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The Homotopy Theory of (∞,1)-Categories

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The homotopy theory of $(\infty, 1)$-categories

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Abstract

The homotopy category of a stable $(\infty, 1)$-category can be endowed with a triangulated structure. The main objective of this thesis is to give a proof of this fact. First it will be discussed some ideas of higher category theory, before $(\infty, 1)$-categories and models of $(\infty, 1)$-categories will be studied. In particular, topological categories and simplicial categories will be mentioned, but the main focus will be on quasi-categories, which all are models for $(\infty, 1)$-categories. The theory of $(\infty, 1)$-categories, which is required in order to define stable $(\infty, 1)$-categories, is then discussed, in particular functors, subcategories, join constructions, undercategories, overcategories, initial objects, terminal objects, limits and colimits are formally discussed for quasi-categories. Finally, the definition of a stable $(\infty, 1)$-category will be discussed. Then the main theorem will be proved, after the required properties of stable $(\infty, 1)$-categories are discussed. Background theory from ordinary categories and simplicial sets are collected in the appendices.

Sammendrag

Homotopikategorien til en stabil $(\infty, 1)$-kategori kan bli gitt en triangulert struktur. Hovedmålet med denne oppgaven er å gi et bevis for dette faktumet. Først vil det bli presentert noen ideer bak høyere kategoriteori, før $(\infty, 1)$-kategorier og modeller for $(\infty, 1)$-kategorier vil bli studert. Spesielt er topologiske kategorier og simplisielle kategorier nevnt, men hovedfokuset er på teorien om kvasikategorier. Teorien som kreves for å definere stabile $(\infty, 1)$-kategorier er deretter diskutert, spesielt er funktorer, underkategorier, join konstruksjoner, kategorier under, kategorier over, initielle objekter, terminelle objekter, grenser og kogrenser formelt diskutert for kvasikategorier. Tilslutt diskuteres definisjonen av en stabil $(\infty, 1)$-kategori sammen med nødvendige egenskaper for å bevise hovedresultatet. Bakgrunnsteori for ordinære kategorier og simplisielle mengder er samlet i appendiksene.

Contents

Preface

This Master Thesis is written under the course code *TMA4900 Mathematics, Master Thesis*, which is the final assessment of the Master program in Applied physics and mathematics, with specialisation in Industrial mathematics. This master project has been supervised by Professor Petter Andreas Bergh and Post-doc Marius Thaule at the Department of Mathematical Sciences, Norwegian University of Science and Technology (NTNU), and involves mainly the fields algebra, algebraic topology and higher category theory. The project was started up by 21^{st} August 2013 and delivered 14^{th} January 2014.

Problem description

The main objective of this thesis is to give a proof of the fact that the homotopy category of a stable $(\infty, 1)$-category is a triangulated category. Moreover, the first part of this thesis is devoted to the theory of $(\infty, 1)$-categories. The second part is devoted to stable $(\infty, 1)$-categories, in particular the homotopy categories of stable $(\infty, 1)$-categories are studied in order to prove the main objective.

Overview

The main theorem in this thesis (the main objective rephrased as a theorem, namely the homotopy category of a stable $(\infty, 1)$-category is a triangulated category) is a well known result to experts in the field. The theorem is presented in for example [Cam13], [Gro10] and [Lur12]. A proof is also presented in [Lur12]. In this thesis there will be discussed enough details from the theory of $(\infty, 1)$-categories and from the theory of stable $(\infty, 1)$-categories in order to understand

and present a proof of the main theorem, and hence respond to the problem description.

More concretely, it will first be sketched some ideas of higher categories and in particular $(\infty,1)$-categories. This discussion of higher categories are mainly based on studies of [Cam13]. In particular, the theory of $(\infty,1)$-categories can conveniently be studied through models. Conceptually, models of $(\infty,1)$-categories can be regarded as formalisations of $(\infty,1)$-categories, some models can be motivated from known examples, that intuitively should capture the structure of $(\infty,1)$-categories. Moreover, in this thesis the models topological categories and simplicial categories are mentioned, but the most of the theory which is discussed in this thesis, is formalised by quasi-categories.

In this thesis, it will be observed that the intuition behind why topological categories and simplicial categories models $(\infty,1)$-categories easily can be motivated from the homotopy hypothesis. But the theory of topological categories, or simplicial categories, has the drawback that many of notions needed in the discussions here may be complicated to describe. An example of such difficulty is limits. However, the theory of quasi-categories has the advantage that many notions from ordinary category theory can intuitively be adopted more or less directly by applying the appropriate structure. In particular for purposes in this thesis, limits and colimits are examples of notions that can be adopted this way. Thus, the theory of quasi-categories is a common theme for this thesis.

The discussion of topological categories and simplicial categories are mainly based on discussions in [Lur09] and [Cam13]. The study of quasi-categories is mainly based on studies of [Cam13], [Gro10] and [Lur09], but also some inspiration is taken from [Joy08]. This note is written by Joyal, which is an authority in the field. Quasi-categories is said (by [Lur09]) to be introduced by Boardman and Vogt under the name weak Kan complexes.

The discussions of the theory of stable $(\infty,1)$-categories are mainly based on studies of [Cam13], [Gro10] and [Lur12]. While the explanations of the theory in [Gro10] have the advantage to be intuitive and well motivated, [Lur12] goes deeper and presents more proofs. When mentioned that some notions in the theory of quasi-categories intuitively can be adopted to ordinary categories, it is referred to the approaches in [Mac98] to for example limits and colimits as universal arrows. Moreover, [Mac98] has also been used for studies of some other notions of ordinary category theory which have been use for in the thesis. Quasi-categories arise from particular simplicial sets. The studies of simplicial sets are mainly based on [GJ09], but the nice and short summary in [Joy08] was also used to supply these studies.

Moreover, the contents of the thesis described above are organised into the following chapters and appendices,

Chapter 1 Here is presented some motivations behind the project and some ideas for higher categories and $(\infty, 1)$-categories.

Chapter 2 This chapter discusses models for $(\infty, 1)$-categories, more concretely topological categories, simplicial categories and quasi-categories are studied, together with some comments why they actually model $(\infty, 1)$-categories, there are also some sketches of comparisons of these models.

Chapter 3 In this chapter some notions in the theory of $(\infty, 1)$-categories are discussed, these notions are formalised by quasi-categories, and they are required to define and understand stable $(\infty, 1)$-categories.

Chapter 4 This chapter discusses the necessary theory of stable $(\infty, 1)$-categories and properties of the homotopy category of a stable $(\infty, 1)$-category, before the main theorem is proved, again the discussion here is formally approached by quasi-categories.

Appendix A Here is presented some definitions and some needed properties for ordinary categories used in the discussions in the thesis.

Appendix B Here is presented some basic theory of simplicial sets mainly in order to understand the definition and properties of quasi-categories.

Summarised, the theory of $(\infty, 1)$-categories in this thesis, or in particular the theory of stable $(\infty, 1)$-categories, are mainly based on and formalised by the theory of quasi-categories. As mentioned, this model for $(\infty, 1)$-categories has the advantage that many required notions, needed for purposes in this thesis, can be adopted intuitively and well-motivated form classical cases. This is the main advantage of this approach to the study of stable $(\infty, 1)$-categories.

Terminology and conventions are mainly explained in the discussions in the thesis. Some definitions with properties are referred to the appendices, often to avoid long technical parts that might disturb the discussions. This is however indicated in the overview of the chapters above.

The theory of $(\infty, 1)$-categories can be approached axiomatically based on works of Toën as indicated [Cam13], instead of approached by models as discussed here. But this will not be paid any attention to in this thesis. This approach to $(\infty, 1)$-categories is however not so relevant for the theory of stable $(\infty, 1)$-categories either. Moreover, since $(\infty, 1)$-categories are conceptually closely related to abstract homotopy theory, in particular to Quillen model categories (see the motivating comments in Section 1.1), the theory of model categories can in some sense be underlying many of the notions discussed in this thesis. But these relationships will however not be discussed here. Nevertheless, the material in this thesis can hopefully give a starting point of further investigations of such underlying subjects. But the thesis can also be a starting point of studies of how stable $(\infty, 1)$-categories can be used in homological algebra and a starting point of further investigations of later chapters in [Lur12].

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Magnus Hellstrøm-Finnsen,
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Chapter 1

Higher categories

The aim for this chapter is to give and discuss some ideas from the higher categorical landscape. First the discussion is about the notion of higher categories in general, or more precise about some basic notions in the higher categorical language, such as, to mention a few, higher morphisms and higher invertible morphisms. The main discussion in this thesis will obviously more concretely be on $(\infty, 1)$-categories, so this chapter has to be read as a preparation in order to get some feeling and some understanding of what the construction of an $(\infty, 1)$-category actually involve. But the first coming is some introductory words consisting of a short motivation and plan for this thesis in addition to the overview given in the Preface.

1.1 Introduction and plan

The notion of an $(\infty, 1)$-category can be thought of as a construction involving objects, morphisms between objects, homotopies between morphisms, etcetera with no upper bound (up to infinity). Homotopies here can be thought of as higher invertible morphisms, and will be discussed later (in Section 1.2.3). But this picture, of an $(\infty, 1)$-category "constructed" this way, is an important sketch to have in mind. This is because many notions in the understanding of $(\infty, 1)$-categories themselves, but also notions in their theory, can be captured intuitively from this picture.

In particular the term 'homotopy' here gives a first psychological clue of what the construction of an $(\infty, 1)$-category involve. In this setting homotopies can be thought of as invertible higher structures connecting lower dimensional homotopies, or connecting morphisms in the 1 dimensional base case. Moreover, the term 'homotopy' give an indication that topological spaces can be involved in the study of $(\infty, 1)$-categories. In fact, a model of $(\infty, 1)$-categories can be obtained by category like constructions with their Hom-objects in topological spaces, namely a category enriched over topological spaces. The reason why this actually can work can be motivated from the homotopy hypothesis (stated in Section 1.2.3). This model for an $(\infty, 1)$-category is called topological categories, and they are defined in the next chapter (in Section 2.1.1).

Similar as in the Preface, at this point it should be remarked that in most practical cases the theory $(\infty, 1)$-categories is studied by models. A model of $(\infty, 1)$-categories can be regarded as a formalisation of $(\infty, 1)$-categories, often obtained from known examples. This can conceptually mean that a model of $(\infty, 1)$-categories should capture the structure of $(\infty, 1)$-categories. Above, topological categories was mentioned to be a model for $(\infty, 1)$-categories, simplicial categories is an other, but in this thesis the study of quasi-categories will be in particular interest.

The advantages of the theory of quasi-categories are that many notions, like limits and colimits, adapt easily and intuitively from classical category theory, which is useful for many purposes in this thesis, as mentioned in the Preface. In particular, the $(\infty, 1)$-categorical notions of initial objects, terminal objects, pullbacks and pushouts are key ingredients in order to define stable $(\infty, 1)$-categories. These promised approaches to required notions in the $(\infty, 1)$-categorical language formalised by quasi-categories are discussed in Chapter 3. The final chapter (Chapter 4) is devoted to the study of stable $(\infty, 1)$-categories, formalised by the established notions in the theory of quasi-categories. Finally in Chapter 4, the overall aim for this thesis is proved, namely that the homotopy category of a stable $(\infty, 1)$-category can be endowed with a triangulated structure (Theorem 4.3.2.4).

Now some underlying motivation will be discussed. One of the motivations behind this project, or a motivation behind studies higher categories overall, is the observation that the rich structures of higher categories capture a lots information. This is information and requirements that else would have to be added for many situations based on ordinary categorical cases. In particular, the understanding of $(\infty, 1)$-categories by a construction consisting of objects, morphisms, homotopies of morphisms, homotopies of these homotopies etcetera, should benefit any situation where some notion of homotopies are involved. Such situations are clearly homotopy theory itself, homological algebra and any theory where Quillen model categories may occur. So an idea is that, in some situations where homotopy theory is involved, for example those situations that were mentioned, these situations can in some cases be studied from properties of $(\infty, 1)$-categories instead of the classical approach by equipping particular ordinary categories with additional data for the homotopical structure.

There is even more to be said about the examples mentioned above. Instead of requiring strict isomorphisms there is required a weaker replacement, namely weak equivalences for model categories and quasi-isomorphisms for homological algebra. This requirement can also be covered by the structure of $(\infty, 1)$-categories by the description that any homotopy (higher morphism) is invertible up to higher homotopies. These weaker notions (than isomorphisms) motivate also for use of the theory of $(\infty, 1)$-categories in other fields, where such weaker notions are frequently used. In particular the field of derived algebraic geometry makes use of $(\infty, 1)$-categories. Derived algebraic geometry can be interpreted as the subject obtained when replacing the meaning of commutative rings in algebraic geometry by commutative differential graded algebras, but concerning about them only up to quasi-isomorphism. So, all these examples can be regarded as a reflection over the same themes, considering a weaker, but homotopically well-behaved, notion than isomorphisms.

Common for the motivating examples discussed so far is that they can be though of as captured by the framework of $(\infty, 1)$-categories, because of the structural picture of higher homotopies sketched for $(\infty, 1)$-categories. In a similar way, triangulated categories can in some sense be thought of as captured by the framework of stable $(\infty, 1)$-categories. Before this perspective is discussed further, it should be mentioned a few words about the homotopy category of an $(\infty, 1)$-category.

The homotopy category of an $(\infty, 1)$-category $\mathscr{C}$ can be though to as a decategorification of $\mathscr{C}$. Conceptually, this can be interpreted as making $\mathscr{C}$ into an ordinary category by strictifying all higher homotopies. This strictifying conceptually means that higher homotopies are turned into isomorphisms. The construction of the homotopy category is formalised by the models for $(\infty, 1)$-categories discussed in this thesis. More details are discussed in Chapter 2.

Although it can be found examples of triangulated categories that do not arises from homotopy categories of stable $(\infty, 1)$-categories, the most natural examples arise this way (as mentioned in [Cam13]). So an idea is that stable $(\infty, 1)$-categories contain the structure of triangulated categories, as mentioned, but stable $(\infty, 1)$-categories are better behaved in many cases, and can sometimes be regarded as a replacement of triangulated categories fixing some of their drawbacks (as mentioned in [Cam13]). Instead of requiring additional data satisfying certain axioms, triangulated structures can possible with this description be viewed as a property of the theory of stable $(\infty, 1)$-categories, which definition is intuitive and well motivated.

The motivation of the studies of stable $(\infty, 1)$-categories can be approached differently, as done in [Lur12]. The formation of the derived category of an abelian category by the usual localisation on quasi isomorphisms may have the drawback that it does not "remember" actually why objects are homotopic to one another. It may be possible to correct this by viewing the derived category as the homotopy category of an underlying $(\infty, 1)$-category. The $(\infty, 1)$-categories that are constructed in this way (in order to satisfy the ideas for derived categories) can argued to have the property of being stable.

The main theorem (Theorem 4.3.2.4) states that the homotopy category of a stable $(\infty, 1)$-category is a triangulated category. Although there will not be a discussion about how the theory of triangulated categories is captured by stable $(\infty, 1)$-categories, the main theorem and its proof can nevertheless be regarded as a first qualified indication of the assertions discussed above may hold in some cases.

In particular, from the proof of the main theorem it can be observed that the octahedron axiom (axiom **(TR4)** in Definition A.2.2.1) follows almost directly by basic properties of stable $(\infty, 1)$-categories. These basic properties can easily be described, which gives an indication of that stable $(\infty, 1)$-categories in fact are "nicer behaved" than triangulated categories, since the most natural examples of triangulated categories arise from homotopy categories of stable $(\infty, 1)$-categories.

These motivating comments can be regarded as underlying motivation for the discussions here. However, the main objective for this thesis is to discuss and establish the required equipments needed in order to understand the statement for and discuss a proof of the main theorem (Theorem 4.3.2.4), and then respond to the Problem Description in the Preface. A short overview of the content in the chapters is also given in the Preface.

1.2 Ideas of higher categories

The aim for this section is to discuss some ideas behind the notion of higher categories. The approach here will be quite conceptual, and can be read as an attempt of motivating for some of the ideas behind the discussions in the next chapters. In particular, the landscape of higher categories will be explored through generalisations of the 2-categorical structure that the category of all (small) categories **Cat** can be equipped with. There will be descriptions of the ideas of higher morphisms and higher homotopies, together with a sketch of some requirements these data have to satisfy in order to give some appropriate meaning for notions of higher categories. Briefly, the intention here is to give some underlying ideas of some of the ingredients in forthcoming discussions in this thesis.

1.2.1 Ideas of higher morphisms

The first example of a higher category is the category of all (small) ordinary categories, **Cat**. Or, more precisely **Cat** can be equipped a 2-categorical structure when considered that **Cat** consists of (small) ordinary categories as objects, functor between categories and natural transformations between functors, together with composition laws that satisfy interchanging and units laws. The two latter ones will be discussed later. First this 2-dimensional structure on **Cat** can be visualised the following way,

- let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be categories,
- let $F_0, F_1, F_2 : \mathscr{A} \to \mathscr{B}$ and $G_0, G_1, G_2 : \mathscr{B} \to \mathscr{C}$ be functors,
- and let $\alpha_0 : F_0 \to F_1$, $\alpha_1 : F_1 \to F_2$, $\beta_0 : G_0 \to G_1$ and $\beta_1 : G_1 \to G_2$ be natural transformations.

These data can be structured into the following diagram,

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left=60, "F_0"] \arrow[r, "F_1" description] \arrow[r, bend right=60, "F_2"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha_0"] \arrow[r, phantom, shift right=3, "\Downarrow\alpha_1"] & \mathscr{B} \arrow[r, bend left=60, "G_0"] \arrow[r, "G_1" description] \arrow[r, bend right=60, "G_2"'] \arrow[r, phantom, shift left=3, "\Downarrow\beta_0"] \arrow[r, phantom, shift right=3, "\Downarrow\beta_1"] & \mathscr{C} \end{tikzcd} .$$

In particular, this additional construction obtained on **Cat** can be equipped with two composition rules. The first rule, called *vertical composition*, is obtained from the usual composition rule of natural transformation, when they are regarded as morphisms in functor categories. Let for now the operation of vertical composition be denoted $\circ$, and observe then that composing $\alpha_1 \circ \alpha_0$ in the diagram sketch above, give a natural transformation $\alpha_1 \circ \alpha_0 : F_0 \to F_2$, whose components are given by $(\alpha_1 \circ \alpha_0)_A = (\alpha_1)_A \circ (\alpha_0)_A : F_0(A) \to F_2(A)$ for any object $A \in \mathscr{A}$. The required associativity relations can easily be checked, details are described in [Mac98].

From the diagram sketch above, when $\alpha_0 : F_0 \to F_1$ and $\beta_0 : G_0 \to G_1$ are natural transformations where the target of F_0 and F_1 coincides with the source of G_0 and G_1, it can be constructed a natural transformation $G_0F_0 \to G_1F_1$ obtained from α_0 and β_0. This composition of α_0 with β_0 is referred to as the *horizontal composition* of α_0 with β_0. For now, let the operation of horizontal composition of natural transformations be denoted $\bullet$. The components of $\beta_0 \bullet \alpha_0$ are given by the diagonals (upper-right or left-bottom composing) in similar diagrams to the following,

$$\begin{tikzcd} G_0(F_0(A)) \arrow[r, "(\beta_0)_{F_0 A}"] \arrow[d, "G_0((\alpha_0)_A)"'] & G_1(F_0(A)) \arrow[d, "G_1((\alpha_0)_A)"] \\ G_0(F_1(A)) \arrow[r, "(\beta_0)_{F_1 A}"'] & G_1(F_1(A)), \end{tikzcd}$$

for all $A \in \mathscr{A}$. These diagrams commute by definition of natural transformations. So for $A \in \mathscr{A}$, the component of $\beta_0 \bullet \alpha_0$ at A can, for example, be chosen to be $(\beta_0 \bullet \alpha_0)_A = ((\beta_0)_{F_1 A}) \circ (G_0((\alpha_0)_A))$. The requirement that horizontal compositions actually give rise to natural transformations can easily be observed, a description can again be found in [Mac98].

In fact, even more is true, the composition rules *interchange* with one another and are *unitary*. The latter means that the vertical composition and the horizontal composition are sharing units, that means the unit for one composition is also a unit for the other. The units are obviously the identity natural transformations. While the interchange law basically means that composing vertically first then horizontally coincides with first composing horizontally then vertically,

$$(\beta_1 \circ \beta_0) \bullet (\alpha_1 \circ \alpha_0) = (\beta_1 \bullet \beta_0) \circ (\beta_0 \bullet \alpha_0). \tag{1.2.1.i}$$

The interchanging law can easily be verified for the 2-dimensional categorical structure on **Cat**, again this verification is indicated in [Mac98]. From the data of categories, functors and natural transformations together with the operations of vertical composition and horizontal composition, which satisfies the required interchanging and unit laws, it can be concluded that **Cat** can be equipped with the structure of a (strict) 2-*category.* This conceptually means that there are notions of vertical composition and horizontal compositions that coincide at the "start points" and "end points" by satisfying the unitary law and interchange law, as for **Cat**.

The term '2-categorical' structure on **Cat** can reflect the following terminology,

- the 0-*morphisms* or the objects are categories
- the 1-*morphisms* are functors connecting the objects, objects are categories
- the 2-*morphisms* are natural transformations connecting 1-morphisms, 1-morphisms are functors

Finally it can also be mentioned that, in this 2-dimensional categorical structure of **Cat**, there are also notions of higher morphisms, 3-morphisms, 4-morphisms etcetera, but all these are regarded as identities in 2-categorical structures. When higher morphisms are so, then they are said to be *trivial*.

The ideas of a 2-category sketched above can be generalised to notions of higher morphisms. The notion of an n-*category*, is a construction that consists of higher morphisms for arbitrary k, but when $k > n$ all higher morphisms are regarded to be trivial. The notion of an n-category will also involve a large amount of operations of compositions that satisfy various appropriate associativity, interchange and unitary laws. The notion of an ∞-*category*[1] reflects that there are no upper bounds from where higher morphisms are trivial.

Moreover, observe now that there were not mentioned any requirements for associative laws for the 2-categorical structure on **Cat**. The reason for that is the indicated strictness property for this 2-categorical structure. In fact any 2-category is equivalent (in some sense) to a strict one ([Cam13]). The ideas of strictness are taken up again later (in Section 1.2.4). But conceptually, strictness properties can be thought of that there are strict equalities between compositions and their candidates of compositions, similar as for ordinary categories. In weaker cases there are weaker notions than equalities, for example homotopies, connecting the composition to a candidate of the composition, but there may be more choices for such candidates. Secondary this weaker notion of compositions have consequences for associativity (composable triples), etcetera.

From this it can be concluded that, when higher categories are thought of this way discussed here, there are a lots of data and requirement to take in account. As mentioned for $(\infty, 1)$-categories, models can be used in order to formalise some ideas of higher categories. Some models of higher categories are obtained from geometric shapes of cells. Morphisms are often in these models represented by cells. The 2-categorical structure on **Cat**, that was visualised a sketched of above, is an example of *globular* shaped cells, because of the geometric shapes of these cells. A globular 2-cell is pictured below,

$$\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

Other shapes of cells are *cubes*, next visualised by a 2-cube,

[1] In [Gro10], [Lur09], [Lur12] the term ∞-category is used for what here will be called a quasi-category, namely, as mentioned, a model for $(\infty, 1)$-categories.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}$$

with 1-cubes as boundary. The cubical shaped cells occur naturally when a model for higher categories is motivated by higher homotopies of topological spaces $X \times [0,1]^n \to Y$. The final shape discussed here is *simplicial* shaped cells,

$$\begin{tikzcd}
& \bullet \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\
\bullet \arrow[ur] \arrow[rr] & {} & \bullet
\end{tikzcd}$$

where the 2-cell visualised above may in some cases be regarded as a 2-morphism assigning a composition to a candidate for the composition. This idea will be taken up again in the discussion of quasi-categories in Section 2.2. There are also other shapes of cells which benefit their situations, some more are listed in [Cam13].

Some other elementary examples of lower dimensional higher categories can be approached as follows. As discussed, **Cat** was the first example of a construction that can intuitively be equipped with a 2-categorical structure. Similarly, any ordinary category is obviously a first example of a 1-category. Any set or any discrete category can be regarded as elementary example of a 0-category.

1.2.2 Inductive interpretation of higher categories

Observe now the following from the example of the 2-categorical structure on **Cat**,

- functors $\mathscr{A} \to \mathscr{B}$ are 1-morphisms in **Cat**, but 0-morphisms (objects) in the functor category $\mathrm{Fun}(\mathscr{A}, \mathscr{B})$
- natural transformations of functors $\mathscr{A} \to \mathscr{B}$ are 1-morphisms in $\mathrm{Fun}(\mathscr{A}, \mathscr{B})$ but 2-morphisms in **Cat**,

so even the collection of 1-morphisms in this 2-categorical structure on **Cat** can be regarded a 1-category itself, namely the category of functors and natural transformations. This observation can indicate that there are inductive ways of thinking about higher categories. As exemplified with **Cat**, 2-categories seem to have their Hom-objects in 1-categories.

These observations can be generalised to an inductive interpretation of the notion of higher categories. In view of enriched categories (Definition A.1.2.1) a strict n-category can be regarded as an enriched category of $(n-1)$-categories. But in order to describe not only the particular cases of strict n-categories, it should be searched for a notion of "weakly enriched". Consequences of these notions for higher categories will be discussed in more details later (in Section 1.2.4).

Now, by following these ideas even further, $(n-1)$-categories can be regarded as weakly enriched over $(n-2)$-categories, and so forth. This iteration terminates at the base cases of 1-categories and 0-categories, which are well known as ordinary categories and sets respectively from discussions previously (Section 1.2.1). So from this, it can be concluded that there are clearly iterative interpretation of higher categories, one of them is the procedure described here.

However, at this point it should be remarked that the notion of ∞-categories can not directly be approached by this inductive interpretation of higher categories. The following problem may occur. In the inductive interpretation an ∞-category can be regarded as "weakly enriched" over $(\infty - 1 = \infty)$-categories, which does not give anything useful.

1.2.3 Higher invertible morphisms and (n, k)-categories

The aim for this part is to discuss the notions of higher invertible morphisms and (n, k)-categories. Conceptually, the notation of an (n, k)-category means an n-category, where all morphisms from level $(k+1)$ and above are invertible in the sense of higher categories, this sense will be discussed in this section. Well, morphisms above level n are clearly invertible, since they are assumed to be trivial, but higher categorical invertibility is a weaker notion.

It can often be convenient to think about higher invertible morphisms in a similar way as homotopies. Conceptually in order to sketch some ideas, a j-morphism α is said to be invertible in the sense of higher categories if there is an other j-morphism β together with invertible (again in the higher categorical sense) $(j+1)$-morphisms connecting compositions of α with β to the suitable identities for this operation of composition. These $(j+1)$-morphisms connecting compositions of α and β to identities are again invertible in the same sense that there are $(j+2)$-morphisms connecting appropriate compositions of these $(j+1)$-morphisms to the suitable identities, etcetera. For (n, k)-categories this notion of invertibility should hold for all j-morphisms, where $k < j \leq n$.

This notion of higher invertible morphisms is not really that mysterious that it might look like at a first sight. Recall that in an ordinary category $\mathscr{C}$ two objects are said to be isomorphic if there is an invertible morphism between them. So passing one level up from objects themselves to morphisms is necessary in order to establish the actual meaning of “two objects that in fact are the same object”.

Moreover, two categories $\mathscr{A}$ and $\mathscr{B}$ in **Cat** are said to equivalent if there are functors $F : \mathscr{A} \rightleftarrows \mathscr{B} : G$ in opposite directions, together with invertible natural transformations, namely natural isomorphisms, which are connecting suitable compositions of the functors with appropriate identities, namely GF with $I_{\mathscr{A}}$ and FG with $I_{\mathscr{B}}$ (where $I_{\mathscr{A}}$ and $I_{\mathscr{B}}$ are the identity functors on $\mathscr{A}$ and $\mathscr{B}$ respectively). This may give a first indication of that the notion of higher invertible morphisms described above is in fact an appropriate generalisation of the notions of isomorphisms and equivalences that can be extracted from ordinary categories and from the 2-categorical structure on **Cat**, respectively.

As indicated, an invertible j-morphism can also be thought of as an invertible morphism “up to homotopy”, namely up to the required invertible $(j+1)$-morphisms which can be thought of as homotopies “deforming” the required compositions of the invertible j-morphisms with its “inverse” to the appropriate identities. Moreover, these homotopoies are again invertible up to higher homotopies etcetera, in fact an invertible j-morphism can be though of as a homotopy itself that connects the appropriate notion of source and target to one another.

So, as mentioned in the introduction, it is often convenient to think about $(\infty, 1)$-categories as a construction consisting of objects, morphisms between objects, homotopies between morphisms, higher homotopies between these homotopies, etcetera, with no upper bound. This interpretation of $(\infty, 1)$-categories can also be regarded as a first generalisation of ordinary categories, by more carefully replacing equalities by homotopies. For example, in $(\infty, 1)$-categories compositions can be required to be connected to candidates of compositions by homotopies, not required to be equal their candidates as for ordinary categories.

Thinking along the same lines, by replacing equality systematically by isomorphism, any set or discrete category can be thought of as formed into an $(\infty, 0)$-category, which frequently will be called a ∞-*groupoid*[2]. Recall that a *groupoid* is often defined to be a category where all morphisms are isomorphisms, then

[2]At this point it should be warned that in [Gro10] and [Lur09] the term ∞-groupoid is defined to be a quasi-category where even all 1-morphisms are invertible, here this notion can be referred to as a “quasi-groupoid”, but the study of these notions will not be discussed that much in this thesis.

the term ∞-groupoid can reflect the construction obtained from a groupoid but where the notion of isomorphisms are replaced by weaker notions of homotopies. Similar, an *n-groupoid* denotes a $(n, 0)$-category.

Now, take up the discussions of the inductive perspective of higher categories from the previous section (Section 1.2.2), in order to use these ideas on (n, k)-categories. From the inductive perspective of higher categories, (n, k)-categories can be thought of as constructions with their mapping objects in $(n-1, k-1)$-categories. Or more precisely, (n, k)-categories can be thought of as (the appropriate sense of) enriched over $(n-1, k-1)$-categories. Similarly, (∞, n)-categories can be thought of having their mapping objects in $(\infty, n-1)$-categories.

While the cases of $(n-1, k-1)$-categories the inductions reduce to the base case of $(n-k, 0)$-categories or $(n-k)$-groupoids, the inductive perspective of (∞, n)-categories reduces to an understanding of $(\infty, 0)$-categories, namely ∞-groupoids, as base case. An understanding of these base cases can be approached by the homotopy hypothesis, which was proposed by Grothendieck, but a version stated in [Cam13] will be used here. The next aim is to discuss this approach.

But first some terminology, recall from algebraic topology that an n-type often denotes a topological space X, whose k-homotopy groups are trivial for all $k > n$ and for all choices of base points, so $\pi_k(X, x) = 0$ for all $k > n$ and for all $x \in X$. Let X be a topological space, the idea of the *fundamental n-groupoid* of X, denoted $\pi_{\leq n}X$, can be interpreted as a higher categorical construction, where objects are points in X, 1-morphisms are continuous paths, 2-morphisms are homotopies of continuous path, 3-morphisms are homotopies of 2-morphisms, etcetera, up to the decision that n-morphisms are said to be equal if they are homotopic with one another. If $n = \infty$, then this means that there is no upper bound for such decisions. The homotopy hypothesis attempts to describe the following connections between fundamental n-groupoids and n-groupoids (defined as $(\infty, 0)$-categories).

The homotopy hypothesis: Any topological space should have a fundamental n-groupoid, $\pi_{\leq n}X$, including $n = \infty$. Any n-groupoid should be equivalent to the fundamental n-groupoid for some topological space X. Furthermore, the theory of n-groupoids should contain the same information as the homotopy theory of n-types. These theories are then said to be equivalent. When $n = \infty$ the 'homotopy theory of n-types' is often called 'homotopy theory' for short.

From the inductive perspective for higher categories, $(\infty, 1)$-categories can be thought of as weakly enriched over ∞-groupoids. Now, the homotopy hypothesis suggests that instead of enriched over ∞-groupoids it may instead be possible to enrich over topological spaces, or simplicial sets whose homotopy theory is known to be equivalent to the homotopy theory of topological spaces. In fact, models of $(\infty, 1)$-categories can be defined this way, since it can be argued that enrichments over ∞-groupoids can be "strictified". These models are called topological categories and simplicial categories respectively, and are discussed further in Section 2.1. In the next section, the discussions about higher categories as weakly enriched categories from Section 1.2.2 will be continued, but now viewed in the light of this section.

1.2.4 Higher categories as "weakly" enriched categories

Recall that any category with finite products can be equipped with a monoidal structure, by taking the tensor product to be the category theoretical product. Such monoidal structures are called *cartesian* monoidal structures.

With motivation from the inductive perspective of higher categories, a *strict* n-category can be defined to be an enriched category over the cartesian monoidal category $\mathbf{StrCat}_{n-1}$. The category $\mathbf{StrCat}_n$ consists of objects all strict n-categories and the morphisms are given by enriched functors. This defines a valid recursion with base cases $\mathbf{StrCat}_1$, which is the 1-category of all categories and functors, namely **Cat**, and $\mathbf{StrCat}_0$, which is the category of sets and functions. The requirement, that there are all finite products in $\mathbf{StrCat}_n$ for $n \geq 0$, is inhered from the usual category theoretic product in the base cases. Hence, $\mathbf{StrCat}_n$ is a cartesian monoidal category as required. This discussion gives an understanding of the notion of strict n-categories (for finite n).

The until now discussed example of a 2-category, namely **Cat**, was mentioned to be an example of a strict 2-category, since it can be regarded as enriched over functor categories with category theoretical products. In fact, any 2-category is equivalent to a strict 2-category. This can argued to be a consequence of MacLane's coherence theorem. This consequence is indicated in some more details in [Cam13].

However, there is at least an example of a 3-category that is not equivalent to a strict one, namely the fundamental 3-groupoid of the 2-sphere $\pi_{\leq 3} S^2$, as discussed in [Cam13]. So, there is at least some need for a notion of weaker enrichments in order to understand higher categories from this perspective, since there are clearly examples of those.

As previously indicated in this chapter, a description of higher categories will often involve various notions of composition laws together with requirements expressed by numerous associativity, interchanging and unitary laws, in order characterise their structure. Sometimes a composition law can be thought of as a certificate or verification that there exists a candidate for a composition of two higher morphisms together with even higher invertible morphisms that are connecting the candidates to the "actual" composition.

These notions can be reflected in an associativity law of a composition in the following way. Let f, g and h be a composable triple by some composition rule of p-morphisms in a certain higher category. Then the expressions $(hg)f$ and $h(gf)$, where the symbol for the composition is omitted, are not claimed to be equal, but the expressions are connected by an invertible $(p+1)$-morphism

$$\alpha : (hg)f \to h(gf), \tag{1.2.4.i}$$

where α often is referred to as an *associator* for this composition.

Now, for any composable quadruple f, g, h and k of p-morphisms, with the same composition rule, there are two different way of relating the compositions $((kh)g)f \to k(h(gf))$, which can be obtained by an argument using appropriate expressions involving the associator and the unitor for this composition. These two ways are displayed as the upper and downer paths in the following diagram,

$$\begin{tikzcd}
((kh)g)f \arrow[r] \arrow[d] & (kh)(gf) \arrow[r] & k(h(gf)) \\
(k(hg))f \arrow[rr] & & k((hg)f). \arrow[u]
\end{tikzcd}$$

Recall that this diagram is similar to the pentagon diagram in the usual definition of monoidal categories. This diagram is required to commute in the monoidal categorical case. But this is not the case for higher categories. In the situation of higher categories the pentagon diagrams commute only up to invertible $(p+2)$-morphisms, called *pentagonators*. A pentagonator should satisfy its own diagram condition up to even higher morphisms, etcetera.

