\sinh(\pi x/h))}{(\cosh(\pi x/h) + \cos(\pi(y/h - \theta)))(\cosh(\pi x/h) + \cos(\pi(y/h + \theta)))},\end{aligned}$$

and the function $(x, y) \mapsto \Phi(x, y) - (4\pi)^{-1} \log(x^2 + (y + (1 - \theta)h)^2)$ *is harmonic and satisfies*

$$\nabla^\perp \left(\Phi - \frac{1}{4\pi} \log(x^2 + (y + (1 - \theta)h)^2) \right) (0, -(1 - \theta)h) = \left(\frac{1}{4h} \cot(\pi\theta), 0 \right).$$

Proof. We will apply Theorem A.10 to prove this result. We thus need a bijective conformal map from the strip $\mathbb{R} \times (-h, 0) \subseteq \mathbb{C}$ to the unit disk, mapping the point $-i(1 - \theta)h$ to the origin. We do this in three steps; first mapping the strip $\mathbb{R} \times (-h, 0)$ to the strip $\mathbb{R} \times (0, 1)$, then the strip to the upper half plane, and finally the upper half plane to the unit disk:

$$\begin{array}{ccccccc} \mathbb{R} \times (-h, 0) & \xrightarrow{z \mapsto (z + ih)/h} & \mathbb{R} \times (0, 1) & \xrightarrow{z \mapsto \exp(\pi z)} & \mathbb{R} \times (0, \infty) & \xrightarrow{z \mapsto \frac{z - \exp(i\pi\theta)}{z - \exp(-i\pi\theta)}} & \mathbb{D} \\ -i(1 - \theta)h & & i\theta & & e^{i\pi\theta} & & 0 \end{array} \tag{6.7}$$

The conformal map for each individual step should be well known from complex analysis, see for instance [Gamelin, 2001, II.7, p. 60]. Hence

$$f(z) := \frac{e^{\pi(z+ih)/h} - e^{i\pi\theta}}{e^{\pi(z+ih)/h} - e^{-i\pi\theta}}$$

defines the desired map from the strip to the unit disk. By the aforementioned theorem, then,

$$
\begin{aligned}
\Phi(x, y) &:= \frac{1}{2\pi} \log(|f(x+iy)|) \\
&= \frac{1}{4\pi} \log\left(\left|\frac{e^{\pi(x+i(y+h))/h} - e^{i\pi\theta}}{e^{\pi(x+i(y+h))/h} - e^{-i\pi\theta}}\right|^2\right) \\
&\ \ \vdots \\
&= \frac{1}{4\pi} \log\left(\frac{\cosh(\pi x/h) + \cos(\pi(y/h - \theta))}{\cosh(\pi x/h) + \cos(\pi(y/h + \theta))}\right)
\end{aligned}
$$

solves Equation (6.6) in $\mathbb{R} \times (-h, 0)$. One can also verify the boundary conditions directly by observing that cos is even and 2π-periodic. Moreover it is immediate that Φ is well defined on all of $\mathbb{R}^2$ (except in the isolated singularities). By the oddness in the y-direction around $y = 0$, Φ satisfies the mean value property on $y = 0$, and is therefore harmonic across it[2]. Hence Φ solves Equation (6.6) also in $\Omega(\eta)$; because of the assumption of $\eta \in \Gamma_\theta$ we will not get additional point vortices in our domain.

The explicit expressions for $\nabla^\perp \Phi$ is found by straightforward differentiation. This will also hold in the distributional sense because of the corresponding calculation we did for the Newtonian potential in Proposition 5.1. Finally, we have

$$
\begin{aligned}
\nabla^\perp\left(\Phi - \frac{1}{4\pi}\log(x^2 + (y - (1-\theta)h)^2)\right)(0, \theta) &= \frac{i}{4\pi}\overline{\left(\frac{f''(i\theta)}{f'(i\theta)}\right)} \\
&\ \ \vdots \\
&= \frac{i}{4\pi}\overline{\left(-\frac{\pi}{h}\frac{e^{i\pi\theta} + e^{-i\pi\theta}}{e^{i\pi\theta} - e^{-i\pi\theta}}\right)} \\
&= \left(\frac{1}{4h}\cot(\pi\theta), 0\right)
\end{aligned}
$$

by the remark after Theorem A.10 in Appendix A, which will be important for the asymptotic velocity of the traveling waves that we shall obtain in Theorem 6.11. □

Note that, as opposed to Ψ, the function Φ is $2h$-periodic in the y-direction. Thus there will arise another phantom point vortex at the point $(0, (1-\theta)h)$, since $-(1+\theta)h + 2h = (1-\theta)h$. This is the reason for the limitation on the maximum height of the surface profiles in the set Γ_θ defined in Equation (6.2).

We can connect Ψ and Φ using the harmonic extension operator. Indeed, it will be clear that

$$
\Phi = \Psi - H(0)\Psi(\cdot, 0) \tag{6.8}
$$

in $\Omega(0) = \mathbb{R} \times (-h, 0)$. We recall from Section 5.4 that both the harmonic extension operator $H(0)$ and the Dirichlet-to-Neumann operator $G(0)$ can be written in terms of Fourier multipliers.

[2]This alternatively follows from f extending to a meromorphic function on $\mathbb{C}$.

The next proposition is crucial, because the traces of Φ and its derivatives on the surface enter in the Zakharov–Sulem–Craig formulation of the problem. Having an explicit expression for Φ enables us to prove the proposition in a quite direct way.

Proposition 6.2. *Suppose that $\eta \in H^s(\mathbb{R}) \cap \Gamma_\theta$, where $s > \frac{1}{2}$. Then*

$$
\begin{gathered}
[x \mapsto \Phi(x, \eta(x))] \in H^s(\mathbb{R}), \\
[x \mapsto \nabla^\perp \Phi(x, \eta(x))] \in H^s(\mathbb{R})^2.
\end{gathered}
$$

Moreover, the dependence on η is analytic.

Proof. Observe that we may rewrite $\Phi(x, \eta(x))$ as

$$
\Phi(x, \eta(x)) = \frac{1}{4\pi} \log\left(1 + \frac{2\sin(\pi\theta)\sin(\pi\eta(x)/h)}{\cosh(\pi x/h) + \cos(\pi(\eta(x)/h + \theta))}\right), \tag{6.9}
$$

where we may further rewrite

$$
\frac{\sin(\pi\eta(x)/h)}{\cosh(\pi x/h) + \cos(\pi(\eta(x)/h + \theta))} = \frac{\sin(\pi\eta(x)/h)\operatorname{sech}(\pi x/h)}{1 + \cos(\pi(\eta(x)/h + \theta))\operatorname{sech}(\pi x/h)}.
$$

Now

$$
\mathscr{F}(\operatorname{sech}(\pi x/h))(\xi) = \frac{h}{\sqrt{2\pi}} \operatorname{sech}\left(\frac{h\xi}{2}\right),
$$

so $[x \mapsto \operatorname{sech}(\pi x/h)]$ is in $H^t(\mathbb{R})$ for any $t \in \mathbb{R}$. Moreover, since sin is smooth (C^∞) and $\sin(0) = 0$, we can use Lemma A.9 to conclude that $[x \mapsto \sin(\pi\eta(x)/h)]$ is in $H^s(\mathbb{R})$. The same is true for

$$
\begin{aligned}
&\cos(\pi(\eta(x)/h + \theta))\operatorname{sech}(\pi x/h) \\
&\quad = [\cos(\pi(\eta(x)/h + \theta)) - \cos(\pi\theta)]\operatorname{sech}(\pi x/h) + \cos(\pi\theta)\operatorname{sech}(\pi x/h),
\end{aligned}
$$

which is also lower bounded away from -1 because of the assumption that $\eta \in \Gamma_\theta$. Here we also used the fact that $H^s(\mathbb{R})$ is an algebra for $s > \frac{1}{2}$, which is Lemma A.7. We may therefore apply Lemma A.7 and Lemma A.8 to deduce that the fraction in the logarithm in Equation (6.9) is a member of $H^s(\mathbb{R})$. Finally, because this fraction is bounded away from -1 from below, again by $\eta \in \Gamma_\theta$, we deduce from Lemma A.9 that the first claim holds. The analyticity also holds by the same lemma, after a small argument.

The corresponding claim for the derivative holds by similar calculations. We can rewrite the derivative $\nabla^\perp \Phi(x, \eta(x))$ as

$$
-\frac{\sin(\pi\theta)}{2h} \frac{[\cos(\pi\theta)\operatorname{sech}(\pi x/h) + \cos(\pi\eta(x)/h), \sin(\pi\eta(x)/h)\tanh(\pi x/h)]\operatorname{sech}(\pi x/h)}{(1 + \cos(\pi(\eta(x)/h - \theta))\operatorname{sech}(\pi x/h))(1 + \cos(\pi(\eta(x)/h + \theta))\operatorname{sech}(\pi x/h))},
$$

where the only extra piece of information we need is that $[x \mapsto \tanh(\pi x/h)\operatorname{sech}(\pi x/h)]$ is clearly in $H^s(\mathbb{R})$; tanh is bounded, and the derivative of tanh is given by $x \mapsto \operatorname{sech}^2(x)$. $\square$

Remark. A completely analogous result to Proposition 6.2 holds for the alternative stream function Ψ. (In the case that one wishes to use that instead.)

As we have seen, because of the reliance on the stream function and the operators $H(\eta)$ and $G(\eta)$, a central problem is the solution of the Laplace equation,

$$\begin{aligned} \Delta\hat{\psi} &= 0 \quad \text{in } \Omega(\eta), \\ \hat{\psi}\Big|_{y=\eta} &= \zeta, \quad \hat{\psi}\Big|_{y=-h} = 0 \end{aligned} \tag{6.10}$$

on the fluid domain, given η and ζ. By Weyl's lemma, Lemma A.2, any distributional solution of these equations can be represented by a function that is harmonic in the pointwise sense. In particular, as we have mentioned previously, this means that any distributional solution will actually be smooth.

We have the following theorem, which is adapted from [Lannes, 2013, Corollary 2.44], and which establishes both existence and uniqueness to Equation (6.10) in suitable Sobolev spaces. Functions on the surface will be identified with functions on the real line as in Equation (5.15).

Theorem 6.3 [Lannes, 2013] (Well-posedness of the Laplace equation)**.** *Suppose that $\eta \in H^s(\mathbb{R}) \cap \Gamma_\theta$ for some $s > \frac{3}{2}$, and that $\zeta \in H^{\frac{3}{2}}(\mathbb{R})$. Then Equation* (6.10) *has a unique solution in $H^2(\Omega(\eta))$.*

Remark. While the natural setting for the velocity potential or the stream function on infinite depth is the Beppo–Levi spaces, used in both [Shatah et al., 2013] and [Lannes, 2013], this is not the case for the stream function on finite depth. Because we require $\hat{\psi}$ to be constant at the bottom, it must necessarily be the case that $\hat{\psi}$ tends to the same constant at infinity. Otherwise, because of the finite depth, $\hat{\psi}_y$ would not decay at infinity (in the sense that $\lim_{|(x,y)|\to\infty} \psi_y(x,y) = 0$), and therefore not describe a localized wave.

Observe that, as $H^s(\mathbb{R}) \hookrightarrow BC^1(\mathbb{R})$ for $s > \frac{3}{2}$, η is in particular Lipschitz in the setting of Theorem 6.3. This yields the following, which establishes the equivalence of the definitions of the Sobolev spaces on $\Omega(\eta)$, by the discussion after Proposition 3.2 in Section 3.2.

Proposition 6.4 (Strong local Lipschitz condition satisfied)**.** *If $\eta \in H^s(\mathbb{R}) \cap \Gamma_\theta$ for some $s > \frac{3}{2}$, then $\Omega(\eta)$ satisfies Definition 3.3.*

Proof. As we mentioned above, we have $\eta \in BC^1(\mathbb{R})$. Let $m \coloneqq -\min \eta < (1-\theta)h, M \coloneqq \max \eta < \theta h$, and $\delta \coloneqq \frac{1}{5}(h-m)$. It is sufficient to use a cover of two open sets,

$$\begin{aligned} U_1 &\coloneqq \mathbb{R} \times (m - 2\delta, M + 2\delta), \\ U_2 &\coloneqq \mathbb{R} \times (-2\delta, 2\delta), \end{aligned}$$

with corresponding Lipschitz mappings $\varphi_1 \coloneqq \eta, \varphi_2 \equiv h$ (where the second requires a rotation). All the conditions in Definition 3.3 are then trivially satisfied. □

6.1 Properties of the boundary operators

Theorem 6.3 enables us to rigorously define the harmonic extension operator described in Definition 5.2 as an operator $H^{\frac{3}{2}}(\mathbb{R}) \to H^2(\Omega(\eta))$, and using this, defining the Dirichlet-to-Neumann operator. The linearity of these operators follows immediately from the linearity of the Laplace equation and uniqueness of solutions; indeed, suppose that ζ_1, ζ_2 are boundary values and define

$$\hat{\psi} := \alpha H(\eta)\zeta_1 + \beta H(\eta)\zeta_2,$$

where $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{gathered} \Delta\hat{\psi} = 0, \\ \hat{\psi}\Big|_{y=\eta} = \alpha\zeta_1 + \alpha\zeta_2, \quad \hat{\psi}\Big|_{y=-h} = 0, \end{gathered}$$

whence $\hat{\psi} = H(\eta)(\alpha\xi_1 + \beta\xi_2)$ by uniqueness.

We shall see that the Dirichlet-to-Neumann operator has some very beneficial properties. The proofs of these theorems are very technical, and so will not be proved here. We refer the reader to [Lannes, 2013], which is a rich source of results for these operators also in more general settings. The results there are proved for the Laplace equation with a Neumann boundary at the bottom, but should be adaptable for the stream function with minor changes (cf. [Lannes, 2013, Theorem 3.49, Theorem A.13] for the case of infinite depth, where the boundary conditions for the Laplace equation for the stream function and velocity potential coincide).

The main idea is to map $\Omega(\eta)$ to the strip $\mathbb{R} \times (0,1)$ using a diffeomorphism with certain properties, since the strip is easier to work on (for instance, one can use the Fourier transform in the horizontal direction). On the strip one can establish the existence of unique variational solutions, prove that they are in fact solutions in the distributional or classical sense under sufficient regularity conditions, and then transfer some of the results back again. Proofs of existence of variational solutions often rely on the Poincaré inequality, which in its original form is only valid for bounded domains. A version still holds for $\Omega(\eta)$ due to boundedness in the vertical direction, see Proposition A.6.

The first theorem is a corollary of [Lannes, 2013, Corollary 2.40]. Note that the harmonic extension operator itself is of less interest than the Dirichlet-to-Neumann operator. We will for the most part only work with the traces on the surface.

Theorem 6.5 [Lannes, 2013] (Harmonic extensions). *Given $\eta \in H^s(\mathbb{R}) \cap \Gamma_\theta$ for some $s > \frac{3}{2}$, the harmonic extension operator $H(\eta)$ is a member of $B(H^{\frac{3}{2}}(\mathbb{R}), H^2(\Omega(\eta)))$. Moreover, its norm is uniformly bounded on subsets of $H^s(\mathbb{R}) \cap \Gamma_\theta$ that are bounded in the norm on $H^s(\mathbb{R})$.*

Before we state the next theorem, which is a combination of [Lannes, 2013, Theorem 3.15, Theorem A.11], we make a definition:

Definition 6.6 (Shape analyticity). We say that G is *shape analytic* if the map $\eta \mapsto G(\eta)\zeta$ is analytic, in the sense of Definition 3.4, for fixed ζ.

Theorem 6.7 [Lannes, 2013] (Dirichlet-to-Neumann)**.** *Assume that $\eta \in H^s(\mathbb{R}) \cap \Gamma_\theta$ for some $s > \frac{3}{2}$. Then*

$$G(\eta) \in B(H^s(\mathbb{R}), H^{s-1}(\mathbb{R})),$$

with norm uniformly bounded on subsets of $H^s(\mathbb{R}) \cap \Gamma_\theta$ that are bounded in the $H^s(\mathbb{R})$-norm. Furthermore, G is shape analytic.

In the same setting as above, the curvature of the surface is well defined. Note that $\kappa(\eta)$ is not necessarily a regular distribution unless $s \geq 2$, because it involves second order derivatives, but this will not cause any trouble for us.

Proposition 6.8 (Curvature)**.** *The curvature operator κ is well defined as an operator $H^s(\mathbb{R}) \to H^{s-2}(\mathbb{R})$ for any $s > \frac{3}{2}$. Moreover, it is analytic in a neighborhood of the origin in $H^s(\mathbb{R})$.*

Proof. We have

$$\kappa(\eta) = \left(\frac{\eta'}{\langle \eta' \rangle} \right)'$$

by definition. For this to be well defined, we only need to check that the expression in the parentheses is a well defined function in $H^{s-1}(\mathbb{R})$. Observe that the function $f \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x\langle x \rangle^{-1}$ is C^∞ and satisfies $f(0) = 0$. As $s - 1 > \frac{1}{2}$, Lemma A.9 ensures that $f(\eta') \in H^{s-1}(\mathbb{R})$. Since f is also analytic, κ is analytic by the same lemma. □

6.2 Functional-analytic setting

We are now almost ready to formulate the functional-analytic setting in which we will look for our solutions. There is one thing we have not yet looked at, namely the steady version of the weak vorticity transport equation, Equation (5.10). We will consider velocity fields of the form

$$w = \nabla^\perp \left(H(\eta)\zeta + \varepsilon\Phi \right), \tag{6.11}$$

cf. Equation (5.11). We know that the part of the stream function that is generated by the point vortex at $(0, -(1-\theta)h)$ is given by the Newtonian potential (recall Proposition 5.1)

$$\frac{\varepsilon}{4\pi} \log(x^2 + (y + (1-\theta)h)^2),$$

whence the vorticity equation reduces to

$$\begin{aligned}
(c, 0) &= \nabla^\perp [H(\eta)\zeta](0, -(1-\theta)h) + \varepsilon \nabla^\perp \left[\Phi - \frac{1}{4\pi} \log(x^2 + (y-\theta)^2) \right] (0, -(1-\theta)h) \\
&= \nabla^\perp [H(\eta)\zeta](0, -(1-\theta)h) + \varepsilon \left(\frac{1}{4h} \cot(\pi\theta), 0 \right),
\end{aligned}$$

where the last equality stems from the final part of Proposition 6.1.

In particular, this means that any solution necessarily must satisfy

$$[H(\eta)\zeta]_x(0, -(1-\theta)h) = 0.$$

For simplicity, we choose to look for η, ζ in appropriately chosen subspaces of $H^s(\mathbb{R})$, such that this condition is automatically satisfied. Specifically, define

$$\begin{aligned} H^s_{\mathrm{odd}}(\mathbb{R}) &\coloneqq \{f \in H^s(\mathbb{R}) : f \text{ is odd}\}, \\ H^s_{\mathrm{even}}(\mathbb{R}) &\coloneqq \{f \in H^s(\mathbb{R}) : f \text{ is even}\}, \end{aligned} \tag{6.12}$$

where parity may need to be understood in the sense of distributions[3]. when $s < 0$. If one defines $\sigma \colon S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ by $(\sigma\varphi)(x) = \varphi(-x)$, and extends this to distributions by $\langle \sigma T, \varphi \rangle = \langle T, \sigma\varphi \rangle$, then one says that a distribution $T \in S'(\mathbb{R}^d)$ is

$$\text{odd, if } \sigma T = -T; \quad \text{even, if } \sigma T = T.$$

One may check that this definition is the correct one for regular distributions. As an example for singular distributions, δ is an even distribution.

Since convergence in $H^s(\mathbb{R}^d)$ implies convergence in the distributional sense, see Lemma A.4, it is clear that the spaces defined in Equation (6.12) are closed subspaces of their respective parent spaces. (This is of course immediate for $s \geq 0$, since L^2-convergence implies pointwise almost everywhere convergence of a subsequence.) Hence they are Hilbert spaces in the inherited norm.

Assume now that $\eta \in H^s_{\mathrm{even}}(\mathbb{R}) \cap \Gamma_\theta$, with $s > \frac{3}{2}$, and that $\zeta \in H^{\frac{3}{2}}_{\mathrm{even}}(\mathbb{R})$. Then it must necessarily be the case that $H(\eta)\zeta$ is even in x. Indeed, $(x, y) \mapsto [H(\eta)\zeta](-x, y)$ yields a solution to Equation (6.10) in $H^s(\Omega(\eta))$ by the evenness of ζ and η, and so by uniqueness one must necessarily have $[H(\eta)\zeta](-x, y) = [H(\eta)\zeta](x, y)$ in $\Omega(\eta)$. By the evenness of $H(\eta)\zeta$ in x, then, $[H(\eta)\zeta]_x$ vanishes along the y-axis and so the vorticity equation reduces further to

$$c = \frac{\varepsilon}{4h}\cot(\pi\theta) - [H(\eta)\zeta]_y(0, -(1-\theta)h), \tag{6.13}$$

meaning that any two of the quantities $c, [H(\eta)\zeta]_y(0, -(1-\theta)h), \varepsilon$ for a solution of this type uniquely determines the third. In particular, we can determine c given the right hand side of Equation (6.13).

