RANDOM WAVE-DRIVEN SEABED SHEAR STRESS ESTIMATION IN SHALLOW WATER BASED ON LONG-TERM VARIATION OF DEEP WATER WAVE CONDITIONS

by

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Abstract

The paper presents a simple analytical method giving estimates of wave-driven bottom stresses for very rough and mud seabeds in shallow water from long-term wave statistics in deep water. The results are exemplified using long-term in-situ wave statistics from the Northern North Sea, and by providing examples representing realistic field conditions. The results can be used to make estimates of the seabed shear stress in estuarine and coastal waters based on e.g. global wave statistics.

Keywords: random waves, bottom friction, large bed roughness, stability of scour protections, laminar flow, erosion and deposition of mud, long-term wave statistics

1. Introduction

Simple and effective descriptions of transport mechanisms in operational estuarine and coastal circulation models are often required, in which the seabed shear stress represents an important component. At shallow and intermediate water depths, e.g. in estuarine and coastal zones, the water particle movements driven by surface waves affect the flow in the water column from the surface to the bottom of the sea. In general, the flow in this region is driven by surface waves and currents. The seabed wave boundary layer is a thin flow region at the bottom dominated by friction arising from the bed roughness. The wave boundary layer flow determines the bottom stresses, affecting, for example, transport of sediments as well as assessment of stability of scour protections in estuarine and coastal waters. The boundary layer flow regime is most commonly rough turbulent. However, the flow over a mud bottom is most commonly laminar and smooth turbulent depending on the bottom sediments and wave activity (Whitehouse et al.[1]; Myrhaug et al.[2]).

The purpose of this study is to demonstrate how long-term wave statistics in deep water may be utilized to provide wave-driven bottom stresses in shallow water. Results are given for the bottom stresses under random surface waves at seabeds which are very rough and for laminar flow applied to mud seabeds, and are primarily based on the previous work by Myrhaug and Holmedal[3] (hereafter referred to as MH[3]) who provided the bottom stress spectrum for large roughnesses and for laminar flow. Examples representing field conditions are presented.

In the rough turbulent regime considered here, the near-bed wave orbital displacement amplitude (A) to the bottom roughness ($z_0 = k_s/30$ where k_s is the Nikuradse sand roughness) ratio (A/z_0) is small; i.e. $A/z_0 < 300$ corresponding to very rough beds (see Section 4 and MH[3] for more details). The results in this flow regime are relevant for e.g. assessment of the stability of scour protection in coastal and estuarine environments for relative large roughnesses compared to the near-bed random wave activity.

Laminar flow near mud (i.e. clay and silt) beds is of practical interest. It appears that e.g. organic (polychlorinated biphenyl (PCBs), etc.) pollutants easily stick to clay particles and organic materials of sediments. Results for laminar flow are relevant for assessing erosion and deposition of mud under random waves (see MH[3] and Myrhaug et al.[2] for more details on the relevance for mud beds).

Details on the background, complexity of the flow and reviews are given in the textbooks of e.g. Nielsen[4], Fredsøe and Deigaard[5], Soulsby[6], Whitehouse et al.[1], Winterwerp and van Kesteren[7].

2. Seabed shear stress beneath random waves in shallow water

2.1 Spectrum of seabed shear stress

Following MH[3] the bottom shear stress spectrum for laminar flow in shallow water $(kh << 1)$ is obtained as (see Eq. (67) in the Appendix)

$$S_{(\tau/\rho)(\tau/\rho)}(\omega,h) = \frac{\nu_f \omega^3}{(kh)^2}\frac{\omega^2 h}{2g} S_{\zeta\zeta}(\omega) \tag{1}$$

Here τ is the seabed shear stress, ρ is the fluid density, ω is the cyclic wave frequency, h is the water depth, ν_f is the kinematic viscosity of the fluid, k is the wave number determined from the dispersion relationship $\omega^2 = gk \tanh kh$ which in shallow water reduces to $\omega^2 = k^2 gh$, g is the acceleration due to gravity, and $S_{\zeta\zeta}(\omega)$ is the deep water wave spectrum.