So, form this discussion it can be concluded that the higher categorical notions of operations of compositions, by assigning to compositions candidates of the compositions, should come together with the data of associators, pentagonators, etcetera, at least in the descriptions given here. Requiring all these data, a composition rule is said to be associative up to *coherent homotopy*, and diagrams in higher categories that consider all these data coming with a composition rule are said to be *homotopy coherent* or diagrams are said to commute up to *coherent homotopy*. The notion of homotopy coherent diagrams is the analogous notion to commutative diagrams for ordinary categories.

Describing higher categories this way, by drawing all these diagrams considering all these data, turns quite complicated, when taking care of all structure following with a composition law. According to [Cam13], writing up a definition of (weak) higher categories this way is properly done only up to 3-categories and 4-categories. This gives additional motivation of introducing models of higher categories. Some models for $(\infty, 1)$-categories will be discussed in the next chapter (Chapter 2).

Chapter 2

Models for $(\infty, 1)$-categories

The aim for this chapter is to define topological categories, simplicial categories and quasi-categories, which are models for $(\infty, 1)$-categories. It will also be discussed how homotopy categories for these models are constructed. This is followed up by some ideas around the comparison of simplicial categories and quasi-categories.

The reason why topological categories and simplicial categories model $(\infty, 1)$-categories can intuitively be motivated from the homotopy hypothesis, as indicated in the previous chapter. This discussion will be taken up again in this chapter. There will also be discussed some motivation before defining quasi-categories, with some further words why they model $(\infty, 1)$-categories.

As mention previously, a model describing $(\infty, 1)$-categories can be thought of as a formalisation of the ideas behind $(\infty, 1)$-categories. Although an axiomatic approach to $(\infty, 1)$-categories has been carried out based on works of Toën (as mentioned in [Cam13, Section 4.1]), the study of $(\infty, 1)$-categories in this thesis is carried out through these models mentioned above.

When quasi-categories have their benefit in this thesis, that many notions and constructions from the theory ordinary categories can be adopted intuitively and well-motivated (this will be discussed in Chapter 3), other models for $(\infty, 1)$-categories may have their benefit for other purposes of studies of $(\infty, 1)$-categories. Other examples of models for $(\infty, 1)$-categories are Segal categories, complete Segal spaces and relative categories. Some more are even mentioned in [Cam13].

2.1 Simplicial and topological categories

In this section will topological categories and simplicial categories be defined. Then the notion of homotopy categories of these models for $(\infty, 1)$-categories will be discussed.

2.1.1 Definition of topological categories

First some motivation, recall from the discussion of higher categories from the inductive perspective, as mentioned some idea of in the previous chapter, that it is natural to search for models for $(\infty, 1)$-categories, that involve weakly enrichments over $(\infty, 0)$-categories, namely ∞-groupoids. The homotopy hypothesis suggests that the theory of ∞-groupoids is equivalent to the homotopy theory for topological spaces. So from this motivation it can be expected to search for a model for $(\infty, 1)$-categories defined as a category weakly enriched over topological spaces. In fact, as mentioned in the previous chapter and in [Cam13, Section 3.1], it turns out that enrichments over ∞-groupoids always can be "strictified". Then it seems that a model for $(\infty, 1)$-categories even can be defined as a category enriched over topological spaces. This consequence gives a much nicer description, than first expected, since 'weakly enriched' may not that well-behaved after all.

Remark 2.1.1.1. The category of compact generated weakly Hausdorff topological spaces will be denoted **CG**. Often the category **CG** will be referred to as the *category of topological spaces* or the *category of spaces* for short. The restriction to compactly generated weakly Hausdorff topological spaces has to do with technicalities and safety reasons in homotopy theory, which will not be discussed further in this thesis. But an idea is that enrichment should be over a sufficiently "nice enough" category in order to avoid these technicalities, if possible.

Definition 2.1.1.2. A *topological category* is a category enriched over compactly generated weakly Hausdorff topological spaces, **CG**. The category of topological categories and enriched functors will be denoted **topCat**.

More concretely, a topological category $\mathscr{C}$ consists of a collection of *objects* and for each ordered pair of objects X, Y a (compactly generated weakly Hausdorff topological) *mapping space* $\mathrm{Map}_{\mathscr{C}}(X, Y)$. The operation of composition is given by a continuous map

$$\mathrm{Map}_{\mathscr{C}}(Y, Z) \times \mathrm{Map}_{\mathscr{C}}(X, Y) \to \mathrm{Map}_{\mathscr{C}}(X, Z), \tag{2.1.1.i}$$

where the product taken over (compactly generated weakly Hausdorff) topological spaces.

Conceptually, functors of $(\infty, 1)$-categories are regarded to map "higher homotopies" to "higher homotopies" in order to preserve the $(\infty, 1)$-categorical structure. So, let $F : \mathscr{C} \to \mathscr{D}$ be a functor of topological categories, in the model topological categories the induced map on the mapping spaces

$$F_{X,Y} : \mathrm{Map}_{\mathscr{C}}(X, Y) \to \mathrm{Map}_{\mathscr{D}}(FX, FY), \tag{2.1.1.ii}$$

where X and Y are objects of $\mathscr{C}$, is then required to be a continuous map, since $F_{X,Y}$ is a map of homotopy types from the description of mapping "higher homotopies" to "higher homotopies". So, the definition of **topCat**, with enriched functors, appears to be the correct one, after all.

The next definition denotes some particular functors of topological categories,

Definition 2.1.1.3. A functor $F : \mathscr{C} \to \mathscr{D}$ of topological categories is said to be a *strong equivalence* if F is an equivalence of enriched categories, that is

- for every pair of objects X, Y in $\mathscr{C}$ the induced map

  $$F_{X,Y} : \mathrm{Map}_{\mathscr{C}}(X, Y) \to \mathrm{Map}_{\mathscr{D}}(FX, FY) \tag{2.1.1.iii}$$

  is a homeomorphism.
- and every object in $\mathscr{D}$ is isomorphic (in $\mathscr{D}$) to an object mapped from $\mathscr{C}$, that is an object of the form $F(X)$, where $X \in \mathscr{C}$.

Moreover, this definition can be used for any kind of enriched functor, just by replacing homeomorphism $F_{X,Y}$ with the appropriate notion of isomorphism. Observe now in the situation of ordinary (small) categories, which can be regarded as categories enriched over **Set**, the notion of strong equivalences coincides with the usual notion of equivalences of categories in the sense of ordinary category theory (fully faithful and essentially surjective functor). As the name strong equivalence suggests, there is a notion of "weak equivalences", that denotes functors that behave like homotopy equivalences on the mapping spaces, but this will be defined later.

2.1.2 The homotopy category of a topological category

Recall that the notion of a "homotopy category" in classical cases is a category, whose collection of objects is the same as the original category, while the Hom-sets are equivalence classes of homotopic morphisms in the original category. As already mentioned, the homotopy category of an $(\infty, 1)$-category $\mathscr{C}$ can be thought of as a decategorification of $\mathscr{C}$ determined by "strictifying" higher homotopies.

This can be thought of as identifying those 1-morphisms that is connected by 2-morphisms. Visually, this means for a topological category $\mathscr{C}$ that 1-morphisms in the same path connected components in their mapping space, say $\mathrm{Map}_{\mathscr{C}}(X, Y)$, are identified with one another. From a geometrical interpretation of topological categories, morphisms in the same path connected component are homotopic with one another. This leads to the next definition.

Definition 2.1.2.1. The *homotopy category* of a topological category $\mathscr{C}$ is defined to be a construction, denoted $\mathrm{h}\mathscr{C}$, whose

- collection of objects in $\mathrm{h}\mathscr{C}$ is same as the collection of objects in $\mathscr{C}$.
- the Hom-sets are determined from the path connected components of the mapping spaces, namely

$$\mathrm{Hom}_{\mathrm{h}\mathscr{C}}(X, Y) = \pi_0(\mathrm{Map}_{\mathscr{C}}(X, Y)) \tag{2.1.2.i}$$

  for each pair of objects X, Y in $\mathrm{h}\mathscr{C}$ (or $\mathscr{C}$).
- composition in $\mathrm{h}\mathscr{C}$ is inhered from the composition in $\mathscr{C}$ by applying the functor π_0.

In particular, $\mathrm{h}\mathscr{C}$ is a category in the ordinary sense.

For the composition rule on $\mathrm{h}\mathscr{C}$ in the definition above the following should be remarked. First the functorality of π_0 follows for example from discussions in [Ark11, p. 50]. Second, it can be shown that π_0 preserves products (as stated in [Lur09]) then this gives rise to the composition rule described in the definition.

So, the intuition of associating 1-morphisms in the same path connected component with one another was used to determine this first approach to the homotopy category of a topological category. The aim now is to present an alternative approach based on the theory of CW-complexes.

But first recall that a continuous map $f : X \to Y$ of topological spaces is said to be a *weak homotopy equivalence* if f is carried to a bijection $\pi_0 X \to \pi_0 Y$ of sets and group isomorphisms $\pi_i(X, x) \to \pi_i(Y, f(x))$ for all $i \geq 1$ and for every point $x \in X$. Further, let $\mathscr{C}_{\mathbf{CW}}$ denote the topological category whose objects are CW-complexes and the mapping spaces $\mathrm{Map}_{\mathscr{C}_{\mathbf{CW}}}(X, Y)$ are the set of continuous maps equipped with the (compactly-generated version of the) compact-open topology. The homotopy category of $\mathscr{C}_{\mathbf{CW}}$ will be denoted $\mathscr{H}$ and called the *homotopy category of spaces.*

As stated in [Lur09, Section 1.1.3], one of the versions of the Whitehead theorem implies that any compactly generated topological space can homotopically be related to a CW-complex. In fact, for any compactly generated space

$X \in \mathbf{CG}$ there exists a CW-complex X' and a weak homotopy equivalence $\phi : X' \to X$, relating X to X'.

Passing further to the homotopy category, $\mathscr{C}_{\mathbf{CW}} \to \mathscr{H}$, defines a functor $\theta : \mathbf{CG} \to \mathscr{H}$ that assigns $X \mapsto [X] = X'$, where $X \in \mathbf{CG}$ and X' is the related CW-complex. This construction is well-defined since weak homotopy equivalences in $\mathscr{C}_{\mathbf{CW}}$ are per definition mapped to isomorphisms in $\mathscr{H}$. Furthermore, observe that any topological category $\mathscr{D}$, which is per definition enriched over $\mathbf{CG}$, can be inverted into a $\mathscr{H}$-enriched category, which will be denoted $\mathfrak{h}\mathscr{D}$ and called the *$\mathscr{H}$-enriched homotopy category* of $\mathscr{D}$. This observation relies on the fact that θ preserves products (as stated in [Lur09, Section 1.1.3]), and hence by the theory of enriched categories ([Lur09, Appendix A.1.4]), for each $\mathbf{CG}$-enriched category θ defines an $\mathscr{H}$-enriched category. Moreover, $\mathfrak{h}\mathscr{D}$ can be described as follows,

- the objects of $\mathfrak{h}\mathscr{D}$ are the same as in $\mathscr{D}$,
- the $\mathscr{H}$-enriched mapping spaces of $\mathfrak{h}\mathscr{D}$ are given by

$$\mathrm{Map}_{\mathfrak{h}\mathscr{D}}(X,Y) = [\mathrm{Map}_{\mathscr{D}}(X,Y)], \tag{2.1.2.ii}$$

- the compositions in $\mathfrak{h}\mathscr{D}$ are inhered from the compositions in $\mathscr{D}$ just by applying the functor $\theta : \mathbf{CG} \to \mathscr{H}$.

In particular, let $\mathscr{C}$ be a topological category, the homotopy category $\mathrm{h}\mathscr{C}$ and the $\mathscr{H}$-enriched homotopy category $\mathfrak{h}\mathscr{C}$ are compatible with one another. The collections of objects are the same from definition. In fact, for any topological space X there is a canonical bijection $\pi_0 X \simeq \mathrm{Map}_{\mathscr{H}}(*, [X])$, which relates the path connected components of X with the "$\mathscr{H}$-enriched elements" in $[X]$. This observation applied on mapping spaces gives,

$$\pi_0 \mathrm{Map}_{\mathscr{C}}(X,Y) = \mathrm{Hom}_{\mathrm{h}\mathscr{C}}(X,Y) \simeq \mathrm{Map}_{\mathscr{H}}(*, [\mathrm{Map}_{\mathscr{C}}(X,Y)]), \tag{2.1.2.iii}$$

and hence $\mathrm{h}\mathscr{C}$ can be understand as the underlying category of the $\mathscr{H}$-enriched category $\mathfrak{h}\mathscr{C}$ (as discussed in [Lur09, Remark 1.1.3.5]). In the most of the discussions here there will not be distinguished between these interpretations of the homotopy category of a topological category, both denoted $\mathrm{h}\mathscr{C}$ clear from the situation if the $\mathscr{H}$-enriched structure is required.

Definition 2.1.2.2. A functor $F : \mathscr{C} \to \mathscr{D}$ of topological categories is said to be a *weak equivalence* (or simply an *equivalence* for short) if the induced functor $\mathrm{h}\mathscr{C} \to \mathrm{h}\mathscr{D}$ is an equivalence of $\mathscr{H}$-enriched categories. This means that, F is an equivalence if and only if

- for each pair of objects X, Y in $\mathscr{C}$ the induced map

$$F_{X,Y} : \mathrm{Map}_{\mathscr{C}}(X,Y) \to \mathrm{Map}_{\mathscr{D}}(FX, FY) \tag{2.1.2.iv}$$

is a weak homotopy equivalence of topological spaces,

- every object $Y \in \mathscr{D}$ is isomorphic to an object $FX \cong Y$ for some $X \in \mathscr{C}$.

A morphism $f : X \to Y$ in any topological category $\mathscr{D}$ is called an *equivalence* if the induced morphism in $\mathrm{h}\mathscr{D}$ is an isomorphism.

2.1.3 Simplicial categories

Simplicial sets are important in many interpretations of higher categories. In the previous chapter it was observed that simplicial sets gave rise to a shape of cells. The definition of quasi-categories in the next section builds further on these ideas. But here, the singular complex and the geometric realisation establish a compatibility between the homotopy theories of topological spaces and of simplicial sets (see the expression in Equation 2.1.3.i below). So motivated from the homotopy hypothesis again, it can be looked for a model of $(\infty, 1)$-categories enriched over simplicial sets.

Definition 2.1.3.1. A *simplicial category* is a category enriched over simplicial sets. The category of simplicial categories and simplicial enriched functors will be denoted by **sCat**.

Remark 2.1.3.2. At this point there are some restrictions that have to be remarked. In this thesis simplicial categories are restricted to actually mean *fibrant* simplicial categories, which means that the mapping spaces are restricted to be Kan complexes (Definition B.2.2.3).

The relationship between topological categories and simplicial categories as models for $(\infty, 1)$-categories will now be discussed in some more details than mentioned above. Moreover, in the interpretation of $(\infty, 1)$-categories as a construction consisting of objects, morphisms of objects, homotopies of morphisms etcetera, it can be search for connections between topological categories and simplicial categories preserving all this structure. Conceptually, in a comparison of topological categories with simplicial categories verifying that they are modelling the same higher categorical notion, it should be searched for "equivalence" maps that preserves all homotopical structure. In fact, this can be argued to follow from properties of the adjunction,

$$(||, \mathrm{Sing}) : \mathbf{sSet} \to \mathbf{CG}, \tag{2.1.3.i}$$

where $||$ denotes the *geometric realisation*, while Sing denoted the *singular complex*. An element wise establishment of these functors can be found for example in [Wei94, Chapter 8], while discussions probably more abstractly can be found in [GJ09, Chapter III]. In particular, there will only be discussed some ideas of the properties that $(||, \mathrm{Sing})$ should satisfy here, while some more details are discussed in [Lur09, Section 1.1.4]. In fact, the unit maps

$$S \to \mathrm{Sing}\,|S| \tag{2.1.3.ii}$$

for the adjunction $(||, \mathrm{Sing})$ and the counit maps

$$|\,\mathrm{Sing}\,X| \to X \tag{2.1.3.iii}$$

are weak homotopy equivalences for every simplicial set S and every (compactly generated) topological space X. This is stated in a theorem of Quillen (as mentioned in [Lur09, Section 1.1.4]).

Now, as stated in [Lur09, Section 1.1.4] the functors $||$ and Sing preserve finite products, then they induce enriched categories in the respectively categories **sCat** and **topCat**. In particular, for any simplicial category $\mathscr{C}$ the geometric realisation $||$ indices a topological category $\mathscr{D}$ whose objects are the same as $\mathscr{C}$, but mapping spaces are given by

$$\mathrm{Map}_{\mathscr{D}}(X,Y) = |\,\mathrm{Map}_{\mathscr{C}}(X,Y)| \tag{2.1.3.iv}$$

and the composition is induced from $\mathscr{C}$. Similar comments applies for Sing, which for any topological category induces a simplicial category. Since it can be shown that the unit and counit maps are homotopy equivalences from the comments above, the induced comparison by $||$ and Sing have to be understand as an equivalence between the theory of topological categories and simplicial categories as models for $(\infty,1)$-categories, as suggested in the motivating comments before this section.

Remark 2.1.3.3. Next observe that the category obtained from inverting weak homotopy equivalences in **CG** is equivalent to the category obtained from inverting weak homotopy equivalences in **sSet**, since the unit and counit maps are weak homotopy equivalences. In both cases this category will be denoted $\mathscr{H}$. This means that the homotopy category of a simplicial category can be regarded as a $\mathscr{H}$-enriched category, similar as for topological categories. Moreover, the notions described in Definition 2.1.2.2 can be adapted more or less directly to simplicial categories, so a functor of simplicial categories is said to be a *weak equivalence* if the induced functor between the homotopy categories is an equivalence.

2.2 Quasi-categories

As indicated previously, quasi-categories is a model of $(\infty, 1)$-categories which arises from simplicial sets with particular properties. Although topological categories or simplicial categories give a nice psychological aid when thinking about the ideas behind the notion of $(\infty, 1)$-categories, there can be argued to be some troublesome technicalities when defining notions like limits and other constructions. But as mentioned, many such construction, needed for purposes in this thesis, can be defined intuitively for quasi-categories. However, this is the theme for the next chapter (Chapter 3).

The aim for this section is to give the precise definition of quasi-categories and discuss the notion of a homotopy category for quasi-categories. While some motivation is given here in this section, in the next section (Section 2.3) there will be discussed some more ideas of why quasi-categories actually model $(\infty, 1)$-categories, together with some ideas from the comparison of quasi-categories with simplicial categories.

2.2.1 Definition of quasi-categories

First some motivation will be discussed before the definition of quasi-categories will be stated. There are certain examples, or more precisely classes of examples, that at least should be contained in any interpretation of $(\infty, 1)$-categories, namely

(a) ordinary categories, and

(b) ∞-groupoids.

The class (a) should be contained in any model of $(\infty, 1)$-categories, since higher morphisms in any ordinary categories are regarded to be trivial, hence much stricter than the interpretation of higher morphisms as homotopies in $(\infty, 1)$-categories. The class (b) should also be contained in any model of $(\infty, 1)$-categories, since all higher morphisms in ∞-groupoids can be regarded as invertible, not only them from “level” 2 and above.

These classes (a) and (b) can be shown to correspond to particular subclasses of simplicial sets. So, these subclasses, that correspond to (a) and (b), give a first indication that there probably can be defined a model of $(\infty, 1)$-categories by a subclass of simplicial sets, that contains both subclasses correspond to (a) and (b). The following discussion explores further how ordinary categories and ∞-groupoids correspond to subclasses of simplicial sets.

First the subclass of simplicial sets that corresponds to ∞-groupoids, namely item (b), will be discussed. Recall the homotopy hypothesis suggests that anything that is modelling homotopy types should also be modelling ∞-groupoids. Without topological spaces themselves, a good model of homotopy types are Kan complexes, as claimed in [Cam13]. Now some thoughts and ideas behind this claim. Recall that a simplicial set K is said to be a *Kan complex* if all horns in K admit fillers

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\forall"] \arrow[d, hook] & S. \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

First observe that any 1-simplex (edge) in K has a "left inverse", let $f : X \to Y$ be a 1-simplex in K, then f together with the identity map on X, namely $\mathrm{id}_X = s_0 X$, give the data of a horn $(\bullet, \mathrm{id}_X, f) : \Lambda^2_0 \to K$, which from assumption can be filled by a 2-simplex $\alpha : \Delta^2 \to K$ as illustrated by

$$\begin{tikzcd} & Y \arrow[dr, dashed, "g"] \arrow[d, phantom, "\Downarrow\alpha"] & \\ X \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_X"'] & {} & X. \end{tikzcd}$$

So, the horn filler property ensures the existence of a 2-simplex α in K, with $d_0\alpha = g$. The 2-simplex α can be regarded as connecting the composition $g \circ f$ to id_X. The arrow g can be thought of as an inverse of f, but an inverse up to a higher morphism, which α can be thought of as an example of. In particular, α, as any 2-morphism, is invertible (up to 3-morphism) by applying the similar argument to an appropriate 3-horn, etcetera. The dual argument, namely with the identity on the other side, ensures that there also exists some arrow that can be thought of as a right inverse of f. Similar arguments can conceivable be applied to higher morphisms as well. The impotence of the horn filler property will be prominent in the forthcoming discussions.

Now to the discussion of how ordinary categories, item (a) above, correspond to a subclass of simplicial sets. Recall from Section B.3.2 that the simplicial sets, which arise as nerves of categories, can be characterised by the following result. A simplicial set S is isomorphic to the nerve $N(\mathscr{C})$ of some category $\mathscr{C}$ if and only if all inner horns in S admit unique fillers. This means that for any horn $\Lambda^n_k \to S$ with $n > 0$ and $0 < k < n$, there exits a unique map $\Delta^n \to S$ such that the following diagram commute,

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & S. \\ \Delta^n \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

An intuitive approach to this result is that any category $\mathscr{C}$ always is equipped with a strict composition rule, namely a unique way to obtain an arrow $h = g \circ f$ for each composable pair f, g. This property can be regarded as certified by the unique filler of the horn $(g, \bullet, f) : \Lambda^2_1 \to N(\mathscr{C})$, which in this case is the filler $\gamma : \Delta^2 \to N(\mathscr{C})$ with boundary $\partial\gamma = (g, h, f)$. A proof of this result can be found after [Lur09, Proposition 1.1.2.2].

So an idea, that can be extracted from these motivating comments, is that it can be searched for a model of $(\infty, 1)$-categories that arises from a subclass of simplicial sets containing at least the examples of (a) ordinary categories and (b) ∞-groupoids. In fact it turns out that it works to assume that all horns admit inner fillers, which is the definition of a quasi-category.

Definition 2.2.1.1. A simplicial set X is said to be a *quasi-category* if any inner horn in X admits a filler. That is for any horn $\Lambda^n_k \to X$ with $n > 0$ and $0 < k < n$ there exists a (not necessarily unique) map $\Delta^n \to X$ such that the following diagram commutes in **sSet**,

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X. \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

The term quasi-category is used by Joyal, for example in [Joy08], which also is the terminology used in [Cam13]. In many other texts, like [Gro10], [Lur09] and [Lur12], the term ∞-category is used, while the pioneers Boardman and Vogt used the term weak Kan complex. But the term ∞-category has already been given another meaning here, so the term quasi-category will be used.

Now some immediate comments to the definition of quasi-categories will be discussed. First observe that the classes (a) and (b) from the motivation before the definition clearly are covered by the definition of quasi-categories, by requiring that any inner horn admits a (not necessarily unique) filler.

Next, the terminology from simplicial sets, and Kan complexes, will be used for quasi-categories. Let $\mathscr{C}$ be a quasi-category. An *object* $X \in \mathscr{C}$ is defined to be an element $X \in \mathscr{C}_0 \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta^0, \mathscr{C})$, an *arrow* $f : X \to Y$ in $\mathscr{C}$ is defined to be an element $f \in \mathscr{C}_1 \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta^1, \mathscr{C})$, with source $d_1 f = X$ and target $d_0 f = Y$, while *identity arrows* in $\mathscr{C}$ are given by $s_0 X = \mathrm{id}_X \in \mathscr{C}_1 \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta^1, \mathscr{C})$ for all objects $X \in \mathscr{C}$. All isomorphisms in the terminology here follow from notation and the Yoneda lemma.

The composition rule of arrows can be regarded as an formalisation of the geometric interpretation for simplicial shaped cells in Section 1.2.1. Let $f : X \to Y$ and $g : Y \to Z$ be arrows in a quasi-category $\mathscr{C}$, then this composable pair can be represented by the horn $(g, \bullet, f) : \Lambda^2_1 \to \mathscr{C}$ in $\mathscr{C}$. From definition of quasi-categories there exists a filler $\alpha : \Delta^2 \to \mathscr{C}$ connecting the composition $g \circ f$ with the arrow $d_1\alpha = h : X \to Z$, which can be regarded as a candidate for the composition. But this filler is not unique, as for ordinary categories. However, the collection of all such choices of candidates for a composition $g \circ f$ can be shown to be a contractible Kan complex. So, composition can be regarded as well-defined up to homotopy after all. This discussion continues more formally in Section 2.3.

Before introducing the homotopy category of quasi-categories it is convenient to discuss the notion of the opposite (quasi-)category of a quasi-category. While the opposite (topological) category of a topological category, or opposite (simplicial) category of simplicial category $\mathscr{C}$, is easy to describe, just by "reversing" the mapping spaces, namely $\mathrm{Map}_{\mathscr{C}^{\mathrm{op}}}(X, Y) = \mathrm{Map}_{\mathscr{C}}(Y, X)$, the definition of the *opposite quasi-category*, or *opposite category* for short, of a quasi-category $\mathscr{C}$ is the same as the definition of the opposite of a simplicial set S, as defined in Definition B.1.2.10, namely $S^{\mathrm{op}}(J) = S(J^{\mathrm{op}})$ for any (free category on a) linear quiver J. Let $\mathscr{C}$ be a quasi-category, then $\mathscr{C}^{\mathrm{op}}$ is a quasi-category. In particular, $\mathscr{C}$ is a quasi-category if and only if $\mathscr{C}^{\mathrm{op}}$ is a quasi-category. This follows from the observation that $\mathscr{C}$ admits some filler for the horn $\Lambda^n_k \to \mathscr{C}$ (with $n > 0$ and $0 < k < n$) if and only if $\mathscr{C}^{\mathrm{op}}$ admits some filler for the horn $\Lambda^n_{n-k} \to \mathscr{C}^{\mathrm{op}}$ (as also indicated in [Lur09, Section 1.2.1]).

When introducing some new objects, it is often convenient to introduce the appropriate notions of maps between them. Functors of quasi-categories can be defined as the following.

Definition 2.2.1.2. Let $\mathscr{C}$ and $\mathscr{D}$ be quasi-categories, then a *functor of quasi-categories* $F : \mathscr{C} \to \mathscr{D}$ is a map of simplicial sets.

One of the great advantages working with quasi-categories is that all functors can be described this way. In particular, the collection of all functors of quasi-categories $\mathscr{C}$ and $\mathscr{D}$ is the simplicial mapping space $\mathrm{Hom}_{\mathbf{sSet}}(\mathscr{C}, \mathscr{D})$ (as indicated in [Cam13]), and will often be denoted $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$. In the next section (Section 2.3) it will be said a few words about why this definition capture the appropriate notions of maps of higher morphisms, which is not yet so much discussed. But, the first aim is to establish the notion of the homotopy category of a quasi-category.

2.2.2 The homotopy category of a quasi-category

Recall that a homotopy category of an $(\infty, 1)$-category can be though of as a decategorification, by stricifying higher homotopies. In terms of quasi-categories, this decategorification can be expressed by relating those 1-morphisms that are "homotopic" to one another, namely the 1-morphisms that are connected by a particular kind of 2-simplices, which give rise the notion of homotopies. The approach to the notion of homotopy categories of quasi-categories presented here in this section build on discussions in [Lur09, Section 1.2.3] and [Gro10, Section 1.1].

Now, with these motivating remarks in mind, the next is the definition of homotopic edges in a quasi-category.

Definition 2.2.2.1. Let $\mathscr{C}$ be a quasi-category and let $f : X \to Y$ and $g : X \to Y$ be two arrows in $\mathscr{C}$ having the same source and target, g is said to be *homotopic* with f, written $g \simeq f$, if there is a 2-simplex $\alpha : \Delta^2 \to \mathscr{C}$ visualised by,

$$\begin{tikzcd}
 & Y \arrow[dr, "\mathrm{id}_Y"] \arrow[d, phantom, "\Downarrow\alpha"] & \\
X \arrow[ur, "g"] \arrow[rr, "f"'] & {} & Y
\end{tikzcd}$$

namely with a 2-simplex α satisfying $d_0\alpha = \mathrm{id}_Y$, $d_1\alpha = f$ and $d_2\alpha = g$. In this situation α is said to be a *homotopy* from g to f.

The quasi-categorical interpretation of homotopy defined above is an equivalence relation, which will be observed next.

Proposition 2.2.2.2. Let $\mathscr{C}$ be a quasi-category and let X and Y be vertices in $\mathscr{C}$. Then the homotopy relation defined above is an equivalence relation on edges from X to Y.

Proof: First observe that the relation is reflexive. Let $g : X \to Y$ be an edge, and let α denote the 2-simplex obtained from $\alpha = s_1 g$. Now use the simplicial identities (Definition B.1.2.4) to show that $d_1\alpha = d_1 s_1 g = g$, $d_2\alpha = d_2 s_1 g = g$ and $d_0\alpha = d_0 s_1 g = s_0 d_0 g = s_0 Y = \mathrm{id}_Y$, which proves that α is a homotopy from g to g.

For symmetry, let $f : X \to Y$ and $g : X \to Y$ be edges and let β be a homotopy from f to g. The aim now is to find a homotopy from g to f. Let α now denote a homotopy form f to f and let $\gamma = s_0 \operatorname{id}_Y = s_1 \operatorname{id}_Y = s_2 \operatorname{id}_Y$ denote the homotopy from id_Y to itself. The homotopies β, α and γ give the data to the horn $(\gamma, \bullet, \alpha, \beta) : \Lambda^3_1 \to \mathscr{C}$ which has a filler $\omega : \Delta^3 \to \mathscr{C}$ from assumption of $\mathscr{C}$ being a quasi-category. Denote $\delta = d_1\omega$ which is clearly a homotopy from g to f. This proves symmetry.

For the remaining transitivity relation, let f, g and h be edges from X to Y, and let β again denote a homotopy from f to g and let ϵ denote a homotopy from g to h. Again let γ denote the homotopy from id_Y to it self. The homotopies β, ϵ and γ give the data of a horn $(\gamma, \epsilon, \bullet, \beta) : \Lambda^3_2 \to \mathscr{C}$ which has a filler $\chi : \Delta^3 \to \mathscr{C}$. The transitivity relation follows from the observation that $\zeta = d_2\chi$ is a homotopy from f to h. This proves transitivity and shows that the quasi-categorical notion of homotopy is an equivalence relation. □

Remark 2.2.2.3. Now it should be remarked that instead of in Definition 2.2.2.1 defining the quasi-categorical notion of a homotopy, from the edge $f : X \to Y$ to the edge $g : X \to Y$ in a quasi-category $\mathscr{C}$, as a 2-simplex λ with boundary $\partial\lambda = (\operatorname{id}_Y, g, f)$, the quasi-categorical notion of a homotopy can be defined as 2-simplex that has the identity on the left side, here exemplified by η a homotopy from f to g, with this interpretation of homotopies, can be visualised by

$$\begin{tikzcd} & X \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\eta"] & \\ X \arrow[ur, "\operatorname{id}_X"] \arrow[rr, "g"'] & & Y, \end{tikzcd}$$

which formally means that $d_0\eta = f$, $d_1\eta = g$ and $d_2\eta = \operatorname{id}_X$. An equivalent statement can be phrased as, if α is a homotopy from f to g then it follows that there is a 2-simplex η in $\mathscr{C}$ as visualised by the diagram above. If so this will also prove that homotopic edges in $\mathscr{C}$ remain homotopic when passing to the opposite category $\mathscr{C}^{\mathrm{op}}$, since passing to the opposite category reversing the arrows in terms of reflecting the indices.

Proof: Let θ denote the homotopy from f to itself, let $\kappa = s_0 f$ an let λ be as in the remark. Since $d_1 s_0 f = \operatorname{id} f = f$, these data defines a horn $(\theta, \lambda, \bullet, \kappa) : \Lambda^3_2 \to \mathscr{C}$ in $\mathscr{C}$, which has a filler $\upsilon : \Delta^3 \to \mathscr{C}$ with $d_2\upsilon = \eta$, as in the statement in the remark. □

In order to define the "homotopy category", denoted $h\mathscr{C}$, of a quasi-category $\mathscr{C}$, there is need for a well-defined notion of compositions. In particular so far define $h\mathscr{C}$ as follows

- the class of objects in $h\mathscr{C}$ which is the same as the set of objects in $\mathscr{C}$,
- the set morphisms $\mathrm{Hom}_{h\mathscr{C}}(X,Y)$ for each pair of objects X and Y, defined to be the set of equivalence classes of homotopic edges $X \to Y$.

Moreover, for a given edge $f : X \to Y$ in $\mathscr{C}$ let the equivalence class of edges homotopic to f be denoted $[f]$, and let $g : Y \to Z$ be another edge in $\mathscr{C}$. The composition law in $h\mathscr{C}$ is induced from the compositions in $\mathscr{C}$ as follows. Let $\sigma : \Delta^2 \to \mathscr{C}$ denote the filler of the horn $(g, \bullet, f) : \Lambda^2_1 \to \mathscr{C}$, then the suggested composition of homotopy classes is defined to be $[g] \circ [f] = [d_1\sigma]$. This will be referred to as the *composition law* in $h\mathscr{C}$. Clearly, there are several well-defined issues concerning this definition, which will now be taken into consideration.

Proposition 2.2.2.4. Let $\mathscr{C}$ be a quasi-category with edges $f : X \to Y$ and $g : Y \to Z$. Then the composition rule $[g] \circ [f] = [d_1\sigma]$ defined above is independent of choice of representative for the equivalence classes $[f]$, $[g]$ and $[d_1\sigma]$.

Proof: First let σ and σ' both be fillers of the horn $(g, \bullet, f) : \Lambda^2_1 \to \mathscr{C}$ with $d_1\sigma = h$ and $d_1\sigma' = h'$, and let π be the homotopy from g to it self, then σ, σ' and π give the data of a horn $(\pi, \bullet, \sigma', \sigma) : \Lambda^3_1 \to \mathscr{C}$ which admits a filler $\mu : \Delta^3 \to \mathscr{C}$. The 2-simplex $d_2\mu$ is then a homotopy from h to h', which proves that all choices of candidates for the composition are homotopic to one another.

Next choose representatives $f, f' \in [f]$ and let ν denote the homotopy from f to f', while $\xi : \Delta^2 \to \mathscr{C}$ denote a filler of horn $(g, \bullet, f) : \Lambda^2_1 \to \mathscr{C}$ and denote $\rho = s_0 g$, then these 2-simplices give the data of a horn $(\rho, \bullet, \xi, \nu) : \Lambda^3_1 \to \mathscr{C}$ and let $\tau : \Delta^3 \to \mathscr{C}$ denote a filler of this horn. Further, denote $\xi' = d_1\tau$, then $d_2\xi' = f'$, $d_0\xi' = g$ and $d_1\xi' = h$ which proves independence of representative for the homotopy class $[f]$. Moreover, the dual argument verifies independence of representative for $[g]$, then the proposition is proved. □

The last verifications needed, in order to prove that $h\mathscr{C}$ defines an ordinary category after all, are existence of identities and associativity of compositions.

