

Consistent regularization and renormalization in models with inhomogeneous phases

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In many models in condensed matter and high-energy physics, one finds inhomogeneous phases at high density and low temperature. These phases are characterized by a spatially dependent condensate or order parameter. A proper calculation requires that one takes the vacuum fluctuations of the model into account. These fluctuations are ultraviolet divergent and must be regularized. We discuss different ways of consistently regularizing and renormalizing quantum fluctuations, focusing on momentum cutoff, symmetric energy cutoff, and dimensional regularization. We apply these techniques calculating the vacuum energy in the Nambu-Jona-Lasinio model in $1 + 1$ dimensions in the large- N_c limit and in the $3 + 1$ dimensional quark-meson model in the mean-field approximation both for a one-dimensional chiral-density wave.

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I. INTRODUCTION

There are many systems in condensed matter and high-energy physics where some of the phases are inhomogeneous. These are phases where an order parameter or a condensate depends on position. The simplest case is where only the phase of the order parameter is varying; the general case is where both magnitude and phase are functions of position. The idea of inhomogeneous phases is rather old going back to the work of Fulde and Ferrell as well as by Larkin and Ovchinnikov in the context of superconductors [1,2], density waves in nuclear matter by Overhauser [3], and pion condensation by Migdal [4]. In recent years, inhomogeneous phases have been studied in, for example, cold atomic gases [5], color superconducting phases [6–8], quarkyonic phases [9,10], as well as chiral condensates [11–22]; see Refs. [23,24] for recent reviews.

In the case of QCD, these inhomogeneous phases exist at large baryon chemical potential μ_B or isospin chemical potential μ_I , and low temperature. In the case of large μ_B , they cannot be studied by lattice simulations due to the infamous sign problem and one must use low-energy models of QCD. Examples of such models are the Nambu-Jona-Lasinio (NJL) model and the quark-meson (QM) model or their Polyakov-loop extended versions, the PNJL and PQM models. Most of the calculations in $3 + 1$ dimensions have been inspired by corresponding calculations in $1 + 1$ dimensions using *Ansätze* for the inhomogeneities that are one dimensional [25–33]. Interestingly,

some of these models in $1 + 1$ dimensions can be solved exactly in the large- N_c limit and show a rich phase diagram [28,29].

When calculating the thermodynamic potential in these models, one faces ultraviolet divergences due to vacuum fluctuations. The ultraviolet divergences in the NJL model are typically regularized using a sharp momentum cutoff Λ [34]. However, in the case of inhomogeneous condensates, a naive application of a momentum cutoff leads to an incorrect expression for the vacuum energy in the limit where the magnitude of the order parameter vanishes [30,31]. Instead, proper time regularization [12], symmetric energy cutoff regularization” [30,31] and Pauli-Villars regularization have been applied [13–16]. In this paper, we will discuss how to use momentum cutoff, symmetric energy cutoff, and dimensional regularization in the case of inhomogeneous phases. In order to obtain a meaningful expression for the vacuum energy, it is necessary to perform a unitary transformation (that depends on the wave vector) on the free Hamiltonian and subtract the vacuum energy of the noninteracting system.

In the NJL model, one cannot throw away the quantum fluctuations since chiral symmetry breaking is induced by them; i.e. there is no symmetry breaking at tree level.¹ This is in contrast to the quark-meson model, where the Higgs mechanism is implemented by choosing a negative mass term in the tree-level potential. In many finite-temperature applications of the quark-meson model, one ignores the

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¹In the NJL model, there is symmetry breaking for a coupling constant larger than a critical one, G_c , which depends on the cutoff Λ .

vacuum fluctuations hoping that their effects on the chiral transition are negligible [13,35]. However, it turns out that vacuum fluctuations are important [36]. In the two-flavor QM model, the chiral transition is first order in the whole μ_B - T plane without vacuum fluctuations. Adding the quantum fluctuations, the transition changes from being first order (in the chiral limit) to being second order at zero baryon chemical potential μ_B , while it remains first order at zero temperature. Thus, including the vacuum fluctuations, the first-order line that starts at $T = 0$ ends at a tricritical point somewhere in the μ_B - T plane. Similarly, when allowing for an inhomogeneous phase such as a chiral-density wave, it exists in the entire μ_B - T plane in the absence of quantum fluctuations. Including quantum fluctuations, the inhomogeneous phase emerges from a tricritical point and exists in a region of low temperatures down to $T = 0$ [16].

Conventionally, dimensional regularization has been used in the context of the quark-meson model. However, *a priori*, there is nothing that prevents us from treating the quark-meson model as a cutoff field theory [37] and regularizing it using a sharp ultraviolet momentum cutoff Λ . Having introduced an ultraviolet cutoff Λ , one can renormalize, i.e. redefine the parameters of the model and take the limit $\Lambda \rightarrow \infty$ at the end. In this case, one trades the ultraviolet cutoff for a renormalization scale μ . In fact, this procedure yields results that are reminiscent of dimensional regularization in which power divergences are set to zero and logarithmic divergences show up as poles. The poles are then removed by renormalization of the parameters of the theory. Either way, there is an ambiguity, since there is a dependence on ultraviolet cutoff Λ or the renormalization scale μ .

The article is organized as follows. In Sec. II we discuss the problem of a simple momentum cutoff in the context of an NJL type model in 1+1 dimensions. We will show that by subtracting the vacuum energy of the free theory after a unitary transformation, one can obtain a meaningful vacuum energy using a momentum cutoff, an energy cutoff, or dimensional regularization. In Sec. III, we show how to apply these techniques to the quark-meson model in three dimensions. In Sec. IV, we summarize and discuss our results.