Remark. One has to be careful with claims about the solution set when $\varepsilon = 0$. Equation (6.13) of course only actually needs to be satisfied if $\varepsilon \neq 0$. This means that if we impose this equation, then we for instance lose the trivial set of solutions

$$(\eta, \zeta, c, \varepsilon) \in \{0\} \times \{0\} \times \mathbb{R} \times \{0\}$$

for the other equations (except for the point $(0, 0, 0, 0)$). This should be kept in mind in any claims of uniqueness, and is even more true after the rescaling that we will make.

Observe also that if $H(\eta)\zeta$ is even in x, then $[H(\eta)\zeta]_x$ is odd in x and $[H(\eta)\zeta]_y$ is even in x. Hence if η and ζ are even, then

$$(-\eta', 1) \cdot \nabla[H(\eta)\zeta]$$

[3]For the specific case of Sobolev spaces one can in fact define parity in terms of that of its Fourier transform (which is always regular, even when $s < 0$), since the Fourier transform preserves this property.

is even in x. All this then means that the Dirichlet-to-Neumann operator $G(\eta)$ is well defined as an operator $H^s_{\text{even}}(\mathbb{R}) \to H^{s-1}_{\text{even}}(\mathbb{R})$ for $\eta \in H^s_{\text{even}} \cap \Gamma_\theta$ and $s > \frac{3}{2}$. We also see directly from the definition of κ that it can be viewed as an operator $H^s_{\text{even}}(\mathbb{R}) \to H^{s-2}_{\text{even}}(\mathbb{R})$.

For convenience, define now the spaces

$$\begin{aligned} X^s &:= H^s_{\text{even}}(\mathbb{R}) \times H^s_{\text{even}}(\mathbb{R}) \times \mathbb{R}, \\ Y^s &:= H^{s-2}_{\text{even}}(\mathbb{R}) \times \partial_x(H^s_{\text{even}})(\mathbb{R}) \times \mathbb{R}, \end{aligned}$$

where $\partial_x(H^s_{\text{even}})(\mathbb{R}) \subseteq H^{s-1}_{\text{odd}}(\mathbb{R})$. Then X^s, Y^s are Banach spaces in the norms[4]

$$\begin{aligned} \|(\eta,\zeta,c)\|_{X^s} &:= \|\eta\|_{H^s(\mathbb{R})} + \|\zeta\|_{H^s(\mathbb{R})} + |c|, \\ \|(f,g,a)\|_{Y^s} &:= \|f\|_{H^{s-2}(\mathbb{R})} + \|g\|_{\partial_x H^s(\mathbb{R})} + |a|. \end{aligned}$$

The intention is to use X^s in the domain, and Y^s in the codomain of the maps that we will soon define.

From the governing equations, Equation (6.13) and Equations (5.17) and (5.18) with Equation (6.11) inserted, we expect that the resulting solutions will be of order $O(\varepsilon)$ as $\varepsilon \to 0$, where we recall that ε is the vortex strength introduced in Equation (6.20). It is thus convenient to make the rescaling

$$(\eta,\zeta,c) = \varepsilon(\tilde{\eta},\tilde{\zeta},\tilde{c}), \tag{6.14}$$

which also means that we need to incorporate Γ_θ into this setting in some way. This leads us to introduce the set

$$U^s_\theta := \{(\tilde{\eta},\tilde{\zeta},\tilde{c},\varepsilon) \in X^s \times \mathbb{R} : \varepsilon\tilde{\eta} \in \Gamma_\theta\},$$

which is clearly open in $X^s \times \mathbb{R}$ for $s > \frac{1}{2}$, since $\Gamma_\theta \cap H^s_{\text{even}}(\mathbb{R})$ is an open neighborhood of the origin in $H^s_{\text{even}}(\mathbb{R})$.

We now proceed to discuss the three maps that together we will form the basis for our argument.

The first map (Bernoulli equation)

Let $s > \frac{3}{2}$, and define $F_1 \colon U^s_\theta \to H^{s-2}_{\text{even}}(\mathbb{R})$ by

$$\begin{gathered} (\tilde{\eta},\tilde{\zeta},\tilde{c},\varepsilon) \\ \mapsto \\ \varepsilon\tilde{c}\left[\frac{\varepsilon\tilde{\eta}'\tilde{\zeta}' + G(\varepsilon\tilde{\eta})\tilde{\zeta}}{\langle\varepsilon\tilde{\eta}'\rangle^2} + \Phi_y\right] + \varepsilon\frac{(\tilde{\zeta}' + \Phi_x + \varepsilon\tilde{\eta}'\Phi_y)^2 + (G(\varepsilon\tilde{\eta})\tilde{\zeta} - \varepsilon\tilde{\eta}'\Phi_x + \Phi_y)^2}{2\langle\varepsilon\tilde{\eta}'\rangle^2} \\ + g\tilde{\eta} - \frac{\alpha^2}{\varepsilon}\kappa(\varepsilon\tilde{\eta}), \end{gathered} \tag{6.15}$$

[4]Sensible norms on X^s, which are all equivalent, are those that arise as a norm on $\mathbb{R}^3$ composed with the map $(\eta,\zeta,c) \mapsto (\|\eta\|_{H^s(\mathbb{R})}, \|\zeta\|_{H^s(\mathbb{R})}, |c|)$; and similarly for Y^s.

where the derivatives of Φ are evaluated at $(x, \varepsilon\tilde{\eta}(x))$. If we take into account Equation (6.11), then solutions of Equation (5.17) in this setting correspond to solutions of $F_1(\tilde{\eta}, \tilde{\xi}, \tilde{c}, \varepsilon) = 0$ (recall the rescaling). Note that while the division by ε in the last term looks ominous, it will cancel against the factor ε in the numerator of $\kappa(\varepsilon\tilde{\eta}')$. Let us verify that F_1 is well defined:

Proposition 6.9. *The map F_1 introduced in Equation* (6.15) *is well defined.*

Proof. There are two things to verify. First, that we end up with the correct Sobolev index; and second, that the result is even.

Since $s > \frac{3}{2}$ and $d = 1$, both $H^s(\mathbb{R})$ and $H^{s-1}(\mathbb{R})$ are algebras by Lemma A.7. We thus see that the numerators in both the fractions in Equation (6.15) are in $H^{s-1}(\mathbb{R})$. It follows by an application of Lemma A.8 that the whole fractions are in $H^{s-1}(\mathbb{R})$. Hence F_1 maps into $H^{s-2}(\mathbb{R})$ (recall that we lose two derivatives because of κ, see Proposition 6.8).

That F_1 maps into $H^{s-2}_{\text{even}}(\mathbb{R})$ follows from the parity of terms, and the properties of $G(\varepsilon\tilde{\eta})$ and κ that were discussed earlier in this section. ☐

Since G is shape analytic and κ is analytic, and since all terms except the last one is a rational function in $\tilde{\eta}', \tilde{\zeta}', G(\varepsilon\tilde{\eta})\tilde{\zeta}, \Phi_x(\cdot, \tilde{\eta}(\cdot)), \Phi_y(\cdot, \tilde{\eta}(\cdot))$, we deduce that the map F_1 is of class C^∞.

The second map (Kinematic boundary condition)

Let $s > \frac{3}{2}$, and define $F_2 \colon U^s_\theta \to \partial_x(H^s_{\text{even}})(\mathbb{R})$ by

$$\begin{gathered} (\tilde{\eta}, \tilde{\zeta}, \tilde{c}, \varepsilon) \\ \downarrow \\ \varepsilon\tilde{c}\tilde{\eta}' + \tilde{\zeta}' + \Phi_x\big|_{y=\varepsilon\tilde{\eta}} + \varepsilon\tilde{\eta}'\ \Phi_y\big|_{y=\varepsilon\tilde{\eta}} . \end{gathered} \tag{6.16}$$

The map is well defined by Proposition 6.2, since

$$\Phi_x(\cdot, \varepsilon\tilde{\eta}(\cdot)) + \varepsilon\tilde{\eta}'(\cdot, \varepsilon\tilde{\eta})\Phi_y = \Phi(\cdot, \varepsilon\tilde{\eta}(\cdot))',$$

and the same proposition yields smoothness (C^∞) of F_2. Solutions of $F_2(\tilde{\eta}, \tilde{\zeta}, \tilde{c}, \varepsilon) = 0$ yield solutions of Equation (5.18), after rescaling and using Equation (6.11).

The third map (Vorticity equation)

The third and final map models the vorticity equation, Equation (6.13), that needs to be satisfied in our setting when $\varepsilon \neq 0$. To that end, if $s > \frac{3}{2}$, define $F_3 \colon U^s_\theta \to \mathbb{R}$ by

$$\begin{gathered} (\tilde{\eta}, \tilde{\zeta}, \tilde{c}, \varepsilon) \\ \downarrow \\ \tilde{c} + [H(\varepsilon\tilde{\eta})\tilde{\zeta}]_y(0, -(1-\theta)h) - \frac{1}{4h}\cot(\pi\theta). \end{gathered} \tag{6.17}$$

The pointwise evaluation in the second term is allowed because $H(\varepsilon\tilde{\eta})\tilde{\zeta}$ is harmonic, and so has continuous derivatives of all orders[5]. In addition to this, we have the following proposition.

Proposition 6.10. *Let $\eta \in H^s(\mathbb{R}) \cap \Gamma_\theta$. Then the map defined by*

$$\zeta \mapsto [H(\eta)\zeta]_y(0, -(1-\theta)h) \tag{6.18}$$

is a continuous linear functional on $H^s(\mathbb{R})$.

Proof. Let $\zeta \in H^s(\mathbb{R})$, define $\psi := H(\eta)\zeta$, and denote the point $(0, -(1-\theta)h)$ by z_0. Proceed to pick a positive number $r > 0$ such that $B_r(z_0) \Subset \Omega(\eta)$. By the mean value property of harmonic functions and the divergence theorem, we have

$$\begin{aligned}
\partial_y \psi(z_0) &= \frac{1}{\pi r^2} \int_{B_r(z_0)} \partial_y \psi(z)\, dz \\
&= \frac{1}{\pi r^2} \int_{\partial B_r(z_0)} (0, \psi(z)) \cdot n\, d\gamma \\
&= \frac{1}{\pi r} \int_0^{2\pi} \psi(re^{i\beta}) \sin(\beta)\, d\beta,
\end{aligned}$$

where, on the second line, n denotes the outward-pointing unit normal. Hence, by taking absolute values,

$$\begin{aligned}
|\partial_y \psi(z_0)| &\leq \frac{4}{\pi r} \|\psi\|_{BC(\overline{B_r(z_0)}, \mathbb{R})} \\
&\leq C \|\psi\|_{H^2(B_r(z_0))} \\
&\leq C \|H(\eta)\|_{B(H^{3/2}(\mathbb{R}), H^2(\Omega(\eta)))} \|\zeta\|_{H^s(\mathbb{R})}
\end{aligned}$$

is obtained for some constant $C > 0$. The embedding $H^2(B_r(z_0)) \hookrightarrow BC(\overline{B_r(z_0)}, \mathbb{R})$ follows by Theorem A.5 in Appendix A and the existence of extension operators. We mention that the constant C only depends on r, and can therefore be uniformly bounded in η as long as the surface is kept uniformly away from z_0. (This is not strictly necessary for the proposition, but motivates the regularity of the evaluation operator in Equation (6.18) with respect to η.) □

Moreover, a similar argument to the one that shows that G is shape analytic, see Theorem 6.7, shows that this evaluation is also shape analytic. Hence F_3 is smooth.

We can now define $F\colon U_\theta^s \to Y^s$ by

$$F := (F_1, F_2, F_3), \tag{6.19}$$

and our task will then be to find solutions of the equation $F(\tilde{\eta}, \tilde{\zeta}, \tilde{c}, \varepsilon) = 0$. One may immediately note that we have the trivial solution

$$F(0, 0, \tilde{c}_0, 0) = 0,$$

[5] That $H(\eta)$ maps into $H^2(\Omega(\eta))$ by Theorem 6.5 is not sufficient to obtain pointwise derivatives from Theorem A.5.

where we have defined

$$\tilde{c}_0 := \frac{1}{4h}\cot(\pi\theta). \tag{6.20}$$

In particular, this means that

$$\operatorname{sgn}(\tilde{c}_0) = \operatorname{sgn}\left(\frac{1}{2} - \theta\right),$$

and that $\tilde{c}_0$ vanishes identically when $\theta = \frac{1}{2}$. It will turn out that in a small neighborhood of the point $(0, 0, \tilde{c}_0, 0)$, there is a unique curve of nontrivial solutions parametrized by the vortex strength parameter ε. Observe that if $\theta = 1 - 1/h$ (when $h > 1$) then

$$\begin{aligned}\tilde{c}_0 &= -\frac{1}{4h}\cot(\pi/h)\\ &= -\frac{1}{4\pi} + O(1/h^2)\end{aligned}$$

as $h \to \infty$. This is in agreement with what was found in [Shatah et al., 2013] for a point vortex situated at $(0, -1)$ on infinite depth.

6.3 Existence

We can finally state and prove the following theorem, establishing the existence of small, localized, traveling wave solutions with a point vortex. For this, we will use an implicit function theorem argument on F. Note that while we do not use the Crandall–Rabinowitz theorem (Theorem 4.13) directly, the situation is very much in the spirit of that theorem. We bifurcate from the family of trivial waves (trivial line of solutions) described in the remark after Equation (6.13) by introducing the scaling and the vorticity equation.

Theorem 6.11 (Traveling waves with a point vortex)**.** *Let $s > \frac{3}{2}$ and let $\theta \in (0, 1)$. Then there exists an open interval $I \ni 0$ and a C^∞-curve*

$$\begin{array}{ccc} I & \to & (H^s_{even}(\mathbb{R}) \cap \Gamma_\theta) \times H^s_{even}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \\ \varepsilon & \mapsto & (\eta(\varepsilon), \zeta(\varepsilon), c(\varepsilon), \varepsilon) \end{array} \tag{6.21}$$

of solutions to the Zakharov–Craig–Sulem formulation for a point vortex of strength ε situated at $(0, -(1 - \theta)h)$. The solutions have the asymptotic form

$$\begin{aligned}\eta(\varepsilon) &= \tilde{\eta}_1\varepsilon^2 + O(\varepsilon^3),\\ \zeta(\varepsilon) &= O(\varepsilon^3),\\ c(\varepsilon) &= \tilde{c}_0\varepsilon + O(\varepsilon^3),\end{aligned}$$

as $\varepsilon \to 0$, where $\tilde{\eta}_1 \in H^s_{even}(\mathbb{R})$ is defined by

$$\tilde{\eta}_1 := -(g - \alpha^2\partial_x^2)^{-1}\chi, \quad \chi := \tilde{c}_0\Phi_y(\cdot, 0) + \frac{1}{2}\Phi_y(\cdot, 0)^2,$$

and where $\tilde{c}_0$ *was defined in Equation* (6.20). *Moreover, there is a neighborhood of* $(0, 0, \tilde{c}_0, 0)$ *in* U^s_θ *such that the curve*

$$\varepsilon \mapsto (\varepsilon^{-1}\eta(\varepsilon), \varepsilon^{-1}\zeta(\varepsilon), \varepsilon^{-1}c(\varepsilon), \varepsilon)$$

describes the only solutions to $F(\tilde{\eta}, \tilde{\zeta}, \tilde{c}, \varepsilon) = 0$ *in that neighborhood.*

Proof. In the scaled variables, we have the trivial solution

$$F(0, 0, \tilde{c}_0, 0) = 0,$$

as remarked before Equation (6.20). In order to apply the implicit function theorem, we require the Fréchet derivative of F at this point. To that end, observe that

$$\begin{bmatrix} F_1(\tilde{\eta}, 0, \tilde{c}_0, 0) & F_1(0, \tilde{\zeta}, \tilde{c}_0, 0) & F_1(0, 0, \tilde{c}_0 + \tilde{c}, 0) \\ F_2(\tilde{\eta}, 0, \tilde{c}_0, 0) & F_2(0, \tilde{\zeta}, \tilde{c}_0, 0) & F_2(0, 0, \tilde{c}_0 + \tilde{c}, 0) \\ F_3(\tilde{\eta}, 0, \tilde{c}_0, 0) & F_3(0, \tilde{\zeta}, \tilde{c}_0, 0) & F_3(0, 0, \tilde{c}_0 + \tilde{c}, 0) \end{bmatrix}$$
$$= \begin{bmatrix} g\tilde{\eta} - \alpha^2\tilde{\eta}'' & 0 & 0 \\ 0 & \tilde{\zeta}' & 0 \\ 0 & [H(0)\tilde{\zeta}]_y(0, -(1-\theta)h) & \tilde{c} \end{bmatrix},$$

where we have used the definition of $\tilde{c}_0$, and the fact that $\Phi_x(\cdot, 0) = 0$ (since Φ is constant on $y = 0$). This implies that

$$B(X^s, Y^s) \ni D_X F(0, 0, \tilde{c}_0, 0) = \begin{bmatrix} g - \alpha^2\partial_x^2 & 0 & 0 \\ 0 & \partial_x & 0 \\ 0 & [H(0)\cdot]_y(0, -(1-\theta)h) & 1 \end{bmatrix}, \tag{6.22}$$

where the subscript X denotes the partial derivative with respect to the variable $(\tilde{\eta}, \tilde{\zeta}, \tilde{c})$ in X^s.

Now, every operator on the diagonal of $D_X F(0, 0, \tilde{c}_0, 0)$ is an isomorphism. Indeed, the operator $g - \alpha^2\partial_x^2$ may be written as

$$g + \alpha^2 D^2$$

in Fourier multiplier notation. Since $\alpha^2 > 0$, we have

$$g + \alpha^2|\xi|^2 \sim \langle\xi\rangle^2,$$

and so $[g - \alpha^2\partial_x^2]\colon H^s_{\text{even}}(\mathbb{R}) \to H^{s-2}_{\text{even}}(\mathbb{R})$ is invertible[6], with inverse given by

$$(g - \alpha^2\partial_x^2)^{-1} = \frac{1}{g + \alpha^2 D^2}.$$

[6]This is not true when $\alpha^2 = 0$. If $\alpha^2 = 0$ we would have $H^{s-1}_{\text{even}}(\mathbb{R})$ as the codomain instead, but this is not sufficient since $H^{s-1}(\mathbb{R}) \subsetneq H^s(\mathbb{R})$.

Moreover, the operator ∂_x is an isomorphism $H^s_{\mathrm{even}}(\mathbb{R}) \to \partial_x(H^s_{\mathrm{even}})(\mathbb{R})$ by definition of the space $\partial_x H^s(\mathbb{R})$. Hence, by Lemma A.1, $D_X F(0,0,\tilde{c}_0,0) \in B(X^s, Y^s)$ is also an isomorphism.