For rough turbulent flow near a seabed with large roughnesses in shallow water the bottom stress spectrum is obtained as (see Eq. (71) in the Appendix)

$$S_{(\tau/\rho)(\tau/\rho)}(\omega,h) = \frac{c^2 z_0^2 \omega^4}{4(kh)^2}\frac{\omega^2 h}{2g} S_{\zeta\zeta}(\omega) \tag{2}$$

Here z_0 is the average bottom roughness, and c is a constant with the two values 9 and 18 reflecting that c is strongly related to the geometry of the roughness elements (see MH[3] for more details). The first term on the right hand side of Eqs. (1) and (2) represents the square of the magnitude of the transfer function between the seabed shear stress τ/ρ and the free surface elevation ζ ; the second term represents the depth correction factor in shallow water, i.e. a correction factor which is used to transform the deep water wave spectrum $S_{\zeta\zeta}(\omega)$ to shallow water (see the Appendix for more details).

By substituting $k^2 = \omega^2 / gh$, Eqs. (1) and (2) are rearranged, respectively, to

$$S_{(\tau/\rho)(\tau/\rho)}(\omega,h) = \frac{1}{2}\nu_f \omega^3 S_{\zeta\zeta}(\omega) \text{ ; laminar} \tag{3}$$

$$S_{(\tau/\rho)(\tau/\rho)}(\omega,h) = \frac{1}{8}(cz_0)^2 \omega^4 S_{\zeta\zeta}(\omega) \text{ ; rough} \tag{4}$$

It appears that the seabed shear stress spectra are given in terms of the deep water wave spectrum and that the dependence on the water depth disappears. This is a consequence of transforming the waves from deep to shallow water and by using the bed shear stress formulas for laminar (mud beds) (Eqs. (60) and (64)) and for very rough beds (Eqs. (60) and (68)).

2.2 Very rough beds

The zeroth spectral moment of the bottom stress spectrum for seabeds with large roughnesses is obtained from Eq. (4) as

$$m_{0\tau/\rho} = \int_0^\infty S_{(\tau/\rho)(\tau/\rho)}(\omega,h)d\omega = \frac{1}{8}(cz_0)^2 m_4 \tag{5}$$

where m_4 is the fourth spectral moment of the wave spectrum in deep water, i.e. the n'th spectral moment of the deep water wave spectrum is defined as

$$m_n = \int_0^\infty \omega^n S_{\zeta\zeta}(\omega) d\omega; n = 0,1,2,--- \tag{6}$$

The most common model wave spectra are proportional to ω^{-5} for large ω, and thus m_4 does not exist. However, m_4 can be expressed in terms of the spectral moments m_0, m_1 and m_2, and consequently in terms of characteristic sea state parameters, as will be described in the following. For deep water waves the significant wave height H_s, the mean spectral wave period T_1 , the mean zero-crossing wave period T_2 , the mean period between maxima T_4, and the spectral bandwidth parameters ε and ν are given in terms of deep water wave the spectral moments as (see e.g. Tucker and Pitt[8])

$$H_s = 4\sqrt{m_0} \tag{7}$$

$$T_1 = 2\pi \frac{m_0}{m_1} \tag{8}$$

$$T_2 = 2\pi \sqrt{\frac{m_0}{m_2}} \tag{9}$$

$$T_4 = 2\pi \sqrt{\frac{m_2}{m_4}} \tag{10}$$

$$\varepsilon^2 = 1 - \frac{m_2^2}{m_0 m_4} = 1 - \frac{T_4^2}{T_2^2} \tag{11}$$

$$\nu^2 = \frac{m_0 m_2}{m_1^2} - 1 = \frac{T_1^2}{T_2} - 1 \tag{12}$$

The last expressions on the right hand side of Eqs. (11) and (12) have been obtained by using Eqs. (8), (9) and (10).