II. NJL MODEL IN 1 + 1 DIMENSIONS

A. Lagrangian and thermodynamic potential

The Lagrangian of the NJL model in 1 + 1 dimensions is

$$\mathcal{L} = \bar{\psi} \left[i\partial - m_0 + \left(\mu + \frac{1}{2} \tau_3 \mu_I \right) \gamma^0 \right] \psi + \frac{G}{N_c} [(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5\tau\psi)^2], \quad (1)$$

where N_c is the number of colors and m_0 is the current quark mass. Moreover ψ is a color N_c -plet, a two-component Dirac spinor as well as a flavor doublet

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}. \quad (2)$$

Here $\mu_B = 3\mu = \frac{3}{2}(\mu_u + \mu_d)$ and $\mu_I = (\mu_u - \mu_d)$ are the baryon and isospin chemical potentials expressed in terms of the quark chemical potentials μ_u and μ_d . The γ -matrices are $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_1$, and $\gamma^5 = \gamma^0\gamma^1 = \sigma_3$, where σ_i are the three Pauli matrices, and τ_a are the three Pauli matrices in flavor space. The Lagrangian (1) has a global $SU(N_c)$ symmetry and for $m_0 = \mu_I = 0$, it is also invariant under $U_B(1) \times SU_L(2) \times SU_R(2)$. For $m_0 \neq 0$ and $\mu_I = 0$, the $SU_L(2) \times SU_R(2)$ symmetry is reduced to $SU_I(2)$. For $m_0 = 0$ and $\mu_I \neq 0$, the symmetry $SU_L(2) \times SU_R(2)$ is reduced to $U_{I_3L}(1) \times U_{I_3R}(1)$, where I_3 is the third component of isospin. If $m_0 \neq 0$ and $\mu_I \neq 0$ the $SU_L(2) \times SU_R(2)$ symmetry is reduced to $U_{I_3}(1)$.

We next introduce the collective sigma and pion fields

$$\sigma = -2 \frac{G}{N_c} \bar{\psi}\psi, \quad (3)$$

$$\pi_a = -2 \frac{G}{N_c} \bar{\psi}i\gamma^5\tau_a\psi. \quad (4)$$

The Lagrangian (1) then becomes

$$\mathcal{L} = \bar{\psi} \left[i\partial - m_0 + \left(\mu + \frac{1}{2} \tau_3 \mu_I \right) \gamma^0 - \sigma - i\gamma^5 \pi_a \tau_a \right] \psi - \frac{N_c(\sigma^2 + \pi_a^2)}{4G}. \quad (5)$$

The chiral condensate that we choose is a chiral-density wave of the form²

$$\langle \sigma \rangle = M \cos(2bz) - m_0, \quad (6)$$

$$\langle \pi_3 \rangle = M \sin(2bz), \quad (7)$$

where b is a wave vector. The Mermin-Wagner-Coleman theorem normally forbids spontaneous symmetry breaking in 1 + 1 dimensions; however, it does not apply in the large- N_c limit [38,39]. We denote the last term in Eq. (5) by $-V_0$ such that V_0 is the tree-level potential. Note that for nonzero z , the crossterm $-\frac{N_c m_0 M \cos(2bz)}{2G}$ averages to zero when the spatial extent L of the system is large enough. This term can then be written as $-\frac{N_c m_0 M \delta_{b,0}}{2G}$ and the expression for V_0 is

²With a nonzero isospin chemical potential, there is also the possibility of a pion condensate Δ . For simplicity, we do not include this in the present analysis.

$$V_0 = \frac{N_c(M^2 + m_0^2 - 2Mm_0\delta_{b,0})}{4G}. \quad (8)$$

In the homogeneous case, $V_0 = \frac{N_c(M-m_0)^2}{4G}$.

With the *Ansatz* (6)–(7), the Dirac operator D can be written as

$$D = \bar{\psi} \left[i\bar{\partial} + \left(\mu + \frac{1}{2}\tau_3\mu_I \right) \gamma^0 - M e^{2i\gamma^5\tau_3bz} \right] \psi. \quad (9)$$

We next redefine the quark fields, $\psi \rightarrow e^{-i\gamma^5\tau_3bz}\psi$ and $\bar{\psi} \rightarrow \bar{\psi}e^{-i\gamma^5\tau_3bz}$. The Dirac operator then reads

$$D = [i\bar{\partial} + (\mu + b'\tau_3)\gamma^0 - M], \quad (10)$$

where $b' = (b + \frac{1}{2}\mu_I)$ and $2b'$ is an effective isospin chemical potential. The transformation of the field ψ amounts to a unitary transformation of the Dirac Hamiltonian, $\mathcal{H} \rightarrow \mathcal{H}' = e^{i\gamma^5\tau_3bz}\mathcal{H}e^{-i\gamma^5\tau_3bz}$. It turns out that there is a spurious dependence on b in the free energy: For some regulators, the free energy depends on b in the limit $M \rightarrow 0$. However, physical quantities cannot depend on the wave vector when the modulus of the condensate is zero. This unphysical behavior of the free energy requires the introduction of a subtraction term that by construction guarantees that the free energy is independent of b in the limit $M \rightarrow 0$. We will return to this issue below. There is an additional complication for nonzero isospin chemical potential since the spurious dependence of b translates into additional dependence on the isospin chemical potential in the free energy. We therefore set $\mu_I = 0$ for now and return to the case of nonzero μ_I at the end of this section.

Going to momentum space, Eq. (10) can be written as

$$D = [\not{p} + (\mu + b\tau_3)\gamma^0 - M]. \quad (11)$$

It is now straightforward to derive the fermionic spectrum in the background (7). It is given by the zeros of the Dirac determinant and reads [30]

$$E_{\pm} = \sqrt{\left(\sqrt{p^2 + M^2} \pm b\right)^2}. \quad (12)$$

We notice that the lower branch, E_- , has zero energy for nonzero momentum, $p = \pm\sqrt{b^2 - M^2}$, if $b > M$. It is this nonmonotonic behavior that allows for inhomogeneous condensates at finite chemical potential by lowering the energy and at the same time populating only the lower branch E_- .

