Multi-Resolution Explicit Model Predictive Control: Delta-Model Formulation and Approximation

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Abstract

This paper deals with the explicit solution and approximation of the constrained linear finite time optimal control problem for systems with fast sampling rates. To this aim, the recently developed explicit model predictive control (eMPC) is reformulated and characterized using the δ-operator to enjoy its promising advantages compared to the time-shift operator. Using the proposed δ-model eMPC formulation, a systematic method is proposed for first designing a low-complexity approximate eMPC solution and then improving its closed loop action without first determining an exact optimal solution that might be of prohibitive complexity. It is shown that the stability and feasibility of the proposed sub-optimal solution is guaranteed.

Index Terms

Explicit Model Predictive Control, Delta-operator, Multi-resolution MPC, Approximate explicit MPC, Multi-Parametric Programming.

I. INTRODUCTION

Model Predictive Control (MPC) has proved its ability to handle constrained optimal control problems in which mathematical models play a crucial role in the design and analysis of the control system. Often such a model is derived from physical laws resulting in continuous-time descriptions. In general it is well known that using continuous-time models gives realistic

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insight into the system due to the fact that the physical systems typically evolve continuously. Unfortunately, the continuous-time models cannot be used directly for implementation in digital computers. A well-known and widely used method for describing discrete-time models is the time-shift operator which is described by $x_{k+1} = qx_k$. In [1] it is shown that not only is there no intuitive connection between discrete-time and continuous-time models but also serious numerical problems arise at the high sampling rate when the shift operator is used to describe a discrete-time model. To overcome this limitation, a very simple but powerful affine mapping named the δ-operator is introduced in [1] as $q = T_s\delta + 1$. Based on the results in [1]-[3] the main properties of the δ-model are: (i) the δ-operator offers a model with almost the same degree of flexibility and simplicity as the shift operator, (ii) the δ-operator provides a more direct insight into the discrete-time system, (iii) the implementation is almost as simple as the shift operator, (iv) many results with the δ-model can be seen as an approximation for continuous-time systems with approximation error of order $\mathcal{O}(T_s)$, (v) the δ-model makes it possible to avoid non-minimum-phase sampling zeros arising in high sampling rates when using the shift operator, (vi) there are numerical advantages compared to the shift operator since not only a finite word-length is required in practice but also fixed point arithmetic is sometimes preferred.

It is well known that the online MPC is mainly limited to the systems with relatively low sampling rate. Recently in order to extend the applicability of MPC for systems with high sampling rate, the power of multi-parametric programming (mpQP) [4] has been exploited to solve the MPC problem offline, so-called explicit MPC [5], [6]. In [5] it has been proved that such a solution is a piecewise affine (PWA) function defined over a polyhedral subdivision of feasible states, mapping the current state to the optimal control. Thus, the complexity of online computation at each time instant mainly depends on how fast one can identify the region in which the current state lies, the so called point location problem (see [7], [8] and references therein). Operating at high sampling rates along with the mentioned characteristics of the δ-operator motivates us to invoke the δ-model description to reformulate and solve the explicit MPC problem which naturally inherits the mentioned advantages. In this regard, many works ([9]-[12]) have exploited the δ-model to solve unconstrained generalized predictive control (GPC). Reference [9] utilizes the δ-model to solve continuous-time emulator-based GPC. Extension of the output end-point weighted GPC of SISO systems to the δ-domain is addressed in [10]. Reference [12] considers δ-domain GPC for both minimum and non-minimum phase linear SISO systems considering nominal stability and performance. In the recent work [11] an exact discrete-time formulation is obtained to solve the unconstrained δ-GPC problem.

In contrast to the mentioned works, this paper deals with the explicit solution to the general constrained linear model predictive control and its approximation based on the δ-model representation (δ-eMPC). Recently there is also a growing interest in approximation of eMPC controllers, and several approaches with different points of view have been reported [13]-[27].In [13] by relaxing the first order Karush-Kuhn-Tucker optimality conditions, an approximate solution to mpQP is proposed. In [14] a suboptimal solution is computed based on sub-division of hypercubes and minimizing the loss in the cost function over hypercubes which provides a priori stability guarantee and performance bound. Also, in [21] the concept of Input-to-State-Stability Lyapunov function is exploited to obtain a priori conditions for asymptotic stability and feasibility of the approximate controller. The interpolation idea is used in [15]-[19]. In [20] an approximate solution is obtained using bilevel optimization with no need to compute the optimal explicit MPC first. However, it can be computationally expensive in certain cases, as an iterative solution of MILP problems in each step is required. Besides, only an a posteriori stability test is provided. In [22] a polynomial approximation of the optimal control law is presented which requires computation of stability tubes to ensure stability and constraint satisfaction. Recently, canonical PWA functions are employed in [23] to obtain an approximate eMPC controller suitable for implementing on chips. However, only a posteriori checks for stability are provided.

The main contribution of this paper is that using the close connection between δ-model and continuous-time system, a multi-resolution design method is introduced which enables us to systematically design a stable low-complexity approximate eMPC solution without first determining an exact high-complexity optimal solution. The key feature of this approach is that the structure of the controller is predetermined using coarse design parameters leading to a low-complexity controller and then an approximate controller is redesigned by solving an optimization problem. Stability and feasibility of the closed loop system with the proposed sub-optimal low-complexity solution is guaranteed a priori.

