2017; -:1–30

Fault tolerant control of uncertain dynamical systems using interval virtual actuators

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SUMMARY

In this paper, a model reference fault tolerant control (FTC) strategy based on a reconfiguration of the reference model, with the addition of a virtual actuator block, is presented for uncertain systems affected by disturbances and sensor noise. In particular, this paper: (i) extends the reference model approach to the use of interval state observers, by considering an error feedback controller which uses the estimated bounds for the error between the real state and the reference state; (ii) extends the virtual actuator approach to the use of interval observers, which means that the virtual actuator is added to the control loop in order to preserve the nonnegativity of the interval estimation errors and the boundedness of the involved signals, in spite of the fault occurrence. In both cases, the conditions to assure the desired operation of the control loop are provided in terms of linear matrix inequalities (LMIs). An illustrative example is used to show the main characteristics of the proposed approach.

Received ...

KEY WORDS: Fault tolerant control (FTC), virtual actuators, linear matrix inequalities (LMIs), model reference control, uncertain systems, interval observers.

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1. INTRODUCTION

Fault tolerant control (FTC) systems aim at conserving the stability and maintaining the overall performances close to the desired ones despite the presence of faults [1]. **The design of these systems, and their integration with fault detection and isolation (FDI) techniques, is a hot topic of research in the recent literature [2, 3].** Following a well-established terminology, FTC approaches are classified into *passive* and *active* [4]. The *passive* approaches use robust control theory to guarantee some fault tolerant capabilities **[5]**. They are simple to design and implement, but are more conservative in terms of performances and allowable faults with respect to active approaches. On the other hand, these latter compensate the faults either by selecting a precalculated control law or synthesizing online a new control strategy **[6]**. In this case, the adaptation of the control law is done using some information about the fault so as to satisfy the control objectives with minimum performance degradation after the fault occurrence.

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In recent years, the *fault-hiding* paradigm has been proposed as an active FTC strategy to obtain fault tolerance [7]. In this paradigm, the faulty plant is reconfigured, instead of the controller/observer, by inserting a reconfiguration block when a fault occurs. The reconfiguration block aims at hiding the fault from the controller point of view, such that it sees (approximately) the same plant as before the fault. In the case of actuator faults, the reconfiguration block is named *virtual actuator*. This active FTC strategy has been extended successfully to many classes of systems, e.g. linear parameter varying (LPV) [8], hybrid [9], Takagi-Sugeno [10] and piecewise affine [11]. Recently, the case of unstable linear systems subject to actuator saturations and fault isolation delays has been considered as well [12].

An important line of research in control systems literature concerns the design of *robust* controllers, which are able to deal with bounded uncertainties [13]. **This problem is currently investigated due to the need of ensuring high accuracy and performances in several applications, e.g. satellite attitude tracking manoeuvres [14], and several solutions have been proposed. For example, [15] have synthesized a control law that uses the estimation provided**

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by sliding-mode observers in order to follow a desired trajectory. On the other hand, [16] have developed a disturbance observer-based integral sliding-mode control approach for systems with mismatched disturbances or uncertainties, where the estimated disturbance is used to counteract the disturbance.

This problem has been considered also in the design of FTC systems, giving rise to robust fault diagnosis and robust FTC strategies. For example, [17] have presented a robust adaptive fault tolerant compensation control via sliding-mode output feedback for uncertain systems with actuator faults and exogenous disturbances, where mismatched disturbance attenuation is performed via $\mathscr{H}_\infty$ norm minimization. [18] have proposed an integrated fault estimation and FTC design for Lipschitz nonlinear systems subject to uncertainty, disturbance and actuator/sensor faults. **The velocity-free uncertain attenuation control for a class of nonlinear systems with external disturbance and multiple actuator faults has been addressed by [19] using an adaptive sliding mode observer. Finally, approaches based on fuzzy adaptive compensation control and adaptive neural networks have been proposed for dealing with actuator faults in nonlinear systems with unmodelled dynamics, see e.g. [20, 21].** R2-4 R3-5

It is well known that, in an uncertainty setting, interval observers are an appealing approach [22] because, under some assumptions, they can provide the set of admissible values for the state at each instant of time. Using the knowledge about the boundedness of the uncertainty, an interval observer computes the lower and upper bounds for the state, which are compatible with the uncertainty [23]. A successful framework for interval observer design is based on the monotone system theory, proposed at first by [24], and further investigated by **[25, 26, 27, 28].** R4-5

An interesting twist on the interval observer approach is its application to control [29]. This means that the control law is designed feeding back the computed lower and upper bounds in order to stabilize the interval observer, ensuring convergence to a vicinity of zero for the bounding variables [30]. **The approach presented in [30] has undoubtedly a strong appeal, but also a few limitations. First of all, the proposed control law ensures convergence to a vicinity of zero for the bounding variables and, as such, the extension to trajectory tracking problems** R1-4 R3-1

is not straightforward. Moreover, the possibility of faults affecting the system is not taken into account, which makes the approach in [30] fragile. The main original contribution of this paper is the extension of the idea developed in [30] to the case of fault tolerant tracking, that allows overcoming the above-mentioned limitations, and which is done taking into account the theoretical results presented in [31]. In order to achieve this:

- **the reference model approach [32] is extended to the use of interval observers, by considering an error feedback controller which uses the estimated bounds for the error between the real state and the reference one;**
- **the virtual actuator approach [31] is extended to the use of interval observers, which means that the virtual actuator is added to the control loop in order to preserve the nonnegativity of the interval estimation errors and the boundedness of the involved signals, in spite of the fault occurrence.**

Both results are advancements in the state-of-the-art of interval observer-based control. It is worth highlighting the fact that the model reference FTC approach described in [31] does not take into account the fact that the controlled system could be affected by undesired effects, such as structural uncertainty, exogenous unknown disturbances, measurement noise and fault estimation errors. In fact, [31] discusses only briefly the effects of fault estimation errors by suggesting the application of perturbation rejection techniques, such as $\mathscr{H}_2/\mathscr{H}_\infty$ norm optimization, in order to enhance the robustness of the FTC system. On the other hand, the approach proposed in this paper considers all the above mentioned effects and, by exploiting the interval formulation, guarantees some theoretical properties (interval estimation and signal boundedness). **R1-1**

This paper is structured as follows. Section 2 presents the interval observer-based model reference control. Section 3 describes the fault tolerant control using virtual actuators. Section 4 demonstrates the application of the proposed technique to an illustrative example. Finally, the main conclusions are summarised in Section 5.

Notation: **Euclidean norm for a vector $x \in \mathbb{R}^n$ and 2-norm for a matrix $A \in \mathbb{R}^{m\times n}$ will be** **R2-5**

denoted as $\|x\|$ and $\|A\|$, respectively. We will denote as $\mathscr{L}_\infty^n$ the set of all signals $\omega : \mathbb{R}^+ \to \mathbb{R}^n$ such that $\bar{\omega} = \sup_{t\in\mathbb{R}_+} \|\omega(t)\| < +\infty$. The symbols I_n and 1_n denote the identity matrix of order n and the n-dimensional vector with all elements equal to 1, respectively. For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $A_1, A_2 \in \mathbb{R}^{n\times m}$, the relationships $x_1 \leq x_2$ and $A_1 \leq A_2$ are understood elementwise. If the matrix $P \in \mathbb{R}^{n\times n}$ is symmetric, then $P \in \mathbb{S}^{n\times n}$. The relation $P \prec 0$ ($P \succ 0$) means that $P \in \mathbb{S}^{n\times n}$ is negative (positive) definite. A matrix $A \in \mathbb{R}^{n\times n}$ is called Metzler, i.e. $A \in \mathbb{M}^{n\times n}$, if all its elements outside the main diagonal are nonnegative. Given a matrix $A \in \mathbb{R}^{m\times n}$, then $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ (similarly for vectors) and the matrix of absolute values of all elements is denoted by $|A|$.

2. INTERVAL OBSERVER-BASED MODEL REFERENCE CONTROL

Let us consider an uncertain system in state-space form, described by the following equations:

$$\dot{x}(t) = [A_0 + \Delta A(\rho(t))]\, x(t) + Bu(t) + d(t) \tag{1}$$

$$y(t) = Cx(t) + v(t) \tag{2}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, $u \in \mathbb{R}^{n_u}$ is the input vector and $y \in \mathbb{R}^{n_y}$ is the output vector. The matrices $A_0 \in \mathbb{R}^{n_x\times n_x}$, $B \in \mathbb{R}^{n_x\times n_u}$ and $C \in \mathbb{R}^{n_y\times n_x}$ are known, whereas the matrix function ΔA depends on the value taken by a varying parameter $\rho \in \Pi \subset \mathbb{R}^{n_\rho}$ which is not available for measurements, where Π is a known bounded set.