We can now integrate over the fermions to obtain the free energy in the mean-field approximation. After integrating over p_0 , this yields the standard expression

$$V = V_0 - N_c \int_{-\infty}^{\infty} \frac{dp}{2\pi} (E_+ + E_-). \quad (13)$$

B. Momentum versus energy cutoff

The starting point is the one-loop correction to the effective potential,

$$V_1 = -N_c \int_{-\infty}^{\infty} \frac{dp}{2\pi} (E_+ + E_-). \quad (14)$$

If we regulate the integral by a simple momentum cutoff Λ , we can write

$$V_1 = -\frac{N_c}{\pi} \int_0^{\Lambda} (E_+ + E_-) dp. \quad (15)$$

It will prove useful to change the variable to $u = \sqrt{p^2 + M^2}$. The integral then becomes

$$V_1 = -\frac{N_c}{\pi} \int_M^{\sqrt{\Lambda^2 + M^2}} (|u + b| + |u - b|) \frac{udu}{\sqrt{u^2 - M^2}}. \quad (16)$$

We next write $V_1 = V_+ + V_-$. In the limit of large Λ , we find

$$\begin{aligned} V_+ &= -\frac{N_c}{\pi} \int_M^{\sqrt{\Lambda^2 + M^2}} |u + b| \frac{udu}{\sqrt{u^2 - M^2}} \\ &= -\frac{N_c}{2\pi} \left[\Lambda^2 + 2b\Lambda + \frac{1}{2}M^2 \left(\log \frac{4\Lambda^2}{M^2} + 1 \right) \right]. \end{aligned} \quad (17)$$

The other contribution V_- is given by

$$V_- = -\frac{N_c}{\pi} \int_M^{\sqrt{\Lambda^2 + M^2}} |u - b| \frac{udu}{\sqrt{u^2 - M^2}}. \quad (18)$$

Here we must be careful distinguishing between the cases $u > b$ and $u < b$. In the large- Λ limit, one finds

$$\begin{aligned} V_- &= -\frac{N_c}{2\pi} \left[\Lambda^2 - 2b\Lambda + \frac{1}{2}M^2 \left(\log \frac{4\Lambda^2}{M^2} + 1 \right) \right] \\ &\quad + \theta(b - M)f(M, b), \end{aligned} \quad (19)$$

where we have defined the function $f(M, b)$

$$f(M, b) = -\frac{N_c}{\pi} \left[b\sqrt{b^2 - M^2} - M^2 \log \frac{b + \sqrt{b^2 - M^2}}{M} \right]. \quad (20)$$

The one-loop contribution to the free energy is then given by the sum of Eqs. (17) and (19). After renormalizing the vacuum energy by removing the term proportional to Λ^2 , the effective potential in the mean-field approximation becomes

$$V = V_0 - \frac{N_c M^2}{2\pi} \left[\log \frac{4\Lambda^2}{M^2} + 1 \right] + \theta(b-M)f(M, b). \quad (21)$$

We note that the terms linear in b cancel and that the final result is an even function of b as it must be; cf. Eq. (16). However, the vacuum energy is unbounded below due to the term $b\sqrt{b^2 - M^2}$ implying that the system is unstable. Moreover, in the limit $M \rightarrow 0$, the effective potential reduces to $V = -\frac{N_c b^2}{\pi}$ (for $m_0 = 0$). This is clearly unphysical; the effective potential must be independent of the wave vector b when the magnitude M of the condensate vanishes. As pointed out in [30,31], the problem is that the cutoff is imposed on the momentum and not the energy. Using a momentum cutoff Λ , the effective cutoff on the energy is $\sqrt{\Lambda^2 + M^2} \pm b$, which is different for the two branches for nonzero b . The idea put forward in [30,31] is to use a cutoff Λ on the energy rather than the momentum of the particles; i.e. one restricts the integration by imposing the same cutoff on the two branches,

$$E_{\pm} < \Lambda. \quad (22)$$

This is referred to as symmetric energy cutoff [30,31]. This restriction can be expressed as an upper limit for the integration variable u and yields $u < \Lambda \mp b$. The contribution V_+ from E_+ in the large- Λ limit then becomes

$$\begin{aligned} V_+ &= -\frac{N_c}{\pi} \int_M^{\Lambda-b} |u+b| \frac{udu}{\sqrt{u^2 - M^2}} \\ &= -\frac{N_c}{2\pi} \left[\Lambda^2 - b^2 + \frac{1}{2} M^2 \left(\log \frac{4\Lambda^2}{M^2} - 1 \right) \right]. \end{aligned} \quad (23)$$

The contribution V_- from E_- is in the large- Λ limit given by

$$\begin{aligned} V_- &= -\frac{N_c}{\pi} \int_M^{\Lambda+b} |u-b| \frac{udu}{\sqrt{u^2 - M^2}} \\ &= -\frac{N_c}{2\pi} \left[\Lambda^2 - b^2 + \frac{1}{2} M^2 \left(\log \frac{4\Lambda^2}{M^2} - 1 \right) \right] \\ &\quad + \theta(b-M)f(M, b). \end{aligned} \quad (24)$$

Note that $V_+ = V_-$ when $M > b$. The final result for the vacuum energy is given by the sum of the tree-level term, Eqs. (23) and (24)

$$\begin{aligned} V &= V_0 - \frac{N_c}{2\pi} \left[-2b^2 + M^2 \left(\log \frac{4\Lambda^2}{M^2} - 1 \right) \right] \\ &\quad + \theta(b-M)f(M, b), \end{aligned} \quad (25)$$

where we again have added the vacuum counterterm $\frac{N_c}{\pi} \Lambda^2$. In the limit $M \rightarrow 0$, the b -dependent terms in Eq. (25) drop out and the effective potential vanishes (again for $m_0 = 0$).

Thus the symmetric-energy cutoff provides us with a well-defined effective potential.

Returning to the momentum cutoff, one can of course simply subtract the term $-\frac{N_c b^2}{\pi}$ from the effective potential, but this requires some justification. The idea is to subtract the vacuum energy of the noninteracting system, i.e. that of a free Fermi gas, as a part of the renormalization prescription [40]. As explained above, the redefinition of the quark fields immediately after Eq. (9) corresponds to a unitary transformation of the Hamiltonian of the system. We must therefore also perform the same transformation on the free Hamiltonian. The subtraction term is then obtained by making the substitution $M \rightarrow m_0$ in Eq. (21). The total effective potential is given by the sum of

$$\begin{aligned} V &= V_0 - \frac{N_c M^2}{2\pi} \left[\log \frac{4\Lambda^2}{M^2} + 1 \right] + \theta(b-M)f(M, b) \\ &\quad - \theta(b-m_0)f(m_0, b), \end{aligned} \quad (26)$$

where we have dropped terms that depend on Λ and m_0 . Taking the limits $m_0 \rightarrow 0$ and $M \rightarrow 0$ (in this order), we see that all the b -dependent terms cancel as they should. We note in passing that the subtraction term is not unique. We could have subtracted the vacuum energy of a massless Fermi gas, $V_{\text{sub}} = -\frac{N_c b^2}{\pi}$, which was done in Ref. [30]. If we use this prescription, the expression for the vacuum energy becomes

$$\begin{aligned} V &= V_0 - \frac{N_c}{2\pi} \left[-2b^2 + M^2 \left(\log \frac{4\Lambda^2}{M^2} + 1 \right) \right] \\ &\quad + \theta(b-M)f(M, b). \end{aligned} \quad (27)$$

