Modelling the Multivariate Dynamic Dependence Structure of Commodity Futures Portfolios

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Abstract

This paper examines the time-varying dependence structure of commodity futures portfolios based on multivariate dynamic copula models. The importance of accounting for time-variation is emphasized in the context of the Basel traffic light system. We enhance the flexibility of this structure by modeling regimes with multivariate mixture copulas and by applying the dynamic conditional correlation model (DCC) to multivariate elliptical copulas. The most suitable dynamic dependence model in terms of in-sample and out-of sample valuation is the dynamic Student-t-Clayton mixture copula, followed by the dynamic Student-t copula, and the dynamic Gaussian-Clayton mixture. In comparison to the multivariate normal model, the dynamic Clayton copula also scales down significantly the number of VaR(99%) violations during the 2007/08 financial crisis period. The predictive performance of our multivariate dynamic copula models confirms its superiority over bivariate regime-switching copula models for various states of the economy.

JEL Classification: $C32$, $C51$, $C53$
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1 Introduction

Commodities have become an important asset class in the portfolios of institutional investors such as pension funds, insurance companies, and hedge funds over the last decade. In particular, since the beginning of the millennium investments into commodities began to increase substantially. The number of open contracts in commodity exchanges almost doubled with volumes of exchange-traded derivatives being 20 to 30 times higher than the physical production of many commodities (Silvennoninen and Thorp (2012); Paraschiv, Mudry, and Andries (2015)). The growing interest in this asset class might be attributed to the perceived opinion that commodities show low correlation with traditional assets, and thus, provide diversification benefits in a mixed-asset portfolio (Paraschiv, Mudry, and Andries (2015)).

There are several reasons why commodity prices behave differently from stock and bond prices: On the one hand, commodities are not only driven by demand and supply originating from the business cycle, but also by event risk like weather, geopolitical influences or macroeconomic factors. For example, the political instability of oil-exporting countries and the lack of governmental control account for further variation in commodity prices, besides the variation coming from demand/supply shocks (see, e.g., Driesprong, Jacobsen, and Maat (2008); Füss, Adams, and Kaiser (2010); Delatte and Lopez (2013)). With respect to the reaction to macroeconomic factors, in contrast to stocks and bonds commodity prices tend to rise when inflation is accelerating, so that they offer a natural hedge against inflation (see, e.g., Geman (2005); Gorton and Rouwenhorst (2006); Fabozzi, Füss, and Kaiser (2008)).

On the other hand, commodities are far from being a homogenous asset class, but differ significantly in their properties. While some commodities like precious and industrial metals are storable, others such as energy and livestock commodities may only be stored at very high costs. As a consequence, several studies confirm that commodities differ in their return distributions from traditional assets. Commodity returns tend to be positively skewed, and thus, exhibit less downside risk, however, they also show fat tails. In addition, the correlation among futures excess returns of commodities is positive, but on average low (see, e.g., Bodie and Rosansky (1980); Kaplan and Lummer (1998); Greer (2000); Kat and Oomen (2007a); Gorton and Rouwenhorst (2006); Kat and Oomen (2007b)).

The introduction of commodity indices along with the increasing number of long-only commodity index traders, who aim at exploiting diversification potentials, had important consequences on the price behavior of commodities. Tang and Xiong (2012) as well as Silvennoninen and Thorp (2012) argue that the growing presence of index funds in commodity markets led to an increasing integration of commodity futures markets with stock and bond markets over the last years. Adams and Glück (2015) show empirically that risk spillovers from stocks to commodities increased significantly since the 2007/08 financial crisis and that financialization has affected the traditional correlation structure among commodities. Furthermore, Henderson, Pearson, and Wang (2014) argue that institutional trades and holdings have influenced commodity prices and return dynamics. They provide empirical evidence for the impact of hedging trades on commodity futures prices. The significant shift in the dynamics of commodity risk and returns highlights the need for the assessment of the joint dynamics of commodity futures prices (Ohashi and Okimoto (2016)), which can serve as a basis for volatility forecasts of commodity prices, and thus, for the risk management of commodity-based portfolios.

Our study aims at contributing to the literature on the pricing (see, e.g., Brooks, Prokopczuk, and Wu (2013)) and volatility forecasting of commodity futures (see, e.g., Sadorsky (2006); Kang and Yoon (2013)), as well as the risk management of commodity portfolios (see, e.g., Chkili, Hammoudeh, and Nguyen (2014); Ghorbel and Trabelsi (2014)). Accordingly, we evaluate the in-sample estimation and forecasting performance of time-varying copula models for portfolios of commodity futures. We test and compare the performance of different multivariate dynamic copula models to capture the joint dynamics of the portfolio components and to forecast commodity futures prices by taking into account asymmetries in the dependence structure between individual commodity returns (see, e.g., Erb, Harvey, and Viskanta (1994), Longin and Solnik (2001); Ang and Bekaert (2002); Ang and Chen (2002)). For instance, we use several mixture copulas to control for the asymmetry in the dependence structure and employ dynamic and regime-switching copulas to account for the fact that the dependence structure of commodity markets is not constant over time, but changes in shape and intensity (see, e.g., Patton (2006b); Longin and Solnik (2001); Ang and Bekaert (2002); Christoffersen (2009)).

To control for the above-mentioned changes in the behavior of commodities, an extensive evaluation of copula models sheds light on the joint dynamics between commodity futures prices and is of great importance for the risk management of commodity-based portfolios. In particular, financial regulators emphasize that the time-varying correlations among portfolio assets are highly relevant for the derivation of consistent risk measures (Basel Committee on Banking Supervision (2011), pp. 10). Some of the recent contributions which employ copulas to capture time-varying dependencies are Okimoto (2008), Ng (2008), Guégan and Zhang (2010), Dias and Embrechts (2010), Silva Filho, Ziegelmann, and Dueker (2012), and De Lira Salvatierra and Patton (2015). However, these studies and the majority of research on copulas are conducted on the bivariate level. In this study, we extend the copula framework to higher dimensions. To model the time-variation in the multivariate dependence structure we propose two approaches: regime-switching technique where the states are modelled by multivariate mixture copulas and dynamic elliptical copulas based on Engle's (2002) dynamic conditional correlation (DCC) model.

The performance of the various models is ranked and discussed in the context of the Basel traffic light system classification. According to the information criteria, the most suitable dynamic in-sample dependence model is the dynamic Student-t followed by the dynamic Student-t-Clayton mixture copula. The best-ranked copulas according to the out-of-sample forecast performance confirm that dynamic copulas generally produce superior forecast accuracy compared to both static and regime-switching copula models. These findings are confirmed when the performance accuracy is evaluated for the 2007/08 financial crisis and the European sovereign debt crisis period. In summary, the empirical results derived from an extensive model comparison support the superiority of our proposed dynamic copula models for commodity futures portfolios.

The remainder of the paper is organized as follows: In Section 2, we discuss the multivariate dynamic copula methodologies. Section 3 introduces the data and presents descriptive statistics. Section 4 shows the in-sample analysis by ranking the fits of different copula model specifications. Section 5 investigates the forecast accuracy for the commodity portfolio's risk by providing out-of-sample backtests. In addition, it evaluates the models' predictive power during the 2007/08 financial crisis as well as the European sovereign debt crisis. Section 6 concludes.

2 Multivariate Dynamic Copula Models

Copula theory is based on the contribution of Sklar (1959), who showed that a multivariate distribution can be divided into its d marginal distributions and a d-dimensional copula, which completely characterizes the dependence structure between the variables. The theorem provides an accessible way to build valid multivariate distributions from known marginals.

Consider $F(y_1, \ldots, y_d)$ to be a continuous d-variate cumulative distribution function with univariate margins $F_i(y_i)$. Sklar's theorem states that there exists a function C named a copula, which maps $[0,1]^d$ into $[0,1]$ such that

$$F(y_1, \ldots, y_d) = C(F_1(y_1), \ldots, F_d(y_d)). \tag{1}$$

Forecasting in a multivariate setting is based on an extension of Sklar's theorem (1) for conditional joint distributions presented in Patton (2006b). Considering some information set $\mathcal{F}_{t-1}$, Patton shows that the conditional distribution $F(y_1, \ldots, y_d|\mathcal{F}_{t-1})$ can be decomposed into its conditional marginal distributions and the conditional copula such that

$$F(y_1, \ldots, y_d|\mathcal{F}_{t-1}) = C(F_1(y_1|\mathcal{F}_{t-1}), \ldots, F_d(y_d|\mathcal{F}_{t-1})|\mathcal{F}_{t-1}). \quad (2)$$

The d-dimensional conditional copula is:

$$C(u_{1,t}, ...u_{d,t}|\mathcal{F}_{t-1}) = F(F_1^{-1}(u_{1,t}|\mathcal{F}_{t-1}), \ldots, F_d^{-1}(u_{d,t}|\mathcal{F}_{t-1})). \quad (3)$$

A valid conditional multivariate distribution based on Sklar's theorem and Patton's extension can thus be created by first estimating the models for each of the conditional marginal distributions, $F_i(y_i|\mathcal{F}_{t-1})$, $i = 1, .., d$, construct the probability integral transformed variables $u_{i,t} = F_i(y_{i,t}|\mathcal{F}_{t-1})$, $i = 1, .., d$, and then consider copula models for the joint distribution of these variables. In analogy to the construction of unconditional copulas, this procedure yields a valid d-dimensional model without the intricacy of a simultaneous specification and estimation.[1]

2.1 Regime-Switching Copulas

There is a broad consensus in the literature regarding the increased dependence structure between assets in times of crises, compared to "normal" markets (see, e.g., Garcia and Tsafack (2011); Baur (2013); Delatte and Lopez (2013); Lombardi and Ravazzolo (2016)). One approach to account for the different levels of dependence is to switch between different copula models. For instance, Stöber and Czado (2012) show that there are structural breaks in the dependence structure of financial variables similar to the clusters in univariate volatilities. Combinations of regime-switching models with bivariate copu-

[1] We do not discuss the main copula models such as the elliptical copulas (Gaussian and Student-t copulas), the Archimedean copulas (Clayton and Frank copulas), and mixture copulas. For an overview of these static copula models we refer the reader to the Internet Appendix A.

las were proposed, for example, by Okimoto (2008), Rodriguez (2007), and Silva Filho, Ziegelmann, and Dueker (2012), who estimate regime-switching copulas for bivariate international stock market data.

However, in our approach we extend the dependence structure beyond the bivariate level. We choose the methodology of Chollete, Heinen, and Valdesogo (2009), Garcia and Tsafack (2011) and Braun (2011) which employ two dependence regimes that are different in intensity and/or shape. The marginal distributions are modeled separately from the dependence structure and are thus not dependent on the regime. This approach allows applying separate copulas for different dependence regimes. Accordingly, the parameters and the families of the copulas remain constant within a regime but differ across the regimes. Switching between the regimes is governed by a latent Markov process which determines the regime probabilities.

To model the dynamics of the data, we follow Hamilton (1989), who proposes a method which allows switching between different density functions. While Hamilton considered univariate time series, our approach focuses on the joint density of multiple time series as described by the copula functions. Since the modeled copulas only diverge with regards to their dependence characteristics, the impact of the different regimes is concentrated on the dependence structure. The model thus expresses different fractions of the joint density of the data by separate copula functions.

Conditional on being in regime j, the joint probability density is

$$f\left(Y_t|Y_{t-1}, s_t = j\right) = c^{(j)}\left(F_1(y_{1,t}), \ldots, F_d(y_{d,t}); \theta_c^{(j)}\right) \prod_{i=1}^{d} f_i\left(y_{i,t}; \theta_{m,i}\right), \tag{4}$$

where $Y_t = (y_{1,t}, ..., y_{d,t})$ with state variable s_t for the regime. The copula density function $c^{(j)}(.)$ in regime j have the corresponding parameter set $\theta_c^{(j)}$, where θ_c denotes the vector of all parameters that describe dependence through the copula (c). F_i is the distribution and f_i the corresponding density function of the marginal y_t with the parameters of the margins $\theta_{m,i} = (\theta_{m,1}, \ldots, \theta_{m,n})$. The model assumes the unobserved state variable to be governed by the transition probability matrix

$$P = Pr(s_t = i|s_{t-1} = j) = p_{i|j}, \tag{5}$$

where $p_{i|j}$ represent the probability that state j will be followed by state i. As the Markov chain is latent and thus not observable, we apply Hamilton's (1989) filter. Accordingly, the transition probability matrix drives the regime probabilities which in turn define the density function of the complete dataset. Explicitly, the filtered process for k regimes obeys

$$\xi_{t|t} = \frac{\xi_{t|t-1} \bigodot \delta_t}{1'(\xi_{t|t-1} \bigodot \delta_t)'}, \tag{6}$$

$$\xi_{t+1|t} = P'\xi_{t|t}, \tag{7}$$

$$\delta_t = \begin{pmatrix} c^{(1)}(F_1\left(y_{1,t}|y_1^{t-1}\right), \ldots, F_d\left(y_{d,t}|y_d^{t-1}\right); \theta_c^{(1)} \\ \vdots \\ c^{(k)}(F_1\left(y_{1,t}|y_1^{t-1}\right), \ldots, F_d\left(y_{d,t}|y_d^{t-1}\right); \theta_c^{(k)} \end{pmatrix}, \tag{8}$$

where $\xi_{t|t}$ is a (k x 1) vector with the regime probabilities at time t, conditional on the observations until time t; 1 is a (k x 1) vector of ones and $\bigodot$ stands for the Hadamard product. The regime probabilities $\xi_{t+1|t}$ at time $t+1$ conditional on information until time t are captured by the transition probability matrix P. The copula densities at time t, conditional on being in one of the regimes are contained in the vector δ_t. While Equation (6) represents a Bayesian updating of the probability to be in a specific regime given all observations δ_t up to the current time, Equation (7) comprises one forward iteration of the Markov chain. With this recursive procedure it is straightforward to forecast the regime probabilities ($\xi_{t+1|t}$).

