

Phase transitions for rock-scissors-paper game on different networks

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Monte Carlo simulations and dynamical mean-field approximations are performed to study the phase transitions in the rock-scissors-paper game on different host networks. These graphs are originated from lattices by introducing quenched and annealed randomness simultaneously. In the resulting phase diagrams three different stationary states are identified for all structures. The comparison of results on different networks suggests that the value of the clustering coefficient plays an irrelevant role in the emergence of a global oscillating phase. The critical behavior of phase transitions seems to be universal and can be described by the same exponents.

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Complex networks have recently been studied extensively as these objects pervade all nature [1–3]. Initial efforts were focused on the characterization and evolution of these graphs. Very recently, the investigations were extended to critical behavior in dynamical models defined on various random networks [4,5]. The effects of the network topology on critical transitions can well be investigated on the small-world networks introduced by Watts and Strogatz [6] because this structure provides a transition from a regular lattice to some random networks. It is shown that modifications in the host topology either change the class of universality [7,8] or the long-range connections prevent complete ordering [9].

In addition, there are examples where the increase of randomness of the host network induces a phase transition toward a state that cannot be observed on lattices. To be specific, the emergence of an oscillating state has been observed by Kuperman and Abramson [10] at a finite rate of randomness in an epidemiological model. The appearance of global oscillation (synchronization) is also observed in some cyclically dominated three-state systems, such as the voluntary prisoner's dilemma [11] and rock-scissors-papers (RSP) games [12]. For the latter models the amplitude of oscillation depends on the randomness, and a second transition may occur when the limit cycle approaches the absorbing states. The application of annealed (temporal) randomness results in qualitatively similar behaviors. In this Brief Report, concentrating on the RSP game, we extend our previous work by combining both types of randomness to explore the global phase diagram and to clarify a universal feature.

For the spatial rock-scissors-paper game the individuals located on site i of a lattice belong to one of the three species ($S_i=1,2,3$) which dominate each other cyclically. This means that the time evolution of this system is governed by the iteration of cyclic invasion processes between two randomly chosen neighboring sites; namely, the pair (1,2) or (2,1) becomes (1,1), (2,3) or (3,2) becomes (2,2), and finally (3,1) or (1,3) evolves to (3,3) with the same rate, defined to be 1. Starting from a random initial state on a square lattice, this system evolves into a self-organizing pattern in which the three species are present with the same average concentration (1/3). This state, maintained by cyclic invasions [13–15], provides protection against some types of invaders

[16,17] and exhibits an unusual response to external perturbation [18,19].

In our previous paper [12] the RSP game was studied on such regular small-world structures where randomly chosen (long-range) links are substituted for a portion Q of the nearest-neighbor links between sites located on a square lattice. A similar algorithm was proposed by Watts and Strogatz [6] as a continuous transition between a lattice and a random network. For the sake of simplicity, our rewiring process [12] conserves regularity, i.e., the coordination number of each site (called the degree) remains unchanged $z=4$. Evidently, for $Q=0$ this structure is equivalent to a square lattice; meanwhile the limit $Q \rightarrow 1$ results in a regular random graph. It is important to note that the restriction of fixed degree does not change the small-world feature of the network: the characteristic (average) path length between two sites scales as the logarithm of the network size. Consequently, the main conclusion is expected to be valid also for standard Watts-Strogatz networks. At the same time, this simplification has allowed us to compare the results from these quenched structures with those obtained from “annealed” structures that can also be investigated analytically. For the annealed structures the (long-range) links are substituted temporarily with a probability P for the standard (nearest-neighbor) links on the square lattice. Naturally, in the limit $P \rightarrow 1$ the dynamics of the system satisfies the mean-field condition. The effect of annealed randomness on rumor propagation [20,21] and for the two-state voter model [22] has already been studied.

For the RSP model on the quenched structures the emergence of global oscillation is observed if Q exceeds a threshold value and the amplitude of oscillation tends to a fixed value in the limit $Q \rightarrow 1$. Such a transition occurs also for annealed structures when the value of P increases. In the latter case, however, the amplitude of global oscillation increases with time and finally the evolution ends in one of the homogeneous (absorbing) states if P exceeds a second threshold value. The different consequences of the two types of randomness have inspired us to combine these randomnesses when starting from different lattices.