C. Dimensional regularization

Let us next consider the vacuum energy using dimensional regularization. The one-loop energy is given by the sum of the two terms

$$V_{\pm} = -N_c \int_p E_{\pm}, \quad (28)$$

where the integral is defined in $d = 1 - 2\epsilon$ dimensions:

$$\int_p = \left(\frac{e^{\gamma_E} \Lambda^2}{4\pi} \right)^{\epsilon} \int \frac{d^d p}{(2\pi)^d}. \quad (29)$$

Here Λ is the renormalization scale associated with the $\overline{\text{MS}}$ renormalization scheme. We first change variables $u = \sqrt{p^2 + M^2}$ and integrate over angles, This yields

$$V_{\pm} = -\frac{N_c (e^{\gamma_E} \Lambda^2)^{\epsilon}}{\sqrt{\pi} \Gamma(\frac{1}{2} - \epsilon)} \int_M^{\infty} |u \pm b| \frac{udu}{(u^2 - M^2)^{\frac{1}{2} + \epsilon}}. \quad (30)$$

We first consider the contribution V_+ to the effective potential from E_+ . We find

$$V_+ = \frac{N_c M^2}{4\pi} \left(\frac{e^{\gamma_E} \Lambda^2}{M^2} \right)^\epsilon \Gamma(-1 + \epsilon). \quad (31)$$

The contribution V_+ is independent of b , which is most easily understood by going back to momentum space in Eq. (30). The b -dependence is then given by an integral over p with no mass scale multiplied by b , and this integral vanishes in dimensional regularization.

The contribution V_- from the negative solution E_- is given by

$$\begin{aligned} V_- &= -\frac{N_c (e^{\gamma_E} \Lambda^2)^\epsilon}{\sqrt{\pi} \Gamma(\frac{1}{2} - \epsilon)} \int_M^\infty \frac{|u-b|udu}{(u^2 - M^2)^{\frac{1}{2} + \epsilon}} \\ &= -\frac{N_c (e^{\gamma_E} \Lambda^2)^\epsilon}{\sqrt{\pi} \Gamma(\frac{1}{2} - \epsilon)} \left[\int_M^\infty \frac{(u-b)udu}{(u^2 - M^2)^{\frac{1}{2} + \epsilon}} \right. \\ &\quad \left. + 2\theta(b-M) \int_M^b \frac{(b-u)udu}{(u^2 - M^2)^{\frac{1}{2} + \epsilon}} \right] \\ &= \frac{N_c M^2}{4\pi} \left(\frac{e^{\gamma_E} \Lambda^2}{M^2} \right)^\epsilon \Gamma(-1 + \epsilon) \\ &\quad + \theta(b-M)f(M, b) \end{aligned} \quad (32)$$

where the second integral has been evaluated directly in one dimension. We note that the first term in Eq. (32) is equal to V_+ .

The total vacuum energy is given by the sum of the tree-level term, Eqs. (31) and (32). Expanding this expression in powers of ϵ , we obtain

$$\begin{aligned} V &= V_0 - \frac{N_c M^2}{2\pi} \left(\frac{\Lambda^2}{M^2} \right)^\epsilon \left[\frac{1}{\epsilon} + 1 \right] \\ &\quad + \theta(b-M)f(M, b). \end{aligned} \quad (33)$$

The pole in ϵ is removed by renormalizing the quark mass m_0 and the constant coupling G . This is carried out by the substitutions $m_0 \rightarrow Z_{m_0} m_0$, and $\frac{1}{G} \rightarrow Z_{G^{-1}} \frac{1}{G}$ where the mass and inverse coupling renormalization constants are

$$Z_{m_0} = \left[1 + \frac{2G}{\pi\epsilon} \right]^{-1}, \quad (34)$$

$$Z_{G^{-1}} = \left[1 + \frac{2G}{\pi\epsilon} \right]. \quad (35)$$

Note that $Z_{G^{-1}} = Z_G^{-1}$ and that the ratio $\frac{m_0}{G}$ is the same for bare and renormalized quantities since $Z_{m_0} Z_{G^{-1}} = 1$. This yields the renormalized effective potential in the mean-field approximation

$$V = V_0 - \frac{N_c M^2}{2\pi} \left[\log \frac{\Lambda^2}{M^2} + 1 \right] + \theta(b-M)f(M, b). \quad (36)$$

We note in passing that the substitutions (34)–(35) correspond to a nonperturbative renormalization. In perturbation theory, it amounts to summing an infinite series of diagrams from all orders of perturbation theory. This can be seen, for example, by analyzing the model in terms of the two-particle irreducible effective action formalism to leading order in the $1/N_c$ expansion in analogy with the bosonic case in three dimensions [41,42]. Moreover, the running coupling G and the running mass m_0 satisfy the renormalization group equations

$$\Lambda \frac{dG}{d\Lambda} = -\frac{4G^2}{\pi}, \quad (37)$$

$$\Lambda \frac{dm_0}{d\Lambda} = -\frac{4m_0 G}{\pi}. \quad (38)$$

However, Eq. (36) is still problematic. Taking the limit $M \rightarrow 0$, we find $V = -\frac{N_c b^2}{\pi}$ (for $m_0 = 0$) which is unphysical. In order to understand the source of the problem, we must go back to the contribution V_\pm and take the limit $M \rightarrow 0$:

$$V_\pm = -\frac{N_c}{\pi} \left(\frac{e^{\gamma_E} \Lambda^2}{4\pi} \right) \int_0^\infty |p \pm b| p^{-2\epsilon} dp. \quad (39)$$