The filtered system needs initial values for the regime probabilities $\xi_{1|0}$ from which the optimization procedure is started. Iterations over the two Equations (6) and (7) yield the likelihood value

$$\log \mathcal{L}(\theta) = \sum_{t=1}^{T} \log(1'(\xi_{t|t-1} \bigodot \delta_t)). \tag{9}$$

Naturally one would like to test the null hypothesis that there are k regimes versus the alternative of $k+1$ regimes. However, Hamilton (2008) points out that likelihood ratio tests of these hypotheses do not comply with the usual regularity conditions. Given for example that there is only one regime, the maximum likelihood estimate for the probability of staying in regime 1 fails to converge to a well-defined population value. Thus, the likelihood ratio test does not have the χ^2-limiting distribution. As a solution, Hamilton (2008) proposes to establish model comparisons based on their ability to forecast.

With the estimated transition probabilities, one can form inference about the dependence regime at date t based on the realized observations at a later date T (the "posteriori" observation date T, when new information becomes available). In order to calculate these inferences for the regime probabilities, the Kim filter is used, which represents a combination of the Kalman filter and the Hamilton filter, particularly designed for Markov-switching models (see Hamilton (1988, 1989, 1994)).

The limitation on a number of different static dependence structures as modeled by the regime-switching copulas may still be too restrictive. To increase the adaptability of the dependence specification one might think of simply increasing the number of regimes. However, a more flexible approach consists in allowing the dependence structure to be dynamic, i.e. vary with every discrete time step.

2.2 Dynamic Copulas

Engle and Sheppard (2001) and Engle (2002) established the basis for the estimation of time-varying dependence structure by introducing the dynamic conditional correlation model. In the field of copulas, the seminal work of Patton (2006b) was among the first to allow copulas to be time-varying.[2]

Dynamic Gaussian Copula. Based on Engle (2002), the correlation matrix Σ_t of the dynamic Gaussian copula is set to evolve through time as follows:

$$Q_t = (1 - \alpha - \beta)\bar{Q} + \alpha z_{t-1} z'_{t-1} + \beta Q_{t-1} \tag{10}$$

$$\Sigma_t = \tilde{Q}_t^{-1} Q_t \tilde{Q}_t^{-1}, \tag{11}$$

where z_t is the vector of transformed standardized residuals $z_{i,t}$ with a skewed t-distribution, $skewed - t_{\nu,\lambda}^{-1}(u_{i,t})$ where ν is the degrees of freedom and λ the asymmetry parameter (see, e.g., Hansen (1994)). $\bar{Q}$ is the sample correlation of z_t, and $\tilde{Q}_t = [\tilde{q}_{ii,t}] = [\sqrt{q_{ii,t}}]$ is the diagonal square matrix with the square root of the ith diagonal element of Q_t on its ith diagonal position. The constraints for the parameters α and β are $\alpha + \beta < 1$, with $\alpha, \beta \in (0, 1)$. Accordingly, the dynamic Gaussian copula is defined as:

$$C_{\Sigma_t}^{Ga}(u_1, ...u_d) = \Phi_{\Sigma_t}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)), \tag{12}$$

where Φ_{Σ_t} is the dynamic cumulative distribution function of the multivariate normal distribution with a mean of zero and a covariance matrix Σ.

Dynamic Student-*t* Copula. The Student-t copula parameters consists of the correlations and the degrees of freedom, ν. The dynamic process which drives the correlations is identical to the one defined for the Gaussian copula in Equations (10) and (11). We further allow the degrees of freedom parameter to vary over time (Jin and Lehnert (2011)). The Student-t copula is therefore not only provided with the capability to adapt the level of dependence, but also the strength of tail dependence over time. Fantazzini (2008) proposes to model the evolution of the degrees of freedom parameter of a bivariate

[2]Some of the contributions to this field on the bivariate level include Guégan and Zhang (2010), Dias and Embrechts (2010) and De Lira Salvatierra and Patton (2015), and on the multivariate level Jin and Lehnert (2011), Braun (2011) and Christoffersen, Errunza, Jacobs, and Langlois (2012).

Student-t copula as

$$\nu_t = \Lambda(\varsigma + \varphi |u_{i,t-1} - u_{j,t-1}|), \tag{13}$$

where Λ is a logistic transformation designed to keep the conditional degrees of freedom in the interval of $[2, 100]$ at all times. The mapping into the authorized domain ($L_\nu = 2, U_\nu = 100$) is ensured by the logistic transformation

$$\Lambda(x) = L_\nu + \frac{(U_\nu - L_\nu)}{1 + e^{-x}}. \tag{14}$$

Even if x is permitted to vary over the entire real line, $\Lambda(x)$ will be constrained to lie in the domain $[L_\nu, U_\nu]$.

We will not partition the observations into multiple clusters as in Braun (2011). However, in this study we will compute the absolute distance (AD) between all observations in time t. Therefore, the number of clusters is set to $k = 1$, which means that the AD_{ℓ_1}-norm is the sum of absolute differences between the observations u_t and their median $\tilde{u}_t$ at time t:

$$AD_{\ell_1} = \sum_{j=1}^{k} \sum_{i=1}^{d} |u_{i,t} - \tilde{u}_{j,t}| = \sum_{i=1}^{d} |u_{i,t} - \tilde{u}_t|. \tag{15}$$

Replacing the bivariate absolute difference in Equation (13) with the multivariate absolute difference AD_{ℓ_1} in (15) yields the dynamic process of the degrees of freedom of a multivariate Student-t copula, given by:

$$\nu_t = \Lambda(\varsigma + \varphi \sum_{i=1}^{d} |u_{i,t-1} - \tilde{u}_{t-1}|). \tag{16}$$

Hence, the dynamic multivariate Student-t copula is defined as

$$C^t_{\nu_t,\Sigma_t}(u_1, ..., u_d) = t_{\nu_t,\Sigma_t}(t^{-1}_{\nu_t}(u_1), ..., t^{-1}_{\nu_t}(u_d)), \tag{17}$$

where t_{ν_t} is the cumulative distribution function of the one dimensional t_ν-distribution and t_{ν_t,Σ_t} is the cumulative distribution function of the multivariate $t_{\nu,\Sigma}$-distribution, both with the same degree of freedom ν.

Dynamic Archimedean Copulas. Patton (2006b) adapts the idea of Engle (2002) to model the dynamics of bivariate Archimedean copulas with an ARMA-type process. He assumes that the functional form of the copula stays fixed over the sample, whereas the transformed copula parameter as Kendall's tau (ρ_τ) varies according to the evolution equation

$$\rho_{\tau_t} = \Lambda\left(\omega + \beta \cdot \rho_{\tau_{t-1}} + \alpha \cdot \frac{1}{10}\sum_{j=1}^{10} |u_{i,t-j} - u_{j,t-j}|\right), \tag{18}$$

where $\Lambda(x) = (1+e^{-x})^{-1}$ is the logistic transformation to keep $\rho_{\tau_t} \in [0,1]$ at all times and $(u_{i,t}, u_{j,t})$ are two observations at time t.[3] The Clayton respectively the Frank copula (see Internet Appendix A.2) parameter in time t can then be obtained using the functional relationship between Kendall's tau and the Archimedean copula parameter

Copula	ρ_τ
C_θ^{Cl}	$\theta/(\theta+2)$
C_θ^{Fr}	$1 - 4\theta^{-1}(1 - D_1(\theta))$,

(19)

where $D_1(\theta)$ is the Debye function of order one $D_1(\theta) = \theta^{-1}\int_0^\theta t/(\exp(t)-1)dt$ (Hofert, Mächler, and McNeil (2013)). For more information, we refer to the Internet Appendix.

The dynamic process of Patton (2006b) in Equation (18) is yet again limited to bivariate applications through the absolute difference term $|u_{t-1} - v_{t-1}|$. To extend Equation (18) to the multidimensional world, this difference term is substituted with the multivariate absolute distance AD_{ℓ_1} in (15). This yields a multivariate extension of Patton

[3] Kendall's tau is the rank correlation for two vectors of random variables Y_1 and Y_2, defined as $\rho_\tau = E(\text{sign}((Y_1 - \tilde{Y}_1)(Y_2 - \tilde{Y}_2)))$, where $(\tilde{Y}_1, \tilde{Y}_2)$ is an independent copy of (Y_1, Y_2) (McNeil, Frey, and Embrechts (2005)).

(2006b)'s parameter evolution process

$$\rho_{\tau_t} = \Lambda\left(\omega + \beta \cdot \rho_{\tau_{t-1}} + \alpha \cdot \frac{1}{10}\sum_{j=1}^{10}\sum_{i=1}^{d}|u_{i,t-j} - \tilde{u}_{t-j}|\right), \tag{20}$$

where $\tilde{u}_t$ is the median of $u_1, ..., u_d$ in time t and $\Lambda(x) = (1+e^{-x})^{-1}$. With Equation (20), the parameter of the multivariate Clayton copula in time t, θ_t^{Cl}, is then given in closed form through Equation (19). The Frank copula parameter θ_t^{Fr} in terms of Kendall's tau, however, is not available in closed form but has to be determined numerically. To achieve an efficient estimation of the dynamic Frank copula, we directly model the dynamics of θ_t^{Fr} as

$$\theta_t^{Fr} = \omega + \beta \cdot \theta_{t-1}^{Fr} + \alpha \cdot \frac{1}{10}\sum_{j=1}^{10}\sum_{i=1}^{d}|u_{i,t-j} - \tilde{u}_{t-j}|, \tag{21}$$

where the constraint $\theta_t^{Fr} \geq 0$ ensures that the parameter remains in the permissible range. Stationarity and invertibility is accounted for with the constraints $|\alpha| < 1$ and $|\beta| < 1$.

Dynamic Mixture Copulas. Ng (2008) adopts the dynamic process of Patton (2006a) to create a time-varying specification of the weight in the mixture copula depending on the natural filtration of the process. He suggests a dynamic bivariate mixture copula model, where the parameters of the copulas are constant, but the weighting parameter is stochastic, following an ARMA-type model for the mixture weight:

$$w_{i,t} = \omega_i + \alpha_i \cdot h_{i,t-1}(.) + \beta \cdot w_{i,t-1}. \tag{22}$$

Equation (22) establishes a linear relationship between the mixture weight w_i at time t and the corresponding lagged value at $t-1$ and $h(.)$, which is a stochastic explanatory variable or a function. In particular, Ng (2008) proposes to model $w_{i,t}$ with the special function being

$$h_{i,t-1}(.) = \frac{1}{10}\sum_{p=1}^{10}|u_{t-p} - v_{t-p}|. \tag{23}$$

However, this model is also limited to the bivariate setting due to the absolute distance measure $|u_{t-p}-v_{t-p}|$. Braun (2011) suggests an extension of Ng (2008)'s concept to higher dimensions by replacing the absolute distance measure with the copula density relative to the sum of all copula densities during the lag period. This results in a special function of the following type:

$$h^c_{i,t-1}(.) = \frac{1}{10}\sum_{p=1}^{10}\left(\frac{c_i(u_{1,t-p},...,u_{d,t-p};\theta_i)}{\sum_{j=1}^{n} c_j(u_{1,t-p},...,u_{d,t-p};\theta_j)}\right). \tag{24}$$

Generating weight forecasts by plugging Equation (24) into Equation (22), Braun (2011) has to impose six different constraints on the weight process parameters in order to keep the resulting weights $w_{i,t}$ within the unit interval. In contrast, we make use of the logistic function $\Lambda(x) = (1 + e^{-x})^{-1}$, which in combination with Equations (22) and (24) results in:

$$w_{i,t-1} = \Lambda\left(\omega_i + \alpha_i h^c_{i,t-1}(.) + \beta_i w_{i,t-1}\right). \tag{25}$$

Hence, the weight parameters are bounded on the unit interval without the need to impose any constraints on the parameters. Note that Equation (25) also nests the static mixture copula with $\alpha = \beta = 0$. Employing the dynamic weights of Equation (25) in the mixture copula yields the complete multivariate dynamic mixture copula model:[4]

$$C\left(u_1,...,u_d;w_{1,t},...,w_{n,t};\theta_1,...,\theta_n\right) = \sum_{j=1}^{n}[w_{j,t}, C_j(u_1,...,u_d,\theta_j)]. \tag{26}$$

It has to be emphasized that the parameters θ_j and $w_{j,t}$ have different functions within the mixture copula construct, allowing a very flexible way of modeling dependence structures. While the association parameter θ controls the degree of dependence, the weight parameter

[4]See the Internet Appendix A.3 for more details.

w determines the structure of the dependence. The advantage of linking the weight parameter to the copula densities lies in the difference of the copula density functions. The Clayton copula for example is capable of modeling lower tail dependence and exhibits its largest density in the lower tail. Thus, the weight parameter in the dynamic mixture structure is directly coupled with the capabilities of the mixture copula constituents to describe the dependence structure during the lag period.

Shifts in the dependence structure are expected to have an immediate effect on the dynamic weights. A rise of one copula's relative density signalizes its enhanced fit to the current dependence pattern. Through the dynamic weighting process in Equation (25), this copula's weight in the mixture setting and its impact on the overall mixture density extends. Calibrating this model using maximum likelihood estimation ensures that the parameters of each copula in the mixture are fitted most accurately to those data fractions, where the dependence structure naturally concurs with the copula's characteristics.[5] Every individual copula thus only captures the dependence in a specific part of the data set in a optimal way, but merging the copulas into a mixture structure governed by the dynamic weight process yields an overall accurate and flexible dependence model.