Assumption 1. The matrix function $\Delta A(\rho(t))$ is bounded by $-\overline{\Delta A} \leq \Delta A(\rho(t)) \leq \overline{\Delta A}$ for all $\rho \in \Pi$ with some known $\overline{\Delta A} \in \mathbb{R}^{n_x\times n_x}$, $\overline{\Delta A} \geq 0$. The initial condition $x(0)$ is bounded by $\underline{x}_0 \leq x(0) \leq \bar{x}_0$ for some known $\underline{x}_0, \bar{x}_0 \in \mathbb{R}^{n_x}$. The unknown disturbance $d(t)$ is bounded by $\underline{d}(t) \leq d(t) \leq \overline{d}(t)$, with $\underline{d}, \overline{d} \in \mathscr{L}_\infty^{n_x}$. Finally, the sensor noise is bounded by $|v(t)| \leq V$, with known constant $V > 0$.

Remark 1: The system (1)-(2) has four sources of uncertainty, which are assumed to belong to known bounded sets through Assumption 1: the unknown value of $\Delta A(\rho(t))$, which belongs to the interval $[-\overline{\Delta A}, \overline{\Delta A}]$; the initial condition $x(0)$, which belongs to the interval $[\underline{x_0}, \overline{x_0}]$; the **R3-2**

unknown disturbance $d(t)$, which belongs to the interval $[\underline{d}(t), \overline{d}(t)]$; and the sensor noise $v(t)$, which belongs to the interval $[-V, V]$.

In this paper, the system (1)-(2) is controlled using a model reference controller [32], composed of feedforward and feedback actions. The following reference model is considered for providing the feedforward action:

$$\dot{x}_{ref}(t) = A_0 x_{ref}(t) + B u^c_{ref}(t) \tag{3}$$

$$y_{ref}(t) = C x_{ref}(t) \tag{4}$$

where $x_{ref} \in \mathbb{R}^{n_x}$ is the reference state vector, $u^c_{ref} \in \mathbb{R}^{n_u}$ is the nominal reference input vector and $y_{ref} \in \mathbb{R}^{n_y}$ is the output vector of the reference model.

The reference model gives the trajectories to be followed by the real system. Thus, considering the error, defined as $e(t) \triangleq x_{ref}(t) - x(t)$, the following *error system* is obtained:

$$\dot{e}(t) = [A_0 + \Delta A(\rho(t))]\, e(t) + B\Delta u_c(t) - \Delta A(\rho(t))\, x_{ref}(t) - d(t) \tag{5}$$

$$\varepsilon_c(t) = Ce(t) - v(t) \tag{6}$$

with $\Delta u_c(t) = u^c_{ref}(t) - u(t)$ and $\varepsilon_c(t) = y_{ref}(t) - y(t)$.

Following the ideas developed in [30], the error feedback control law is chosen as:

$$\Delta u_c(t) = \underline{K}\underline{e}(t) + \overline{K}\overline{e}(t) \tag{7}$$

where $\underline{K}, \overline{K} \in \mathbb{R}^{n_u \times n_x}$ are two feedback gains to be designed in order to stabilize (5)-(6), and $\underline{e}, \overline{e} \in \mathbb{R}^{n_x}$ are estimated bounds for the error variable $e(t)$, such that:

$$\underline{e}(t) \leq e(t) \leq \overline{e}(t) \tag{8}$$

In particular, $\underline{e}(t)$ and $\overline{e}(t)$ are provided by the following *interval error observer*:

$$\dot{\underline{e}}(t) = [A_0 - LC]\,\underline{e}(t) + B\Delta u_c(t) + L\varepsilon_c(t) - |L|\, 1_{n_y} V - \overline{d}(t) - \overline{\Delta A}\left|x_{ref}(t)\right| - \phi(t) \tag{9}$$

$$\dot{\overline{e}}(t) = [A_0 - LC]\,\overline{e}(t) + B\Delta u_c(t) + L\varepsilon_c(t) + |L|\, 1_{n_y} V - \underline{d}(t) + \overline{\Delta A}\left|x_{ref}(t)\right| + \phi(t) \tag{10}$$

where $L \in \mathbb{R}^{n_x \times n_y}$ is the observer gain matrix, to be designed in order to ensure nonnegativity of the estimation error dynamics **[33]**, and: **R2-5**

$$\phi(t) = \overline{\Delta A}\left(\overline{e}^+(t) + \underline{e}^-(t)\right) \tag{11}$$

Theorem 1

Let Assumption 1 be satisfied, $\underline{e}(0) = x_{ref}(0) - \bar{x}_0$, $\overline{e}(0) = x_{ref}(0) - \underline{x}_0$, and the observer gain L be chosen such that $A_0 - LC \in \mathbb{M}^{n_x \times n_x}$. Then, the relation (8) is satisfied for (5)-(6) and (9)-(10). In addition, define $\mathscr{A}$ as the set of matrices A_d for which:

$$\begin{bmatrix} 2\overline{\Delta A}\,(\overline{e}^{+} + \underline{e}^{-}) \\ LCe_x \end{bmatrix} = A_d \begin{bmatrix} e_d \\ e_a \end{bmatrix} \tag{12}$$

with $\underline{e} = e_a - 0.5e_d$, $\overline{e} = e_a + 0.5e_d$ and values of e_x such that $-0.5e_d \leq e_x \leq 0.5e_d$. Then, if $x_{ref} \in \mathscr{L}_\infty^{n_x}$ and there exist $P, Q \in \mathbb{S}^{2n_x \times 2n_x}$, $P, Q \succ 0$, $K_d, K_a \in \mathbb{R}^{n_u \times n_x}$ and a constant $\nu > 0$ such that the following matrix inequality is verified:

$$G^T P + PG + \nu P + Q \prec 0 \tag{13}$$

$$G = \begin{bmatrix} A_0 - LC & 0 \\ B_d K_d & A_0 + BK_a \end{bmatrix} + A_d \tag{14}$$

$\forall A_d \in \mathscr{A}$, then $\underline{e}, \overline{e} \in \mathscr{L}_\infty^{n_x}$ if $\underline{K} = 0.5\,[K_a - 2K_d]$ and $\overline{K} = 0.5\,[2K_d + K_a]$.

Proof: Let us consider the dynamics of the interval estimation errors $\underline{\eta}(t) = e(t) - \underline{e}(t)$ and $\overline{\eta}(t) = \overline{e}(t) - e(t)$:

$$\dot{\underline{\eta}}(t) = [A_0 - LC]\,\underline{\eta}(t) + \sum_{i=1}^{4} \underline{w}_i(t) \tag{15}$$

$$\dot{\overline{\eta}}(t) = [A_0 - LC]\,\overline{\eta}(t) + \sum_{i=1}^{4} \overline{w}_i(t) \tag{16}$$

where:

$$\underline{w}_1(t) = \overline{\Delta A}\left(\overline{e}^{+}(t) + \underline{e}^{-}(t)\right) + \Delta A\,(\rho(t))\,e(t) \qquad \overline{w}_1(t) = \overline{\Delta A}\left(\overline{e}^{+}(t) + \underline{e}^{-}(t)\right) - \Delta A\,(\rho(t))\,e(t)$$

$$\underline{w}_2(t) = \overline{\Delta A}\left|x_{ref}(t)\right| - \Delta A\,(\rho(t))\,x_{ref}(t) \qquad \overline{w}_2(t) = \overline{\Delta A}\left|x_{ref}(t)\right| + \Delta A\,(\rho(t))\,x_{ref}(t)$$

$$\underline{w}_3(t) = |L|\,1_{n_y} V + Lv(t) \qquad \overline{w}_3(t) = |L|\,1_{n_y} V - Lv(t)$$

$$\underline{w}_4(t) = \overline{d}(t) - d(t) \qquad \overline{w}_4(t) = d(t) - \underline{d}(t)$$

According to **[33]**, if the gain L is designed such that $A_0 - LC \in \mathbb{M}^{n_x \times n_x}$, then nonnegativity of **R2-5** the signals $\underline{\eta}(t)$ and $\overline{\eta}(t)$, i.e. (8), holds as long as $\underline{w}_i(t) \geq 0$ and $\overline{w}_i(t) \geq 0$ for $i = 1, \ldots, 4$, **which** **R2-6**

can be easily demonstrated to be true by considering that Assumption 1 holds (see [34] for a comparison).