Proposition 2.2.2.5. Let $\mathscr{C}$ be a quasi-category, then the operation of composition in $h\mathscr{C}$ is associative and has two-sided units.

Proof: First let $f : W \to X$ be a 1-simplex in $\mathscr{C}$, it follows directly from the composition rule that $[\mathrm{id}_X] \circ [f] = [f]$, since the 2-simplex $s_1 f$ is a filler of the horn $(\mathrm{id}_X, \bullet, f) : \Lambda^2_1 \to \mathscr{C}$. This proves that $[\mathrm{id}_X]$ is a left unit of $[f]$. The equivalence class $[\mathrm{id}_X]$ is also a right unit of equivalence classes $[g]$ represented by edges $g : X \to Y$ in $\mathscr{C}$, this follows by applying the dual argument. This proves the unit law for the composition.

Moreover let f and g be as above and $h : Y \to Z$ be a third edge in $\mathscr{C}$. Define the following

(a) let $\alpha : \Delta^2 \to \mathscr{C}$ denote a filler of the horn $(g, \bullet, f) : \Lambda^2_1 \to \mathscr{C}$, denote $d_1\alpha = i$

(b) let $\beta : \Delta^2 \to \mathscr{C}$ denote a filler of the horn $(h, \bullet, g) : \Lambda^2_1 \to \mathscr{C}$, denote $d_1\beta = j$

(c) let $\gamma : \Delta^2 \to \mathscr{C}$ denote a filler of the horn $(h, \bullet, i) : \Lambda^2_1 \to \mathscr{C}$, denote $d_1\gamma = k$

These data define the horn $(\beta, \gamma, \bullet, \alpha) : \Lambda^3_2 \to \mathscr{C}$ which admits a filler $\omega : \Delta^3 \to \mathscr{C}$. In particular, denote $\delta = d_2\omega$. Now observe $d_0\delta = j$, $d_1\delta = k$ and $d_2\delta = f$, which proves $[k] = [j] \circ [f] = ([h] \circ [g]) \circ [f]$ where the last bracket follows from (b) above. Recall from (c) and (a) that $[k] = [h] \circ [i] = [h] \circ ([g] \circ [f])$. Hence the composition rule on $h\mathscr{C}$ is associative. $\square$

From all this it can be concluded that $h\mathscr{C}$ is a category in the ordinary sense.

Definition 2.2.2.6. Let $\mathscr{C}$ be a quasi-category. Then $h\mathscr{C}$ as defined above is called the *homotopy category* of $\mathscr{C}$.

Moreover, some terminology from topological categories and simplicial categories can be adopted to quasi-categories. An arrow $f : X \to Y$ in a quasi-category $\mathscr{C}$ is said to be an *equivalence* if $[f] : X \to Y$ is an isomorphism in $h\mathscr{C}$.

2.3 Model for $(\infty, 1)$-categories

Essentially all properties needed in order to determine the homotopy category of a quasi-category followed from the inner horn filler property in the previous section. The aim for this section is to discuss the inner horn filler property even further in order to sketch some additional ideas of why quasi-categories actually model $(\infty, 1)$-categories. Moreover, like in many other mathematical disciplines, in order to understand the theory of quasi-categories themselves it is important to understand properties of the functors between them. Therefore a discussion of functors will be merged with the discussions of higher morphisms, mapping spaces and homotopy coherent diagrams in this section.

2.3.1 Mapping spaces and higher morphisms

Recall that $(\infty, 1)$-categories conveniently can be thought of as a construction consisting of objects, morphisms between objects, homotopies between morphisms, homotopies between these, etcetera, with no upper bound. So, there are clearly notions of morphisms of arbitrary dimensions, which from level 2 and above are invertible, and such invertible morphisms can be regarded as homotopies. When quasi-categories model $(\infty, 1)$-categories, there can be searched for how these notions are covered by the definition of quasi-categories. The aim for this section is to explore some of these ideas. The discussion here is mainly inspired by the approach in [Gro10, Section 1.1], but also by [Cam13, Section 3.2] and [Lur09, Section 1.2.2].

But first, the following attempts to structuring the ideas sketched in the previous paragraph. This way of thinking about higher morphisms may be convenient in this section. Higher morphisms in an $(\infty, 1)$-category can be regarded as homotopies of lower dimensional morphisms. In particular, n-morphisms can be regarded as $(n-1)$-homotopies of $(n-1)$-morphisms, these $(n-1)$-morphisms can again be regarded as $(n-2)$-homotopies of $(n-2)$-morphisms, for $n \geq 2$, 1-morphisms are not assumed invertible.

The first aim is to give an explanation of the notion of higher morphisms in a quasi-category $\mathscr{C}$. Here it will be appealed to a globular understanding of higher morphisms, mainly in order to give some ideas and illustrations. Some ideas and an illustration of globular shaped cells can be recalled from the short mentioning in Section 1.2.1. The discussion here does not give any formal details, but the main objective of the discussion is to give some ideas.

The aim now is to develop some terminology, which can help to make the notions discussed so far in this section more precise. Let $\Delta^{\{i_0, i_1, \ldots, i_k\}} \subseteq \Delta^n$ denote the k dimensional subsimplex of Δ^n spanned by the vertices $i_0, i_1, \ldots, i_k$ of Δ^n. For example $\Delta^{\{0,2,3\}} \subseteq \Delta^3$ denote the 2 dimensional subsimplex illustrated by

$$\begin{tikzcd} & 2 \arrow[dr] \arrow[d, Rightarrow] & \\ 0 \arrow[ur] \arrow[rr] & {} & 3, \end{tikzcd}$$

namely the front simplex (opposite of 1) in the following illustration of Δ^3,

$$\begin{tikzcd} & 1 \arrow[dr] \arrow[dd, dash] & \\ 0 \arrow[ur] \arrow[rr] \arrow[dr] & & 2 \arrow[dl] \\ & 3. & \end{tikzcd}$$

Moreover, for any simplicial set S with some vertex X, let $\mathrm{cons}_X^n : \Delta^n \to S$ denote the constant n-simplex on X, which means that any vertices of cons_X^n is equal X, any edge is equal id_X, etcetera. For example, the simplex $\Delta^2 \to S$ visualised by

$$\begin{tikzcd} & X \arrow[dr, "\mathrm{id}_X"] \arrow[d, phantom, "\Downarrow"] & \\ X \arrow[ur, "\mathrm{id}_X"] \arrow[rr, "\mathrm{id}_X"'] & & X, \end{tikzcd}$$

attempts to illustrate cons_X^2.

Next, some familiar examples will be discussed. First recall and observe that any edge $f : X \to Y$ in a quasi-category $\mathscr{C}$ is given by a 1-simplex $f : \Delta^1 \to \mathscr{C}$ with $f|_{\Delta^{\{0\}}} = X$ and $f|_{\Delta^{\{1\}}} = Y$. Secondly, let $f, g : X \rightrightarrows Y$ be parallel edges in $\mathscr{C}$ then observe that a (1-)homotopy α of from f to g (if such exists) is defined to be a 2-simplex $\alpha : \Delta^2 \to \mathscr{C}$ with $\alpha|_{\Delta^{\{0\}}} = X$ and $\alpha|_{\Delta^{\{1,2\}}} = \mathrm{cons}_Y^1$. Homotopies of parallel edges, such as α, can be regarded as 2-morphisms from X to Y in this interpretation of quasi-categories as models for $(\infty, 1)$-categories.

The idea now is to generalise these established observations for homotopies in order to obtain the appropriate notions of higher morphisms, and higher homotopies. In particular, taking up the discussion from the introductory words of this section, the notion of an n-morphism from X to Y, which should coincide with notion of a $(n-1)$-homotopy, can be understand as an n-simplex $\omega : \Delta^n \to \mathscr{C}$ with $\omega|_{\Delta^{\{0\}}} = X$ and $\omega|_{\Delta^{\{1,2,\ldots,n\}}} = \mathrm{cons}_Y^{n-1}$. Then ω can be regarded as an $(n-1)$-homotopy of $(n-1)$-morphisms between X and Y, by thinking in terms of globular cells.

In order to illustrate these ideas by a concrete example, next is the case of $n = 3$ discussed. Let $\tau : \Delta^3 \to \mathscr{C}$ illustrate the notion of a 3-morphism between the edges X and Y, then by the discussion above $\tau|_{\Delta^{\{0\}}} = X$ while $\tau|_{\Delta^{\{1,2,3\}}} = \mathrm{cons}_Y^2$. Then τ can be visualised by,

$$\begin{tikzcd} & Y \arrow[dr, "\mathrm{id}_Y"] \arrow[dd, "\mathrm{id}_Y"'] & \\ X \arrow[ur] \arrow[rr] \arrow[dr] & & Y \arrow[dl, "\mathrm{id}_Y"] \\ & Y. & \end{tikzcd}$$

Furthermore, denote $d_3\tau = \alpha$, $d_2\tau = \gamma$ and $d_1\tau = \beta$. Observe then that α, β and γ all are 1-homotopies and hence they can be regarded as 2-morphisms from X to Y. In particular, τ can be regarded as a 2-homotopy from α to β.

Moreover, give name to the edges such that the 2-simplices above determine the following,

$$\alpha : f \simeq g, \qquad \beta : g \simeq h, \qquad \gamma : f \simeq h. \tag{2.3.1.i}$$

Now with α, β and cons^2_Y given a priori, τ is clearly a filler of the horn $(\mathrm{cons}^2_Y, \beta, \bullet, \alpha) : \Lambda^3_2 \to \mathscr{C}$. This was in fact the argument that was used to verify the transitivity of the homotopy relation from the proof of Proposition 2.2.2.2 in the previous section. But now in the discussion here, τ can also be regarded as a 2-homotopy assigning the "vertical" composition of α with β to a candidate of this "vertical" composition, since τ is as a filler of the horn $(\mathrm{cons}^2_Y, \beta, \bullet, \alpha) : \Lambda^3_2 \to \mathscr{C}$. Here let this vertical composition can probably be denoted $\beta \bullet \alpha \approx \gamma$. From this the higher inner horn filler property can also be regarded as a certificate for this notion of composition.

The notion of higher homotopies, described above, should be invertible, since quasi-categories are models for $(\infty, 1)$-categories, where higher morphisms are assumed to be invertible. For 1-homotopies, which correspond to 2-morphisms, this can be indicated by the verification of that the homotopy relation on quasi-categories is symmetric (from the proof of Proposition 2.2.2.2). Hence, for a given 1-homotopy, say $\alpha : f \simeq g$ as above, a filler $\upsilon : \Delta^3 \to \mathscr{C}$ of the horn $(\mathrm{cons}^2_Y, \bullet, \kappa, \alpha) : \Lambda^3_1 \to \mathscr{C}$, where κ denotes the homotopy from f to itself, verifies the existence of a 1-homotopy denoted $\epsilon = d_1\upsilon$, that can be regarded as a candidate for the "vertical" composition $\epsilon \bullet \alpha \approx \kappa$. It is more or less clear that $\kappa : f \simeq f$ can be regarded as an identity 2-morphism for this vertical composition, since $\alpha \bullet \kappa \approx \alpha$ (now for arbitrary $\alpha : f \simeq g$) and $\kappa \bullet \epsilon \approx \epsilon$ (now for arbitrary $\epsilon : g \simeq f$). Since κ can be regarded as an identity, the observation $\epsilon \bullet \alpha \approx \kappa$ indicates that 1-homotopies in quasi-categories are invertible in this interpretation.

The notion of higher homotopies can also probably be argued to be invertible by applying an appropriate horn filler generalising the previous discussion of 1-homotopies. However, this can be approached by properties of the collection of all n-morphisms for $n \geq 1$ between fixed vertices. This notion will corresponds to what later will be referred to as mapping spaces in quasi-categories. The next aim is to give some conceptual ideas of these notions.

Let $\mathscr{C}$ be a quasi-category and let X and Y be vertices in $\mathscr{C}$. For all possible choices of n, the collection of n-morphisms from X to Y, as defined above, is denoted $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(X, Y)$ and referred to as the set of *left morphisms* in

$\mathscr{C}$. Moreover, it can be argued for that $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(X,Y)$ is a simplicial set itself, where the n-cells, $\Delta^n \to \mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(X,Y)$, are represented by the set of all $(n+1)$-morphisms. In fact, it can be proved that $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(X,Y)$ is a Kan complex (as indicated in [Lur09, Proposition 1.2.2.3]).

As indicated before (for example in Section 2.2.1) a consequence of being a Kan complex is that any simplex $(n \geq 1)$ in $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(X,Y)$ is invertible, firstly indicated since all horns can be filled, secondly known for sure since Kan complexes are known model ∞-groupoids. All (higher) homotopies are then invertible, since a (higher) homotopy is, from construction, a simplex in $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(X,Y)$, in particular an n-morphism is an $(n-1)$-simplex in $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(X,Y)$. Hence the requirement that every higher morphism should be invertible seems to be satisfied in this interpretation of quasi-categories.

Observe now that there is an obvious dual notion to the left morphisms. With the same setup as above, the space of *right morphisms*, denoted $\mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(X,Y)$, is constructed by using the dual interpretation of homotopies "with the identity on the left". The space of right morphisms is given by the collection of n-simplices $\beta : \Delta^n \to \mathscr{C}$ for various $n \geq 1$ with $\beta|_{\Delta^{\{0,\dots,n-1\}}} = \mathrm{cons}^{n-1}_X$ while $\beta|_{\Delta^{\{n\}}} = Y$, which can, dually to $\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(X,Y)$, be shown to be a Kan complex.

To summarise, motivated by intuition of what higher homotopies should be, two candidates for the notion of mapping spaces in quasi-categories have been discussed. But, at this point it should be remarked that there is no obvious ("horizontal") composition rule

$$\circ : \mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(Y,Z) \times \mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(X,Z) \tag{2.3.1.ii}$$

for these descriptions of mapping spaces. This concern is also formulated in [Lur09, Remark 1.2.2.4].

However, a third candidate for the quasi-categorical notion of a mapping space can be obtained from the pullback of a diagram of simplicial sets assigning edges to the fixed vertices X and Y in a quasi-category $\mathscr{C}$. This approach is the perhaps most natural way of thinking of a mapping space of all morphisms from X to Y, since this pullback can be thought of as containing the information of simplices in $\mathscr{C}$ that "fit with" X and Y. In order to do so, recall that for any simplicial sets S and T the internal Hom, or the simplicial mapping space $\mathrm{Map}_{\mathbf{sSet}}(S,T)$, is itself a simplicial set, where the n-simplices in $\mathrm{Map}_{\mathbf{sSet}}(S,T)$ are given by $\mathrm{Map}_{\mathbf{sSet}}(S,T)_n = \mathrm{Hom}_{\mathbf{sSet}}(S \times \Delta^n, T)$. A discussion of simplicial mapping spaces can be found at [GJ09, p. 20].

In fact, even more is true, if $\mathscr{C}$ is a quasi-category then the simplicial mapping space $\mathrm{Map}_{\mathbf{sSet}}(S,\mathscr{C})$ is a again a quasi-category for any arbitrary simplicial set S. This is a special case of [Lur09, Proposition 1.2.7.3]. Often when the property of $\mathrm{Map}_{\mathbf{sSet}}(S,\mathscr{C})$ of being a quasi-category is important, then the notation $\mathrm{Fun}(S,\mathscr{C}) = \mathrm{Map}_{\mathbf{sSet}}(S,\mathscr{C})$ will be used, while when the simplicial structure on this simplicial mapping space is important the already established notation $\mathrm{Map}_{\mathbf{sSet}}(S,\mathscr{C})$ will be used. For other purposes when no particular structure is important the notation $\mathrm{Hom}_{\mathbf{sSet}}(S,\mathscr{C})$ will be used.

Now back to the promised third description of mapping spaces. Edges in a quasi-category $\mathscr{C}$ can be represented by vertices of the simplicial mapping space $\mathrm{Map}_{\mathbf{sSet}}(\Delta^1,\mathscr{C})$, while pairs of vertices in $\mathscr{C}$ can be represented by vertices of the simplicial mapping space $\mathrm{Map}_{\mathbf{sSet}}(\Delta^0 \coprod \Delta^0,\mathscr{C}) \cong \mathscr{C}\times\mathscr{C}$. Moreover, the canonical projection

$$\pi : \mathrm{Map}_{\mathbf{sSet}}(\Delta^1,\mathscr{C}) \to \mathrm{Map}_{\mathbf{sSet}}(\Delta^0 \coprod \Delta^0,\mathscr{C}) \tag{2.3.1.iii}$$

can be constructed by sending each edge in $\mathscr{C}$ to the pair consisting of its source and target, that is, the projection π corresponds to the expression $d_1 \times d_0$. Furthermore, let

$$(X,Y) : \Delta^0 \to \mathrm{Map}_{\mathbf{sSet}}(\Delta^0 \coprod \Delta^0,\mathscr{C}) \tag{2.3.1.iv}$$

denote the simplicial map that corresponds to the vertex in $\mathrm{Map}_{\mathbf{sSet}}(\Delta^0 \coprod \Delta^0,\mathscr{C})$ that is represented by the pair (X,Y) of vertices in $\mathscr{C}$. Now, the third description of mapping space in $\mathscr{C}$ is given by the pullback

$$\begin{tikzcd}
\mathrm{Map}_{\mathscr{C}}(X,Y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathscr{C}}(\Delta^1,\mathscr{C}) \arrow[d, "\pi"] \\
\Delta^0 \arrow[r, "{(X,Y)}"'] & \mathrm{Map}_{\mathscr{C}}(\Delta^0 \coprod \Delta^0,\mathscr{C})
\end{tikzcd}$$

over **sSet**. This pullback is known to exists, since **sSet**, as every **Set**-valued presheaf category, is both cartesian and cocartesian (Proposition B.1.2.2). Moreover, it is more conceivable what a "horizontal" composition in this interpretation of mapping spaces, by using the properties of **sSet** over again.

Again this third approach to the interpretation of the mapping spaces $\mathrm{Map}_{\mathscr{C}}(X,Y)$ is perhaps the most intuitive, since it is constructed by using the properties of the simplicial structure of $\mathscr{C}$. By construction $\mathrm{Map}_{\mathscr{C}}(X,Y)$ can be thought of as the simplicial set of all morphisms starting in X and ending in Y. The simplicial set $\mathrm{Map}_{\mathscr{C}}(X,Y)$ has in fact the correct homotopy type, as indicated at [Cam13, p. 14], so then this description of the quasi-categorical mapping space seems to work after all. Moreover, some further discussions of this perspective of the quasi-categorical mapping spaces can be found in [Cam13, Section 3.2].

Moreover, for a quasi-category $\mathscr{C}$, the last notion of a mapping space, which has already been given the notation $\mathrm{Map}_{\mathscr{C}}(X, Y)$, is however related to the spaces of left morphisms and right morphisms. In fact there are natural inclusions,

$$\mathrm{Hom}^{\mathrm{L}}_{\mathscr{C}}(X, Y) \hookrightarrow \mathrm{Map}_{\mathscr{C}}(X, Y) \hookleftarrow \mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(X, Y), \tag{2.3.1.v}$$

which both can be shown to be homotopy equivalences. This result is indicated in [Lur09, Section 1.2.2] with references to [Lur09, Corollary 4.2.1.8], which understanding of requires more details than will be given here. So, the three descriptions of mapping spaces model the same homotopy types, which again is indicated in [Cam13] to be the correct one, after all.

Definition 2.3.1.1. Let $\mathscr{C}$ be a quasi-category with vertices X and Y. Often the term *mapping space* will be used for the third description, namely $\mathrm{Map}_{\mathscr{C}}(X, Y)$, while the space of *left morphisms* or the space of *right morphisms* will be used for the others, as already decided. But, when clear from situation, the notation $\mathrm{Map}_{\mathscr{C}}(X, Y)$ will denote all these cases, all referred to as the *mapping space*, since they all model the same homotopy type.

So, in this section it has been discussed how higher homotopies and mapping spaces in quasi-categories can be described from intuition and generalisation. These notions supply the understanding of quasi-categories as models for $(\infty, 1)$-categories, for example by the notion of higher homotopies and that mapping spaces can be shown to be Kan complexes, which are known to model ∞-groupoids. In the next section the notion of functors and homotopy coherent diagrams will be discussed for quasi-categories.

2.3.2 Functors and homotopy coherent diagrams

Recall from Definition 2.2.1.2 that any functor of quasi-categories is a map of simplicial sets. In this section there will be sketched some ideas why this definition is natural and works well after all. Homotopy coherent diagrams, regarded as a special examples of functor, will also be discussed.

First observe that any simplicial map $f : S \to T$ preserves the simplicial structure in the sense that vertices are mapped to vertices, arrows to arrows, n-simplices to n-simplices, etcetera. This follows easily from the forthcoming interpretation. Let $\omega : \Delta^n \to S$ be an n-simplex in S. The following commutative diagram

$$
\begin{tikzcd}
\Delta^n \arrow[r, "\omega"] \arrow[dr, "f\omega"'] & S \arrow[d, "f"] \\
& T,
\end{tikzcd}
$$

indicates that the corresponding element $\omega \in S_n$ (by applying the Yoneda lemma) is mapped to an element in T_n, which corresponds to $f\omega$ in the diagram (by applying the Yoneda lemma again). Moreover, any simplicial map $f : S \to T$ preserves also faces and degeneracies of any n-simplex $\omega : \Delta^n \to S$. This follows directly since simplicial maps are natural transformations. These arguments ensure that simplicial maps preserve all expected structure.

Now some comments of what these structure preservations actually mean for simplicial maps of quasi-categories. Let $\mathscr{C}$ and $\mathscr{D}$ be quasi-categories, then it is required for a simplicial map $F : \mathscr{C} \to \mathscr{D}$ to send objects to objects, arrows to arrows, higher homotopies to higher homotopies, etcetera, in order to give rise to a functor of quasi-categories, as a model for $(\infty,1)$-categories. But this follows directly from the structure preservation of simplicial maps sketched above. For example, higher homotopies in $\mathscr{C}$ can be thought of as particular simplices that are constant on "one side", as argued for in the previous section. Since degeneracies are preserved, constant simplices are sent to the constant simplices, etcetera. This, together with all preservations of structure, indicates conceptually that homotopies are mapped to homotopies as required.

Following the same thoughts, it can be argued that horns in $\mathscr{C}$ will be sent to horns in $\mathscr{D}$. Moreover, let $\alpha : \Delta^2 \to \mathscr{C}$ denote a filler of the horn $(g, \bullet, f) : \Lambda^2_1 \to \mathscr{C}$, then $F(\alpha)$ is also a filler of the horn $F(g, \bullet, f) = (F(g), \bullet, F(f)) : \Lambda^2_1 \to \mathscr{D}$. So the essence of all these ideas is that the description of functors of quasi-categories as simplicial maps is naturally motivated from the discussion so far. So from this, the functors of quasi-categories are conceivable to be exactly the simplicial maps, so the collection of all such functors can be regarded as the simplicial mapping space $\mathrm{Map}_{\mathbf{sSet}}(\mathscr{C}, \mathscr{D}) = \mathrm{Fun}(\mathscr{C}, \mathscr{D})$, as previously stated.

As previously discussed, the fact that inner horns in a quasi-category $\mathscr{C}$ can be filled gives the appropriate notion of compositions in quasi-categories. For edges $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$ the inner horn filler property claims that there is a 2-simplex α with boundary, say $\partial\alpha = (g, h, f)$, assigning the composition gf to a candidate h for the composition. All these data are going into the notion of a triangle shaped diagram for quasi-categories. Recall also, as usual there can be more 2-simplices filling the horn $(g, \bullet, f) : \Lambda^2_1 \to \mathscr{C}$, which give rise to "the same" triangle shaped diagram, in the meaning that all choices of candidates for the composition are homotopic to one another, so there are

2-homotopies connecting such certificates, invertible up to higher homotopies, etcetera. This can also be viewed a consequence of the proof of the proposition asserting that compositions in the homotopy category of a quasi-category are well-defined (Proposition 2.2.2.4).

Moreover, the notion of triangle shaped diagrams above can be generalised to strings of composable arrows in $\mathscr{C}$, which can be encoded by concatenations of Λ^2_1-horns of required length. Formally, let

$$P_n = \Delta^1 \coprod_{\Delta^0} \cdots \coprod_{\Delta^0} \Delta^1, \tag{2.3.2.i}$$

denote the concatenation of n standard edges Δ^1, then a string of length n of composable edges in $\mathscr{C}$ can be obtained from a simplicial map $P_n \to \mathscr{C}$. A string of length $n = 3$ is used in the proof of Proposition 2.2.2.5 in order to prove that the induced composition rule on $h\mathscr{C}$ is associative. The obtained homotopy associating $h(gf)$ with $(hg)f$ can be understand as the associator for the composition in $\mathscr{C}$, in view of the notion of associator from the first chapter (in Section 1.2.4). Similarly, the pentanator can be obtained from strings of length $n = 4$, etcetera.

Recall the notion of homotopy coherence from the first chapter (Section 1.2.4), with all the additional data of associators, pentagonators etcetera, diagrams in higher categories are often said to commute up to coherent homotopy. The aim now is to give some ideas how homotopy coherent diagrams are expressed for quasi-categories as models for $(\infty, 1)$-categories. So, for a triangle shaped diagram, as seen above, the inner horn fillers property ensures the existence of a 2-simplex assigning a composition to a candidate for the composition, with all choices homotopic to one another.

In particular, a diagram of arbitrary (non-cyclic) shape in a quasi-category $\mathscr{C}$, which always can be visualised as a shape build up by vertices and edges, can be interpreted as a simplicial map $F : X \to \mathscr{C}$, where X is the shape of the diagram, or more precisely, X is the nerve of the free category on the underlying directed graph of the diagram visualised by vertices and edges in $\mathscr{C}$. The discussion about strings of arbitrary lengths in quasi-categories above can be generalised to any diagram of arbitrary (non-cyclic) shape (as indicated in [Cam13]). This means that any diagram in $\mathscr{C}$ commutes up to coherent homotopy, as required for a model for $(\infty, 1)$-categories. So, from this it can be concluded that the inner horn filler property gives not only the appropriate notion of compositions, but also the appropriate notion of homotopy coherent diagrams in quasi-categories.

Equivalently, the idea of homotopy coherent diagrams can also be interpreted as a lifting property. This perspective of homotopy coherent diagrams is discussed in [Lur09, Section 1.2.6]. In particular, let X denote the nerve of the free category of the directed graph with the shape of an arbitrary diagram, as described above. A homotopy coherent diagram in a quasi-category $\mathscr{C}$ can be interpreted as a functor $F : X \to h\mathscr{C}$ (by forget the nerve of X) together with all additional data of higher homotopies available in order to lift F to a functor $\overline{F} : X \to \mathscr{C}$ of quasi-categories. These additional data are basically the same as the data mentioned for the first perspective from discussion.

2.3.3 Contractable spaces of choices

Recall again that all choices for fillers of a horn $\Lambda_1^2 \to \mathscr{C}$ give homotopic candidates for a composition. Thinking geometrically, all such candidates for a composition represented by a fixed horn are in the same path connected component of an appropriate notion of a "space" of candidates, since they are all homotopic to one another. Hence, composition can be said to be "defined up to a contractable space of choices". The aim for this section is to give some ideas of this assertion.

But first, the following result characterises the simplicial sets that have the property of being quasi-categories.

Proposition 2.3.3.1. A simplicial set $\mathscr{C}$ is a quasi-category if and only if the map

$$\mathrm{Map}_{\mathbf{sSet}}(\Delta^2, \mathscr{C}) \to \mathrm{Map}_{\mathbf{sSet}}(\Lambda_1^2, \mathscr{C}), \tag{2.3.3.i}$$

induced from the inclusion $\Lambda_1^2 \hookrightarrow \Delta^2$, is a trivial Kan fibration.

This proposition is also stated in [Cam13, Proposition 3.4 and Remark 3.5] and in [Gro10, Theorem 1.11]. The result is due to Joyal. In particular, the proposition implies that the fibers over the map in Equation 2.3.3.i are contractable Kan complexes (as stated in [Cam13, Proposition 3.4]). Although both directions in the proposition are true, it will mainly be assumed that $\mathscr{C}$ is a quasi-category here. The aim now is to see use of this proposition in some examples in order to formalising some of ideas and expectations discussed previously.

A filler of the horn $\Lambda_1^2 \to \mathscr{C}$ can be interpreted as a certificate that ensures the existence of a candidate of a composition, as discussed various places previously. Fibers over the map in Equation 2.3.3.i can be interpreted as choices for certificates of composition, homotopies of these certificates, etcetera, analogously to the data obtained under the discussion of homotopy coherent diagrams previously. For example, as indicated in the discussion of mapping spaces (Section 2.3.1), a composable pair is given by some vertex

$$\Delta^0 \to \mathrm{Map}_{\mathbf{sSet}}(\Lambda_1^2, \mathscr{C}). \tag{2.3.3.ii}$$

The fiber over $\mathrm{Map}_{\mathbf{sSet}}(\Delta^2, \mathscr{C}) \to \mathrm{Map}_{\mathbf{sSet}}(\Lambda^2_1, \mathscr{C})$ (Equation 2.3.3.i) at this vertex is given by the following pullback in **sSet**,

$$\begin{tikzcd}
\mathrm{Fib} \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbf{sSet}}(\Delta^2, \mathscr{C}) \arrow[d] \\
\Delta^0 \arrow[r] & \mathrm{Map}_{\mathbf{sSet}}(\Lambda^2_1, \mathscr{C})
\end{tikzcd}$$

which can be interpreted as the space of all possible candidates of composition of the composable pair determined by the vertex $\Delta^0 \to \mathrm{Map}_{\mathbf{sSet}}(\Lambda^2_1, \mathscr{C})$. From Proposition 2.3.3.1 this space is a contractable Kan complex. Geometrically, contractable can be interpreted as the space Fib in the diagram above has trivial homotopy type. Hence all choices of candidates of the composition can be regarded as homotopic to one another, as discussed previously. Moreover, the assertion that the simplicial set Fib is a Kan complex guarantees that all simplices ($n \geq 1$), which for the simplicial set Fib can be interpreted as certificates of compositions, homotopies of these etcetera, are invertible, as discussed. So, for a quasi-category $\mathscr{C}$ the requirement that $\mathrm{Map}_{\mathbf{sSet}}(\Delta^2, \mathscr{C}) \to \mathrm{Map}_{\mathbf{sSet}}(\Lambda^2_1, \mathscr{C})$ is a trivial Kan fibration formalise the properties of the notions of compositions and candidates for them discussed previously.

Furthermore as indicted in [Cam13], Proposition 2.3.3.1 can be generalised (or at least one of the direction in the proposition) to strings of composable morphisms of arbitrary length. Let $\mathscr{C}$ be a quasi-category and let P_n be defined as previously (in Equation 2.3.2.i), namely

$$P_n = \Delta^1 \coprod_{\Delta^0} \cdots \coprod_{\Delta^0} \Delta^1. \tag{2.3.3.iii}$$

Then the simplicial map $P_n \to \mathscr{C}$ represents a string of n composable morphisms in $\mathscr{C}$. Then the map

$$\mathrm{Map}_{\mathbf{sSet}}(\Delta^n, \mathscr{C}) \to \mathrm{Map}_{\mathbf{sSet}}(P_n, \mathscr{C}), \tag{2.3.3.iv}$$

obtained from the inclusion $P_n \hookrightarrow \Delta^n$ can be shown to be a trivial Kan fibration ([Cam13]). This result can be interpreted as giving a precise meaning to the situations of composable strings of arbitrary lengths, as discussed previously. For example the case where $n = 3$ describes associativity of the composition, the result can be interpreted as specify a precise meaning of compositions being associative ([Cam13]). The fiber over the map $\mathrm{Map}_{\mathbf{sSet}}(\Delta^3, \mathscr{C}) \to \mathrm{Map}_{\mathbf{sSet}}(P_3, \mathscr{C})$ at Δ^0 is a contractable Kan complex, so all higher simplices can be regarded as invertible and homotopic to one another, which basically also have been formulated previously for the notion of associativity.

Even more generally Proposition 2.3.3.1 can be extended to arbitrary diagrams in $\mathscr{C}$, as stated more precisely in [Cam13, Proposition 3.6]. The idea behind this generalisation is to interpret arbitrary diagrams as a functor $F : X \to \mathscr{C}$, where X denotes the nerve of the free category of the directed graph describing the shape of the diagram, in the same way as discussed for homotopy coherent diagrams in Section 2.3.2. Similarly as for associativity discussed above, this generalisation can be interpreted as specify a precise sense of the notion of homotopy coherent diagrams in quasi-categories.

Summarised, the Proposition 2.3.3.1 and its generalisations gives more precise interpretations of many of the notions discussed previously, which are particular important when discussing limits and other universal constructions for quasi-categories in Chapter 3. The nest section sketches some ideas in the comparison of simplicial categories and quasi-categories.

2.3.4 Comparison of quasi-categories with simplicial categories

Up until now in this part, some $(\infty, 1)$-categorical properties have been discussed for quasi-categories. The aim now is to discuss some ideas of a comparison of quasi-categories with simplicial categories. Moreover, from the comparison of topological categories with simplicial categories in Section 2.1.3 recall that it was searched for constructions that preserved the homotopical structure. But this followed easily, since the unit maps and the counit maps for the adjunction $(||, \mathrm{Sing}) : \mathbf{sSet} \to \mathbf{CG}$ can be proved to be weak homotopy equivalences. The aim for this section is to give some ideas of the comparison of quasi-categories and simplicial categories. Finally, it will be discussed some short ideas of alternative approaches to mapping spaces and the homotopy categories for quasi-categories. In particular, it will first be given a definition of $\mathfrak{C}[\bullet]$ and sN. In fact $\mathfrak{C}[\bullet]$ and sN define an adjoint pair

$$(\mathfrak{C}[\bullet], \mathrm{sN}) : \mathbf{sSet} \to \mathbf{sCat}, \tag{2.3.4.i}$$

which can be shown (as stated in [Gro10, Theorem 1.27]) to be a Quillen-equivalence, where **sSet** and **sCat** are equipped with appropriate model structures, which will be referred to as the “Joyal model structure” and “Bergner model structure” respectively. The discussion here has been inspired by [Gro10, Section 1.2] and [Lur09, Section 1.1.5].

First the definition of $\mathfrak{C}[\bullet] : \mathbf{sSet} \to \mathbf{sCat}$. Evaluated on representable presheaves, say Δ^p, $\mathfrak{C}[\Delta^p]$ is defined to be the simplicial category, where

- objects are $\{0, 1, 2, \ldots, p\}$, namely the standard vertices in Δ^p and

- mapping spaces are determined by

$$\mathrm{Map}_{\mathfrak{C}[\Delta^p]}(i,j) = \begin{cases} NP_{i,j} & \text{for} \quad i \leq j \\ \emptyset & \text{for} \quad i > j, \end{cases} \tag{2.3.4.ii}$$

  where $P_{i,j}$ denotes partially ordered sets of subsets $\{i,j\} \subseteq S \subseteq \{i,\dots,j\}$ which are ordered by the relation $\leq$ where $S' \leq S$ if $S \subseteq S'$

Compositions are induced from unions of subsets. Sketches presented in [Gro10, pp. 10-11] visualise these constructions for lower dimensional simplices. Moreover, for a morphism $f : [p] \to [q]$ in $\mathbf{\Delta}$, the simplicial functor $\mathfrak{C}[f] : \mathfrak{C}[p] \to \mathfrak{C}[q]$ is defined by

- $\mathfrak{C}[i] = f(i) \in \mathfrak{C}[\Delta^q]$ for ever object $i \in \mathfrak{C}[\Delta^p]$
- for $i \leq j$ in $\mathfrak{C}[\Delta^p]$ the map $\mathrm{Map}_{\mathfrak{C}[\Delta^p]}(i,j) \to \mathrm{Map}_{\mathfrak{C}[\Delta^q]}(f(i), f(j))$ induced from f is obtained by the nerve of the map $P_{i,j} \to P_{f(i),f(j)}$ that assigns $S \mapsto f(S)$

A detailed description of the functor $\mathfrak{C}[\bullet]$ is however given in [Lur09, Section 1.1.5]. Now, as indicated at [Lur09, p. 23], since **sCat** admits (small) colimits, the functor $\mathfrak{C}[\bullet]$ can be shown to preserve colimits. Then since $\mathfrak{C}[\bullet]$ up until now has been defined on representable presheaves, it can be extended uniquely, up to unique isomorphism, to a functor of arbitrary simplicial sets, $\mathfrak{C}[\bullet] : \mathbf{sSet} \to \mathbf{sCat}$. Recall that arbitrary simplicial sets can be regarded as colimits of representable simplicial sets.