3 Data and Summary Statistics

The commodity data set used in this paper consists of commodity futures subindices of the Standard & Poors Goldman Sachs Commodity Index (SPGSCI), which together with the Bloomberg Commodity Index (former DJ-UBSCI) is by far the most influential commodity index. The series were chosen based on the length of their data history and their weight in the main index. The data set contains the following commodity futures excess return indices: crude oil (OIL), heating oil (HOL), unleaded gasoline (GAS), gold

[5] We estimate the model parameters and the corresponding standard errors for the regime-switching model and for the dynamic multivariate copulas by using a multi-stage maximum likelihood estimation procedure as shown in Internet Appendix B.

(GLD), silver (SLV), copper (CPP), wheat (WHT), as well as corn (CRN). It thus covers the commodity sectors energy, precious and industrial metals, as well as agriculture. The excess return measures the return from investing in nearby commodity futures and rolling them forward each month to avoid the cost of holding physical commodities. In this way, the selected commodity indices yield returns comparable to passive long positions in listed commodity futures contracts.[6]

All returns are computed as continuously compounded returns, $\log(P_t/P_{t-1})$, where P_t is the value of the index at time t. The data consists of Wednesday weekly returns, to avoid any day-of-the-week effects. The sample covers the period from June 30, 1988 until June 5, 2013, yielding 1'300 weekly returns. Table 1 presents the descriptive statistics of the commodity futures data. Energy sector index returns display the largest standard deviations while gold returns have the smallest standard deviation among the commodity indices. With the exception of wheat and corn, all weekly commodity returns are negatively skewed.All series display excess kurtosis ranging from 4.73 to 7.64. The Jarque-Bera test clearly rejects the hypothesis of a normal distribution for all commodity index returns.

[**Table 1 about here**]

The largest unconditional sample correlations are detected between the fossil fuel returns, followed by the correlation between the agricultural returns. Further tests shown in Table 1 reveal that returns are stationary series and we find clear evidence for ARCH-effects. In order to remove the heteroscedasticity from our return series, each individual risk factor is modeled by a GARCH specification. The univariate model for each index return series is determined by selecting the AIC and BIC optimal model considering ARMA(p,q) specifications for the conditional mean up to order (p=3, q=3) and

[6]More precisely, the return consists of a spot and a roll component. The spot return is the percentage change in the near-month futures contract. To keep a long futures position the futures contracts are rolled forward to the next-month futures contract. The roll return is positive when the market is in backwardation and negative when the market is in contango. The roll return therewith comes from rolling up or down the term structure of futures prices.

GARCH(P,Q), EGARCH(P,Q), and GJR-GARCH(P,Q) volatility models up to order (P=3, Q=3). The initial autocorrelation present in the squared returns has been successfully removed by the GARCH models. To account for potential asymmetries in the distribution of standardized residuals we employ Hansen's (1994) skewed Student-t distribution.[7]

4 In-Sample Analysis

In this section, we investigate the joint dynamics of commodity futures within a portfolio framework. We firstly estimate three *static* mixture copulas to the data by combining the asymmetric Clayton copula with three symmetric copulas: Gaussian, Student-t, and Frank copulas.[8] In particular, we examine the six months rolling Kendall's tau computed via the one-to-one mapping of Frank's multivariate copula parameter θ_F and Kendall's tau (see Figure 1). We find that there are substantial changes in both structure and level of dependence during the observation period. Neglecting this time-variation might result in inaccurate risk forecasts. To capture the dynamics in the dependence structure among a portfolios' assets, the two proposed multivariate copula models, regime-switching and dynamic copula model, are evaluated.

4.1 Static Copulas

To calibrate the static copulas, the filtered standardized residuals from the univariate models are transformed to uniform variates by inversion using the corresponding cumulative skewed-t distribution function.[9] Three static mixture copulas are constructed by combining the asymmetric Clayton copula with the other three (symmetric) copulas. The

[7]To conserve space, the results on GARCH specifications and the skewed Student-t distribution of standardized residuals are not shown here. However, they are available in the Internet Appendix C.1.
[8]Note that a comparative in-sample evaluation of the performance of static mixture copulas is not in the scope of this study. However, the results are available from the authors upon request.
[9]The results of the univariate models are shown in the Internet Appendix C.

combination of the two Archimedean copulas into a mixture construct yields a parsimonious model which is able of capturing lower tail dependence. Mixing the Clayton with the Gaussian copula combines the parameter plurality of the elliptical copula with the lower tail dependence feature of the Clayton copula and creates a flexible model which is capable of capturing asymmetries in the dependence structure. Adding the Clayton to the Student-t copula finally results in an adaptive model capable of modeling different degrees of upper and lower tail dependence. The results are shown in Table 2.

Using the information criteria to rank the fit of the static models, the Student-t-Clayton mixture fits best according to both AIC and BIC, followed by the Student-t copula and the Gauss-Clayton mixture. Overall, the in-sample analysis shows that the best fitting static dependence model is the Student-t-Clayton mixture copula. The two models with the second-best fit are the Gaussian-Clayton mixture copula and the Student-t copula. All Archimedean copulas rank far behind with the stand-alone Archimedean copulas having the largest AIC/BIC values and the lowest likelihood value, respectively.

Following Dias and Embrechts (2010) and Guégan and Zhang (2010), the information criteria are used to rank the fit of the different models. The comparison of the fit of the two Archimedean copulas reveals that the Clayton copula is more suitable to describe the dependence structure compared to the Frank copula, which indicates that the commodity index returns are lower tail dependent. There is a large difference in the likelihood values and the two information criteria values when comparing the purely Archimedean copulas with the other dependence models based on an elliptical copula. The Archimedean copulas' likelihood values range from 661 for the Frank copula to 788 for the Frank-Clayton mixture while the other copulas' likelihood are more than three times as large, all exceeding the value of 2746. This substantial difference can be attributed to the fact that the standalone Archimedean and pure Archimedean mixture copulas have to capture the dependence structure with only one or three parameters, respectively. Thus, the reason lies in the dependence structure which is diverse as indicated by the large difference in parameter values ranging between 0.082 (OIL:WHT) and 0.875 (OIL:HOL). The low likelihood values of the Archimedean copulas express the difficulty of these dependence models to capture such a diverse structure with only one or three parameters. Based on a correlation matrix, the rest of the models have at least 28 parameters to characterize the dependence structure among the eight indices. This allows a more precise fit which materializes in higher likelihood values and in lower values of both information criteria.

Overall, the ranking of the model fit based on both the AIC and BIC criterion is as follows: the Student-t-Clayton mixture is best capable to characterize the commodity indices' dependence followed by the Student-t copula and in third place the Gaussian-Clayton mixture. The degrees of freedom of the Student-t copula of 16.804 indicate tail dependence which is substantiated by the improved fit (i.e., lower AIC and BIC values) of the Student-t compared to the Gaussian copula. In the Student-t-Clayton mixture ν is estimated as 18.579 which shows that the Clayton copula, even though it only accounts for 2.7% of the overall mixture, covers a part of the lower tail dependence. This part allows for increased degrees of freedom in the Student-t fraction. Although θ_C is lower compared to the Student-t-Clayton mixture, the Gaussian-Clayton mixture substantiates the conclusion of lower tail dependence in the data with a higher mixture weight w_C.

[**Table 2 about here**]

4.2 Regime-Switching Copulas

In this sub-section, we show the estimation results of two-regime and three-regime switching copula models calibrated to the entire sample data. All copula models are estimated using the same residuals which result from the filtering with univariate EGARCH models. We derive three regime-switching copula models which combine the elliptical copulas into a two-regime setup. One Gaussian/Gaussian (G/G) model version allows for two regimes with different levels of dependence, but it does not capture tail dependence in any of them. The Gaussian/Student-t (G/T) copula allows for tail dependence in one regime. The G/G model describes presumably tranquil periods without tail dependence in returns, while the G/T version is more suitable in presumably turbulent periods with tail dependence. The Student-t/Student-t (T/T) regime-switching model then allows for tail dependence in both regimes.

To allow for asymmetry, the Gaussian/Student-t model is enhanced by mixing one of them with the asymmetric Clayton which firstly results in the Gaussian/Student-t-Clayton mixture (G/TC). This setup is thus capable of capturing different levels of lower and upper tail dependence in one regime. Secondly, the Student-t/Gaussian-Clayton mixture copula (T/GC) allows for symmetric (lower and upper) tail dependence in one regime and an asymmetric dependence with a probability of joint negative extreme returns in the other regime.

In addition to the two-state models, regime-switching copulas with three separate regimes are constructed. Firstly, the Gaussian/Clayton/Frank copula (G/C/F), which has an elliptical and an Archimedean regime modeling the interrelation between the returns without tail dependence and one asymmetric regime with lower tail dependence. Secondly, the Gaussian/Student-t/Clayton copula (G/T/C), which has one regime with asymptotic independence in the tails (Gaussian), a second regime with equal lower and upper tail dependence (Student-t), and a third regime with only lower tail dependence (Clayton).

Table 3 presents the parameter estimates of two- and three-regime models of which three contain only elliptical copulas. The regimes' parameters are listed in the order indicated by the abbreviated name, i.e. for the G/T regime switch copula, the Gaussian regime parameters are listed under *Regime 1* and the Student-t copula parameters under *Regime 2*. In all commodity regime switching models with two states, the parameter estimates indicate one high and one low dependence regime. Some copula correlations which are low already in the static Gaussian copula are now close to zero in the low dependence regime. This is the case for example for the copula correlation between crude oil and wheat in the Gaussian regime of the G/T model, where it is as low as 0.025. The degrees of freedom of 9.557 in the high dependence Student-t regime of the G/T copula suggests that tail dependence is a feature of this state of the economy. This is consistent with the results shown in Figure 1. However, in the T/T model both ν parameters are equally low suggesting tail dependence in both regimes.

For the G/C/F model, the θ_C of 0.304 transformed into a rank correlation measure, which yields a Kendall's tau of 0.1319. The Frank copula parameter of $\theta_F = 0.506$ corresponds to a value of 0.056, while the average Gaussian copula correlation translates to a Kendall's tau amounting to 0.184. The second regime (Clayton) thus forms the midpoint dependence regime. However, the importance of the Clayton regime is negligible which can be seen in both the probability $p_{2|2}$ of virtually zero and the according minimal expected regime duration listed in Table 3. This suggests that the Clayton regime in the G/C/F structure is irrelevant and that the other two regimes would be sufficient to capture the dependence structure of the commodity futures indices. This finding is confirmed by the two information criteria, which display the highest values for the G/C/F model, as well as the low log-likelihood value.

[**Table 3 about here**]

As shown in Tabel 3, the ranking of the model fit according to both BIC and AIC is: Student-t/Gaussian-Clayton mixture in the first, the Gaussian/Student-t-Clayton mixture in the second and the Gaussian/Student-t in the third place. The expected regime durations in weeks for each model are shown in the lower part of Table 3. We observe that the persistence of the regimes in the purely elliptical models is identical for the G/G and the T/T, while being only one week apart in the G/T copula. As soon as asymmetric dependence is introduced to the model, the differences in expected duration become larger.

Figure 1 depicts the level of dependence of commodity returns over time. The evolution can be fragmented into two distinct periods: from 1988 until about 2005, the level of dependence remained very low, at around 0.1 with little variation. From the year 2005 onwards, variations in the level of dependence surged, along with a significant increase of the rank correlation. The two periods are also clearly shown by the lower tail dependence coefficient over time in the lower panel of Figure 1. Until about 2005, there was virtually no lower tail dependence among the returns of the commodity futures indices. Starting in 2005, substantial spikes in tail dependence can be observed, indicating that along with the level of dependence, the structure of dependence was as well subject to change.

[**Figure 1 about here**]

The recent literature on the financialization of commodities ascribes this change to the emergence of commodities as an asset class, which has become increasingly held by institutional investors in search for diversification benefits (see, e.g., Basak and Pavlova, 2016; Büyükşahin and Robe, 2014; Singleton, 2014). Indeed, beginning in the year 2004, institutional investors have been building up substantial positions in commodity futures. The U.S. Commodity Futures Trading Commission (CFTC) estimates in its staff report (2008) that institutional holdings have increased from USD 15 billion in 2003 to over USD 200 billion in 2008. Many of the institutional investors hold commodities through commodity futures indices, such as the Standard & Poors Goldman Sachs Commodity Indices (SPGSCI) (Basak and Pavlova, 2016).

Figure 2 depicts the Kim filtered evolution of the state probabilities over the entire sample period. The G/T regime probability paths in Figure 2 differentiates itself as it gives the least clear idea about which regime the system was in at any point in time until about 2007. At first sight it becomes apparent that the plot for the G/C/F model is different to the other six, as the probability of the Gaussian regime is close to one almost all the time. The Clayton and the Frank copula probabilities in this G/C/F model are virtually zero, which is also reflected by their minimal expected regime duration in Table

3. With two of three regimes being insignificant, the G/C/F regime switch copula is evidently an inadequate model for the data at hand. The G/T regime probability paths in Figure 2 differentiates itself as it gives the least clear idea about which regime the system was in at any point in time until about 2007. However, all regime probability paths capture the spikes in dependence identified in Figure 1 by the high dependence regime. The regime switching models do not show a consensus shift from the low to the high dependence regime over the observation period even if such a shift could be expected from analyzing Figure 1.