II. PROBLEM FORMULATION

A. δ-Model Specifications

Consider a continuous-time LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

Where $x \in R^n, u \in R^m, A \in R^{n\times n}, B \in R^{n\times m}$ and $C \in R^{p\times n}$. Using the δ-domain representation proposed in [28], the corresponding state space formulation is given by:

$$\begin{aligned} \delta x(t) &= A_\delta x(t) + B_\delta u(t) \\ y(t) &= C_\delta x(t) \end{aligned} \tag{2}$$

Where $\delta = \frac{q-1}{T_s}$ and T_s represents the associated sampling time. Based on this definition it is easy to verify that the system matrices are related to the continuous time model as $A_\delta = \frac{e^{T_s A} - I}{T_s}$, $B_\delta = \frac{\int_0^{T_s} e^{A(T_s - \tau)} B d\tau}{T_s}$ and $C_\delta = C$. Hereafter to simplify notation we use x_t instead of $x(tT_s)$.

Lemma 1 ([2]): The quantities appearing in the δ-domain representation converge to the corresponding continuous time quantities; i.e. (**i**) $\lim\limits_{(T_s A)\to 0} A_\delta = A$, and (**ii**) $\lim\limits_{(T_s A)\to 0} B_\delta = B$.

Remark 1: The slight modification in lemma 1 is that, we have replaced $(T_s \to 0)$ by $(T_s A \to 0)$ to emphasize that the high sampling rate assumption does not refer to its literal $(T_s \to 0)$ but could be measured relative to the system bandwidth, i.e. $(T_s A \to 0)$. Using Taylor expansion formula one easily obtains $A_\delta = A(I + \sum\limits_{j=1}^{\infty} \frac{(T_s A)^j}{(j+1)!})$, then it is obvious that $A_\delta \to A$ as $(T_s A) \to 0$.

B. δ-model based explicit MPC (δ-eMPC)

Consider the problem of regulating to the origin the LTI δ-model (2) with constraints $y_{\min} \leq y_t \leq y_{\max}$ and $u_{\min} \leq u_t \leq u_{\max}, \forall t \geq 0$. Then, δ-model based MPC solves the following optimization problem, where $J^*(x_t) = \min\limits_U J(x_t)$:

$$\begin{aligned} J^*(x_t) = \min_U & \left\{ \|x_{t+N}\|_2^{P_\delta} + \sum_{k=0}^{N-1} \|x_{t+k}\|_2^Q + \|u_{t+k}\|_2^R \right\} \\ s.t.\ \ & y_{\min} \leq y_{t+k} \leq y_{\max}, \qquad k = 1, ..., N \\ & u_{\min} \leq u_{t+k} \leq u_{\max}, \qquad k = 0, ..., N-1 \\ & x_{t+N} \in \mathcal{O}_\delta^\infty,\ u_{t+k} = K_\delta x_{t+k}, \qquad k \geq N \\ & \delta x_{t+k} = A_\delta x_{t+k} + B_\delta u_{t+k}, \\ & y_{t+k} = C_\delta x_{t+k}, \qquad k \geq 0 \end{aligned} \tag{3}$$

where $U = \left[u_t^T, u_{t+1}^T, \cdots, u_{t+N-1}^T\right]^T \in R^{mN}$ is the optimization vector, $\|x_t\|_2^Q = x_t^T Q x_t$, $\mathcal{O}_\delta^\infty = \{x_t \in R^n | G_f x_t \leq h_f\}$ indicates the maximal output admissible set associated with the infinite horizon LQR controller (see e.g. [32]), and $P_\delta \succ 0$ is the solution of the following algebraic Riccati equation:

$$\begin{aligned} K_\delta &= -\widehat{R}^{-1} B_\delta^T P_\delta \left(I + T_s A_\delta\right) \\ \mathbf{0} &= Q + A_\delta^T P_\delta + P_\delta A_\delta + T_s A_\delta^T P_\delta A_\delta - K_\delta^T \widehat{R} K_\delta \end{aligned} \tag{4}$$

where $\widehat{R} = (R + T_s B_\delta^T P_\delta B_\delta)$ (see theorem 11.2.1 in [1] and letting $N \to \infty$).

Remark 2: Based on the results in [2], [29]-[31], (4) can be seen as an approximation to the continuous time LQR with approximation error of order $\mathcal{O}(T_s A)$. More importantly, this feature is not sensitive to the chosen sampling rate. Consequently, the approximation will remain accurate for a wide range of sampling rates and is still valid for arbitrary sampling rate with approximation error of $\mathcal{O}(T_s A)$. Note that this is not true when using the time-shift operator.

Lemma 2 provides properties we need to adapt the optimization problem (3) to the mpQP formulation studied in [5].

Lemma 2: Define $X = \left[x_{t+1}^T, \cdots, x_{t+N}^T\right]^T, Y = \left[y_{t+1}^T, \cdots, y_{t+N}^T\right]^T, \tilde{X} = \left[\delta^1 x_t^T, \cdots, \delta^N x_t^T\right]^T$, $\tilde{Y} = \left[\delta^1 y_t^T, \cdots, \delta^N y_t^T\right]^T$ and $\tilde{U} = \left[\delta^0 u_t^T, \cdots, \delta^{N-1} u_t^T\right]^T$, then the following properties are held. (**i**) $\delta^k x_t = A_\delta^k \delta^0 x_t + \sum_{j=0}^{k-1} A_\delta^{k-1-j} B_\delta \delta^j u_t$, and $\delta^0 x_t = x_t$, (**ii**) $x_{t+k} = \sum_{j=0}^{k} \Upsilon_j^k \delta^j x_t$, where $\Upsilon_j^k = \frac{k!}{j!(k-j)!} T_s^j$, (**iii**) $\tilde{X} = \tilde{A}\delta^0 x_t + \tilde{B}\tilde{U}$, and $\tilde{Y} = \tilde{C}\tilde{X}$, (**iv**) $X = \Lambda_0 \delta^0 x_t + \Lambda_x \tilde{X}$, and $Y = \tilde{C}X$, (**v**) $U = \Lambda_u \tilde{U}$. Where I_n represents the identity matrix of dimension n, $\Lambda_0 = [I_n, \cdots, I_n]^T \in R^{nN \times n}$, $\tilde{A} \in R^{nN \times n}$ is a vertical block matrix where its i-th building block is A_δ^i, $\tilde{C} = Ndiag(C_\delta, N)$[1]. $\tilde{B} \in R^{nN \times mN}$, $\Lambda_x \in R^{nN \times nN}$ and $\Lambda_u \in R^{mN \times mN}$ are lower triangular block matrices where the (i,j)-th $(i \geq j)$ building blocks are ${A_\delta}^{i-1} B_\delta$, $\Upsilon_j^i I_n$, and $\Upsilon_{j-1}^{i-1} I_m$, respectively.