Following [30], let us introduce the new variables $e_d(t) = \overline{e}(t) - \underline{e}(t)$ (the estimated interval length) and $e_a(t) = 0.5\,(\underline{e}(t) + \overline{e}(t))$ (the interval average), as well as $K_d = 0.5\left[\overline{K} - \underline{K}\right]$ and $K_a = \underline{K} + \overline{K}$. Then, taking into account (6), we obtain:

$$\dot{e}_d(t) = [A_0 - LC]\, e_d(t) + \phi_d(t) + \delta_d(t) \tag{17}$$

$$\dot{e}_a(t) = [A_0 - LC + BK_a]\, e_a(t) + BK_d e_d(t) + LCe(t) + \delta_a(t) \tag{18}$$

where:

$$\phi_d(t) = 2\overline{\Delta A}\left(\overline{e}^{+}(t) + \underline{e}^{-}(t)\right) \tag{19}$$

is globally Lipschitz with respect to $\xi(t) = \left[e_d(t)^T, e_a(t)^T\right]^T$, and: **R2-6**

$$\delta_d(t) = 2\,|L|\,1_{n_y} V + \overline{d}(t) - \underline{d}(t) + 2\overline{\Delta A}\left|x_{ref}(t)\right| \tag{20}$$

$$\delta_a(t) = -0.5\left(\underline{d}(t) + \overline{d}(t)\right) - Lv(t) \tag{21}$$

are bounded due to Assumption 1 and the fact that $x_{ref} \in \mathscr{L}_\infty^{n_x}$.

On the other hand, it is straightforward to show that $e(t)$ **can be rewritten as $e_a(t) + e_x(t)$ with** **R2-6** **an unknown $e_x(t)$, but bounded by $-0.5e_d(t) \leq e_x(t) \leq 0.5e_d(t)$, and** (18) **becomes:**

$$\dot{e}_a(t) = [A_0 + BK_a]\, e_a(t) + BK_d e_d(t) + LCe_x(t) + \delta_a(t) \tag{22}$$

Then, taking into account that the existence of the set of matrices $\mathscr{A}$ for which (12) holds is ensured by the fact that $\phi_d(t)$ is Lipschitz [35], the following is true:

$$\dot{\xi}(t) = G(t)\xi(t) + \delta(t) \tag{23}$$

where $\delta(t) = \left[\delta_d(t)^T, \delta_a(t)^T\right]^T$, and $G(t)$ is defined as:

$$G(t) = \begin{bmatrix} A_0 - LC & 0 \\ BK_d & A_0 + BK_a \end{bmatrix} + A_d(t) \tag{24}$$

with $A_d(t)$ unknown, but belonging to the known set $\mathscr{A}$.

By considering a Lyapunov function $V(t) = \xi(t)^T P \xi(t)$, and taking into account that (13) **holds $\forall A_d \in \mathscr{A}$, then the following is obtained:** **R2-6**

$$\dot{V}(t) \leq \nu^{-1} \delta(t)^T P \delta(t) - \xi(t)^T Q \xi(t) \tag{25}$$

that, by input to state stability reasoning [36] means that $e_d, e_a \in \mathscr{L}_\infty^{n_x}$ and, therefore, $\underline{e}, \overline{e} \in \mathscr{L}_\infty^{n_x}$. □

The matrix inequality (13) can be rewritten easily in a way that allows finding the controller gains K_d and K_a (hence, $\underline{K}$ and $\overline{K}$). In fact, by pre- and post-multiplying (13) by P^{-1}, and using the change of variable $Q_p = P^{-1} Q P^{-1}$, (13) is equivalent to:

$$P^{-1} G^T + G P^{-1} + \nu P^{-1} + Q_p \prec 0 \tag{26}$$

For a fixed value of ν, (26) can be brought to an LMI form through an additional change of variable. To this end, let us rewrite G as follows:

$$G = \begin{bmatrix} A_0 - LC & 0 \\ 0 & A_0 \end{bmatrix} + A_d + \begin{bmatrix} 0 \\ B \end{bmatrix} K \tag{27}$$

with:

$$K = \begin{bmatrix} K_d & K_a \end{bmatrix} \tag{28}$$

Then, by considering a convex set containing $\mathscr{A}$, with vertices $A_d^1, \ldots, A_d^N$ **(see [35] for details about the computation of A_d^i, $i = 1, \ldots, N$)**, (26) becomes a set of LMIs (one for each of these vertices) through the change of variable $W = K P^{-1}$, such that a solution can be found using available toolboxes and solvers, e.g. YALMIP [37] and SeDuMi [38], and the controller gains can be easily obtained as $K = WP$. **R1-3**

Remark 2: As proposed in [39], the requirement that $A_0 - LC \in \mathbb{M}^{n_x \times n_x}$ can be relaxed by means of a change of coordinates with a nonsingular transformation matrix that can be found using the results developed in [40] and [41].

3. FAULT TOLERANT CONTROL USING INTERVAL VIRTUAL ACTUATORS

In this paper, actuator losses are considered such that the nominal state equation of the system (1) is changed by the fault appearance, as follows:

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$$\dot{x}(t) = [A_0 + \Delta A(\rho(t))]\, x(t) + B_f(\gamma(t))\, u(t) + d(t) + f(t) \tag{29}$$

where $f \in \mathbb{R}^{n_x}$ denotes the additive fault vector, which may represent effects such as process faults or unexpected biases in the actuators' actions. The multiplicative actuator faults are embedded in the matrix $B_f(\gamma(t))$, as follows:

$$B_f(\gamma(t)) = B\, diag(\gamma_1(t), \ldots, \gamma_{n_u}(t)) \tag{30}$$

where $0 \leq \gamma_i(t) \leq 1$ represents the effectiveness of the i-th actuator, such that the extreme values $\gamma_i = 0$ and $\gamma_i = 1$ represent a total failure of the i-th actuator and the healthy i-th actuator, respectively.

It is assumed that estimations of $f(t)$ and $\gamma(t)$ are available, denoted in the following as $\hat{f}(t)$ and $\hat{\gamma}(t)$, respectively, such that: **R2-2**

$$f(t) = \hat{f}(t) + \Delta f(t) \tag{31}$$

$$B_f(\gamma(t)) = B_f(\hat{\gamma}(t)) + B_f(\Delta\gamma(t)) \tag{32}$$

where $\Delta f(t)$ and $\Delta\gamma(t)$ denote the fault estimation errors for $f(t)$ and $\gamma(t)$, respectively.

Assumption 2. **$\Delta f(t)$ and $B_f(\Delta\gamma(t))$ are bounded by $-\overline{\Delta f}(t) \leq \Delta f(t) \leq \overline{\Delta f}(t)$ and $-\overline{\Delta B_f} \leq B_f(\Delta\gamma(t)) \leq \overline{\Delta B_f}$ with some known $\overline{\Delta f} \in \mathscr{L}_\infty^{n_x}$, $\overline{\Delta B_f} \in \mathbb{R}^{n_x \times n_u}$, $\overline{\Delta f}(t) \geq 0$, $\overline{\Delta B_f} \geq 0$.**

Remark 3: The virtual actuator-based FTC strategy proposed in this paper does not depend on the specific fault estimation technique which is used for obtaining $\hat{f}(t)$ and $\hat{\gamma}(t)$. Notice that bounded-error approaches for achieving a set-membership fault estimation in uncertain systems with unknown inputs, disturbances and noise have been proposed in the recent literature, see e.g. [42], which could be a valid choice for obtaining $\Delta f(t)$ and $\Delta\gamma(t)$.