V_+ can be written as a sum of two integrals in which there is no mass scale. These integrals are then set to zero in dimensional regularization. In V_- , the wave vector b is the only scale in the integral and according to the rules of dimensional regularization, the integral will be proportional to the appropriate power of b .³ Dimensional analysis gives $V_- \sim b^{2-2\epsilon}$. The coefficient is finite and in the limit $\epsilon \rightarrow 0$ one finds $V_- = -\frac{N_c b^2}{\pi}$. This expression is exactly the vacuum energy of N_c massless fermions after a unitary transformation of the Hamiltonian. As in the case of a momentum cutoff, we subtract the vacuum energy in the normal phase as part of the renormalization procedure. This is given by the second and third term in Eq. (36) after the substitution $M \rightarrow m_0$. Dropping trivial terms that depend on m_0 and Λ , we find

$$\begin{aligned} V &= V_0 - \frac{N_c M^2}{2\pi} \left[\log \frac{\Lambda^2}{M^2} + 1 \right] + \theta(b-M)f(M, b) \\ &\quad - \theta(b-m_0)f(m_0, b). \end{aligned} \quad (40)$$

Our results for the vacuum energy in the three regularization schemes are given by Eqs. (25), (26),

³Alternatively, we note that V_- is proportional to $\int_0^\infty (p-b)p^{-2\epsilon} dp + 2 \int_0^b (b-p)p^{-2\epsilon} dp$, where only the latter integral is nonzero.

and (40). In the case of nonzero isospin chemical potential, these results are still somewhat problematic. Consider the free energy Eq. (25) in the case $b = 0$. It does not reduce to the free energy of a massive Fermi gas at $T = 0$ due to the extra term $-\frac{N_c}{4\pi}\mu_I^2$. The problem can be solved simply by adding the b -independent term $\frac{N_c}{4\pi}\mu_I^2$ to the vacuum energy [30]. This term or $\theta(\frac{1}{2}\mu_I - m_0)f(m_0, \frac{1}{2}\mu_I)$ should be added to the vacuum energy calculated in the momentum cutoff scheme (26) and dimensional regularization (40).

We close this section by noting that in the chiral limit, the gap equation $\frac{dV}{dM} = 0$, in the vacuum where $b = 0$, has two solutions, either $M_0 = 0$ or

$$M_0 = \Lambda e^{-\frac{\pi}{4\sigma}}. \quad (41)$$

Using the renormalization group equation (37), it is easy to verify that M_0 is renormalization group invariant. The nonanalytic behavior of M_0 as a function of G shows that it is a nonperturbative result. As mentioned above, it corresponds to summing an infinite series of diagrams from all order of perturbation theory.

Equation (41) can be used to trade the cutoff or the renormalization scale for the mass scale M_0 . In dimensional regularization, we find

$$V = -\frac{N_c M^2}{2\pi} \left[\log \frac{M_0^2}{M^2} + 1 \right]. \quad (42)$$

The RG invariance of M_0 implies the RG invariance of V in (42). As pointed out in [30], the unrenormalized expression for the vacuum energy contains a dimensionless parameter G , while the renormalized result (42) contains a dimensionful mass scale M_0 . This is an example of dimensional transmutation.

III. QUARK-MESON MODEL

A. Lagrangian and thermodynamic potential

The Euclidean Lagrangian of the two-flavor quark-meson model is

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} [(\partial_\mu \sigma)^2 + (\partial_\mu \boldsymbol{\pi})^2] + \frac{1}{2} m^2 (\sigma^2 + \boldsymbol{\pi}^2) \\ & - h\sigma + \frac{\lambda}{24} (\sigma^2 + \boldsymbol{\pi}^2)^2 \\ & + \bar{\psi} \left[\not{\partial} - \left(\mu + \frac{1}{2} \tau_3 \mu_I \right) \gamma^0 + g(\sigma + i\gamma^5 \boldsymbol{\tau} \cdot \boldsymbol{\pi}) \right] \psi, \end{aligned} \quad (43)$$

where ψ is a color N_c -plet, a four-component Dirac spinor as well as a flavor doublet

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix}. \quad (44)$$

Here $\mu_B = 3\mu = \frac{3}{2}(\mu_u + \mu_d)$ and $\mu_I = (\mu_u - \mu_d)$ are the baryon and isospin chemical potentials expressed in terms of the quark chemical potentials μ_u and μ_d , τ_i ($i = 1, 2, 3$) are the Pauli matrices in flavor space, and $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$.

Apart from the global $SU(N_c)$ symmetry, the Lagrangian (43) has a $U(1)_B \times SU(2)_L \times SU(2)_R$ symmetry for $h = 0$ and a $U(1)_B \times SU(2)_V$ symmetry for $h \neq 0$. When $\mu_u \neq \mu_d$, this symmetry is reduced to $U(1)_B \times U_{I_3L}(1) \times U_{I_3R}(1)$ for $h = 0$ and $U(1)_B \times U_{I_3}(1)$ for $h \neq 0$. In the remainder of this paper we take $h = 0$, i.e. we work in the chiral limit. We also set $\mu_u = \mu_d$.

In order to study inhomogeneous phases, we must make an *Ansatz* for the space-time dependence of the mesonic mean fields. In the literature, mainly one-dimensional modulations have been considered, for example, chiral-density waves (CDW) and soliton lattices. Since the results seem fairly independent of the modulation [16], we opt for the simplest, namely a chiral-density wave. The *Ansatz* is

$$\sigma(z) = \phi_0 \cos(qz), \quad \pi_3(z) = \phi_0 \sin(qz), \quad (45)$$

where ϕ_0 is the magnitude of the condensate and q is the wave vector. The mean fields can be combined into a complex order parameter as $M(z) = g[\sigma(z) + i\pi_3(z)]$ or

$$M(z) = \Delta e^{iqz}, \quad (46)$$

where we have introduced $\Delta = g\phi_0$. After averaging over a sufficiently large volume V_3 in three dimensions, the tree-level effective potential is then

$$V_0 = \frac{1}{2} q^2 \frac{\Delta^2}{g^2} + \frac{1}{2} m^2 \frac{\Delta^2}{g^2} + \frac{\lambda}{24} \frac{\Delta^4}{g^4}. \quad (47)$$

In analogy with the previous example, we can derive the spectrum by finding the zeros of the Dirac determinant. The result is [43]

$$E_{\pm} = \sqrt{\left(\sqrt{p_{\parallel}^2 + \Delta^2} \pm \frac{q}{2} \right)^2 + p_{\perp}^2}, \quad (48)$$

where $p_{\perp}^2 = p_1^2 + p_2^2$, and $p_{\parallel} = p_3$. Note that the lower branch has a vanishing minimum, $E_- = 0$, for nonzero momentum $p_{\parallel} = \pm \sqrt{\frac{q^2}{4} - \Delta^2}$ and $p_{\perp} = 0$ in the case $\frac{q}{2} > \Delta$. It is this nonmonotonic behavior that allows for inhomogeneous condensates at finite density; it may be energetically favorable for the system to develop a nonzero value of q and populate only the lower branch E_- . Although inhomogeneous phases are possible only for nonzero chemical potentials, the vacuum energy is independent of μ_f so the chemical potentials play no role in the calculation below.