Proof: Since $\delta^0 x_t = x_t$, the first proposition is immediate by forward multiplication of δ-operator on $\delta x_t = A_\delta x_t + B_\delta u_t$ and backward substitution for $j = 1, ..., k$. Definition $q = T_s \delta + 1$ yields $x_{t+1} = (1 + T_s \delta) x_t$ and then $x_{t+2} = (1 + T_s \delta) x_{t+1} = (1 + T_s \delta)^2 x_t$. By similar successive operations one can obtain $x_{t+k} = (1 + T_s \delta)^k x_t$ and invoking the binomial formula yields (ii). Using (i) and (ii) for $k = 1, ..., N$ it is straightforward to verify (iii) and (iv), noting that $\delta^k y_t = C_\delta \delta^k x_t$. Finally with a same reasoning in (ii) one can obtain $u_{t+k} = \sum_{j=0}^{k} \Upsilon_j^k \delta^j u_t$ and then use this to verify proposition (v). ■

[1] $Ndiag(Q, n)$ indicates n times block diagonal concatenation of Q.

Substituting x_{t+k} in (3) by utilizing the results of Lemma 2, the optimization problem (3) can by algebraic manipulations be reformulated as:

$$\begin{aligned} V^*(x_t) = \min_{\tilde{U}} \ & \tfrac{1}{2}\tilde{Z}^T\tilde{H}\tilde{Z} \\ s.t.\ & \tilde{G}\tilde{Z} \le \tilde{W} + \tilde{S}x_t \end{aligned} \tag{5}$$

where $V^*(x_t) = J^*(x_t) - x_t^T(\tilde{\Gamma} - \frac{1}{2}\tilde{F}\tilde{H}^{-1}\tilde{F}^T)x_t, \tilde{Z} = \tilde{U} + \tilde{H}^{-1}\tilde{F}^T x_t, \tilde{\Gamma} = \bar{\Lambda}_0^T\bar{Q}\bar{\Lambda}_0 + Q, \tilde{H} = 2(\bar{\Lambda}_u + \tilde{B}^T\bar{\Lambda}_x\tilde{B}), \tilde{F} = 2(\bar{\Lambda}_0^T\bar{Q}\Lambda_x\tilde{B}), \bar{\Lambda}_0 = \Lambda_0 + \Lambda_x\tilde{A}, \bar{\Lambda}_x = \Lambda_x^T\bar{Q}\Lambda_x, \bar{\Lambda}_u = \Lambda_u^T\bar{R}\Lambda_u, \bar{R} = Ndiag(R,N), \bar{Q} = diag(Ndiag(Q,N-1),P_\delta)$ [2], $\tilde{G} = [(\tilde{C}\Lambda_x\tilde{C}\tilde{B})^T, (-\tilde{C}\Lambda_x\tilde{C}\tilde{B})^T, (\Lambda_u)^T, (-\Lambda_u)^T, (G_f\underline{\Lambda}_x\tilde{B})^T]^T$, $\tilde{W} = [Y_{max}^T, -Y_{min}^T, U_{max}^T, -U_{min}^T, h_f^T]^T, \tilde{S} = \tilde{E} + \tilde{G}\tilde{H}^{-1}\tilde{F}^T$, and $\tilde{E} = [(-\tilde{C}\bar{\Lambda}_0)^T, (\tilde{C}\bar{\Lambda}_0)^T, \mathbf{0}, \mathbf{0}, (-G_f - G_f\underline{\Lambda}_x\tilde{A})^T]^T$, $\underline{\Lambda}_x$ denotes the last n rows of Λ_x, and $Y_{max/min}$ ($U_{max/min}$) is similar to Y (U), replacing all y_{t+k} ($\delta^{k-1}u_t$) for all $k = 1, ..., N$, with $y_{max/min}$ ($u_{max/min}$).
The proof of derivation is omitted due to the lack of space and its similarity to that of [5].

Theorem 1 fully describes the solution characteristics of the δ-eMPC problem (5).

Definition 1 (feasible set $\mathcal{X}_{feas}$): The feasible set $\mathcal{X}_{feas} \subset R^n$ is defined as the set of all states $x_t \in R^n$ for which the optimization problem (5) is feasible, i.e. $\mathcal{X}_{feas} = \{x_t \in R^n | \exists \tilde{Z} \in R^{mN}, \tilde{G}\tilde{Z} \le \tilde{W} + \tilde{S}x_t\}$. The feasible is naturally computed by orthogonal projection of polytope $\{\tilde{G}\tilde{Z} - \tilde{S}x_t \le \tilde{W}\}$ onto the x-coordinate. More computationally efficient approach is proposed in [33] to compute exact and inner approximate feasible set.