Thus we can use the implicit function theorem to conclude that there is an open interval I containing zero and an open set $V \subseteq X^s$ containing $(0,0,\tilde{c}_0)$, such that $V \times I \subseteq U^s_\theta$, and a map $f \in C^\infty(I, V)$ such that for $(\tilde{\eta}, \tilde{\zeta}, \tilde{c}, \varepsilon) \in V \times I$, we have

$$F(\tilde{\eta}, \tilde{\zeta}, \tilde{c}, \varepsilon) = 0 \iff (\tilde{\eta}, \tilde{\zeta}, \tilde{c}) = f(\varepsilon).$$

Furthermore, since

$$\begin{bmatrix} F_1(0,0,\tilde{c}_0,\varepsilon) \\ F_2(0,0,\tilde{c}_0,\varepsilon) \\ F_3(0,0,\tilde{c}_0,\varepsilon) \end{bmatrix} = \begin{bmatrix} \varepsilon\chi \\ 0 \\ 0 \end{bmatrix},$$

where we have defined

$$\chi \coloneqq \tilde{c}_0 \Phi_y(\cdot, 0) + \frac{1}{2}\Phi_y^2(\cdot, 0),$$

we have

$$D_\varepsilon F(0,0,\tilde{c}_0,0) = \begin{bmatrix} \chi \\ 0 \\ 0 \end{bmatrix}.$$

Thus we obtain

$$\begin{aligned}
Df(0) &= -D_X F(0,0,\tilde{c}_0,0)^{-1} D_\varepsilon F(0,0,\tilde{c}_0,0) \\
&= -\begin{bmatrix} (g - \alpha^2\partial_x^2)^{-1} & 0 & 0 \\ 0 & \partial_x^{-1} & 0 \\ 0 & -[H(0)\partial_x^{-1}\cdot]_y(0, -(1-\theta)h) & 1 \end{bmatrix} \begin{bmatrix} \chi \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -(g - \alpha^2\partial_x^2)^{-1}\chi \\ 0 \\ 0 \end{bmatrix},
\end{aligned}$$

where the formula for $Df(0)$ in the first line comes from the implicit function theorem. Now, from Taylors theorem, we have

$$f(\varepsilon) = f(0) + Df(0)\varepsilon + O(\varepsilon^2), \quad \text{as } \varepsilon \to 0$$

in Y^s, so, writing $f \coloneqq (\tilde{\eta}, \tilde{\zeta}, \tilde{c})$, one has

$$\begin{aligned}
&\tilde{\eta}(\varepsilon) = \varepsilon\tilde{\eta}_1 + O(\varepsilon^2), \quad \text{where } \tilde{\eta}_1 = -(g - \alpha^2\partial_x^2)^{-1}\chi, \\
&\tilde{\zeta}(\varepsilon) = O(\varepsilon^2), \\
&\tilde{c}(\varepsilon) = \tilde{c}_0 + O(\varepsilon^2)
\end{aligned}$$

asymptotically as $\varepsilon \to 0$. This concludes the proof of the theorem. $\square$

Remark. At the beginning of Section 5.3, we claimed that there was a real reason why we could not use the velocity potential instead of the stream function for the irrotational part. The reason for this is that, while the equations obtained are very similar, we would require invertibility of the operator $\widetilde{G}(0) = |D|\tanh(h|D|)$ defined on $\dot{H}^s(\mathbb{R})$ (for the velocity potential) instead of ∂_x on the diagonal of the derivative in Equation (6.22). While $\widetilde{G}(\eta)$ will be injective for any $\eta \in H^s(\mathbb{R})$, it is not at all clear which subspace of $H^{s-1}(\mathbb{R})$ the images of the family of operators is contained in, or if the operator $\widetilde{G}(0)$ is surjective on this space. For more on this subject, see [Lannes, 2013, Appendix A.3].

Because Theorem 6.11 holds for any $s > 3/2$, we can get arbitrary regularity on the solutions, by possibly making the interval I smaller. We cannot conclude that they are smooth, however, because it could be that the interval is forced to shrink to a point as $s \to \infty$.

Observe that, because $\tilde{c}_0$ changes sign at $\theta = 1/2$, the direction that the waves obtain in Theorem 6.11 will travel (for small ε) depends on whether or not the point vortex is situated above the line $y = -h/2$. This effect is something that does not come into play for waves on infinite depth. Since $\tilde{c}_0$ vanishes when $\theta = 1/2$, it could also be interesting to know the next term in the expansion for $c(\varepsilon)$, but we have not performed this calculation here.

Written out, we have

$$\begin{aligned}\chi(x) &= \frac{1}{8h^2}\left(\frac{\cos(\pi\theta)}{\cosh(\pi x/h)+\cos(\pi\theta)} + \frac{\sin^2(\pi\theta)}{(\cosh(\pi x/h)+\cos(\pi\theta))^2}\right)\\ &= \frac{1}{8h^2}\frac{1+\cos(\pi\theta)\cosh(\pi x/h)}{(\cosh(\pi x/h)+\cos(\pi\theta))^2},\end{aligned} \tag{6.23}$$

for the function χ defined in the statement of Theorem 6.11. We will have use for the fact that χ has an elementary antiderivative $\chi^\sharp$ and a double antiderivative $\chi^{\sharp\sharp}$ given by

$$\begin{aligned}\chi^\sharp(x) &= \frac{1}{8\pi h}\frac{\sinh(\pi x/h)}{\cosh(\pi x/h)+\cos(\pi\theta)},\\ \chi^{\sharp\sharp}(x) &= \frac{1}{8\pi^2}\log(\cosh(\pi x/h)+\cos(\pi\theta)),\end{aligned} \tag{6.24}$$

respectively. While there, maybe not so unexpectedly, seems to be no nice closed form of the leading order surface profile

$$\tilde{\eta}_1 = -(g-\alpha^2\partial_x^2)^{-1}\chi$$

obtained in Theorem 6.11 in general, one can still use numerical tools to visualize it. Note that since $\chi \in H^s(\mathbb{R})$ for every $s > 0$, so is the function $\tilde{\eta}_1$, which in particular means that $\tilde{\eta}_1$ is smooth. In Proposition 6.13 we give a series expansion for $\tilde{\eta}_1$ in powers of $e^{-\pi|x|/h}$. Furthermore, perhaps more surprisingly, we *can* find an explicit expression for $\tilde{\eta}_1$ in terms of elementary functions whenever

$$m := \frac{\sqrt{g}h}{\pi\alpha}. \tag{6.25}$$

is a natural number. If $m \in \mathbb{N}$, then $e^{\pm\sqrt{g}x/\alpha} = e^{\pm m\pi x/h}$ are integral powers of $e^{\pm\pi x/h}$, which would appear on the right side of Equation (6.23) if we had written out $\cosh(\pi x/h)$ and $\sinh(\pi x/h)$. Since $x \mapsto e^{\pm\sqrt{g}x/\alpha}$ are eigenvectors of $g - \alpha^2\partial_x^2$ with eigenvalue 0 (although not members of any Sobolev space), this motivates integral values of m being special.

Before we state Proposition 6.13 and Theorem 6.14, we need a lemma that simplifies some expressions.

Lemma 6.12. *For $m \in (0,\infty) \setminus \mathbb{N}$ and $\theta \in (0,1)$, we have*

$$\frac{1}{m} + 2m\sum_{k=1}^{\infty}(-1)^k\frac{\cos(k\pi\theta)}{m^2-k^2} = \pi\frac{\cos(m\pi\theta)}{\sin(m\pi)}, \tag{6.26}$$

which, moreover, is equal to

$$\int_0^{\infty} y^{m-1}\frac{\cos(\pi\theta)y+1}{y^2+2\cos(\pi\theta)y+1}\,dy \tag{6.27}$$

whenever $m \in (0,1)$. Furthermore, for $m \in \mathbb{N}$

$$\frac{1}{m} + 2m\sum_{\substack{k=1\\k\neq m}}^{\infty}(-1)^k\frac{\cos(k\pi\theta)}{m^2-k^2} = -(-1)^m\left(\frac{\cos(m\pi\theta)}{2m} + \pi\theta\sin(m\pi\theta)\right). \tag{6.28}$$

Proof. We begin with Equation (6.26), as Equation (6.28) will follow as a corollary. Consider the parameter $\theta \in (0,1)$ to be fixed. The series on the left hand side of Equation (6.26) converges uniformly on any $K \Subset \mathbb{C} \setminus (\mathbb{Z} \times \{0\})$ because of the squares in the denominators, meaning that it defines a meromorphic function on $\mathbb{C}$ with simple poles in the points $\mathbb{Z} \times \{0\}$. This is because uniform limits of analytic functions are analytic (this is a consequence of Morera's theorem, [Markushevich, 1965a, Theorem 14.6]). The right hand side of Equation (6.26) also defines a meromorphic function on the same set, with the same poles. In order to show that the sides are equal on $\mathbb{C} \setminus (\mathbb{Z} \times \{0\})$ it is thus sufficient to exhibit a set containing non-isolated points on which they are equal, by the remark after Theorem 3.7.

Suppose now that $m \in (0,1)$, which indeed consists of non-isolated points. We will calculate the integral in Equation (6.27) in two different ways, which will yield Equation (6.26). Observe that the fraction in the integrand has two simple poles $-e^{\pm i\pi\theta}$ on the unit circle, meaning that we can expand the fraction in one Laurent series valid inside the unit circle, and one that is valid outside the unit circle. These are

$$\frac{\cos(\pi\theta)y+1}{y^2+2\cos(\pi\theta)y+1} = \begin{cases} \sum_{k=0}^{\infty}(-1)^k\cos(k\pi\theta)y^k & |y|<1 \\ -\sum_{k=1}^{\infty}(-1)^k\cos(k\pi\theta)y^{-k} & |y|>1 \end{cases},$$

found by partial fraction decomposition of the left hand side, which implies that

$$\begin{aligned}\int_0^x y^{m-1}\frac{\cos(\pi\theta)y+1}{y^2+2\cos(\pi\theta)y+1}\,dy &= \sum_{k=0}^{\infty}(-1)^k\frac{\cos(k\pi\theta)}{m+k}x^{m+k}, \quad \text{when } x\in(0,1),\\ \int_x^\infty y^{m-1}\frac{\cos(\pi\theta)y+1}{y^2+2\cos(\pi\theta)y+1}\,dy &= \sum_{k=1}^{\infty}(-1)^k\frac{\cos(k\pi\theta)}{m-k}x^{m-k}, \quad \text{when } x\in(1,\infty).\end{aligned} \tag{6.29}$$

Because of the bound

$$\sup_{n\in\mathbb{N}}\left|\sum_{k=1}^{n}(-1)^k\cos(k\pi\theta)\right| \le \sec(\pi\theta/2)<\infty, \quad \theta\in(0,1)$$

we can use Dirichlet's test, [Hardy, 1952, p. 379], to deduce that the series in Equation (6.29) converge when $x=1$, and hence that equality holds also when $x=1$ by Abel's limit theorem (see [Markushevich, 1965a, Theorem 17.14]). It follows that the integral in Equation (6.27) is equal to the left hand side of Equation (6.26).

We can also use the residue theorem (see for instance [Markushevich, 1965b, Theorem 2.1]) to calculate the same integral. The residues at the poles of the integrand, which we denote by f, are

$$\operatorname*{Res}_{y=-e^{\pm i\pi\theta}} f = -\frac{1}{2}e^{im\pi}e^{\pm im\pi\theta},$$

respectively. We use the branch of $y\mapsto y^{m-1}$ on $\mathbb{C}\setminus[0,\infty)$ that agrees with the standard definition of y^{m-1} for real y as y approaches the real axis from the upper half plane. By integrating f around the keyhole contour $\varepsilon\mathbb{S}^1\cup(\varepsilon,R)\cup R\mathbb{S}^1\subseteq\mathbb{C}$ and letting $\varepsilon\downarrow 0$ and $R\uparrow\infty$, one finds

$$(1-e^{2im\pi})\int_0^\infty f(y)\,dy = 2\pi i\left(\operatorname*{Res}_{y=-e^{i\pi\theta}} f+\operatorname*{Res}_{y=-e^{-i\pi\theta}} f\right),$$

which leads to integral being equal to the right hand side of Equation (6.26). The integrals of f around the circular portions of the contour are easily found to vanish as $\varepsilon\downarrow 0$ and $R\uparrow\infty$; the former because $m>0$, and the latter relying on the fact that $m<1$.

For Equation (6.28), fix $m_0\in\mathbb{N}$, and observe that Equation (6.26) shows that

$$\frac{1}{m}+2m\sum_{\substack{k=1\\k\neq m_0}}\frac{(-1)^k\cos(k\pi\theta)}{m^2-k^2} = \pi\frac{\cos(m\pi\theta)}{\sin(m\pi)}-2m\frac{(-1)^{m_0}\cos(m_0\pi\theta)}{m^2-m_0^2}$$

in a punctured neighborhood of m_0. The left hand side is analytic at m_0, and by letting $m\to m_0$ we obtain Equation (6.28) for m_0. The limit on the right hand side can be calculated by applying L'Hôpital's rule twice. □

Proposition 6.13 (Expansion for $\tilde{\eta}_1$). *If the number m in Equation* (6.25) *satisfies $m\in(0,\infty)\setminus\mathbb{N}$, then the leading order term of the surface profile from Theorem 6.11 is* *given by*

$$\tilde{\eta}_1(x) = \frac{1}{8\pi^2\alpha^2}\Bigg[\log(\cosh(\pi x/h) + \cos(\pi\theta)) - \pi|x|/h + \log(2) \\ - \pi\frac{\cos(m\pi\theta)}{\sin(m\pi)}e^{-\sqrt{g}|x|/\alpha} + 2m^2\sum_{k=1}^{\infty}(-1)^k\frac{\cos(k\pi\theta)}{k(m^2-k^2)}e^{-k\pi|x|/h}\Bigg],$$

while if $m \in \mathbb{N}$*, then*

$$\tilde{\eta}_1(x) = \frac{1}{8\pi^2\alpha^2}\Bigg[\log(\cosh(\pi x/h) + \cos(\pi\theta)) - \pi|x|/h + \log(2) \\ + (-1)^m\left(\frac{3\cos(m\pi\theta)}{2m} + \pi\theta\sin(m\pi\theta)\right)e^{-\sqrt{g}|x|/\alpha} \\ + (-1)^m\cos(m\pi\theta)(\pi|x|/h)e^{-\sqrt{g}|x|/\alpha} + 2m^2\sum_{\substack{k=1\\k\neq m}}^{\infty}(-1)^k\frac{\cos(k\pi\theta)}{k(m^2-k^2)}e^{-k\pi|x|/h}\Bigg].$$

The series converge uniformly[7]*, and absolutely in* $H^s(\mathbb{R})$ *for* $s < 3/2$*. Moreover, when* $m \in \mathbb{N}$*, the function* $\tilde{\eta}_1$ *is given explicitly in terms of elementary functions by*

$$\tilde{\eta}_1(x) = \frac{1}{8\pi^2\alpha^2}\Bigg[\frac{1}{m} + 2\sum_{k=1}^{m-1}(-1)^{m-k}\frac{\cos((m-k)\pi\theta)}{k}\cosh((m-k)\pi x/h) \\ + r(e^{\pi x/h}) + r(e^{-\pi x/h})\Bigg],$$

where $r\colon (0,\infty) \to \mathbb{R}$ *is defined by*

$$r(x) := \frac{1}{2}(-1)^m\cos(m\pi\theta)x^{-m}\log((x+\cos(\pi\theta))^2 + \sin(\pi\theta)^2) \\ + (-1)^m\sin(m\pi\theta)x^{-m}(\arctan(\cot(\pi\theta) + \csc(\pi\theta)x) - \pi(1/2-\theta)).$$

Proof. It follows from

$$\mathscr{F}(e^{-a|\cdot|})(\xi) = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+\xi^2}, \quad a > 0,$$

and the definition of $\tilde{\eta}_1$, that we may write $\tilde{\eta}_1$ as the convolution

$$\begin{aligned}\tilde{\eta}_1(x) &= -\frac{1}{2\alpha\sqrt{g}}(e^{-\sqrt{g}|\cdot|/\alpha} * \chi)(x) \\ &= -\frac{1}{2\alpha\sqrt{g}}\left(e^{-\sqrt{g}x/\alpha}\int_{-\infty}^{x} e^{\sqrt{g}y/\alpha}\chi(y)\,dy + e^{\sqrt{g}x/\alpha}\int_{x}^{\infty} e^{-\sqrt{g}y/\alpha}\chi(y)\,dy\right), \qquad (6.30)\end{aligned}$$

[7] In fact, so do the series for the derivative if one uses the left- and right-sided derivatives at the origin.

or, equivalently,

$$\tilde{\eta}_1(x) = \frac{1}{2\alpha^2}\left(e^{-\sqrt{g}x/\alpha}\int_{-\infty}^{x} e^{\sqrt{g}y/\alpha}\chi^{\sharp}(y)\,dy - e^{\sqrt{g}x/\alpha}\int_{x}^{\infty} e^{-\sqrt{g}y/\alpha}\chi^{\sharp}(y)\,dy\right) \tag{6.31}$$

$$\begin{aligned} &= \frac{1}{\alpha^2}\chi^{\sharp\sharp}(x) \qquad\qquad (6.32)\\ &\quad - \frac{\sqrt{g}}{2\alpha^3}\left(e^{-\sqrt{g}x/\alpha}\int_{-\infty}^{x} e^{\sqrt{g}y/\alpha}\chi^{\sharp\sharp}(y)\,dy + e^{\sqrt{g}x/\alpha}\int_{x}^{\infty} e^{-\sqrt{g}y/\alpha}\chi^{\sharp\sharp}(y)\,dy\right)\end{aligned}$$

through integration by parts, where $\chi^{\sharp}$ and $\chi^{\sharp\sharp}$ are the antiderivatives defined in Equation (6.24).

We first use Equation (6.31) to obtain an explicit expression for $\tilde{\eta}_1$ when $m \in \mathbb{N}$. By using the substitution $x \mapsto e^{\pi x/h}$ in the first integral, and the substitution $x \mapsto e^{-\pi x/h}$ in the second, we find that

$$\tilde{\eta}_1(x) = \frac{1}{16\pi^2\alpha^2}[f_1(e^{\pi x/h}) + f_1(e^{-\pi x/h})], \quad f_1(x) \coloneqq x^{-m}\int_0^x z^{m-1}\frac{z^2-1}{z^2+2\cos(\pi\theta)z+1}\,dz.$$

The fraction in the integrand in the definition of f_1 has partial fraction decomposition

$$\frac{z^2-1}{z^2+2\cos(\pi\theta)z+1} = 1 - \frac{e^{i\pi\theta}}{z+e^{i\pi\theta}} - \frac{e^{-i\pi\theta}}{z+e^{-i\pi\theta}},$$

and since

$$z^{m-1}\frac{a}{z+a} = -\frac{(-a)^m}{z+a} - \sum_{k=0}^{m-2}(-a)^{m-k-1}z^k, \quad a \in \mathbb{C}, z \neq -a,$$

this means that

$$\begin{aligned} f_1(x) = \frac{1}{m} &+ (-1)^m e^{im\pi\theta}x^{-m}\log(x+e^{i\pi\theta}) + (-1)^m e^{-im\pi\theta}x^{-m}\log(x+e^{-i\pi\theta})\\ &+ 2(-1)^m\pi\theta\sin(m\pi\theta)x^{-m} + 2\sum_{k=1}^{m-1}\frac{(-1)^k\cos(k\pi\theta)}{m-k}x^{-k},\end{aligned}$$

where $\log(\cdot)$ denotes the principal branch of the logarithm. The result now follows by using the identity

$$\log(x+e^{i\pi\theta}) = \frac{1}{2}\log((x+\cos(\pi\theta))^2+\sin(\pi\theta)^2) - i(\arctan(\cot(\pi\theta)+\csc(\pi\theta)x) - \pi/2),$$

valid for all $x \in \mathbb{R}$.

For the series representation of $\tilde{\eta}_1$, we use Equation (6.32), because this leads to a series that converges much more rapidly than the one we would get from Equation (6.31). We will assume that $m \in (0,\infty)\setminus\mathbb{N}$; the case for $m \in \mathbb{N}$ is similar, except that one needs to use Equation (6.28) instead of Equation (6.26). We use the same substitutions as before to arrive at

$$\tilde{\eta}_1(x) = \frac{1}{\alpha^2}\chi^{\sharp\sharp}(x) - \frac{1}{16\alpha^2\pi^2}(f_2(e^{\pi x/h}) + f_2(e^{-\pi x/h})), \tag{6.33}$$

where $f_2\colon (0,\infty) \to \mathbb{R}$ is defined by

$$f_2(x) \coloneqq m x^{-m} \int_0^x z^{m-1} \log((z^{-1}+z)/2 + \cos(\pi\theta))\, dz.$$

One may check that one has

$$\log((z^{-1}+z)/2 + \cos(\pi\theta)) = -\log(2) - \log(z) - 2\sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\pi\theta)}{k} z^k$$

for $z \in (0,1)$ and

$$\log((z^{-1}+z)/2 + \cos(\pi\theta)) = -\log(2) + \log(z) - 2\sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\pi\theta)}{k} z^{-k}$$

for $z \in (1,\infty)$, which can be found by using the well known expansion for $z \mapsto \log(1+z)$ in the unit disk.