[**Figure 2 about here**]

4.3 Dynamic Copulas

We again start to calibrate the dynamic copula models to the residuals obtained from the univariate EGARCH models. While in the regime switching models the copula parameters remain static, the parameters of dynamic copulas will change in discrete time steps. Dynamic copula are built on the well-known static copulas from the literature by adding the time-varying feature. We employ the dynamic versions of both Archimedian and elliptical copulas, and further built dynamic mixture copulas of those.

The estimation results for the portfolio returns derived from the dynamic copula models are listed in Table 4. The comparison of the Akaike and Bayesian information criteria with the values of their static counterparts in Table 2, reveals that the dynamic version of each of the copulas has a better fit to the portfolio than the version with constant parameters. As with the static versions, the purely Archimedean copulas attain significantly lower likelihood values compared to the models with elliptical copulas. This results in higher AIC and BIC values indicating an inferior fit.

For the elliptical copulas, both the lower standard errors and their lower AIC / BIC values indicate that these are better models for the commodities portfolio. The standard errors shows that the dynamic mixture weight parameters are not significantly different from zero. Mixing the dynamic copulas with a static weight may hence be sufficient to capture the dynamics in the dependence structure.

The degrees of freedom parameters for the dynamic Student-t copula induce an evolution ranging between $\nu = 16.493$ and $\nu = 21.870$, with a mean of 18.703. The static Student-t copulas ν is with 16.804 (see Table 2) below the average dynamic ν providing support for the conclusion that ignoring time-variation in the copula parameters might induce spuriously increased conditional tail dependence.

[**Table 4 about here**]

[**Table 5 about here**]

Table 5 shows the overall ranking of the in-sample model fit among the static, regime-switching, and dynamic copulas according to AIC and BIC. The criteria agree on the first four ranks, and indicate superiority of the time-varying copulas' in-sample fit compared to the static versions for the portfolio. The only static copula to attain a top five ranking is the static Student-t-Clayton mixture, which ranks fifth according to the BIC. The dynamic Student-t and the dynamic Student-t-Clayton mixture stand out as they dominate the top two ranks for the portfolio (according to both AIC and BIC). This result indicates the importance of accounting for time-variation, and highlight that positive tail dependence is a crucial feature of a well-fitting model for a commodity portfolio.

5 Forecast Evaluation

To test the predictive power of the different copula models, we present the methodology and results of out-of-sample backtests. Monte Carlo simulations are performed to obtain forecasted profit and loss distributions. The same univariate models are coupled with different copulas, so that differences in the return distribution forecasts are attributable to the copula functions only. In this way, we can assess which copula function is most suitable for describing the multivariate dependence structure. In the context of backtesting scheme, we compare forecasts based on individual copula models with the observed historical portfolio returns. We use a broad range of backtesting risk measures to evaluate the predictive performance. These risk measures are *VaR*, *Unconditional Coverage Test* and *Independence Test* (and a *Joint Test* of both), *Basel Three-Zone Approach*, as well as *Expected Shortfall Evaluation*. We also backtest the *Entire* and *Entire Lower Tail* forecasted profit and loss distribution derived from these risk models.[10]

5.1 Backtesting Procedure

The backtesting procedure is based on a rolling window scheme with 520 returns. The univariate models and the copula functions are calibrated to the $t-520$ until $t-1$ returns with the multi-stage maximum likelihood estimation.[11] Subsequently, dependent uniform variates are simulated with specific copulas. These are further transformed to obtain standardized residuals which will be employed as *i.i.d.* noise processes of the corresponding GARCH-models. We thus simulated 10'000 weekly returns for each portfolio component and computed the portfolio's profit and loss. The return of the simulated equally weighted portfolio is further compared to the historical value. Taking advantage of the entire forecasted portfolio return distributions, both risk measure forecasts and density forecasts are evaluated.[12]

For the commodity data set, the GJR specification replaces the EGARCH as optimal

[10] A detailed description of these backtesting risk measures together with the related literature are provided in Internet Appendix D.

[11] See Internet Appendix B for the estimation procedure and the standard error computation.

[12] The univariate models employed in the backtesting procedure differ from the ones outlined in Section 4. While the latter were found using the entire data set, the univariate models for the backtesting procedure are determined by choosing those specifications which reveal the lowest AIC/BIC value for the first 520 returns in the respective data set.

specification for the heating oil, unleaded gasoline, silver, copper and wheat series. The information criteria select a lag of one to be optimal for all univariate commodity series (See Internet Appendix D.1).

5.2 Overall Forecast Performance

We use a multivariate normal model as benchmark for the backtest results. The model employs the same univariate GARCH processes but assumes the resulting standardized residuals to be distributed according to a multivariate normal distribution. The dependence structure between the different series is described by a static Gaussian copula. The results for the static models are listed in Table 6.[13]

[**Table 6 about here**]

According to the hit ratios and the expected shortfall ratios, the static mixture copulas are ranked in the rearmost positions. None of the mixture passes the unconditional coverage or the joint test, and all static mixture models are rated as *red* according to the model classification of the Basel regulatory framework. Despite the Frank-Clayton and the Student-t-Clayton mixture copulas, which both pass the χ^2-test, the comparatively high test statistics of the Anderson-Darling, Kolmogorov-Smirnov, and Christoffersen's Lower Tail test confirm that for our portfolio static mixture copulas are not suitable to forecast neither the entire profit and loss distribution nor its lower tail. The static Frank and the Gaussian copula are two further models with no predictive power of the commodity data's return distribution. Both traffic lights are *red* and even though the violation of their VaR forecasts are independent in time according to the independence coverage tests, they both fail the remaining tests altogether.

[13]To conserve space, the histograms of the probability integral of the forecasted portfolio profit and loss distribution are not shown here, however, they are available from the authors upon request. The histogram shows that the static mixtures fail to adequately forecast both lower and upper tail of the profit and loss distribution.

The backtest comparison of the static Gaussian model with the multivariate normal benchmark model indicates that modeling the standardized residuals with a skewed-t distribution does not improve the hit ratios at any of the three confidence levels. The benefit of the more elaborate marginals comes forward with the improved expected shortfall ratios, which are particularly meaningful on $\alpha = 99\%$ and $\alpha = 95\%$, since the two compared models have identical hit ratios for these confidence levels. Among the two *yellow* classified models Clayton and Student-t, the former is substantially more successful in forecasting the lower tail of the portfolio's return distribution. This is documented by the hit and ES ratios at confidence levels of $\alpha = 99\%$ and $\alpha = 95\%$, and by the fact that the Clayton is the only static model to pass the Lower Tail test.

While incorporating skewness and kurtosis in the marginal distributions does not improve the 90% and 95% hit ratios for the commodity data (documented by the comparison of the Gaussian copula with multivariate normal model results), the capability of modeling lower tail dependence (as in the Student-t and Clayton copula) results in substantially better hit ratios at the 99% confidence level. Applying the static Clayton copula setup instead of the multivariate normal model reduces the 99% hit ratio by more than 50%. The χ^2-test even indicates that the Clayton copula is the best static model to forecast the entire profit and loss distribution of the commodity futures index portfolio.

The backtesting results of the regime-switching models are listed in Table 7. Forecasting the profit and loss distribution of the commodity futures index portfolio by means of the regime-switching copulas with three states yields worse results than with two-state models. The inclusion of a mixture copula to characterize one of the two regimes does not produce better forecasts compared to the regime-switching structures with two standalone copulas.[14]

[Table 7 about here]

[14]The histograms of the three-state regime-switching models visualize also their incapability to forecast either tail of the portfolio return distribution. To conserve space, the histograms are not shown but are available upon request.

The G/G and the T/T regime-switching copulas are the only two Markov-chain models to achieve a *yellow* rating. The former's hit ratios compared to those of the static Gaussian model in Table 6 show that allowing for regime-switches results in preferable VaR forecasts. The results of the χ^2-tests further suggest that the two-state Gaussian setup yields better forecasts for the entire return distribution than the one-state Gaussian model and the multivariate normal benchmark. The regime-switching copula models are not accurately forecasting the lowest 1% and 5% quantiles of the commodities' profit and loss distribution as they fail the unconditional coverage test and the joint test on the corresponding α levels and further do not pass the Lower Tail test. The backtesting outcomes of the G/G and the G/T models at the confidence level of $\alpha = 90\%$, however, allude to their predictive power for the VaR at the conventional significance level. Note that the information criteria for the in-sample fit relegate the latter two models to the rear positions among the regime-switching models whilst the models with the best in-sample fit perform rather poorly in the backtesting procedure. This result again points to a limited use of the in-sample rankings to gauge the predictive power.

Next, the performance of the dynamic models is shifted into focus with the results listed in Table 8. Comparing the backtest performance of the dynamic copulas to those of their static counterparts in Table 6 shows that allowing for time-variation in the dependence structure improves most of the backtest results for all the copula models under consideration. Four models which are classified as *red* in their static versions rank as *yellow* under the Basel regulatory framework in their dynamic specification. The dynamic Frank copula, however, does not produce materially different results than its static version, indicating that the Frank copula is in neither static nor dynamic form an appropriate dependence model to forecast commodity portfolio returns. Adding the Clayton to the Frank copula in the form of a dynamic convex combination of both models improves the performance to the degree that the dynamic Frank-Clayton mixture reaches a *yellow* traffic light ranking, but still yields the second poorest results among the dynamic models.[15]

The model with the highest predictive power for the negative extreme returns of the portfolio is the dynamic Clayton copula. Even though the dynamic version only improves the VaR forecasts at the 90% confidence level compared to the static Clayton copula, the dynamic version passes the Lower Tail test with a p-value twice as large. This outperformance of the multivariate Clayton copula, both static and dynamic, in forecasting the lowest portfolio return quantiles is due to its capability to model lower tail dependence.[16] This highlights the importance of modeling lower tail dependence to forecast the risk of a commodity futures index portfolio. The second poorest in-sample fit of the dynamic Clayton copula among the dynamic models according to both information criteria put the usefulness of AIC and BIC rankings to identify a powerful risk forecasting model into question.

[**Table 8 about here**]

5.3 Crisis Forecast Performance

Risk management models employed by the financial industry turned out to be inadequate during times of financial stress (see, e.g., Skoglund, Chen, and Erdman (2010); Das, Embrechts, and Fasen (2013)), leading to a sharp increase of the number of VaR violations. In this sub-section, we assess the performance of the various copula models during the last financial crisis and the European sovereign debt crisis. To ensure comparability, we use the same backtest procedures including the same univariate model specifications as in Sub-section 5.2. The performance of the models is analyzed in the light of the

[15] Furthermore, the histograms of the probability integral of forecasted profit and loss distribution derived from the dynamic models (not shown here) indicates that the forecasted return distribution of the dynamic Frank-Clayton mixture copula concentrates too much probability mass in the center of the distribution.

[16] The histogram of the dynamic copula models visualizes the Clayton's superiority in this regard with the low bar for the 2% quantile. The results are not shown here, however, they are available from the authors upon request

Basel supervisory framework (Basel Committee on Banking Supervision (2013)), which demands backtests of the risk model based on the VaR measure at the 99% confidence level.

Firstly, we show the out-of-sample accuracy of the risk forecasts of the different models during the crisis period from January 7, 2007 until January 5, 2011. The time period covers the unfolding of the financial turmoil from the disruptions in the subprime mortgage market to the virulent global financial crisis as well as the subsequent European debt crisis. Secondly, the changes in VaR violations of the presented models in reaction to the outbreak of the crises are investigated over time.

Table 9 summarizes the models' backtest performance results for all presented copula models during the crisis period. The therein reported measures refer to the 99%-confidence level with the exception of the Lower Tail test, which assesses the model's capability to forecast the density of the profit and loss distribution below the 10%-quantile. None of the copula models attains a *green* classification according to the Basel regulatory framework. 19 out of 21 tested models rank *red*, which means that the accuracy of their VaR forecasts for the commodities between 2007 and 2011 is not acceptable. Even though all are labeled *red*, the differences in the results of the static Gaussian, G/G regime-switching and dynamic Gaussian copula highlight the advantages of a time-varying specification of the Gaussian copula as the static version shows the most inaccurate hit ratio among the three models. The G/G regime-switching version and the dynamic version of the Gaussian copula produce VaR forecasts of identical accuracy. However, the value of dynamic Gaussian's ES ratio is closer to 1 and its Lower Tail test statistic shows that the dynamic version's forecast of the profit and loss distribution's density in the lowest quantiles is more accurate. For the Student-t copula, the ability to switch between two states (T/T) does not result in a better hit ratio compared to the static setup. The dynamic form, however, clearly beats the static and regime-switching form in terms of hit ratio, ES ratio, and Lower Tail test statistic.

The best models, both achieve a *yellow* classification, are the static and the dynamic Clayton copula model. The dominance of this asymmetric copula is manifested by far more accurate hit ratios compared to all other models. Furthermore, both static and dynamic Clayton are the only two copulas to pass the Lower Tail test, which shows the importance of capturing lower tail dependence. Indeed, ranks three and four in terms of hit ratio and Lower Tail test statistic are taken by the dynamic Gaussian-Clayton mixture and the Student-t Clayton mixture, whose Clayton component enables them to model asymmetries and lower tail dependence.