Theorem 1: Consider the mpQP (5) with $\tilde{H} \succ 0$. Suppose $\tilde{G}$ has full row rank. Then, **(i)** the feasible set $\mathcal{X}_{feas}$ is convex, **(ii)** the optimal solution $\tilde{Z}^*(x_t)$ (and $\tilde{U}^*(x_t)$) is continuous PWA function of x_t, **(iii)** $J^*(x_t)$ is a convex and continuous PWQ function on $\mathcal{X}_{feas}$, and **(iv)** the critical region $CR \subset \mathcal{X}_{feas}$ where the solution is optimal is given by $CR = \{x \in \mathcal{X}_{feas} | \tilde{G}\tilde{H}^{-1}\tilde{G}^T(\tilde{G}^\beta\tilde{H}^{-1}\tilde{G}^T)^{-1}(\tilde{W}^\beta + \tilde{S}^\beta x_t) \le \tilde{W} + \tilde{S}x_t, \& - (\tilde{G}^\beta\tilde{H}^{-1}\tilde{G}^T)^{-1}(\tilde{W}^\beta + \tilde{S}^\beta x_t) \ge 0\}$, and $(\tilde{G}^\beta, \tilde{W}^\beta, \tilde{S}^\beta)$ be the rows of active constraints.

Proof: The proof is direct implication of the results, mutatis mutandis, in [5] (theorems 2 and 4) to the underlying optimization problem given in (5). ■

[2] $diag(Q,P)$ indicates block diagonal concatenation of Q and P.

III. Approximation of the δ-eMPC

A. Approximate δ-eMPC

The main drawback of explicit MPC is that the number of polyhedral regions in the partition may increase rapidly when the number of constraints involved in the mpQP is large. Several references attempt to handle complexity of eMPC via approximation [14]-[19]. In contrast to the existing works, in this paper a different framework and approximation criterion is proposed. To this aim, taking the physical system information in the continuous MPC into account, we assume that an appropriate prediction horizon is known and is about T time units. Then we approximate the continuous model with a corresponding δ-model as characterized in chapter II. Then we will propose a two-steps multi-resolution δ-eMPC solution which provides a trade-off between complexity and optimality. At the first step, we choose the time prediction horizon ($T_h \approx T$) and assume that if the sampling interval is chosen equal or less than a certain value T_f, then a favorable approximation is achieved (i.e. fine solution with desired performance). This fine δ-model is represented as:

$$\Delta_f : \begin{cases} \delta_f x_t = A_{\delta_f} x_t + B_{\delta_f} u_t \\ y_t = C_{\delta_f} x_t \end{cases} \tag{6}$$

Note that, in general achieving a desirable fine model may lead to a high complexity solution of the corresponding mpQP problem. To handle this limitation we may need to approximate the solution using a coarse sampling time $T_c > T_f$ to achieve less complexity. To parameterize the level of approximation, we define a parameter α denoting the approximation level and let $2T_f < 2T_c < T_h$. Then for given parameters (T_h, T_f, α) we have $N_f = \lceil {}^{T_h}\!/\!{}_{T_f} \rceil + 1$ fine discretization points on the horizon while the coarse approximation is characterized by:

$$(i) N_c = \lceil \frac{N_f}{\alpha} \rceil + 1, (ii) T_c = \frac{T_h}{N_c - 1}. \tag{7}$$

Then, the coarse model (Δ_c) is represented by $(A_{\delta_c}, B_{\delta_c}, C_{\delta_c})$. Assume the coarse optimal solution $\tilde{Z}_c^*(x_t)$ (and $\tilde{U}_c^*(x_t)$) defined on the bounded polyhedral regions $\{\mathcal{X}_j^c\}_{j=1}^{N_P}$ is obtained using the coarse model Δ_c and leading to a stable closed loop system. In the following we introduce a procedure which takes the information of a finite number of optimal control inputs corresponding to the fine controller to improve the coarse controller gains while preserving the simpler structure of the coarse controller, feasibility and stability.

Remark 3: Note that, in the following in order to handle infeasibility of QPs when $\tilde{Z}_f^*(v_i^j)$ is calculated (see Theorem 2), we spilt the coarse model critical region $\mathcal{X}_j^c$ as $\{\widehat{\mathcal{X}}_j^c = \mathcal{X}_j^c \cap \mathcal{X}_{feas}^f, \breve{\mathcal{X}}_j^c = \mathcal{X}_j^c \backslash \widehat{\mathcal{X}}_j^c\}, \forall j = 1, ..., N_p$ when $\mathcal{X}_j^c \not\subset \mathcal{X}_{feas}^f$ (or $\mathcal{X}_j^c \backslash \mathcal{X}_{feas}^f \neq \emptyset$). Note that if $\breve{\mathcal{X}}_j^c$ is not a convex region then it is naturally split to a union of some convex regions with the same affine control gains. Then $\mathbf{X}_c = \widehat{\mathbf{X}}_c \cup \breve{\mathbf{X}}_c$ where $\widehat{\mathbf{X}}_c = \left\{\widehat{\mathcal{X}}_j^c\right\}_{j=1}^{N_p}, \breve{\mathbf{X}}_c = \left\{\breve{\mathcal{X}}_j^c\right\}_{j=1}^{\overline{N}_p}$ and $\overline{N}_p$ denotes all regions including those regions added when $\breve{\mathcal{X}}_j^c$ is not convex.

Theorem 2: Given the PWA solution $\tilde{Z}_c^*(x_t) = \mathcal{K}_j^c[x_t^T, 1]^T, \forall x_t \in \mathcal{X}_j^c$, where $\mathcal{K}_j^c = [F_j^c, G_j^c]$ denotes the optimal coarse gain. Assume any bounded polyhedral region $\widehat{\mathcal{X}}_j^c \in \widehat{\mathbf{X}}_c$ with vertices $\{v_1^j, ..., v_{M_j}^j\}$ and a tolerance $\xi \geq 0$. Then, the improved gain obtained from the least-square optimization problem (8) guarantees the solution $\hat{\tilde{Z}}_c(x_t, \xi) = \hat{\mathcal{K}}_j^c[x_t^T, 1]^T, \forall x_t \in \widehat{\mathcal{X}}_j^c$ is feasible.