For a narrow-band wave process $\nu = \varepsilon / 2$ (Longuet-Higgins[9]), which combined with Eqs (7), (11) and (12) gives

$$m_4 = \frac{\pi^4 H_s^2}{T_2^2 (5T_2^2 - 4T_1^2)} \tag{13}$$

The spectral peak period T_p is frequently used as a characteristic wave period. For a given deep water wave spectrum there exists a relationship between T_p and T_1 , and T_p and T_2, i. e.

$$\gamma_1 = \frac{T_1}{T_p} \tag{14}$$

$$\gamma_2 = \frac{T_2}{T_p} \tag{15}$$

which substituted in Eq. (13) gives

$$m_4 = \frac{\pi^4 H_s^2}{\gamma_2^2 (5\gamma_2^2 - 4\gamma_1^2) T_p^4} \tag{16}$$

Thus, by combining Eqs. (5) and (16) the significant value of the shear stress height is obtained as

$$H_{s\tau/\rho} = 4\sqrt{m_{0\tau/\rho}} = \Delta_R \frac{H_s}{T_p^2} \tag{17}$$

$$\Delta_R = \sqrt{2} c z_0 \frac{\pi^2}{\gamma_2 \sqrt{5\gamma_2^2 - 4\gamma_1^2}} \tag{18}$$

2.3 Laminar flow

The zeroth spectral moment of the laminar flow bottom stress spectrum is obtained from Eq. (3) as

$$m_{0\tau/\rho} = \frac{\nu_f}{2} m_3 \tag{19}$$

The wave period T_3 related to the surface Stokes drift velocity is given as (see e.g. Webb and Fox-Kemper[10])

$$T_3 = 2\pi(\frac{m_0}{m_3})^{1/3} \tag{20}$$

which combined with Eq. (7) gives

$$m_3 = \frac{\pi^3 H_s^2}{2T_3^3} \tag{21}$$

For a given deep water wave spectrum $\gamma_3 = T_3 / T_p$. Thus, by combining this with Eqs. (19) and (21) the significant value of the shear stress height is obtained as

$$H_{s\tau/\rho} = 4\sqrt{m_{0\tau/\rho}} = \Delta_L \frac{H_s}{T_p^{3/2}} \tag{22}$$

$$\Delta_L = 2\sqrt{\nu_f}(\frac{\pi}{\gamma_3})^{3/2} \tag{23}$$

2.4 Summary of bottom stresses for large roughness and laminar flow

The results in Sections 2.2 and 2.3 can be summarized as follows

$$t = \frac{H_s}{T_p^n};\ n = \begin{cases} 2, \text{rough}, \ t = t_R \\ \frac{3}{2}, \text{laminar}, t = t_L \end{cases} \tag{24}$$

where

$$t_R \equiv \frac{H_{s\tau/\rho}}{\Delta_R} = \frac{H_s}{T_p^2}\ ;\ \text{rough} \tag{25}$$

$$t_L \equiv \frac{H_{s\tau/\rho}}{\Delta_R} = \frac{H_s}{T_p^{3/2}}\ ;\ \text{laminar} \tag{26}$$

Thus, the seabed shear stress in shallow water is given in terms of the sea state parameters H_s and T_p in deep water.

3 Statistical properties of bottom shear stress

3.1 Joint distribution of H_s and T_p

Many models representing the joint probability density function (*pdf*) of H_s and T_p are found in the literature. Examples are Haver[11] and Moan et al.[12]. In the present paper the statistical properties of the seabed shear stress in shallow water are exemplified by using the joint *pdf* of H_s and T_p proposed by Moan et al.[12] based upon 29 year of wave data in the Northern North Sea. This *pdf* is given as

$$p(H_s, T_p) = p(T_p \mid H_s)\, p(H_s) \tag{27}$$

where $p(H_s)$ is the marginal *pdf* of H_s given by the following combined lognormal and Weibull distributions (this type of distribution was first suggested by Haver[11])

$$p(H_s)=\begin{cases}\dfrac{1}{\sqrt{2\pi}\kappa H_s}\exp\left[-\dfrac{(\ln H_s-\theta)^2}{2\kappa^2}\right];\ H_s\le 3.25\text{meter}\\ \beta\dfrac{H_s^{\beta-1}}{\zeta^{\beta}}\exp\left[-\left(\dfrac{H_s}{\zeta}\right)^{\beta}\right];\ H_s>3.25\text{meter}\end{cases}\tag{28}$$

Here $\theta=0.801$, $\kappa^2=0.371$ are the mean value and the variance, respectively, of $\ln H_s$, and $\zeta=2.713$, $\beta=1.531$ are the Weibull parameters.