Due to the fault appearance, if the reference model (3)-(4) is used without applying any fault tolerance mechanism, the error system dynamics would be described by:

$$\dot{e}(t) = [A_0 + \Delta A(\rho(t))]\, e(t) + B\Delta u_c(t) - \Delta A(\rho(t))\, x_{ref}(t) - d(t) + \left[B - B_f(\gamma(t))\right] u(t) - f(t) \tag{33}$$

which means that additional terms $\underline{w}_5(t) = -\overline{w}_5(t) = \left[B - B_f(\gamma(t))\right] u(t) + f(t)$ would appear in the interval estimation error dynamics for $\underline{\eta}(t)$ and $\overline{\eta}(t)$. Since one of these terms

must necessarily be negative (unless $f(t) = \left[B_f(\gamma(t)) - B\right] u(t)$), the proof of the first part of Theorem 1 is invalidated, which means that (8) **could not hold anymore. This fact can potentially destabilize the whole control system, since if** (8) **does not hold, $e(t)$ cannot be rewritten as $e_a(t) + e_x(t)$ with an unknown but bounded $e_x(t)$. Hence, some FTC strategy is needed in order to guarantee the correct operation of the control system.**

In this work, following the ideas developed in [31], the used FTC strategy is based on a reconfiguration of the reference model state equation (3), and the addition of a virtual actuator block. **As explained more in detail by [1], the main idea of the virtual actuator is that instead of adapting the controller to the faulty plant, a reconfiguration block is used to adapt the faulty plant to the nominal controller. This solution tries to apply a minimal change to the control loop, by letting the nominal controller and observer to be unchanged blocks in the control loop. In this way, it is possible to add the fault tolerance property to an already existing control loop. The reconfiguration block is called *virtual actuator* because it acts like the faulty actuators by replacing their effect using the control input of the other actuators appropriately.** A conceptual scheme with the main components and involved signals of the proposed FTC strategy is shown in Fig. 1. It is worth remarking that the proposed FTC strategy does not depend on the specific method employed to perform the fault estimation.

R4-2

At first, the reference model state equation (3) **is slightly modified to take into account the actuator faults, as follows:**

$$\dot{x}_{ref}(t) = A_0 x_{ref}(t) + B_f(\hat{\gamma}(t))\, u_{ref}(t) + \hat{f}(t) \tag{34}$$

where $u_{ref} \in \mathbb{R}^{n_u}$ is the reconfigured reference input vector. Hence, the error system becomes:

$$\dot{e}(t) = [A_0 + \Delta A(\rho(t))]\, e(t) + B_f(\hat{\gamma}(t))\, \Delta u(t) - B_f(\Delta\gamma(t))\, u(t) - \Delta A(\rho(t))\, x_{ref}(t) - d(t) - \Delta f(t) \tag{35}$$

where $\Delta u(t) = u_{ref}(t) - u(t)$.

The reconfiguration of the reference model is useful to bring the error system equation into a form that is suitable for defining the virtual actuator, but still not enough on its own to achieve

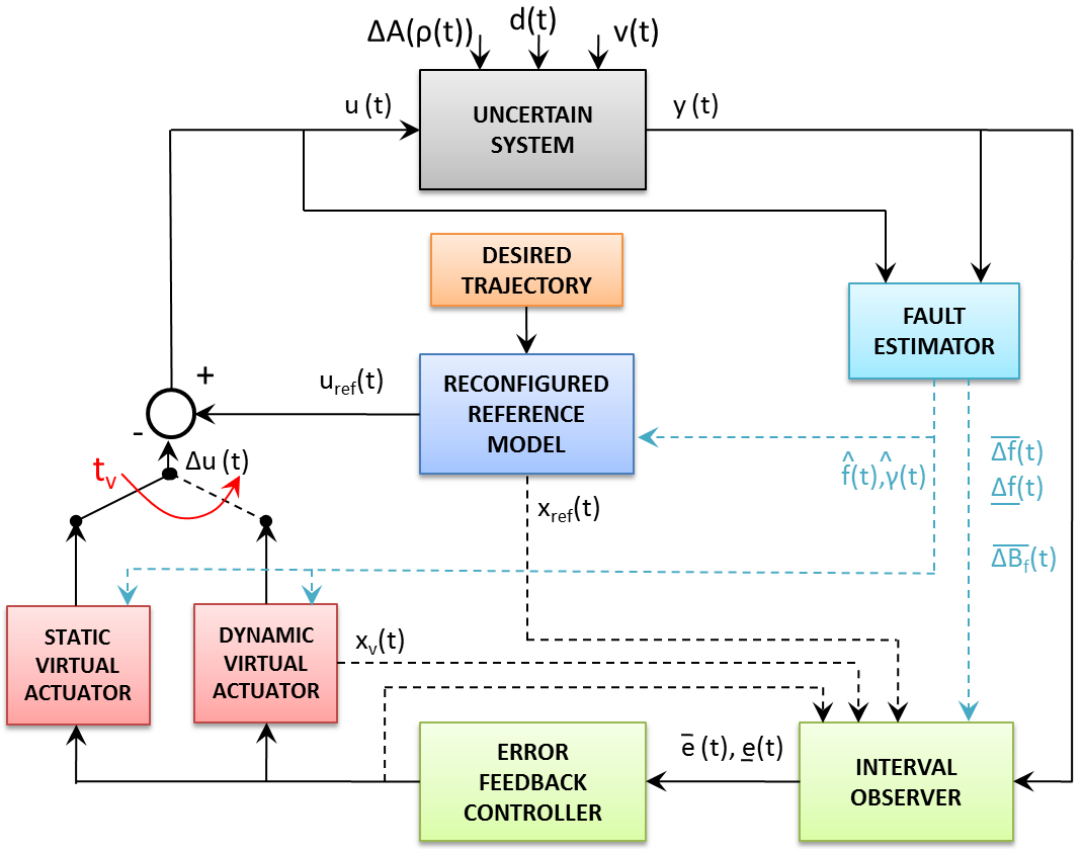

Figure 1. Virtual actuator-based reconfiguration scheme.

fault tolerance, since terms $\underline{w}_5(t) = B_f(\hat{\gamma}(t))\Delta u(t) - B\Delta u_c(t) - B_f(\Delta\gamma(t))u(t) - \Delta f(t)$ **and** $\overline{w}_5(t) = B\Delta u_c(t) + B_f(\Delta\gamma(t))u(t) + \Delta f(t) - B_f(\hat{\gamma}(t))\Delta u(t)$ **would still appear in the dynamics of** $\underline{\eta}(t)$ **and** $\overline{\eta}(t)$**.**

Hence, the concept of virtual actuator described in [1] is applied to the error model (35), by adapting it to an interval formulation. The main objective is to add a reconfiguration block that allows keeping the nominal control law and the nominal interval observer without need for retuning the gains $\underline{K}$, $\overline{K}$ and L.

The virtual actuator can be either a static or a dynamic block, depending on the satisfaction of the following rank condition: **R2-1**

$$rank(B_f(\gamma(t))) = rank(B) \tag{36}$$

In fact, like other FTC strategies, the virtual actuator technique requires redundancy in order to achieve fault tolerance, which is related to the rank condition (36)**. When** (36) **holds, it means that there exists some kind of hardware redundancy with respect to the fault. For example, the fault might have only changed the actuator gain without completely break it; alternatively, there exists a set of healthy actuators that can deliver to the system the same effect as the faulty actuators. In such cases, the reconfiguration structure is static and can be expressed as:**

$$\Delta u(t) = N(\hat{\gamma}(t))\,\Delta u_c(t) \tag{37}$$

where $\Delta u_c(t)$ is the nominal controller output given by (7)**, and the matrix $N(\hat{\gamma}(t))$ is given by:**

$$N(\hat{\gamma}(t)) = B_f(\hat{\gamma}(t))^{\dagger} B \tag{38}$$

where † denotes the pseudo-inverse.

On the other hand, when (36) **does not hold, the only way to achieve fault tolerance is by exploiting the so-called analytical redundancy, i.e. the knowledge about the system's dynamical behavior, which is done through a dynamic reconfiguration structure. These cases should be described through the matrix[†]:**

$$B^* = B_f(\hat{\gamma}(t))\,N(\hat{\gamma}(t)) \tag{39}$$

with a reconfiguration structure expressed by:

$$\Delta u(t) = N(\hat{\gamma}(t))\,(\Delta u_c(t) - M_v x_v(t)) \tag{40}$$

where M_v is the virtual actuator gain and $x_v \in \mathbb{R}^{n_x}$ is the virtual actuator state, which is calculated as:

$$x_v(t) = (A_0 + B^* M_v)\,x_v(t) + (B - B^*)\,\Delta u_c(t) \tag{41}$$

[†]**Notice that the matrix B^* does not depend on $\hat{\gamma}(t)$ because the matrix $N(\hat{\gamma}(t))$ eliminates the effects of partial faults.**

Then, the interval error observer (9)-(10) is also modified by feeding back $\varepsilon_v(t)$ and $\phi_v(t)$ into (9) and (10), respectively, instead of $\varepsilon_c(t)$ and $\phi(t)$, where:

$$\varepsilon_v(t) = \varepsilon_c(t) + Cx_v(t) \tag{42}$$

$$\phi_v(t) = \overline{\Delta A}\left(\overline{e_v}^{+}(t) + \underline{e_v}^{-}(t)\right) + \overline{\Delta f}(t) + \overline{\Delta B_f}\,|u(t)| \tag{43}$$

with $\underline{e_v}(t) = \underline{e}(t) - x_v(t)$ and $\overline{e_v}(t) = \overline{e}(t) - x_v(t)$.