It then follows by termwise integration that

$$f_2(x) = \frac{1}{m} - \log(2) - \log(x) - 2m \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\pi\theta)}{k(m+k)} x^k$$

on $(0,1)$. In fact, since the series converges (absolutely) when $x = 1$, this holds for $x \in (0,1]$ (this is Abel's limit theorem, [Markushevich, 1965a, Theorem 17.14]). For $x \in [1,\infty)$ we thus have

$$\begin{aligned} f_2(x) &= f_2(1)x^{-m} + m x^{-m} \int_1^x z^{m-1} \log((z+z^{-1})/2 + \cos(\pi\theta))\, dz \\ &= -\frac{1}{m} - \log(2) + \log(x) - 2m \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\pi\theta)}{k(m-k)} x^{-k} \\ &\qquad - \left(\frac{2}{m} + 4m \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\pi\theta)}{m^2-k^2} \right) x^{-m}. \end{aligned}$$

Employing Equation (6.33), we find that $\tilde{\eta}_1$ is given by

$$\begin{aligned} \tilde{\eta}_1(x) = \frac{1}{8\pi^2\alpha^2} \Bigg[&\log(\cosh(\pi x/h) + \cos(\pi\theta)) - \pi|x|/h + \log(2) \\ &- \left(\frac{1}{m} + 2m \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\pi\theta)}{m^2-k^2} \right) e^{-\sqrt{g}|x|/\alpha} \\ &+ 2m^2 \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\pi\theta)}{k(m^2-k^2)} e^{-k\pi|x|/h} \Bigg] \end{aligned} \tag{6.34}$$

for all $x \in \mathbb{R}$, by using that $\tilde{\eta}_1$ is even and observing that for $x \geq 0$ we have $e^{\pi x/h} \in [1,\infty)$ and $e^{-\pi x/h} \in (0,1]$. If we now apply Equation (6.26) from Lemma 6.12 in order to get a closed-form expression for the coefficient in front of $e^{-\sqrt{g}|x|/\alpha}$, we arrive at the expansion.

Let us finally show convergence in $H^s(\mathbb{R})$ for an appropriate Sobolev exponent s. Observe that if $a > 0$, then

$$\|e^{-a|\cdot|}\|^2_{H^s(\mathbb{R})} = \frac{a^2}{2\pi}\int_{\mathbb{R}} \frac{(1+\xi^2)^s}{(a^2+\xi^2)^2}\, d\xi,$$

which is clearly finite if and only if also $s < 3/2$. For exponents s satisfying $1 \le s < 3/2$, we have, by the substitution $\xi = ax$,

$$\begin{aligned}\|e^{-a|\cdot|}\|^2_{H^s(\mathbb{R})} &= \frac{a^{-3}}{2\pi}\int_{\mathbb{R}} \frac{(1+a^2x^2)^s}{(1+x^2)^2}\, dx \\ &\le \frac{2^{s-1}a^{-3}}{2\pi}\left(\int_{\mathbb{R}} \frac{dx}{(1+x^2)^2} + a^{2s}\int_{\mathbb{R}} \frac{x^{2s}}{(1+x^2)^2}\right),\end{aligned}$$

where we have used the inequality $(1+a^2x^2)^s \le 2^{s-1}(1+a^{2s}x^{2s})$. Both the integrals on the last line are finite, whence if $1 \le s < 3/2$ we have

$$\|e^{-a|\cdot|}\|_{H^s(\mathbb{R})} \le C(a^{-3/2} + a^{s-3/2}), \quad a > 0,$$

where the constant $C > 0$ only depends on s. This also shows absolute convergence of the series for $\tilde{\eta}_1$ in $H^s(\mathbb{R}^d)$ when $s < 3/2$, as the coefficient in front of $e^{-k\pi|x|/h}$ in Equation (6.34) is $O(k^{-3})$ as $k \to \infty$. □

Remark. The only obstacle to convergence of the series given in Proposition 6.13 is the origin; thanks to the exponential factor $e^{-k\pi|x|/h}$, the convergence is rapid away from the origin. It should also be noted that, while Equation (6.23) seems to suggest that $\tilde{\eta}_1$ should be expandable in a series in powers of $\operatorname{sech}(\pi x/h)$ by equating coefficients in the differential equation defining it, this seems to lead to a series that does not converge. We have kept the series expansion for $\tilde{\eta}_1$ also when $m \in \mathbb{N}$, because the expression in terms of elementary functions is unwieldy, and very prone to numerical errors even for small values of m.

The expressions found in Proposition 6.13 have well defined pointwise limits as $\theta \uparrow 1$ (for $x \neq 0$) and $\theta \downarrow 0$. In particular, when $m = 1$ these are given by

$$\begin{aligned}\lim_{\theta\downarrow 0} \tilde{\eta}_1(x) &= \frac{1}{8\pi^2\alpha^2}[1 - e^{\pi x/h}\log(1+e^{-\pi x/h}) - e^{-\pi x/h}\log(1+e^{\pi x/h})] \\ \lim_{\theta\uparrow 1} \tilde{\eta}_1(x) &= \frac{1}{8\pi^2\alpha^2}[1 + e^{\pi x/h}\log|1-e^{-\pi x/h}| + e^{-\pi x/h}\log|1-e^{\pi x/h}|],\end{aligned}$$

which can can be seen as graphs drawn with thicker lines in Figure 6.2, together with $\tilde{\eta}_1$ for various values of the parameter θ.

We see from Figure 6.2 that one gets a depression at the origin, which becomes more pronounced the closer the point vortex is situated to the surface. The profile when the point vortex is close to the surface is very similar to the the profile for the infinite depth case, found in [Shatah et al., 2013]. However, a feature which was not seen on infinite depth is that there is a significant difference between the case $\theta \le \frac{1}{2}$ and the case $\theta > \frac{1}{2}$.

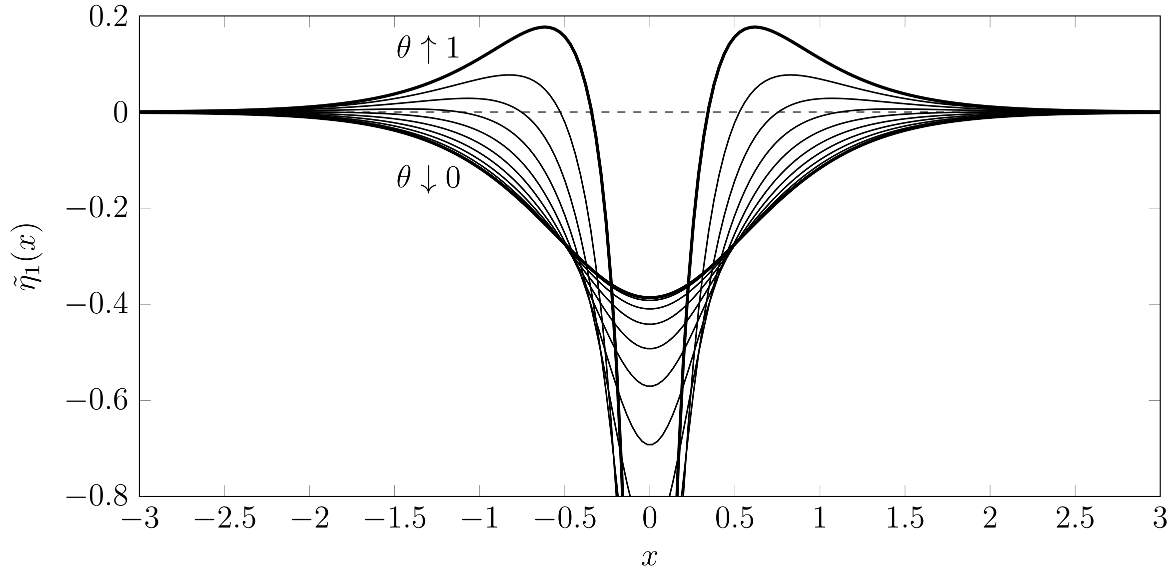


Figure 6.2: The leading order term in $\eta(\varepsilon)$, with $h = 1, \alpha^2 = 1/(8\pi^2), m = 1$. The values of θ shown are $\theta = 0.1, 0.2, \ldots, 0.9$, together with the thicker lower and upper limits $\theta \downarrow 0$ and $\theta \uparrow 1$.

This is in addition to the fact that the waves, for small ε, move in opposite directions depending on whether $\theta < \frac{1}{2}$ or $\theta > \frac{1}{2}$, which we mentioned after Theorem 6.11. For $\theta \leq \frac{1}{2}$ there is a single trough at the origin, and $\tilde{\eta}_1$ is strictly negative. When $\theta > \frac{1}{2}$ one in addition gets crests on either side of the origin (one can show this by using the explicit expression given in Proposition 6.13, but we show a more general result in Theorem 6.14). As we can see from Figure 6.2, the positions of these crests depend on the position of the point vortex.

Some of what we have just discussed is not limited to the specific choice of constants that were used in Figure 6.2, and which yielded an explicit expression for $\tilde{\eta}_1$ in Proposition 6.13. We will see that $m = 1$ plays an important role in the asymptotic behavior of $\tilde{\eta}_1$, however. More precisely, we have the following theorem:

Theorem 6.14 (Properties of $\tilde{\eta}_1$)**.** *The leading order surface term $\tilde{\eta}_1$ always satisfies $\tilde{\eta}_1(0) < 0$ and $\tilde{\eta}_1''(0) > 0$, meaning that the origin is a depression. When $\theta \leq \frac{1}{2}$, the function $\tilde{\eta}_1$ is also negative, and strictly increasing on $[0, \infty)$. For $\theta > \frac{1}{2}$, we have two cases, depending on the number m defined in Equation* (6.25)*:*

(i) If $m > \frac{1}{2\theta}$, then $\tilde{\eta}_1(x)$ is positive for sufficiently large $|x|$. In particular, $\tilde{\eta}_1$ has crests on either side of the origin.

(ii) If $m \leq \frac{1}{2\theta}$, then $\tilde{\eta}_1(x)$ is negative for sufficiently large $|x|$.

Furthermore, $\tilde{\eta}_1$ has the following asymptotic properties for any $\theta \in (0, 1)$:

(i) For $m > 1$

$$\lim_{x\to\infty} \tilde{\eta}_1(x) e^{\pi x/h} = -\frac{2}{m^2-1}\frac{\cos(\pi\theta)}{8\pi^2\alpha^2}. \tag{6.35}$$

(ii) If $m = 1$*, then*

$$\lim_{x\to\infty} \tilde{\eta}_1(x) \frac{e^{\pi x/h}}{\pi x/h} = -\frac{\cos(\pi\theta)}{8\pi^2\alpha^2}. \tag{6.36}$$

(iii) For $m < 1$

$$\lim_{x\to\infty} \tilde{\eta}_1(x) e^{\sqrt{g}x/\alpha} = -\frac{\pi}{\sin(m\pi)}\frac{\cos(m\pi\theta)}{8\pi^2\alpha^2}. \tag{6.37}$$

Proof. We first prove that $\tilde{\eta}_1(0) < 0$ and $\tilde{\eta}_1''(0) > 0$, which holds for all values of m and θ. By inserting $x = 0$ in Equation (6.30), and using the evenness of χ, we find

$$\begin{aligned}\tilde{\eta}_1(0) &= -\frac{1}{\alpha\sqrt{g}}\int_0^\infty e^{-\sqrt{g}y/\alpha}\chi(y)\,dy \\ &= -\frac{1}{\alpha^2}\int_0^\infty \underbrace{e^{-\sqrt{g}y/\alpha}\chi^\sharp(y)}_{>0 \text{ on } (0,\infty)}\,dy < 0,\end{aligned}$$

where the second equality is integration by parts and the function $\chi^\sharp$ was defined in Equation (6.24). Since $\tilde{\eta}_1 = -(g - \alpha^2\partial_x^2)^{-1}\chi$, we also have

$$\begin{aligned}\tilde{\eta}_1''(0) &= \frac{1}{\alpha^2}(g\tilde{\eta}_1(0) + \chi(0)) \\ &= \frac{1}{\alpha^2}\left(-\frac{\sqrt{g}}{2\alpha}\int_{-\infty}^\infty e^{-\sqrt{g}|y|/\alpha}\chi(y)\,dy + \chi(0)\right) \\ &= \frac{\sqrt{g}}{2\alpha^3}\int_{-\infty}^\infty e^{-\sqrt{g}|y|/\alpha}(\chi(0)-\chi(y))\,dy \\ &> 0,\end{aligned}$$

as χ achieves a global maximum at the origin.

The proof for the properties in the $\theta \le \frac{1}{2}$ case is fairly simple. Like in Proposition 6.13, we use the fact that $\tilde{\eta}_1$ may be written as the convolution

$$\tilde{\eta}_1 = -\frac{1}{2\alpha\sqrt{g}}(e^{-\sqrt{g}|\cdot|/\alpha} * \chi), \tag{6.38}$$

which shows that $\tilde{\eta}_1$ is strictly negative, since χ is strictly positive when $\theta \le \frac{1}{2}$. Moreover, some manipulations of the above formula shows that we may write the derivative of $\tilde{\eta}_1$ as

$$\tilde{\eta}_1'(x) = -\frac{1}{\alpha\sqrt{g}}\left[\sinh\left(\frac{\sqrt{g}}{\alpha}x\right)\int_x^\infty e^{-\frac{\sqrt{g}}{\alpha}y}\chi'(y)\,dy + e^{-\frac{\sqrt{g}}{\alpha}x}\int_0^x \sinh\left(\frac{\sqrt{g}}{\alpha}y\right)\chi'(y)\,dy\right],$$

where we have used the fact that χ is even. One may check that χ' is strictly negative for $x > 0$ when $\theta \le \frac{1}{2}$. This shows that $\tilde{\eta}_1'$ is strictly positive for $x > 0$, and so $\tilde{\eta}_1$ is strictly increasing on $[0, \infty)$ by the mean value theorem.

Before we consider the case $\theta > 1/2$, we prove the asymptotic properties for $\tilde{\eta}_1$ listed in Equations (6.35) to (6.37). They follow by multiplying each side in Equation (6.30) with the appropriate factor and taking limits. For instance, suppose that $m > 1$; meaning that $\sqrt{g}/\alpha > \pi/h$. For the integral in

$$e^{\pi x/h}\left(e^{-\sqrt{g}x/\alpha}\int_{-\infty}^{x} e^{\sqrt{g}y/\alpha}\chi(y)\,dy\right) = \frac{\int_{-\infty}^{x} e^{\sqrt{g}y/\alpha}\chi(y)\,dy}{e^{(\sqrt{g}/\alpha-\pi/h)x}}$$

there are two possibilities: If $\theta = 1/2$, then it is possible that the integrand is integrable on the entire real line, meaning that the limit as $x \to \infty$ is zero. Otherwise, the integral tends to $\pm\infty$, and so

$$\begin{aligned}\lim_{x\to\infty}\frac{\int_{-\infty}^{x} e^{\sqrt{g}y/\alpha}\chi(y)\,dy}{e^{(\sqrt{g}/\alpha-\pi/h)x}} &= \lim_{x\to\infty}\frac{e^{\sqrt{g}x/\alpha}\chi(x)}{(\sqrt{g}/\alpha-\pi/h)e^{(\sqrt{g}/\alpha-\pi/h)x}}\\ &= \frac{1}{\sqrt{g}/\alpha-\pi/h}\frac{\cos(\pi\theta)}{4h^2}\end{aligned}$$

by L'Hôpital's rule. The other limits can be treated in a similar way, with one difficulty: The procedure will show that when $m < 1$, we have

$$\begin{aligned}\lim_{x\to\infty}\tilde{\eta}_1(x)e^{\sqrt{g}x/\alpha} &= -\frac{1}{2\alpha\sqrt{g}}\int_{-\infty}^{\infty} e^{\sqrt{g}y/\alpha}\chi(y)\,dy\\ &= -\frac{1}{8\pi^2\alpha^2 m}\int_0^\infty y^m\frac{\cos(\pi\theta)y^2+2y+\cos(\pi\theta)}{(y^2+2\cos(\pi\theta)y+1)^2}\,dy\\ &= -\frac{1}{8\pi^2\alpha^2}\int_0^\infty y^{m-1}\frac{\cos(\pi\theta)y+1}{y^2+2\cos(\pi\theta y)+1}\,dy\end{aligned}$$

where the second equality follows from the substitution $y \mapsto e^{\pi y/h}$ and the third equality is integration by parts. The result now follows since the integral on the final line is equal to the right hand side of Equation (6.26) by Lemma 6.12.

Finally, we consider the case of $\theta > \frac{1}{2}$, which is not as easy to describe completely, as the integrand in the convolution in Equation (6.38) changes sign. Observe that the claims on the sign of $\tilde{\eta}_1(x)$ for sufficiently large x follows for $m \neq \frac{1}{2\theta}$ from the limits in Equations (6.35) to (6.37). An additional argument is needed for the edge case $m = \frac{1}{2\theta}$, because the limit in Equation (6.37) vanishes. It turns out that Equation (6.35) also holds in the special case $m = \frac{1}{2\theta}$, which can be shown with the same method we used to show the other limits. Hence $\tilde{\eta}_1$ is negative for sufficiently large x when $m = \frac{1}{2\theta}$, which exhausts the values of m. □

Remark. If $m \geq 1$ and $\theta = \frac{1}{2}$, or if $\theta > \frac{1}{2}$ and $m = \frac{1}{2\theta}$, then the appropriate limit in Equations (6.35) to (6.37) vanishes. This means that $\tilde{\eta}_1$ decays more rapidly, and one could prove stronger versions of the limits (for instance, we remarked in the proof that Equation (6.35) also holds when $\theta > \frac{1}{2}$ and $m = \frac{1}{2\theta}$). We also mention that it is likely that $\tilde{\eta}_1$ has similar properties to the $\theta \leq \frac{1}{2}$ case when $\theta > \frac{1}{2}$ and $m \leq \frac{1}{2\theta}$, but we have been unable to prove this.

6.4 The original variables and their recovery

The existence theorem, Theorem 6.11, yields a family of traveling waves in terms of the variables from the Zakharaov–Craig–Sulem formulation. For this to be useful, we need to be able to recover the original variables in the Euler equations, and verify that these do in fact satisfy Equations (5.5) to (5.9) and Equation (5.10). The recovery is chiefly done by running the arguments in Section 5.3 in reverse, but can still be illustrative. Suppose therefore that $s > 3/2$ and that we have $\eta \in H^s_{\mathrm{even}}(\mathbb{R}) \cap \Gamma_\theta$, $\zeta \in H^s_{\mathrm{even}}(\mathbb{R})$ and $c \in \mathbb{R}$ from Theorem 6.11, corresponding to a point vortex of strength $\varepsilon \in \mathbb{R} \setminus \{0\}$ at $(0, -(1-\theta)h)$.

The velocity field is recovered by defining w through Equation (6.11). Because

$$\begin{aligned}\hat{w}(0,-(1-\theta)h) &= \nabla^\perp \left[H(\eta)\zeta + \varepsilon \left(\Phi - \frac{1}{4\pi} \log(x^2 + (y + (1-\theta)h)^2) \right) \right] (0, -(1-\theta)h) \\ &= (c - \varepsilon F_3(\varepsilon^{-1}\eta, \varepsilon^{-1}\zeta, \varepsilon^{-1}c, \varepsilon), [H(\eta)\zeta]_x(0, -(1-\theta)h)) \\ &= (c, 0),\end{aligned}$$

the velocity w satisfies the vorticity equation, Equation (5.10). We have here used that η and ζ are even in order to deduce that the second component vanishes (see the discussion before Equation (6.13)). Next, the velocity satisfies Equation (5.6) because

$$\nabla \cdot \nabla^\perp = -\partial_x \partial_y + \partial_y \partial_x = 0$$

for distributions. It also satisfies Equation (5.7) by definition of the function Φ (in Equation (6.6)) and the harmonic extension operator $H(\eta)$. The kinematic boundary condition, Equation (5.8), is satisfied because

$$\begin{aligned}\varepsilon F_2(\varepsilon^{-1}\eta, \varepsilon^{-1}, \varepsilon^{-1}c, \varepsilon) &= c\eta' + \zeta' + \varepsilon \left(\Phi_x|_{y=\eta} + \eta' \, \Phi_y|_{y=\eta} \right) \\ &= c\eta' + \frac{\zeta' - \eta' G(\eta)\zeta}{\langle \eta' \rangle^2} + \frac{\eta'\zeta' + G(\eta)\zeta}{\langle \eta' \rangle^2}\eta' + \varepsilon \left(\Phi_x|_{y=\eta} + \eta' \, \Phi_y|_{y=\eta} \right) \\ &= c\eta' + v|_{y=\eta} - \eta' \, u|_{y=\eta} \\ &= 0,\end{aligned}$$

where we have used Equation (5.16).

Since, by Theorem 6.5, the norm of $H(\eta)$ is uniformly bounded on bounded subsets of $H^s(\mathbb{R}) \cap \Gamma_\theta$, it follows that the velocity field corresponding to the solution curve obtained in Theorem 6.11 satisfies

$$\|w(\varepsilon) - \varepsilon \nabla^\perp \Phi\|_{H^2(\Omega(\eta(\varepsilon)))} = O(\varepsilon^3), \quad \text{as } \varepsilon \to 0.$$

Hence, in this sense, Figure 6.1(b) illustrates the leading order term of $w(\varepsilon)$.