Next the construction of the *simplicial nerve functor* $\mathrm{sN} : \mathbf{sCat} \to \mathbf{sSet}$ will be discussed. For any simplicial category $\mathscr{C}$, the simplicial set $\mathrm{sN}(\mathscr{C})$ is defined to have n simplices

$$\mathrm{sN}(\mathscr{C})_n \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, \mathrm{sN}(\mathscr{C})) = \mathrm{Map}_{\mathbf{sCat}}(\mathfrak{C}[\Delta^n], \mathscr{C}) \in \mathbf{sSet}, \tag{2.3.4.iii}$$

where the isomorphism in the expression above is obtained by the Yoneda lemma as usual, the equality is the definition. The construction of the simplicial nerve establishes the adjunction

$$(\mathfrak{C}[\bullet], \mathrm{sN}) : \mathbf{sSet} \to \mathbf{sCat}, \tag{2.3.4.iv}$$

by definition. This is as also stated in [Gro10, Remark 1.20].

The aim now is to give some ideas why this adjunction determines the first step of a comparison of simplicial categories with quasi-categories. In fact, by following the discussion in [Gro10, Section 1.2], **sSet** can (with some requirements) be equipped with a model structure, choosing fibrant objects to be quasi-categories. This is often called the *Joyal model structure* on simplicial sets, named after Joyal.

Moreover, **sCat** can (with some requirements) be equipped with a model structure taking fibrant objects to be those simplicial categories whose mapping spaces are Kan complexes. This is often referred to as the *Bergner model structure* on simplicial categories, named after Bergner. Recall that the definition of simplicial categories is often restricted to these fibrant simplicial categories in order to model $(\infty, 1)$-categories, as stated in Remark 2.1.3.2.

With these model structures, the adjunction (Equation 2.3.4.iv) can be shown to be a Quillen equivalence ([Gro10, Theorem 1.27]). The idea is that this Quillen equivalence solve the comparison problem, since this can conceptually be regarded as implying that quasi-categories behave homotopically similar as fibrant simplicial categories. Then they model the same higher categorical concept, namely $(\infty, 1)$-categories.

With the established connection between simplicial categories and quasi-categories, some notions for quasi-categories can be approached differently. In particular, one of the benefits with simplicial categories is that the mapping spaces can easily be described, since simplicial categories is defined to be categories enriched over **sSet**. Let $\mathscr{D}$ be a quasi-category, then the mapping spaces in $\mathscr{D}$ can be described by the construction $\mathrm{Map}_{\mathfrak{C}[\mathscr{D}]}(X, Y)$ for any vertices X and Y, as indicated [Lur09, p. 27]. It turns out that this approach can in some situations be used to a procedure for strictifying compositions in quasi-categories by using simplicial enrichments ([Cam13, p. 15]). However, it turns out that this construction has some serious drawbacks, so it is not usually used in other practical situations ([Lur09, p. 27]).

The construction of homotopy categories was easily obtained in the simplicial categorical setting, simply by applying the geometric realisation functor. The construction of homotopy categories of quasi-categories can be approached by this construction from simplicial categories. In order to do so, let $\mathrm{h} : \mathbf{sCat} \to \mathbf{Cat}$ denotes the functor that to each simplicial category assign its homotopy category. The homotopy category of a quasi-category $\mathscr{C}$ can be obtained by applying the composition of the functors

$$\mathbf{sSet} \xrightarrow{\mathfrak{C}[\bullet]} \mathbf{sCat} \xrightarrow{\mathrm{h}} \mathbf{Cat}\,. \tag{2.3.4.v}$$

Restrict the composition of the functors above to quasi-categories and denote this composition by $\overline{\mathrm{h}}$. This description of the homotopy category, which can be denoted $\overline{\mathrm{h}}\mathscr{C}$, of a quasi-category $\mathscr{C}$ is naturally equivalent to the description given earlier, which was denoted $\mathscr{h}\mathscr{C}$, in fact $\overline{\mathrm{h}}\mathscr{C}$ and $\mathscr{h}\mathscr{C}$ are isomorphic as categories, as proved in [Lur09, Proposition 1.2.3.9]. In both these cases the homotopy category will then be denoted $\mathcal{h}\mathscr{C}$.

Recall that the nerve of an ordinary category is a quasi-category. Moreover, the nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}$ can be shown to commute with the composition $\mathbf{Cat} \xrightarrow{i} \mathbf{sCat} \xrightarrow{\mathrm{sN}} \mathbf{sSet}$, where i denotes the inclusion of $\mathbf{Cat}$ in to $\mathbf{sCat}$, so $N = \mathrm{sN} \circ i$, as indicated in the proof of [Lur09, Proposition 1.2.3.1]. In fact, $(\mathrm{h}, i) : \mathbf{sCat} \to \mathbf{Cat}$ is an adjoint pair, which can be seen from the adjunction $(\pi_0, j) : \mathbf{sSet} \to \mathbf{Set}$ used on the enrichments, where j denotes the inclusion $\mathbf{Set} \to \mathbf{sSet}$, again indicated in the proof of [Lur09, Proposition 1.2.3.1]. By composing with the adjunction $(\mathfrak{C}[\bullet], \mathrm{sN}) : \mathbf{sSet} \to \mathbf{sCat}$ establishes the adjunction $(\mathrm{h} \circ \mathfrak{C}[\bullet], N = \mathrm{sN} \circ i) : \mathbf{sSet} \to \mathbf{Cat}$. In fact for a (fibrant) simplicial category $\mathscr{C}$ the descriptions $\mathrm{h}\mathscr{C}$ and $\mathcal{h}\mathrm{sN}\mathscr{C}$ are equivalent ([Lur09, Warning 1.2.3.3]). All ideas and sketches given here are more formally and detailed discussed in [Lur09, Section 1.2.3].

However, as already mentioned, it turns out that many notions in the $(\infty, 1)$-categorical language, like limits and colimits, are easily obtained in the quasi-categorical setting. In the next chapter there will be description of some of these notions that are used further in the discussions of stable $(\infty, 1)$-categories in the final chapter.

Chapter 3

Notions in the language of $(\infty, 1)$-categories

The aim for this chapter is to adapt some of the notions from the theory of ordinary categories to the language of $(\infty, 1)$-categories. In particular, notions that will be used in the theory of stable $(\infty, 1)$-categories in the next chapter will be discussed here. These notion will formally be discussed for quasi-categories, since quasi-categories have the advantage that many of these necessary notions can be adapted intuitively and well motivated from the ordinary categorical language. First the discussion of functors of quasi-categories will be continued from the previous chapter. Then constructions that are used in the establishments of for example terminal objects and limits will be discussed. In particular, these constructions are join constructions for quasi-categories together with overcategories and undercategories for quasi-categories. Finally, initial objects, terminal objects, limits and colimits will be discussed. Moreover, a particular attention will be given to pullbacks and pushouts, since they are key constructions in the theory of stable $(\infty, 1)$-categories that will be discussed in the next chapter. The insight in quasi-categories will be made a practical use in the establishment of stable $(\infty, 1)$-categories in the next chapter.

3.1 Functors and subcategories

In this section the discussion of functors will be continued from the previous chapter. Finally, there will be described a notion of a sub-$(\infty, 1)$-category $\mathscr{C}' \subseteq \mathscr{C}$

of an $(\infty, 1)$-category $\mathscr{C}$, but in the setting quasi-categories.

3.1.1 Functors of $(\infty, 1)$-categories

Recall that a functor of quasi-categories $F : \mathscr{C} \to \mathscr{D}$ was defined to be a map of simplicial sets. For topological categories and simplicial categories functors were defined to be enriched functors. However, there are some difficulties in giving a description of the collection of all functors $\mathscr{C} \to \mathscr{D}$, when $\mathscr{C}$ and $\mathscr{D}$ are topological categories or simplicial categories. This is exemplified in [Lur09, Remark 1.2.7.1]. A problem that may occur is that the "obvious" descriptions of such collections of functors do not have enough natural transformations, in order to be the appropriate description.

However, for quasi-categories $\mathscr{C}$ and $\mathscr{D}$ the collection of functors $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$, namely the simplicial mapping space $\mathrm{Map}_{\mathbf{sSet}}(\mathscr{C}, \mathscr{D})$, is a quasi-category itself, as mentioned in the previously. Even the simplicial mapping space $\mathrm{Fun}(S, \mathscr{C}) = \mathrm{Map}_{\mathbf{sSet}}(S, \mathscr{C})$ is a quasi-category for any arbitrary simplicial set S.

Proposition 3.1.1.1. Let $\mathscr{C}$ be a quasi-category and let S be a simplicial set, then the simplicial set $\mathrm{Fun}(S, \mathscr{C}) = \mathrm{Map}_{\mathbf{sSet}}(S, \mathscr{C})$ is a quasi-category.

This is also mentioned in [Cam13, Section 3.2] and is one of the statements in [Lur09, Proposition 1.2.7.3] and in [Gro10, Proposition 2.2]. This easy description of the collection of functors has some advantages for quasi-categories. For example, recall that diagrams in a quasi-category $\mathscr{C}$ can be thought of as a functor $F : X \to \mathscr{C}$, where X is the nerve of the free category on the directed graph with the shape of the diagram, then some appropriate properties or constructions of quasi-categories can be inhered to $\mathrm{Fun}(X, \mathscr{C})$, which represents the quasi-category of all diagrams in $\mathscr{C}$ with this shape given by X. It will be seen in the next chapter that the "full subcategory" of the quasi-category of square shaped diagrams $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$ spanned by the squares that are pushout squares is important in the discussion of stable $(\infty, 1)$-categories. The forthcoming section discusses the notion of subcategories for quasi-categories.

But now, the notion of equivalences of quasi-categories will be defined. Recall from Section 2.3.4 that the homotopy category of a quasi-category can be regarded as an $\mathscr{H}$-enriched category.

Definition 3.1.1.2. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor of quasi-categories. Then F is said to be an *categorical equivalence*, or *equivalence* for short, if the induced functor on the homotopy categories $h\mathscr{C} \to h\mathscr{D}$ is an equivalence of $\mathscr{H}$-enriched categories.

In fact, the notion of categorical equivalences can also be defined for arbitrary maps $f : S \to T$ of simplicial sets by approach the comparisons and the notion of $\mathscr{H}$-enriched categories slightly more generally than discussed previously. This approach is the one considered in [Lur09], but [Gro10] seems to approach this through model categories and Dwyer-Kan equivalences. However, with this notion established, the following can be shown to be true.

Proposition 3.1.1.3. First, let $F : \mathscr{C} \to \mathscr{D}$ be a categorical equivalence of quasi-categories and S an arbitrary simplicial set, then the induced map $\mathrm{Fun}(S, \mathscr{C}) \to \mathrm{Fun}(S, \mathscr{D})$ is an categorical equivalence of quasi-categories. Second, let now $f : S \to T$ be a categorical equivalence of simplicial sets and let $\mathscr{C}$ be a quasi-category, then the induced map $\mathrm{Fun}(T, \mathscr{C}) \to \mathrm{Fun}(S, \mathscr{C})$ is a categorical equivalence of quasi-categories.

This result is again a special case of [Lur09, Proposition 1.2.7.2], similar as [Gro10, Proposition 2.2], and is due to Joyal, a proof will often require far more theory and details than will be discussed here. However, from this proposition above observe that the formation of functor categories is an invariant notion under equivalences, as expected for quasi-categories.

3.1.2 Subcategories of quasi-categories

The aim for this section is to give some ideas of the notion of a sub-$(\infty, 1)$-category of an $(\infty, 1)$-category, but in the quasi-categorical setting. In fact, the general axiomatic setting of sub-$(\infty, 1)$-categories involves quite much structure, in particular it can be carried out by a recursive matter, with base cases involving equivalences $\mathscr{C} \to \mathscr{D}$. But this will not be discussed here.

However, the discussion here basically follows the interpretation in [Lur09, Section 1.2.11] for quasi-categories. In particular, let $\mathscr{C}$ be a quasi-category, the interpretation of a sub-quasi-category here is obtained from a subcategory of the homotopy category $h\mathscr{C}$ by "pulling back" the nerve of this inclusion in $\mathscr{C}$. The precise definition of a sub-quasi-category is the following.

Definition 3.1.2.1. Let $\mathscr{C}$ be a quasi-category and let $h\mathscr{D} \hookrightarrow h\mathscr{C}$ denote the inclusion of a subcategory $h\mathscr{D}$ of the homotopy category $h\mathscr{C}$ of $\mathscr{C}$. A *sub-quasi-category*, or *subcategory* for short, is the pullback of $\mathscr{C}$ with the nerve of the inclusion above, namely the limit of $N(h\mathscr{D}) \to N(h\mathscr{C}) \leftarrow \mathscr{C}$ over simplicial sets,

$$\begin{tikzcd}
\mathscr{D} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\
N(h\mathscr{D}) \arrow[r] & N(h\mathscr{C}).
\end{tikzcd}$$

Then $\mathscr{D}$ obtained this way is referred to as the *sub-quasi-category of* $\mathscr{C}$ *spanned by* $h\mathscr{D}$, or the *subcategory of* $\mathscr{C}$ *spanned by* $h\mathscr{D}$ for short.

Conceptually, this construction of $\mathscr{D}$ can visually be understand of being the simplicial subset of $\mathscr{C}$, that has the same objects as $h\mathscr{D}$, and the mapping spaces are obtained from connected components of the mapping space of $\mathscr{C}$, whose homotopy equivalence classes are equivalent to the Hom-sets of $h\mathscr{D}$.

Definition 3.1.2.2. Let $\mathscr{C}$ be a quasi-category. If $h\mathscr{D}$ in the construction of subcategories in Definition 3.1.2.1 above is a full subcategory of $h\mathscr{C}$, then $\mathscr{D}$ is said to be the *full subcategory of* $\mathscr{C}$ *spanned by the objects of* $h\mathscr{D}$.

As mentioned, full subcategories of certain functor categories, for example of square shaped diagrams that are spanned by the squares with the property of being quasi-categorical pushout squares, are important parts of the constructions that go in to the discussion of the theory of stable quasi-categories in the next chapter.

3.2 Constructions

The main aim for this section is to define notions of initial objects and terminal objects together with limits and colimits in the $(\infty, 1)$-categorical language, which formally are discussed for quasi-categories. This involves a discussion of joins of quasi-categories and overcategories and undercategories for quasi-categories in order to prepare for the main objectives. The discussion here is based on studies of [Gro10] and [Lur09, Section 2.2], but inspiration is also taken from [Cam13, Section 5], which the latter seems to give a somewhat slightly more general approach.

The title of the section reflects that there are constructed "new" gadgets from others, for example a join of two quasi-categories is a "new" quasi-category constructed from the two original ones, while limits can be defined as particular 'universal' diagrams or functors. The term 'universal' should really in the quasi-categorical setting, or in any $(\infty, 1)$-categorical setting, as usual be interpreted as 'universal up to homotopy' or 'universal up to a contractable space of different choices of candidates', as discussed various places previously. In particular, pullback diagrams and pushout diagrams will be discussed because of their importance in the next chapter, as already said a few words of.

3.2.1 Join constructions

Next aim is to define the notion of "joins of quasi-categories". But first some motivation from ordinary categories will be discussed. For ordinary categories $\mathscr{C}$ and $\mathscr{D}$, the *join* of $\mathscr{C}$ and $\mathscr{D}$ is defined to be a category denoted $\mathscr{C} \star \mathscr{D}$ where

- an object of $\mathscr{C} \star \mathscr{D}$ is either an object of $\mathscr{C}$ or an object of $\mathscr{D}$
- the morphism sets are given as follows, for objects X and Y of $\mathscr{C} \star \mathscr{D}$,

$$\mathrm{Hom}_{\mathscr{C} \star \mathscr{D}}(X,Y) = \begin{cases} \mathrm{Hom}_{\mathscr{C}}(X,Y) & \text{if} \quad (X \in \mathscr{C}) \wedge (Y \in \mathscr{C}) \\ \mathrm{Hom}_{\mathscr{D}}(X,Y) & \text{if} \quad (X \in \mathscr{D}) \wedge (Y \in \mathscr{D}) \\ \emptyset & \text{if} \quad (X \in \mathscr{D}) \wedge (Y \in \mathscr{C}) \\ \{*\} & \text{if} \quad (X \in \mathscr{C}) \wedge (Y \in \mathscr{D}). \end{cases} \tag{3.2.1.i}$$

Composition of morphisms is determined from definition, since $\mathscr{C}$ and $\mathscr{D}$ are full subcategories of $\mathscr{C} \star \mathscr{D}$ by construction.

These motivating comments continue now with some examples of joins for ordinary categories. First let $\mathscr{C}$ be an arbitrary ordinary category, while $[0]$ denotes the category with one object and only identity morphism as usual. Remark that $[0]$ is the terminal object in **Cat**. The join $\mathscr{C} \star [0]$ is often denoted $\mathscr{C}^{\triangleright}$ and called the *right cone* on $\mathscr{C}$. From definition of join, the right cone $\mathscr{C}^{\triangleright}$ is constructed from adjoining an object ∞ to $\mathscr{C}$, such that for all objects X in $\mathscr{C}$ there exists a unique arrow $X \to \infty$. So, the shape of $\mathscr{C}^{\triangleright}$ can be visualised as a cone (formed by various arrows) ending in ∞. The construction of $\mathscr{C}^{\triangleright}$ has some relations to colimits viewed as a universal arrows, as colimits are approached in [Mac98, Chapter III], since $\mathscr{C}^{\triangleright}$ can be viewed as a "completion" of diagrams $F : X \to \mathscr{C}$ of certain shapes. These notions will be discussed further in the definitions of limits and colimits in Section 3.2.4.

In order to give an illustration of the join constructions for ordinary categories, take for example the category $\mathscr{C} = (\bullet \leftarrow \bullet \to \bullet)$ in consideration. The right cone $\mathscr{C}^{\triangleright} \cong [1]^2$ can be illustrated by the following commutative diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

completing $\mathscr{C}$ into a square, where the arrow going diagonally is captured by either the upper-right composition or the left-lower composition.

For a category $\mathscr{C}$, the dual notion $\mathscr{C}^{\triangleleft} = [0] \star \mathscr{C}$ is called the *left cone* on $\mathscr{C}$, and is constructed by adjoining an object $-\infty$ to $\mathscr{C}$ such that for all objects X in $\mathscr{C}$ there is a unique arrow $-\infty \to X$. Considering the dual of the example above, let $\mathscr{C}$ denote the category $(\bullet \to \bullet \leftarrow \bullet)$, then the construction of the left cone $\mathscr{C}^{\triangleleft}$ completes $\mathscr{C}$ to a square. Dually, left cones have some relations to diagrams and limits viewed as universal arrows, which also are discussed further in Section 3.2.4.

There is a corresponding notion of joins of simplicial sets, which are defined in the following way.

Definition 3.2.1.1. Let S and T be simplicial sets. The *join* of S and T denoted $S \star T$ is defined to be the simplicial set with

$$(S \star T)_n = S_n \cup \left(\bigcup_{i+j=n-1} S_i \times T_j \right) \cup T_n. \tag{3.2.1.ii}$$

In order to prepare for the next proposition there will first be introduced some notation. Let $\mathbf{sSet}_{S/}$ denote the category of simplicial sets under S (the definition of undercategories can be found at the beginning of Section 3.2.2). Then the join operation induces a functor

$$S \star (-) : \mathbf{sSet} \to \mathbf{sSet}_{S/} \tag{3.2.1.iii}$$

obtained by sending $T \mapsto S \star T$, while a map $f : T \to V$ of simplicial sets is sent to the morphism $S \star T \to S \star V$ obtained from the identity when restricted to the S part and f restricted to the T part of $S \star T$, which obviously gives a morphism in $\mathbf{sSet}_{S/}$. Similar comments can be applied to the induced functor

$$(-) \star S : \mathbf{sSet} \to \mathbf{sSet}_{S/}\,. \tag{3.2.1.iv}$$

Moreover, the join operation on simplicial sets can be characterised by the following properties.

Proposition 3.2.1.2. Let S and T be simplicial sets, then

(i) the induced functors $S \star (-) : \mathbf{sSet} \to \mathbf{sSet}_{S/}$ and $(-) \star S : \mathbf{sSet} \to \mathbf{sSet}_{S/}$ preserves colimits

(ii) for representable simplicial sets Δ^k there are natural isomorphisms $\phi_{i,j} : \Delta^{i-1} \star \Delta^{j-1} \to \Delta^{(i+j)-1}$ for all $i, j \geq 0$.

In fact, the isomorphisms $\phi_{i,j}$ equips **sSet** with a monoidal structure with tensor unit given by the empty simplicial set Δ^{-1}.

This proposition is similar to [Gro10, Proposition 2.6], and corresponding notions are also discussed in [Lur09, Section 1.2.8].

The ordinary nerve functor carries in fact joins of ordinary categories over to joins of simplicial sets. More precisely, there are natural isomorphisms

$$N(\mathscr{C} \star \mathscr{D}) \cong N(\mathscr{C}) \star N(\mathscr{D}) \tag{3.2.1.v}$$

for ordinary categories $\mathscr{C}$ and $\mathscr{D}$, as stated in [Lur09, Section 1.2.8]. This gives the first indication that the join construction on simplicial sets is the appropriate notion of join construction for quasi-categories. In fact, a join of quasi-categories is again a quasi-category, and hence quasi-categories are closed under joins.

Proposition 3.2.1.3. If $\mathscr{C}$ and $\mathscr{D}$ are quasi-categories, then the join construction $\mathscr{C} \star \mathscr{D}$ is again a quasi-category.

Remark 3.2.1.4. A similar proposition is stated in [Lur09, Proposition 1.2.8.3]. This proposition is due to Joyal, as stated before [Lur09, Proposition 1.2.8.3]. Now it will be sketched some ideas of the proof, but a slightly more rigid discussion can be found in the proof of [Lur09, Proposition 1.2.8.3].

"Proof" of Proposition 3.2.1.3: So, the inner horn filler property has to be checked for $\mathscr{C} \star \mathscr{D}$ in order to prove the proposition. Let $p : \Lambda^n_i \to \mathscr{C} \star \mathscr{D}$ be a horn in $\mathscr{C} \star \mathscr{D}$, where $0 < i < n$ and $n > 0$. There are two mutually exclusive cases to consider, (a) if p carries the horn entirely to either $\mathscr{C}$ or $\mathscr{D}$, or (b) if p does not,

(a) the required horn filler property follows immediately, since both $\mathscr{C}$ and $\mathscr{D}$ are quasi-categories from assumption

(b) assume without loss of generality that p carries the vertices $\{0, 1, \ldots, j\}$ into the $\mathscr{C}$ part and the vertices $\{j+1, j+2, \ldots, n\}$ into the $\mathscr{D}$ part of $\mathscr{C} \star \mathscr{D}$. Now, restrict p to these vertices in order to obtain maps $\Delta^{\{0,1,\ldots,j\}} \to \mathscr{C}$ and $\Delta^{\{j+1,j+2,\ldots,n\}} \to \mathscr{D}$ which together determine a map $\omega : \Delta^n \to \mathscr{C} \star \mathscr{D}$. This map ω is then a filler for p, which proves the required horn filler property.

The mutually exclusive cases (a) and (b) prove the proposition. □

The definition of left cones and right cones are inhered from join constructions of ordinary categories to the join construction of simplicial sets,

Definition 3.2.1.5. Let S be a simplicial set, the *right cone on* S is the join $S^{\triangleright} := S \star \Delta^0$ while the *left cone on* S is the join $S^{\triangleleft} := \Delta^0 \star S$.

The discussion of join constructions closes up with some comments about square shaped diagrams in quasi-categories, which take some ideas from the discussion for square shaped ordinary categories at the beginning of this section. Let $\mathscr{C}$ be a quasi-category, recall that the notion of a diagram in $\mathscr{C}$ should be interpreted as a homotopy coherent diagram. Hence, a square in $\mathscr{C}$ should be interpreted as a simplicial map $\Delta^1 \times \Delta^1 \to \mathscr{C}$ visualised by the following vertices and edges forming this square shaped diagram in $\mathscr{C}$,

$$\begin{tikzcd} X' \arrow[r, "p'"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ Y' \arrow[r, "p"'] & Y, \end{tikzcd}$$

as illustrated in [Lur09, Section 4.4.2]. Such diagram actually contains the information of an edge going "diagonally" $r : X' \to Y$ together with homotopies $\alpha : r \simeq p \circ q'$ and $\beta : r \simeq q \circ p'$ satisfying their higher relations, etcetera.

Analogously with square example for ordinary categories, where the ordinary category $(\bullet \to \bullet \leftarrow \bullet)$ can be made into a commutative square by the right cone, the horn Λ^2_0 in $\mathscr{C}$ can visually be made into a square by the right cone $(\Lambda^2_0)^{\triangleright} \cong (\Delta^1)^2$ in $\mathscr{C}$, since the data of this right cone and the square in fact is the same. Similarly, the horn Λ^2_2 in $\mathscr{C}$ can also be made into a square by the left cone $(\Lambda^2_2)^{\triangleleft} \cong (\Delta^1)^2$ in $\mathscr{C}$, analogously with that the category $(\bullet \leftarrow \bullet \to \bullet)$ can be made into a square by the left cone. Later, the description $(\Lambda^2_0)^{\triangleright} \cong (\Delta^1)^2 \cong (\Lambda^2_2)^{\triangleleft}$ will be used for the discussions of colimits and limits, in particular of pushouts and pullbacks.

3.2.2 Overcategories and undercategories

The aim of this part is to introduce the notions of overcategories and undercategories for $(\infty, 1)$-categories, which ideally should be called over-$(\infty, 1)$-categories and under-$(\infty, 1)$-categories, and frequently will be referred to as $(\infty, 1)$-categorical slice constructions. As usual, the discussion here builds on intuition from the notions of overcategories and undercategories form ordinary category theory, and will be formalised in the theory of quasi-categories. So, let $\mathscr{C}$ be a category.

Recall from ordinary category theory that the *category of objects over* $X \in \mathscr{C}$ or the *overcategory*, denoted $\mathscr{C}_{/X}$, is a category defined by,

- the objects in $\mathscr{C}_{/X}$ are morphisms in $\mathscr{C}$ of the form $Y \to X$ ending in X
- the morphisms in $\mathscr{C}_{/X}$ are given by morphisms in $\mathscr{C}$ making triangles of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d, equal] \\ Z \arrow[r] & X \end{tikzcd}$$

  commutative.

Composition is defined the obvious way.

However, this definition can be rephrased in terms of joins of ordinary categories. Let $x : [0] \to \mathscr{C}$ denote the functor that picks out $X \in \mathscr{C}$. The overcategory $\mathscr{C}_{/X}$ can be described by the following universal property, for any category $\mathscr{D}$ there is a bijection

$$\mathrm{Hom}(\mathscr{D}, \mathscr{C}_{/X}) \cong \mathrm{Hom}_x(\mathscr{D} \star [0], \mathscr{C}) = \mathrm{Hom}_x(\mathscr{D}^{\triangleright}, \mathscr{C}), \tag{3.2.2.i}$$

where $\mathrm{Hom}_x(\mathscr{D}^{\triangleright}, \mathscr{C})$ denotes the subset $\mathrm{Hom}_x(\mathscr{D}^{\triangleright}, \mathscr{C}) \subseteq \mathrm{Hom}(\mathscr{D}^{\triangleright}, \mathscr{C})$ that only contains functors $F : \mathscr{D}^{\triangleright} \to \mathscr{C}$ restricted to the cone (to the adjoined object ∞) coincides with the map x, namely $\mathrm{Hom}_x(\mathscr{D}^{\triangleright}, \mathscr{C}) = \{F : \mathscr{D}^{\triangleright} \to \mathscr{C} \quad | \quad F|_{\infty} = x\}$. The characterisation of $\mathscr{C}_{/X}$ in Equation 3.2.2.i can be observed directly from the categorical constructions of cones and overcategories, as indicated in [Gro10, Section 2.3].

Alternatively, form definition, $\mathrm{Hom}_x(\mathscr{D}^{\triangleright}, \mathscr{C})$ can be obtained from a pullback

$$\begin{tikzcd} \mathrm{Hom}_x(\mathscr{D}^{\triangleright}, \mathscr{C}) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathscr{D}^{\triangleright}, \mathscr{C}) \arrow[d] \\ \{x : [0] \to \mathscr{C}\} \arrow[r] & \mathrm{Hom}([0], \mathscr{C}) \end{tikzcd}$$

over **Set**, which can give a slightly more "category theoretical" approach to the interpretation of the characterisation of $\mathscr{C}_{/X}$ in Equation 3.2.2.i.

More generally, given an arbitrary functor $f : \mathscr{I} \to \mathscr{C}$ of ordinary categories that can be thought of as a diagram in $\mathscr{C}$ as usual, then the *category over*

f, denoted $\mathscr{C}_{/f}$, can be defined to be the category that is characterised by the following property

$$\mathrm{Hom}(\mathscr{D}, \mathscr{C}_{/f}) \cong \mathrm{Hom}_f(\mathscr{D} \star \mathscr{I}, \mathscr{C}), \tag{3.2.2.ii}$$

for any other category $\mathscr{D}$, where the set $\mathrm{Hom}_f(\mathscr{D} \star \mathscr{I}, \mathscr{C})$ denotes the set of functors $\mathscr{D} \star \mathscr{I} \to \mathscr{C}$ that agrees with f when restricted to $\mathscr{I}$. This can also be described from a pullback, similar to the one above.

The appropriate notion of overcategories for quasi-categories can be adapted more or less directly from the characterisations in Equation 3.2.2.i and Equation 3.2.2.ii above. This was done by Joyal, but the discussion here is based on studies of [Lur09, Section 1.2.9] and [Gro10, Section 2.3].

First adapt the notation used previously from ordinary categories to simplicial sets. For simplicial sets S, T, K and a simplicial map $p : K \to S$, the set $\mathrm{Hom}_p(T \star K, S)$ denotes the set of simplicial maps $T \star K \to S$ that restricted to K coincide with p. The set $\mathrm{Hom}_p(T \star K, S)$ can also be regarded as obtained from an appropriate pullback, similarly as in the ordinary category theoretical case above. The next proposition ensures existence of a simplicial set with similar characterisation as ordinary categorical overcategories in Equation 3.2.2.ii.

Proposition 3.2.2.1. Let K and S be simplicial sets and let $p : K \to S$ be a map of simplicial sets. Then there exits a simplicial set $S_{/p}$ characterised by the following universal property, for any simplicial set T there are bijections

$$\mathrm{Hom}_{\mathbf{sSet}}(T, S_{/p}) \cong \mathrm{Hom}_p(T \star K, S), \tag{3.2.2.iii}$$

where $\mathrm{Hom}_p(T \star K, S)$ is defined just as above.

Remark 3.2.2.2. This proposition is given as [Lur09, Proposition 1.2.9.2] and [Gro10, Proposition 2.9], and the following "proof" here sketches some ideas.

"Proof" of Proposition 3.2.2.1: First consider the case where $T = \Delta^n$ is a representable presheaf. This gives the following description of the n-simplices in $S_{/p}$,

$$(S_{/p})_n \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, S_{/p}) \cong \mathrm{Hom}_p(\Delta^n \star K, S), \tag{3.2.2.iv}$$

where the first isomorphism is the Yoneda lemma and the last is the required characterisation in the proposition. So from this it is clear that n-simplices $(S_{/p})_n$ obviously exist from definition of $\mathrm{Hom}_p(\Delta^n \star K, S)$. In fact, the middle and the right entry in Equation 3.2.2.iv above are compatible with colimits in the entry $T = \Delta^n$, which fact is stated in the proof of [Lur09, Proposition 1.2.9.2]. This fact extends the result for representable presheaves carried out in Equation 3.2.2.iv to arbitrary simplicial sets T, which proves the proposition. $\square$

Next concern is whether or not quasi-categories are closed under slice constructions, or more precisely this can be phrased as, if S in the proposition above is a quasi-category and $p : K \to S$ a map of simplicial sets, is then the slide construction $S_{/p}$ again a quasi-category? This is discussed in [Lur09, Proposition 1.2.9.3], but its proof requires some more details from [Lur09, Chapter 2], and will not be presented here. However, next is presented a special case of [Lur09, Proposition 1.2.9.3].

Proposition 3.2.2.3. Let $\mathscr{C}$ be a quasi-category, let K be a simplicial set and let $p : K \to \mathscr{C}$ be a map simplicial sets. Then the simplicial set $\mathscr{C}_{/p}$ from Proposition 3.2.2.1 is a quasi-category.

Definition 3.2.2.4. Let $\mathscr{C}$ be a quasi-category and let $p : K \to \mathscr{C}$ be a map of simplicial sets. Then the quasi-category $\mathscr{C}_{/p}$, that arises from the arguments in Proposition 3.2.2.3 above, is called a *quasi-category* over p, an *over-quasi-category* or simply an *overcategory* for short.

This interpretation of overcategories for quasi-categories can be motivated by the fact that it can be shown to exists canonical isomorphisms between nerves of over-ordinary-categories and over-quasi-categories of nerves. More precisely, for an object X in an ordinary category $\mathscr{C}$ there can be obtained a canonical isomorphism $N(\mathscr{C}_{/X}) \cong (N(\mathscr{C}))_{/X}$, as stated in [Lur09, Remark 1.2.9.6]. This establishes the first example of a over-quasi-category, and gives a first clue that the definition above is an appropriate one after all.

The notion of an overcategory (in the quasi-categorical sense) can be dualised by interchanging $(T \star K)$ with $(K \star T)$ in the definition. Hence, let $\mathscr{C}$ be a quasi-category and let $p : K \to \mathscr{C}$ be a simplicial map, then the *undercategory* denoted $\mathscr{C}_{p/}$ is defined to be the quasi-category that arises from the following universal property,

$$\mathrm{Hom}_{\mathbf{sSet}}(T, \mathscr{C}_{p/}) \cong \mathrm{Hom}_p(K \star T, \mathscr{C}), \tag{3.2.2.v}$$

for every simplicial set T.

The constructions of overcategories and undercategories will be used in the establishments of initial objects, terminal objects, limits and colimits for quasi-categories, that will follow in the next sections.

3.2.3 Initial objects and terminal objects

The aim for this section is to define the notions of initial and terminal objects for $(\infty, 1)$-categories. Recall that for an ordinary category $\mathscr{C}$, an object I in $\mathscr{C}$

is said to be initial if $\mathrm{Hom}_{\mathscr{C}}(I,Y)$ consists of a single element for all objects Y in $\mathscr{C}$, that is $\mathrm{Hom}_{\mathscr{C}}(I,Y)$ is trivial for all objects Y in $\mathscr{C}$. Dually, an object T is said to be terminal if $\mathrm{Hom}_{\mathscr{C}}(Y,T)$ is trivial for all other objects Y, namely that T is initial in the opposite category $\mathscr{C}^{\mathrm{op}}$. The aim now is to adapt these ideas to the setting of $(\infty,1)$-categories.