The dynamic version of the Clayton copula does not increase the accuracy of commodity portfolio return forecasts compared to its static version. The former yields a less accurate hit ratio and further overforecasts the loss given a VaR violation. Figure 3 depicts the evolution of the hit ratios for selected models from the outbreak to the aftermath of the last financial crisis. The depicted hit ratios based on commodity index data are taken from those models which qualified as *green* according to the Basel regulatory framework at the outbreak of the financial crisis: the dynamic Gaussian, the dynamic Student-t, and the dynamic Clayton with their static counterparts, as well as the regime-switching models with two regimes (Gaussian/Gaussian, Gaussian/Student-t, and Student-t/Student-t), and finally the dynamic Gaussian-Clayton mixture. None of these models preserves its Basel traffic light category during the crisis, however, there are subtle differences in the reaction to the outbreak of the financial turmoil.

Only two of the ten models do not become classified as *red* in their immediate reaction to the beginning of the financial turmoil: the dynamic and static Clayton copula. While the dynamic Clayton was better capable of handling the initial impacts of the crisis from 2008 to 2010, the static version shows the best hit ratio in the aftermath of the crisis. Note that these two models are also the closest to the expected hit ratio of 1% in the first half of the year 2008. The dynamic Gaussian-Clayton mixture also faces a deterioration from *green* to *yellow* and manages to maintain the *yellow* classification until the end of

2010, when it becomes red for most of the remaining observation period. All the other models in the figure deteriorated from an acceptable model (*green*) to an unacceptable one (*red*) in just a few months. The most pronounced deterioration during the outbreak of the crisis is shown by the Gaussian and the regime-switching Gaussian/Student-t copula models, which prove to be least capable to appropriately forecast the commodity portfolios' returns with the onset of financial crisis.

Among all considered dependence structures, the three models which performed best for our commodity futures portfolio in terms of hit ratio during the times of crises are asymmetric models, all of which are capable of capturing lower tail dependence. Note that the models whose hit ratio deteriorated drastically at the outbreak of the financial crisis are all combinations of elliptical copulas. The superiority of the static and dynamic Clayton copula established in the analysis of the backtesting results over the entire data set, covering an out-of-sample period of more than 16 years, is thus confirmed by the analysis of models' reaction to the outbreak of financial crises.

[**Table 9 about here**]

[**Figure 3 about here**]

Table 10 relates the results of the models for the financial crisis period (upper panel) to the overall performance of the models documented in the previous Sub-section 5.2 (lower panel) by listing the top rankings of the models for all portfolios according to the accuracy of their VaR(99%) forecasts. The comparison shows that the models' ranking during the financial crisis is largely consistent with the overall ranking, where the Clayton copula stands out. One can see that the static Clayton is ranked first followed by the dynamic copulas.

[**Table 10 about here**]

6 Conclusion

In this study, we evaluate the importance of modeling time-variation and asymmetries in the dependence structure of a commodity futures portfolio to account for the substantial change in the dynamics of commodity prices over the last decade. We firstly implement a regime-switching copula model to account for time-variation in the dependence structure of portfolio constituents. To enhance the flexibility of this set-up, we employ multivariate mixture copulas to characterize the different regime states. As a second approach, we account for time-varying dependencies by introducing multivariate dynamic copula models. Finally, the dynamic copulas were combined into dynamic mixture structures.

The in-sample analysis reveals the superiority of dynamic copulas compared to their static counterparts for our commodity portfolio. This result is of great importance, as it confirms the findings in the Basel III regulatory framework, that rigorous risk measures should be based on a model which accounts for time-varying dependence structures. While the static Student-t-Clayton mixture is the only static copula to attain a top five ranking, the dynamic Student-t and the dynamic Student-t-Clayton mixture stand out as they dominate the top two ranks for the chosen portfolio. The dominance of these models highlights not only the importance of time-variation, but also accentuates that positive tail dependence is a crucial feature of a well-fitting model for a commodity portfolio.

Comparing the backtest performance of the dynamic copulas to those of their static counterparts reveals that allowing for time-variation in the dependence structure improves most of the backtest results for all the copula models under consideration. Four models which are classified as *red* in their static versions rank as *yellow* under the Basel regulatory framework in their dynamic specification. Finally, the most accurate forecasts for the commodity futures index portfolio during the financial crisis were produced by the static Clayton copula, followed by the dynamic Clayton model. Both yield the single most accurate forecasts among all models for the commodity data at all times during the crisis.

Their hit ratios only displayed a minor increase in reaction to the outbreak of the global financial crisis, while the hit ratios of most other models surged drastically.

Our results bring evidentiary support for the critics in Basel III concerning the overreliance on historical correlations: in times of financial stress, the correlations between the risk factors of one commodity-based portfolio have a different structure than in normal regimes, which requires risk managers to recalibrate the models on a regularly base. However, it becomes obvious that, overall, dynamic copula models are more flexible tools to describe asymmetries and the time-varying feature of dependence structures and lead to more robust backtesting results. The comparative assessment of the performance of copula models offers portfolio risk managers an important indication of accurately forecast profits and losses. This is a very important input for the implementation of rigorous risk measures. Furthermore, the ranking in- and out-of-sample of various static, dynamic, regime-switching copula models shed more transparency on the Basel traffic light system classification, which bridges the link between regulators and practitioners.

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Tables and Figures

Table 1: Summary Statistics

	OIL	HOL	GAS	GLD	SLV	CPP	WHT	CRN
Mean	0.001	0.001	0.002	0.000	0.000	0.002	-0.002	-0.002
Std	0.046	0.044	0.047	0.022	0.039	0.035	0.037	0.035
Max	0.232	0.205	0.241	0.129	0.148	0.170	0.190	0.153
Min	-0.318	-0.276	-0.266	-0.132	-0.295	-0.171	-0.177	-0.169
Skew	-0.498	-0.278	-0.286	-0.242	-0.660	-0.109	0.297	0.001
Kurt	6.774	5.667	5.891	7.518	7.638	5.399	4.734	5.411
JB	825.0^{a}	401.9^{a}	470.6^{a}	1118.2^{a}	1259.8^{a}	314.4^{a}	182.0^{a}	315.0^{a}
Correlations								
OIL		0.882	0.858	0.261	0.264	0.246	0.113	0.163
HOL			0.833	0.243	0.244	0.225	0.123	0.164
GAS				0.212	0.217	0.231	0.104	0.140
GLD					0.731	0.284	0.157	0.193
SLV						0.331	0.164	0.220
CPP							0.174	0.171
WHT								0.594
Stationarity Tests								
LMC(1)	0.067^{c}	0.059^{c}	0.047^{c}	0.050^{c}	0.050^{c}	0.144^{b}	0.051^{c}	0.062^{c}
LMC(5)	0.043^{c}	0.043^{c}	0.036^{c}	0.090^{c}	0.058^{c}	0.112^{b}	0.054^{c}	0.059^{c}
LMC(10)	0.038^{c}	0.057^{c}	0.039^{c}	0.196^{a}	0.073^{c}	0.082^{c}	0.060^{c}	0.045^{c}
ADF(1)	-37.2^{a}	-36.4^{a}	-36.9^{a}	-36.7^{a}	-37.1^{a}	-36.3^{a}	-36.2^{a}	-36.5^{a}
ADF(5)	-14.3^{a}	-15.1^{a}	-14.5^{a}	-18.6^{a}	-17.6^{a}	-14.2^{a}	-16.0^{a}	-15.5^{a}
ADF(10)	-9.6^{a}	-10.1^{a}	-11.2^{a}	-13.2^{a}	-12.7^{a}	-9.9^{a}	-11.5^{a}	-10.0^{a}
Heteroscedasticity Tests								
LBQ(1)	67.8^{a}	72.5^{a}	107.4^{a}	31.5^{a}	11.5^{a}	69.3^{a}	60.6^{a}	29.4^{a}
LBQ(5)	140.0^{a}	154.7^{a}	184.2^{a}	181.2^{a}	75.5^{a}	356.8^{a}	239.3^{a}	129.3^{a}
LBQ(10)	312.9^{a}	237.1^{a}	297.9^{a}	272.4^{a}	132.9^{a}	463.3^{a}	288.8^{a}	241.0^{a}
ELM(1)	67.8^{a}	72.5^{a}	107.4^{a}	31.5^{a}	11.5^{a}	69.3^{a}	60.6^{a}	29.4^{a}
ELM(5)	140.0^{a}	154.7^{a}	184.2^{a}	181.2^{a}	75.5^{a}	356.8^{a}	239.3^{a}	129.3^{a}
ELM(10)	312.9^{a}	237.1^{a}	297.9^{a}	272.4^{a}	132.9^{a}	463.3^{a}	288.8^{a}	241.0^{a}

This table shows the summary statistics of the weekly returns over the full sample period from June 30, 1988 to June 5, 2013 for the Standard & Poors Goldman Sachs Commodity excess return subindices: crude oil (OIL), heating oil (HOL), unleaded gasoline (GAS), gold (GLD), silver (SLV), copper (CPP), wheat (WHT), and corn (CRN). Mean, Std, Skew, and Kurt denote the mean, standard deviation, skewness, and kurtosis for the different commodity futures indices. JB is the test statistic of the Jarque-Bera test for normality of the unconditional distribution of the returns. The correlations report Pearson's linear unconditional sample correlations between the weekly returns over the full sample period. LMC(k) is the statistic of Leybourne and McCabe's (1999) test assessing the null hypothesis of a trend stationary AR(k) process against the alternative of a nonstationary ARIMA(k,1,1) process. ADF(k) is the statistic of the augmented Dickey-Fuller (1979) test for a unit root against a trend-stationary alternative augmented with k lagged difference terms. LBQ(k) is the statistic of the Ljung-Box (1978) portmanteau Q-test assessing the null hypothesis of no autocorrelation in the squared (mean-subtracted) residuals at k lags. ELM(k) is Engle's (1982) Lagrange multiplier statistic for heteroscedasticity obtained by regressing the squared returns on k lags. Significance is denoted by superscripts at the 1% (^{a}), 5% (^{b}) and 10% (^{c}) levels.

Table 2: Static Copula Parameters

Copula	Clayton	Frank	Gaussian	Student-t	FC Mix	GC Mix	TC Mix
OIL:HOL			0.875 (0.010)	0.879 (0.010)		0.892 (0.010)	0.887 (0.011)
OIL:GAS			0.846 (0.014)	0.852 (0.013)		0.866 (0.013)	0.864 (0.014)
OIL:GLD			0.222 (0.038)	0.231 (0.037)		0.236 (0.038)	0.231 (0.039)
OIL:SLV			0.214 (0.036)	0.224 (0.038)		0.234 (0.038)	0.231 (0.039)
OIL:CPP			0.207 (0.043)	0.208 (0.044)		0.210 (0.045)	0.208 (0.045)
OIL:WHT			0.082 (0.031)	0.088 (0.032)		0.080 (0.034)	0.086 (0.033)
OIL:CRN			0.120 (0.031)	0.122 (0.032)		0.115 (0.034)	0.119 (0.033)
HOL:GAS			0.817 (0.015)	0.822 (0.015)		0.837 (0.015)	0.831 (0.016)
HOL:GLD			0.209 (0.036)	0.213 (0.035)		0.215 (0.038)	0.213 (0.037)
HOL:SLV			0.200 (0.037)	0.206 (0.037)		0.212 (0.038)	0.212 (0.038)
HOL:CPP			0.196 (0.042)	0.191 (0.044)		0.197 (0.044)	0.191 (0.044)
HOL:WHT			0.101 (0.030)	0.111 (0.030)		0.107 (0.032)	0.111 (0.031)
HOL:CRN			0.126 (0.033)	0.127 (0.033)		0.121 (0.035)	0.124 (0.034)
GAS:GLD			0.167 (0.037)	0.175 (0.036)		0.186 (0.038)	0.178 (0.038)
GAS:SLV			0.172 (0.036)	0.183 (0.037)		0.194 (0.038)	0.190 (0.038)
GAS:CPP			0.190 (0.039)	0.191 (0.040)		0.199 (0.041)	0.193 (0.041)
GAS:WHT			0.085 (0.029)	0.091 (0.031)		0.087 (0.032)	0.091 (0.031)
GAS:CRN			0.109 (0.032)	0.111 (0.032)		0.108 (0.034)	0.110 (0.033)
GLD:SLV			0.728 (0.022)	0.739 (0.020)		0.763 (0.020)	0.755 (0.021)
GLD:CPP			0.251 (0.035)	0.266 (0.035)		0.267 (0.039)	0.271 (0.037)
GLD:WHT			0.127 (0.032)	0.133 (0.033)		0.136 (0.033)	0.134 (0.033)
GLD:CRN			0.164 (0.033)	0.169 (0.034)		0.169 (0.036)	0.171 (0.034)
SLV:CPP			0.287 (0.033)	0.297 (0.033)		0.305 (0.037)	0.305 (0.035)
SLV:WHT			0.141 (0.032)	0.143 (0.033)		0.145 (0.034)	0.144 (0.034)
SLV:CRN			0.171 (0.035)	0.172 (0.035)		0.173 (0.037)	0.175 (0.036)
CPP:WHT			0.133 (0.035)	0.136 (0.037)		0.135 (0.038)	0.134 (0.037)
CPP:CRN			0.127 (0.041)	0.132 (0.041)		0.131 (0.043)	0.132 (0.042)
WHT:CRN			0.598 (0.026)	0.599 (0.025)		0.618 (0.027)	0.606 (0.026)
ν				16.804 (1.804)			18.579 (2.556)
θ_F		1.565 (0.143)			0.632 (1.745)		
θ_C	0.321 (0.029)				0.548 (0.133)	0.163 (0.091)	0.303 (0.215)
w_C					0.598 (0.069)	0.043 (0.013)	0.027 (0.012)
$\log \mathcal{L}$	717	661	2747	2815	788	2785	2824
AIC	-1432	-1319	-5439	-5574	-1570	-5510	-5586
BIC	-1427	-1314	-5294	-5423	-1555	-5355	-5426

This table shows the estimates of the static copulas with standard errors in parentheses. The mixture copulas are abbreviated: Frank-Clayton mixture (FC Mix), Gaussian-Clayton mixture (GC Mix) and Student-t-Clayton mixture (TC Mix). ν, θ_F, θ_C, and w_C denote the degrees of freedom parameter of the Student-t copula, the Frank and Clayton copula parameter, as well as the weight of the Clayton copula in the mixture, respectively.