$$
\begin{aligned}
&\Psi(\xi) = \arg\min_{\hat{\mathcal{K}}_j^c} \sum_{i=1}^{M_j} \mathcal{L}_{\rm i}^T \mathcal{W} \mathcal{L}_{\rm i} \\
&s.t.\ \tilde{G}\hat{\mathcal{K}}_j^c \mathcal{V}_i^j \leq [\tilde{S}, \tilde{W}]\mathcal{V}_i^j, \quad i \in \{1, ..., M_j\} \\
&\begin{pmatrix} I & \hat{H}\hat{\mathcal{K}}_j^c \\ (\hat{H}\hat{\mathcal{K}}_j^c)^T & \mathcal{K}_H \end{pmatrix} \succeq 0, \begin{pmatrix} \xi I & (\hat{\mathcal{K}}_j^c - \mathcal{K}_j^c)^T \\ \hat{\mathcal{K}}_j^c - \mathcal{K}_j^c & I \end{pmatrix} \succeq 0,
\end{aligned}
\tag{8}
$$

Where $\hat{\mathcal{K}}_j^c = \Psi(\xi), \mathcal{V}_i^j = [(v_i^j)^T, 1]^T$, and $\mathcal{L}_{\rm i} = \tilde{Z}_f^*(v_i^j) - \hat{\mathcal{K}}_j \mathcal{V}_i^j$ is the error of vertex i. $\tilde{H} = \hat{H}^T\hat{H}$ and $\mathcal{K}_H = \|\mathcal{K}_j^c\|_2^{\tilde{H}}$.

Proof: Note that (8) is itself feasible, since $\hat{\tilde{Z}}_c(x_t, \xi) = \tilde{Z}_c^*(x_t) = \mathcal{K}_j^c[x_t^T, 1]^T, \forall x_t \in \widehat{\mathcal{X}}_j^c$ is a feasible but not necessarily optimal solution for (8). When $\hat{\mathcal{K}}_j^c = \mathcal{K}_j^c$, then using the Schur complement lemma it is straightforward to check that the second and third constraints are satisfied for any $\xi \geq 0$. This implies that the non-improved controller gain $\mathcal{K}_j^c$ is itself feasible. On the other hand, expanding first constraint reveals that the feasibility of the $\hat{\mathcal{K}}_j^c$ is explicitly imposed for all vertices $\{v_1^j, ..., v_{M_j}^j\}$ for each region $\widehat{\mathcal{X}}_j^c \subset \mathcal{X}_{feas}^c$, i.e. $\tilde{G}(\hat{F}_j^c v_i^j + \hat{G}_j^c) \leq \tilde{W} + \tilde{S}v_i^j, \forall i \in \{1, ..., M_j\}$. Also, by convexity of regions, any state vector $x_t \in \widehat{\mathcal{X}}_j^c$ can be represented by a convex combination of the associated vertices as $x_t = \sum_{r=1}^{M_j} \mu_r v_r^j$ where $\sum_{r=1}^{M_j} \mu_r = 1$. Then we obtain $\hat{\tilde{Z}}_c(x_t, \xi) = \hat{F}_j^c \sum_{r=1}^{M_j} \mu_r v_r^j + \hat{G}_j^c = \sum_{r=1}^{M_j} \mu_r \hat{F}_j^c v_r^j + \sum_{r=1}^{M_j} \mu_r \hat{G}_j^c = \sum_{r=1}^{M_j} \mu_r (\hat{F}_j^c v_r^j + \hat{G}_j^c)$. Accordingly we get $\tilde{G}\hat{\tilde{Z}}_c(x_t, \xi) = \sum_{r=1}^{M_j} \mu_r \tilde{G}(\hat{F}_j^c v_r^j + \hat{G}_j^c) \leq \sum_{r=1}^{M_j} \mu_r (\tilde{W} + \tilde{S}v_r^j) = \tilde{W} + \tilde{S}x_t$. This implies the feasibility of $\hat{\tilde{Z}}_c(x_t, \xi) = \hat{\mathcal{K}}_j^c[x_t^T, 1]^T, \forall x_t \in \widehat{\mathcal{X}}_j^c$. ■

Remark 4: We emphasize that, knowing the complete fine explicit solution in Theorem 2 is not required, which may be computationally very expensive. Rather, one needs to compute optimal fine solutions in some finite points $\{v_1^j, ..., v_{M_j}^j\}$ using numerical QP solvers. Then the convex optimization problem (8) can be solved by of-the-shelf software, like CVX [34].

Remark 5: The parameter ξ in (8) enables us to control deviation of the suboptimal gain $\hat{\mathcal{K}}_j^c$ from the optimal gain $\mathcal{K}_j^c$, where smaller ξ implies less deviation permission and vice versa. We also note that, although the second and third convex constraints in (8) may make the gain computations conservative, but are introduced for reasons that will be clear later (Theorem 4).

Remark 6: Since in the MPC law, only the first m elements of $\tilde{U}_f^*(x_t)$ (or $\tilde{Z}_f^*(x_t)$) are applied to the system, then a sensible choice for $\mathcal{W}$ is $diag(\rho I_m, \mathbf{0})$ which takes only the effect of $u_f^*(x_t)$ into account.