$p(T_p\,|\,H_s)$ is the conditional *pdf* of T_p given H_s, given by the lognormal distribution

$$p(T_p\,|\,H_s)=\frac{1}{\sqrt{2\pi}\sigma T_p}\exp\left[-\frac{\left(\ln T_p-\mu\right)^2}{2\sigma^2}\right]\tag{29}$$

where μ and σ^2 are the mean value and the variance, respectively, of $\ln T_p$, given by (Gao[13])

$$\mu=a_1+a_2H_s^{a_3}\ ;\ \left(a_1,a_2,a_3\right)=\left(1.780,0.288,0.474\right)\tag{30}$$

$$\sigma^2=b_1+b_2e^{b_3H_s}\ ;\ \left(b_1,b_2,b_3\right)=\left(0.001,0.097,-0.255\right)\tag{31}$$

3.2 Statistical properties of seabed shear stress

Statistical properties of t (from which the statistical properties of the bottom stresses for large roughness and laminar flow can be derived) are calculated from the joint *pdf* of H_s and T_p, e.g. yielding the joint *pdf* of t and H_s; this is derived from Eq. (27) by changing variables from $\left(H_s,T_p\right)$ to $\left(H_s,t\right)$, giving

$$p\left(H_s,t\right)=p\left(t\,|\,H_s\right)p\left(H_s\right) \tag{32}$$

This only affects $p\left(T_p\,|\,H_s\right)$ since $T_p=(H_s\,/\,t)^{1/n}$ according to Eq. (24), giving a lognormal *pdf* of *t* given H_s as (i.e. applying the Jacobian $|\partial T_p/\partial t|=H_s^{1/n}/(nt^{1+1/n})$)

$$p\left(t\,|\,H_s\right)=\frac{1}{\sqrt{2\pi}\sigma_t t}\exp\left[-\frac{\left(\ln t-\mu_t\right)^2}{2\sigma_t^2}\right] \tag{33}$$

where μ_t and σ_t^2 are the conditional mean value and the conditional variance, respectively, of $\ln t$, given as

$$\mu_t=\ln H_s-n\mu \quad ; \quad \sigma_t^2=(n\sigma)^2 \tag{34}$$

where μ and σ^2 are given in Eqs. (30) and (31), respectively.

The joint *pdf* of T_p and *t* is derived from Eq. (27) by changing variables from $\left(H_s,T_p\right)$ to (T_p,t) applying the Jacobian $|\partial H_s/\partial t|=T_p^n$ according to Eq. (24). However, this will not be elaborated further here.

The conditional cumulative distribution function (*cdf*) of t given H_s is derived from

$$P\left(t\,|\,H_s\right)=\Phi\left[\frac{\ln t-\mu_t}{\sigma_t}\right] \tag{35}$$

where Φ is the standard Gaussian *cdf*, i.e.

$$\Phi\left(r\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{r}e^{-t^2/2}dt \tag{36}$$

The conditional expected value of t given H_s is (Bury[14])

$$E\left[t\,|\,H_s\right]=\exp\left(\mu_t+\frac{1}{2}\sigma_t^2\right) \tag{37}$$

The conditional standard deviation of t given H_s is (Bury[14])

$$\sigma[t \mid H_s] = \left[\left(e^{\sigma_t^2} - 1 \right) \exp\left(2\mu_t + \sigma_t^2 \right) \right]^{1/2} \tag{38}$$

4 Examples of results

The practical application of the results are illustrated by two examples using data representing realistic field conditions: very rough seabeds with roughness $z_0 = 0.0094\,\text{m}$ (cobble; Soulsby[6]), and mud beds with median grain diameter $d_{50} = 0.03\,\text{mm}$ (medium silt; Soulsby[6]).