B. Energy and momentum cutoff

The vacuum part of the one-loop contribution to the effective potential is given by the expression

$$V_1 = -2N_c \int_p (E_+ + E_-), \quad (49)$$

where the integral is in three spatial dimensions. We first use an energy cutoff to evaluate Eq. (49). In the case of E_- ,

$$\begin{aligned} V_+ &= -2N_c \int_p E_+ \\ &= -\frac{2N_c}{(4\pi)^2} \left[\frac{1}{6} \sqrt{\left(\Lambda - \frac{q}{2}\right)^2 - \Delta^2} \left[\left(\Lambda - \frac{q}{2}\right) (12\Lambda^2 + 4\Lambda q + q^2) - \Delta^2 (6\Lambda + 13q) \right] \right. \\ &\quad \left. - (\Delta^4 + \Delta^2 q^2) \log \frac{(\Lambda - \frac{q}{2} + \sqrt{(\Lambda - \frac{q}{2})^2 - \Delta^2})}{\Delta} \right], \end{aligned} \quad (50)$$

$$\begin{aligned} V_- &= -2N_c \int_p E_- \\ &= -\frac{2N_c}{(4\pi)^2} \left[\frac{1}{6} \sqrt{\left(\Lambda + \frac{q}{2}\right)^2 - \Delta^2} \left[\left(\Lambda + \frac{q}{2}\right) (12\Lambda^2 - 4\Lambda q + q^2) - \Delta^2 (6\Lambda - 13q) \right] \right. \\ &\quad \left. - (\Delta^4 + \Delta^2 q^2) \log \frac{(\Lambda + \frac{q}{2} + \sqrt{(\Lambda + \frac{q}{2})^2 - \Delta^2})}{\Delta} \right] + \theta\left(\frac{q}{2} - \Delta\right) f(\Delta, q), \end{aligned} \quad (51)$$

where the function $f(\Delta, q)$ is defined as

$$f(\Delta, q) = \frac{N_c}{3(4\pi)^2} \left[q \sqrt{\frac{q^2}{4} - \Delta^2} (26\Delta^2 + q^2) - 12\Delta^2 (\Delta^2 + q^2) \log \frac{q + 2\sqrt{\frac{q^2}{4} - \Delta^2}}{2\Delta} \right]. \quad (52)$$

In the limit $\Delta \rightarrow 0$, $V_+ + V_-$ reduces to $-\frac{8N_c}{(4\pi)^2} \Lambda^4$ showing that the thermodynamic potential is independent of q in this limit. Subtracting this term then corresponds to a trivial renormalization of the vacuum energy. In the limit $\Lambda \rightarrow \infty$, the sum of (50) and (51) behaves as

$$\begin{aligned} V_+ + V_- &= -\frac{2N_c}{(4\pi)^2} \left[4\Lambda^4 - 4\Lambda^2 \Delta^2 - q^2 \Delta^2 \left[\log \frac{4\Lambda^2}{\Delta^2} - 2 \right] \right. \\ &\quad \left. - \Delta^4 \left[\log \frac{4\Lambda^2}{\Delta^2} - \frac{1}{2} \right] + \frac{1}{12} q^4 \right] + f(\Delta, q), \end{aligned} \quad (53)$$

in agreement with the result first obtained by Broniowski and Kutschera [44].

Let us briefly discuss the calculation of the vacuum energy using a momentum cutoff Λ . Integrating Eqs. (50) and (51) and taking the limit $\Delta \rightarrow 0$, we find

we must distinguish between the cases $\sqrt{p_{\parallel}^2 + \Delta^2} - \frac{q}{2} > 0$, and $\sqrt{p_{\parallel}^2 + \Delta^2} - \frac{q}{2} < 0$. As in 1 + 1 dimensions, there is an extra term in the case $\sqrt{p_{\parallel}^2 + \Delta^2} - \frac{q}{2} < 0$, which we denote by $f(\Delta, q)$. We first integrate over p_{\perp} from zero to $p_{\perp}^{\max} = \sqrt{\Lambda^2 - (u \pm \frac{q}{2})^2}$, and then integrate over u ($u = \sqrt{p_{\parallel}^2 + \Delta^2}$) from $u = \Delta$ to $u = \Lambda \mp \frac{q}{2}$ (upper sign for E_+ and lower sign for E_-). The expressions for the integrals are

$V_+ + V_- = -\frac{8N_c}{(4\pi)^2} \left(\Lambda^4 + \frac{2}{3} q^2 \Lambda^2 + \frac{1}{15} q^4 \right)$, which must be subtracted. For large Λ , the final result is

$$\begin{aligned} V_+ + V_- &= -\frac{2N_c}{(4\pi)^2} \left\{ 4\Lambda^2 \Delta^2 - \Delta^2 q^2 \left[\log \frac{4\Lambda^2}{\Delta^2} - \frac{5}{3} \right] \right. \\ &\quad \left. - \Delta^4 \left[\log \frac{4\Lambda^2}{\Delta^2} - \frac{1}{2} \right] + \frac{1}{12} q^4 \right\} + f(\Delta, q). \end{aligned} \quad (54)$$

Comparing Eqs. (53) and (54), we see that the coefficients of some of the terms are different. However, the coefficients of the logarithmic terms are identical.