B. Bounds on the sub-optimality and stability

We emphasize that the suboptimal controller (i.e. $\hat{\tilde{U}}_c$) does not naturally inherit the nominal stability properties of the optimal solution (i.e. $\tilde{U}_c^*$) although its feasibility is guaranteed as shown in Theorem 2. However, since the optimal coarse controller $\tilde{U}_c^*(x_t)$ is stabilizing and the associated optimal cost $J_c^*(x_t)$ is a Lyapunov function, then the distance from this stabilizing solution might be used as a measure to guarantee stability of the approximate controller. To this aim, let $\tilde{Z}_c^*(x_t)$ denotes the the optimal solution of an arbitrary region $\widehat{\mathcal{X}}_j^c \subseteq \widehat{\mathbf{X}}_c$. The corresponding optimal cost is given by $J_c^*(x_t) = 0.5\tilde{Z}_c^*(x_t)\tilde{H}\tilde{Z}_c^*(x_t) + x_t^T P_\delta x_t$. For the same region, assume $\hat{\tilde{Z}}_c(x_t, \xi)$ is any feasible approximate solution obtained from (8) with the suboptimal cost $\hat{J}_c(x_t, \xi) = 0.5\hat{\tilde{Z}}_c^T(x_t, \xi)\tilde{H}\hat{\tilde{Z}}_c(x_t, \xi) + x_t^T P_\delta x_t$. Define the upper cost tolerance $\varepsilon_J(\xi) = \max_{x_t \in \mathcal{X}_j^c}(\hat{J}_c(x_t, \xi) - J_c^*(x_t))$. Then, the asymptotic stability of the origin under the approximate δ-eMPC controller given in Algorithm 1 can be guaranteed by imposing conditions on the tolerance ε_J similar to the results in [14] (Theorem 5) and [13] (Theorem 5.2).

Theorem 3: Consider the mpQP problem (5) with $\tilde{H} \succ 0$. Define $\Sigma = Q + K_\delta^T R K_\delta$, assume $\Sigma \succ 0$, and let $\gamma > 0$ be the largest number for which the ellipsoid $\mathbb{E} = \{x_t \in \mathcal{X}_{feas}^c | x_t^T \Sigma x_t \leq \gamma\}$ is contained in $\mathcal{O}_\delta^\infty$ for the coarse problem. If in the proposed Algorithm 1, at each step corresponding to the region $\mathcal{X}_0$, the error tolerance ε_J is less than $\bar{\varepsilon} = 0.5\left(\gamma + x_0^T \Sigma x_0\right)$, where $x_0 = \arg\min_{x_t \in \mathcal{X}_0} x_t^T \Sigma x_t$, then the approximate controller makes the origin asymptotically stable for all $x_t \in \mathcal{X}_{feas}^c$, while guaranteeing the feasibility of the state and input trajectories.

Theorem 4: Assume any approximate solution $\hat{\mathcal{K}}_j^c = \Psi(\xi)$ obtained from (8) for a given $\xi \geq 0$. Then, (i) the optimization problem $\varepsilon_J(\xi) = \max_{x_t \in \mathcal{X}_j^c}(\hat{J}_c(x_t, \xi) - J_c^*(x_t))$ is guaranteed to be concave, and (ii) the following optimization problem is feasible and the associated solution $\hat{\mathcal{K}}_j^c = \Psi(\xi_0)$ is asymptotically stabilizing:

$$\begin{aligned} \xi_0 = \max_{\xi \geq 0} \xi \\ s.t.\ \ \varepsilon_J(\xi) \leq \bar{\varepsilon} \end{aligned} \tag{9}$$

Proof: Since $\tilde{Z}_c^*(x_t)$ and $\hat{\tilde{Z}}_c(x_t)$ belong to the same region $\widehat{\mathcal{X}}_j^c$, then $\hat{J}_c(x_t)$ and $J_c^*(x_t)$ are quadratic and convex $\forall x_t \in \widehat{\mathcal{X}}_j^c$. This implies the difference is also quadratic and $\varepsilon_J(\xi) = 0.5 \max_{x_t \in \mathcal{X}_j^c} (\hat{\tilde{Z}}_c^T(x_t,\xi)\tilde{H}\hat{\tilde{Z}}_c(x_t,\xi) - \tilde{Z}_c^*(x_t)\tilde{H}\tilde{Z}_c^*(x_t))$. Substituting from Theorem 2 gives $\varepsilon_J(\xi) = 0.5 \max_{x_t \in \mathcal{X}_j^c}([x_t^T, 1]H_\varepsilon[x_t^T, 1]^T)$, where $H_\varepsilon = (\hat{\mathcal{K}}_j^{cT}\tilde{H}\hat{\mathcal{K}}_j^c - \mathcal{K}_j^{cT}\tilde{H}\mathcal{K}_j^c)$. On the other hand, using the Schur complement of the second constraint in (8) results $(\mathcal{K}_H - (\hat{H}\hat{\mathcal{K}}_j^c)^T I^{-1}(\hat{H}\hat{\mathcal{K}}_j^c)) \succeq 0$. Equivalently, $(\mathcal{K}_j^{cT}\tilde{H}\mathcal{K}_j^c - \hat{\mathcal{K}}_j^{cT}\tilde{H}\hat{\mathcal{K}}_j^c) \succeq 0$, where $\tilde{H} = \hat{H}^T\hat{H}$ and $\mathcal{K}_H = \mathcal{K}_j^{cT}\tilde{H}\mathcal{K}_j^c$. Comparing this with H_ε implies $H_\varepsilon \preceq 0$ which proves (i). The feasibility of (9) can be easily verified by choosing $\xi_0 = 0$. Then, using the Schur complement of the third constraint in (8) results $0 - (\hat{\mathcal{K}}_j^c - \mathcal{K}_j^c)^T I^{-1}(\hat{\mathcal{K}}_j^c - \mathcal{K}_j^c) \succeq 0$. This evidently implies $\hat{\mathcal{K}}_j^c = \mathcal{K}_j^c$ and thus $\varepsilon_J(0) = 0 \leq \bar{\varepsilon}$. Finally, any feasible solution ξ_0 obtained from (9) guarantees that $\varepsilon_J(\xi_0) \leq \bar{\varepsilon}$. Choosing $\bar{\varepsilon}$ as in Theorem 3, guarantees the asymptotic stability of the approximate controller. ■

Theorem 5: Algorithm 1 terminates after a finite iterations resulting in a feasible sub-optimal solution $\hat{\tilde{Z}}_c(x_t)\forall x_t \in \widehat{\mathbf{X}}_c$ with associated cost $\hat{J}_c(x_t)$ that satisfies $0 \leq \hat{J}_c(x_t) - J_c^*(x_t) \leq \bar{\varepsilon}$.