We proceed to define the pressure through

$$p := -gy + cu - \frac{1}{2}|w|^2, \tag{6.39}$$

which has to be understood in the pointwise sense. It should be emphasized that this does *not* define a distribution on $\Omega(\eta)$, because the term $\frac{1}{2}|w|^2$ is not locally integrable at the point vortex. Indeed, recall that before Proposition 6.1 we pointed out that $\nabla^{\perp}\Psi$ is not locally square integrable at the point vortex; and the distribution Φ only differs from the distribution Ψ by a harmonic function. In order to show that conservation of momentum, Equation (5.5), holds pointwise[8] (except at the point vortex), we now need to use the fact that we may express w as the gradient of a velocity field φ on any simply connected domain contained in $\Omega(\eta) \setminus \{(0, -(1-\theta)h)\}$. By straightforward differentiation, we then find

$$\begin{aligned}\nabla p &= -ge_2 + c\nabla(\varphi_x) - \nabla\left(\frac{1}{2}|w|^2\right)\\ &= -ge_2 + c(\nabla\varphi)_x - (w\cdot\nabla)w - \omega(v,-u)\\ &= -ge_2 + cw_x - (w\cdot\nabla)w,\end{aligned}$$

except at the point vortex, which is Equation (5.5). We have here used that the vorticity is supported in the point $(0, -(1-\theta)h)$, and also the identity given in Footnote 6 in Section 5.3.

Finally, we need to verify that the pressure assumes the correct value at the surface, i.e., Equation (5.9). This follows from the definition of p in Equation (6.39) and $F_3(\varepsilon^{-1}\eta, \varepsilon^{-1}\zeta, \varepsilon^{-1}\zeta, \varepsilon) = 0$, after using Equation (5.16). Subtracting the two equations at the surface yields Equation (5.9). Note that, if we only include the leading order terms for w in Equation (6.39), the deviation from the hydrostatic pressure is approximately

$$p + gy \approx -\varepsilon^2\left(\frac{1}{4h}\cot(\pi\theta)\Phi_y + \frac{1}{2}|\nabla\Phi|^2\right)$$

for small ε.

We finish our exposition on a single point vortex with a short discussion on the paths that the fluid particles follow. Observe that if $(x(t), y(t))$ denotes the position of a fluid particle at time t, then

$$(\dot{x}(t), \dot{y}(t)) = w(x(t), y(t), t), \tag{6.40}$$

before the new variables in Section 5.2. After introducing the steady variables (and dropping the tildes), Equation (6.40) becomes

$$(\dot{x}(t), \dot{y}(t)) = w(x(t), y(t)) - (c, 0), \tag{6.41}$$

meaning that if we only keep the first order terms for w and c from Theorem 6.11, we obtain (keeping the same notation for the paths)

$$(\dot{x}(t), \dot{y}(t)) = \varepsilon\nabla^{\perp}\left(\Phi + \tilde{c}_0 y\right)(x(t), y(t)). \tag{6.42}$$

We have used this to obtain Figure 6.3, which shows streamlines in the steady frame moving with the wave. The portraits for θ and $1-\theta$ can be obtained from each other by a 180° rotation. When $\theta = 1/2$, the phase portrait coincides with Figure 6.1(b), in that

[8] The expression $(w\cdot\nabla)w$ is not well defined in the distributional sense, and p is not a distribution, so this is the best we can hope for.

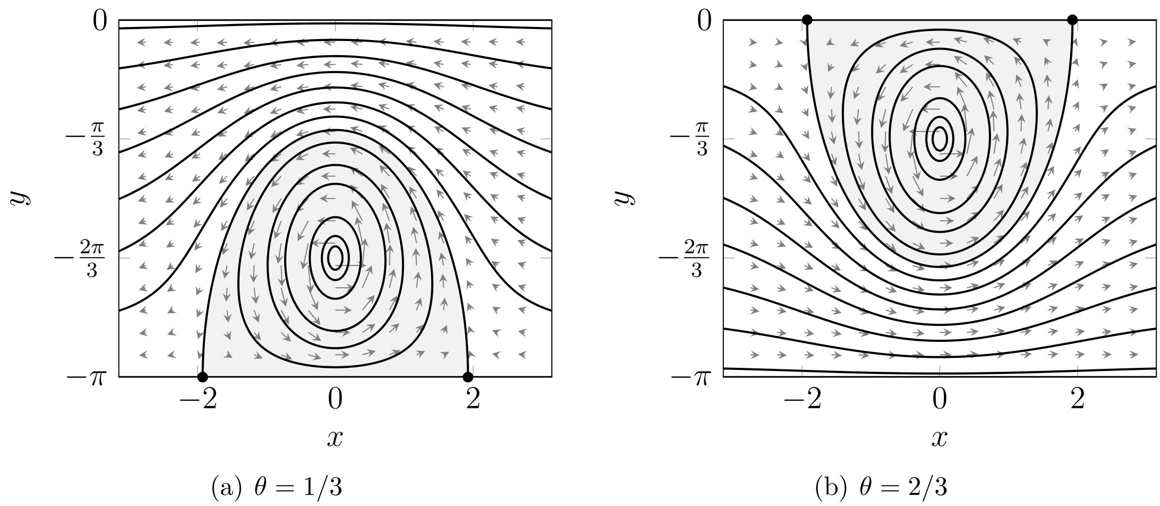


Figure 6.3: Streamlines in the frame of reference traveling with the wave, for $h = \pi$ and $\varepsilon > 0$. The wave corresponding to $\theta = 1/3$ propagates to the right, while the wave corresponding to $\theta = 2/3$ propagates to the left. The arrows illustrating the vector field on the right hand side of Equation (6.42) have been scaled here for visibility, and only their direction is accurate.

all the streamlines are closed, so we will focus on the case $\theta \neq 1/2$. The lines $y = -h$ and $y = 0$ are nullclines for the system in Equation (6.42), and the points (x, y) with

$$x = \pm h/\pi \operatorname{arcosh}(|2\sin(\pi\theta)\tan(\pi\theta) + \cos(\pi\theta)|), \quad y = \begin{cases} -h & \theta < \frac{1}{2} \\ 0 & \theta > \frac{1}{2} \end{cases} \tag{6.43}$$

are equilibrium points, corresponding to stagnation points if they are in the fluid. One may check that

$$h/\pi \operatorname{arcosh}(2\sin(\pi\theta)\tan(\pi\theta) + \cos(\pi\theta)) = \sqrt{3}h\theta + O(\theta^5)$$

as $\theta \downarrow 0$, meaning that the distance between the equilibria is very close to linear in θ for small θ (a corresponding statement holds for $1 - \theta$ small). They go off to infinity as $\theta \to \frac{1}{2}$ from either side. The heteroclinic orbit connecting the two equilibrium points described in Equation (6.43) forms a critical layer[9], enclosing a region of closed streamlines known as a cat's eye vortex. Outside this region the particles always move in the same direction with respect to the steady frame. This direction is either to the left or right depending on the sign of $\cot(\pi\theta)$ and ε.

Because only the first order terms in ε have been kept in Equation (6.42), we do not make any claim about the accuracy of the phase portraits in Figure 6.3 for the full system in Equation (6.41). That would require further and more thorough analysis. Still, the

[9] This curve can be expressed explicitly in terms of arcosh when $\theta < 1/2$, by solving the equation $\Phi(x, y) + \tilde{c}_0 y = \tilde{c}_0(-h)$ for x in terms of y (and similarly for $\theta > 1/2$).

phase portraits can give some indication as to how these waves look beneath the surface. One feature will remain the same for Equation (6.41): Because of the singularity of Φ at $(0, -(1-\theta)h)$, the streamlines will always remain closed sufficiently close to the point vortex.

6.5 Several point vortices

We can extend the existence result for traveling waves with a single point vortex in Theorem 6.11 to a finite number of point vortices on the y-axis without making drastic changes to the argument. However, as opposed to the single vortex case, where we could choose θ freely, there will be limitations on the positions that the point vortices can occupy. We will return to this. Suppose that $1 > \theta_1 > \theta_2 > \cdots > \theta_n > 0$, and that we wish to establish the existence of a traveling wave with point vortex at the points

$$(0, -(1-\theta_1)h), \ldots, (0, -(1-\theta_n)h),$$

the situation being otherwise similar to that of a single point vortex. The admissible surface profiles are those in Γ_{θ_1}, as the uppermost point vortex is the most restrictive.

For $\eta \in \Gamma_{\theta_1}$ we may define

$$\Phi^\gamma := \sum_{j=1}^n \gamma_j \Phi^j, \quad \Phi^j(x,y) := \frac{1}{4\pi} \log\left(\frac{\cosh(\pi x/h) + \cos(\pi(y/h - \theta_j))}{\cosh(\pi x/h) + \cos(\pi(y/h + \theta_j))}\right)$$

in $\Omega(\eta)$ for $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{R}^n$. We will seek solutions of the form

$$w = \nabla^\perp[H(\eta)\zeta + \varepsilon\Phi^\gamma],$$

cf. Equation (6.11) for a single point vortex.

The main difference from the single point vortex case is of course the vorticity equation, Equation (5.10), which needs to be imposed for each of the point vortices. For the ith point vortex, the vorticity equation reduces to

$$\begin{aligned}
(c,0) = {}& \nabla^\perp[H(\eta)\zeta](0, -(1-\theta_i)h) \\
& + \varepsilon\nabla^\perp\left[\gamma_i\left(\Phi^i - \frac{1}{4\pi}\log(x^2 + (y + (1-\theta_i)h)^2)\right) + \sum_{\substack{j=1\\ j\neq i}}^n \gamma_j\Phi^j\right](0, -(1-\theta_i)h) \\
= {}& \nabla^\perp[H(\eta)\zeta](0, -(1-\theta_i)h) \\
& + \frac{\varepsilon}{4h}\left(\gamma_i\cot(\pi\theta_i) + \sum_{\substack{j=1\\ j\neq i}}^n \gamma_j\left[\cot\left(\pi\frac{\theta_i+\theta_j}{2}\right) - \cot\left(\pi\frac{\theta_i-\theta_j}{2}\right)\right], 0\right),
\end{aligned}$$

which, if we assume that η and ζ are even (see the discussion before Equation (6.13)), can be written more succinctly as

$$c\mathbf{1} = -([H(\eta)\zeta]_y(0, -(1-\theta_i)h))_{i=1}^n + \varepsilon\Theta\gamma. \tag{6.44}$$

Here, we have defined $\mathbf{1} \coloneqq (1, \dots, 1) \in \mathbb{R}^n$ and the matrix $\Theta \in \mathbb{R}^{n \times n}$ by

$$\Theta_{i,j} = \begin{cases} \frac{1}{4h} \cot(\pi\theta_i) & i = j \\ \frac{1}{4h} \left(\cot\left(\pi \frac{\theta_i + \theta_j}{2}\right) - \cot\left(\pi \frac{\theta_i - \theta_j}{2}\right)\right) & i \neq j \end{cases} \tag{6.45}$$

for $1 \leq i, j \leq n$.

If we scale the variables as before, Equation (6.44) becomes

$$\tilde{c}\mathbf{1} = -([H(\varepsilon\tilde{\eta})\tilde{\zeta}](0, -(1-\theta_i)h))_{i=1}^n + \Theta\gamma,$$

where one observes that using $\tilde{c}$ as a variable is of course no longer sufficient in general when $n > 1$, since this would involve satisfying n scalar equations by varying one scalar variable. We will therefore use the vortex strengths γ as a variable instead, and set $\tilde{c} \coloneqq 1$. Fixing $\tilde{c} = 1$ only affects the parametrization of the solutions, except in the case that $\tilde{c}$ vanishes when $\varepsilon = 0$.

We now make the necessary redefinitions

$$\begin{aligned} X^s &\coloneqq H^s_{\text{even}}(\mathbb{R}) \times H^s_{\text{even}}(\mathbb{R}) \times \mathbb{R}^n, \\ Y^s &\coloneqq H^{s-2}_{\text{even}}(\mathbb{R}) \times \partial_x(H^s_{\text{even}})(\mathbb{R}) \times \mathbb{R}^n, \\ U^s_{\theta_1} &\coloneqq \left\{ (\tilde{\eta}, \tilde{\zeta}, \gamma, \varepsilon) \in X^s \times \mathbb{R} : \varepsilon\tilde{\eta} \in \Gamma_{\theta_1} \right\}, \end{aligned}$$

and proceed to define, for $s > \frac{3}{2}$, the map $F_1 \colon U^s_{\theta_1} \to H^{s-2}_{\text{even}}(\mathbb{R})$ by

$$\begin{gathered} (\tilde{\eta}, \tilde{\zeta}, \gamma, \varepsilon) \\ \downarrow \\ \varepsilon \left[\frac{\varepsilon\tilde{\eta}'\tilde{\zeta}' + G(\varepsilon\tilde{\eta})\tilde{\zeta}}{\langle \varepsilon\tilde{\eta}' \rangle^2} + \Phi^\gamma_y \right] + \varepsilon \frac{(\tilde{\zeta}' + \Phi^\gamma_x + \varepsilon\tilde{\eta}'\Phi^\gamma_y)^2 + (G(\varepsilon\tilde{\eta})\tilde{\zeta} - \varepsilon\tilde{\eta}'\Phi^\gamma_x + \Phi^\gamma_y)^2}{2\langle \varepsilon\tilde{\eta}' \rangle^2} \\ + g\tilde{\eta} - \frac{\alpha^2}{\varepsilon}\kappa(\varepsilon\tilde{\eta}), \end{gathered}$$

the map $F_2 \colon U^s_{\theta_1} \to \partial_x(H^s_{\text{even}})(\mathbb{R})$ by

$$F_2(\tilde{\eta}, \tilde{\zeta}, \gamma, \varepsilon) \coloneqq \varepsilon\tilde{\eta}' + \tilde{\zeta}' + \Phi^\gamma_x + \varepsilon\tilde{\eta}'\Phi^\gamma_y,$$

and finally the map $F_3 \colon U^s_{\theta_1} \to \mathbb{R}^n$ by

$$F_3(\tilde{\eta}, \tilde{\zeta}, \gamma, \varepsilon) \coloneqq \Theta\gamma - \mathbf{1} - ([H(\varepsilon\tilde{\eta})\tilde{\zeta}](0, -(1-\theta_i)h))_{i=1}^n.$$

In all of these definitions, the derivatives of the function Φ^γ are evaluated at $(x, \varepsilon\tilde{\eta}(x))$, which is suppressed for readability.

One can now define the map $F \coloneqq (F_1, F_2, F_3) \colon U^s_\theta \to Y^s$, which has the trivial zero

$$F(0, 0, \gamma_0, 0) = 0,$$

where

$$\gamma_0 \coloneqq \Theta^{-1}\mathbf{1},$$

under the assumption that the matrix Θ is invertible. It should be emphasized that the matrix is *often* invertible, which we will expand on in Theorem 6.17, but that there always are configurations of n point vortices that yield singular Θ (Proposition 6.18). We have already seen such a configuration, albeit a trivial one: For the case $n = 1$, one has $\Theta = 0$ when $\theta = \frac{1}{2}$, and 0 is certainly singular.

We are led to the following analog of Theorem 6.11 for several point vortices, establishing the existence of a family of small, localized solutions, assuming that Θ is nonsingular.

Theorem 6.15 (Traveling waves with several point vortices)**.** *Let $s > \frac{3}{2}$, and let $1 > \theta_1 > \theta_2 > \cdots > \theta_n > 0$. Suppose that the matrix Θ defined in Equation (6.45) is invertible. Then there exists an open interval $I \ni 0$ and a C^∞-curve*

$$\begin{array}{ccc} I & \to & (H^s_{even}(\mathbb{R}) \cap \Gamma_{\theta_1}) \times H^s_{even}(\mathbb{R}) \times \mathbb{R}^n \times \mathbb{R} \\ \varepsilon & \mapsto & (\eta(\varepsilon), \zeta(\varepsilon), \gamma(\varepsilon), \varepsilon) \end{array} \tag{6.46}$$

of solutions with velocity $c(\varepsilon) = \varepsilon$ to the Zakharov–Craig–Sulem formulation for point vortices of strengths $\varepsilon\gamma_1(\varepsilon), \dots, \varepsilon\gamma_n(\varepsilon)$ situated at $(0, -(1-\theta_1)h), \dots, (0, -(1-\theta_n)h)$. The solutions have the asymptotic form

$$\begin{aligned} \eta(\varepsilon) &= \tilde{\eta}_1 \varepsilon^2 + O(\varepsilon^3), \\ \zeta(\varepsilon) &= O(\varepsilon^3), \\ \gamma(\varepsilon) &= \gamma_0 + O(\varepsilon^2), \end{aligned}$$

as $\varepsilon \to 0$, where $\tilde{\eta}_1 \in H^s_{even}(\mathbb{R})$ is defined by

$$\tilde{\eta}_1 := -(g - \alpha^2 \partial_x^2)^{-1} \chi^{\gamma_0}, \quad \chi^{\gamma_0} := \tilde{c}_0 \Phi^{\gamma_0}_y(\cdot, 0) + \frac{1}{2} \Phi^{\gamma_0}_y(\cdot, 0)^2.$$

Moreover, there is a neighborhood of $(0, 0, \gamma_0, 0)$ in U^s_θ such that the curve

$$\varepsilon \mapsto (\varepsilon^{-1}\eta(\varepsilon), \varepsilon^{-1}\zeta(\varepsilon), \gamma(\varepsilon), \varepsilon)$$

describes the only solutions to $F(\tilde{\eta}, \tilde{\zeta}, \gamma, \varepsilon) = 0$ in that neighborhood.

Proof. As for a single point vortex, we wish to apply the implicit function theorem at the trivial solution $(0, 0, \gamma_0, 0)$. By the same considerations as in the proof of Theorem 6.11, one finds

$$B(X^s, Y^s) \ni D_X F(0, 0, \gamma_0, 0) = \begin{bmatrix} g - \alpha^2 \partial_x^2 & 0 & 0 \\ 0 & \partial_x & 0 \\ 0 & -([H(0)\cdot]_y(0, -(1-\theta_i)h))_{i=1}^n & \Theta \end{bmatrix},$$

where $([H(0)\cdot]_y(0, -(1-\theta_i)h))_{i=1}^n$ means the operator $H^s_{\mathrm{even}}(\mathbb{R}) \to \mathbb{R}^n$ defined by

$$\tilde{\zeta} \mapsto ([H(0)\tilde{\zeta}]_y(0, -(1-\theta_i)h))_{i=1}^n.$$

Recalling that $g - \alpha^2\partial_x^2$ and ∂_x are invertible on the spaces under consideration (see the discussion after Equation (6.22)), and Θ being invertible by assumption, $D_X F(0,0,\gamma_0,0)$ is an isomorphism by lemma A.1.

Hence we can use the implicit function theorem to deduce the existence of an open interval I around zero, an open set $V \subseteq X^s$ such that $V \times I \subseteq U^s_{\theta_1}$ and a map $f \in C^\infty(I, V)$ such that for $(\tilde{\eta}, \tilde{\zeta}, \gamma, \varepsilon) \in V \times I$, we have

$$F(\tilde{\eta}, \tilde{\psi}, \gamma, \varepsilon) = 0 \iff (\tilde{\eta}, \tilde{\zeta}, \gamma) = f(\varepsilon).$$

From the implicit function theorem we also obtain

$$\begin{aligned} Df(0) &= -D_X F(0,0,\tilde{c}_0,0)^{-1} D_\varepsilon F(0,0,\tilde{c}_0,0) \\ &= \begin{bmatrix} -(g-\alpha^2\partial_x^2)^{-1}\chi^{\gamma_0} \\ 0 \\ 0 \end{bmatrix}, \end{aligned}$$

where

$$\chi^{\gamma_0} := \Phi_y^{\gamma_0}(\cdot,1) + \frac{1}{2}\Phi_y^{\gamma_0}(\cdot,1)^2.$$

If we now write $f := (\tilde{\eta}, \tilde{\zeta}, \gamma)$, then it follows by Taylor's theorem that

$$\begin{aligned} &\tilde{\eta}(\varepsilon) = \varepsilon\tilde{\eta}_1, \quad \text{where } \tilde{\eta}_1 := -(g-\alpha^2\partial_x^2)^{-1}\chi^{\gamma_0}, \\ &\tilde{\zeta}(\varepsilon) = O(\varepsilon^2), \\ &\gamma(\varepsilon) = \gamma_0 + O(\varepsilon^2). \end{aligned}$$

as $\varepsilon \to 0$. The wave speed $c(\varepsilon) = \varepsilon$ follows from the choice $\tilde{c} = 1$ made after Equation (6.45). This concludes the proof. $\square$

Remark. An extension of the existence result in Theorem 6.15 to point vortices that are not all on the same vertical line would require a different argument than the one we have used. The main issue is that we can no longer automatically satisfy the vertical component of the vorticity equations as we did in Equation (6.44) by assuming that η, ζ are even. It may be that one can show existence for certain configurations with sufficient symmetry, but we have not pursued this here.