C. Dimensional regularization

We next consider dimensional regularization. The integrals needed are

$$V_{\pm} = -2N_c \int_p E_{\pm}, \quad (55)$$

where the integral is in $d = 3 - 2\epsilon$ dimensions,

$$\begin{aligned} \int_p &= \left(\frac{e^{\gamma_E} \Lambda^2}{4\pi}\right)^\epsilon \int \frac{d^d p}{(2\pi)^d} \\ &= \left(\frac{e^{\gamma_E} \Lambda^2}{4\pi}\right)^\epsilon \int_{p_{\perp}} \frac{d^{d-1} p_{\perp}}{(2\pi)^{d-1}} \int_{p_{\parallel}} \frac{dp_{\parallel}}{2\pi}. \end{aligned} \quad (56)$$

We first integrate over angles in the (p_1, p_2) -plane and introduce the variable $u = \sqrt{p_{\parallel}^2 + \Delta^2}$. The integral then becomes

$$\begin{aligned} V_{\pm} &= -\frac{N_c (e^{\gamma_E} \Lambda^2)^\epsilon}{\pi^2 \Gamma(1-\epsilon)} \int_{\Delta}^{\infty} \frac{udu}{\sqrt{u^2 - \Delta^2}} \\ &\quad \times \int_0^{\infty} dp_{\perp} \sqrt{\left(u \pm \frac{q}{2}\right)^2 + p_{\perp}^2} p_{\perp}^{1-2\epsilon}. \end{aligned} \quad (57)$$

In contrast to calculation in the 1 + 1 dimensional NJL model, we were not able to calculate directly in dimensional regularization the vacuum energy given by $V_1 = V_+ + V_-$. We therefore use another strategy. In order to isolate the ultraviolet divergences, we expand the integrand in powers of q and identify appropriate subtraction terms. This yields

$$\begin{aligned} \sqrt{\left(u \pm \frac{q}{2}\right)^2 + p_{\perp}^2} &= \sqrt{u^2 + p_{\perp}^2} \pm \frac{uq}{2\sqrt{u^2 + p_{\perp}^2}} \\ &\quad + \frac{q^2 p_{\perp}^2}{8(u^2 + p_{\perp}^2)^{\frac{3}{2}}} \mp \frac{q^3 p_{\perp}^2 u}{16(u^2 + p_{\perp}^2)^{\frac{5}{2}}} \\ &\quad + \frac{q^4 p_{\perp}^2 (4u^2 - p_{\perp}^2)}{128(u^2 + p_{\perp}^2)^{\frac{7}{2}}} + \dots \end{aligned} \quad (58)$$

We denote the right-hand side of (58) by $\text{sub}_{\pm}(u, p_{\perp})$ and write the integrals in (57) as

$$V_{\pm} = V_{\text{div}\pm} + V_{\text{fin}\pm} - V_{\text{fin}\pm}, \quad (59)$$

where

$$\begin{aligned} V_{\text{div}\pm} &= -\frac{N_c (e^{\gamma_E} \Lambda^2)^\epsilon}{\pi^2 \Gamma(1-\epsilon)} \int_{\Delta}^{\infty} \frac{udu}{\sqrt{u^2 - \Delta^2}} \\ &\quad \times \int_0^{\infty} \text{sub}_{\pm}(u, p_{\perp}) p_{\perp}^{1-2\epsilon} dp_{\perp}, \end{aligned} \quad (60)$$

$$\begin{aligned} V_{\text{fin}\pm} &= -\frac{N_c (e^{\gamma_E} \Lambda^2)^\epsilon}{\pi^2 \Gamma(1-\epsilon)} \int_{\Delta}^{\infty} \frac{udu}{\sqrt{u^2 - \Delta^2}} \\ &\quad \times \int_0^{\infty} \left[\sqrt{\left(u \pm \frac{q}{2}\right)^2 + p_{\perp}^2} - \text{sub}_{\pm}(u, p_{\perp}) \right] \\ &\quad \times p_{\perp}^{1-2\epsilon} dp_{\perp}. \end{aligned} \quad (61)$$

The integral $V_{\text{fin}\pm}$ can now be calculated directly in three dimensions. After integrating over p_{\perp} , we find

$$V_{\text{fin}\pm} = -\frac{N_c}{3\pi^2} \int_{\Delta}^{\infty} \frac{udu}{\sqrt{u^2 - \Delta^2}} \left(u \pm \frac{q}{2}\right)^2 \left[\left(u \pm \frac{q}{2}\right) - \left|u \pm \frac{q}{2}\right| \right]. \quad (62)$$

Thus $V_{\text{fin}+}$ vanishes identically and $V_{\text{fin}-}$ becomes

$$\begin{aligned} V_{\text{fin}-} &= -\frac{2N_c}{3\pi^2} \int_{\Delta}^{\infty} \frac{udu}{\sqrt{u^2 - \Delta^2}} \left(u - \frac{q}{2}\right)^3 \theta\left(\frac{q}{2} - \Delta\right) \\ &= \frac{N_c}{3(4\pi)^2} \left[q \sqrt{\frac{q^2}{4} - \Delta^2} (26\Delta^2 + q^2) \right. \\ &\quad \left. - 12\Delta^2 (\Delta^2 + q^2) \log \frac{q + 2\sqrt{\frac{q^2}{4} - \Delta^2}}{2\Delta} \right] \theta\left(\frac{q}{2} - \Delta\right) \\ &= f(\Delta, q). \end{aligned} \quad (63)$$

We next integrate $V_{\text{div}\pm}$ using dimensional regularization. This is done by first integrating over p_{\perp} and then over u . This yields

$$\begin{aligned} V_{\text{div}} &= V_{\text{div}+} + V_{\text{div}-} \\ &= \frac{2N_c}{(4\pi)^2} \left(\frac{e^{\gamma_E} \Lambda^2}{\Delta^2}\right)^\epsilon \left[2\Delta^4 \Gamma(-2+\epsilon) + q^2 \Delta^2 \Gamma(\epsilon) \right. \\ &\quad \left. + \frac{q^4}{12} (-1+\epsilon) \Gamma(1+\epsilon) \right]. \end{aligned} \quad (64)$$