Henceforth, both types of randomness are applied to derive a global phase diagram on the P - Q plane. To explore the robust (universal) behaviors we also analyze what happens for other regular structures with degrees of $z=3, 4$, and 6. These lattices are Kagomé (where $z=4$ as for the square

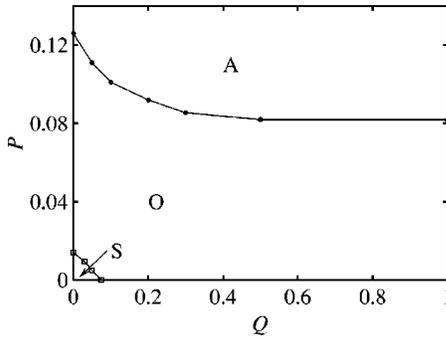


FIG. 1. Phase diagram for the RSP game on a network derived from the honeycomb lattice. The system exhibits self-organizing (S), oscillating (O), and absorbing (A) phases. Symbols represent MC results. The solid lines are guides to the eye.

lattice), honeycomb ($z=3$), triangle ($z=6$), cubic ($z=6$), and ladder-shape ($z=3$) lattices. For the ladder-shape structure two parallel chains are connected by interchain bonds.

Our analysis is based on systematic Monte Carlo (MC) simulations. First we create a quenched random regular structure starting from one of the above mentioned lattices. That is, a portion P of the nearest-neighbor links are replaced by randomly chosen links in a way that conserves the regularity (for details, see [12]). The MC simulations are started from a random (uncorrelated) initial state where the three species take their place with the same probability ($1/3$). Keeping this structure fixed the time evolution is governed by invasions between neighbors with a probability $1-P$ or along a (random) long-range link chosen with a probability P . In the simulation the number of lattice points is varied from 10^3 to 10^7 . The large sizes are used in the close vicinity of transition points to reduce the undesired effect of fluctuations.

The main features of the steady-state phase diagram, which is generally valid for all structures, can be summarized as follows. For small values of Q and P , the stationary state is characterized by a self-organizing strategy distribution denoted by S . In this self-organizing pattern the strategies alternate cyclically at each site, these local oscillations are not synchronized by the short-range interactions, and the average concentrations are the same ($1/3$). For the opposite limit—when both structural parameters Q and P are close to 1—the system evolution is characterized by growing spiral trajectories [12], and finally the evolution ends in one of the three absorbing (homogeneous) states (A) containing only one strategy. Evidently, this absorbing phase is threefold degenerate due to the cyclic symmetry. These two phases (S and A) are separated by the region of global oscillation (O) on the P - Q plane. In this oscillating phase the behavior is characterized by a limit cycle which is quantified by an order parameter Φ defined as the ratio of the area of the limit cycle and area of the full triangle in the ternary phase diagram [12]. This order parameter is $O(1)$ in the state $S(A)$ and varies from 0 to 1 for the occurrence of global oscillation (phase O). Figure 1 shows the phase boundaries obtained by MC simulations for $z=3$ on the P - Q plane.

A striking quantitative difference is found in the behavior of these systems when varying the value of z ; namely, global

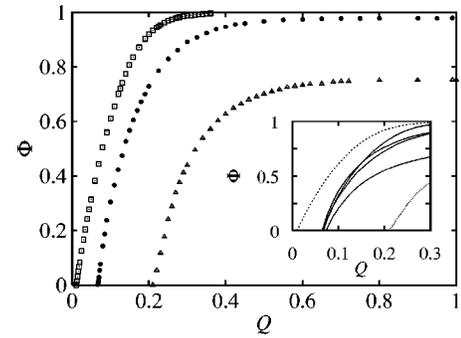


FIG. 2. The order parameter as a function of Q for quenched networks ($P=0$) derived from cubic (box), square (bullet), and ladder-shape (triangle) lattices. The corresponding critical value of Q where the oscillatory phase emerges is $Q_1(P=0)=0.0103(1)$, $0.067(1)$, and $0.210(2)$, respectively. The curves from left to right in the inset show the $\Phi(Q, P=0)$ functions for small-world networks derived from cubic, triangular, square, Kagomé, honeycomb, and ladder-shape lattices.

oscillation (phase O) can be observed on random regular graphs ($Q=1, P=0$) for $z=3$ and $z=4$. On the contrary, the system evolution terminates in one of the absorbing states (A) on random regular graphs for $z=6$, and similar behavior is expected for $z>6$. In order to illustrate the relevance of z , Fig. 2 shows the Q dependence of Φ for quenched structures ($P=0$) created by the rewiring technique from the above mentioned lattices.