Remark 4: Notice that the static reconfiguration block obtained when the rank condition (36) holds is akin to the idea of virtual control used in control allocation-based FTC. For further details about this technique, the reader is referred to [43, 44, 45] and the references therein.

Remark 5: Since the fault $\gamma(t)$ is unknown, it is not possible to check (36). Hence, the choice between the static and the dynamic reconfiguration structure should be done depending on whether it holds $\forall f \in [\hat{f}(t) - \overline{\Delta f}(t), \hat{f}(t) + \overline{\Delta f}(t)]$. In practice, this can be done by checking (36) using a value of $\gamma(t)$ obtained from $\gamma(t) = 1_{n_u}$ by replacing with 0s the elements corresponding to actuators for which a complete loss is compatible with the above interval.

Assumption 3. When the dynamic reconfiguration structure (40)-(41) becomes active, i.e. at time $t = t_v$, (8) still holds.

It is worth noting that the satisfaction of Assumptions 1-2 until time t_v is a sufficient condition for Assumption 3 to hold. **The above assumption is needed in order to guarantee the theoretical properties stated by the following theorem.** **R4-3**

Theorem 2

Let Assumptions 1-3 be satisfied, and consider the augmented system that includes the error system (35) and output equation (6), the virtual actuator (40)-(41), the error feedback control law (7) and the interval error observer (9)-(10) fed by (42)-(43) instead of $\varepsilon_c(t)$ and $\phi(t)$. Then, if $x_v(t_v) = 0$, the following relation is satisfied:

$$\underline{e_v}(t) \leq e(t) \leq \overline{e_v}(t) \tag{44}$$

In addition, define $\mathscr{A}_v$ as the set of matrices for which:

$$\begin{bmatrix} 2\overline{\Delta A}\left(\overline{e_v}^{+}+\underline{e_v}^{-}\right) \\ LCe_x \\ 0 \end{bmatrix} = A_d^v \begin{bmatrix} e_d \\ e_a \\ x_v \end{bmatrix} \tag{45}$$

with $\underline{e} = e_a - 0.5e_d$, $\overline{e} = e_a + 0.5e_d$ and values of e_x such that $-0.5e_d \leq e_x \leq 0.5e_d$. Then, if $x_{ref} \in \mathscr{L}_\infty^{n_x}$ and there exist $P_v, Q_v \in \mathbb{S}^{3n_x \times 3n_x}$, $P_v, Q_v \succ 0$ and a constant $\nu_v > 0$ such that the following matrix inequality is verified:

$$G_v^T P_v + P_v G_v + \nu_v P_v + Q_v \prec 0 \tag{46}$$

where:

$$G_v = \begin{bmatrix} A_0 - LC & 0 & 0 \\ BK_d & A_0 + BK_a & 0 \\ (B-B^*)K_d & (B-B^*)K_a & A_0 + B^* M_v \end{bmatrix} + A_d^v \tag{47}$$

$\forall A_d^v \in \mathscr{A}_v$, then $\underline{e_v}, \overline{e_v}, x_v \in \mathscr{L}_\infty^{n_x}$ if $\underline{K} = 0.5\left[K_a - 2K_d\right]$ and $\overline{K} = 0.5\left[2K_d + K_a\right]$.

Proof: Let us consider the dynamics of the interval estimation errors $\underline{\eta_v}(t) = e(t) - \underline{e_v}(t)$ and $\overline{\eta_v}(t) = \overline{e_v}(t) - e(t)$:

$$\dot{\underline{\eta_v}}(t) = \left[A_0 - LC\right]\underline{\eta_v}(t) + \sum_{i\in\{1,5,6\}} \underline{w_i^v}(t) + \sum_{i=2}^{4} \underline{w_i}(t) \tag{48}$$

$$\dot{\overline{\eta}}_v(t) = \left[A_0 - LC\right]\overline{\eta_v}(t) + \sum_{i\in\{1,5,6\}} \overline{w_i^v}(t) + \sum_{i=2}^{4} \overline{w_i}(t) \tag{49}$$

where:

$$\underline{w_1^v}(t) = \overline{\Delta A}\left(\overline{e_v}^{+}(t) + \underline{e_v}^{-}(t)\right) + \Delta A\left(\rho(t)\right)e(t) \qquad \overline{w_1^v}(t) = \overline{\Delta A}\left(\overline{e_v}^{+}(t) + \underline{e_v}^{-}(t)\right) - \Delta A\left(\rho(t)\right)e(t)$$

$$\underline{w_5^v}(t) = \overline{\Delta B_f}\,|u(t)| - B_f\left(\Delta\gamma(t)\right)u(t) \qquad \overline{w_5^v}(t) = \overline{\Delta B_f}\,|u(t)| + B_f\left(\Delta\gamma(t)\right)u(t)$$

$$\underline{w_6^v}(t) = \overline{\Delta f}(t) - \Delta f(t) \qquad \overline{w_6^v}(t) = \overline{\Delta f}(t) + \Delta f(t)$$

and $\underline{w_i}(t), \overline{w_i}(t)$, $i = 2,3,4$, are defined as in Section 2. The fact that (44) holds for $t \geq t_v$ follows the same reasoning as in the proof of Theorem 1, taking into account that, since Assumptions 2-3 hold, and due to the choice $x_v(t_v) = 0$, then $\underline{w_i^v}(t_v) \geq 0$ and $\overline{w_i^v}(t_v) \geq 0$, $i = 1,5,6$.

Let us show that the variables $\underline{e_v}$, $\overline{e_v}$ and x_v stay bounded. For this purpose, let us consider the variables $e_d(t) = \overline{e}(t) - \underline{e}(t)$ and $e_a(t) = 0.5\,(\underline{e}(t) + \overline{e}(t))$ which, from (42) and (43) and replacing $\varepsilon_c(t)$ and $\phi(t)$ with (42) and (43), respectively, follow:

$$\dot{e}_d(t) = (A_0 - LC)\, e_d(t) + \phi_d^v(t) + \delta_d(t) \tag{50}$$

$$\dot{e}_a(t) = (A_0 - LC + BK_a)\, e_a(t) + BK_d e_d(t) + LCx_v(t) + LCe(t) + \delta_a(t) \tag{51}$$

where:

$$\phi_d^v(t) = 2\overline{\Delta A}\left(\overline{e_v}^{+}(t) + \underline{e_v}^{-}(t)\right) \tag{52}$$

and $\delta_d(t)$, $\delta_a(t)$ are defined as in (20)-(21).

The inputs $\delta_d(t)$ and $\delta_a(t)$ are bounded due to Assumption 1 and the fact that $x_{ref} \in \mathscr{L}_\infty^{n_x}$, while $\phi_d^v(t)$ is globally Lipschitz. On the other hand, from (44)**, it follows that $e(t)$ can be rewritten as $e_a(t) - x_v(t) + e_x(t)$ with an unknown $e_x(t)$, but bounded by $-0.5e_d(t) \le e_x(t) \le 0.5e_d(t)$, and** (51) **becomes:** **R2-6**

$$\dot{e}_a(t) = (A_0 + BK_a)\, e_a(t) + BK_d e_d(t) + LCe_x(t) + \delta_a(t) \tag{53}$$

Finally, taking into account that $\Delta u_c(t) = K_d e_d(t) + K_a e_a(t)$, (41) becomes:

$$x_v(t) = (A_0 + B^* M_v)\, x_v(t) + (B - B^*)\, K_d e_d(t) + (B - B^*)\, K_a e_a(t) \tag{54}$$