In the first approach, in order to adapt these ideas to the $(\infty,1)$-categorical language, it should be searched for an appropriate notion of "trivial" mapping spaces. So, with the homotopical structure of $(\infty,1)$-categories in mind, it should be searched for a notion of mapping spaces being homotopic to a point. More precisely, let $\mathscr{C}$ for example be a topological category, then an object T is said to be *terminal* if the mapping spaces $\mathrm{Map}_{\mathscr{C}}(X,T)$ are (weakly) contractable for all objects X in $\mathscr{C}$. Geometrically, replacing a terminal object T by an equivalent one mapping spaces remain contractable, hence this suggestion of notion of terminal objects can be regarded as an invariant notion under equivalences.

For a topological category $\mathscr{C}$ with a terminal object T, when passing to the homotopy category $\mathrm{h}\mathscr{C}$, it is clear that T now is terminal in the ordinary categorical sense in $\mathrm{h}\mathscr{C}$. This follows from construction of the homotopy category by identifying morphisms that lies in the same path connected components. In fact, T is terminal in the ordinary categorical sense in $\mathrm{h}\mathscr{C}$ if and only if $\mathrm{Map}_{\mathscr{C}}(X,T)$ is (weakly) contractable for all $X \in \mathscr{C}$. This first approach to the definition of terminal objects is taken from [Lur09, Definition 1.2.12.1].

Definition 3.2.3.1. Let $\mathscr{C}$ be a topological category or a simplicial category, an object T in $\mathscr{C}$ is said to be *terminal* if T is terminal in the homotopy category $\mathrm{h}\mathscr{C}$, regarded as $\mathscr{H}$-enriched category. Equivalently, T is terminal if the mapping spaces $\mathrm{Map}_{\mathscr{C}}(X,T)$ are (weakly) contractable for all $X \in \mathscr{C}$.

The second approach to terminal objects is obtained from slice constructions. Again, the motivation behind this approach is taken from properties in the theory of ordinary categories. If T is a terminal object in an ordinary category $\mathscr{C}$, then the category of objects over T is equivalent to the category itself, $\mathscr{C}_{/T} \cong \mathscr{C}$. This follows directly from definition of T being terminal, since for any object $X \in \mathscr{C}$ observe that there is a unique $u_X : X \to T$, hence for any $f : X \to Y$ the composition $u_Y f = u_X : X \to T$. These observations coincide precisely with the definition of $\mathscr{C}_{/T}$.

This gives some motivation for the approach to terminal objects for simplicial sets, since there is defined a notion of slice constructions for these. But, before that it should be remarked that the notion of 'isomorphism' is a too strict notion in order to define an appropriate analogy for quasi-categories, or simplicial sets, as usual. Analogous to the first approach based on mapping spaces, there should be searched for a quasi-categorical similarity to isomorphism, which here turns out to be trivial Kan fibrations, as usual for quasi-categories.

Definition 3.2.3.2. Let S be a simplicial set, a vertex T in S is said to be *strongly terminal* if the projection $S_{/T} \to S$ is a trivial Kan fibration.

The next proposition, which is also stated as [Lur09, Proposition 1.2.12.4], gives the first connection between these two approaches to terminal objects, but for quasi-categories.

Proposition 3.2.3.3. Let $\mathscr{C}$ be a quasi-category and let T be an object in $\mathscr{C}$. The object T is strongly final if and only if for every object X in $\mathscr{C}$ the Kan complex $\mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(X,T)$ is contactable.

Remark 3.2.3.4. With inspiration from the proof of the similar result in [Lur09, Proposition 1.2.12.4], some ideas of the direction "$\Rightarrow$" will be discussed next, while the other direction "$\Leftarrow$" possibly requires some results from [Lur09, Chapter 2], which not will be discussed here.

"Proof" of Proposition 3.2.3.3: First, recall that the space of right morphisms $\mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(X,T)$ is defined to be the set of simplicial maps $\Delta^n \to \mathscr{C}$, which restricted to $\Delta^{\{0,1,\dots,n-1\}}$ maps to cons_X^{n-1}, while restricted to $\Delta^{\{n\}}$ maps to T. From construction, it can easily be shown that $\mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(X,T)$ can be obtained by a pullback $\mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(X,T) = \{X\} \times_{\mathscr{C}} \mathscr{C}_{/T}$,

$$\begin{tikzcd}
\{X\} \times_{\mathscr{C}} \mathscr{C}_{/T} \arrow[r] \arrow[d] & \mathscr{C}_{/T} \arrow[d] \\
\{X\} \arrow[r] & \mathscr{C}
\end{tikzcd}$$

over **sSet**. But this pullback precisely defines a fiber over $\mathscr{C}_{/T} \to \mathscr{C}$ (at $\{X\}$).

Now for "$\Rightarrow$" assume that T is strongly final, then $\mathscr{C}_{/T} \to \mathscr{C}$ is a trivial fibration. Now since $\mathscr{C}_{/T} \to \mathscr{C}$ is assumed to be a trivial Kan fibration this implies that its fibers are contractable Kan complexes. Hence, $\mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(X,T)$ is then a contractable Kan complex, since it arises as a fiber over $\mathscr{C}_{/T} \to \mathscr{C}$ (at $\{X\}$). $\square$

First observe that for simplicial sets strongly terminal objects are terminal. The aim now is to state some ideas from [Lur09, Corollary 1.2.12.5] in order to see this. Let S be a simplicial set with a strongly final object T then there exists a split epimorphism $f : S^{\triangleright} \to S$ mapping the cone point $f : \infty \mapsto T$. By functorality, f is carried over to a split epimorphism $\overline{f} : \mathrm{h}S^{\triangleright} \to \mathrm{h}S$, which maps the cone point $\overline{f} : \infty \mapsto T$. Hence T is terminal in $\mathrm{h}S$, and by definition terminal in S.

Secondly observe that for quasi-categories the converse is also true, terminal objects are strongly terminal. This is also stated in [Lur09, Corollary 1.2.12.5]. To see this, let $\mathscr{C}$ be a quasi-category and let T be a terminal object in $\mathscr{C}$. Since the Kan complex of right morphisms represents the mapping space of $\mathscr{C}$ (up to homotopy), then $\mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(X, T)$ is contractable from definition of terminal object. The proof of Proposition 3.2.3.3 above establishes that $\mathrm{Hom}^{\mathrm{R}}_{\mathscr{C}}(X, T)$ is the fiber over $\mathscr{C}_{/T} \to \mathscr{C}$, which then is a trivial Kan fibration. Finally, T is strongly terminal, since $\mathscr{C}_{/T} \to \mathscr{C}$ is a trivial Kan fibration.

Definition 3.2.3.5. Let $\mathscr{C}$ be a topological category, a simplicial category or a quasi-category. The dual interpretation of terminal objects are called *initial* objects.

As usual the behaviour of nerves of terminal (or initial) objects gives the first indication if the notion of terminal (or initial) objects determined for quasi-categories is the appropriate one. In fact, as exemplified in [Lur09, Example 1.2.12.7], let $\mathscr{C}$ be an ordinary category and let T be an object in $\mathscr{C}$, then T is terminal in $\mathscr{C}$ if and only if it is terminal in $N(\mathscr{C})$. Similar remarks applies if T happens to be initial.

In ordinary categories, initial objects, and terminal objects, are unique up to unique isomorphism. As usual the analogy to this notion of uniqueness for quasi-categories is unique up to a contractible space of choices. More precisely, the collection of all terminal objects in a quasi-category $\mathscr{C}$, if there are any, should be a contractable Kan complex. This is stated in the next proposition, which is due to Joyal. A proof is presented in [Lur09, Proposition 1.2.12.9]. Similar notions is also discussed in [Gro10, Remark 2.13].

Proposition 3.2.3.6. Let $\mathscr{C}$ be a quasi-category and let $\mathscr{D}$ denote the full subcategory spanned by the terminal objects of $\mathscr{C}$. Then $\mathscr{D}$ is either empty or a contractable Kan complex.

The notion of initial and terminal objects plays a central role in the discussion of colimits and limits for $(\infty, 1)$-categories, which is coming next. Moreover, existence of zero objects, which are both initial and terminal, is one of the key ingredients in the definition of stable $(\infty.1)$-categories in the next chapter.

3.2.4 Limits and colimits

Similar as in the presentation of initial objects, there will be discussed two approaches to the notions of limits and colimits in the theory of $(\infty, 1)$-categories. In particular, the first approach is obtained from a characterisation by using homotopy limits and homotopy colimits, which is primarily motivated by considering the theory of simplicial categories and topological categories. While the second approach is obtained from terminal objects and initial objects in slice categories of certain diagram functors, which is motivated by considering the theory of quasi-categories. Let in any of the approaches $\varprojlim(p)$ and $\varinjlim(p)$ denote the appropriate notion of $(\infty, 1)$-categorical limit and colimit respectively of a diagram $p : \mathscr{I} \to \mathscr{C}$.

Although the notions of terminal objects and initial objects in a topological category, or a simplicial category, $\mathscr{C}$ seem to work well when defined as initial objects and terminal objects in the homotopy category $\mathrm{h}\mathscr{C}$, a similar approach, by defining the notions of $(\infty, 1)$-categorical limits and colimits via the homotopy category $\mathrm{h}\mathscr{C}$ does not always work. Conceptually, a problem with this approach can be that such constructions on the homotopy category do not require enough structure to obtain the appropriate notions. This can be regarded as a consequence of the difference between the notions of homotopy commutativity and homotopy coherency in the $(\infty, 1)$-categorical setting, where the first notion may not consider all required higher structure.

The appropriate $(\infty, 1)$-categorical notions of limits and colimits are often referred to as *homotopy limits* and *homotopy colimits* respectively, in order to distinguish to the ordinary notions. There will be no explicit description of homotopy limits and homotopy colimits for topological categories here, such as given in [Lur09, Example 1.2.13.1 and Example 1.2.13.2] with further discussions in [Lur09, Appendix A.2.8], but rather a characterisation of these notions in relation to homotopy limits and homotopy colimits in topological spaces. This characterisation is discussed in the following remark.

Remark 3.2.4.1. Let $\mathscr{C}$ be a topological category and let $p : \mathscr{I} \to \mathscr{C}$ be functor representing a diagram in $\mathscr{C}$. The homotopy limit $\varprojlim(p)$ of p in $\mathscr{C}$, if it exists, can be characterised by a homotopy limit in topological spaces, denoted $\mathrm{holim}(-)$, up to equivalence, by existence of a weak homotopy equivalence

$$\mathrm{Map}_{\mathscr{C}}(Y, \varprojlim(p)) \xrightarrow{\simeq} \mathrm{holim}_{\alpha\in\mathscr{I}}\{\mathrm{Map}_{\mathscr{C}}(Y, p(\alpha))\}, \tag{3.2.4.i}$$

which is natural in $Y \in \mathscr{C}$. Moreover, the homotopy colimit $\varinjlim(p)$ of p in $\mathscr{C}$ if it exists can be characterised, up to equivalence, by existence of a weak homotopy equivalence

$$\mathrm{Map}_{\mathscr{C}}(\varinjlim(p), Y) \xrightarrow{\simeq} \mathrm{holim}_{\alpha\in\mathscr{I}^{\mathrm{op}}}\{\mathrm{Map}_{\mathscr{C}}(p(\alpha), Y)\}, \tag{3.2.4.ii}$$

which is natural in $Y \in \mathscr{C}$.

This characterisation of limits and colimits for topological categories can be motivated from the characterisation of limits and colimits in ordinary category theory obtained by the following bijections

$$\mathrm{Hom}_{\mathscr{C}}(Y, \lim(p)) \to \lim_{\alpha \in \mathscr{I}}\{\mathrm{Hom}_{\mathscr{C}}(Y, p(\alpha))\} \tag{3.2.4.iii}$$

$$\mathrm{Hom}_{\mathscr{C}}(\mathrm{colim}(p), Y) \to \lim_{\alpha \in \mathscr{I}^{\mathrm{op}}}\{\mathrm{Hom}_{\mathscr{C}}(p(\alpha), Y)\}, \tag{3.2.4.iv}$$

which are natural in Y.

The generalisation to the characterisations in Remark 3.2.4.1 above can be regarded as a first expected approach to the notion of limits (and colimits) for $(\infty, 1)$-categories. Geometrically, one of the ideas behind the notion of homotopy limits for topological spaces can be regarded as the construction obtained when replacing the strict notion of isomorphism in ordinary categorical limits by the homotopical corresponding notion, which can be regarded as paths. The characterisations in Remark 3.2.4.1 are also stated in [Cam13, Section 5.2] and [Lur09, Remark 1.2.13.3] with references to a further discussion in [Lur09, Remark A.3.3.13].

The aim now is to exemplify the characterisations in Remark 3.2.4.1 by the established notions of initial objects and terminal objects. Recall that initial objects and terminal objects can be regarded as trivial colimits and trivial limits respectively in ordinary categories, which means that the index category $\mathscr{I}$ is trivial. If $\mathscr{I}$ is trivial in the characterisation in Remark 3.2.4.1, then it can easily be observed that the mapping spaces have the same homotopy type as the point for all Y, and they are then contractable, as required.

It turns out that the characterisations in Remark 3.2.4.1 are important in proving that the homotopy category of a stable $(\infty, 1)$-category can be regarded as an enriched category over abelian groups. However, the next aim is to discuss the promised second approach to limits and colimits, which can be motivated from the interpretation of limits and colimits in ordinary categories as universal arrows, which is described in [Mac98, Chapter III].

Let $p : \mathscr{I} \to \mathscr{C}$ be a functor of ordinary categories representing a diagram in $\mathscr{C}$ and let $\delta : \mathscr{C} \to \mathrm{Fun}(\mathscr{I}, \mathscr{C})$ be the diagonal map. Let $C \in \mathscr{C}$, then a natural transformation $\tau : \delta C \to F$ is called a *cone from the vertex C to the base p*. This can be visualised by the shape of various components of τ all starting at C spreading out to targets of p evaluated on arrows in $\mathscr{I}$, for example

$$\tau_J = p(u)\tau_I : C \to p(J) \tag{3.2.4.v}$$

for all arrows $u : I \to J$ in $\mathscr{I}$.

Moreover with the same notation as in the previous paragraph, for $C \in \mathscr{C}$ all cones $\delta C \to p$ can be organised into a category $\mathrm{cone}(\mathscr{C}, p)$, which is precisely equivalent to the description of the slice category $\mathscr{C}_{/p}$ by construction. Now a limit (in the ordinary sense) is a *universal cone* from the vertex denoted $\lim(p)$ to the base p, namely $\delta \lim(p) \to p$ is a universal cone, which means that for any other cone $\delta C \to p$ there exists a unique natural transformation $\delta C \to \delta \lim(p)$ in $\mathrm{cone}(\mathscr{C}, p)$. This implies that $\lim(p) \to p$ can be regarded as terminal in the slice $\mathscr{C}_{/p}$. Similarly, an (ordinary category theoretical) colimit of a diagram $p : \mathscr{I} \to \mathscr{C}$ can be regarded as an initial object in the slice $\mathscr{C}_{p/}$. This motivates for definitions of limits and colimits for quasi-categories, due to Joyal, but the next definition is stated similarly as [Lur09, Definition 1.2.13.4].

Definition 3.2.4.2. Let $\mathscr{C}$ be a quasi-category and let $p : K \to \mathscr{C}$ be a map of simplicial sets. A *colimit* for p is defined to be an initial object of the undercategory $\mathscr{C}_{p/}$ denoted $\varinjlim(p)$, while a *limit* for p is defined to be a terminal object of the overcategory $\mathscr{C}_{/p}$ denoted $\varprojlim(p)$.

Remark 3.2.4.3. Let $p : K \to \mathscr{C}$ be a diagram in a quasi-category $\mathscr{C}$. From definition a limit $\varprojlim(p)$, if it exists, is a terminal object in $\mathscr{C}_{/p}$, so in particular $\varprojlim(p)$ is an object in $\mathscr{C}_{/p}$, which all can be represented by maps $\Delta^0 \to \mathscr{C}_{/p}$. Recall that the slice $\mathscr{C}_{/p}$ can be characterised by, for any simplicial set T, those simplicial maps $T \star K \to \mathscr{C}$ which agree with p when restricted to T, namely $\mathrm{Hom}_{\mathbf{sSet}}(T, \mathscr{C}_{/p}) \cong \mathrm{Hom}_p(T \star K, \mathscr{C})$. In particular, choose $T = \Delta^0$, as a special case limits $\varprojlim(p)$ correspond to simplicial maps $\Delta^0 \star K \to \mathscr{C}$ when restricted to K coincide with p. Generally simplicial maps of the form $K^{\lhd} \to \mathscr{C}$ are referred to as *limiting diagrams*. The dual notion of a *colimiting diagrams* are simplicial maps $K^{\rhd} \to \mathscr{C}$ when restricted to K coincide with p.

In ordinary category theory limits and colimits are uniquely determined up to unique isomorphism. For quasi-categories, recall from the previous section that the collection of initial objects is either empty or a contractable Kan complex (Proposition 3.2.3.6). Similar result applies for the collection of terminal objects. Now, when limits and colimits are defied to be terminal objects and initial objects respectively in appropriate slice categories, then the same proposition can be applied. So, let $p : K \to \mathscr{C}$ be a simplicial map and $\mathscr{C}$ a quasi-category, then the collection of all limits $\varprojlim(p)$ is either empty or a contractable Kan complex. Similar for the collection of all colimits $\varinjlim(p)$ is either empty or a contractable Kan complex.

Similar as for the ordinary categories a *pullback* is defined to be a limit of the diagram visualised by $\alpha = (\bullet \to \bullet \leftarrow \bullet)$. In Section 3.2.1 it was observed that square shaped diagrams completing this α can be visualised as a simplicial map $p : (\Lambda^2_2)^{\lhd} \to \mathscr{C}$, so a pullback can then be regarded as a terminal object in the slice $\mathscr{C}_{/p}$. Similarly a *pushout* can be regarded as an initial object of $\mathscr{C}_{q/}$, where $q : (\Lambda^2_0)^{\rhd} \to \mathscr{C}$. Pullbacks and pushouts for $(\infty, 1)$-categories, now formalised by quasi-categories, are important in the definition and theory of stable $(\infty, 1)$-

categories discussed in the next chapter.

Chapter 4

Stable $(\infty, 1)$-categories

Recall from the introduction in Section 1.1 that stable $(\infty, 1)$-categories can in some situations be regarded as a better behaved replacement for triangulated categories, which may fix some drawbacks in their theory ([Cam13]). Moreover, many prominent examples of triangulated categories arise as homotopy categories of stable $(\infty, 1)$-categories ([Cam13]). The discussion in this chapter will not go into these motivating comments. But, the overall aim for this thesis after all is to prove that the homotopy category of stable $(\infty, 1)$-categories is in fact a triangulated category. The discussion here is mainly based on studies of [Lur12, Section 1.1.1 and Section 1.1.2], with inspiration from [Cam13, Section 5.4.2] and [Gro10, Section 5.1].

Moreover, first objective here is to define the notion of stable $(\infty, 1)$-categories, which is a definition closely related to classical homotopy theory. Similarly, fiber sequences and cofiber sequences arise as particular pullback diagrams and pushout diagrams respectively. In the setting of quasi-categories it is known from Section 3.2.4 how these can be constructed. Then the next objective is to define suspension and loop functors for $(\infty, 1)$-categories that admit cofibers and fibers respectively. The last objective is to present a proof of the main result, namely that a homotopy categories of a stable $(\infty, 1)$-categories is triangulated.

4.1 The notion of $(\infty, 1)$-categorical stability

The aim for this section is to discuss the data that go into the definition of a stable $(\infty, 1)$-category. These are motivated by notions in classical stable homotopy theory.

4.1.1 Pointed $(\infty, 1)$-categories

The idea of a pointed quasi-category is similar to the idea for pointed topological spaces or the unit element in groups, namely existence of an object that is both initial and terminal. The first definition corresponds to the notion of zero objects for $(\infty, 1)$-categories.

Definition 4.1.1.1. An object in an $(\infty, 1)$-category is said to be a *zero object* if it is both initial and terminal. If an $(\infty, 1)$-category contains a zero object it is said to be *pointed*. A zero object is often denoted by the symbol 0.

Recall the definitions of initial objects and terminal objects for topological categories, or simplicial categories, and quasi-categories from Section 3.2.3. Moreover, let for example $\mathscr{C}$ be a topological category that contains zero objects. From definition of zero objects of both being initial and terminal, it follows directly that the mapping spaces $\mathrm{Map}_{\mathscr{C}}(X, 0)$ and $\mathrm{Map}_{\mathscr{C}}(0, X)$ are both contractable, where $0 \in \mathscr{C}$ is a zero object. In particular, let $\mathscr{C}$ be a quasi-category, in Proposition 3.2.3.6 (or in [Lur09, Proposition 1.2.12.9]) it was stated that the full subcategory of $\mathscr{C}$ spanned by the initial objects of $\mathscr{C}$, or dually the full subcategory of $\mathscr{C}$ spanned by the terminal objects of $\mathscr{C}$, is either empty or a contractable Kan complex. The full subcategory spanned by zero objects is either empty or a contractable Kan complex as well. As usual determination up to a contractable Kan complex is the $(\infty, 1)$-categorical correspondence to unique up to isomorphism for the ordinary categorical setting.

The following remark, which also is given in [Lur12, Remark 1.1.1.2], states some conditions that the zero objects in quasi-categories should satisfy. These conditions are similar to the notion of zero objects in the theory of ordinary categories.

Remark 4.1.1.2. Let $\mathscr{C}$ be a quasi-category (topological category or simplicial category), then $\mathscr{C}$ is pointed if and only if the following conditions are satisfied,

(1) $\mathscr{C}$ has an initial object $\emptyset$

(2) $\mathscr{C}$ has a terminal object 1

(3) there exists a morphism $f : 1 \to \emptyset$ in $\mathscr{C}$

First observe that the "only if part $\Rightarrow$" is clear from the definition of zero objects being both initial and terminal. While for the converse "$\Leftarrow$" assume that (1), (2) and (3) are satisfied. From the assumption that $\emptyset$ is initial, the mapping space

$\mathrm{Map}_{\mathscr{C}}(\emptyset, 1)$ is contractable, so it is known for sure that there exists at least one morphism $g : \emptyset \to 1$ and that all morphisms $\emptyset \to 1$ are homotopic. Now, since $\emptyset$ is initial it follows that $f \circ g \simeq \mathrm{id}_{\emptyset}$, and since 1 is terminal it follows that $g \circ f \simeq \mathrm{id}_1$. This follows since the identity is clearly a map $\mathrm{id}_1 : 1 \to 1$ together with the universal properties of 1 being terminal. Similar comment applies for $\emptyset$. From this it follows that f is an equivalence with a homotopy inverse g, then $\emptyset$ is also a terminal object of $\mathscr{C}$, hence $\mathscr{C}$ is pointed as required. □

Similar as for the notion of zero morphisms in the theory of ordinary categories, in a pointed $(\infty, 1)$-category there is a homotopy class of morphisms between any two objects X and Y factoring through the zero object. The morphisms that are represented by this homotopy class are called *zero morphisms*. Again, zero morphisms are uniquely determined in the appropriate $(\infty, 1)$-categorical sense. The next remark, which is similar to [Lur12, Remark 1.1.1.3], concerns about the notion of zero morphisms in topological categories (simplicial categories) and quasi-categories.

Remark 4.1.1.3. Let $\mathscr{C}$ be a pointed quasi-category (topological category or simplicial category) and let 0 denote a zero object in $\mathscr{C}$. For any objects X and Y in $\mathscr{C}$ the natural map,

$$\mathrm{Map}_{\mathscr{C}}(0, Y) \times \mathrm{Map}_{\mathscr{C}}(X, 0) \to \mathrm{Map}_{\mathscr{C}}(X, Y) \tag{4.1.1.i}$$

has a contractable domain, which corresponds to the notion of zero morphisms in the $(\infty, 1)$-categorical setting. When passing to the homotopy category $h\mathscr{C}$ this gives a well-defined morphism $X \to Y$, analogously to the zero morphism in the sense of ordinary categories.

4.1.2 Fibers and cofibers

Now when the notion of pointed $(\infty, 1)$-categories is established the next building block towards the definition of stable $(\infty, 1)$-categories is the notion of fibers and cofibers, which can be obtained from particular pullback and pushout squares respectively. This is analogous to classical theory. But the first aim is to establish some notation for the particular square diagram which gives rise to fibers and cofibers afterwards.

Definition 4.1.2.1. Let $\mathscr{C}$ be a pointed $(\infty, 1)$-category and let 0 denote a zero object. A *triangle* in $\mathscr{C}$ is a square shaped diagram which can be visualised by

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z. \end{tikzcd}$$

A triangle is said to be a *fiber sequence* if it is a pullback square, while a triangle is said to be a *cofiber sequence* if it is a pushout square.

Recall the discussions about square shaped diagrams for quasi-categories from Section 3.2.1 and Section 3.2.4. Some additional comments will now follow for triangles in pointed quasi-categories. So let $\mathscr{C}$ be a pointed quasi-category, a triangle in $\mathscr{C}$ is a square shaped diagram $\Delta^1 \times \Delta^1 \to \mathscr{C}$ that can be visualised by

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z. \end{tikzcd}$$

Now, the established isomorphisms

$$(\Lambda^2_0)^{\triangleright} \cong (\Delta^1)^2 \cong (\Lambda^2_2)^{\triangleleft} \tag{4.1.2.i}$$

can give a convenient understanding of the required pushout and pullback squares in the definition of cofiber sequences and fiber sequences respectively. Moreover, a triangle in $\mathscr{C}$ can from construction be described by the following data,

(1) a composable pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$

(2) a 2-simplex $\alpha : \Delta^2 \to \mathscr{C}$ visualised by the diagram

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z \end{tikzcd}$$

where h is a candidate of the composition of f with g.

(3) a 2-simplex $\beta : \Delta^2 \to \mathscr{C}$ visualised by

$$\begin{tikzcd} & 0 \arrow[dr] & \\ X \arrow[ur] \arrow[rr, "h'"'] & & Z \end{tikzcd}$$

where h clearly is homotopic with h, $h' \simeq h$, hence h' will be written h. This 2-simplex β can be viewed as a *null homotopy* of h, namely that this 2-simplex β connects h with the zero morphism.

Often a triangle is indicated by specifying the data of item (1), namely an inner horn $\Lambda_1^2 \to \mathscr{C}$ visualised by

$$X \xrightarrow{f} Y \xrightarrow{g} Z, \tag{4.1.2.ii}$$

with (2) and (3) implicitly being assumed.

Next definition establishes some more notions concerning the theory of stable $(\infty,1)$-categories. Again these notions are taken more or less directly from classical cases.

Definition 4.1.2.2. Let $\mathscr{C}$ be a pointed quasi-category, let 0 denote a zero object and let $g : X \to Y$ be a morphism in $\mathscr{C}$.

- A *fiber* of g is a fiber sequence of the form

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

  in $\mathscr{C}$, namely a pullback in $\mathscr{C}$. But often, it is the object W that is referred to as the fiber of g, and write $W = \mathrm{fib}(g)$.

- Dually, a *cofiber* of g is a cofiber sequence of the form

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

  in $\mathscr{C}$, namely a pushout in $\mathscr{C}$. Similarly, the object Z is often referred to as the cofiber of g, and write $Z = \mathrm{cofib}(g)$.

It should be clear from situation what interpretation of fibers and cofibers that is intended.

Since fibers and cofibers are determined by pullbacks and pushouts respectively they are uniquely determined up to equivalence, or more precisely. The next aim is to follow some of the ideas in [Lur12, Remark 1.1.1.7] in order to make this more precise for cofibers. The dual ideas apply for fibers.

Let $\mathscr{C}$ be a pointed quasi-category that contains a morphism $f : X \to Y$. The statement that a cofiber of f if it exists is uniquely determined up to equivalence, can here in the quasi-categorical setting be interpreted as, the map, that assigns to each morphism in $\mathscr{C}$ a cofiber, is a trivial Kan fibration. This interpretation has been seen previously for other constructions. In order to give some ideas behind this result, define $\mathscr{E}$ to be the full subcategory $\mathscr{E} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$ spanned by all square diagrams $\Delta^1 \times \Delta^1 \to \mathscr{C}$ having the property of being cofiber sequences. So, the full subcategory $\mathscr{E}$ is spanned by the triangles in $\mathscr{C}$ that have the property of being pushouts. Let

$$\theta : \mathscr{E} \to \mathrm{Fun}(\Delta^1, \mathscr{C}) \tag{4.1.2.iii}$$

denote the forgetful functor that maps each object in $\mathscr{E}$ to its top row, namely the pushout diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & Y \arrow[d, "h"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

is mapped to $g : X \to Y$, so θ forgets everything in the pushout square except for the top arrow, as illustrated. By using the result [Lur09, Proposition 4.3.2.15] twice it can be shown (as stated in [Lur12, Remark 1.1.1.7]) that $\theta : \mathscr{E} \to \mathrm{Fun}(\Delta^1, \mathscr{C})$ is a trivial Kan fibration, whose fibers are either contractable, when g happens to admit a cofiber, or empty, if g does not. In particular, if every morphism in $\mathscr{C}$ is in addition assumed to admit a cofiber, it then turns out that θ is a trivial Kan fibration. If so, since θ is a trivial Kan fibration, it clearly admits a section, which will be denoted

$$\mathrm{cofib}' : \mathrm{Fun}(\Delta^1, \mathscr{C}) \to \mathscr{E}\,. \tag{4.1.2.iv}$$

Visually the map cofib' takes a morphism $g : X \to Y$ in $\mathscr{C}$ and send it to its cofiber sequence

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z, \end{tikzcd}$$

which is uniquely determined up to a contractable space formed by the possible choices for cofiber sequences for g. Moreover, let T denote the terminal object of $\in \Delta^1 \times \Delta^1$ and let $\mathrm{ev}_T : \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C}) \to \mathscr{C}$ denote the evaluation map at the terminal T. Visually, ev_T sends each diagram in $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$ to its right-down entry in $\mathscr{C}$. The notation cofib will often be used for the composition,

$$\mathrm{Fun}(\Delta^1, \mathscr{C}) \xrightarrow{\mathrm{cofib}'} \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C}) \xrightarrow{\mathrm{ev}_T} \mathscr{C} \tag{4.1.2.v}$$

namely $\mathrm{cofib} = \mathrm{ev}_T \circ \mathrm{cofib}'$. So, visually this means that a morphism $g : X \to Y$ will be sent by cofib' to a pushout diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

namely the cofiber sequence to g, let this diagram be denoted $\alpha \in \mathscr{E}$, then evaluating α at T, it will eventually be mapped to $Z = \mathrm{cofib}(\alpha)$. Hence, $g \mapsto \alpha \mapsto Z = \mathrm{cofib}(g)$, which is a formalisation of the notation described in Definition 4.1.2.2.

Remark 4.1.2.3. In the discussion above, the fact that θ could be shown to be a trivial Kan fibration relied on the result [Lur09, Proposition 4.3.2.15]. This proposition establishes a general criteria when certain maps of simplicial sets are trivial Kan fibrations (as stated in [Gro10]). So, under the assumptions determining θ, it can be chosen a section of θ, which was called cofib'. The proposition cited here ([Lur09, Proposition 4.3.2.15]) is also central in the determinations of loop functor and suspension that will follows later.

By [Lur12, Remark 1.1.1.8] the functor $\mathrm{cofib} : \mathrm{Fun}(\Delta^1, \mathscr{C}) \to \mathscr{C}$ can also be identified as a left adjoint to the left Kan extension functor $\mathscr{C} \simeq \mathrm{Fun}(\{1\}, \mathscr{C}) \to \mathrm{Fun}(\Delta^1, \mathscr{C})$, which to each object X in $\mathscr{C}$ associates a zero morphism $0 \to X$. From [Lur09, Proposition 5.2.3.5] it can be shown that the functor cofib preserves all colimits that happen to exist in $\mathrm{Fun}(\Delta^1, \mathscr{C})$.

4.1.3 Definition of stability

With the data considered so far, the next aim is to define stable $(\infty, 1)$-categories.

Definition 4.1.3.1. An $(\infty, 1)$-category $\mathscr{C}$ is said to be *stable* if it satisfies the following conditions,

(1) there exists a zero object 0 in $\mathscr{C}$, hence $\mathscr{C}$ is pointed

(2) every morphism in $\mathscr{C}$ admits a fiber and a cofiber

(3) a triangle in $\mathscr{C}$ is a fiber sequence if and only if it is a cofiber sequence, hence a triangle is a pullback if and only if it is a pushout.

Now some relations to classical cases will be mentioned. From algebraic topology recall that a *spectrum* consists of a infinite sequence of pointed topological spaces $\{X_i\}_{i \geq 0}$ together with homeomorphisms $X_i \simeq \Omega X_{i+1}$, where Ω denotes the loop space functor. As indicated in [Lur12, Example 1.1.1.11], the collection of spectra can be organised into a stable quasi-category, which will be denoted Sp. The construction Sp can be regarded as the first canonical example of a stable quasi-category, which motivates the terminology in the definition of a stable quasi-category. A stable quasi-category can be regarded as structured somewhat similarly as Sp. In fact, the homotopy category $\mathscr{h}\mathrm{Sp}$ can be identified with the classical *stable homotopy category*.

As mentioned in [Lur12, Example 1.1.1.12], for an abelian category $\mathscr{A}$ it can be constructed a stable quasi-category, denoted $\mathscr{D}(\mathscr{A})$, under some requirements. The homotopy category $\mathscr{h}\mathscr{D}(\mathscr{A})$ of $\mathscr{D}(\mathscr{A})$ can in fact be identified with the usual derived category of $\mathscr{A}$ as it is defined in homological algebra.

So far in this section it has been seen that notions in the theory of quasi-categories formalise the ideas of what stable $(\infty, 1)$-categories should be to a rigid and appropriate definition, since the notions of pushouts, pullbacks and zero objects are well established for quasi-categories. From the formal definition it can also be observed that stability can be regarded as a property of $(\infty, 1)$-categories. The property of stability for $(\infty, 1)$-categories, as such properties should be, does not, at a first sight, involve any additional data, that should satisfies various axioms, etcetera. Definitions by properties are often attractive descriptions, since they are often much easier to deal with, without concerning all these additional data. For example additive and abelian categories can be regarded as descriptions by properties, but the definition of triangulated categories requires the additional data of a class of distinguished triangles (Definition A.2.2.1).

4.2 Constructions of suspension functor and loop functor

The objective for this section is to construct and discuss the meaning of "suspension functor" and "loop functor" for $(\infty, 1)$-categories, formalised by the theory of quasi-categories. These functors are major parts of the triangulated structure, which construction and proof of is again the overall aim for this chapter. The discussions here are mainly inspired by studies of [Gro10, Section 5.1] and [Lur12, Section 1.1.2]

But first the notion of suspension and loop functor for based topological spaces will be recalled. Let X be a based topological space, then the suspension of X, denoted ΣX, is constructed from to be the homotopy pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X, \end{tikzcd}$$

where $*$ denotes the base point. Dually, the loop functor, denoted ΩX, is constructed by the homotopy pullback

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X. \end{tikzcd}$$

More details of these notions are described in [Ark11].

4.2.1 Suspension functor

First the construction of suspension functor will be discussed. First assume that $\mathscr{C}$ is a pointed quasi-category. Define

$$\mathscr{M}^{\Sigma} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C}) \tag{4.2.1.i}$$

to be the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$ spanned by diagrams of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & Y \end{tikzcd}$$

all having the property of being pushout squares in $\mathscr{C}$, where 0 and $0'$ are zero objects in $\mathscr{C}$. Now let

$$\mathrm{ev}_I : \mathscr{M}^{\Sigma} \to \mathscr{C} \tag{4.2.1.ii}$$

denote the functor obtained from evaluating a diagram $\alpha \in \mathscr{M}^{\Sigma}$ at the initial object I of $\Delta^1 \times \Delta^1$. The functor ev_I can visually be thought of as mapping the diagram above to the upper-left entry X, namely "taking out" the "initial object" in the diagram.