Proof: The finite termination of the algorithm is immediate implication of the fact that the corresponding polyhedral partition has finite number of regions. Also, the feasibility guarantee of all improved controllers is explicitly imposed in the least squares problem (8) (see Theorem 2). Finally, since the cost $J_c^*(x_t)$ is optimal, $\hat{J}_c(x_t)$ is itself an upper bound on $J_c^*(x_t)$ and the error upper bound is explicitly enforced in Alg.1 step (v), such that $0 \leq \hat{J}_c(x_t) - J_c^*(x_t) \leq \bar{\varepsilon}$. ■

Remark 7: Note that the proposed low-complexity control law is continuous (see Theorem 1) while the resulting improved control law is possibly discontinuous as some other approximate approaches [14], [21]. However, since the asymptotic stability of $\mathbf{0} \in \mathcal{O}_\delta^\infty$ and the final set constraint $x_{t+N} \in \mathcal{O}_\delta^\infty$ are guaranteed, then the control signal would not chatter for a long time (i.e. $> N_c T_c$).

Algorithm 1 : Approximate δ-eMPC

Given a coarse optimal solution $\tilde{Z}_c^*(x_t), x_t \in \mathbf{X}_c$.

i: Calculate $\mathcal{X}_{feas}^c$ and $\mathcal{X}_{feas}^f$ as in Definition 1 and $\mathcal{O}_\delta^\infty$ for the coarse solution. Then calculate γ as in Theorem 3.

ii: For $j = 1, ..., N_p$, if $\mathcal{X}_j^c \not\subset \mathcal{X}_{feas}^f$, then update $\mathbf{X}_c$ and $\mathcal{X}_j^c$ according to Remark 3 as $\mathbf{X}_c = \widehat{\mathbf{X}}_c \cup \breve{\mathbf{X}}_c$. Then suppose without loss of generality that $\widehat{\mathcal{X}}_1^c$ is associated with the unconstrained LQR controller and $\mathbf{0} \in \widehat{\mathcal{X}}_1^c$. Mark $\widehat{\mathcal{X}}_1^c$ as explored and all other regions $\widehat{\mathcal{X}}_j^c \in \widehat{\mathbf{X}}_c$ as unexplored.

iii: Select any unexplored region $\mathcal{X}_0 \in \widehat{\mathbf{X}}_c$ with vertices $\{v_1, ..., v_M\}$, mark this region as explored. If there is no such a region, go to step **vi**.

iv: Solve the QP (5) considering the fine model for all $x_t \in \{v_1, ..., v_M\}$. Then calculate corresponding optimal fine solution $\tilde{Z}_f^*(x_t)$. Also calculate $\bar{\varepsilon}$ as in Theorem 3.

v: Solve the optimization problem (9) together with (8) to obtain ξ_0 and then $\hat{\mathcal{K}}_0^c = \Psi(\xi_0)$. Go to step **iii**.

vi: Collect and return all approximate controllers' gain, then terminate.

C. Simulation Example

Consider the second order system $y(s) = \frac{1.01s+1.515}{s^2+0.7s+1.11}u(s)$ with the fine parameters $(T_h, T_f, \alpha) = (0.2, 0.01, 2)$. Then, using (7) the corresponding coarse δ-model is obtained as

$$\delta x(t) = \begin{bmatrix} -0.5068 & 0.9936 \\ -1.0035 & -0.2088 \end{bmatrix} x(t) + \begin{bmatrix} -0.9863 \\ 1.0173 \end{bmatrix} u(t), \tag{10}$$

Where $y(t) = x_2(t)$. The employed design parameters are $Q = I_{2\times 2}, R = 0.1, \rho = 10, \|u(t)\|_\infty \leq 1, \|y(t)\|_\infty \leq 1$, P_δ is obtained using (4) and $\gamma = 0.179$ associated with the coarse model is obtained from Theorem 3. The exact solution to the coarse δ-eMPC problem contains 729 polyhedral regions having overlap with the fine feasible region. In order to compare, the exact Fine δ-eMPC solution is also computed which contains 2739 polyhedral regions. It can be seen that in the cost of sub-optimality the complexity of the coarse solution has been reduced to a large extent ($\approx 70\%$) compared to the fine solution. Note that, this complexity reduction costs performance degradation which can be moderated in the next step by applying the proposed improvement procedure in Algorithm 1. To verify efficiency of the improvement procedure, control actions corresponding to the coarse and improved-coarse controllers are calculated for 3759 feasible state variables and compared to the fine controller. According to the results in this case $(\alpha = 2)$, the mean square error in the improved control law is about 50% less than the original coarse solution. Despite the fact that the complexity reduction is achieved in the cost of performance degradation, we observe that the closed loop action in the improved controller is fairly close to the fine controller. This fact is illustrated and compared in Fig.1 for $x(0) = [-1, -1]^T$. Also, the output trajectory difference is compared in Fig.2 for the same simulation. Several simulations are employed to illustrate how the reduction factor (α) affects the complexity of solution. In Table 1 the simulation results are given for different values of α and also for two extra MIMO examples. Therein N_v denotes the number of feasible points and $Mean\,(J)$ denotes the normalized mean value of all cost functions in (3) associated with all regions. The results illustrate that the parameter α can be used as a tuning knob to trade-off between complexity and performance of the resulting controller. The last column in the Table 1 denotes the performance of the improvement procedure in the sense of objective function (3) when applied to the sub-optimal coarse solution to make its action as close as possible to the fine optimal solution. The design parameters associated with the next two examples are as follows. In example 2 a second order MIMO system is considered as $y_1(s) = \frac{1}{p(s)}\left[(7.5\mathrm{s} - 46.87)\,u_1(s) + (22.5\mathrm{s} + 18.76)\,u_2(s)\right], y_2(s) = \frac{1}{p(s)}\left[(7.497\mathrm{s} + 9.373)\,u_1(s) + (-0.0012\mathrm{s} + 18.76)\,u_2(s)\right]$, where $p(s) = s^2 + 1.249s + 6.252$. The corresponding design parameters are $(T_h, T_f) = (0.4, 0.02)\,, Q = 0.3I_2, R = I_2, \rho = 0.1, \|u(t)\|_\infty \leq 10, \|y(t)\|_\infty \leq 5$, and $\gamma = 0.198$ is obtained using Theorem 3. The last example deals with the so-called Ball $\&$ Plate system of [35] in the form of regulating to the origin. The design parameters are $(T_h, T_f) = (0.2, 0.03)\,, Q = diag\,([6, 0.1, 500, 100])\,, R = 1, \rho = 0.1, \|u(t)\|_\infty \leq 10, |y_1| \leq 30, |y_2| \leq 15, |y_2| \leq 15, |y_3| \leq 0.26, |y_4| \leq 1$, and $\gamma = 0.264$ is obtained using Theorem 3.