One may note that the sign reversal of the wave velocity about the midpoint $\theta = 1/2$ that we saw with the single point vortex, can be seen also here, albeit in a different manner. If the matrix Θ corresponds to $1 > \theta_1 > \cdots > \theta_n > 0$, and we reflect the vortices across the line $y = -h/2$ by considering $\vartheta_i := 1 - \theta_i, 1 \leq i \leq n$ instead (without reordering them), then the new matrix is $-\Theta$, which can be seen from Equation (6.45) and the identity $\cot(\pi(1-x)) = -\cot(\pi x)$, $x \in (0,1)$. This causes a swap of sign on the leading order vortex strengths, $\gamma_0 = \Theta^{-1}\mathbf{1}$.

We have pointed out that the matrix Θ is not invertible for all configurations of point vortices, and gave the trivial example of $\theta = \frac{1}{2}$ for a single point vortex. This example, together with Theorem 6.11, also shows that invertibility of Θ is not a necessary condition for the existence of a traveling wave with point vortices in those points.

The only case for multiple point vortices on the y-axis where we can feasibly describe the admissible positions directly is for $n = 2$, where the determinant of Θ is given by

$$\det(\Theta) = \frac{1}{16h^2}\left[\cot(\pi\theta_1)\cot(\pi\theta_2) + \cot\left(\pi\frac{\theta_1-\theta_2}{2}\right)^2 - \cot\left(\pi\frac{\theta_1+\theta_2}{2}\right)^2\right]. \tag{6.47}$$

In fact, we give a complete description of when Θ is invertible in Proposition 6.16. See also Figure 6.4, which presents this result graphically. One may observe that the midpoint between the bottom and surface is important also here.

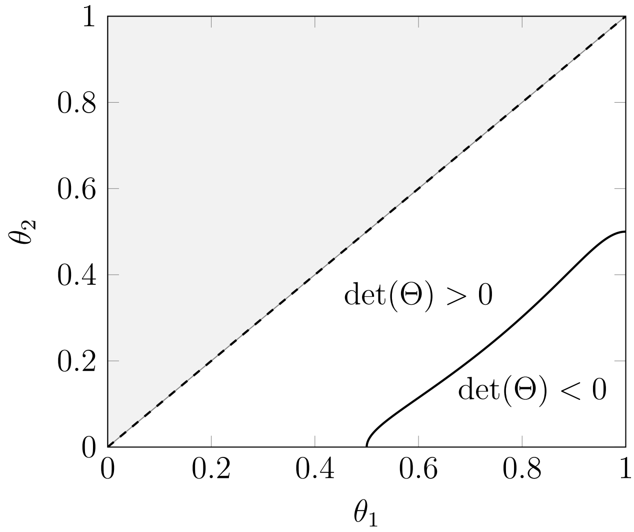


Figure 6.4: The determinant of Θ for the case $n = 2$ as a function of (θ_1, θ_2). The determinant vanishes along the solid black curve, which is given explicitly as a parametrization in Proposition 6.16. (In the figure, the level curve for $\det(\Theta) = 0$ is computed numerically.)

Proposition 6.16 (Θ for $n = 2$)**.** *For two point vortices, we have the following:*

(i) If $\theta_1 \leq \frac{1}{2}$, then Θ is invertible for all $\theta_2 \in (0, \theta_1)$.

(ii) If $\theta_1 > \frac{1}{2}$, then Θ is invertible for all $\theta_2 \in (0, \theta_1)$ except for exactly one value, $0 < \hat{\theta}_2(\theta_1) < \frac{1}{2}$. The graph of $\hat{\theta}_2 \colon \left(\frac{1}{2}, 1\right) \to \left(0, \frac{1}{2}\right)$ is described by the curve

$$\begin{array}{ccc} \left(\frac{1}{4}\pi, \frac{3}{4}\pi\right) & \to & \left(\frac{1}{2}, 1\right) \times \left(0, \frac{1}{2}\right) \\ t & \mapsto & \frac{1}{\pi}(t + f(t), t - f(t)) \end{array},$$

where $f \colon \left(\frac{1}{4}\pi, \frac{3}{4}\pi\right) \to \mathbb{R}$ is given by

$$f(x) := \arccos\left(\sqrt{\frac{1}{2}\left(5 - \cos^2(x) - 2\sec^2(x) + \tan^2(x)\sqrt{\cos^4(x) - 8\cos^2(x) + 4}\right)}\right)$$

for $x \neq \frac{\pi}{2}$, and which is made analytic on the interval by defining $f(\frac{\pi}{2}) := \frac{\pi}{4}$.

Proof. It is useful to rewrite the determinant in Equation (6.47) as

$$\det(\Theta) = \frac{1}{16h^2}\left[\cot(\pi\theta_1)\cot(\pi\theta_2) + \frac{4\sin(\pi\theta_1)\sin(\pi\theta_2)}{(\cos(\pi\theta_2)-\cos(\pi\theta_1))^2}\right]. \tag{6.48}$$

One immediately observes that the second term is always strictly positive. If $\theta_1 \leq \frac{1}{2}$, then one has in addition that the first term is nonnegative for any $\theta_2 \in (0,\theta_1) \subseteq (0,\frac{1}{2})$. This proves the first part of the proposition.

For the second point, let us first prove that there is exactly one value of θ_2 for each $\theta_1 \in (1/2,1)$ that makes Θ singular, and that this value lies in the interval $(0,1/2)$. For fixed $\theta_1 \in (1/2,1)$ the determinant is strictly increasing in θ_2. This is clear for the first term in the parentheses; for the second term, observe that it has the partial derivative

$$(\theta_1,\theta_2) \mapsto 4\pi\sin(\pi\theta_1)\frac{2-\cos^2(\pi\theta_2)-\cos(\pi\theta_1)\cos(\pi\theta_2)}{(\cos(\pi\theta_2)-\cos(\pi\theta_1))^3}$$

with respect to θ_2. This function is strictly positive for $\theta_2 \in (0,\theta_1)$, since the cosines in the numerator have absolute value strictly less than 1.

Thus, there is at most one value of θ_2, given $\theta_1 \in (1/2,1)$, that makes $\det(\Theta)$ vanish. Since the determinant clearly tends to $-\infty$ as $\theta_2 \downarrow 0$ and to ∞ as $\theta_2 \uparrow \theta_1$, there is exactly one such value, say $\hat{\theta}_2(\theta_1)$. Because the determinant is positive when $\theta_2 = \frac{1}{2}$, as the first term in Equation (6.48) is zero, this value must necessarily lie in the interval $(0,1/2)$.

We now move to the parametrization of the graph of the map $\hat{\theta}_2\colon (1/2,1) \to (0,1/2)$. It should first be pointed out that it is possible to find an explicit expression for the function $\hat{\theta}_2$, as the equation

$$\det(\Theta) = 0$$

can be written as a cubic equation in $\cos(\pi\theta_2)$ by the use of trigonometric identities. Such equations are of course solvable in radicals of their coefficients, but the caveat is that this yields very unwieldy expressions. So let us rather focus on parametrizing it instead, which is simpler.

One may note from Figure 6.4 that there is symmetry across the diagonal line $\{(\theta_1,\theta_2) \in (0,1)^2 : \theta_1 > \theta_2, \theta_1+\theta_2 = 1\}$, which suggests making a change of variables. By letting

$$\phi_1 := \pi\frac{\theta_1+\theta_2}{2}, \quad \phi_2 := \pi\frac{\theta_1-\theta_2}{2}, \tag{6.49}$$

we can write the determinant in the form

$$\det(\Theta) = \frac{1}{16h^2}\left[\frac{\cos^2(\phi_1)+\cos^2(\phi_2)-1}{\cos^2(\phi_2)-\cos^2(\phi_1)} + \frac{\cos^2(\phi_2)}{1-\cos^2(\phi_2)} - \frac{\cos^2(\phi_1)}{1-\cos^2(\phi_1)}\right],$$

which leads us to solve the quadratic equation

$$ax^2 + (a^2-5a+2)x + (2a-1), \quad a := \cos^2(\phi_1), \quad x := \cos^2(\phi_2)$$

for x, given a. Doing this yields the parametrization, by using ϕ_1 as the parameter (some care has to be used to ensure that one picks the right branches of the functions involved) and going back to the original variables by inverting Equation (6.49). Note that the singularity at $x = \frac{\pi}{2}$ in the function f given in the statement of the proposition is removable . $\square$

While the set of configurations that make $\det(\Theta)$ vanish is hard to describe in general when $n > 2$, some observations can be made. Of course, if $n \geq 2$, and as long as the derivative of $\det(\Theta)$ with respect to the variable $(\theta_1, \ldots, \theta_n)$ does not vanish at a point where $\det(\Theta) = 0$, the zero set[10] of $\det(\Theta)$ is locally a smooth manifold of dimension $n-1$ around that point by the implicit function theorem. When $n = 2$ the zero set is actually the graph of a smooth function in θ_1 by Proposition 6.16, and numerical evidence suggests that the zero set is the graph of a smooth function in (θ_1, θ_2) when $n = 3$. Actually checking that the derivative does not vanish is hard, but we have the following theorem:

Theorem 6.17. *The subset of configurations of point vortices in*

$$\{(\theta_1, \ldots, \theta_n) \in (0,1)^n : 1 > \theta_1 > \theta_2 > \cdots > \theta_n > 0\}$$

such that Θ is not invertible has measure zero.

Proof. We will apply Theorem 3.7 to prove this theorem. By the Leibniz formula for the determinant, we have that

$$\det(\Theta) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \Theta_{i,\sigma(i)},$$

where each entry in Θ is real analytic in each θ_i for $\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n$ fixed. It follows that $\det(\Theta)$ also has this property, when viewed as a function

$$U \coloneqq \{(\theta_1, \ldots, \theta_n) \in (0,1)^n : 1 > \theta_1 > \theta_2 > \cdots > \theta_n > 0\} \to \mathbb{R}.$$

In fact, it makes sense to view the determinant as a function

$$U_{\mathbb{C}} \coloneqq \{(z_1, \ldots, z_n) \in ((0,1) \times \mathbb{R})^n \subseteq \mathbb{C}^n : 1 > \operatorname{Re} z_1 > \operatorname{Re} z_2 > \cdots > \operatorname{Re} z_n > 0\} \to \mathbb{C}$$

which is complex analytic in each z_i, for $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n$ fixed. By Hartog's theorem, a deep theorem on functions of several complex variables which can be found in [Hörmander, 1990, Theorem 2.2.8], $\det(\Theta)$ is jointly complex analytic[11] on $U_{\mathbb{C}}$. In particular, it is jointly real analytic when restricted to the set $U \subseteq U_{\mathbb{C}}$.

In order to conclude, we need to show that $\det(\Theta)$ does not vanish identically on U. To that end, fix $\frac{1}{2} > \tilde{\theta}_1 > \tilde{\theta}_2 > \cdots > \tilde{\theta}_n > 0$ and consider $\theta_1 = \varepsilon\tilde{\theta}_1, \ldots \theta_n = \varepsilon\tilde{\theta}_n$ for

[10] The zero set of a real analytic function (which we will see that the determinant is) can still be exceedingly complicated, as evidenced by [Krantz and Parks, 2002, Theorem 6.3.3] (Lojasiewicz' theorem).

[11] The corresponding claim for separately real analytic functions is false. In fact, they need not even be continuous. The standard example is $(x, y) \mapsto xy/(x^2 + y^2)$ on $\mathbb{R}^2$ (defined to be 0 at the origin), which is certainly separately real analytic, but not continuous at the origin. There even exist C^∞-functions that are separately analytic, but not analytic. An example is $(x, y) \mapsto xy \exp(-1/(x^2 + y^2))$.

$1 > \varepsilon > 0$. The purpose of the upper bound of $\frac{1}{2}$ is to make sure that $\tan(\pi\theta_i)$ is well defined for all $1 \le i \le n$. Observe now that

$$h[\mathrm{diag}(\tan(\pi\theta_1), \ldots, \tan(\pi\theta_n))\Theta]_{ij} = \begin{cases} \frac{1}{4} & i = j \\ \frac{1}{4}\tan(\pi\theta_i)\left(\cot\left(\pi\frac{\theta_i+\theta_j}{2}\right) - \cot\left(\pi\frac{\theta_i-\theta_j}{2}\right)\right) & i \neq j \end{cases}$$

where

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0}\frac{1}{4}\tan(\varepsilon\pi\tilde{\theta}_i)&\left(\cot\left(\varepsilon\pi\frac{\tilde{\theta}_i+\tilde{\theta}_j}{2}\right) - \cot\left(\varepsilon\pi\frac{\tilde{\theta}_i-\tilde{\theta}_j}{2}\right)\right) \\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{4}\frac{\tan(\varepsilon\pi\tilde{\theta}_i)}{\varepsilon}\left(\varepsilon\cot\left(\varepsilon\pi\frac{\tilde{\theta}_i+\tilde{\theta}_j}{2}\right) - \varepsilon\cot\left(\varepsilon\pi\frac{\tilde{\theta}_i-\tilde{\theta}_j}{2}\right)\right) \\
&= \frac{1}{4}\cdot\pi\tilde{\theta}_i\cdot\frac{2}{\pi}\left(\frac{1}{\tilde{\theta}_i+\tilde{\theta}_j} - \frac{1}{\tilde{\theta}_i-\tilde{\theta}_j}\right) \\
&= -\frac{\tilde{\theta}_i\tilde{\theta}_j}{\tilde{\theta}_i^2-\tilde{\theta}_j^2}
\end{aligned}$$

for $i \neq j$. The limits on the second line can be calculated using the definition of the derivative. Since all norms on finite-dimensional spaces are equivalent, it follows that $\mathrm{diag}(\tan(\pi\theta_1), \ldots, \tan(\pi\theta_n))\Theta$ has a limit in $B(\mathbb{R}^n)$ as $\varepsilon \downarrow 0$, and that this limit is

$$\lim_{\varepsilon\downarrow 0}\mathrm{diag}(\tan(\pi\theta_1), \ldots, \tan(\pi\theta_n))\Theta = \frac{1}{4h}(I_{\mathbb{R}^n} - B), \tag{6.50}$$

where we have defined $B \in \mathbb{R}^{n\times n}$ by

$$B_{i,j} := \begin{cases} 0 & i = j \\ 4\tilde{\theta}_i\tilde{\theta}_j(\tilde{\theta}_i^2 - \tilde{\theta}_j^2)^{-1} & i \neq j \end{cases}. \tag{6.51}$$

In particular, B is skew-symmetric ($B^T = -B$), which implies that $I_{\mathbb{R}^n} - B$ is invertible. To see this, assume that $x \in \mathbb{R}^n$ is in the kernel of $I_{\mathbb{R}^n} - B$; meaning that $Bx = x$. Then

$$|x|^2 = \langle x, x\rangle_{\mathbb{R}^n} = \langle Bx, x\rangle_{\mathbb{R}^n} = \langle x, -Bx\rangle_{\mathbb{R}^n} = \langle x, -x\rangle_{\mathbb{R}^n} = -|x|^2,$$

implying that $x = 0$. Thus $I_{\mathbb{R}^n} - B$ is invertible, and since the set of invertible operators is open, see Lemma 3.5, so is the matrix

$$\mathrm{diag}(\tan(\pi\theta_1), \ldots, \tan(\pi\theta_n))\Theta$$

for sufficiently small ε, which in turn means that Θ is invertible for such ε.

Finally, the set U is convex; indeed, if $(\theta_1, \ldots, \theta_n), (\hat{\theta}_1, \ldots, \hat{\theta}_n) \in U$, then it follows by adding the defining inequalities (multiplied by $1 - t$ and t, respectively) that

$$1 > (1-t)\theta_1 + t\hat{\theta}_1 > \cdots > (1-t)\theta_n + t\hat{\theta}_n > 0$$

for any $t \in (0, 1)$. In particular, this implies that U is connected. Hence, since we know that $\det(\Theta)$ is analytic in the variable $(\theta_1, \ldots, \theta_n)$ and does not vanish identically, we can apply Theorem 3.7 to infer that the subset of U on which $\det(\Theta)$ vanishes has measure zero. □

In general we cannot do better than Theorem 6.17, in the sense that for any $n \geq 1$ there will always be a configuration of n point vortices that makes $\det(\Theta)$ vanish.

Proposition 6.18. *There are always configurations of point vortices in*

$$\{(\theta_1, \ldots, \theta_n) \in (0,1)^n : 1 > \theta_1 > \theta_2 > \cdots > \theta_n > 0\}$$

where $\det(\Theta)$ *has opposite signs. In particular, since the set is connected and* $\det(\Theta)$ *is continuous on this set, there exists a configuration for which* $\det(\Theta)$ *vanishes.*

Proof. Not only is the matrix appearing on the right hand side of Equation (6.50) in the proof of Theorem 6.17 invertible, it has a positive determinant. Indeed, the matrix B defined in Equation (6.51) is skew-symmetric, so its spectrum is purely imaginary. Moreover, since the matrix is real, the eigenvalues are either zero or appear in complex conjugate pairs. Say that the first m eigenvalues $\lambda_1, \ldots, \lambda_m$ are zero and that

$$\lambda_{m+2j-1} = \overline{\lambda_{m+2j}} = i\mu_j, \quad j = 1, \ldots, (n-m)/2,$$

where the μ_j are real.

It then follows that

$$\begin{aligned} \det\left(\frac{1}{4h}(I_{\mathbb{R}^n} - B)\right) &= \frac{1}{(4h)^n}\det(I_{\mathbb{R}^n} - B) \\ &= \frac{1}{(4h)^n}(1+\mu_1^2)(1+\mu_2^2)\cdots(1+\mu_{(n-m)/2}^2), \end{aligned}$$

because the determinant of a matrix is equal to the product of its eigenvalues (taking algebraic multiplicity into account). By Equation (6.50) we then have

$$\det(\Theta)\prod_{j=1}^{n}\tan(\pi\theta_j) > 0 \tag{6.52}$$

for small $\varepsilon > 0$ (as in the proof of Theorem 6.17) by continuity of the determinant. Since all the tangents are also positive, this implies that $\det(\Theta) > 0$ for small $\varepsilon > 0$.

It remains to exhibit a configuration where $\det(\Theta) < 0$. To that end, fix $\frac{1}{2} > \tilde{\theta}_1 > \tilde{\theta}_2 > \tilde{\theta}_3 > \cdots > \tilde{\theta}_n > 0$ and consider $\theta_1 = 1 - \varepsilon\tilde{\theta}_1, \theta_2 = \varepsilon\tilde{\theta}_2, \ldots \theta_n = \varepsilon\tilde{\theta}_n$ for $1 > \varepsilon > 0$. Proceeding as in the proof of Theorem 6.17 we find

$$\lim_{\varepsilon \downarrow 0} \operatorname{diag}(\tan(\pi\theta_1), \ldots, \tan(\pi\theta_n))\Theta = \frac{1}{4h}(I_{\mathbb{R}^n} - \tilde{B}), \tag{6.53}$$

where we have defined $\tilde{B} \in \mathbb{R}^{n\times n}$ by

$$\tilde{B}_{i,j} \coloneqq \begin{cases} 0 & i = j \text{ or } i = 1 \text{ or } j = 1 \\ 4\tilde{\theta}_i\tilde{\theta}_j(\tilde{\theta}_i^2 - \tilde{\theta}_j^2)^{-1} & \text{otherwise} \end{cases}.$$

This matrix is still skew-symmetric like B, and so the right side of Equation (6.53) has a positive determinant, as before. Hence Equation (6.52) holds for small ε. However, now $\tan(\pi\theta_1)$ is negative and the rest of the tangents are positive, meaning that $\det(\Theta)$ must be negative. $\square$

7 Periodic waves

As for the solitary waves we worked with in Chapter 6, there are two stream functions that are of interest for periodic waves: One which is constant at the surface when $\eta = 0$, and one which is not (corresponding to the functions Φ and Ψ in Equations (6.3) and (6.5), respectively). However, there are two differences in the periodic case. One is that we may no longer expect in general that this constant is the same at the bottom and surface. For the localized case we argued by going to infinity, but we can no longer do this here. The second is that, as opposed to the case of the localized waves, the stream function that is constant also on the surface is not possible to write down with elementary functions. This is true even for the special case where the point vortex is placed at the midpoint between the bottom and surface (corresponding to $\theta = 1/2$). We will apply Theorem A.11 in order to explicitly construct the periodic version of the function Ψ for one point vortex in each period, at any depth.