Table 3: Regime Switching Copula Parameters

Copula	G/G	G/T	T/T	T/GC	G/TC	G/T/C	G/C/F
				Regime 1			
OIL:HOL	0.797 (0.071)	0.819 (0.059)	0.946 (0.070)	0.782 (0.054)	0.936 (0.030)	0.942 (0.034)	0.894 (0.010)
OIL:GAS	0.751 (0.095)	0.771 (0.082)	0.924 (0.077)	0.763 (0.062)	0.910 (0.040)	0.923 (0.039)	0.867 (0.014)
OIL:GLD	0.102 (0.153)	0.093 (0.113)	0.301 (0.114)	0.163 (0.126)	0.284 (0.083)	0.272 (0.093)	0.234 (0.038)
OIL:SLV	0.090 (0.138)	0.030 (0.100)	0.332 (0.112)	0.187 (0.117)	0.247 (0.081)	0.252 (0.086)	0.231 (0.038)
OIL:CPP	0.143 (0.114)	0.093 (0.103)	0.217 (0.112)	0.242 (0.129)	0.156 (0.090)	0.153 (0.092)	0.208 (0.045)
OIL:WHT	0.060 (0.069)	0.025 (0.064)	0.083 (0.072)	0.153 (0.085)	0.055 (0.060)	0.039 (0.057)	0.097 (0.034)
OIL:CRN	0.104 (0.062)	0.050 (0.058)	0.095 (0.065)	0.160 (0.073)	0.047 (0.056)	0.034 (0.054)	0.111 (0.034)
HOL:GAS	0.725 (0.092)	0.746 (0.076)	0.886 (0.076)	0.729 (0.065)	0.878 (0.039)	0.887 (0.042)	0.842 (0.015)
HOL:GLD	0.108 (0.139)	0.093 (0.101)	0.269 (0.107)	0.162 (0.108)	0.260 (0.076)	0.263 (0.082)	0.210 (0.039)
HOL:SLV	0.079 (0.131)	0.028 (0.092)	0.310 (0.109)	0.169 (0.106)	0.227 (0.077)	0.226 (0.081)	0.207 (0.039)
HOL:CPP	0.149 (0.107)	0.090 (0.102)	0.203 (0.120)	0.215 (0.122)	0.143 (0.095)	0.146 (0.096)	0.196 (0.045)
HOL:WHT	0.082 (0.070)	0.043 (0.067)	0.083 (0.070)	0.216 (0.083)	0.064 (0.059)	0.045 (0.059)	0.125 (0.033)
HOL:CRN	0.107 (0.060)	0.053 (0.058)	0.086 (0.063)	0.168 (0.072)	0.051 (0.056)	0.038 (0.055)	0.117 (0.036)
GAS:GLD	0.056 (0.140)	0.065 (0.113)	0.239 (0.111)	0.084 (0.112)	0.245 (0.075)	0.238 (0.080)	0.186 (0.038)
GAS:SLV	0.051 (0.138)	0.020 (0.106)	0.294 (0.120)	0.099 (0.120)	0.241 (0.072)	0.245 (0.077)	0.192 (0.039)
GAS:CPP	0.124 (0.114)	0.091 (0.104)	0.197 (0.119)	0.192 (0.118)	0.169 (0.088)	0.156 (0.087)	0.196 (0.042)
GAS:WHT	0.081 (0.072)	0.029 (0.070)	0.056 (0.073)	0.184 (0.087)	0.045 (0.061)	0.027 (0.057)	0.101 (0.033)
GAS:CRN	0.094 (0.064)	0.040 (0.063)	0.082 (0.066)	0.150 (0.079)	0.034 (0.052)	0.014 (0.052)	0.103 (0.035)
GLD:SLV	0.612 (0.103)	0.663 (0.082)	0.688 (0.068)	0.820 (0.061)	0.692 (0.046)	0.696 (0.042)	0.765 (0.020)
GLD:CPP	0.091 (0.182)	0.145 (0.111)	0.210 (0.112)	0.301 (0.125)	0.237 (0.087)	0.243 (0.092)	0.270 (0.039)
GLD:WHT	0.093 (0.086)	0.081 (0.097)	0.097 (0.092)	0.178 (0.103)	0.085 (0.077)	0.069 (0.075)	0.136 (0.034)
GLD:CRN	0.089 (0.098)	0.081 (0.084)	0.143 (0.068)	0.254 (0.083)	0.102 (0.073)	0.123 (0.070)	0.170 (0.037)
SLV:CPP	0.125 (0.168)	0.161 (0.107)	0.227 (0.100)	0.344 (0.116)	0.244 (0.082)	0.279 (0.086)	0.306 (0.036)
SLV:WHT	0.127 (0.083)	0.110 (0.101)	0.104 (0.091)	0.187 (0.096)	0.097 (0.082)	0.081 (0.077)	0.143 (0.036)
SLV:CRN	0.119 (0.095)	0.096 (0.096)	0.152 (0.075)	0.261 (0.086)	0.105 (0.071)	0.136 (0.071)	0.175 (0.038)
CPP:WHT	0.153 (0.081)	0.092 (0.086)	0.161 (0.082)	0.210 (0.097)	0.095 (0.075)	0.082 (0.074)	0.151 (0.039)
CPP:CRN	0.057 (0.110)	0.030 (0.094)	0.162 (0.091)	0.181 (0.114)	0.069 (0.085)	0.091 (0.087)	0.129 (0.043)
WHT:CRN	0.649 (0.123)	0.675 (0.111)	0.480 (0.130)	0.728 (0.089)	0.515 (0.071)	0.483 (0.070)	0.620 (0.026)
ν_1			10.304 (1.135)	9.729 (1.060)			

Continued on next page

This table shows regime-switching copula parameters with standard errors in parentheses. The copulas are abbreviated as follows: Frank (F), Clayton (C), Gaussian (G), Student-t (T), Gaussian-Clayton mixture (GC) and Student-t-Clayton mixture (TC). ν_1, ν_2 are the degrees of freedom parameters of the Student-t copula in the two regimes and w_C denotes the weight of the Clayton copula in the mixture. The forward slash indicates the separate regimes i.e. G/T/C and G/C/F are three-state models. $p_{i|i}$ denotes the probability of staying in regime i.

Table 3 continued

				Regime 2			
OIL:HOL	0.946 (0.073)	0.945 (0.062)	0.788 (0.070)	0.942 (0.049)	0.778 (0.065)	0.768 (0.061)	
OIL:GAS	0.935 (0.098)	0.939 (0.069)	0.761 (0.078)	0.918 (0.050)	0.765 (0.074)	0.741 (0.067)	
OIL:GLD	0.365 (0.149)	0.380 (0.110)	0.091 (0.100)	0.243 (0.117)	0.120 (0.135)	0.165 (0.140)	
OIL:SLV	0.359 (0.134)	0.416 (0.100)	0.057 (0.103)	0.226 (0.105)	0.173 (0.131)	0.209 (0.128)	
OIL:CPP	0.269 (0.124)	0.324 (0.115)	0.190 (0.096)	0.147 (0.104)	0.266 (0.148)	0.252 (0.148)	
OIL:WHT	0.140 (0.074)	0.145 (0.079)	0.023 (0.061)	0.046 (0.072)	0.151 (0.110)	0.148 (0.102)	
OIL:CRN	0.138 (0.063)	0.177 (0.071)	0.064 (0.054)	0.049 (0.064)	0.192 (0.090)	0.145 (0.083)	
HOL:GAS	0.904 (0.096)	0.907 (0.071)	0.718 (0.079)	0.890 (0.056)	0.733 (0.075)	0.732 (0.073)	
HOL:GLD	0.335 (0.131)	0.349 (0.096)	0.106 (0.092)	0.224 (0.104)	0.119 (0.110)	0.174 (0.118)	
HOL:SLV	0.338 (0.122)	0.388 (0.091)	0.066 (0.100)	0.216 (0.099)	0.182 (0.115)	0.231 (0.113)	
HOL:CPP	0.242 (0.114)	0.297 (0.112)	0.151 (0.103)	0.131 (0.106)	0.236 (0.144)	0.234 (0.140)	
HOL:WHT	0.159 (0.075)	0.162 (0.076)	0.068 (0.060)	0.051 (0.072)	0.204 (0.106)	0.224 (0.100)	
HOL:CRN	0.143 (0.060)	0.178 (0.070)	0.088 (0.051)	0.052 (0.062)	0.200 (0.087)	0.167 (0.080)	
GAS:GLD	0.297 (0.134)	0.299 (0.107)	0.044 (0.097)	0.211 (0.105)	0.050 (0.113)	0.081 (0.115)	
GAS:SLV	0.306 (0.135)	0.349 (0.112)	0.012 (0.114)	0.219 (0.096)	0.093 (0.129)	0.111 (0.128)	
GAS:CPP	0.253 (0.121)	0.294 (0.115)	0.152 (0.101)	0.153 (0.100)	0.191 (0.141)	0.180 (0.137)	
GAS:WHT	0.116 (0.079)	0.135 (0.082)	0.070 (0.060)	0.036 (0.073)	0.180 (0.106)	0.181 (0.098)	
GAS:CRN	0.120 (0.061)	0.159 (0.076)	0.074 (0.052)	0.037 (0.062)	0.192 (0.090)	0.160 (0.086)	
GLD:SLV	0.840 (0.104)	0.823 (0.070)	0.796 (0.066)	0.700 (0.058)	0.826 (0.062)	0.832 (0.055)	
GLD:CPP	0.403 (0.184)	0.382 (0.121)	0.247 (0.094)	0.221 (0.115)	0.314 (0.144)	0.311 (0.146)	
GLD:WHT	0.185 (0.094)	0.168 (0.097)	0.110 (0.090)	0.079 (0.100)	0.220 (0.119)	0.219 (0.119)	
GLD:CRN	0.257 (0.097)	0.247 (0.066)	0.146 (0.068)	0.094 (0.085)	0.245 (0.100)	0.212 (0.091)	
SLV:CPP	0.436 (0.167)	0.434 (0.108)	0.279 (0.090)	0.254 (0.110)	0.400 (0.129)	0.324 (0.130)	
SLV:WHT	0.179 (0.084)	0.158 (0.095)	0.138 (0.086)	0.090 (0.100)	0.242 (0.125)	0.222 (0.118)	
SLV:CRN	0.241 (0.088)	0.243 (0.075)	0.157 (0.070)	0.096 (0.090)	0.265 (0.106)	0.229 (0.098)	
CPP:WHT	0.167 (0.080)	0.166 (0.094)	0.061 (0.073)	0.082 (0.092)	0.228 (0.118)	0.186 (0.117)	
CPP:CRN	0.205 (0.104)	0.216 (0.106)	0.032 (0.077)	0.072 (0.104)	0.198 (0.139)	0.155 (0.135)	
WHT:CRN	0.564 (0.123)	0.539 (0.089)	0.667 (0.123)	0.506 (0.106)	0.735 (0.083)	0.763 (0.085)	
ν_2		9.557 (0.954)	9.132 (1.153)		8.132 (1.063)	9.527 (0.848)	
θ_C				0.572 (0.295)	0.317 (0.295)	0.309 (0.239)	0.304 (0.579)
θ_F							0.506 (0.647)
w_C				0.027 (0.020)	0.046 (0.038)		
$p_{1\|1}$	0.738 (0.087)	0.825 (0.110)	0.876 (0.082)	0.893 (0.142)	0.900 (0.074)	0.860 (0.074)	0.959 (0.012)
$p_{2\|2}$	0.764 (0.088)	0.785 (0.132)	0.878 (0.072)	0.922 (0.105)	0.867 (0.149)	0.794 (0.159)	0.000 (0.200)
$p_{3\|3}$						0.205 (0.073)	0.176 (0.158)
$\log \mathcal{L}$	2880	2883	2879	2904	2899	2892	2789
AIC	-5644	-5648	-5638	-5686	-5676	-5662	-5512
BIC	-5344	-5343	-5328	-5371	-5361	-5347	-5341
$E(D_{R_1})$	4	6	**8**	9	**10**	**7**	**24**
$E(D_{R_2})$	**4**	**5**	8	**13**	8	5	1
$E(D_{R_3})$						1	1

This table shows regime-switching copula parameters with standard errors in parentheses and the expected regime durations (in weeks) under the regime switching models. The expected duration of the high dependence regime is marked in bold. The copulas are abbreviated as follows: Clayton (C), Gaussian (G), Student-t (T), Gaussian-Clayton mixture (GC) and Student-t-Clayton mixture (TC). ν_1, ν_2 are the degrees of freedom parameters of the Student-t copula in the two regimes and w_C denotes the weight of the Clayton copula in the mixture. The forward slash indicates the separate regimes. $p_{i|i}$ denotes the probability of staying in regime i.