First, common features of the two examples are described. Many standard spectral formulations can be used, but here a Phillips spectrum is taken as the deep water wave spectrum from which analytical expressions can be obtained (see e.g. Tucker and Pitt[8]; Massel[15])

$$S(\omega) = \alpha \frac{g^2}{\omega^5} \quad , \quad \omega \geq \omega_p \tag{39}$$

where $\alpha = 0.0081$ is the Phillips constant and ω_p is the spectral peak frequency. Based on the definition of the spectral moments in Eq. (6) it follows for waves in deep water that

$$H_s = 4\sqrt{m_0} = 2\sqrt{\alpha} \frac{g}{\omega_p^2} \tag{40}$$

$$T_1 = 2\pi \frac{m_0}{m_1} = \frac{3\pi}{2\omega_p} \tag{41}$$

$$T_2 = 2\pi \sqrt{\frac{m_0}{m_2}} = \frac{\sqrt{2}\pi}{\omega_p} \tag{42}$$

$$T_3 = 2\pi \left(\frac{m_0}{m_3} \right)^{1/3} = \frac{2^{\frac{1}{3}}\pi}{\omega_p} \tag{43}$$

With $T_p = 2\pi / \omega_p$ this gives

$$\gamma_1 = \frac{T_1}{T_p} = \frac{3}{4} \tag{44}$$

$$\gamma_2 = \frac{T_2}{T_p} = 2^{-1/2} \tag{45}$$

$$\gamma_3 = \frac{T_3}{T_p} = 2^{-2/3} \tag{46}$$

Then it follows from Eqs. (18) and (23), respectively, that

$$\Delta_R = 4\pi^2 c z_0 \tag{47}$$

$$\Delta_L = 4\pi^{3/2} \sqrt{\nu_f} \tag{48}$$

Furthermore, the Phillips spectrum transformed to shallow water becomes (Massel[15])

$$S_{\zeta\zeta}(\omega, h) = \frac{\omega^2 h}{2g} S_{\zeta\zeta}(\omega) = \frac{\omega^2 h}{2g} \cdot \alpha \frac{g^2}{\omega^5} = \alpha \frac{gh}{2\omega^3} \; ; \; \omega \geq \omega_p \tag{49}$$

which gives the following shallow water significant wave height

$$H_{sh} = 4\sqrt{m_{0h}} = \frac{2\sqrt{\alpha g h}}{\omega_p} \; ; \text{ shallow water} \tag{50}$$

where $m_{0h} = \int_0^\infty S_{\zeta\zeta}(\omega, h) d\omega$. It should be noted that the wave transformation effect included here is based on extending the concept of the Phillips equilibrium range from deep to shallow water, i.e. that the frequency dependence is in the range ω^{-5} in deep water to ω^{-3} in shallow water (Massel[15]). In this equilibrium range of the spectrum the wave dissipation is due to wave breaking, i.e whitecapping (Tucker and Pitt[8]).

4.1 Example 1: very rough beds

The flow conditions are

- Water depth, h = 3 m
- Spectral peak period, $T_p = 8.2\text{s}$, corresponding to $\omega_p = 2\pi / T_p = 0.766\,\text{rad/s}$

The calculated quantities are:

- Deep water significant wave height, $H_s = 2\sqrt{\alpha g} / \omega_p^2 = 3\text{m}$
- Shallow water significant wave height, $H_{sh} = 2\sqrt{\alpha g h} / \omega_p = 1.27\,\text{m}$
- k_p from the shallow water dispersion relationship corresponding to $\omega_p, k_p = \omega_p / \sqrt{gh} = 0.141\,\text{rad/m}$
- Peak near-bed orbital displacement amplitude, $A_p = H_{sh} / (2k_p h) = 1.50\,\text{m}$
- $A_p / z_0 = 160 < 300$, i.e. being in the range of the data of both Myrhaug et al.[16] and Dixen et al.[17] (see MH[3] for more details). Thus both c = 9 and c = 18 are used.
- $$\Delta_R = 4\pi^2 z_0 \begin{cases} 9 \\ 18 \end{cases} = \begin{cases} 3.34\,\text{m} \\ 6.68\,\text{m} \end{cases} \tag{51}$$

  according to Eq. (47)