Evidently, in the limit $Q \rightarrow 1$ the quenched structures become independent of the original lattice. Consequently, starting from either the square or the Kagomé lattice the order parameter $\Phi(Q, P=0)$ tends monotonically to the same value [$\Phi_1(z=4)=0.980(5)$] in the limit $Q \rightarrow 1$. For $z=3$ (triangular and ladder-shape lattices) we have obtained a lower limit value, namely, $\Phi_1(z=3)=0.750(5)$. For $z=6$, however, the system ends in the phase A if Q exceeds a threshold value Q_2 . According to our simulations on the quenched structures ($P=0$) $Q_2=0.378(8)$ if the random regular structure is created from a cubic lattice and $Q_2=0.405(10)$ for the triangular lattice.

The topological structure of the original lattice affects the value of Q_1 because the global oscillation occurs for small Q for all the investigated lattices. The inset in Fig. 2 demonstrates clearly that the lowest Q_1 appears for the three-dimensional cubic lattice, while the highest value is found for the one-dimensional ladder-shape structures. It is underlined that the values of Q_1 are very close to each other for all the small-world structures created from the two-dimensional lattices.

The present RSP system undergoes two subsequent phase transitions when increasing the randomness (Q and/or P) of the backgrounds. The first transition (from S to O) is a Hopf bifurcation that is well studied by mean-field type approaches [23]. Notice that the order parameter Φ vanishes linearly for all the structures (see Fig. 2), in agreement with theory.

Our results indicate clearly that the global oscillation emerges just above a threshold value of quenched randomness although the small-world feature characterizes this type

of network at any small rate of disorder. The same phenomenon was observed by Kuperman and Abramson [10] when considering the transition to an oscillating phase in a three-state (susceptible-infected-refractory) epidemiological model where the initial lattice was the traditional one-dimensional ring. They have conjectured that the emergence of global oscillation is related to the variation in the clusterization. In the present model, however, our results suggest that the clustering coefficient (C) cannot play a significant role because the corresponding values of C are very different for the structures studied. We cite as an example the case where $C=0$ for all initial lattices except the triangle and Kagomé where $C=2/5$ and $1/3$, respectively. Furthermore, the value of C ranges from 0.0004 to 0.3 for different topologies when the system enters into the globally oscillating phase. The inset of Fig. 2 shows the order parameter as a function of Q for all networks studied here. These results suggest that Q_1 depends basically on the dimension of the initial lattice; namely, $Q_1=0.068(6)$ for all networks derived from a two-dimensional lattice independently of the value of coordination number. On the contrary, Q_1 is much smaller for the graph derived from the cubic lattice and substantially larger for the network originating from the ladder-shape (practically one-dimensional) lattice.

The second phase transition (from state O to A) can be studied more efficiently (or with a higher accuracy) if P rather than Q is varied, because the limit cycle is affected by the quenched structural randomness even for large N . This discrepancy can be avoided by averaging over many runs on different structures whose creation is very time consuming. For temporary randomness ($P>0$), however, the technical difficulties can be overcome more easily and the numerical analysis is executable with adequate accuracy. This is the reason why henceforth we concentrate on the second phase transition occurring with the increase of P .

First we study what happens on random regular graphs (limit $Q\rightarrow 1$) for $z=3$ when increasing P . This choice was motivated by the simplicity of this treelike structure on which the dynamical cluster technique can be applied for sufficiently large clusters. In Fig. 3 the MC results show that the order parameter Φ tends very slowly to 1. Despite the large sizes in simulations we could not study the very close vicinity of the transition point (P_2) because of the fluctuations yielding an occasional transition from the noisy limit cycle to one of the homogeneous states (A). Our MC data can be well approximated by a power law behavior, namely, $1-\Phi\propto(P_2-P)^\gamma$ with an exponent $\gamma=3.3(4)$.