We mention that it is possible to write down the periodic version of Φ when $\theta = 1/2$ in terms of the so-called Jacobi elliptic functions, by using a slight modification of Theorem A.11, but we will not discuss this here. Our goal in this chapter is Section 7.2, where we will apply our knowledge of the periodic Ψ in order to get more explicit expressions for solutions found in [Shatah et al., 2013].

7.1 Construction of the periodic Ψ

Suppose that we wish to find a stream function $\psi \colon \mathbb{R} \times (-h, \infty) \to \mathbb{R}$ corresponding to equally spaced point vortices of unit strength at the points $l\mathbb{Z} \times \{-(1-\theta)h\} \subseteq \mathbb{R}^2$, and which is such that this stream function vanishes at the bottom, $\mathbb{R} \times \{-h\}$. It is natural to look for a function that is even and l-periodic in x and by symmetry that ψ_x vanishes on $(l/2 + l\mathbb{Z}) \times (-h, \infty)$. This leads us to the equation

$$\Delta\psi = \delta_\theta, \quad \psi|_{y=-h} = 0, \quad \psi_x|_{x=\pm l/2} = 0, \tag{7.1}$$

on $(-l/2, l/2) \times (-h, \infty)$, which is precisely the kind of setting that Theorem A.11 deals with. We will see that the solution of this equation obtained from the theorem has a periodic extension to $\mathbb{R} \times (-h, \infty)$ that has the properties that we seek.

We require a conformal map from $(-l/2, l/2) \times (-h, \infty)$ to the slit unit disk that satisfies the requirements of Theorem A.11. This is done in several steps, where each step should be well known from complex analysis (see e.g. [Gamelin, 2001, II.6, XI.1]). By $z \mapsto \sqrt{z}$ we will mean the principal branch of the square root on $\mathbb{C} \setminus [0, \infty)$, and we also define the constant

$$a \coloneqq \tanh(\pi\theta h/l).$$

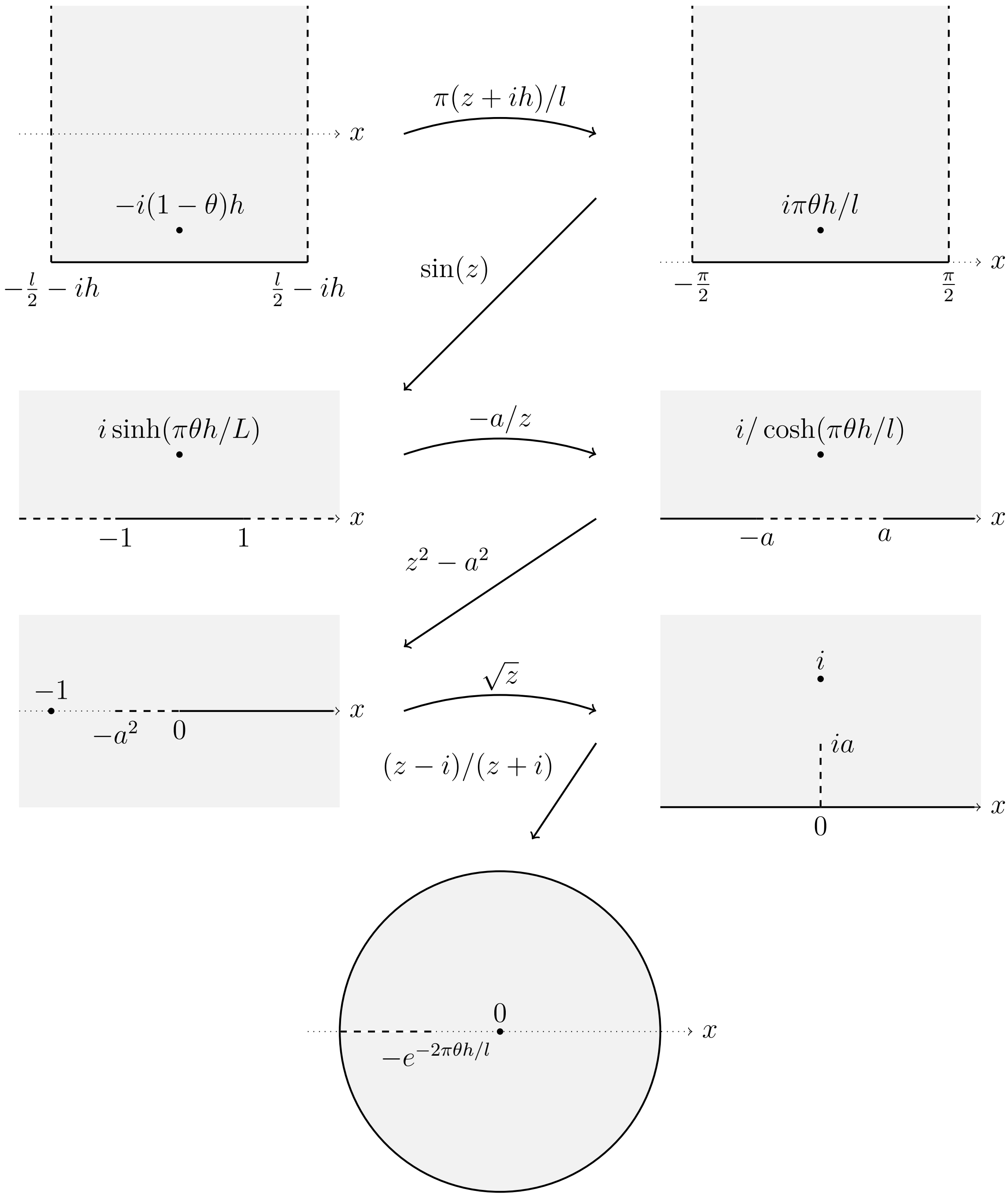


Figure 7.1: The conformal map from $(-l/2, l/2) \times (-h, \infty)$ to a slit unit disk for use in Theorem A.11. By $\sin(z)$ we mean $z \mapsto \sin(z)$, et cetera.

Figure 7.1, together with Theorem A.11, and observing that the resulting solution extends to a periodic function, proves the following proposition.

Proposition 7.1. *The map* $\Psi\colon (-l/2, l/2) \times (-h, \infty) \to \mathbb{R}$ *defined by*

$$\Psi(x, y) := \frac{1}{4\pi} \log\left(\frac{\cos(2\pi x/l) - \cosh(2\pi(y + (1-\theta)h)/l)}{\cos(2\pi x/l) - \cosh(2\pi(y + (1+\theta)h)/l)}\right)$$

satisfies Equation (7.1)*, and extends to a function on* $\mathbb{R} \times (-h, \infty)$ *that is* l*-periodic and even in* x *and harmonic on* $(\mathbb{R} \times (-h, \infty)) \setminus (l\mathbb{Z} \times \{-(1-\theta)h\})$*.*

Proof. The composition of the maps in Figure 7.1 shows that

$$f(z) := \frac{\tanh(\pi\theta h/l) + \tanh(i\pi(z + ih)/l)}{\tanh(\pi\theta h/l) - \tanh(i\pi(z + ih)/l)}$$

defines the required conformal map, which yields the expression for Ψ after taking its modulus and applying Theorem A.11. It is clear that this extends to a function that is periodic in x. □

7.2 Explicit expressions for infinite depth

Remark. In this section we adopt the notation and conventions from [Shatah et al., 2013]. The fluid domain is $\mathbb{R} \times (-\infty, 1)$ and the waves have period $l = 2\pi L$. The stream function for a single point vortex at the origin, vanishing at the surface, will be denoted by G.

Observe that the map constructed in Proposition 7.1 also provides the stream function for periodic solutions on infinite depth (vanishing at the surface) by a shift and reversal of the vertical direction. Indeed, from a slight modification of the procedure in Figure 7.1, one has that

$$f(z) := \frac{\tanh(1/(2L)) - \tanh((1 + iz)/(2L))}{\tanh(1/(2L)) + \tanh((1 + iz)/(2L))} \tag{7.2}$$

defines a bijective conformal map from the half strip $(-\pi L, \pi L) \times (-\infty, 1)$ onto the slit unit disk $\mathbb{D} \setminus ((0, \exp(-1/L)) \times \{0\})$, and which is such that

(i) The origin is fixed.

(ii) The surface is mapped to $\mathbb{S}^1$.

(iii) The sides $\{\pm\pi L\} \times (-\infty, 1)$ are mapped to the slit.

It follows, after taking the logarithm the modulus of the map f in Equation (7.2), that we can explicitly write the stream function G used in [Shatah et al., 2013] as

$$\begin{aligned} G(x, y) &= \frac{1}{4\pi} \sum_{k=-\infty}^{\infty} \log\left(\frac{(x - 2k\pi L)^2 + y^2}{(x - 2k\pi L)^2 + (y-2)^2}\right) \\ &= \frac{1}{4\pi} \log\left(\frac{\cos(x/L) - \cosh(y/L)}{\cos(x/L) - \cosh((y-2)/L)}\right). \end{aligned}$$

From the remark after Theorem A.10 we also learn that

$$\begin{aligned}\tilde{c}_0 &= -\frac{1}{4\pi}\sum_{k=-\infty}^{\infty}\frac{1}{k^2L^2\pi^2+1}\\ &= \frac{i}{4\pi}\overline{\left(\frac{f''(0)}{f'(0)}\right)}\\ &= -\frac{1}{4\pi L}\coth(1/L),\end{aligned}$$

where we mention that an alternative method of arriving at a closed form of the series given for $\tilde{c}_0$ is using the Poisson summation formula ([Stein and Shakarchi, 2003, Theorem 3.1], with the remark after) on the function $x \mapsto e^{-2|x|/L}$. Observe that

$$\lim_{L\to\infty}\tilde{c}_0 = -\frac{1}{4\pi},$$

which coincides with the speed of the localized case on infinite depth (this can also be seen from the series).

We can also give a Fourier series expansion for the leading order surface profile term, η_*. From [Shatah et al., 2013] we know that

$$\eta_* = -(g-\alpha^2\partial_x^2)^{-1}\left(\chi - \frac{1}{2\pi L}\int_{-\pi L}^{\pi L}\chi\,d\mu\right), \tag{7.3}$$

where we have defined χ by

$$\chi(x) \coloneqq \tilde{c}_0 G_y(x,1) + \frac{1}{2}G_y(x,1)^2.$$

Observe that

$$\partial_y G(x,1) = -\frac{1}{2\pi L}\frac{\sinh(1/L)}{\cos(x/L)-\cosh(1/L)},$$

so that

$$\begin{aligned}\chi(x) &= \frac{1}{8\pi^2L^2}\left[\frac{\cosh(1/L)}{\cos(x/L)-\cosh(1/L)} + \frac{\sinh^2(1/L)}{(\cos(x/L)-\cosh(1/L))^2}\right]\\ &= \frac{1}{8\pi^2L^2}\frac{\cosh(1/L)\cos(x/L)-1}{(\cos(x/L)-\cosh(1/L))^2}\\ &= -\frac{1}{8\pi^2 L}\left[z\mapsto\frac{\sin(z/L)}{\cos(z/L)-\cosh(1/L)}\right]'(x).\end{aligned}$$

In particular, this means that

$$\begin{aligned}\int_{-\pi L}^{\pi L}\chi\,d\mu &= -\frac{1}{8\pi^2L}\left(\frac{\sin(\pi)}{\cos(\pi)-\cosh(1/L)} - \frac{\sin(-\pi)}{\cos(-\pi)-\cosh(1/L)}\right)\\ &= 0,\end{aligned}$$

so that Equation (7.3) reduces to

$$\eta_* = -(g - \alpha^2 \partial_x^2)^{-1} \chi. \tag{7.4}$$

Consider now the rational function f_a on $\mathbb{C}$ defined by

$$f_a(z) := \frac{z^2 - 1}{z^2 - 2az + 1}, \quad a > 1,$$

and which has partial fraction decomposition

$$\frac{z^2 - 1}{z^2 - 2az + 1} = 1 + \frac{b}{z - b} + \frac{b^{-1}}{z - b^{-1}},$$

where $b := a - \sqrt{a^2 - 1}$ and $b^{-1} = a + \sqrt{a^2 - 1}$. Observe that $b < 1 < b^{-1}$, so that f can be expanded in a Laurent series on the annulus

$$\{z \in \mathbb{C} : b < |z| < b^{-1}\},$$

which contains the unit circle. Explicitly, we have

$$\begin{aligned} f_a(z) &= 1 + \frac{b}{z}\frac{1}{1 - \frac{b}{z}} - \frac{1}{1 - bz} \\ &= -\sum_{n=-\infty}^{\infty} \operatorname{sgn}(n) b^{|n|} z^n \end{aligned}$$

in the annulus.

In particular

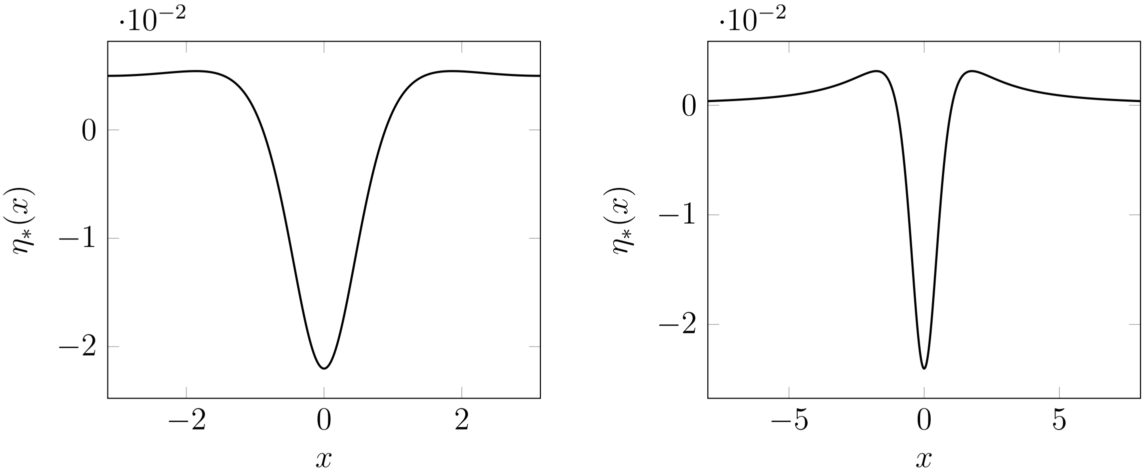
$$\begin{aligned} \frac{\sin(x/L)}{\cos(x/L) - \cosh(1/L)} &= \frac{1}{i} f_a(e^{ix/L}), \quad a := \cosh(1/L) \\ &= i \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n) e^{-|n|/L} e^{inx/L}, \end{aligned}$$

since $a = \cosh(1/L)$ yields $b = e^{-1/L}$. Hence, by termwise differentiation (justified by the rapid decay of the coefficients), we obtain

$$\begin{aligned} \chi(x) &= \frac{1}{8\pi^2 L^2} \sum_{n=-\infty}^{\infty} |n| e^{-|n|/L} e^{inx/L} \\ &= \frac{1}{4\pi^2 L^2} \sum_{n=1}^{\infty} n e^{-n/L} \cos(nx/L). \end{aligned}$$

Finally, by Equation (7.4), this implies that the leading order term in the expansion for the surface can be written as the Fourier series

$$\eta_*(x) = -\frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{n}{gL^2 + \alpha^2 n^2} e^{-n/L} \cos(nx/L).$$



(a) $L = 1$ (Whole period) (b) $L = 100$ (Only part of a period)

Figure 7.2: The leading order surface profile term, η_*, when $g = 1, \alpha^2 = 0.01$, c.f. Figure 1 in [Shatah et al., 2013].

When L is large, this is very similar to the surface in the localized case, see Figure 7.2(b). At the other extreme, one has

$$e^{1/L}\left|\eta_*(x) + \frac{1}{4\pi^2(gL^2+\alpha^2)}e^{-1/L}\cos(nx/L)\right| \leq \frac{1}{8\pi^2\alpha^2(e^{1/L}-1)},$$

meaning that η_* will very rapidly tend to the first term in the Fourier series as $L \downarrow 0$. Figure 7.2(a) illustrates this tendency.

A Useful results

We start with a lemma that generalizes a well known fact about triangular matrices.

Lemma A.1 (Invertibility of a lower triangular operator)**.** *Suppose that $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ are Banach spaces and that $T_{i,j} \in B(X_j, Y_i)$ for all $1 \leq i \leq j \leq n$. If $T_{i,i}$ is invertible for $1 \leq i \leq n$, then the operator $T \in B(\prod_{i=1}^n X_i, \prod_{i=1}^n Y_i)$ defined by*

$$T := \begin{bmatrix} T_{1,1} & & 0 \\ \vdots & \ddots & \\ T_{n,1} & \cdots & T_{n,n} \end{bmatrix}$$

is invertible.

Proof. It is sufficient to consider the case of $n = 2$, as a simple induction argument will account for $n > 2$. For $n = 2$ it is trivial to verify by direct calculation that the operator $S \in B(Y_1 \times Y_2, X_1 \times X_2)$ defined by

$$S := \begin{bmatrix} T_{1,1}^{-1} & 0 \\ -T_{2,2}^{-1} T_{2,1} T_{1,1}^{-1} & T_{2,2}^{-1} \end{bmatrix} \tag{A.1}$$

is the inverse of T. □

Remark. Explicit inverses for $n > 2$ can be found by multiple applications of the formula in Equation (A.1). It should also be clear that a corresponding version of this lemma holds for upper triangular operators.

Lemma A.2 (Weyl's lemma)**.** *Suppose that $\Omega \subseteq \mathbb{R}^d$ is open and that $f \in D'(\Omega)$ is harmonic in the sense of distributions. Then $f \in L^1_{loc}(\Omega)$ and has a harmonic (in the pointwise sense) representative.*

Proof. This is a special case of more general regularity theorems for distributional solutions of elliptic partial differential equations. See e.g. [Folland, 1999, Theorem 9.26] for one such elliptic regularity theorem that applies in this case. □

Lemma A.3 ($BC^k(U, Y)$ is Banach)**.** *Let X, Y be Banach spaces, and let $U \subseteq X$ be an open set. Then $BC^k(U, Y)$ is a Banach space for any $k \geq 0$.*

Proof. That $BC(U,Y)$ is a Banach space follows by uniform limits of continuous functions being continuous. Suppose thus that $k \geq 1$ and let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $BC^k(U,Y)$. By the definition of the norm in $BC^k(U,Y)$, Equation (2.1), the sequences $(D^j f_n)_{n\in\mathbb{N}} \subseteq BC(U, B^j(X,Y))$, $1 \leq j \leq k$, are also Cauchy. Hence, there is a function $g_0 \in BC(U,Y)$ and functions $g_j \in BC(U, B^j(X,Y))$ such that

$$\begin{aligned} f_n &\to g_0, \\ D^j f_n &\to g_j, \quad 0 \leq j \leq k, \end{aligned} \tag{A.2}$$

in $BC(U,Y)$ and $BC(U, B^j(X,Y))$, respectively. It remains to show that g_0 is k times differentiable, and that $D^j g_0 = g_j$ for each j. If we can show this, then Equation (A.2) shows that $f_n \to g_0$ in $BC^k(U,Y)$.