Denote $\xi_v(t) = \left[e_d(t)^T, e_a(t)^T, x_v(t)^T\right]^T$ and $\delta_v(t) = \left[\delta_d(t)^T, \delta_a(t)^T, 0\right]^T$. Then, taking into account that the existence of the set of matrices $\mathscr{A}_v$ for which (45) holds is ensured by the fact that $\phi_d^v(t)$ is Lipschitz [35], the following is true:

$$\dot{\xi}_v(t) = G_v(t)\xi_v(t) + \delta_v(t) \tag{55}$$

where $G_v(t)$ is defined as:

$$G_v(t) = \begin{bmatrix} A_0 - LC & 0 & 0 \\ BK_d & A_0 + BK_a & 0 \\ (B - B^*)\, K_d & (B - B^*)\, K_a & A_0 + B^* M_v \end{bmatrix} + A_d^v(t) \tag{56}$$

with $A_d^v(t)$ unknown, but belonging to the known set $\mathscr{A}_v$. Then, by considering a Lyapunov function $V_v(t) = \xi_v(t)^T P_v \xi_v(t)$ and through input to state stability reasoning [46], it follows that if (46) holds, then $e_d, e_a, x_v \in \mathscr{L}_\infty^{n_x}$ and, therefore, $\underline{e}, \overline{e}, x_v \in \mathscr{L}_\infty^{n_x}$ (see proof of Theorem 1 for comparison). □

By keeping the nominal controller gains $\underline{K}$, $\overline{K}$ designed using Theorem 1, the matrix inequality (46) can be rewritten easily in a way that allows finding the virtual actuator gain M_v. In fact, by pre- and post-multiplying (46) by P_v^{-1}, and using the change of variable $Q_{vp} = P_v^{-1} Q_v P_v^{-1}$, (46) is equivalent to:

$$P_v^{-1} G_v^T + G_v P_v^{-1} + \nu_v P_v^{-1} + Q_{vp} \prec 0 \tag{57}$$

For a fixed value of ν_v, (57) can be brought to an LMI form through an additional change of variable. To this end, let us rewrite G_v as follows:

$$G_v = \begin{bmatrix} A_0 - LC & 0 & 0 \\ BK_d & A_0 + BK_a & 0 \\ (B - B^*) K_d & (B - B^*) K_a & A_0 \end{bmatrix} + A_d^v + \begin{bmatrix} 0 \\ 0 \\ B^* \end{bmatrix} M \tag{58}$$

with:

$$M = \begin{bmatrix} 0 & 0 & M_v \end{bmatrix} \tag{59}$$

Then, by considering a convex set containing $\mathscr{A}_v$, with vertices $A_d^{v,1}, \ldots, A_d^{v,N_v}$ **(see [35] for details about the computation of $A_d^{v,i}$,** $i = 1, \ldots, N_v$**)**, (57) becomes a set of LMIs (one for each of these vertices) through the change of variable $W_v = M P_v^{-1}$, with P_v^{-1} chosen as a block-diagonal matrix variable and W_v defined with the first $2n_x$ columns equal to zero. Then, the controller gains can be obtained as $M = W P_v$. **R1-3**

Remark 6: In general, it is desirable that the modes introduced by the virtual actuator are faster than the dominant modes of the nominal closed-loop system, in order to avoid performance degradation. It is possible to enforce this behaviour by using pole clustering LMI-based techniques, see e.g. [47, 48]. **R3-3**

4. EXAMPLE

Let us consider an uncertain system as in (1)-(2) with:

$$A_0 = \begin{bmatrix} 10 & 2 \\ -0.1 & -10 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\overline{\Delta A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \overline{d} = -\underline{d} = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix} \quad V = 0.01$$

Following the first part of Theorem 1, it is necessary to choose the observer gain L such that $A_0 - LC \in \mathbb{M}^{2\times 2}$. It is easy to check that the choice:

$$L = \begin{bmatrix} 14 \\ -0.1 \end{bmatrix}$$

satisfies this requirement. In order to apply the second part of Theorem 1, and enforce boundedness of the signals $\underline{e}, \overline{e}$, it is necessary to calculate a convex representation for the set $\mathscr{A}$ of matrices A_d for which (12) holds. **In order to do so, let us notice that:** **R1-3**

$$\begin{bmatrix} 2\overline{\Delta A}\left(\overline{e}^{+} + \underline{e}^{-}\right) \\ LCe_x \end{bmatrix} = \begin{bmatrix} 2\left(\overline{e_1}^{+} + \underline{e_1}^{-}\right) \\ 2\left(\overline{e_2}^{+} + \underline{e_2}^{-}\right) \\ 14e_{x,1} \\ -0.1e_{x,1} \end{bmatrix}$$

where $\overline{e_1}$ and $\underline{e_1}$ ($\overline{e_2}$ and $\underline{e_2}$) denote the first (second) element of $\overline{e}$ and $\underline{e}$, respectively, while $e_{x,1}$ denotes the first element of e_x. By taking into account that:

$$2\left(\overline{e_1}^{+} + \underline{e_1}^{-}\right) = \begin{cases} 2\overline{e_1} = e_{d,1} + 2e_{a,1} & if\ \overline{e_1} \geq 0, \underline{e_1} \geq 0 \\ 2\left(\overline{e_1} - \underline{e_1}\right) = 2e_{d,1} & if\ \overline{e_1} \geq 0, \underline{e_1} < 0 \\ -2\underline{e_1} = e_{d,1} - 2e_{a,1} & if\ \overline{e_1} < 0, \underline{e_1} < 0 \end{cases}$$

where $e_{d,1}$ and $e_{a,1}$ are the first elements of e_d and e_a, respectively, the following is true:

$$2\left(\overline{e_1}^{+} + \underline{e_1}^{-}\right) = a_{11}e_{d,1} + a_{13}e_{a,1}$$

with $(a_{11}, a_{13}) \in \{(1,2),(2,0),(1,-2)\}$. By applying the same reasoning to $2\left(\overline{e_2}^{+} + \underline{e_2}^{-}\right)$, and a similar reasoning to LCe_x, it is obtained that the set $\mathscr{A}$ is made up by matrices with the

following structure:

$$A_d^i = \begin{bmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & 0 & a_{24} \\ a_{31} & 0 & 0 & 0 \\ a_{41} & 0 & 0 & 0 \end{bmatrix}$$

with $i = 1,\ldots,N = 18$, obtained by considering all the possible combinations of elements belonging to the following sets: $(a_{11}, a_{13}) \in \{(1,2),(2,0),(1,-2)\}$, $(a_{22}, a_{24}) \in \{(1,2),(2,0),(1,-2)\}$ and $(a_{31}, a_{41}) \in \{(-7,-0.05),(7,0.05)\}$.

By applying Theorem 1, the following nominal controller gains are calculated:

$$K_d = \begin{bmatrix} -0.0014 & 0.0002 \\ 0.0008 & 0.0004 \end{bmatrix} \quad K_a = \begin{bmatrix} -31.0306 & -2.6853 \\ 18.5748 & 0.0076 \end{bmatrix}$$

On the other hand, using (39), the matrices B^* can be calculated for the cases of total loss of the first and the second actuator (B_1^* and B_2^*, respectively):

$$B_1^* = \begin{bmatrix} 0.8 & 1 \\ 1.6 & 2 \end{bmatrix} \quad B_2^* = \begin{bmatrix} 2 & 1.6 \\ 1 & 0.8 \end{bmatrix}$$

In order to apply Theorem 2, the set of matrices $\mathscr{A}_v$ of matrices A_d^v for which (45) holds must be calculated. In this case, $\mathscr{A}_v$ is made up by 18 matrices with the following structure:

$$A_d^v = \begin{bmatrix} a_{11} & 0 & a_{13} & 0 & a_{15} & 0 \\ 0 & a_{22} & 0 & a_{24} & 0 & a_{26} \\ a_{31} & 0 & 0 & 0 & 0 & 0 \\ a_{41} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

obtained by considering all the combinations of elements belonging to the following sets: $(a_{11}, a_{13}, a_{15}) \in \{(1,2,-2),(2,0,0),(1,-2,2)\}$, $(a_{22}, a_{24}, a_{26}) \in \{(1,2,-2),(2,0,0),(1,-2,2)\}$ and $(a_{31}, a_{41}) \in \{(-7,-0.05),(7,0.05)\}$.