IV. CONCLUSIONS

In this paper the explicit MPC problem is reformulated and characterized using the δ-model. It is discussed in section II that in the proposed method the system model and solution properties have close connections to the continuous-time system, which is not the case in the explicit MPC based on the shift operator. Using this close connection to the continuous-time domain, an approximation and improvement procedures were proposed, leading to a low complexity PWA controller. Two features of the current method comparing to some existing approximate approaches is that, the final structure of the approximate controller in this method is already known thanks to the coarse polyhedral partition. Moreover, the constraints satisfaction and stability of the proposed approximate controller were guaranteed a priori and does not require post-processing of cost functions. Extensive simulation results illustrate the potential abilities and performance of the proposed method. It is worthwhile remarking that the proposed improvement procedure can be used, mutatis mutandis, to recover approximate solution of other existing approximate methods. Finally, we would remark that the proposed δ-eMPC framework can be extended naturally to handle situations where reference tracking, disturbance rejection, soft constraints and variable constraints are required.

TABLE I

SIMULATION RESULTS OF PROPOSED APPROACH COMPARED TO THE FINE SOLUTION.

Example	α	$\mathbf{N_v}$	**Regions**		$\frac{\sqrt{\sum_{i=1}^{N_v} \left\| \frac{u_f^*(x_i)-u(x_i)}{\|u\|_\infty} \right\|_2^2}}{N_v}$		$Mean(J)$	
			Fine	**Coarse**	**Coarse** $(\mathbf{u}=\mathbf{u_c})$	**Improved** $(\mathbf{u}=\hat{\mathbf{u}}_\mathbf{c})$	**Coarse** $(\mathbf{J}=\mathbf{J_c})$	**Improved** $(\mathbf{J}=\hat{\mathbf{J}}_\mathbf{c})$
$\#\mathbf{1}$	2	3759	2739	729	0.0011	0.0006	0.0206	0.0018
	3	2039		387	0.0030	0.0009	0.0264	0.0022
	4	1165		213	0.0059	0.0021	0.0428	0.0031
	5	801		141	0.0073	0.0031	0.0338	0.0037
	6	509		83	0.0117	0.0044	0.0604	0.0078
	8	317		45	0.0201	0.0080	0.1283	0.0201
	11	167		15	0.0336	0.0275	0.4334	0.2841
$\#\mathbf{2}$	2	1399	1349	279	0.0793	0.0312	0.1146	0.0163
	3	1239		247	0.0914	0.0391	0.1324	0.0338
	4	1109		221	0.1042	0.0577	0.1669	0.0589
	5	959		191	0.1233	0.0799	0.1865	0.0787
	6	779		155	0.1561	0.1188	0.2554	0.1291
$\#\mathbf{3}$	2	13121	10993	8830	0.0515	0.0397	0.0524	0.0479
	3	4874		4182	0.1184	0.0991	0.0678	0.0631
	4	1904		1346	0.1765	0.1746	0.1997	0.1972

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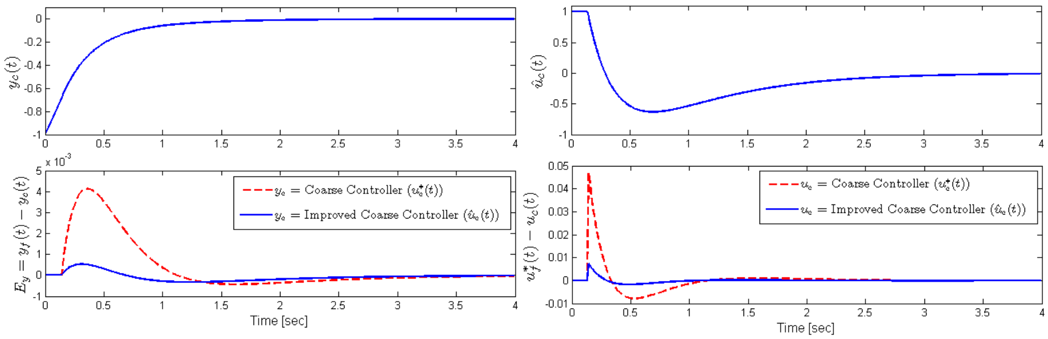

Fig. 1. Upper right/left: Improved coarse control/output signal. Lower right/left: Control/output difference signals of coarse (dashed) and improved-coarse controllers (solid).

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