Further, assume that $\mathscr{C}$ is a pointed quasi-category that admits cofibers, which again means that any morphism in $\mathscr{C}$ admits a cofiber. By applying [Lur09, Proposition 4.3.2.15] twice the following can be shown.

Proposition 4.2.1.1. Let $\mathscr{C}$ be a pointed quasi-category which admits cofibers. Then the evaluation map $\mathrm{ev}_I : \mathscr{M}^{\Sigma} \to \mathscr{C}$ is a trivial Kan fibration.

This proposition is a special case of [Gro10, Proposition 5.3], a result which also is indicated in the discussion in [Lur12, Section 1.1.2]. Moreover, let $s_{\Sigma} : \mathscr{C} \to \mathscr{M}^{\Sigma}$ denote a section of ev_I,

$$\begin{tikzcd} \mathscr{M}^{\Sigma} \arrow[r, shift left, "\mathrm{ev}_I"] & \mathscr{C} \arrow[l, shift left, dashed, "s_{\Sigma}"] \end{tikzcd}$$

which exists by the fact that ev_I is a trivial Kan fibration.

Now, let $\mathrm{ev}_T : \mathscr{M}^{\Sigma} \to \mathscr{C}$ denote the evaluation at the terminal object T of $\Delta^1 \times \Delta^1$, which can be visualised by "taking out" the lower-right object Y of the pushout diagram above. The suspension is defined to be the following composition of functors.

Definition 4.2.1.2. Let $\mathscr{C}$ be a pointed quasi-category which admits cofibers. The *suspension functor* on $\mathscr{C}$ is defined to be the composition

$$\mathscr{C} \xrightarrow{s_{\Sigma}} \mathscr{M}^{\Sigma} \xrightarrow{\mathrm{ev}_T} \mathscr{C},$$

namely $\Sigma = \mathrm{ev}_T \circ s_{\Sigma} : \mathscr{C} \to \mathscr{C}$.

The choice of section for ev_I in the definition of Σ arises the usual well-defined issue in the theory of quasi-categories. The suspension functor is well-defined up to a contractable space of choices (as stated in [Gro10, Remark 5.5]), which is enough for the most quasi-categorical purposes as usual.

Finally, it can be observed that the construction of suspension functor, which is now determined for quasi-categorical in a precise way, is visually analogous to the suspension functor from classical homotopy theory for based topological spaces, which was mentioned in the opening comments of this section. This picture, by taking the homotopy pushout of $(* \leftarrow X \rightarrow *)$, can conveniently be kept in mind when the $(\infty, 1)$-categorical suspension functor is discussed. In this section it has been shown that this picture can be made precisely for pointed quasi-categories which admit cofibers.

4.2.2 Loop functor

The aim now is to construct the dual notion of suspension functor, namely loop functor on a quasi-category with appropriate properties. Again, first let $\mathscr{C}$ be a pointed quasi-category and define

$$\mathscr{M}^{\Omega} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C}) \tag{4.2.2.i}$$

to be the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$ spanned by diagrams of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & Y, \end{tikzcd}$$

which have the property of being pullback squares in $\mathscr{C}$, again 0 and $0'$ are zero objects in $\mathscr{C}$. In order to construct the loop functor on $\mathscr{C}$, apply the same arguments as above to the dual notions. So, let $\mathscr{C}$ in addition admits fibers (every morphism admits a fiber) and let

$$\mathrm{ev}_T : \mathscr{M}^{\Omega} \rightarrow \mathscr{C} \tag{4.2.2.ii}$$

denote the evaluation on the terminal object T of $\Delta^1 \times \Delta^1$, similar as above. Then by applying [Lur09, Proposition 4.3.2.15] twice the following can be shown.

Proposition 4.2.2.1. Let $\mathscr{C}$ be a pointed quasi-category which admits fibers. Then the functor $\mathrm{ev}_T : \mathscr{M}^{\Omega} \rightarrow \mathscr{C}$ defined above is a trivial Kan fibration.

This proposition is the other special case of [Gro10, Proposition 5.3], similar result is also discussed in [Lur12, Section 1.1.2]. Since the functor ev_T now is argued to be a trivial Kan fibration it admits a section. Let $s_\Omega : \mathscr{C} \to \mathscr{M}^\Omega$ denote a section of ev_T,

$$\begin{tikzcd} \mathscr{M}^\Omega \arrow[r, shift left, "\mathrm{ev}_T"] & \mathscr{C} \arrow[l, shift left, dashed, "s_\Omega"] \end{tikzcd} .$$

Then the loop functor on $\mathscr{C}$ is defined the following way.

Definition 4.2.2.2. Let $\mathscr{C}$ be a pointed quasi-category which admits fibers. The *loop functor* on $\mathscr{C}$ is defined to be the composition

$$\mathscr{C} \xrightarrow{s_\Omega} \mathscr{M}^\Omega \xrightarrow{\mathrm{ev}_I} \mathscr{C}$$

where $\mathrm{ev}_I : \mathscr{M}^\Omega \to \mathscr{C}$ again is the evaluation functor at the initial object I of $\Delta^1 \times \Delta^1$. Hence, the loop functor is defined to be $\Omega = \mathrm{ev}_I \circ s_\Omega : \mathscr{C} \to \mathscr{C}$.

The same well-defined issue should be formulated for the loop functor, as for the suspension functor. The loop functor is well-defined up to a contractable space of choices for the section of the trivial Kan fibration ev_T.

Finally, it should be remarked that the construction of the loop functor of quasi-categories can visually be seen to be analogous to the determination of the loop functor for classical homotopy theory. This picture, which now has been made precise for quasi-categories, should be kept in mind when the loop functor on an $(\infty, 1)$-category is studied.

4.2.3 Suspension functor and loop functor on stable $(\infty, 1)$-categories

The aim for this section is to discuss some immediate properties for suspension functor and loop functor on stable $(\infty, 1)$-categories. As usual the discussion here is based on the formalisations obtained from properties of quasi-categories.

But, first assume that $\mathscr{C}$ is a pointed quasi-category which admits cofibers and fibers, such that both the suspension functor Σ and the loop functor Ω are defined, but $\mathscr{C}$ is not necessarily stable. Although an appropriate notion of adjoint pairs has not been defined for quasi-categories, or even not any ideas for $(\infty, 1)$-categorical notion of adjunctions have been discussed after all, it can be shown that the suspension functor and the loop functor

$$(\Sigma, \Omega) : \mathscr{C} \to \mathscr{C} \tag{4.2.3.i}$$

is an adjoint pair with the determinations discussed previously. This result is indicated in [Gro10, Proposition 5.6] and [Lur12, Remark 1.1.2.8]. Moreover, this result is similar to the classical result for suspension and loop functor on topological spaces. The establishment of this adjunction for based topological categories is discussed in [Ark11, Section 2.3].

Assume now that $\mathscr{C}$ is a stable $(\infty, 1)$-category. Then the suspension functor and loop functor on $\mathscr{C}$ are mutually inverse equivalences in the sense of $(\infty, 1)$-categories. The objective now is to explore this assertion formally for quasi-categories. Let $\mathscr{C}$ be a stable quasi-category, then the subcategories $\mathscr{M}^{\Sigma}$ and $\mathscr{M}^{\Omega}$ obtained in the previous sections are equivalent,

$$\mathscr{M}^{\Sigma} \simeq \mathscr{M}^{\Omega}. \tag{4.2.3.ii}$$

The latter follows since a triangle in a stable quasi-category is a fiber sequence if and only if it is a cofiber sequence, hence $\mathscr{M}^{\Sigma}$ and $\mathscr{M}^{\Omega}$ spans the same full subcategories under $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$. With this fact established, the result that $(\Sigma, \Omega) : \mathscr{C} \to \mathscr{C}$ are mutually inverse equivalences can visually be shown to follow from the next diagram, which is also stated in [Gro10, Proposition 5.8],

$$\begin{tikzcd} X \arrow[d, mapsto] & \Sigma : & \mathscr{C} \arrow[r, dashed, "s_{\Sigma}"] \arrow[d, equal] & \mathscr{M}^{\Sigma} \arrow[r, "\mathrm{ev}_T"] \arrow[d, equal] & \mathscr{C} \arrow[d, equal] & & \Sigma\Omega Y \\ \Omega\Sigma X & & \mathscr{C} & \mathscr{M}^{\Omega} \arrow[l, "\mathrm{ev}_I"'] & \mathscr{C} \arrow[l, dashed, "s_{\Omega}"'] & : \Omega & Y \arrow[u, mapsto] \end{tikzcd}$$

where the left assignment $X \mapsto \Omega\Sigma X$ is obtained by following the upper-lower path in the diagram, while the right assignment $Y \mapsto \Sigma\Omega Y$ is obtained from the lower-upper path in the diagram. The left and right double edges indicate the equivalences

$$\Omega\Sigma = (\mathrm{ev}_I\, s_{\Omega})(\mathrm{ev}_T\, s_{\Sigma}) \simeq I_{\mathscr{C}} \quad \text{and} \tag{4.2.3.iii}$$

$$\Sigma\Omega = (\mathrm{ev}_T\, s_{\Sigma})(\mathrm{ev}_I\, s_{\Omega}) \simeq I_{\mathscr{C}}, \tag{4.2.3.iv}$$

where $I_{\mathscr{C}}$ is the identity functor on $\mathscr{C}$, which are determined from the upper-lower and lower-upper paths respectively, while the middle double arrow is determined from the equivalence in Equation 4.2.3.ii discussed above. The equivalences determined in Equation 4.2.3.iii and Equation 4.2.3.iv follow directly, since s_{Ω} is a section of ev_T and s_{Σ} is a section of ev_I from construction. This deduce that the suspension and loop $(\Sigma, \Omega) : \mathscr{C} \to \mathscr{C}$ are a pair of inverse equivalences ([Gro10, Proposition 5.8]).

Now some notation will be defined, mainly taken from from [Lur12]. If it happens to not be clear from the context which $(\infty, 1)$-category the suspension functor and loop functor are taken over, then the notations $\Sigma_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ and $\Omega_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ are used to indicate that they are taken over $\mathscr{C}$. This is for example used in order to distinguish between suspension on $(\infty, 1)$-categories and topological spaces. For an $(\infty, 1)$-category $\mathscr{C}$ and $n \geq 0$, the notation $X \mapsto X[n]$ is used to denote the nth power of the suspension functor $\Sigma : \mathscr{C} \to \mathscr{C}$, namely $\Sigma^n : \mathscr{C} \to \mathscr{C}$. So for example the assignment $X \mapsto X[1]$ means $X \mapsto \Sigma X$. If n is negative, $n \leq 0$, then $X \mapsto X[n]$ is used to denote the $(-n)$th power of the loop functor $\Omega : \mathscr{C} \to \mathscr{C}$, namely $\Omega^{-n} : \mathscr{C} \to \mathscr{C}$. For example the assignment $X \mapsto X[-1]$ means $X \mapsto \Omega X$. The same notation is used on the level of the homotopy category $h\mathscr{C}$.

Finally in this part the following should be commented. Often in the following discussions it will be assumed that an $(\infty, 1)$-category $\mathscr{C}$ is pointed and admits cofibers where the suspension functor $\Sigma : \mathscr{C} \to \mathscr{C}$ is an equivalence of $(\infty, 1)$-categories. It is clear that if $\mathscr{C}$ is a stable $(\infty, 1)$-category, then it satisfies these requirements. The first two assertions, pointed and admits cofibers, follow from definition of stability, the last follows since the suspension functor $\Sigma : \mathscr{C} \to \mathscr{C}$ and loop functor $\Omega : \mathscr{C} \to \mathscr{C}$ were shown to be mutually inverse equivalences for quasi-categories, then in the conceivable case of suspension functor of $(\infty, 1)$-categories it can be regarded as an equivalence of $(\infty, 1)$-categories. Then stable $(\infty, 1)$-categories is an example of pointed $(\infty, 1)$-categories that admit cofibers, where the suspension functor is an equivalence. So the property of being a stable $(\infty, 1)$-category implies the property of being a pointed $(\infty, 1)$-category which admits cofibers, where the suspension functor is an equivalence.

In fact, the implication the other way can also be argued to be true. As indicated in [Lur12, Remark 1.1.2.15], which can be regarded as implied by [Lur12, Corollary 1.4.2.27], a pointed $(\infty, 1)$-category which admits cofibers where the suspension functor is an equivalence can be shown to be a stable $(\infty, 1)$-category. The idea behind these results requires much more theory and details than will be be discussed here. Although the statements seem to be equivalent, the notion of stable $(\infty, 1)$-categories gives probably a more intuitive approach, and this statement can hence in many situations be preferred.

4.3 Properties of the homotopy category of stable $(\infty, 1)$-categories

The main objective for the section is to discuss the necessary properties of the homotopy category of a stable $(\infty, 1)$-category in order to prove the main theorem. First the required additive structure will be discussed before the triangulated structure will be proved.

4.3.1 Additive structure

Recall the notion of an additive category from Appendix A.2.1. First aim for this section is to observe that the homotopy category of a stable $(\infty, 1)$-category can be regarded as an enriched category over abelian groups. This observation builds on classical homotopy theory of spaces used on the mapping spaces. The remaining requirement in order to deduce that the homotopy category of a stable $(\infty, 1)$-category is an additive category is to show existence of all finite coproducts, this will be observed in the final part of the section.

First let $\mathscr{C}$ be a pointed $(\infty, 1)$-category. For each pair of objects X and Y in $\mathscr{C}$ there is a natural choice of base point in the mapping space $\mathrm{Map}_{\mathscr{C}}(X, Y)$, namely a zero morphism $X \to 0 \to Y$.

Furthermore, if $\mathscr{C}$ as above in addition admits cofibers, then the suspension functor $\Sigma_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ can be characterised by the existence of the following natural homotopy equivalences,

$$\mathrm{Map}_{\mathscr{C}}(\Sigma_{\mathscr{C}} X, Y) \xrightarrow{\simeq} \Omega \, \mathrm{Map}_{\mathscr{C}}(X, Y), \tag{4.3.1.i}$$

where $\mathscr{C}$ conveniently can be regarded as a topological category and Ω denotes the usual loop functor on pointed topological spaces from classical homotopy theory, or loop functor on other appropriate categories when other models of $(\infty, 1)$-categories are considered.

The determination of this characterisation (Equation 4.3.1.i) will now be discussed. Recall that the suspension functor on $\mathscr{C}$ can be thought of as the $(\infty, 1)$-functor constructed by sending each object X in $\mathscr{C}$ to the cofiber $\mathrm{cofib}(X \to 0)$. So, as mentioned the following picture of a pushout square can be convenient to have in mind

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & \Sigma_{\mathscr{C}} X \end{tikzcd}$$

where 0 and $0'$ are zero-objects in $\mathscr{C}$. This picture was formalised by the construction of suspension functor for quasi-categories discussed in Section 4.2.1. Let this pushout diagram be denoted β.

As discussed under the introductory word of Section 4.2, in classical homotopy theory the loop space functor on pointed topological spaces can be obtained from homotopy pullbacks. So the loop functor on the mapping space $\Omega \operatorname{Map}_{\mathscr{C}}(X,Y)$ from Equation 4.3.1.i can be obtained from the following homotopy pullback over pointed spaces,

$$\begin{tikzcd} \Omega \operatorname{Map}_{\mathscr{C}}(X,Y) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \operatorname{Map}_{\mathscr{C}}(X,Y). \end{tikzcd}$$

Let this homotopy pullback be denoted γ.

Next observe that $\operatorname{Map}_{\mathscr{C}}(0,Y) \simeq *$ and $\operatorname{Map}_{\mathscr{C}}(0',Y) \simeq *$ for any object $Y \in \mathscr{C}$ from definition of initial objects, which 0 and $0'$ are since they are zero objects. Then the diagram above called γ can be rewritten as

$$\begin{tikzcd} \Omega \operatorname{Map}_{\mathscr{C}}(X,Y) \arrow[r] \arrow[d] & \operatorname{Map}_{\mathscr{C}}(0,Y) \arrow[d] \\ \operatorname{Map}_{\mathscr{C}}(0',Y) \arrow[r] & \operatorname{Map}_{\mathscr{C}}(X,Y), \end{tikzcd}$$

since a homotopy commutative square equivalent to a homotopy pullback square is again a homotopy pullback square, as precisely stated in [Ark11, Proposition 6.3.2].

The homotopy pullback square γ over pointed spaces obtained above is not at all arbitrary with respect to the discussion here. By applying $\operatorname{Map}_{\mathscr{C}}(-,Y)$ to the pushout square β the following commutative square is obtained in pointed spaces,

$$\begin{tikzcd} \operatorname{Map}_{\mathscr{C}}(\Sigma_{\mathscr{C}} X,Y) \arrow[r] \arrow[d] & \operatorname{Map}_{\mathscr{C}}(0,Y) \arrow[d] \\ \operatorname{Map}_{\mathscr{C}}(0',Y) \arrow[r] & \operatorname{Map}_{\mathscr{C}}(X,Y). \end{tikzcd}$$

Let this diagram be denoted α. In fact, α is a homotopy pullback over pointed spaces by the characterisation of limits and colimits for $(\infty, 1)$-categories from Section 3.2.4. Namely, let $F : \mathscr{I} \to \mathscr{C}$ be a functor, then the following expressions are homotopy equivalences,

$$\mathrm{Map}_{\mathscr{C}}(X, \varprojlim(F)) \to \mathrm{holim}_{i \in \mathscr{I}} \mathrm{Map}_{\mathscr{C}}(X, F(i)) \tag{4.3.1.ii}$$

$$\mathrm{Map}_{\mathscr{C}}(\varinjlim(F), X) \to \mathrm{holim}_{i \in \mathscr{I}^{\mathrm{op}}} \mathrm{Map}_{\mathscr{C}}(F(i), X). \tag{4.3.1.iii}$$

The last expression (Equation 4.3.1.iii) proves that α is a homotopy pullback square after all. The homotopy pullback squares α and β are equivalent to each other, which proves the required homotopy equivalence in the characterisation formulated in Equation 4.3.1.i.

Moreover, let now $\mathscr{C}$ be a pointed $(\infty, 1)$-category which admits fibers. By a similar argument it can be shown that

$$\mathrm{Map}_{\mathscr{C}}(X, \Omega_{\mathscr{C}} Y) \simeq \Omega \, \mathrm{Map}_{\mathscr{C}}(X, Y) \tag{4.3.1.iv}$$

which follows from applying $\mathrm{Map}_{\mathscr{C}}(X, -)$ to the pullback square

$$\begin{tikzcd} \Omega_{\mathscr{C}} Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & Y, \end{tikzcd}$$

which is visualising the loop functor $\Omega_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ applied on $Y \in \mathscr{C}$. Again this follows from the characterisation of $(\infty, 1)$-limits stated in Equation 4.3.1.ii.

Now with the established equivalences characterising suspension and loop formulated in Equation 4.3.1.i and Equation 4.3.1.iv, respectively, above in mind, the next aim is, for a pointed $(\infty, 1)$-category which admits cofibers, to determine the following equivalence,

$$\pi_0 \, \mathrm{Map}_{\mathscr{C}}(\Sigma_{\mathscr{C}} X, Y) \simeq \pi_1 \, \mathrm{Map}_{\mathscr{C}}(X, Y). \tag{4.3.1.v}$$

For now, let the category of pointed spaces, which the mapping spaces of $\mathscr{C}$ require to belong to, be denoted $\mathbf{Top}_*$. So, in this discussion it is perhaps convenient to think about $\mathscr{C}$ as a topological category. First write out the definition of π_0 for a space,

$$\pi_0 \, \mathrm{Map}_{\mathscr{C}}(\Sigma_{\mathscr{C}} X, Y) \simeq \mathrm{Hom}_{\mathrm{h}\mathbf{Top}_*}(S^0, \mathrm{Map}_{\mathscr{C}}(\Sigma_{\mathscr{C}} X, Y)), \tag{4.3.1.vi}$$

where S^0 denotes 0-sphere (which is homeomorphic to two points, the boundary of the unit interval, in $\mathbf{Top}_*$) and $\mathrm{h}\mathbf{Top}_*$ denotes the homotopy category of pointed spaces. Then apply the equivalence in Equation 4.3.1.i to obtain the following,

$$\mathrm{Hom}_{\mathrm{h}\mathbf{Top}_*}(S^0, \mathrm{Map}_{\mathscr{C}}(\Sigma_{\mathscr{C}} X, Y)) \simeq \mathrm{Hom}_{\mathrm{h}\mathbf{Top}_*}(S^0, \Omega \, \mathrm{Map}_{\mathscr{C}}(X, Y)). \tag{4.3.1.vii}$$

Now apply the adjunction $(\Sigma, \Omega) : \mathbf{Top}_* \to \mathbf{Top}_*$ from classical homotopy theory (as discussed in [Ark11, pp. 47-48]) which gives

$$\mathrm{Hom}_{\mathrm{h}\mathbf{Top}_*}(S^0, \Omega \operatorname{Map}_{\mathscr{C}}(X, Y)) \simeq \mathrm{Hom}_{\mathrm{h}\mathbf{Top}_*}(\Sigma S^0, \operatorname{Map}_{\mathscr{C}}(X, Y)). \quad (4.3.1.\mathrm{viii})$$

Suspension in $\mathbf{Top}_*$ of the sphere can often be thought of as increasing its dimension by a simple geometric argument. The homeomorphism $S^n \cong \Sigma S^{n-1}$ is determined in [Ark11, Proposition 2.3.9], this gives

$$\mathrm{Hom}_{\mathrm{h}\mathbf{Top}_*}(\Sigma S^0, \operatorname{Map}_{\mathscr{C}}(X, Y)) \cong \mathrm{Hom}_{\mathrm{h}\mathbf{Top}_*}(S^1, \operatorname{Map}_{\mathscr{C}}(X, Y)). \quad (4.3.1.\mathrm{ix})$$

Furthermore, the nth homotopy group can be defined as $\pi_n = \mathrm{Hom}_{\mathrm{h}\mathbf{Top}_*}(S^n, -)$, which gives

$$\mathrm{Hom}_{\mathrm{h}\mathbf{Top}_*}(S^1, \operatorname{Map}_{\mathscr{C}}(X, Y)) \simeq \pi_1 \operatorname{Map}_{\mathscr{C}}(X, Y). \quad (4.3.1.\mathrm{x})$$

This proves the equivalence stated in Equation 4.3.1.v.

The importance of this result is that the set of path connected components $\pi_0 \operatorname{Map}_{\mathscr{C}}(\Sigma_{\mathscr{C}} X, Y)$ is involved with the homotopically same group structure as the fundamental group $\pi_1 \operatorname{Map}_{\mathscr{C}}(X, Y)$. Now, using the adjunction $(\Sigma, \Omega) : \mathbf{Top}_* \to \mathbf{Top}_*$ twice establishes the homotopy equivalence

$$\pi_0 \operatorname{Map}_{\mathscr{C}}(\Sigma^2_{\mathscr{C}} X, Y) \simeq \pi_2 \operatorname{Map}_{\mathscr{C}}(X, Y), \quad (4.3.1.\mathrm{xi})$$

which equips the set of path connected components, $\pi_0 \operatorname{Map}_{\mathscr{C}}(\Sigma^2_{\mathscr{C}} X, Y)$, with the structure of an abelian group, homotopically the same group structure as for $\pi_2 \operatorname{Map}_{\mathscr{C}}(X, Y)$. This follows since $\pi_n X$ is an abelian group for all $n \geq 2$, which again is a result from classical homotopy theory, and is stated in [Ark11, p. 50] this relies on [Ark11, Proposition 2.3.8].

Now, if in addition the suspension functor $\Sigma_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ is an equivalence of $(\infty, 1)$-categories, then for every object Z in $\mathscr{C}$ it can be chosen an object X in $\mathscr{C}$ such that $\Sigma^2_{\mathscr{C}} X \simeq Z$. This choice deduce that for every Z in $\mathscr{C}$ there exists an abelian group structure on the Hom-sets $\mathrm{Hom}_{h\mathscr{C}}(Z, Y) = \pi_0 \operatorname{Map}_{\mathscr{C}}(Z, Y)$ in the homotopy category $h\mathscr{C}$, which is, by use of the adjunction $(\Sigma, \Omega) : \mathbf{Top}_* \to \mathbf{Top}_*$, determined from

$$\pi_0 \operatorname{Map}_{\mathscr{C}}(Z, Y) \cong \pi_0 \operatorname{Map}_{\mathscr{C}}(\Sigma^2_{\mathscr{C}} X, Y) \simeq \pi_2 \operatorname{Map}_{\mathscr{C}}(X, Y). \quad (4.3.1.\mathrm{xii})$$

The aim now is to sketch the final ideas in order to prove that the homotopy category $h\mathscr{C}$ of a pointed $(\infty, 1)$-category $\mathscr{C}$ which admits cofibers where the suspension functor is an equivalence is enriched over abelian groups. The final requirement in order to prove this assertion is that the composition should be bilinear over the integers, since the abelian group structure on the Hom-sets already is established from the previous discussions. Let $f, f' : X \to Y$ and $g, g' : Y \to Z$ be arbitrary arrows in $\mathscr{C}$, and regard $\mathscr{C}$ perhaps most conveniently as a simplicial category, or topological category, or a quasi-category. By applying the enriched version of the Yoneda embedding (as it is stated in [Lur09, p. 316]) gives a fully faithful functor $Y' : \mathscr{C}^{\mathrm{op}} \to \mathbf{sSet}^{\mathscr{C}}$, which maps

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] & \mathrm{Map}_{\mathscr{C}}(X, -) \\ & \\ Y \arrow[r, mapsto] & \mathrm{Map}_{\mathscr{C}}(Y, -) \arrow[uu, "f^*"'] \end{tikzcd} \qquad f \longmapsto f^*$$

where f^* is the natural transformation which components are given by the usual precomposing. Hence evaluated on $g : Y \to Z$ gives the following commutative diagram in $\mathbf{sSet}$,

$$\begin{tikzcd} \mathrm{Map}_{\mathscr{C}}(X, Y) \arrow[r, "g_*"] & \mathrm{Map}_{\mathscr{C}}(X, Z) \\ \mathrm{Map}_{\mathscr{C}}(Y, Y) \arrow[r, "g_*"'] \arrow[u, "f^*"] & \mathrm{Map}_{\mathscr{C}}(Y, Z), \arrow[u, "f^*"'] \end{tikzcd}$$

where the left and right vertical arrows both named f^* actually denotes the components $(f^*)_Y$ and $(f^*)_Z$ respectively, which coincide with the percomposing maps obtained from applying $\mathrm{Map}_{\mathscr{C}}(-, Y)$ and $\mathrm{Map}_{\mathscr{C}}(-, Z)$, respectively, to $f : X \to Y$. While, the upper and lower horizontal arrows is the postcomposing maps obtained from applying $\mathrm{Map}_{\mathscr{C}}(X, -)$ and $\mathrm{Map}_{\mathscr{C}}(Y, -)$, respectively, to $g : Y \to Z$. Since $\Sigma_{\mathscr{C}}$ is an equivalence of $(\infty, 1)$-categories there exists an object $X' \in \mathscr{C}$ such that $X \simeq \Sigma_{\mathscr{C}} X'$, hence use the top arrow from the diagram above to obtain,

$$g_* : \mathrm{Map}_{\mathscr{C}}(\Sigma_{\mathscr{C}} X', Y) \to \mathrm{Map}_{\mathscr{C}}(\Sigma_{\mathscr{C}} X', Z), \tag{4.3.1.xiii}$$

which induces the following map of loop spaces

$$g_* : \Omega \,\mathrm{Map}_{\mathscr{C}}(X', Y) \to \Omega \,\mathrm{Map}_{\mathscr{C}}(X', Z). \tag{4.3.1.xiv}$$

The functorality of π_0 gives that

$$\pi_0 g_* = \overline{g}_* : \mathrm{Hom}_{h\mathscr{C}}(X', Y) \to \mathrm{Hom}_{h\mathscr{C}}(X', Z) \tag{4.3.1.xv}$$

is a homomorphism of abelian groups, which again shows that the composition

$$\overline{g}_*(\overline{f} + \overline{f'}) = \overline{g}(\overline{f} + \overline{f'}) = \overline{g}\overline{f} + \overline{g'f}, \tag{4.3.1.xvi}$$

as required.

The similar argument now applied to g with the enriched Yoneda embedding $Y : \mathscr{C} \to \mathbf{sSet}^{\mathscr{C}^{\mathrm{op}}}$ gives the mappings,

$$\begin{tikzcd} Y \arrow[r, mapsto] \arrow[d] & \mathrm{Map}_{\mathscr{C}}(-,Y) \arrow[d] \\ Z \arrow[r, mapsto] & \mathrm{Map}_{\mathscr{C}}(-,Z) \end{tikzcd} \qquad g \longmapsto g_*$$

applied to f gives the commutative diagram of simplicial sets,

$$\begin{tikzcd} \mathrm{Map}_{\mathscr{C}}(Y,Y) \arrow[r, "f^*"] \arrow[d, "g_*"'] & \mathrm{Map}_{\mathscr{C}}(X,Y) \arrow[d, "g_*"] \\ \mathrm{Map}_{\mathscr{C}}(Y,Z) \arrow[r, "f^*"'] & \mathrm{Map}_{\mathscr{C}}(X,Z). \end{tikzcd}$$

This gives rise to group homomorphism,

$$\overline{f}^* : \mathrm{Hom}_{h\mathscr{C}}(Y,Z) \to \mathrm{Hom}_{h\mathscr{C}}(X,Z), \tag{4.3.1.xvii}$$

which takes $\overline{g} + \overline{g'}$ to $\overline{f g} + \overline{f g'}$. Then it can be concluded that that the additive structure on $h\mathscr{C}$ is functorial in all objects $X, Z \in h\mathscr{C}$, and $h\mathscr{C}$ is an enriched category over abelian groups. The discussion here is inspired from [Lur12, p. 20].

Proposition 4.3.1.1. Let $\mathscr{C}$ be a pointed $(\infty,1)$-category which admits cofibers where the suspensions functor $\Sigma_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ is an equivalence. Then the homotopy category $h\mathscr{C}$ is an enriched category over abelian groups in the canonical way as described previously.

With the same setup as above, let $\mathscr{C}$ be a pointed $(\infty,1)$-category which admits cofibers where the suspension functor $\Sigma_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ is an equivalence, the forthcoming aim is to give some thoughts about why the induced functor $\overline{\Sigma} : h\mathscr{C} \to h\mathscr{C}$ is in fact an additive functor. Since $\Sigma_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ at least is a functor of $(\infty,1)$-categories, then the induced map

$$\Sigma_{X,Y} : \mathrm{Map}_{\mathscr{C}}(X,Y) \to \mathrm{Map}_{\mathscr{C}}(\Sigma_{\mathscr{C}}X, \Sigma_{\mathscr{C}}Y), \tag{4.3.1.xviii}$$

for any pairs of object X and Y in $\mathscr{C}$, is a map of homotopy types. So the aim is to give some ideas why these $\Sigma_{X,Y}$ induce group homomorphisms when passing to the homotopy category. In an appropriate model the induced maps on the mapping spaces $(\Sigma_{X,Y})$ can be thought of for example as maps of simplicial sets or continuous maps of (sufficiently nice enough) topological spaces. Let the inverse functor of $\Sigma_{\mathscr{C}}$ be denoted $\Sigma_{\mathscr{C}}^{-1}$. Since $\Sigma_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ is an equivalence of $(\infty, 1)$-categories it can always be chosen an object $X' \simeq \Sigma_{\mathscr{C}}^{-1} X$ in $\mathscr{C}$. Then the map $\Sigma_{X,Y}$ of homotopy types above induces a maps of loop spaces

$$\Sigma_{X,Y} : \Omega \operatorname{Map}_{\mathscr{C}}(\Sigma_{\mathscr{C}}^{-1} X, Y) \to \Omega \operatorname{Map}_{\mathscr{C}}(X, \Sigma_{\mathscr{C}} Y), \qquad (4.3.1.\text{xix})$$

by using the characterisation of colimits in $(\infty, 1)$-categories. Since this is a map of loop spaces, applying π_0 induces a group homomorphism on the Hom-sets by functorality, from discussions previously. This proves that the induced map $\overline{\Sigma} : h\mathscr{C} \to h\mathscr{C}$ is an additive functor, in fact an additive equivalence from the assumption that Σ is an equivalence of $(\infty, 1)$-categories.

Until now it has been observed that if $\mathscr{C}$ is a pointed $(\infty, 1)$-category which admits cofibers where the suspension functor is an equivalence, then the homotopy category $h\mathscr{C}$ is a category enriched over abelian groups and the induced functor $\overline{\Sigma} : h\mathscr{C} \to h\mathscr{C}$ is an additive equivalence. Next lemma, which is similar to [Lur12, Lemma 1.1.2.9], states that, with the assumptions above, the homotopy category $h\mathscr{C}$ is not just an enriched category over abelian groups, but also an additive category. In the “proof” it is sketched some ideas of the proof of [Lur12, Lemma 1.1.2.9].

Lemma 4.3.1.2. Let $\mathscr{C}$ be a pointed $(\infty, 1)$-category which admits cofibers and suppose the suspension functor is an equivalence. Then the homotopy category $h\mathscr{C}$ is an additive category.

“Proof”: Since it is already shown that $h\mathscr{C}$ is enriched over abelian groups, then it is left with proving that $h\mathscr{C}$ admits all finite coproducts. Moreover, it turns out, even stronger, that $\mathscr{C}$ itself admits all finite coproducts. First it will be proved that $\mathscr{C}$ itself admits all finite coproduct then a short argument proving that $h\mathscr{C}$ admits finite coproducts after all.

Now since $\mathscr{C}$ contains initial objects, without loss of generality, it should be sufficient to consider pairwise coproduct. So, for any two arbitrary objects X and Y in $\mathscr{C}$ the objective now is to show that there exists a coproduct of them in $\mathscr{C}$. By adding up more objects to this construction, this will show that there exists arbitrary coproducts in $\mathscr{C}$, but this iterative procedure has to start at some point, namely at some initial object, in order to prove the appropriate notion. Any coproduct with an initial object should be isomorphic (in the appropriate notion) to the object itself, an empty coproduct is isomorphic to an initial object.