Then it follows from Eqs. (24), (25), (37) and (51) that

$$E[H_{s\tau/\rho} \mid H_s = 3\text{m}] = \Delta_R E[t_R \mid H_s = 3\text{m}] = \begin{cases} 0.119\,\text{m}^2/\text{s}^2 \\ 0.238\,\text{m}^2/\text{s}^2 \end{cases} \tag{52}$$

and from Eqs. (24), (25), (38) and (51) that

$$\sigma[H_{s\tau/\rho} \mid H_s = 3\text{m}] = \Delta_R \sigma[t_R \mid H_s = 3\text{m}] = \begin{cases} 0.0534\,\text{m}^2/\text{s}^2 \\ 0.107\,\text{m}^2/\text{s}^2 \end{cases} \tag{53}$$

Thus the intervals corresponding to the mean value $\pm$ one standard deviation are given by $(0.0656\,\mathrm{m}^2/\mathrm{s}^2,\, 0.172\,\mathrm{m}^2/\mathrm{s}^2)$ and $(0.131\mathrm{m}^2/\mathrm{s}^2, 0.345\,\mathrm{m}^2/\mathrm{s}^2)$ corresponding to $c=9$ and $c=18$, respectively.

The critical shear stress for movement of the bottom material is given by Soulsby[4] as

$$(\frac{\tau}{\rho})_c = 0.055(s-1)g\,d_{50} \tag{54}$$

with $s = 2.65$ as the sediment to fluid density ratio for quartz sand, and $d_{50} = 12z_0$. Thus, $(\tau/\rho)_c = 0.1\mathrm{m}^2/\mathrm{s}^2$, showing that the bottom material will be partly unstable/stable for $c=9$ and unstable for $c=18$.

4.2 Example 2: Mud beds

The given flow conditions are:

- Water depth, h = 3 m
- Spectral peak period, $T_p = 4.8\,\mathrm{s}$, corresponding to $\omega_p = 2\pi/T_p = 1.308$ rad/s
- Density of water, $\rho = 1027\,\mathrm{kg/m}^3$
- Kinematic viscosity of water at temperature $10^\circ\,\mathrm{C}$ and salinity 35 $^{\mathrm{o}}/_{\mathrm{oo}}$, $\nu_f = 1.36\cdot 10^{-6}\,\mathrm{m}^2/\mathrm{s}$

The calculated quantities are

- Deep water significant wave height, $H_s = 2\sqrt{\alpha}g/\omega_p^2 = 1.03\,\mathrm{m}$
- Shallow water significant wave height, $H_{sh} = 2\sqrt{\alpha g h}/\omega_p = 0.75\,\mathrm{m}$
- k_p from the dispersion relationship in shallow water corresponding to $\omega_p, k_p = \omega_p/\sqrt{gh} = 0.241\mathrm{rad/m}$

- Peak near-bed orbital displacement amplitude, $A_p = H_{sh}/(2k_p h) = 0.52\,\mathrm{m}$
- The wave Reynolds number, $\mathrm{Re} = \dfrac{U_p A_p}{\nu_f} = \dfrac{\omega_p A_p^2}{\nu_f} = 2.6\cdot 10^5$, i.e. laminar flow $(\mathrm{Re} \lesssim 3\cdot 10^5)$

  (Soulsby[6]) and $U_p = \omega_p A_p$
- $\Delta_L = 4\pi^{3/2}\sqrt{\nu_f} = 0.0260\,\mathrm{m/s}^{1/2}$ (55)

  according to Eq. (48).

Then it follows from Eqs. (24), (26) (37) and (55) that

$$\mathrm{E}[H_{s\tau/\rho} \,|\, H_s = 1.03\,\mathrm{m}] = \Delta_L E[t_L \,|\, H_s = 1.03\,\mathrm{m}] = 0.00131\,\mathrm{m}^2/\mathrm{s}^2 \qquad (56)$$

and from Eqs. (24), (26), (38) and (55) that

$$\sigma[H_{s\tau/\rho} \,|\, H_s = 1.03\,\mathrm{m}] = \Delta_L E[t_L \,|\, H_s = 1.03\,\mathrm{m}] = 0.00056\,\mathrm{m}^2/\mathrm{s}^2 \qquad (57)$$

Thus the interval corresponding to the mean value $\pm$ one standard deviation is given by $(0.00075\,\mathrm{m}^2/\mathrm{s}^2, 0.00187\,\mathrm{m}^2/\mathrm{s}^2)$.