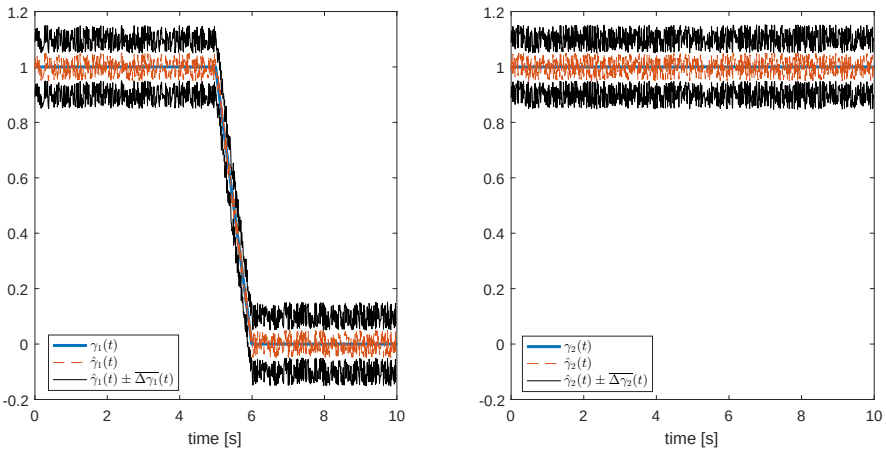

Figure 2. Fault $\gamma(t)$, its estimation $\hat{\gamma}(t)$ and estimated bounds $\hat{\gamma}(t) \pm \Delta\gamma(t)$.

By applying Theorem 2, the virtual actuator gains M_1 and M_2 corresponding to the faults described by B_{f1} and B_{f2}, respectively, are calculated as:

$$M_1 = \begin{bmatrix} -555.0474 & 0.8196 \\ 0 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} -19.9568 & 2.2688 \\ 0 & 0 \end{bmatrix}$$

In order to assess the behaviour of the proposed approach, let us consider a simulation lasting $10\,s$, where the system starts from the initial state $x(0) = [50, 50]^T$ and is required to track a constant reference state chosen as $x_{ref}(t) = [5, -5]^T$. Notice that the initial error is given by $e(0) = x_{ref}(0) - x(0) = [-45, -55]^T$. Let us choose the initial estimated bounds for the error as $\underline{e}(0) = [-100, 100]^T$ and $\overline{e}(0) = [100, 100]^T$. It is assumed that the system is working under nominal conditions up to $t = 5\,s$. Then, after $t = 5\,s$, it is affected by **an incipient fault that degrades linearly the effectiveness of the first actuator, which is completely lost starting from time $t = 6\,s$. The results shown hereafter have been obtained using values of $\Delta A(\rho(t))$, $d(t)$, $v(t)$ which have been changed randomly within their bounds every $0.01\,s$. On the other hand, Fig. 2 shows the fault estimation $\hat{f}(t)$, together with the bounds given by $\hat{f}(t) - \overline{\Delta f}(t)$ and $\hat{f}(t) + \overline{\Delta f}(t)$, used for the simulations.** R1-2 R2-2 R4-1

Figs. 3-4 show the simulation results obtained without applying the proposed FTC strategy. Under nominal conditions, the interval controller stabilizes the system, such that the state tracks the desired reference signal (see Fig. 3) and, at the same time, the tracking error $e(t)$ is contained within the

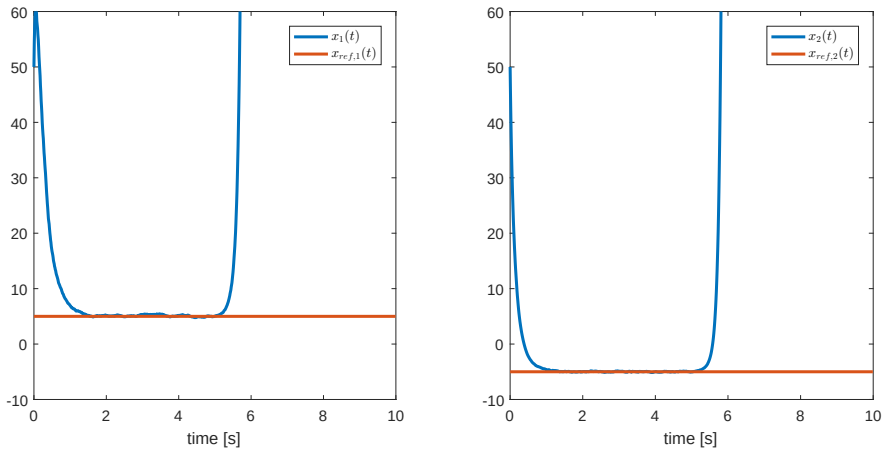

Figure 3. State $x(t)$ and reference state $x_{ref}(t)$ (without FTC).

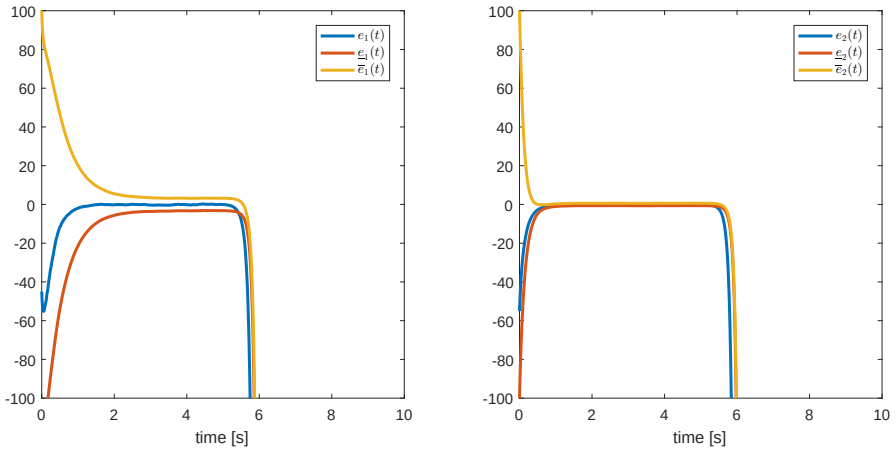

Figure 4. Error $e(t)$ and estimated bounds $\underline{e}(t)$, $\overline{e}(t)$ (without FTC).

estimated bounds $\underline{e}(t)$ and $\overline{e}(t)$ (see Fig. 4). However, due to the fault in the first actuator, the state diverges after the fault occurrence, since the closed-loop system becomes unstable if no fault tolerance mechanism is implemented. Notice also that, as expected from the theory and discussed in Section 3, due to the fault occurrence $e(t)$ exits the interval between the estimated lower and upper bound.

Figs. 5-6 show the simulation results obtained applying the proposed FTC strategy. Notably, at time $t_v = 5.87\,s$, $\hat{f}_1(t) - \overline{\Delta f}_1(t) < 0$, such that the dynamic virtual actuator is activated. Since at time t_v Assumption 3 holds, the theory ensures that the closed-loop system augmented with the

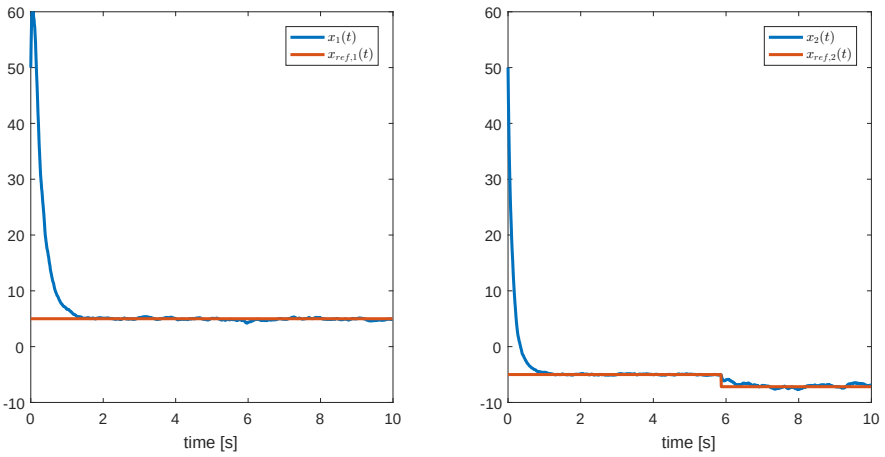

Figure 5. State $x(t)$ and reference state $x_{ref}(t)$ (with FTC).

dynamic virtual actuator will operate correctly. It is worth mentioning that, when the first actuator is lost, the state $[5,-5]^T$ is not anymore a valid equilibrium point for the system. Hence, after t_v, $x_{ref,1}(t) = 5$ is considered, while $x_{ref,2}(t)$ is set to the only value that is compatible with the $x_{ref,1}$ and the condition $\dot{x}_{ref} = 0$. Fig. 5 shows that, thanks to the reconfiguration brought by the virtual actuator, the closed-loop system maintains the tracking stability despite the fault occurrence. Also, Fig. 6 shows that, contrarily to the results without FTC, the tracking error $e(t)$ is always contained within the estimated bounds $\underline{e_v}(t)$ and $\overline{e_v}(t)$ (before t_v, $x_v(t) = 0$, so $\underline{e_v}(t) = \underline{e}(t)$ and $\overline{e_v}(t) = \overline{e}(t)$). Fig. 7 shows the control inputs and in particular the fact that after the activation of the dynamic virtual actuator, the control effort corresponding to the first actuator is completely redistributed on the second actuator (the healthy one).