Table 4: Estimation Results of Multivariate Dynamic Copula Models

Copula	DC	DF	DG	DT	DFC	DGC	DTC
α			0.019	0.019		0.019	0.020
			(0.005)	(0.005)		(0.006)	(0.005)
β			0.948	0.951		0.947	0.951
			(0.051)	(0.045)		(0.056)	(0.048)
α_C	-0.542				-0.659	-2.221	-2.032
	(0.295)				(0.661)	(3.089)	(3.364)
β_C	-0.385				-8.581	2.691	1.737
	(1.073)				(4.964)	(3.830)	(3.859)
ω_C	-0.987				-0.411	2.212	0.519
	(0.516)				(1.103)	(4.643)	(4.618)
α_F		-0.078			-0.054		
		(0.862)			(1.987)		
β_F		0.981			0.850		
		(0.848)			(0.530)		
ω_F		0.143			0.965		
		(0.273)			(0.227)		
α_W					-2.746	6.081	0.321
					(22.250)	(7.048)	(30.436)
β_W					-1.024	0.415	-3.036
					(1.678)	(14.788)	(31.267)
ω_W					-0.891	4.464	7.926
					(1.497)	(14.443)	(28.447)
ς				-0.862			-1.748
				(0.432)			(0.412)
φ				0.162			0.153
				(0.278)			(0.264)
$\log \mathcal{L}$	737	686	2856	2910	889	2853	2907
AIC	-1468	-1365	-5708	-5811	-1759	-5689	-5795
BIC	-1452	-1350	-5698	-5791	-1713	-5648	-5743

This table shows the parameters of dynamic copula parameters. Standard errors are listed in parentheses. The prefixed D stands for 'dynamic' and the copula models are abbreviated as follows: Frank (F), Clayton (C), Gaussian (G), Student-t (T), Frank-Clayton mixture (FC), Gaussian-Clayton mixture (GC), and Student-t-Clayton mixture (TC). The subscript W indicates the parameters of the dynamic mixture weight process. ς and φ are the parameters of the dynamic Student-t copula's degrees of freedom process.

Table 5: In-Sample Model Fit Ranking

	1.	2.	3.	4.	5.
AIC	DT	DTC	DG	DGC	T/GC
BIC	DT	DTC	DG	DGC	TC

This table presents the top in-sample model fit rankings for the static, regime-switching, and dynamic copulas according to the Akaike (AIC) and the Bayesian (BIC) information criteria. The copulas are abbreviated as follows: Gaussian (G), Student-t (T), Clayton (C), Gaussian-Clayton mixture (GC) and Student-t-Clayton mixture (TC). The forward slash indicates the separation of the regimes in the Markov switching models while the prefixed D denotes dynamic copula models.

Table 6: Static Models Backtest Results

	α	Benchmark	F	C	G	T	FC	GC	TC
Hit Ratio	99%	0.037	0.056	0.018	0.037	0.029	0.099	0.100	0.097
	95%	0.078	0.100	0.077	0.078	0.078	0.172	0.172	0.164
	90%	0.122	0.144	0.141	0.127	0.122	0.227	0.233	0.228
ES Ratio	99%	0.902	0.847	1.025	0.989	0.956	0.794	0.793	0.803
	95%	0.918	0.839	1.044	0.935	0.964	0.804	0.800	0.787
	90%	0.943	0.857	1.060	0.975	0.973	0.807	0.825	0.819
Traffic Light		Red	Red	Yellow	Red	Yellow	Red	Red	Red
Ind. Cov.	99%	2.567	2.281	5.264	2.567	4.780	1.792	3.698	3.200
		(0.109)	(0.131)	(0.022)	(0.109)	(0.029)	(0.181)	(0.054)	(0.074)
	95%	3.806	2.605	1.369	2.328	2.197	3.049	2.269	3.269
		(0.051)	(0.107)	(0.242)	(0.127)	(0.138)	(0.081)	(0.132)	(0.071)
	90%	3.131	5.009	4.625	2.950	4.298	0.042	0.547	0.100
		(0.077)	(0.025)	(0.032)	(0.086)	(0.038)	(0.837)	(0.460)	(0.752)
Unc. Cov.	99%	34.351	81.570	4.028	34.351	19.644	220.606	225.388	215.854
		(0.000)	(0.000)	(0.045)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	11.232	32.221	10.295	11.232	11.232	153.523	153.523	137.393
		(0.001)	(0.000)	(0.001)	(0.001)	(0.001)	(0.000)	(0.000)	(0.000)
	90%	3.877	14.715	13.111	5.840	3.877	106.751	116.647	108.701
		(0.049)	(0.000)	(0.000)	(0.016)	(0.049)	(0.000)	(0.000)	(0.000)
Joint Test	99%	36.918	83.851	9.292	36.918	24.424	222.399	229.086	219.053
		(0.000)	(0.000)	(0.010)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	15.038	34.826	11.664	13.559	13.429	156.572	155.792	140.662
		(0.001)	(0.000)	(0.003)	(0.001)	(0.001)	(0.000)	(0.000)	(0.000)
	90%	7.008	19.724	17.736	8.790	8.175	106.793	117.193	108.801
		(0.030)	(0.000)	(0.000)	(0.012)	(0.017)	(0.000)	(0.000)	(0.000)
χ^2-Test		42.118	78.975	36.276	54.768	53.183	64.295	72.233	65.630
		(0.088)	(0.000)	(0.363)	(0.005)	(0.008)	(0.015)	(0.003)	(0.011)
AD Test		5.441	14.708	17.319	5.859	5.605	118.621	123.705	118.060
		(0.002)	(0.000)	(0.000)	(0.001)	(0.001)	(0.000)	(0.000)	(0.000)
KS Test		0.056	0.065	0.079	0.061	0.063	0.175	0.174	0.175
		(0.014)	(0.002)	(0.000)	(0.006)	(0.004)	(0.000)	(0.000)	(0.000)
Lower Tail		97.262	259.651	4.548	66.008	52.095	474.543	518.550	427.286
		(0.000)	(0.000)	(0.208)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

This table reports the backtest evaluation results for the static copula models. The copula models are abbreviated as follows: Benchmark model, Frank (F), Clayton (C), Gaussian (G), Student-t (T), Frank-Clayton mixture (FC), Gaussian-Clayton mixture (GC), and Student-t-Clayton mixture (TC). α denotes the confidence level of VaR(α). The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns, when VaR(α) is violated, corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. The mid and lower panel lists test statistics and p-values (in parentheses) for multiple backtesting evaluation tests. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the joint test for conditional coverage. The lower panel reports the test statistics with p-values in parentheses of density forecast evaluation tests. χ^2-Test is Pearson's χ^2-test with 10 evenly spaced bins. AD and KS are the tests of Anderson-Darling and Kolmogorov-Smirnov. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the entire lower tail (losses below the 10%-quantile) of the P&L distribution.

Table 7: Regime-Switching Models Out-of-Sample Backtest Results

	α	G/G	G/T	T/T	T/GC	G/TC	G/T/C	G/C/F
Hit Ratio	99%	0.029	0.036	0.031	0.082	0.053	0.094	0.095
	95%	0.077	0.078	0.078	0.141	0.092	0.164	0.168
	90%	0.114	0.115	0.126	0.195	0.145	0.232	0.231
ES Ratio	99%	0.922	0.927	0.973	0.832	0.840	0.766	0.787
	95%	0.930	0.940	0.954	0.799	0.836	0.779	0.797
	90%	0.929	0.928	0.975	0.826	0.905	0.823	0.822
Traffic Light		Yellow	Red	Yellow	Red	Red	Red	Red
Ind. Cov.	99%	4.780	5.754	4.343	2.845	3.157	1.597	1.492
		(0.029)	(0.016)	(0.037)	(0.092)	(0.076)	(0.206)	(0.222)
	95%	2.600	2.328	2.328	2.569	3.140	3.269	2.359
		(0.107)	(0.127)	(0.127)	(0.109)	(0.076)	(0.071)	(0.125)
	90%	3.959	3.613	1.442	1.047	3.534	0.190	0.109
		(0.047)	(0.057)	(0.230)	(0.306)	(0.060)	(0.663)	(0.741)
Unc. Cov.	99%	19.644	31.706	21.891	161.202	71.123	201.772	206.436
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	10.295	11.232	11.232	93.153	23.779	137.393	145.375
		(0.001)	(0.001)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	1.656	1.964	5.314	62.910	15.548	114.639	112.645
		(0.198)	(0.161)	(0.021)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	99%	24.424	37.460	26.234	164.047	74.280	203.369	207.928
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	12.894	13.559	13.559	95.723	26.919	140.662	147.734
		(0.002)	(0.001)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)
	90%	5.615	5.578	6.756	63.958	19.082	114.829	112.754
		(0.060)	(0.061)	(0.034)	(0.000)	(0.000)	(0.000)	(0.000)
χ^2-Test		33.755	49.988	34.938	44.981	58.597	84.563	91.784
		(0.383)	(0.029)	(0.286)	(0.271)	(0.010)	(0.000)	(0.000)
AD Test		4.773	5.478	6.287	82.530	15.924	121.344	121.996
		(0.004)	(0.002)	(0.001)	(0.000)	(0.000)	(0.000)	(0.000)
KS Test		0.058	0.057	0.066	0.142	0.074	0.179	0.177
		(0.010)	(0.012)	(0.002)	(0.000)	(0.000)	(0.000)	(0.000)
Lower Tail		65.859	60.530	35.474	344.719	212.651	562.397	519.691
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

This table reports the backtest evaluation results for the regime-switching copula models. The copula models are abbreviated with Gaussian (G), Student-t (T), Frank (F), Clayton (C), Gaussian-Clayton mixture (GC), and Student-t-Clayton mixture (TC). G/T/C and G/C/F are three-regime models. α denotes the confidence level of VaR(α). The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns, when VaR(α) is violated, corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. The mid and lower panel lists test statistics and p-values (in parentheses) for multiple backtesting evaluation tests. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the joint test for conditional coverage. The lower panel reports the test statistics with p-values in parentheses of density forecast evaluation tests. χ^2-Test is Pearson's χ^2-test with 10 evenly spaced bins. AD and KS are the tests of Anderson-Darling and Kolmogorov-Smirnov. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the entire lower tail (losses below the 10%-quantile) of the P&L distribution.

Table 8: Dynamic Models Out-of-Sample Backtest Results

	α	DF	DC	DG	DT	DFC	DGC	DTC
Hit Ratio	99%	0.053	0.018	0.029	0.024	0.033	0.026	0.026
	95%	0.104	0.078	0.079	0.072	0.082	0.074	0.078
	90%	0.136	0.135	0.122	0.118	0.138	0.119	0.122
ES Ratio	99%	0.839	1.065	0.972	0.954	0.953	0.950	0.983
	95%	0.863	1.049	0.965	0.939	0.942	1.072	0.975
	90%	0.851	1.052	0.974	0.960	0.981	1.011	0.984
Traffic Light		Red	Yellow	Yellow	Yellow	Yellow	Yellow	Yellow
Ind. Cov.	99%	3.157	5.264	4.780	6.870	3.557	2.702	6.290
		(0.076)	(0.022)	(0.029)	(0.009)	(0.059)	(0.100)	(0.012)
	95%	1.835	0.395	3.305	3.687	1.609	3.039	2.328
		(0.176)	(0.530)	(0.069)	(0.055)	(0.205)	(0.081)	(0.127)
	90%	1.941	4.201	2.136	1.982	2.253	2.677	2.136
		(0.164)	(0.040)	(0.144)	(0.159)	(0.133)	(0.102)	(0.144)
Unc. Cov.	99%	71.123	4.028	19.644	11.595	26.639	13.458	13.458
		(0.000)	(0.045)	(0.000)	(0.001)	(0.000)	(0.000)	(0.000)
	95%	36.831	11.232	12.204	6.914	14.254	8.530	11.232
		(0.000)	(0.001)	(0.000)	(0.009)	(0.000)	(0.003)	(0.001)
	90%	10.159	9.475	3.877	2.656	11.592	3.038	3.877
		(0.001)	(0.002)	(0.049)	(0.103)	(0.001)	(0.081)	(0.049)
Joint Test	99%	74.280	9.292	24.424	18.465	30.196	16.160	19.748
		(0.000)	(0.010)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	95%	38.666	11.627	15.509	10.600	15.863	11.569	13.559
		(0.000)	(0.003)	(0.000)	(0.005)	(0.000)	(0.003)	(0.001)
	90%	12.099	13.676	6.013	4.638	13.845	5.716	6.013
		(0.002)	(0.001)	(0.049)	(0.098)	(0.001)	(0.057)	(0.049)
χ^2-Test		71.852	65.422	38.637	38.480	42.313	44.958	31.116
		(0.000)	(0.002)	(0.163)	(0.200)	(0.128)	(0.064)	(0.410)
AD Test		13.972	15.284	4.910	4.745	14.748	4.108	4.801
		(0.000)	(0.000)	(0.003)	(0.004)	(0.000)	(0.008)	(0.004)
KS Test		0.063	0.079	0.058	0.059	0.087	0.057	0.060
		(0.004)	(0.000)	(0.009)	(0.008)	(0.000)	(0.013)	(0.007)
Lower Tail		302.755	2.295	41.068	36.010	57.616	37.133	21.853
		(0.000)	(0.513)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

This table reports the backtest evaluation results for the dynamic copula models. The models are abbreviated as follows: Dynamic (D), Frank (F), Clayton (C), Gaussian (G), Student-t (T), Frank-Clayton mixture (FC), Gaussian-Clayton mixture (GC), and Student-t-Clayton mixture (TC). α denotes the confidence level of VaR(α). The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns, when VaR(α) is violated, corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. The mid and lower panel lists test statistics and p-values (in parentheses) for multiple backtesting evaluation tests. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the joint test for conditional coverage. The lower panel reports the test statistics with p-values in parentheses of density forecast evaluation tests. χ^2-Test is Pearson's χ^2-test with 10 evenly spaced bins. AD and KS are the tests of Anderson-Darling and Kolmogorov-Smirnov. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the entire lower tail (losses below the 10%-quantile) of the P&L distribution.