As exemplified in Whitehouse et al.[1] (i.e. example 4.2 for erosion and example 8.1 for deposition), the critical bed shear stress for erosion is $\tau_e = 0.197\,\mathrm{N/m}^2$ and for deposition it is $\tau_d = 0.08\,\mathrm{N/m}^2$. Thus it follows that $\tau_e/\rho = 0.00019\,\mathrm{m}^2/\mathrm{s}^2$ and $\tau_d/\rho = 0.000078\,\mathrm{m}^2/\mathrm{s}^2$, showing that the bed is exposed to erosion for this flow condition.

In general, mud beds exhibit cohesive properties and the details of the flow can only be understood by including a number of complex transport mechanisms; see e.g. Whitehouse et al.[5] and Winterwerp and van Kesteren[7] for further details. The flow over muds is not necessarily laminar, but will depend on the wave Reynolds number *Re*, which can be large enough corresponding to turbulent flow over smooth (or mud) beds, i.e. $\mathrm{Re} \gtrsim 3\cdot 10^5$. Furthermore, the results in Fig 2.13 in Fredsøe and Deigaard[5] as well as in Ch. 4.5 in Soulsby[6] can be used to distinguish between laminar and turbulent flow for different combinations of grain size and Reynolds number.

5. Discussion

Here the present method versus a commonly used practice in coastal engineering is briefly discussed. For calculating random wave-driven bottom shear stress in shallow water common practice would be to start from available data on joint statistics of H_s and T_p (or other characteristic wave periods) at a nearby offshore (deep water) location; then to transform this applying an appropriate wave simulation model yielding the joint statistics of H_s and T_p at the shallow water site; then using this result as input for calculating the seabed shear stress. Alternatively, this paper provides a simple analytical method giving first estimates of random wave-driven bottom stresses for very rough and mud seabeds from observed deep water sea surface wave statistics with an example based on in-situ data obtained from the Northern North Sea. The Phillips deep water wave frequency spectrum is chosen as an example and is transformed to shallow water using the narrow-band and shallow water assumptions. It is also assumed to be a smooth transition from deep to shallow water excluding an evolving bathymetry with varying shallow water depths. Thus, an analytical estimate of the associated bottom shear stresses is obtained. The narrow-band assumption is justified since the waves with the frequencies close to the spectral peak frequency are the most energetic contributing to the random wave-driven seabed shear stresses in shallow water. Such simple methods are useful to be able to quickly make estimates which can be used to compare and verify more complete computational methods. It might also be useful e.g. under field conditions when there is limited time and access to computational resources. However, the accuracy of the method versus common approaches should be assessed, but this is only possible to quantify by comparing with such methods covering a wide parameter range, which is beyond the scope of the present work.

6. Summary

A simple analytical method which can be used to give estimates of the random wave-driven seabed shear stress for very rough beds and mud beds in shallow water based on long-term observations of wave conditions in deep water is provided. Results are exemplified by using long-term in-situ wave statistics from the Northern North Sea and by giving examples representing realistic field conditions. The results might serve as a useful tool for assessing e.g. stability of scour protections in coastal and estuarine environments where the roughness elements are large compared to the near-bed wave activity, as well as for assessing erosion and deposition of mud. The method should also represent a useful representation of the seabed shear stress often required in operational estuarine and coastal circulation models based on, for example, available global wave statistics.

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APPENDIX Brief summary of the MH^3 seabed shear stress spectrum in shallow water

A horizontally uniform oscillatory seabed wave boundary layer flow is considered with the free stream velocity outside the boundary layer as

$$u(t) = Ue^{i\omega t} \tag{58}$$

where complex notation is used, U is the near-bed orbital velocity amplitude, t is the time, ω is the cyclic wave frequency, and $i = (-1)^{1/2}$ is the complex unity. The seabed shear stress is

$$\tau(t) = \tau_m e^{i(\omega t + \varphi)} \tag{59}$$

where φ is the phase angle between $\tau(t)$ and $u(t)$, and τ_m is the maximum seabed shear stress given by

$$\frac{\tau_m}{\rho} = \frac{1}{2} f_w U^2 \tag{60}$$

with ρ as the fluid density and f_w as the wave friction factor.