Finally, in order to complete the assessment of the characteristics of the proposed method, let us compare the model reference control using interval observers with a more traditional model reference control that uses a classical observer, i.e. that does not take into account the possible uncertainty. To this end, the design of the interval controller gains $\underline{K}$ and $\overline{K}$ is done with $\overline{\Delta A} = diag(1.5, 4.5)$ and the simulations are obtained with $\Delta A(\rho(t)) = 0$ (denoted as *nominal*) and $\Delta A(\rho(t)) = diag(1.5, -4.5)$ (denoted as *uncertain*). The results are shown in R3-4

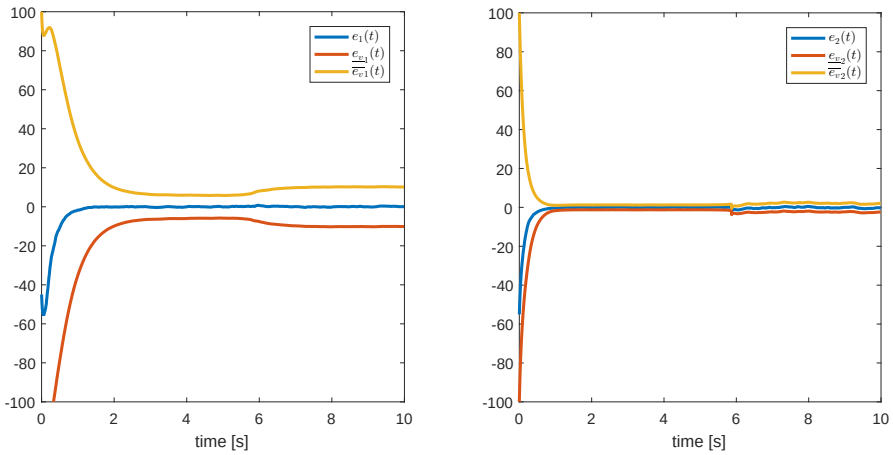

Figure 6. Error $e(t)$ and estimated bounds $\underline{e_v}(t)$, $\overline{e_v}(t)$ (with FTC).

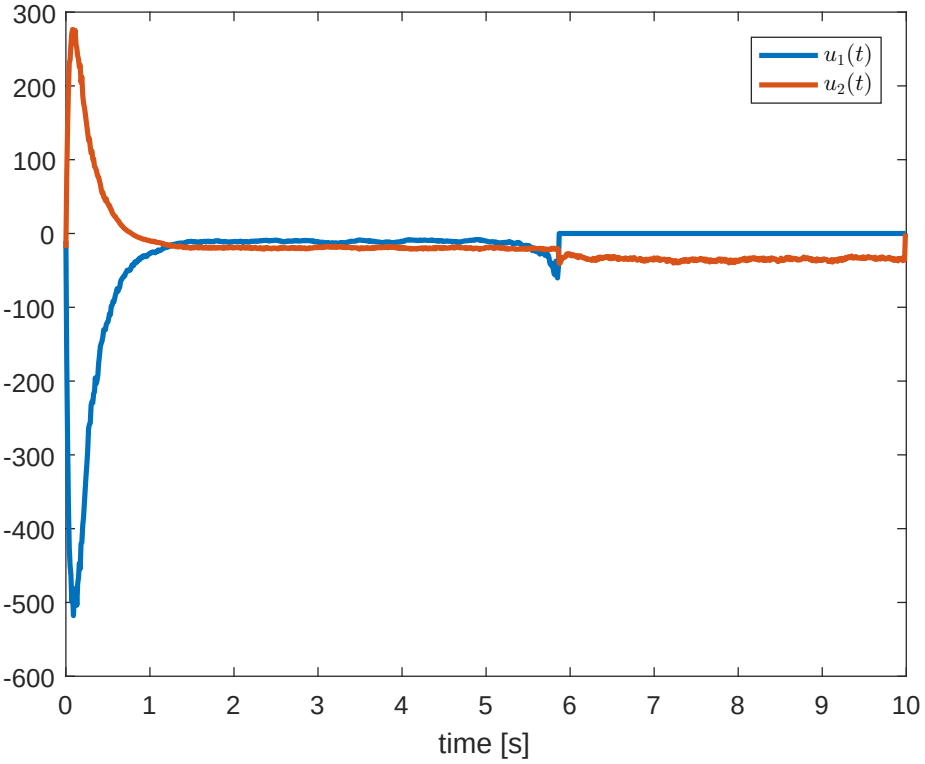

Figure 7. Control inputs $u_1(t)$ and $u_2(t)$ (with FTC activated at $t = 5.05\,s$).

Fig. 8. It can be seen that in absence of uncertainty, both the interval (blue line) and the non-interval approach (red line) guarantee boundedness of the tracking error. On the other hand, when uncertainty affects the system, while the interval approach still assures boundedness of

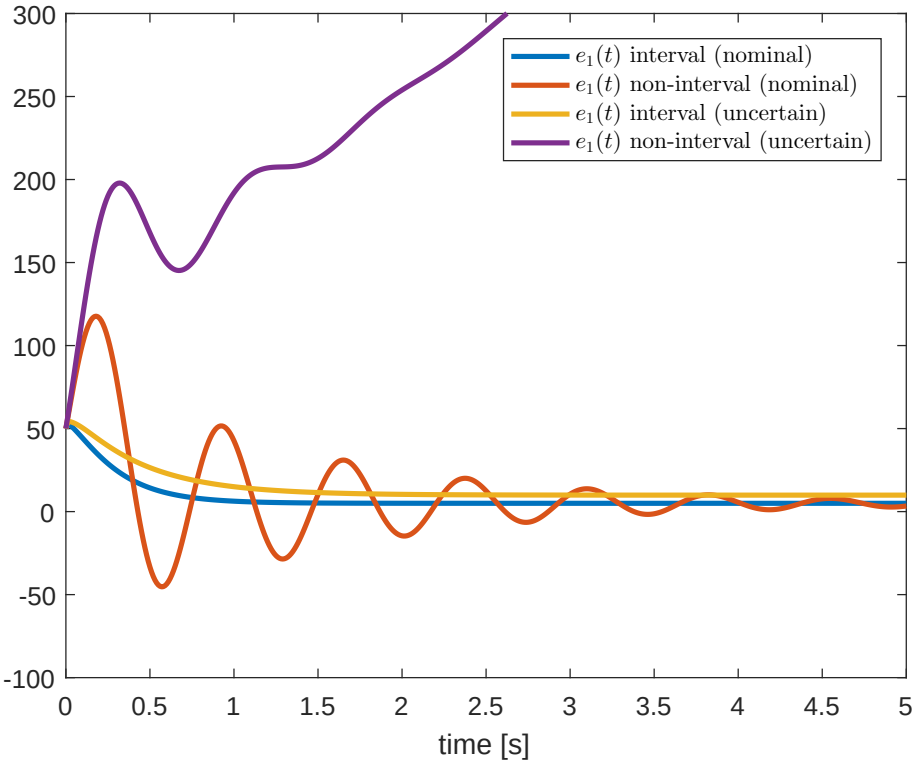

Figure 8. Comparison between the model reference control using interval observers and the model reference control using a classical observer.

the tracking error by design (yellow line), the non-interval approach can potentially lead to undesired phenomena, e.g. instability (purple line).

5. CONCLUSIONS

In this paper, an FTC strategy based on model reference control and virtual actuators has been proposed for uncertain systems. In particular, both the reference model approach and the virtual actuator technique have been extended to an interval formulation in order to ensure cooperativity of the estimation error dynamics and boundedness of the interval signals both in nominal and faulty situation. The conditions that assure the correct operation of the control loop have been provided in terms of Metzler property of the matrix $A - LC$ and of linear matrix inequalities derived using appropriate Lyapunov functions and input-to-state stability reasoning. The potential

and performance of the proposed approach have been demonstrated using an illustrative example, showing promising results.

6. ACKNOWLEDGEMENTS

This work has been supported by the Research Council of Norway through the Centres of Excellence funding scheme (ref. 223254 - AMOS). Damiano Rotondo is also supported by the ERCIM Alain Bensoussan Fellowship programme.

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