So, let $X, Y \in \mathscr{C}$ and let $\operatorname{cofib} : \operatorname{Fun}(\Delta^1, \mathscr{C}) \to \mathscr{C}$ be the functor that to any morphism assigns its cofiber, as usual. The aim now is to determine the following equivalences,

$$X \simeq \operatorname{cofib}(X[-1] \xrightarrow{u} 0) \quad \text{and} \tag{4.3.1.xx}$$

$$Y \simeq \operatorname{cofib}(0 \xrightarrow{v} Y), \tag{4.3.1.xxi}$$

where $X[-1]$ means the inverse functor of $\Sigma_{\mathscr{C}}$ (which is assumed to be an equivalence) denoted $\Sigma_{\mathscr{C}}^{-1}$, applied on X, namely $\Sigma_{\mathscr{C}}^{-1} : X \mapsto X[-1]$. The first equivalence can be obtained from definition of a cofiber sequence, namely a pushout square,

$$\begin{tikzcd}
X[-1] \arrow[r, "u"] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \operatorname{cofib}(u) \simeq X
\end{tikzcd}$$

so by definition $\Sigma_{\mathscr{C}} X[-1] \simeq X$. The second equivalence can be obtained from the pushout

$$\begin{tikzcd}
0 \arrow[r, "v"] \arrow[d] & Y \arrow[d] \\
0 \arrow[r] & \operatorname{cofib}(v) \simeq Y.
\end{tikzcd}$$

Now by applying [Lur09, Proposition 5.1.2.2] to u and v it can be shown that u and v admit a coproduct in $\operatorname{Fun}(\Delta^1, \mathscr{C})$ as indicated in [Lur12, Lemma 1.1.2.9]. This coproduct can be shown to be a zero map $X[-1] \xrightarrow{0} Y$. Since the functor cofib can be shown to preserve colimits, since it can be identified with the left adjoint to the left Kan extension functor $\mathscr{C} \simeq \operatorname{Fun}(\{1\}, \mathscr{C}) \to \operatorname{Fun}(\Delta^1, \mathscr{C})$, with some additional theory for adjoints and Kan extensions for $(\infty, 1)$-categories, it can be concluded that there is a coproduct of X and Y in $\mathscr{C}$ constructed as a cofiber of the map $X[-1] \to Y$, namely as the following pushout in $\mathscr{C}$,

$$\begin{tikzcd}
X[-1] \arrow[r, "0"] \arrow[d] & Y \arrow[d] \\
0 \arrow[r] & \operatorname{coprod}(X, Y).
\end{tikzcd}$$

Finally some comments that it is in fact stronger to prove that $\mathscr{C}$ itself admits finite coproducts than $h\mathscr{C}$, or more precisely, if $\mathscr{C}$ admits finite coproducts, so do $h\mathscr{C}$. But recall the characterisation of colimits in $(\infty, 1)$-categories in

Equation 4.3.1.iii, together with fact that π_0 preserves products ([Lur09, Example 1.2.13.1]) it follows obviously that $\mathrm{coprod}(X, Y)$ is a coproduct in $h\mathscr{C}$. $\square$

The next objective is to discuss a characterisation of the additive structure described previously. More precisely, first let $\mathscr{C}$ is a pointed $(\infty, 1)$-category which admits cofibers where the suspension functor is an equivalence, the next result gives a characterisation of additive inverse for a morphism $\theta \in \mathrm{Hom}_{h\mathscr{C}}(X, Y)$. First observe that any diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & Y \end{tikzcd}$$

which belongs to $\mathscr{M}^{\Sigma}$ determines a canonical isomorphism $X[1] = \Sigma X \to Y$ in the homotopy category $h\mathscr{C}$. In particular for quasi-categories this follows directly form the construction of the suspension functor via a pushout. From the uniqueness property of $(\infty, 1)$-categorical pushouts, $X[1]$ and Y are both in the same path connected component of an appropriate space of choices, hence they are isomorphic in $h\mathscr{C}$.

Further, consider any square shaped diagram of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f'"'] & 0 \arrow[d] \\ 0' \arrow[r] & Y \end{tikzcd}$$

in $\mathscr{C}$, where 0 and $0'$ are zero objects. Let this diagram visualised above be denoted δ. This square diagram δ classifies a morphism $\theta \in \mathrm{Hom}_{h\mathscr{C}}(X[1], Y)$. This classification follows since $X[1]$ is (in quasi-categories) obtained from a pushout square of similar form as δ, so from properties of pushouts, for example regarded as a initial object in an appropriate slice $\mathscr{C}_{p/}$ (Section 3.2.4), there exist morphisms $X[1] \to Y$ homotopic to one another. This determines a well-defined morphism $\theta \in \mathrm{Hom}_{h\mathscr{C}}(X[1], Y)$ as desired.

The next result is similar to [Lur12, Lemma 1.1.2.10]. There will not be give any ideas of a proof here, but a proof can be found at [Lur12, Lemma 1.1.2.10].

Lemma 4.3.1.3. Let $\mathscr{C}$ be a pointed quasi-category which admits cofibers where the suspension functor is an equivalence, and let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f'"'] & 0 \arrow[d] \\ 0' \arrow[r] & Y \end{tikzcd}$$

be a diagram in $\mathscr{C}$ classifying a morphism $\theta \in \mathrm{Hom}_{h\mathscr{C}}(X[1], Y)$, here 0 and $0'$ are zero morphisms in $\mathscr{C}$. Then the transposed diagram

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "f"'] & 0' \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

classifies a morphism $-\theta \in \mathrm{Hom}_{h\mathscr{C}}(X[1], Y)$, which is the additive inverse of θ with respect to the group structure determined by the identification

$$\mathrm{Hom}_{h\mathscr{C}}(X[1], Y) = \pi_0 \operatorname{Map}_{\mathscr{C}}(X[1], Y) \cong \pi_1 \operatorname{Map}_{\mathscr{C}}(X, Y), \tag{4.3.1.xxii}$$

as obtained previously.

With the formalities determined by the theory of topological categories, or simplicial categories, and quasi-categories, the additive structure on the homotopy category of a stable $(\infty,1)$-category has now been established. In the next section the triangulated structure will be discussed.

4.3.2 Triangulated structure

The aim for this part is to establish the triangulated structure on the homotopy category of a stable $(\infty,1)$-category $\mathscr{C}$. The discussion here is mainly inspired from studies of [Cam13, Section 5.4.2], [Gro10, Section 5.1] and [Lur12, Section 1.1.2]. First the distinguished triangles in $\mathscr{C}$ will be described, before it will be shown that $\mathscr{C}$ with these distinguished triangles satisfies Verdier's axioms.

Let $\mathscr{C}$ be a pointed $(\infty,1)$-category which admits cofibers where the suspension functor is an equivalence, then the distinguished triangles can be described as particular cofiber sequences, often formed by concatenations of particular pushout squares representing cofiber sequences. Then the following two-out-of-three property for quasi-categorical pushouts is useful, similar to an analogous two-out-of-three property for ordinary categorical pushouts.

Proposition 4.3.2.1. Let $\mathscr{C}$ be a quasi-category and let $\sigma : \Delta^1 \times \Delta^2 \to \mathscr{C}$ be a simplicial map representing a diagram in $\mathscr{C}$, which can be visualised by

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z', \end{tikzcd}$$

and suppose that the left square is a pushout. Then the right square is also a pushout if and only if the outer square is a pushout.

This result is the same as [Lur09, Lemma 4.4.2.1], where also a proof is presented. The next definition describes the distinguished triangles of the triangulated structure on the homotopy category of a stable $(\infty, 1)$-category. Again the distinguished triangles are formally described for quasi-categories.

Definition 4.3.2.2. Let $\mathscr{C}$ be a pointed quasi-category which admits cofibers. Given a diagram of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \tag{4.3.2.i}$$

in the homotopy category $h\mathscr{C}$. Let this diagram be denoted τ. The diagram τ is said to be a *distinguished triangle* if there exists a diagram $\sigma : \Delta^1 \times \Delta^2 \to \mathscr{C}$ in $\mathscr{C}$ which can be visualised by

$$\begin{tikzcd} X \arrow[r, "\widetilde{f}"] \arrow[d] & Y \arrow[r] \arrow[d, "\widetilde{g}"'] & 0 \arrow[d] \\ 0' \arrow[r] & Z \arrow[r, "\widetilde{h}"] & W \end{tikzcd}$$

consisting of and satisfying the following data and conditions,

(i) the objects 0 and $0'$ are zero objects in $\mathscr{C}$,

(ii) both of the visualised squares above are pushout squares in $\mathscr{C}$,

(iii) the following arrows in the diagram can be identified with the following morphisms in τ when passing to $h\mathscr{C}$,

$$\widetilde{f} \mapsto f \quad \text{and} \quad \widetilde{g} \mapsto g, \tag{4.3.2.ii}$$

(iv) the map $h : Z \to X[1]$ in τ is determined from the composition of the corresponding homotopy class containing $\widetilde{h}$ in the diagram, with the equivalence $W \simeq X[1]$ which is obtained from the outer triangle of the diagram above

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & X[1] \end{tikzcd}$$

this diagram is known to be a pushout from Proposition 4.3.2.1.

As usual, with the formal details established for quasi-categories, this picture with concatenations of pushouts is what having in mind when distinguished triangles in an $(\infty, 1)$-categorical setting is considered.

The next lemma gives an immediate consequence of Lemma 4.3.1.3 considering the definition of distinguished triangles. Similar result is stated in [Lur12, Lemma 1.1.2.13].

Lemma 4.3.2.3. Let $\mathscr{C}$ be a stable quasi-category, given a diagram $\omega : \Delta^2 \times \Delta^1 \to \mathscr{C}$ visualised by

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & 0' \arrow[d] \\ Y \arrow[r, "g"] \arrow[d] & Z \arrow[d, "h"] \\ 0 \arrow[r] & W \end{tikzcd}$$

where both squares are pushouts and the objects 0 and $0'$ are zero objects. Observe now that ω has the shape of the transposed diagram of the previous "double" pushout σ in Definition 4.3.2.2. Then the diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{-h'} W \tag{4.3.2.iii}$$

is a distinguished triangle in $h\mathscr{C}$ determined from ω in the following way. The maps f and g denote the obvious homotopy classes from ω. The map h' denotes the composition of h with the isomorphism $W \cong X[1]$ in $h\mathscr{C}$ which is determined by the outer square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & 0' \arrow[d] \\
0 \arrow[r] & X[1]
\end{tikzcd}$$

and $-h'$ denotes the composition of this h' with the map

$$-\mathrm{id}_{X[1]} \in \mathrm{Hom}_{\mathscr{C}}(X[1], X[1]) \simeq \pi_1 \operatorname{Map}_{\mathscr{C}}(X, X[1]), \tag{4.3.2.iv}$$

which is determined by the transposed diagram directly above.

With the formal details for stable quasi-categories established in the Lemma above, the transposed diagram of σ in Definition 4.3.2.2 gives rise to a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{-h'} W \tag{4.3.2.v}$$

for stable $(\infty, 1)$-categories. Now, let $\mathscr{C}$ be a stable $(\infty, 1)$-category. From the construction of distinguished triangles and the Lemma above a following sequence can be obtained in the homotopy category $h\mathscr{C}$,

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"'] & 0 \arrow[d] & & \\
0 \arrow[r] & Z \arrow[r, "h"] \arrow[d] & X[1] \arrow[r] \arrow[d, "-f[1]"'] & 0 \arrow[d] & \\
& 0 \arrow[r] & Y[1] \arrow[r, "-g[1]"] \arrow[d] & Z[1] \arrow[r] \arrow[d, "-h[1]"'] & 0 \arrow[d] \\
& & 0 \arrow[r] & X[2] \arrow[r, "f[2]"] & \ddots
\end{tikzcd}$$

where all squares are determined by cofiber sequences in $\mathscr{C}$.

When all necessary language and equipments defined and discussed, the next aim is to state and prove the main theorem for this thesis.

Theorem 4.3.2.4. Consider the following data,

- let $\mathscr{C}$ be a stable $(\infty, 1)$-category
- let $\mathscr{T}$ denote the collection of distinguished triangles defined above (in Definition 4.3.2.2)

- let the translation functor assigning $X \mapsto X[1]$ be the suspension functor $\Sigma_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$

Then these data endow the homotopy category $h\mathscr{C}$ of $\mathscr{C}$ with the structure of a triangulated category.

Remark 4.3.2.5. Before the proof of the main theorem recall from the discussion finally of Section 4.2.3, that stable $(\infty, 1)$-categories can in fact be regarded as equivalent to pointed $(\infty, 1)$-categories which admit cofibers, where the suspension functor is supposed to be an equivalence. The hypotheses of the main theorem here uses stable $(\infty, 1)$-category, since this assumptions is perhaps slightly more intuitive that the other, but still equivalent. The similar results stated in [Cam13, Section 5.4.2] and [Gro10, Theorem 5.10] assume also stable $(\infty, 1)$-categories, but the hypothesis in [Lur12, Theorem 1.1.2.14] uses pointed $(\infty, 1)$-categories which admit cofibers where the suspension functor is assumed to be an equivalence.

Proof of Theorem 4.3.2.4: The aim now is to verify the axioms of a triangulated category (Definition A.2.2.1) with the data in the hypothesis of the main theorem.

(TR1) Let $\mathscr{E} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^2, \mathscr{C})$ denote the full subcategory spanned by diagrams of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & Z \arrow[r] & W \end{tikzcd}$$

which give rise to distinguished triangles, namely having the property that both of the squares are pushouts. Let the diagram visualised above be denoted $\alpha \in \mathscr{E}$. Now let $e : \mathscr{E} \to \mathrm{Fun}(\Delta^1, \mathscr{C})$ be the map that visually sends a diagram in $\mathscr{E}$ to the diagram in $\mathrm{Fun}(\Delta^1, \mathscr{C})$ which corresponds to its upper left horizontal arrow, so in the example of the diagram above $e : \alpha \mapsto f$. Now by a repeated use of the argument in [Lur09, Proposition 4.3.2.15], which establishes a general criterion when simplicial maps are trivial Kan fibrations (as remarked in Remark 4.1.2.3), it can be shown (as stated in the proof of [Lur12, Theorem 1.1.2.14]) that e is a trivial fibration, let its section be denoted s,

$$\mathscr{E} \underset{s}{\overset{e}{\rightleftarrows}} \mathrm{Fun}(\Delta^1, \mathscr{C}).$$

Visually, this means that any diagram in $\mathrm{Fun}(\Delta^1, \mathscr{C})$, which corresponds to a map f in $\mathscr{C}$, can be sent to a diagram corresponding to a distinguished triangle by the map s. This proves **(TR1)** part (a), (b) and (c) by the following consideration,

(a) let f be a morphism in $h\mathscr{C}$, by applying s in the arguments above f can be extended to a distinguished triangle

(b) the collection of distinguished triangles $\mathscr{T}$ is closed under isomorphism, since distinguished triangles are obtained from pushout squares in $\mathscr{C}$, hence any diagram of the appropriate form in $\mathrm{Fun}(\Delta^1 \times \Delta^2)$ equivalent to a diagram in $\mathscr{E}$ is again in $\mathscr{E}$

(c) for any object $X \in h\mathscr{C}$, choose $f = \mathrm{id}_X$ in (a), then id_X will be mapped by s to a diagram corresponding to a distinguished triangle where the third entry (corresponding to the entry Z above), namely equivalent to the cofiber of the identity, can from construction be taken to be a zero object of $\mathscr{C}$.

(TR2) Suppose that

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \tag{4.3.2.vi}$$

is a distinguished triangle in $h\mathscr{C}$ which is obtained from the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & Z \arrow[r] & W. \end{tikzcd}$$

Let the diagram visualised above will be denoted $\sigma \in \mathscr{E}$. Now extend σ to a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & Z \arrow[r] \arrow[d] & W \arrow[d, "u"] \\ & 0'' \arrow[r] & V \end{tikzcd}$$

where $0''$ is a zero object of $\mathscr{C}$ and the lower right square is a pushout in $\mathscr{C}$. Denote this diagram μ. Now, from this diagram μ consider the squares

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0' \arrow[r] & W, \end{tikzcd}$$

obtained by concatenation horizontally, and

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0'' \arrow[r] & V, \end{tikzcd}$$

obtained by concatenation vertically. From the large diagram μ it is obvious that the two previous diagrams are connected by maps (in an appropriate functor category), since in μ visually there are a map $X \to Y$, clearly $0 = 0$, a map $0' \to X \to 0''$ and a map $W \xrightarrow{u} V$ with the required commutativity properties in order to induce the following commutative diagram in $h\mathscr{C}$,

$$\begin{tikzcd} W \arrow[r, "\cong"] \arrow[d, "u"] & X[1] \arrow[d, "{f[1]}"'] \\ V \arrow[r, "\cong"] & Y[1] \end{tikzcd}$$

where the horizontal arrows are isomorphisms, which follows from the usual classification since the large diagram μ is obtained by concatenation of pushouts. With these observations and by apply Lemma 4.3.2.3 to the part

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ Z \arrow[r] \arrow[d] & W \arrow[d] \\ 0'' \arrow[r] & V \end{tikzcd}$$

of the large diagram μ above to conclude that

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1] \tag{4.3.2.vii}$$

is a distinguished triangle in $h\mathscr{C}$.

For the converse direction in this axiom suppose that

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1] \tag{4.3.2.viii}$$

is a distinguished triangle in $h\mathscr{C}$. Since $\Sigma_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ is an equivalence conclude that

$$Y[-2] \xrightarrow{g[-2]} Z[-2] \xrightarrow{h[-2]} X[-1] \xrightarrow{-f[-1]} Y[-1] \tag{4.3.2.ix}$$

is a distinguished triangle, since it is isomorphic to a distinguished triangle. Now use the already obtained rotation from the first part of this axiom five times to determine a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \tag{4.3.2.x}$$

as desired. This proves **(TR2)**.

(TR3) Suppose that the following diagrams are distinguished triangles in $h\mathscr{C}$,

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \tag{4.3.2.xi}$$

$$X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} X'[1] \tag{4.3.2.xii}$$

which are induced by the diagrams $\sigma \in \mathscr{E}$ and $\sigma' \in \mathscr{E}$ respectively (up to contractable choices, which turn into isomorphism classes in $h\mathscr{C}$, so the assumption with σ and σ' is without loss of generality after all), where $\mathscr{E}$ is defined as in **(TR1)**. Any commutative diagram of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

in the homotopy category $h\mathscr{C}$ can be lifted (not necessary uniquely) to a square, say α in $\mathscr{C}$, or more precisely $\alpha \in \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathscr{C})$. Moreover, α can be identified with a morphism $\phi : e(\sigma) \to e(\sigma')$ in $\mathrm{Fun}(\Delta^1, \mathscr{C})$, which is best visualised by the diagram above, where $e : \mathscr{E} \to \mathrm{Fun}(\Delta^1, \mathscr{C})$ is defined as in **(TR1)**. Since e can be shown to be a trivial fibration of simplicial sets, ϕ can be lifted to a morphism $\sigma \to \sigma'$ in $\mathscr{E}$ by applying $s : \mathrm{Fun}(\Delta^1, \mathscr{C}) \to \mathscr{E}$ to ϕ, where s denotes the section of e as defined in **(TR1)**. This determines a morphism (natural transformation) of distinguished triangles

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X[1] \arrow[d] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}
$$

in the homotopy category $h\mathscr{C}$, as desired.

(TR4) In order to show the octahedron axiom let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathscr{C}$. Since $e : \mathscr{E} \to \mathrm{Fun}(\Delta^1, \mathscr{C})$ can be shown to be a trivial fibration, any distinguished triangle in $h\mathscr{C}$ beginning with any f, g or $g \circ f$ is uniquely determined up to non-unique isomorphisms. This can be seen by applying the section $s : \mathrm{Fun}(\Delta^1, \mathscr{C}) \to \mathscr{E}$ of e in order to place any morphism (object in $\mathrm{Fun}(\Delta^1, \mathscr{C})$) on the beginning of a diagram which determines a distinguished triangle. From construction a distinguished triangle is determined by a concatenation of pushouts, which are uniquely determined up to contractable space of choices. Then distinguished triangles are uniquely determined up to non-unique isomorphism by the first arrow.

Since every distinguished triangle is determined by its first morphism, it is sufficiently to prove that there exists *some* triple of distinguished triangles starting with f, g and $g \circ f$ which satisfy the required conclusion of **(TR4)**, namely existence of a fourth distinguished triangle fitting in with the commutativity property of the braid diagram. In order to prove this, construct a diagram such as the following visualised in $\mathscr{C}$,

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Y/X \arrow[r] \arrow[d] & Z/X \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & 0 \arrow[r] & Z/Y \arrow[r] & Y' \arrow[r] & (Y/X)'
\end{tikzcd}
$$

where 0 denotes a zero object of $\mathscr{C}$ and each square is a pushout. Or perhaps more precisely the previous diagram can be obtained by a repeatedly use of [Lur09, Proposition 4.3.2.15] in order to construct a diagram visualised in $\mathscr{C}$ from a map from the nerve of an appropriate (free category on a) quiver into $\mathscr{C}$. Moreover, since $\mathscr{C}$ admits cofibers together with the two-out-of-three property for pushouts formulated in Remark 4.3.2.1, even the squares in the diagram above that are not in "contact" with some zero object are also pushout squares and then, such diagram exists in $\mathscr{C}$ for sure. Let this diagram be denoted ξ.

Now by restricting the attention to the appropriate rectangles in the diagram ξ above the isomorphisms

$$X' \cong X[1], \qquad Y' \cong Y[1], \qquad (Y/X)' \cong (Y/X)[1] \tag{4.3.2.xiii}$$

can be obtained in $h\mathscr{C}$, where the first isomorphism follows from horizontal concatenation of upper row, the second from the large middle square, and the last from concatenation horizontally of the lower row. The diagram ξ and the isomorphisms together with appropriate compositions determine the following four distinguished triangles

$$X \xrightarrow{f} Y \to Y/X \to X[1] \tag{4.3.2.xiv}$$
$$Y \xrightarrow{g} Z \to Z/Y \to Y[1] \tag{4.3.2.xv}$$
$$X \xrightarrow{g \circ f} Z \to Z/X \to X[1] \tag{4.3.2.xvi}$$
$$Y/X \to Z/X \to Z/Y \to (Y/X)[1]. \tag{4.3.2.xvii}$$

in $h\mathscr{C}$. Just by rotation and shrink the zero maps in the large diagram ξ above, the these distinguished triangles can be now putted together into the required braid diagram

$$\begin{tikzcd}
X \arrow[rr, bend left, "g \circ f"] \arrow[dr] & & Z \arrow[rr, bend left] \arrow[dr, dashed] & & Z/Y \arrow[rr, bend left, "\theta"] \arrow[dr] & & (Y/X)[1] \\
& Y \arrow[ur] \arrow[dr] & & Z/X \arrow[ur, dashed] \arrow[dr, dashed] & & Y[1] \arrow[ur] & \\
& & Y/X \arrow[ur, dashed] \arrow[rr, bend right] & & X[1] \arrow[ur] & &
\end{tikzcd}$$

which for sure commutes in $h\mathscr{C}$, since even stronger the large diagram ξ above is commutative (homotopy coherently) in $\mathscr{C}$.

The verification of Verdier's axioms proves the main theorem. $\square$

4.3.3 Enclosing comments

The main theorem (Theorem 4.3.2.4) above proves that the homotopy category of a stable $(\infty, 1)$-category is triangulated. Recall the from the introduction in the first chapter (Section 1.1) that the perhaps most interesting examples of triangulated categories can be regarded as examples of homotopy categories of stable $(\infty, 1)$-categories. But, the definition of a stable $(\infty, 1)$-category can be regarded as easier, and especially the definition of stable $(\infty, 1)$-categories can be regarded as a definition by properties, perhaps somewhat unlike definitions considering additional data. So, stable $(\infty, 1)$-categories are particularly interesting to study.

Moreover, in the proof of the main theorem it can be observed that the axioms where satisfied almost directly by properties of $(\infty, 1)$-categories. For example the octahedron axiom **(TR4)** followed almost directly from quasi-categorical principles. By the easy and well motivated principles of stable $(\infty, 1)$-categories, or in particular with the formal theory developed for stable quasi-categories, they may in some appropriate situations be preferred before triangulated categories themselves.

Appendix A

Notions in the ordinary categorical language

The objective for this appendix is to discuss some of the notions in the ordinary categorical language that will be used in the thesis.

A.1 Monoidal and enriched categories

The aim for this section is to establish the notions of monoidal categories and enriched categories used in this thesis.

A.1.1 Definition of monoidal categories

Monoidal categories can be thought of as categories with an "intern product functor", like a generalisation or formalisation of the tensor product $\otimes_{\mathbb{Z}}$ of abelian groups or vector spaces. But before the definition, recall that any object, say e in any category say $\mathscr{A}$ can be regarded as a functor $e : [0] \to \mathscr{A}$, where $[0]$ is the category with one object and identity morphism and a *bifunctor* is a functor from a product category $\mathscr{A} \times \mathscr{B} \to \mathscr{C}$.

Definition A.1.1.1. A *strict monoidal category* is a triple $\langle \mathscr{A}, \otimes, e \rangle$ consisting of the following data

- a category $\mathscr{A}$

- a bifunctor $\otimes : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$
- an object e in $\mathscr{A}$.

These data should satisfy the following properties,

- the bifunctor $\otimes$ is (strictly) associative, which can be view as the commutativity of the following diagram

$$\begin{tikzcd}
\mathscr{A} \times \mathscr{A} \times \mathscr{A} \arrow[r, "I_{\mathscr{A}} \times \otimes"] \arrow[d, "\otimes \times I_{\mathscr{A}}"'] & \mathscr{A} \times \mathscr{A} \arrow[d, "\otimes"] \\
\mathscr{A} \times \mathscr{A} \arrow[r, "\otimes"'] & \mathscr{A}
\end{tikzcd}$$

  in **Cat**, where $I_{\mathscr{A}}$ is the identity functor.

- the object e is a left and right unit of the tensor product, which can be viewed as commutativity of the following diagram

$$\begin{tikzcd}
{[0]} \times \mathscr{A} \arrow[r, "e \times I_{\mathscr{A}}"] \arrow[d, "\cong"'] & \mathscr{A} \times \mathscr{A} \arrow[d, "\otimes"] & \mathscr{A} \times [0] \arrow[l, "I_{\mathscr{A}} \times e"'] \arrow[d, "\cong"] \\
\mathscr{A} \arrow[r, equal] & \mathscr{A} \arrow[r, equal] & \mathscr{A} .
\end{tikzcd}$$

A strict monoidal category is in fact an example of the more general notion of a monoidal category.

Definition A.1.1.2. A *monoidal category* is a category $\mathscr{A}$ equipped with a (coherent) associative bifunctor, called *tensor product* $\otimes : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ together with a unit object e and natural isomorphisms α, λ and ρ. The data of the monoidal category $\langle \mathscr{A}, \otimes, e, \alpha, \lambda, \rho \rangle$ is given by and should satisfy the following,

- the natural isomorphism $\alpha : \otimes(I_{\mathscr{A}} \times \otimes) \to \otimes(\otimes \times I_{\mathscr{A}})$ has components $\alpha_{a,b,c} : a \otimes (b \otimes c) \cong (a \otimes b) \otimes c$ which is natural in a, b and c such that the *pentagon diagram*

$$\begin{tikzcd}
a \otimes (b \otimes (c \otimes d)) \arrow[r, "\alpha"] \arrow[d, "I_{\mathscr{A}} \otimes \alpha"'] & (a \otimes b) \otimes (c \otimes d) \arrow[r, "\alpha"] & ((a \otimes b) \otimes c) \otimes d \\
a \otimes ((b \otimes c) \otimes d) \arrow[rr, "\alpha"] & & (a \otimes (b \otimes c)) \otimes d \arrow[u, "\alpha \otimes I_{\mathscr{A}}"']
\end{tikzcd}$$

  is commutative for all objects a, b, c and d in $\mathscr{A}$.

- the natural isomorphism $\lambda : e \otimes - \to -$ has components $\lambda_a : e \otimes a \cong a$ and the natural isomorphism $\rho : - \otimes e \to -$ has components $\rho_a : a \otimes e \cong a$, with the property that the following diagram commutes for all objects a and c in $\mathscr{A}$,

$$\begin{tikzcd} a \otimes (e \otimes c) \arrow[r, "\alpha"] \arrow[d, "I_{\mathscr{A}} \otimes \lambda_c"'] & (a \otimes e) \otimes c \arrow[d, "\rho_a \otimes I_{\mathscr{A}}"] \\ a \otimes c \arrow[r, equal] & a \otimes c, \end{tikzcd}$$

  moreover it requires that $\lambda_e = \rho_e : e \oplus e \cong e$, which can immediately be observed from the diagram.

The natural isomorphisms α, λ and ρ are often called *associator*, *left unitor* and *right unitor* respectively for the established monoidal structure on $\mathscr{A}$.

Observe now that a monoidal category $\mathscr{A}$ is a strict monoidal category if the given natural isomorphisms α, λ and ρ (which belong to the monoidal structure on $\mathscr{A}$) are identities. Moreover, the commutativity of the diagrams above should imply that all diagrams involving only the natural isomorphisms α, λ and ρ are commutative as indicated in [Lur09] and [Mac98]. In [Lur09] this is called MacLane's coherence theorem, and can further be shown to imply that any monoidal category is (monoidally) equivalent to a strict monoidal category.

Any category with finite products $\mathscr{A}$ admits the structure of a monoidal category by taking the operation $\otimes$ to be the category theoretical product, the unit object to be the terminal object in $\mathscr{A}$ and the required natural transformations to be the natural transformations that have their components constructed from the universal properties of the category theoretical product. Hence, this construction now described gives rise to monoidal structure on $\mathscr{A}$, by then $\mathscr{A}$ is frequently called a *cartesian* monoidal category.

Examples of monoidal categories occurs quite frequently, since any category that admits finite products can be equipped with a cartesian monoidal structure. As mentioned examples of non-cartesian monoidal categories are the category of abelian groups and the category of vector spaces with the usual tensor product.

A.1.2 Enriched categories

The conceptual idea behind "enriched" categories is that this category like construction instead of having the Hom-objects in the category **Set** as for ordinary categories, Hom-objects are now taken to be objects in a monoidal category and the composition is given by the tensor product. Before stating the definition of enriched categories, it should be mentioned that additive categories are examples of categories enriched over abelian groups, just by the idea that the Hom-sets are abelian groups and the composition is bilinear over $\mathbb{Z}$, which is captured by the tensor product $\otimes_{\mathbb{Z}}$.

Definition A.1.2.1. Let $\mathscr{B}$ be a monoidal category. A *$\mathscr{B}$-enriched category $\mathscr{A}$* or a *category enriched over $\mathscr{B}$* consists of and satisfies the following data

- there is a collection of objects in $\mathscr{A}$
- for each ordered pair of objects (x, y) in $\mathscr{A}$ a *mapping object* denoted $\mathrm{Map}_{\mathscr{A}}(x, y)$, which is an object of $\mathscr{B}$
- for each ordered triple of objects (x, y, z) in $\mathscr{A}$ an operation of *composition*
$$\mathrm{Map}_{\mathscr{A}}(y, z) \otimes \mathrm{Map}_{\mathscr{A}}(x, y) \to \mathrm{Map}_{\mathscr{A}}(x, z) \tag{A.1.2.i}$$
given by the tensor product $\otimes$ on $\mathscr{B}$.
- for every object x in $\mathscr{A}$ a unit map $u_x : e \to \mathrm{Map}_{\mathscr{A}}(x, x)$, where e is the tensor unit of $\mathscr{B}$, the unit map is associated with the identity morphism on x.

These data are required to satisfy the following properties,

- the composition is associative, that is for every ordered quadruple of objects (w, x, y, z) in $\mathscr{A}$ the following diagram commutes
$$\begin{tikzcd}
\mathscr{A}(y,z) \otimes \mathscr{A}(x,y) \otimes \mathscr{A}(w,x) \arrow[r] \arrow[d] & \mathscr{A}(y,z) \otimes \mathscr{A}(w,y) \arrow[d] \\
\mathscr{A}(x,z) \otimes \mathscr{A}(w,x) \arrow[r, no head] & \mathscr{A}(w,z)
\end{tikzcd}$$
where the notation $\mathscr{A}(w, x)$ is short for $\mathrm{Map}_{\mathscr{A}}(w, x)$
- the composition respects the unit law, which means that for each object x in $\mathscr{A}$ the unit map $u_x : e \to \mathrm{Map}_{\mathscr{A}}(x, x)$ satisfies commutativity of the following diagrams with the identity functor on $\mathscr{B}$ in the horizontal arrows
$$\begin{tikzcd}
e \otimes \mathscr{A}(y,x) \arrow[r, "u_x \otimes I_{\mathscr{B}}"] \arrow[d] & \mathscr{A}(x,x) \otimes \mathscr{A}(y,x) \arrow[d] \\
\mathscr{A}(y,x) \arrow[r, equal] & \mathscr{A}(y,x)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathscr{A}(x,y)\otimes e \arrow[r, "I_{\mathscr{B}}\otimes u_x"] \arrow[d] & \mathscr{A}(x,y)\otimes\mathscr{A}(x,x) \arrow[d] \\
\mathscr{A}(x,y) \arrow[r, equal] & \mathscr{A}(x,y),
\end{tikzcd}$$

where the vertical arrows are natural isomorphisms from the monoidal structure on $\mathscr{B}$.

The models topological categories and simplicial categories for $(\infty,1)$-categories, which are discussed in Section 2.1, are defined to be enriched categories over topological spaces and simplicial sets, respectively. When mapping objects are in such categories they are often referred to as *mapping spaces*. A trivial example of an enriched category is ordinary categories, which can be regarded as enriched over **Set** with the cartesian monoidal structure.

When some "new" constructions now have been defined, it is naturally to obtain the appropriate notion of maps between them.

Definition A.1.2.2. Let $\mathscr{C}$ and $\mathscr{D}$ both be $\mathscr{B}$-enriched categories, a *$\mathscr{B}$-enriched functor* $F:\mathscr{C}\to\mathscr{D}$ consists of an object map assigning objects of $\mathscr{C}$ to objects of $\mathscr{D}$ and a collection of morphism maps

$$F_{x,y}:\mathrm{Map}_{\mathscr{C}}(x,y)\to\mathrm{Map}_{\mathscr{D}}(Fx,Fy) \tag{A.1.2.ii}$$

preserving the enriched structure, that is,

- for all objects x in $\mathscr{C}$ the map

  $$e\to\mathrm{Map}_{\mathscr{C}}(x,x)\xrightarrow{F_{x,y}}\mathrm{Map}_{\mathscr{D}}(Fx,Fx), \tag{A.1.2.iii}$$

  where e is the tensor unit of $\mathscr{B}$, coincides with the unit map for Fx, similar to functors preserving identities.

- for every ordered triple (x,y,z) of objects in $\mathscr{C}$, the diagram

  $$\begin{tikzcd}
  \mathrm{Map}_{\mathscr{C}}(y,z)\otimes\mathrm{Map}_{\mathscr{C}}(x,y) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathscr{C}}(x,z) \arrow[d] \\
  \mathrm{Map}_{\mathscr{D}}(Fy,Fz)\otimes\mathrm{Map}_{\mathscr{D}}(Fx,Fy) \arrow[r] & \mathrm{Map}_{\mathscr{D}}(Fx,Fz)
  \end{tikzcd}$$

  commutes, similar to functors preserving compositions.

There is an enriched notion of the Yoneda embedding. Let $\mathscr{C}$ be a $\mathscr{B}$-enriched category, then there is a fully faithful functor $y : \mathscr{C} \to \mathscr{B}^{\mathscr{C}^{\mathrm{op}}}$, where $\mathscr{B}^{\mathscr{C}^{\mathrm{op}}}$ denote the category of functors $\mathscr{C}^{\mathrm{op}} \to \mathscr{B}$. Similar as for ordinary category theory, $y : \mathscr{C} \to \mathscr{B}^{\mathscr{C}^{\mathrm{op}}}$ is obtained from assigning to each object x in $\mathscr{C}$ the functor $x \mapsto \mathrm{Map}_{\mathscr{C}}(-, x) : \mathscr{C}^{\mathrm{op}} \to \mathscr{B}$. The dual states that there is a fully faithful functor $y' : \mathscr{C}^{\mathrm{op}} \to \mathscr{B}^{\mathscr{C}}$ by assigning $x \mapsto \mathrm{Map}_{\mathscr{C}}(x, -) : \mathscr{C} \to \mathscr{B}$. These notions are discussed in [Lur09, Section 5.1.3].