Expanding V_{div} to zeroth order in powers of ϵ , we obtain

$$V_{\text{div}} = \frac{2N_c}{(4\pi)^2} \left(\frac{\Lambda^2}{\Delta^2}\right)^\epsilon \left[\left(\frac{1}{\epsilon} + \frac{3}{2}\right) \Delta^4 + \frac{1}{\epsilon} \Delta^2 q^2 - \frac{q^4}{12} + \mathcal{O}(\epsilon) \right]. \quad (65)$$

The one-loop effective potential is then given by the sum of Eqs. (63) and (65). It contains poles in ϵ , which are removed by coupling-constant and wave-function renormalization. This amounts to making the substitutions $m^2 \rightarrow Z_{m^2} m^2$, $\lambda \rightarrow Z_{\lambda} \lambda$, and $g^2 \rightarrow Z_{g^2} g^2$, where

$$\begin{aligned} Z_{m^2} &= 1 + \frac{4N_c g^2}{(4\pi)^2 \epsilon}, & Z_{\lambda} &= 1 + \frac{8N_c}{(4\pi)^2 \epsilon} [\lambda g^2 - 6g^4], \\ Z_{g^2} &= 1 + \frac{4N_c g^2}{(4\pi)^2 \epsilon}. \end{aligned} \quad (66)$$

After renormalization, the effective potential in the mean-field approximation reads

$$V = \frac{1}{2}q^2 \frac{\Delta^2}{g^2} + \frac{1}{2}m^2 \frac{\Delta^2}{g^2} + \frac{\lambda}{24} \frac{\Delta^4}{g^4} + \frac{2N_c q^2 \Delta^2}{(4\pi)^2} \log \frac{\Lambda^2}{\Delta^2} + \frac{2N_c \Delta^4}{(4\pi)^2} \left[\log \frac{\Lambda^2}{\Delta^2} + \frac{3}{2} \right] - \frac{N_c q^4}{6(4\pi)^2} + f(\Delta, q). \quad (67)$$

In contrast to the example in $1 + 1$ dimensions, we need not subtract a term proportional to the appropriate power of the wave vector (here q^4) to obtain an effective potential with the right properties. The reason is simply that the vacuum energy is independent of q .

We close this section by discussing how dimensional regularization can be used in conjunction with a Landau-Ginzburg (GL) analysis of the quark-meson model. In this case we expand the effective potential in powers of Δ and its derivatives. Up to a temperature-dependent constant, we find

$$V = \frac{1}{2}q^2 \frac{\Delta^2}{g^2} + \frac{1}{2}m^2 \frac{\Delta^2}{g^2} + \frac{\lambda}{24} \frac{\Delta^4}{g^4} + \beta_1 \Delta^2 + \beta_2 \Delta^4 + \beta_3 (\nabla \Delta)^2 + \dots, \quad (68)$$

where the coefficients are

$$\beta_1 = -4N_c \int_{\{P\}} \frac{1}{P^2}, \quad (69)$$

$$\beta_2 = 2N_c \int_{\{P\}} \frac{1}{P^4}, \quad (70)$$

$$\beta_3 = -N_c \int_{\{P\}} \left[\frac{4p_{\parallel}^2}{P^6} - \frac{3}{P^4} \right]. \quad (71)$$

Here, the sum integral is defined by

$$\int_{\{P\}} = \left(\frac{e^{\gamma_E \Lambda^2}}{4\pi} \right)^\epsilon T \sum_{\{P_0\}} \int \frac{d^d p}{(2\pi)^d}, \quad (72)$$

where $P_0 = (2n + 1)\pi T + i\mu$ are the fermionic Matsubara frequencies with $n = 0, \pm 1, \pm 2, \dots$. Using integrating by parts in $d = 3 - 2\epsilon$ dimension, it is straightforward to show that $\beta_2 = \beta_3$. This result was first obtained in [16] using Pauli-Villars regularization. For the special value of the sigma mass $m_\sigma = 2m_q$, it was shown in [16] that this implies that the tricritical point is actually a Lifschitz point. In the NJL model this is always the case when using a regulator where the total derivative vanishes [13]. Because of infinite surface terms, such an expansion is problematic

in the case of a momentum cutoff. This problem is avoided in the NJL model in $1 + 1$ dimensions, since the coefficients in the GL functional are finite.

IV. SUMMARY AND DISCUSSION

In this paper, we have for the first time discussed momentum cutoff regularization, symmetric energy cutoff regularization, and dimensional regularization in the context of one-dimensional inhomogeneities in the NJL and QM models. We have shown that all regularization schemes can be used to define a physically meaningful vacuum energy. In the case of symmetric energy cutoff regularization, the result is independent of the wave vector when the magnitude of the condensate vanishes, while in the other cases one must subtract a wave-vector-dependent term. We propose to subtract such a term for all regularizations as a part of the renormalization procedure. In the examples considered in this paper, an appropriate term is the Hamiltonian of a free Fermi gas after a unitary transformation. After this subtraction, one must also add a term that depends on the isospin chemical potential in order to obtain the correct expression for the free energy and isospin density in the limit $b = 0$.

We have also briefly discussed finite temperature and a Ginzburg-Landau analysis of critical points. Because of the absence of surface terms in the coefficients of the GL functional, dimensional regularization can always be used in the analysis of critical points. The application of momentum cutoff or symmetric energy cutoff at finite temperature is restricted to the cases where the GL coefficients are finite, for example the NJL model in $1 + 1$ dimensions. Results for the phase diagram of the $1 + 1$ dimensional NJL model is presented in [45].

There are other regularization schemes that we have briefly mentioned, namely Schwinger's proper time regularization and Pauli-Villars regularization. The latter method was successfully applied to the problem of inhomogeneous phases in the NJL model [13] and the QM model [16], where the equality of the two coefficients β_2 and β_3 was shown. In other words, Pauli-Villars regularization has the same virtues as dimensional regularization although the final expressions for renormalized quantities are not so compact.

It is often argued that since the NJL model in three dimensions is "nonrenormalizable," one cannot use dimensional regularization but is forced to use cutoff (momentum or energy) regularization or Pauli-Villars regularization. We disagree with this view. Nonrenormalizability alone cannot be an argument against applying dimensional regularization since it has been applied successfully to nonrenormalizable models. For example, it has been used in chiral perturbation theory [46] and in the theory of weakly interacting Bose gases and Bose condensation, both involving nonrenormalizable field theories [47,48].

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