For periodic structures the dynamical cluster technique has proved to be a very efficient method to describe different phenomena in several nonequilibrium models [12,24–26]. In the limit $N\rightarrow\infty$ the random regular graph becomes locally treelike, and it can be considered as a Bethe lattice on which this technique works well too [27]. Using this technique one can determine the probability of each configuration occurring on a k -site cluster by solving a suitable set of master equations (details are given in [12,28]). The cluster size k is a crucial parameter. For one-site clusters this technique is equivalent to the mean-field approximation predicting concentric orbits independent of P [23]. The pair approximation

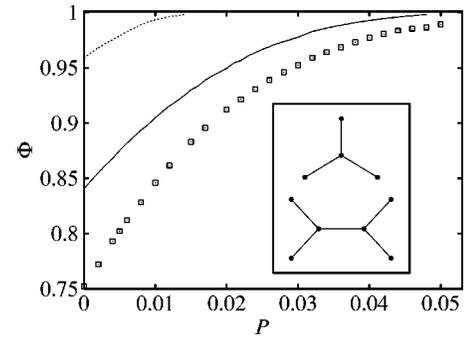


FIG. 3. The order parameter as a function of P for the three-degree random regular graph. Boxes indicate the results of MC simulation. The estimated critical value of the transition to an absorbing state is $P_2=0.079(2)$. The dashed (solid) line represents the prediction of the dynamical cluster technique at the levels of the four- and six-site approximations. Inset shows the shape of four- and six-site clusters used in these approximations.

($k=2$) gives spiral trajectories reaching the edge of triangles (or the absorbing states) for arbitrary P . Choosing four- and six-site clusters (see the inset in Fig. 3) this method was capable of reproducing the appearance of limit cycles below a threshold value of P . The quantitative predictions of this method are compared with the MC results in Fig. 3. Obviously, the increase of cluster size improves the estimation. Notice that, according to the six-site approximation, the order parameter Φ also tends to 1 very slowly (see Fig. 4). The extrapolated critical values are $P_2^{(4s)}=0.019(2)$ and $P_2^{(6s)}=0.067(4)$.

The above value of the exponent γ agrees (within statistical error) with those we observed on the square lattice in our previous work concentrating only on the effect of annealed randomness. This coincidence inspired us to study the robustness of this transition. For this purpose the MC analyses of the second transition were carried out on some two-dimensional lattices (square, triangle, and honeycomb for $Q=0$). The results, summarized in Fig. 4, seem to confirm that the transition from the global oscillation (phase O) to the absorbing states (A) is universal.

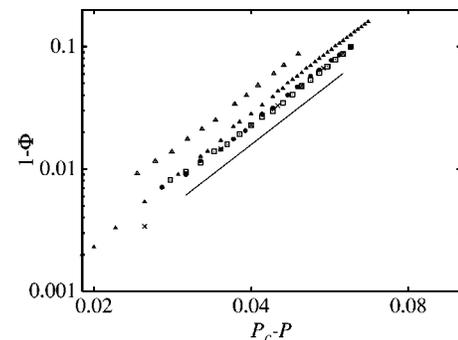


FIG. 4. Comparison of the continuous transitions to the absorbing phase for different networks. The MC data are obtained on honeycomb (triangle), square (cross), and triangular (bullet) lattices, and on a random regular graph with a degree of $z=3$ (box). Filled triangle represents the result of dynamical cluster approximation at six-site level for the same graph. The solid line represents the slope of $\beta=3.3$.

To summarize, we have studied the effect of host lattice randomness on the stationary state for a simple rock-scissors-paper system. The quenched and annealed randomness of the regular background is characterized by two parameters Q and P varying from 0 (corresponding to a lattice with a dimension of 1, or 2, or 3) to 1 (random regular graph and/or mean-field condition). This system displays two subsequent transitions if the randomness is increased. A self-organizing pattern can be observed if these randomness parameters remain within a region of the P - Q plane. When crossing the boundary of this region a global oscillation (limit cycle) occurs via a Hopf bifurcation. The transition point (for $P=0$) depends strongly on the dimension d of the original lattice; meanwhile it is hardly affected by the clustering coefficient. For the global oscillation the area of limit cycle (as well as the “amplitude”) increases with randomness up to its saturation value. Thus, above a second threshold value [more precisely for $P > P_2(Q)$] the system sooner or later terminates in one of the homogeneous absorbing states. Our simulations indicate that this second transition has also a universal fea-

ture in the slow tendency toward the saturation value. It also turned out that the global oscillation is stable ($\Phi < 1$) on quenched random regular structures if the number of neighbors is not larger than 4 ($z \leq 4$). It would be interesting to see how the synchronization (as well as the above mentioned two transitions) emerges on other random networks. The present rock-scissors-paper model involves two crucial features. On the one hand, the dynamical rule is cyclically symmetric; on the other hand, the invasion fronts become very irregular even on two-dimensional lattices because the invasion between two neighboring sites is not affected by their neighborhood. Further investigations are required to clarify what happens if the dynamical rules are not cyclically symmetric and/or the moving interfaces are smoothed by local interactions.

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