Moreover, let $\mathscr{C}$ and $\mathscr{D}$ be ordinary categories which admit finite products, so they can both be equipped with a cartesian monoidal structure. If a functor $F : \mathscr{C} \to \mathscr{D}$ preserves products, then from a $\mathscr{C}$-enriched category $\mathscr{A}$ the functor F defines a $\mathscr{D}$ enriched category denoted $\mathscr{B}$ by taking the collection of objects in $\mathscr{B}$ to be the same as $\mathscr{A}$, the mapping spaces are given by $\mathrm{Map}_{\mathscr{B}}(x, y) = F(\mathrm{Map}_{\mathscr{A}}(x, y))$ and the composition is induced from $\mathscr{A}$, which works since F preserve products. This construction is used in the definition of the homotopy category of a topological category via π_0, and in the "comparison" of topological categories with simplicial categories via $(||, \mathrm{Sing}) : \mathbf{sSet} \to \mathbf{CG}$. Some more details for this construction is remarked in [Lur09, Appendix A.1.4].

A.2 Additive and triangulated categories

In this section the definitions of additive and triangulated categories will be presented.

A.2.1 Additive categories

The homotopy category of a stable $(\infty, 1)$-category is an additive category (Section 4.3.1). The aim for this section is to define additive categories. First the standard definition.

Definition A.2.1.1. An (ordinary) category $\mathscr{A}$ is said to be *additive* if it satisfies the following axioms,

(1) the Hom-sets in $\mathscr{A}$ are abelian groups

(2) the composition in $\mathscr{A}$ is bilinear over the integers

(3) the category $\mathscr{A}$ contains a zero object denoted 0

(4) the category $\mathscr{A}$ admits all finite coproducts

The map $f : X \to Y$ that factors (uniquely) through 0, $X \to 0 \to Y$ is called the *zero morphism*. As indicated in [Mac98] if $\mathscr{A}$ is an additive category it is easy to show that any finite coproduct is isomorphic to a finite product, this coinciding of universal constructions is frequently called *biproducts*. Moreover, any map of finite biproducts $\phi : \bigoplus_{i \in \mathscr{I}} X_i \to \bigoplus_{j \in \mathscr{J}} Y_j$ can be written a matrix, where the entries are maps $\phi_{ij} : X_i \to Y_j$. Composition of maps of biproducts can be proved to be the usual matrix multiplication.

An other interpretation of additive categories is given in [Lur12, Definition 1.1.2.1]. The first two axioms in this interpretation states that an additive category admits finite products and coproducts together with a zero object. The third axiom states that for each pair of objects X and Y the map $X \coprod Y \to X \times Y$ described by the matrix

$$\begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{pmatrix} : X \coprod Y \to X \times Y \tag{A.2.1.i}$$

is an isomorphism, the inverse is denoted $\psi_{X,Y}$. An additive operation on morphisms can be formulated by

$$X \to X \times X \to Y \times Y \to Y \coprod Y \to Y \tag{A.2.1.ii}$$

which equips $\mathrm{Hom}(X, Y)$ with the structure of a commutative monoid, the identity for this operation is the zero morphism. Moreover, the fourth axiom asserts that for every arrow $f : X \to Y$ there exists an other arrow $(-f) : X \to Y$. This gives an abelian group structure to the commutative monoid $\mathrm{Hom}(X, Y)$.

With the observations above, the two formulations of additive categories can be regarded as reformulations of one another. For example the bilinearity of the composition over the integers of the formulation in second interpretation follows from abelian groups being regarded as modules over $\mathbb{Z}$ (with $na = a + a + \cdots + a$). Moreover, the additive operation can be verified in the first description by,

$$X \xrightarrow{\begin{pmatrix} \mathrm{id}_X \\ \mathrm{id}_X \end{pmatrix}} X \oplus X \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}} Y \oplus Y \xrightarrow{\begin{pmatrix} \mathrm{id}_Y & \mathrm{id}_Y \end{pmatrix}} Y, \tag{A.2.1.iii}$$

namely

$$\begin{pmatrix} \mathrm{id}_Y & \mathrm{id}_Y \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} \mathrm{id}_X \\ \mathrm{id}_X \end{pmatrix} = f + g. \tag{A.2.1.iv}$$

Remark A.2.1.2. Observe now that an additive category can be regarded as an enriched category over abelian groups, with in addition having all finite coproducts. Here the category of abelian groups is equipped with the usual tensor product over $\mathbb{Z}$. The bilinearity of the composition can clearly be described by property of the tensor product and by definition of enriched categories. This final observation is used when the property of being additive categories is verified in the thesis.

A.2.2 Definition of triangulated categories

The main objective for this thesis is to give a proof of the fact that the homotopy category of a stable $(\infty, 1)$-category is a triangulated category. So, the final aim in this appendix is to give a definition of triangulated categories. The axioms given here are reformulations of the axioms originally stated by Verdier, which are reprinted in [Ver96].

Definition A.2.2.1. A *triangulated category* consists of the following data,

- an additive category $\mathscr{D}$,
- a translation functor $\mathscr{D} \to \mathscr{D}$ which is an additive equivalence of additive categories, this functor will be identified with the notation $X \mapsto X[1]$,
- a collection $\mathscr{T}$ of diagrams consisting of a sequence of objects and morphisms of the from

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \tag{A.2.2.i}$$

  denoted (f, g, h), the members of this collection are called *distinguished triangles.*

These data are required to satisfy the following axioms,

(TR1)(a) every morphism $f : X \to Y$ in $\mathscr{D}$ can be completed by a distinguished triangle in $\mathscr{T}$ of the form (f, g, h),

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1], \tag{A.2.2.ii}$$

(TR1)(b) any sequence of objects and arrows (f, g, h) which is isomorphic to a distinguished triangle is again a distinguished triangle, hence the collection $\mathscr{T}$ of distinguished triangles can be said to be closed under isomorphism,

(TR1)(c) for any object $X \in \mathscr{D}$ there is a distinguished triangle of the form

$$X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]. \tag{A.2.2.iii}$$

(TR2) A diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \tag{A.2.2.iv}$$

is a distinguished triangle if and only if the rotated diagram

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1] \tag{A.2.2.v}$$

is a distinguished triangle.

(TR3) Given a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r, "h"] & X[1] \arrow[d] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1], \end{tikzcd}$$

where the horizontal rows are distinguished triangles, then there exists a (not necessarily) unique morphism $Z \to Z'$ making the following completed diagram commutative

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[r, "h"] \arrow[d] & X[1] \arrow[d] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1]. \end{tikzcd}$$

(TR4) Given three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{u} Y/X \xrightarrow{d} X[1] \tag{A.2.2.vi}$$

$$Y \xrightarrow{g} Z \xrightarrow{v} Z/Y \xrightarrow{d'} Y[1] \tag{A.2.2.vii}$$

$$X \xrightarrow{g\circ f} Z \xrightarrow{w} Z/X \xrightarrow{d''} X[1], \tag{A.2.2.viii}$$

where Y/X is just some unnatural notation, then there is a fourth distinguished triangle

$$Y/X \xrightarrow{\phi} Z/X \xrightarrow{\psi} Z/Y \xrightarrow{\theta} (Y/X)[1] \tag{A.2.2.ix}$$

making the following diagram commute

$$
\begin{tikzcd}
X \arrow[rr, bend left, "g \circ f"] \arrow[dr, "f"'] & & Z \arrow[rr, bend left, "v"] \arrow[dr, dashed, "w"] & & Z/Y \arrow[rr, bend left, "\theta"] \arrow[dr, "d'"'] & & (Y/X)[1] \\
& Y \arrow[ur, "g"'] \arrow[dr, "u"'] & & Z/X \arrow[ur, dashed, "\psi"] \arrow[dr, dashed, "d''"] & & Y[1] \arrow[ur, "u[1]"'] & \\
& & Y/X \arrow[ur, dashed, "\phi"] \arrow[rr, bend right, "d"'] & & X[1] \arrow[ur, "f[1]"'] & &
\end{tikzcd}
$$

The previous diagram will be referred to as the *braid diagram*, and this last axiom is often called the *octahedron axiom*.

Appendix B

Simplicial sets

Simplicial sets can be interpreted as a model for higher categories, by considering simplicial shaped cells, as indicated in [Cam13]. However, quasi-categories, which is the model of $(\infty, 1)$-categories that mainly will be used here in this thesis, can be defined as a particular class of simplicial sets satisfying the 'inner horn filler property'. The main objective for this appendix is to define and establish some of the required equipments behind the notions discussed in this thesis, such as the 'inner horn filler property'. This material is mainly studied from [GJ09], [Joy08], [Lur09] and [Mac98].

B.1 Definition of simplicial sets

The aim for this section is to give some background information and define the notion of simplicial sets.

B.1.1 The delta-category

Let X be a directed graph. A *free category* on X is a category $F(X)$, where the objects of $F(X)$ are the nodes in X, the morphisms of $F(X)$ are the arrows of X, the composition in $F(X)$ is obtained from concatenation of arrows in X and identities are obtained from adjoining a "lazy" arrow to each note in X. Moreover, a *linear graph* or a *linear quiver* is a directed graph of the form

$$0 \to 1 \to \ldots \to n = [n] \tag{B.1.1.i}$$

for an arbitrary $n \geq 0$. The convention $[-1] = \emptyset$ will be used.

Definition B.1.1.1. The category $\mathbf{\Delta}$, called the *delta-category*, is defined to be the full subcategory of **Cat** spanned by the free categories on linear quivers. That is,

- the objects of $\mathbf{\Delta}$ are the collection of free categories on $[n]$ for various $n \geq 0$, these objects will also be denoted $[n]$ (not $F([n])$) and called *standard simplices*, while
- for objects $[n]$ and $[m]$ in $\mathbf{\Delta}$ the set of morphisms are all functors $[n] \to [m]$,

$$\mathrm{Hom}_{\mathbf{\Delta}}([n],[m]) = \mathrm{Hom}_{\mathbf{Cat}}([n],[m]). \tag{B.1.1.ii}$$

Equivalently, the category $\mathbf{\Delta}$ can be described by the following. The collection of objects are totally orders numbers $\{0 \leq 1 \leq \cdots \leq n\}$, while the morphisms are order preserving maps $f : [n] \to [m]$. These descriptions can be regarded as equivalent by the following, arrows $\to$ can be translated as relations $\leq$, while functors are ordering preserving, since they commutes with compositions. With this interpretation of arrows as relations, a partially ordered set can be regarded as a free category on a acyclic quiver. It should also be remarked that $\mathbf{\Delta}$ is equivalent to the (possible larger) category of all free categories over finite linear quivers

$$c_0 \to c_1 \to \cdots \to c_n \tag{B.1.1.iii}$$

and functors, just by choosing equivalences wisely. Often it is actually this (possibly larger) subcategory of **Cat** that is referred to in the literature when Δ is mentioned ([Lur09]).

The next aim is to establish the cosimplicial identities, but first are the canonical inclusions an canonical projections of simplices defined.

Definition B.1.1.2. The *coface* map is the canonical inclusion $d^i : [n-1] \to [n]$ that do not hit a fixed object $i \in [n]$, namely this coface map is determined by

$$d^i(j) = \begin{cases} j & j < i, \\ j+1 & j \geq i \end{cases} \tag{B.1.1.iv}$$

on objects, while the arrows are sent to their only possible images in $[n]$ determined the induced function on objects.

Definition B.1.1.3. The *codegeneracy* map $s^i : [n+1] \to [n]$ is defined to be the canonical projection that sends exactly two elements onto i for $0 \leq i \leq n$, namely the codegeneracy map is determined by the object function

$$s^i(j) = \begin{cases} j & j \leq i, \\ j-1 & j > i, \end{cases} \tag{B.1.1.v}$$

while the arrows (or relations) are sent to their obvious images in $[n]$, determined by where the objects are sent to.

Remark B.1.1.4. The coface and codegeneracy map are related by the following identities

$$\begin{array}{llll}
\text{if} & i < j & \text{then} & d^j d^i = d^i d^{j-1} \qquad \text{(B.1.1.vi)} \\
\text{if} & i < j & \text{then} & s^j d^i = d^i s^{j-1} \qquad \text{(B.1.1.vii)} \\
\text{if} & i > j+1 & \text{then} & s^j d^i = d^{i-1} s^j \qquad \text{(B.1.1.viii)} \\
\text{if} & i \leq j & \text{then} & s^j s^i = s^i s^{j+1} \qquad \text{(B.1.1.ix)} \\
\text{while} & & & s^j d^j = \mathrm{id} = s^j d^{j+1}. \qquad \text{(B.1.1.x)}
\end{array}$$

by an appropriate change of domains and codomains differing from the definition. These identities can easily be verified just by writing up their definitions in the right meanings.

Definition B.1.1.5. The identities (Equations B.1.1.vi–B.1.1.x) in Remark B.1.1.4 are called the *cosimplicial identities.*

Remark B.1.1.6. Every map in $\mathbf{\Delta}$ admits a monic-epic factorisation. That is, let $u : [n] \to [n']$ be a map in $\mathbf{\Delta}$, then u can be written as $u = ds$ where

$$d = d^{i_1} d^{i_2} \cdots d^{i_k} \quad \text{and} \quad s = s^{j_1} s^{j_2} \cdots s^{j_h} \tag{B.1.1.xi}$$

satisfying

$$\begin{aligned} n - h + k = n', \quad &\text{with} \quad 0 \leq j_1 < \cdots < j_h < n \\ &\text{and} \quad 0 \leq i_k < i_{k-1} < \cdots < i_1 < n'. \end{aligned}$$

This can be observed from first, the elements $i_k < i_{k-1} < \cdots < i_1$ can be viewed as objects in $[n']$ which are not in the image of u, and second that the elements $j_1 < j_2 < \cdots < j_k$ can be viewed as objects in $[n]$ where u is non-increasing. Then u can be viewed as the composition,

$$[n] \twoheadrightarrow [n-h] \cong [n'-k] \rightarrowtail [n'] \tag{B.1.1.xii}$$

where $\twoheadrightarrow$ can be interpreted as objects which are sent onto the same object and $\rightarrowtail$ extending the image in $[n-h]$ to the appropriate image in $[n']$. By using the cosimplicial identities, it can be observed that such a factorisation is unique. More details are discussed in [Mac98, Section VII.5].

Remark B.1.1.7. From the uniqueness of factorisation it follows that $\mathbf{\Delta}$ can be obtained from the category with objects free categories on linear quivers freely generated by the arrows d^i and s^i with respect to the cosimplicial identities (as formulated in [Mac98, Proposition 2, Section VII.5]).

B.1.2 Definition of simplicial sets

First aim is to define presheaf categories.

Definition B.1.2.1. Let $\mathscr{A}$ and $\mathscr{B}$ be categories, in this thesis a *presheaf on $\mathscr{A}$ with values in $\mathscr{B}$* or a *$\mathscr{B}$-valued presheaf on A* is defined to be a contravariant functor from $\mathscr{A}$ to $\mathscr{B}$. As usual all $\mathscr{B}$-valued presheaves can be organised into a category, namely the functor category $\mathscr{B}^{\mathscr{A}^{\mathrm{op}}}$ where the objects are $\mathscr{B}$-valued presheaves on $\mathscr{A}$ and morphisms are natural transformations. In particular for any category $\mathscr{A}$, **Set**-valued presheaves will simply often referred to as *presheaves* and the category $\mathbf{Set}^{\mathscr{A}^{\mathrm{op}}}$ will be referred to as the *presheaf category on $\mathscr{A}$*.

At this point the following should be remarked.

Remark B.1.2.2. Let $\mathscr{A}$ and $\mathscr{B}$ be categories. If $\mathscr{B}$ admits all (small) limits or all (small) colimits then so do the category of presheaves $\mathscr{B}^{\mathscr{A}^{\mathrm{op}}}$. An idea behind this observation is to form the required construction in $\mathscr{B}^{\mathscr{A}^{\mathrm{op}}}$ by using the appropriate construction on the $\mathscr{B}$-values. In particular, any **Set**-valued presheaf category admits all (small) limits and colimits, then **Set**-valued presheaves are said to be both *cartesian* and *cocartesian.* More precise discussion can be found in [KS06, pp. 405-407].

For any (small) category $\mathscr{A}$ the first class of examples of **Set**-valued presheaves on $\mathscr{A}$ should be those that arise from the contravariant Hom-functor.

Definition B.1.2.3. Let $\mathscr{A}$ be a (small) category. Any **Set**-valued presheaf on $\mathscr{A}$ isomorphic to $\mathrm{Hom}_{\mathscr{C}}(-, X) : \mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$ is called a *representable presheaf on $\mathscr{A}$*.

For presheaves on $\mathbf{\Delta}$ the following terminology is used.

Definition B.1.2.4. Let $\mathscr{C}$ be a category, a $\mathscr{C}$-valued presheaf on $\mathbf{\Delta}$, namely a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathscr{C}$ is called a *simplicial object* in $\mathscr{C}$. Moreover, any $\mathscr{C}$-valued presheaf X on $\mathbf{\Delta}$ evaluated at any simplex $[n] \in \mathbf{\Delta}$ gives an object in $\mathscr{C}$ denoted $X_n = X([n])$. The image of the coface maps in $\mathbf{\Delta}$ are called *face* maps denoted $d_i = X(d^i) : X_n \to X_{n-1}$, while $s_i = X(s^i) : X_n \to X_{n+1}$ are called *degeneracy* maps. Moreover, the *simplicial identities* are obtained from the cosimplicial identities (Definition in B.1.1.5), by reversing compositions.

In particular, for **Set**-valued presheaves on $\mathbf{\Delta}$ the following terminology is used.

Definition B.1.2.5. A **Set**-valued presheaf on $\mathbf{\Delta}$ is called a *simplicial set*, that is, a simplicial set is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$. The functor category of simplicial sets is denoted **sSet**. Representable simplicial sets are will often be denoted $\Delta^n = \mathrm{Hom}_{\mathbf{\Delta}}(-, [n]) : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ for any $[n] \in \mathbf{\Delta}$.

At this point it should be observed that for a simplicial set X (even for arbitrary functor categories of simplicial objects) the face maps and degeneracy maps form chains in **Set** (or in an arbitrary value)

$$\begin{tikzcd} X_0 \arrow[r, shift left] & X_1 \arrow[l, shift left] \arrow[l, shift left=2] \arrow[r, shift left] \arrow[r, shift left=2] & X_2 \arrow[l, shift left] \arrow[l, shift left=2] \arrow[l, shift left=3] & \cdots \arrow[l, shift right=2] \arrow[l, shift left=2] \end{tikzcd}$$

which is characteristic of **sSet** (or for images of arbitrary simplicial objects).

Now some notation for simplicial sets.

Definition B.1.2.6. Let X be a simplicial set. Elements in X_n are referred to as n-simplices. In particular 0-simplices, or elements in X_0, are referred to as *objects* or *vertices* of X, while 1-simplices, or elements in X_1, are referred to as *arrows* or *edges*, or even in some situations *morphisms* of X. Let $f \in X_1$ be an arrow of X, then $d_1(f)$ is called the *source* of f, while $d_0(f)$ is called the target of f.

A 1-simplex f in a simplicial set S can be visualised by $X \xrightarrow{f} Y$, where $d_1(f) = X$ and $d_0(f) = Y$. Moreover a 2-simplex $\alpha \in S_2$ can be visualised by a solid triangle

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z, \end{tikzcd}$$

where $d_0(\alpha) = g$, $d_1(\alpha) = h$ and $d_2(\alpha) = f$. So, this "face rule" can be generalised to arbitrary n-simplex ω by saying that $d_i(\omega)$ is the $(n-1)$-simplex opposite of vertex i.

Now with this terminology for simplicial sets established, the aim now is to take up the discussion of representable presheaves on $\mathbf{\Delta}$.

Remark B.1.2.7. Firstly, let X be a simplicial set. The n-simplices in X can be classified by the following set of natural transformations in **sSet** by the Yoneda lemma ([Mac98, Section III.2]),

$$\mathrm{Nat}(\Delta^n, X) = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X) \cong X_n. \tag{B.1.2.i}$$

The notation in this thesis do not distinguish if a n-simplex β is regarded as an element $\beta \in X_n$ or a natural transformation $\beta : \Delta^n \to X$, in the discussed case this should be clear from the situation.

Remark B.1.2.8. Secondly, representable presheaves are in particular interest since any **Set**-valued presheaf arises as a colimit of representable presheaves. This is very intuitive in view of the Yoneda lemma, the colimit of representable "glue" together all values of any **Set**-valued presheaf. A more rigid proof of these ideas can be found in [Mac98, Theorem 1, Section III.7], the result is also encoded in [KS06, Section 17.1].

Remark B.1.2.9. Thirdly, let $y : \mathbf{\Delta} \to \mathbf{sSet}$ denote the fully faithful Yoneda embedding sending standard simplices to their representing presheaf, for objects $[n] \mapsto \Delta^n = \mathrm{Hom}_{\Delta}(-, [n])$ and any morphism $u : [m] \to [n]$ in $\mathbf{\Delta}$ is sent to a natural transformation $u \mapsto \Delta(u) = \mathrm{Hom}_{\mathbf{\Delta}}(-, u) : \Delta^m \to \Delta^n$ whose components coincides with the contravariant Hom-functor and will often be denoted $\mathrm{Hom}_{\Delta}(-, [n]) = u^*$. In particular, any coface map $d^i : [n-1] \to [n]$ give rise to a natural transformation $d^i \mapsto \Delta(d^i) : \Delta^{n-1} \to \Delta^n$.

The natural transformations of the form $\Delta(d^i)$ will in particular be used in to the constructions of simplicial spheres and simplicial horns in the next section (Section B.2). But first the notion of a opposite of a simplicial set will be defined.

Definition B.1.2.10. The *opposite* of a simplicial set S is determined from $S_n^{\mathrm{op}} = S^{\mathrm{op}}([n]) = S([n]^{\mathrm{op}})$, where $[n]^{\mathrm{op}}$ is the linear quiver $[n]$ with reversed arrows. Or more precisely let $\tau : \mathbf{\Delta} \to \mathbf{\Delta}$ be the auto functor that reverses all quivers, then $S^{\mathrm{op}} = S\tau : \mathbf{\Delta} \to \mathbf{Set}$.

The face and degeneration maps in the opposite simplicial set of S can easily be observed to be,

$$(d^i : S_n^{\mathrm{op}} \to S_{n-1}^{\mathrm{op}}) = (d^{n-i} : S_n \to S_{n-1}) \tag{B.1.2.ii}$$
$$(s^i : S_n^{\mathrm{op}} \to S_{n+1}^{\mathrm{op}}) = (d^{n-i} : S_n \to S_{n+1}), \tag{B.1.2.iii}$$

just by reversing the arguments.

B.2 Horn and spheres

The aim for this section is to establish the constructions of simplicial spheres and simplicial horns. These simplicial sets are important in many constructions important in this thesis.

B.2.1 Construction of simplicial spheres

Let S be a simplicial set. The idea of a simplicial n-sphere in S, is a union or "gluing" of $(n-1)$-simplices in order to obtain the "boundary" of an n-simplex $X : \Delta^n \to S$ in S. The aim for this section is to give a formal discussion of these ideas. But now to the notion of a simplicial subset will be defined.

Definition B.2.1.1. Let S be a simplicial set, a subfunctor $T \subseteq S$ is said to be a *simplicial subset* of S.

The simplicial set $\partial_i \Delta^n : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ is defined to be the simplicial subset of Δ^n obtained from image of $\Delta(d_i) : \Delta^{n-1} \to \Delta^n$, namely determined by sending $[k]$ to the image of the component $\Delta(d^i)_{[k]}$,

$$[k] \mapsto \operatorname{im} \Delta(d^i)_{[k]} \subseteq \Delta^n_k \tag{B.2.1.i}$$

and each map $u : [k] \to [l]$ are sent to the restricted map

$$u^*|_{\operatorname{im}(\Delta d^i)_{[l]}} = \operatorname{Hom}_\Delta([l], d^i)|_{\operatorname{im}(\Delta d^i)_{[l]}} : \operatorname{im}(\Delta d^i)_{[l]} \to \operatorname{im}(\Delta d^i)_{[l]}, \tag{B.2.1.ii}$$

as indicated by the sketch

$$\begin{tikzcd}
{[k]} \arrow[r, mapsto] \arrow[d, "u"'] & \operatorname{im} \Delta(d^i)_{[k]} \subseteq \Delta^n_k \\
u \arrow[r, mapsto] & u^*|_{\operatorname{im} \Delta(d^i)_{[l]}} \\
{[l]} \arrow[r, mapsto] & \operatorname{im} \Delta(d^i)_{[l]} \subseteq \Delta^n_l . \arrow[uu]
\end{tikzcd}$$

This is well-defined since $\Delta(d_i)$ is a natural transformation.

Definition B.2.1.2. The simplicial sphere is the simplicial set $\partial\Delta^n \subseteq \Delta^n$ obtained from the union of $\partial_i \Delta^n$ over $i \in [n]$. Or more concretely, the simplicial sphere $\partial\Delta^n : \Delta^{\mathrm{op}} \to \mathbf{Set}$ is the simplicial set constructed from sending

$$[k] \mapsto \bigcup_{i \in [n]} \operatorname{im} \Delta(d^i)_{[k]}, \tag{B.2.1.iii}$$

and arrows are now sent to the their contravariant Hom-functors now restricted to the union, then $u : [k] \to [l]$ will be sent to

$$u \mapsto u^*|_{\bigcup_{i\in[n]} \operatorname{im}\Delta(d^i)_{[k]}}, \tag{B.2.1.iv}$$

which gives rise to a functor, hence $\partial\Delta^n$ a simplicial set in its own right.

Definition B.2.1.3. Let S be a simplicial set. A simplicial map $\gamma : \partial\Delta^n \to S$ is said to be an *n-sphere in S*, and γ is said to *admit a filler* if there exists a n-simplex $\delta : \Delta^n \to S$ such that the following diagram commute

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\gamma"] \arrow[d, hook] & S \\ \Delta^n \arrow[ur, "\delta"'] & \end{tikzcd}$$

in **sSet**, δ is said to be a *filler* of the sphere γ.

Moreover, let S be a simplicial set, a simplicial sphere $\epsilon : \partial\Delta^n \to S$ is determined by a sequence of its faces $\epsilon = (x_0, x_1, \ldots, x_n) = (\epsilon d_0, \epsilon d_1, \ldots, \epsilon d_n) : \partial\Delta^n \to S$. The *boundary* of any n-simplex $\zeta : \Delta^n \to S$ is an n-sphere $\partial\zeta : \partial\Delta^n \to S$ obtained from $\partial\zeta = (\partial_0\zeta, \partial_1\zeta, \ldots, \partial_n\zeta) = (\zeta d_0, \zeta d_1, \ldots, \zeta d_n) : \partial\Delta^n \to S$. For example consider the 2-simplex $\eta : \Delta^2 \to S$ visualised from

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z, \end{tikzcd}$$

then the boundary of η is $(g, h, f) : \partial\Delta^2 \to S$.

B.2.2 Definition of simplicial horns

When simplicial n-spheres were thought of as removing the interior of an n-simplex, the concept of the simplicial kth-n-horn go even further, and can be thought of as removing the interior and the $(n-1)$-simplex opposite of the vertex k.

Definition B.2.2.1. The *k-th-n-horn* where $0 \le h \le n$ denoted $\Lambda^n_k : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ is defined to be the simplicial set that sends simplices, say $[l]$, to the same union of images as the sphere, except $(\Delta d_k)_{[l]}$, namely

$$[l] \mapsto \bigcup_{\substack{0\leq i\leq n \\ i\neq k}} \operatorname{im}\Delta(d^i)_{[l]} = (\Lambda^n_k)_l, \tag{B.2.2.i}$$

and a map $u : [l] \to [m]$ will be sent to

$$u \mapsto u^*|_{\bigcup_{\substack{0\leq i\leq n \\ i\neq k}} \operatorname{im}\Delta(d^i)_{[k]}} : (\Lambda^n_k)_m \to (\Lambda^n_k)_l, \tag{B.2.2.ii}$$

hence the kth-n-horn gives rise to a functor and Λ^n_k is a simplicial set in its own right.

By following the same ideas as the previous section, the aim now is to define the notion of a horn in simplicial sets.

Definition B.2.2.2. Let S be a simplicial set, a *kth-n-horn* is a map $\kappa : \Lambda^n_k \to S$ where $0 \leq k \leq n$. A horn in S is said to be a *inner horn* if $0 < k < n$, while a horn in S is said to be a *outer horn* if $k = 0 \vee k = n$. A horn $\kappa : \Lambda^n_k \to S$ is said to *admit a filler* if there exits an n-simplex $\sigma : \Delta^n \to S$ such that the following diagram commute

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\kappa"] \arrow[d, hook] & S, \\
\Delta^n \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

in **sSet**, σ is said to be a *filler* for κ.

Similar for simplicial spheres a horn in a simplicial set S is determined by a sequence of faces $\tau = (x_0, \ldots, x_{k-1}, \bullet, x_{k+1}, \ldots, x_n) : \Lambda^n_k \to S$, while a filler visually can be thought of as being a n-simplex with boundary "filling" the kth entry $\bullet$. Consider now the example of a inner horn $\upsilon : \Lambda^2_1 \to S$ which can be visualised as

$$\begin{tikzcd}
& Y \arrow[dr, "g"] & \\
X \arrow[ur, "f"] & & Z,
\end{tikzcd}$$

then $\upsilon = (g, \bullet, f) : \Lambda^2_1 \to S$. The next definition classifies some simplicial sets, which have filler for certain horns.

Definition B.2.2.3. A *Kan complex* is a simplicial set X, where all horns admit fillers, that is for all horns Λ^n_k with $n > 0$ and $0 \le k \le n$ admit a filler,

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "\forall"] \arrow[d, hook] & S. \\
\Delta^n \arrow[ur, "\exists"'] &
\end{tikzcd}$$

In fact simplicial sets that arises as "nerves" (which will be defined in the next Section B.3) of ordinary categories, will admit *unique inner horn fillers.* While "quasi-categories" or "weak Kan complexes" are simplicial sets that admit inner fillers, this notions are defined in Chapter 2

B.3 Nerves

The aim for this section is to defined the nerve functor and state a result classifying the simplicial sets that arises as nerves of categories.

B.3.1 The nerve functor

The idea of the nerve functor is to determine a simplicial set from an ordinary category, where its n-simplices are given as functors $[n] \to \mathscr{C}$. This is formulated in the next definition.

Definition B.3.1.1. The *nerve functor* is the fully faithful functor $N : \mathbf{Cat} \to \mathbf{sSet}$ that to each ordinary category $\mathscr{C}$ assign a simplicial set

$$N(\mathscr{C}) = \mathrm{Hom}_{\mathbf{Cat}}(-, \mathscr{C}) : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}, \tag{B.3.1.i}$$

while any functor $F : \mathscr{C} \to \mathscr{D}$ will be sent to the natural transformation $\mathrm{Hom}_{\mathbf{Cat}}(-, F)$, which components are covariant Hom-functors.

Now this definition will be studied. The n-simplices $N(\mathscr{C})_n$, or set of maps $\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, N(\mathscr{C}))$, are given by the set $\mathrm{Hom}_{\mathbf{Cat}}([n], \mathscr{C})$, while maps $u : [n] \to [m]$ in $\mathbf{\Delta}$ are sent to the contravariant Hom-functor

$$\mathrm{Hom}_{\mathbf{Cat}}(u, \mathscr{C}) = u^* : \mathrm{Hom}_{\mathbf{Cat}}([m], \mathscr{C}) \to \mathrm{Hom}_{\mathbf{Cat}}([n], \mathscr{C}). \tag{B.3.1.ii}$$

Visually, the nerve functor can be thought of as the images of each $[n]$ in $\mathscr{C}$, namely chain of composable arrows in $\mathscr{C}$,

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} C_i \xrightarrow{f_{i+1}} C_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} C_n. \tag{B.3.1.iii}$$

Moreover, the i-th degeneracy map $s_i : [n] \to [n+1]$ can be viewed as a map that carries the previous chain to a chain of the form

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} C_i \xrightarrow{\mathrm{id}_{C_i}} C_i \xrightarrow{f_{i+1}} C_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} C_n. \tag{B.3.1.iv}$$

Similarly, the face map $d_i : [n] \to [n-1]$ carries the first chain to a chain of the form

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} C_{i-1} \xrightarrow{f_{i+1}f_i} C_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} C_n. \tag{B.3.1.v}$$

B.3.2 Relations between categories and nerves

The aim for this section is to observe some relations between the category itself and its nerve.

Remark B.3.2.1. Any category $\mathscr{C}$ can be described by its nerve $N(\mathscr{C})$,

- the objects of $\mathscr{C}$ can be described by $N(\mathscr{C})_0 = \mathrm{Hom}_{\mathbf{Cat}}([0], \mathscr{C})$, namely all canonical functors sending the simplex $[0]$, which is the category with only one object, to objects in $\mathscr{C}$.
- the arrows of $\mathscr{C}$ can be described by the images of $N(\mathscr{C})_1$ which is functors from the simplex $[1]$ to $\mathscr{C}$. The image of such a functor, say $x = f : C_0 \to C_1$, will represent an arrow f in $\mathscr{C}$ with its domain and codomain represented by face maps $\mathrm{dom}(f) = d_1(x) = C_0$ and $\mathrm{cod}(f) = d_0(x) = C_1$
- the identity of an object C can be described by the degeneracy map from $[0]$ to $[1]$, visually mapping C to $\mathrm{id}_C : C \to C$
- composition of two arrows $\phi : C_0 \to C_1$ and $\psi : C_1 \to C_2$ can be viewed as a image y of a functor from $[2]$ to $\mathscr{C}$ with $d_0(y) = \psi : C_1 \to C_2$, $d_2(y) = \phi : C_0 \to C_1$ and $d_1(y) = \psi\phi : C_0 \to C_2$

Remark B.3.2.2. Conversely, a simplicial set X is isomorphic to the nerve for an ordinary category $\mathscr{C}$ if every inner horn in X can uniquely be filled,

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\forall"] \arrow[d, hook] & X. \\ \Delta^n \arrow[ur, "\exists!"'] & \end{tikzcd}$$

The idea behind this result is that there are a unique way of composing edges, a proof can be found after [Lur09, Proposition 1.1.2.2]. This result is also discussed in [Cam13, Proposition 3.2].

From these two remarks the following proposition can be stated (similar to [Lur09, Proposition 1.1.2.2]).

Proposition B.3.2.3. A simplicial set S is isomorphic to the nerve of a (small) ordinary category $\mathscr{C}$, $S \cong N(\mathscr{C})$ if and only if all inner horns in S admit *unique* fillers.

B.4 Model structure on simplicial set

Although model categories and homotopical algebra lies under many of the ideas and notions in this thesis, there will not be deduced a particular interpretation of these. However, some of the terminology will be used, which some of will be presented here.

B.4.1 The Kan model structure

The idea of a model category is a ordinary category which is equipped with three distinguished classes of morphisms

- cofibrations
- fibrations
- weak equivalences

satisfying certain axioms (see for example [Lur09, Definition A.2.8.1]). The aim here is to describe these three classes for simplicial sets, but a verification of the axioms can for example be found in [GJ09].

Definition B.4.1.1. The Kan model structure on simplicial sets is given by the following distinguished classes of simplicial maps,

- a map $f : X \to Y$ of simplicial sets is a *cofibration* if it is a monomorphims, which means that the induced map $X_n \to Y_n$ is injective for all $n \geq 0$
- a map $f : X \to Y$ of simplicial sets is a *fibration* if it is a Kan fibration, which means that for any diagram of the form

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

where the outer square commute, there exists a dashed arrow as displayed in the diagram such that the diagram commute.

- a map $f : X \to Y$ of simplicial sets is said to be a *weak equivalence* if the geometric realised map $|f| : |X| \to |Y|$ is a weak homotopy equivalence of topological spaces.

Moreover *trivial fibrations* are both fibrations and weak equivalences.

However **sSet** can also be equipped with an other model structure, called the *Joyal model structure*, where fibrations and weak equivalences are chosen differently. These notions are carried out in [Lur09, Chapter 2].

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