Fix a point $x \in U$ and let $r > 0$ be such that $B_r(x) \subseteq U$. Now, for any $h \in B_r(0)$ and j satisfying $0 \leq j \leq k-1$, we have[1]

$$\begin{aligned} &D^j f_n(x+h) - D^j f_n(x) - D^{j+1} f_n(x)h \\ &\qquad = \frac{1}{\|h\|_X} \int_0^{\|h\|_X} (D^{j+1} f_n(x + t\|h\|_X^{-1} h) - D^{j+1} f_n(x))h\, dt, \end{aligned}$$

whence taking the limit as $n \to \infty$ and using the uniform convergence yields

$$g_j(x+h) - g_j(x) - g_{j+1}(x)h = \frac{1}{\|h\|_X} \int_0^{\|h\|_X} (g_{j+1}(x + t\|h\|_X^{-1} h) - g_{j+1}(x))h\, dt.$$

If we denote the right hand side by $r(h)$, we find

$$\begin{aligned} \|r(h)\|_{B^j(X,Y)} &\leq \sup_{\substack{\tilde{h}\in X \\ \|\tilde{h}\|_X \leq \|h\|_X}} \|g_{j+1}(x+\tilde{h}) - g_{j+1}(x)\|_{B^{j+1}(X,Y)} \|h\|_X \\ &= o(\|h\|_X), \quad \text{as } h \to 0, \end{aligned}$$

due to the continuity of g_{j+1} at x. This shows that g_j is differentiable at the point x and that $Dg_j(x) = g_{j+1}(x)$ for all $0 \leq j \leq k-1$. Hence $g_0 \in BC^k(U,Y)$ with $D^j g_0 = g_j$ for each j. □

Lemma A.4 (Convergence in $H^s(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$)**.** *Let $f \in H^s(\mathbb{R}^d)$, and assume that $(f_n)_{n\in\mathbb{N}}$ is a sequence converging to f in $H^s(\mathbb{R}^d)$. Then $(f_n)_{n\in\mathbb{N}}$ also converges to f in the topology on $S'(\mathbb{R}^d)$.*

Proof. By definition we have (σ is the reflection operator introduced after Equation (6.12))

$$\begin{aligned} \langle f - f_n, \varphi\rangle &= \langle \mathscr{F}(f - f_n), \mathscr{F}^{-1}\varphi\rangle \\ &= \langle \langle\cdot\rangle^s \mathscr{F}(f-f_n), \sigma(\langle\cdot\rangle^{-s}\mathscr{F}\varphi)\rangle_{L^2(\mathbb{R}^d)}, \end{aligned}$$

[1] The integral is the so-called Bochner integral, generalizing the Lebesgue integral to Banach-space-valued functions.

for any $\varphi \in S(\mathbb{R}^d)$, whence the Cauchy–Schwarz inequality yields

$$|\langle f - f_n, \varphi\rangle| \leq \|f - f_n\|_{H^s(\mathbb{R}^d)} \|\varphi\|_{H^{-s}(\mathbb{R}^d)}, \quad \varphi \in S(\mathbb{R}^d).$$

This implies convergence of $(f_n)_{n\in\mathbb{N}}$ to f in $S'(\mathbb{R}^d)$. $\square$

Theorem A.5 (Sobolev embedding in BC^k)**.** *Suppose that* $s > \frac{d}{2}$. *Then*

$$H^s(\mathbb{R}^d) \hookrightarrow BC^{\lceil s - \frac{d}{2}\rceil - 1}(\mathbb{R}^d),$$

with continuous embedding.

Proof. It is well known that $S(\mathbb{R}^d)$ is dense in $H^s(\mathbb{R}^d)$. Let $l := \lceil s - \frac{d}{2}\rceil - 1$. Now observe that, the result will follow if we can show that there is a constant $C \geq 0$, such that

$$|D^\alpha \psi(x)| \leq C\|\psi\|_{H^s(\mathbb{R}^d)}$$

for all multi-indices $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq l$, $x \in \mathbb{R}^d$ and $\psi \in S(\mathbb{R}^d)$. Indeed, given $f \in H^s(\mathbb{R}^d)$, choose a sequence $(\varphi_n)_{n\in\mathbb{N}}$ such that $\varphi_n \to f$ in $H^s(\mathbb{R}^d)$. Then the above inequality would imply that this sequence was Cauchy in $BC^l(\mathbb{R}^d)$, hence converging to some $\varphi \in BC^l(\mathbb{R}^d)$. One must necessarily have $f = \varphi$ a.e. since L^2-convergence implies pointwise almost everywhere convergence of a subsequence. The continuity of the embedding also follows from the same bound.

We find (with some abuse of notation)

$$D^\alpha \psi(x) = \mathscr{F}^{-1}[i^{|\alpha|}\xi^\alpha \mathscr{F}\psi](x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} i^{|\alpha|}\xi^\alpha (\mathscr{F}\psi)(\xi) e^{ix\cdot\xi} d\xi,$$

whence we can use $|\xi^\alpha| \leq \left(\frac{\alpha!}{|\alpha|!}\right)^{\frac{1}{2}} \langle\xi\rangle^{|\alpha|}$, following from the multinomial theorem, to obtain

$$\begin{aligned} |D^\alpha \psi(x)| &\leq C_{\alpha,d}\|\langle\cdot\rangle^{|\alpha|}\mathscr{F}\psi\|_{L^1(\mathbb{R}^d)}, \quad C_{\alpha,d} := \left(\frac{\alpha!}{(2\pi)^d|\alpha|!}\right)^{\frac{1}{2}} \\ &\leq C_{\alpha,d}\|\langle\cdot\rangle^{-(s-l)}\|_{L^2(\mathbb{R}^d)}\|\psi\|_{H^s(\mathbb{R}^d)}, \end{aligned}$$

where the last inequality is the Hölder inequality. By changing to polar coordinates (e.g. [Folland, 1999, Theorem 2.49]) and using that $s - l > d/2$, one sees that the first norm is finite. In fact, one may check that

$$\begin{aligned} \|\langle\cdot\rangle^{-a}\|_{L^2(\mathbb{R}^d)} &= \left(\sigma(\mathbb{S}^{d-1}) \int_0^\infty \frac{r^{d-1}\,dr}{(1+r^2)^a}\right)^{\frac{1}{2}} \\ &= \left(\pi^{d/2}\frac{\Gamma(a - d/2)}{\Gamma(a)}\right)^{\frac{1}{2}} \end{aligned}$$

when $a > d/2$, where Γ is the gamma function, by writing the radial integral in terms of the so-called beta function. The measure σ is the standard surface measure on $\mathbb{S}^{d-1}$. $\square$

Remark. We cannot do better than this in general, see for instance [Adams and Fournier, 2003, 4.40].

Proposition A.6 (Poincaré inequality for $\Omega(\eta)$)**.** *Let $\eta \in \mathrm{Lip}(\mathbb{R},\mathbb{R}) \cap BC(\mathbb{R},\mathbb{R})$ be such that* $\inf \eta > -h$*. Then*

$$\|f\|_{L^2(\Omega(\eta))} \leq (h + \sup \eta)\|\partial_y f\|_{L^2(\Omega(\eta))}$$

for every $f \in H_0^1(\Omega(\eta))$, where $H_0^1(\Omega(\eta))$ is the closure of $D(\Omega)$ in $H^1(Omega)$.

Proof. Suppose first that $f \in D(\Omega(\eta))$. We have

$$\|f\|^2_{L^2(\Omega(\eta))} = \int_{\mathbb{R}} \int_{-h}^{\eta(x)} f(x,y)^2 \, dy \, dx,$$

where for each fixed x, we can use integration by parts on the inner integral to obtain

$$\int_{-h}^{\eta(x)} f(x,y)^2 dy = -\int_{-h}^{\eta(x)} \left(y - \frac{\eta(x) - h}{2}\right) 2f(x,y)\partial_y f(x,y) dy.$$

Here, the constant in the integrand is chosen in order to get a tighter bound in the final inequality. Taking absolute values now yields

$$\int_{-h}^{\eta(x)} f(x,y)^2 dy \leq (\eta(x) + h) \int_{-h}^{\eta(x)} |f(x,y)\partial_y f(x,y)| dy,$$

whence

$$\begin{aligned}\|f\|^2_{L^2(\Omega(\eta))} &\leq (h + \sup \eta)\|f\partial_y f\|_{L^1(\Omega(\eta))} \\ &\leq (h + \sup \eta)\|f\|_{L^2(\Omega(\eta))}\|\partial_y f\|_{L^2(\Omega(\eta))}\end{aligned}$$

by the Hölder inequality. Hence the inequality holds when $f \in D(\Omega)$.

Assume now that $f \in H_0^1(\Omega(\eta))$. Then there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq D(\Omega(\eta))$ such that $f_n \to f$ in $H^1(\Omega(\eta))$. For each $n \in \mathbb{N}$, we have

$$\|f_n\|_{L^2(\Omega(\eta))} \leq (h + \sup \eta)\|\partial_y f_n\|_{L^2(\Omega(\eta))},$$

which, since $f_n \to f$ and $\partial_y f_n \to \partial_y f$ in $L^2(\Omega(\eta))$, means that the result follows by letting $n \to \infty$. ☐

Lemma A.7 (Multiplication in Sobolev spaces)**.** *Suppose that $f \in H^s(\mathbb{R}^d), g \in H^t(\mathbb{R}^d)$, where $s,t > 0$ and $\max(s,t) > \frac{d}{2}$. Then $fg \in H^{\min(s,t)}(\mathbb{R}^d)$ and*

$$\|fg\|_{H^{\min(s,t)}(\mathbb{R}^d)} \leq C\|f\|_{H^s(\mathbb{R}^d)}\|g\|_{H^t(\mathbb{R}^d)}$$

for some real constant $C > 0$ depending only on s,t,d. In particular, if $s > \frac{d}{2}$, then $H^s(\mathbb{R}^d)$ is an algebra under pointwise multiplication.

Proof. See [Runst and Sickel, 1996, 4.6.1 Theorem 1]. □

Lemma A.8 [Lannes, 2013] (Division in Sobolev spaces)**.** *If* $s > \frac{d}{2}$*, and* $f, g \in H^s(\mathbb{R}^d)$ *with* $\min g > -1$*, then*

$$\frac{f}{1+g} \in H^s(\mathbb{R}^d).$$

Moreover, for fixed g*, one has*

$$\left\| \frac{f}{1+g} \right\|_{H^s(\mathbb{R}^d)} \leq C \|f\|_{H^s(\mathbb{R}^d)}$$

for some real constant $C > 0$ *that is independent of* f*. The constant can be uniformly bounded on bounded subsets of* $H^s(\mathbb{R}^d)$ *that uniformly satisfy the lower bound on* g*.*

Proof. See for instance [Lannes, 2013, Proposition B.4]. □

Lemma A.9 (Nemytskii operator on $H^s(\mathbb{R}^d)$)**.** *Assume that* $f \in C^\infty(\mathbb{R}, \mathbb{R})$ *and that* $f(0) = 0$*. If* $s > \frac{d}{2}$*, then*

$$F(g) \coloneqq f \circ g$$

defines a C^∞*-operator* $H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$*. Moreover, if* f *is analytic in a neighborhood of the origin, then* F *is analytic in a neighborhood of the origin.*

Proof. The proof of the first result, and generalizations thereof, can be found in [Runst and Sickel, 1996, 5.3, 5.5], by noting that $H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ for $s > \frac{d}{2}$. The analyticity of the operator F follows from [Runst and Sickel, 1996, 5.5.3 Theorem 4]. □

Theorem A.10 (Green's functions in $\mathbb{R}^2$)**.** *Suppose that* $\Omega \subsetneq \mathbb{R}^2$ *is a simply connected domain and that* $z_0 \in \Omega$*. Furthermore, suppose that* $f \colon \Omega \to \mathbb{D}$ *is a bijective conformal map onto the open unit disk, extending continuously to a function* $\overline{\Omega} \to \overline{\mathbb{D}}$ *and satisfying* $f(z_0) = 0$*. Then the function* $\varphi \colon \Omega \to \mathbb{R}$ *defined by*

$$\varphi(z) \coloneqq \frac{1}{2\pi} \log(|f(z)|)$$

is in $L^1_{loc}(\Omega)$*, extends continuously to the boundary of* Ω*, and satisfies*

$$\begin{aligned} \Delta \varphi &= \delta_{z_0}, \\ \varphi|_{\partial\Omega} &= 0. \end{aligned}$$

Furthermore, the harmonic function h *defined by*

$$h(z) \coloneqq \frac{1}{2\pi} \log\left(|f(z)|\right) - \frac{1}{2\pi} \log\left(|z - z_0|\right)$$

satisfies

$$\nabla h(z_0) = \frac{1}{4\pi} \overline{\left(\frac{f''(z_0)}{f'(z_0)} \right)}$$

after identifying $\mathbb{R}^2$ *and* $\mathbb{C}$ *via* $(x, y) \mapsto x + iy$*.*

Proof. We first check the boundary values of the function φ. By assumption, f extends continuously to $\partial\Omega$, and every point on $\partial\Omega$ must necessarily be mapped to $\mathbb{S}^1$. It is thus immediate that φ also extends continuosly to the boundary, and moreover, vanishes there.

Identify now $\mathbb{R}^2$ and $\mathbb{C}$. Observe that since $f(z_0) = 0$, we have

$$f(z) = g(z)(z - z_0), \quad z \in \Omega$$

for some (complex) analytic function g, where $|g| > 0$. Indeed, we must have $g(z_0) = f'(z_0) \neq 0$ because f is injective, and the injectivity of f also ensures that there can be no other roots. Thus

$$\varphi(z) = \frac{1}{2\pi} \log\left(|z - z_0|\right) + h(z),$$

where

$$h(z) \coloneqq \frac{1}{2\pi} \operatorname{Re} \log\left(g(z)\right)$$

is harmonic by $|g| > 0$ and the Cauchy-Riemann equations. Hence, by Proposition 5.1, the function φ is L^1_{loc} and satisfies

$$\Delta\varphi = \delta_{z_0}.$$

The last assertion follows by observing that one must necessarily have $g'(z_0) = \frac{1}{2} f''(z_0)$, meaning that

$$\left(\frac{1}{2\pi} \log(g(\cdot))\right)'(z_0) = \frac{1}{2\pi} \frac{g'(z_0)}{g(z_0)} = \frac{1}{4\pi} \frac{f''(z_0)}{f'(z_0)},$$

whence we deduce from the Cauchy-Riemann equations that

$$\nabla h(z_0) = \frac{1}{4\pi} \overline{\left(\frac{f''(z_0)}{f'(z_0)}\right)}. \qquad \square$$

Remark. One can always find a bijective conformal map between a simply connected domain $\Omega \subsetneq \mathbb{C}$ and the unit disk. This is the famous Riemann mapping theorem. That the conformal map extends continuously to the boundary is not always true, but it is true under mild assumptions on the regularity of the boundary of the domain (Carathéodory's theorem, see [Markushevich, 1967, Theorems 2.24, 2.25]). We will only use Theorem A.10 on domains where we can check this condition directly. One also has

$$\nabla^{\perp} h(z_0) = \frac{i}{4\pi} \overline{\left(\frac{f''(z_0)}{f'(z_0)}\right)}$$

by a rotation.

There are several variations of Theorem A.10, depending on the desired boundary conditions of the equation. The following one is useful for us, because it can be used to provide stream functions for the periodic case in certain situations. For clarity, we have included Figure A.1.

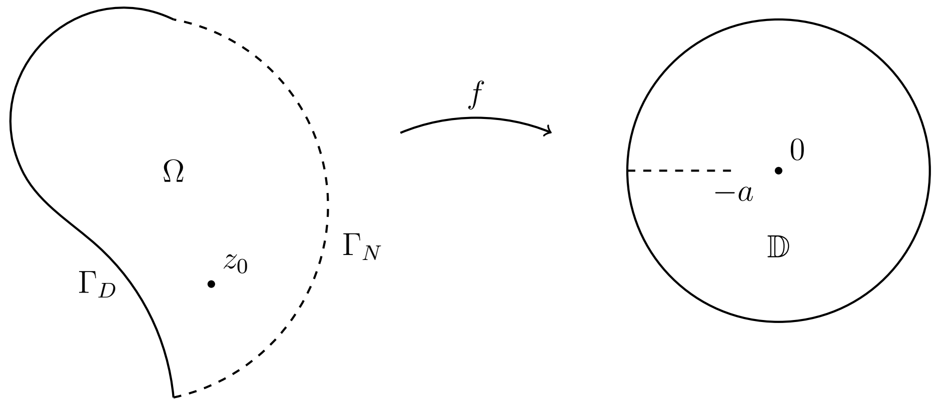


Figure A.1: The situation in Theorem A.11 illustrated. Γ_D and its image are shown with solid lines, while Γ_N and its image are dashed. The curve Γ_N is "wrapped" around the slit in the disk.

Theorem A.11 (Green's functions in $\mathbb{R}^2$, mixed)**.** *Suppose that $\Omega \subsetneq \mathbb{R}^2$ is a simply connected domain and that $z_0 \in \Omega$. Furthermore, assume that $\partial\Omega = \Gamma_D \sqcup \Gamma_N$, where Γ_N is C^1 and open in $\partial\Omega$. Finally, suppose that $f\colon \Omega \to \mathbb{D} \setminus ((-1,-a] \times \{0\})$, where $a > 0$, is a bijective conformal map of Ω onto the unit disk with a slit, satisfying $f(z_0) = 0$ and extending continuously to the boundary. This map should send Γ_D to $\mathbb{S}^1$ and Γ_N to the interval $(-1, a] \times \{0\}$, and should extend analytically across Γ_N (when viewed as a map on $\mathbb{C}$). Then the function $\varphi\colon \Omega \to \mathbb{R}$ defined by*

$$\varphi(z) := \frac{1}{2\pi} \log\left(|f(z)|\right)$$

is in $L^1_{loc}(\Omega)$, extends continuously to the boundary and satisfies

$$\begin{aligned} \Delta\varphi &= \delta_{z_0}, \\ \varphi|_{\Gamma_D} &= 0, \\ \partial_n \varphi|_{\Gamma_N} &= 0, \end{aligned}$$

where ∂_n denotes the normal derivative.

Proof. The fact that φ is in $L^1_{\mathrm{loc}}(\Omega)$, extends continuously to the boundary and satisfies

$$\begin{aligned} \Delta\varphi &= \delta_{z_0} \\ \varphi|_{\Gamma_D} &= 0, \end{aligned}$$

follows by a slight modification of the proof of Theorem A.10. We need therefore only check the Neumann condition on Γ_N.

To that end, let w_0 be any point on Γ_N that is *not* mapped to the point $(-a, 0)$ (there is precisely one point that is mapped to $(-a, 0)$) and which is such that $f'(w_0) \neq 0$ (this can only happen in isolated points since f is analytic). Since Γ_N is C^1 and open in $\partial\Omega$, we can describe an open neighborhood of w_0 in $\partial\Omega$ as a simple C^1-curve $\gamma\colon (-\varepsilon,\varepsilon) \to \mathbb{C}$ with $\gamma(0) = w_0$ and $\gamma'(0) \neq 0$. Choose now another simple C^1-curve $\gamma_\perp\colon [0,\varepsilon) \to \overline{\Omega}$ with $\gamma_\perp(0) = w_0$, $|\gamma_\perp'(0)| = 1$ and which is such that

$$\gamma_\perp'(0) \perp \gamma'(0).$$

Since we assumed that f extends analytically across Γ_N and that $f'(w_0) \neq 0$, it is conformal at w_0, and so

$$(f \circ \gamma_\perp)'(0) \perp (f \circ \gamma)'(0),$$

which implies that $(f \circ \gamma_\perp)'(0)$ is purely imaginary (and nonzero), since $(f \circ \gamma)'(0)$ is purely real, as Γ_N is mapped to the interval $(-1,-a] \times \{0\}$ by f. Hence

$$\begin{aligned}
\partial_n \varphi(w_0) = (\varphi \circ \gamma_\perp)'(0) &= \frac{1}{2\pi}(\log|f \circ \gamma_\perp|)'(0) \\
&= \operatorname{Re}\left(\frac{1}{2\pi}(\log(f \circ \gamma_\perp))'(0)\right) \\
&= \operatorname{Re}\left(\frac{1}{2\pi}\frac{(f \circ \gamma_\perp)'(0)}{f(w_0)}\right) \\
&= 0,
\end{aligned}$$

where we have used that $(f \circ \gamma_\perp)'(0)$ is imaginary and $f(w_0)$ is real. Since f extends analytically across Γ_N, this must also hold in the point which is mapped to $(-a,0)$ and the points where f' vanishes, by the resulting continuity of $\nabla\varphi$. □

Remark. The assertion about the gradient of the harmonic function in Theorem A.10 is clearly still valid in this setting. The theorem is also valid for multiple slits, which need not necessarily be lying on the real line, as long as Γ_N is still mapped to the slits. This holds because, while $(f \circ \gamma_\perp)'(0)$ and $f(w_0)$ (in the proof of Theorem A.11) may no longer be imaginary and real, respectively, the fraction $(f \circ \gamma_\perp)'(0)/f(w_0)$ will still be purely imaginary, because $(f \circ \gamma_\perp)'(0)$ and $f(w_0)$ are orthogonal when viewed as elements of $\mathbb{R}^2$.

It should also be stressed that Theorem 6.15 says nothing about the existence of such a conformal map. If Ω and z_0 are particularly simple, we can sometimes construct the map explicitly. In general, this is not an easy task, and it may not even exist.

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