Table 9: Financial Crisis Out-of-sample Backtest Results (for $\alpha = 99\%$)

Static	F	C	G	T	FC	GC	TC
Hit Ratio	0.086	0.024	0.072	0.062	0.120	0.124	0.115
ES Ratio	0.807	1.050	1.010	0.978	0.702	0.723	0.716
Traffic Light	Red	Yellow	Red	Red	Red	Red	Red
Ind. Cov.	3.459	9.187	2.859	4.326	0.509	0.299	0.779
	(0.063)	(0.002)	(0.091)	(0.038)	(0.475)	(0.584)	(0.377)
Unc. Cov.	46.951	2.944	34.129	26.288	80.901	86.145	75.749
	(0.000)	(0.086)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	50.410	12.131	36.989	30.614	81.411	86.444	76.528
	(0.000)	(0.002)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Lower Tail	162.158	5.792	39.675	32.527	221.283	215.249	196.192
	(0.000)	(0.122)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Regime-Switching	G/G	G/T	T/T	T/GC	G/TC	G/T/C	G/C/F
Hit Ratio	0.053	0.072	0.067	0.120	0.115	0.086	0.095
ES Ratio	0.956	0.952	1.015	0.713	0.677	0.928	0.787
Traffic Light	Red	Red	Red	Red	Red	Red	Red
Ind. Cov.	6.258	5.992	3.542	0.509	0.779	3.459	1.492
	(0.012)	(0.014)	(0.060)	(0.475)	(0.377)	(0.063)	(0.222)
Unc. Cov.	19.105	34.129	30.132	80.901	75.749	46.951	206.436
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Joint Test	25.363	40.122	33.674	81.411	76.528	50.410	207.928
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Lower Tail	41.192	49.911	26.440	254.503	259.490	61.174	519.691
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Dynamic	DF	DC	DG	DT	DFC	DGC	DTC
Hit Ratio	0.081	0.029	0.053	0.048	0.062	0.038	0.043
ES Ratio	0.798	1.144	1.002	1.009	0.993	0.986	1.035
Traffic Light	Red	Yellow	Red	Red	Red	Yellow	Yellow
Ind. Cov.	4.203	7.440	6.258	7.451	4.770	4.924	8.837
	(0.040)	(0.006)	(0.012)	(0.006)	(0.029)	(0.026)	(0.003)
Unc. Cov.	42.547	4.910	19.105	15.795	26.288	9.827	12.694
	(0.000)	(0.027)	(0.000)	(0.000)	(0.000)	(0.002)	(0.000)
Joint Test	46.750	12.350	25.363	23.245	31.058	14.750	21.532
	(0.000)	(0.002)	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)
Lower Tail	176.480	5.981	28.031	20.838	63.014	13.762	18.770
	(0.000)	(0.113)	(0.000)	(0.000)	(0.000)	(0.003)	(0.000)

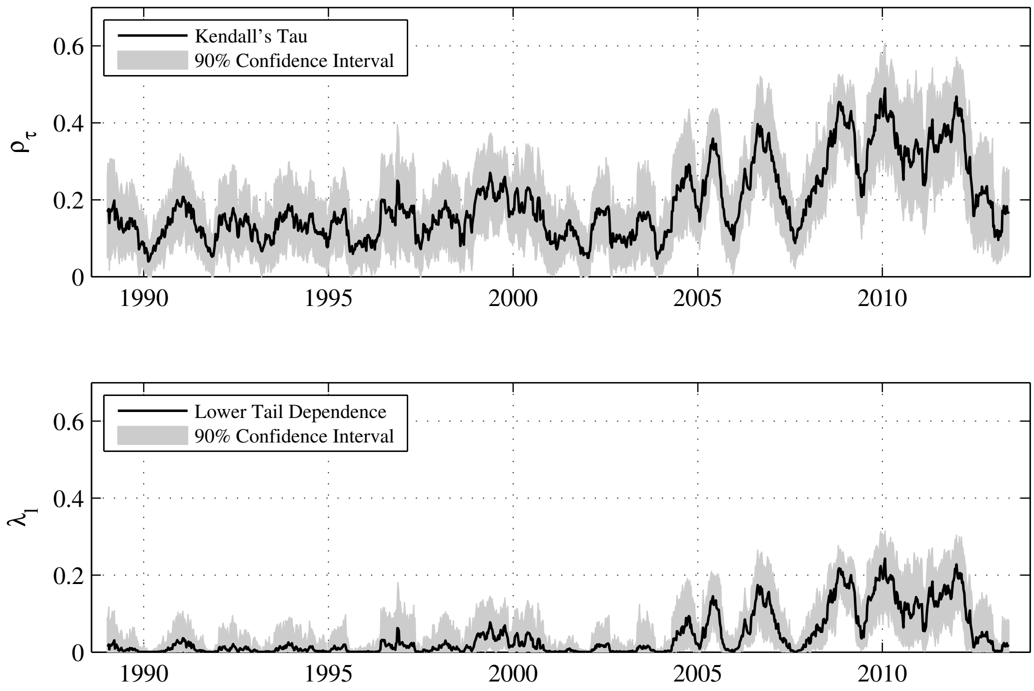
This table reports the backtest evaluation results for the forecasting period from January 2007 until January 2011 for all copula models. The copula models are abbreviated as follows: Gaussian (G), Student-t (T), Frank (F), Clayton (C), Frank-Clayton mixture (FC), Gaussian-Clayton mixture (GC), and Student-t-Clayton mixture (TC). The prefixed D denotes dynamic copulas. The confidence level of VaR is $\alpha = 99\%$. The hit ratio reflects the percentage of times when the portfolio return exceeds VaR(α). ES ratio shows whether the mean of the returns, when VaR(α) is violated, corresponds to the average expected shortfall in these weeks. The traffic light is the model classification of the Basel regulatory framework. Independence (unconditional) coverage is abbreviated with Ind. Cov. (Unc. Cov.). Joint Test is the test for conditional coverage. Lower Tail is the test of Christoffersen (2012) assessing the models' ability to forecast the lowest decile of the P&L distribution. P-values are given in parentheses.

Table 10: Out-of-sample Forecast Accuracy Ranking

	1.	2.	3.	4.	5.
Financial Crisis	C	DC	DGC	DTC	DG
Overall	C	DC	DT	DTC	DGC

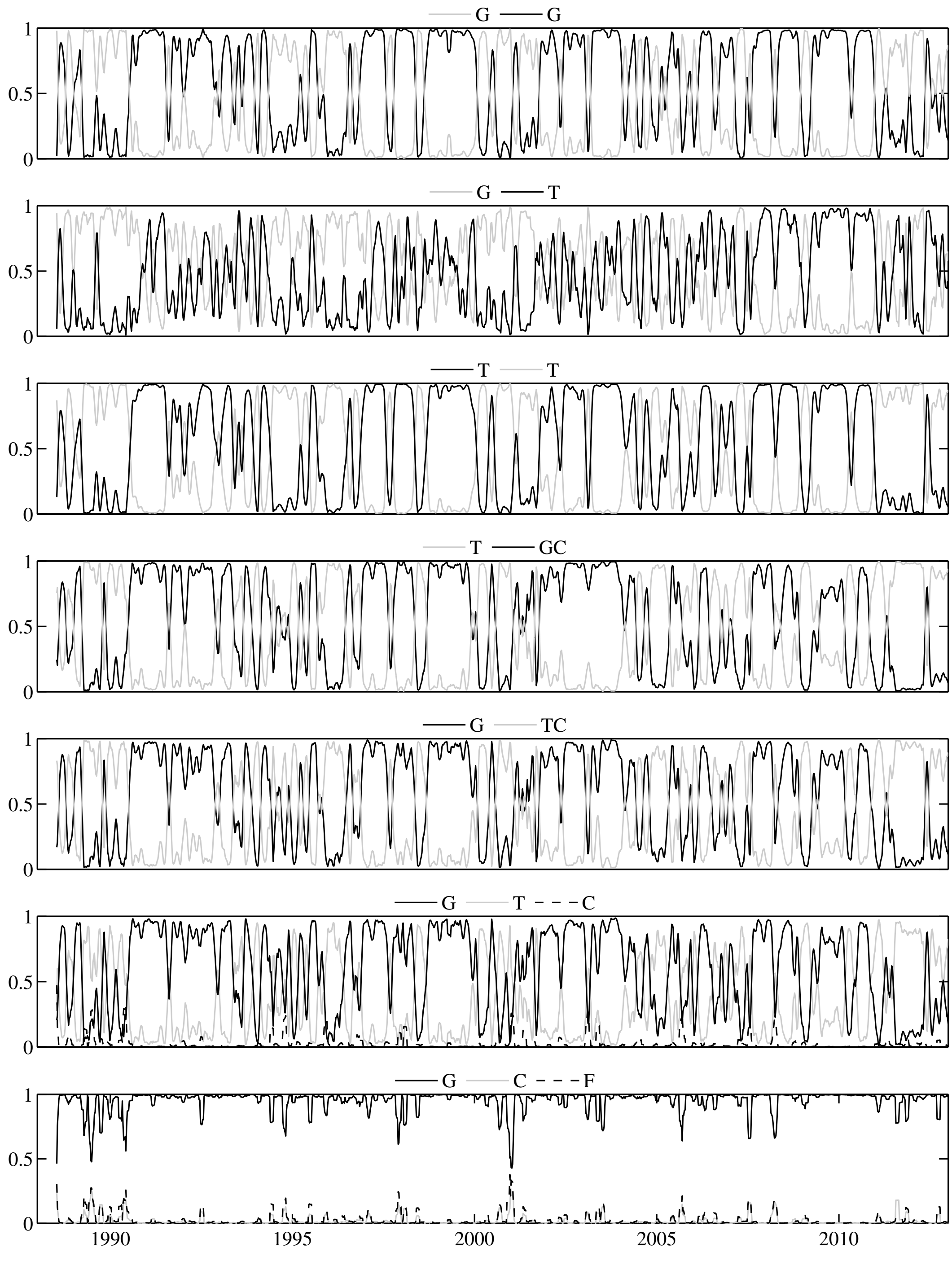
This table presents the top rankings of the out-of-sample VaR(99%) forecast accuracy for the static, regime-switching, and dynamic copulas. The rankings for models with identical hit ratios are determined by the accuracy of their ES ratio. The upper panel refers to the financial crisis out-of-sample performance (see Sub-section 5.3) while the lower panel refers to the overall out-of-sample performance of the models (see Sub-section 5.2). The copulas are abbreviated as follows: Gaussian (G), Student-t (T), Clayton (C), Frank-Clayton mixture (FC), Gaussian-Clayton mixture (GC) and Student-t-Clayton mixture (TC). The prefixed D denotes dynamic copula models.

Figure 1: Dependence Level and Lower Tail Dependence over Time



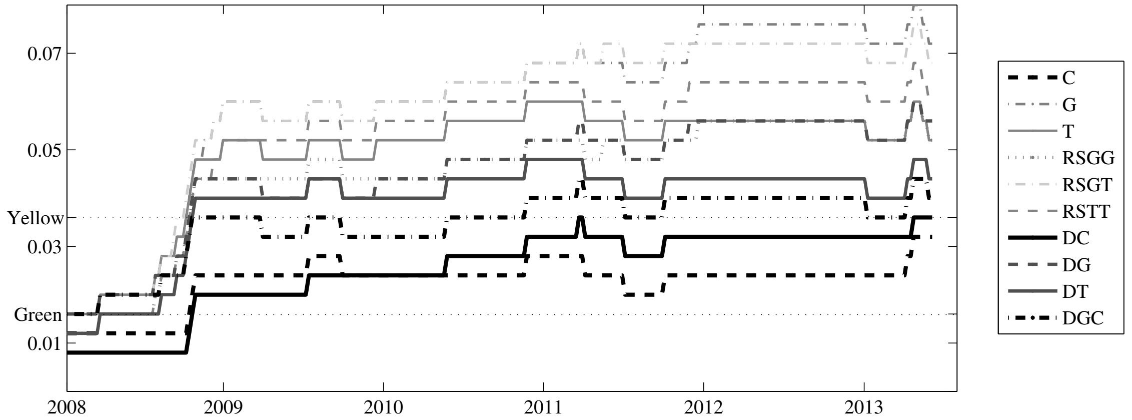
This figure shows Kendall's tau implied by the multivariate Frank copula (upper graph) and lower tail dependence implied by the multivariate Clayton copula (lower graph) of the commodity portfolio over a six months rolling window along with 90% bootstrap confidence intervals obtained from 500 bootstrap replications of the data.

Figure 2: Regime Probabilities



This figure shows Kim filtered regime probabilities of the various regime-switching copula models over the entire sample period. The copulas are abbreviated as follows: Gaussian (G), Student-t (T),Gaussian-Clayton mixture (GC), Student-t-Clayton mixture (TC), Clayton (C) and Frank (F). The solid black line marks the high dependence regime.

Figure 3: Hit Ratio Evolution during the 2007/08 Financial and European Debt Crisis



The plot shows the evolution of the hit ratios of several copula models starting from the outbreak of the financial crisis. Depicted are the hit ratios at the 99% level over a rolling window of 250 returns of all those models which classified as "Green" by the Basel II framework at the beginning of 2008. Hit ratios below the dotted line labeled "Green" respectively "Yellow" are classified accordingly by the Basel regulatory framework. The models with ratios above the line "Yellow" are categorized as "Red" according to the Basel traffic light approach.