The relationship between $u(t)$ and the free surface elevation $\zeta(t)$ with wave amplitude ζ_A is

$$u(t) = \frac{\omega\zeta(t)}{\sinh kh} \quad ; \quad \zeta(t) = \zeta_A e^{i\omega t} \tag{61}$$

where h is the water depth, and k is the wave number determined from the dispersion relationship for linear waves as given in Section 2.1. From Eqs. (58) and (61) it follows that

$$U = \frac{\omega\zeta_A}{\sinh kh} \tag{62}$$

The wave spectrum in finite water, $S_{\zeta\zeta}(\omega, h)$, is obtained by multiplying the deep water wave spectrum, $S_{\zeta\zeta}(\omega)$, with a depth correction factor, $\Psi(\omega, h)$, which in shallow water $(kh << 1)$ is (Section 7.3 in Massel[15])

$$\Psi(\omega, h) = \frac{\omega^2 h}{2g} \tag{63}$$

For laminar flow the wave friction factor and the phase angle are given as

$$f_w = \mathrm{Re}^{-0.5} \;;\quad \mathrm{Re} = UA/\nu_f \;;\qquad \varphi = \frac{\pi}{4} \tag{64}$$

where Re is the Reynolds number associated with the wave motion, $A = U/\omega$, and ν_f is the kinematic viscosity of the fluid. By using Eqs. (58), (60), (61), and (64), Eq. (59) becomes

$$\frac{\tau(t)}{\rho} = \frac{\sqrt{\nu\omega}\omega}{\sinh kh}\zeta_A e^{i(\omega t+\frac{\pi}{4})} \tag{65}$$

Then the Response Amplitude Operator (RAO) = ratio between the amplitude of the free surface elevation, $\zeta(t)$, and the amplitude of the seabed shear stress, $\tau(t)/\rho$, is obtained as (i.e. corresponding to the magnitude of the transfer function)

$$RAO = \frac{(\nu_f\omega)^{1/2}\omega}{kh} \tag{66}$$

Consequently, the seabed shear stress spectrum for laminar flow is

$$S_{(\tau/\rho)(\tau/\rho)}(\omega,h) = \frac{\nu_f\omega^3}{(kh)^2}\frac{\omega^2 h}{2g}S_{\zeta\zeta}(\omega) \tag{67}$$

For rough turbulent flow the transfer function between the free surface elevation and the seabed shear stress at beds with very large roughness can be found analytically. In this roughness regime the wave friction factor is given as

$$f_w = c\left(\frac{A}{z_0}\right)^{-1} \tag{68}$$

where c is a constant, using the two values 9 and 18 (see MH[3] for further details).

By combining Eqs. (58), (60), (61), $A = U/\omega$ and (68), Eq. (59) becomes

$$\frac{\tau(t)}{\rho} = \frac{c z_0 \omega^2}{2\sinh kh} \zeta_A e^{i(\omega t+\varphi)} \tag{69}$$

and the *RAO* between the free surface elevation and the seabed shear stress is

$$RAO = \frac{c z_0 \omega^2}{2kh} \tag{70}$$

Consequently, the seabed shear stress spectrum for rough turbulent flow over a bed with very large roughness is

$$S_{(\tau/\rho)(\tau/\rho)}(\omega, h) = \frac{c^2 z_0^2 \omega^4}{4(kh)^2} \frac{\omega^2 h}{2g} S_{\zeta\zeta}(\omega) \tag{71}$$

It should be noted that only the waves with wavelengths longer than approximately two times the water depth will give wave activity at the seabed. By using the deep water dispersion relationship $\omega^2 = gk$, this means that the waves at the seabed have frequencies below $(\pi g / h)^{1/2}$.

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