

Distributed Nonlinear Model Predictive Control by Sequential Linearization and Accelerated Gradient Method

Alexandra Grancharova*, Tor A. Johansen**, Valeria Petrova*

* Department of Industrial Automation, University of Chemical Technology and Metallurgy, Bul. Kl. Ohridski 8, Sofia 1756, Bulgaria (e-mails: alexandra.grancharova@abv.bg, valeria.petrova.petrova@abv.bg)

** Center for Autonomous Marine Operations and Systems, Department of Engineering Cybernetics, Norwegian University of Science and Technology, 7491 Trondheim, Norway, (e-mail: Tor.Arne.Johansen@itk.ntnu.no)

Abstract: A suboptimal approach to distributed NMPC for nonlinear interconnected systems subject to constraints is proposed. The objective is to develop a computationally efficient approach. The suggested method is based on a sequential linearization of the nonlinear system dynamics and finding a suboptimal solution of the resulting Quadratic Programming problem by using distributed iterations of the dual accelerated gradient method. The benefits of the approach are reduced complexity of the on-line computations, and simple software implementation, which makes it appropriate for embedded distributed convex NMPC. The proposed method is illustrated with simulations on the model of a quadruple-tank system.

Keywords: Predictive control, Distributed control, Interconnected systems, Nonlinear systems, Constraints.

1. INTRODUCTION

Model predictive control (MPC) involves the solution at each sampling instant of a finite horizon optimal control problem subject to the system dynamics, and state and input constraints. The centralized solution of MPC problems for large-scale systems may be impractical due to the topology of the plant and data communication, and the large number of decision variables. Recently, several methods for distributed/decentralized MPC have been developed (Christofides et al. (2013), Maestre and Negenborn (2014)).

In Venkat et al. (2008), Alessio et al. (2011), Giselsson and Rantzer (2010), Giselsson et al. (2013), Giselsson and Rantzer (2014), approaches for distributed/decentralized MPC for systems consisting of *linear* interconnected subsystems have been developed. The approach in Giselsson and Rantzer (2010) is based on the dual decomposition methods (Cohen and Miara (1990)), where large-scale optimization problems are handled by using Lagrange multipliers to relax the couplings between the sub-problems. In Giselsson et al. (2013), a distributed optimization algorithm based on accelerated gradient methods using dual decomposition is proposed and its performance is evaluated in distributed MPC.

Also, approaches to distributed MPC for systems composed of several *nonlinear* subsystems have been proposed. Some of them assume the subsystems are coupled through their dynamics (e.g. Raimondo et al. (2007), Dunbar (2007), Grancharova and Johansen (2014), Heidarinejad et al. (2011)), while others suppose that the dynamics of the nonlinear subsystems are not interconnected, but the cost function and/or constraints couple the dynamical behavior of the subsystems (e.g. Keviczky et al. (2006)). In Grancharova and Johansen (2014), a suboptimal approach to *distributed* NMPC for a class of systems consisting of *nonlinear* constrained subsystems with separable coupled dynamics has been

proposed. It applies the dynamic dual decomposition method (Cohen and Miara (1990), Giselsson and Rantzer (2010)) and reformulates the *centralized* NMPC problem into a *distributed* quasi-NMPC problem by linearization of the nonlinear system dynamics. The approach is based entirely on *distributed* on-line optimization, involving the solution of Nonlinear Programming (NLP) and Quadratic Programming (QP) sub-problems associated with the interacting subsystems.

In this paper, a new suboptimal approach to *distributed* NMPC is proposed. It differs from the one in Grancharova and Johansen (2014) in two aspects. First, it does not involve an exact on-line solution of NLP and QP sub-problems. Instead, a suboptimal solution of the QP problem resulting from the linearization of the system dynamics is obtained by distributed iterations of the dual accelerated gradient method applied for linear MPC (Giselsson et al. (2013)). This leads to reduced complexity of the on-line computations and simple software implementation. For this reason, the suggested approach is appropriate for embedded distributed NMPC. Second, NMPC problems for a more general class of interconnected nonlinear systems can be solved distributedly, since the approach does not assume separability of the couplings between the subsystems. The objective of the paper is related to computational advantages of the suggested approach without considering possible limitations related to lack of guarantees of feasibility, stability and optimality of the closed-loop system.

2. MODEL PREDICTIVE CONTROL PROBLEM FOR NONLINEAR INTERCONNECTED SYSTEMS

Consider a system composed by the interconnection of M subsystems with overall state and overall control input:

$$x(t) = [x_1(t), x_2(t), \dots, x_M(t)] \in \mathbb{R}^n, \quad n = \sum_{i=1}^M n_i \quad (1)$$

$$u(t) = [u_1(t), u_2(t), \dots, u_M(t)] \in \mathbb{R}^m, \quad m = \sum_{i=1}^M m_i \quad (2)$$

where $x_i(t) \in \mathbb{R}^{n_i}$ and $u_i(t) \in \mathbb{R}^{m_i}$ are the state and the control input, related to the i -th subsystem. Let the dynamics of the subsystems be described by the nonlinear discrete-time models:

$$x_i(t+1) = f_i(x(t), u(t)), \quad i = 1, 2, \dots, M \quad (3)$$

where $f_i: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{n_i}$ is a nonlinear function.

The constraints imposed on the subsystems are:

$$x_i(t) \in \mathcal{X}_i, \quad u_i(t) \in \mathcal{U}_i, \quad i = 1, 2, \dots, M \quad (4)$$

where \mathcal{X}_i and \mathcal{U}_i are the admissible sets, and the following assumptions are made:

A1. The functions f_i , $i = 1, \dots, M$ are continuously differentiable with $f_i(0, 0) = 0$.

A2. The admissible sets \mathcal{X}_i and \mathcal{U}_i are bounded polyhedral sets, i.e. they are defined by:

$$\mathcal{X}_i = \{x_i \in \mathbb{R}^{n_i} \mid C_i^x x_i \leq d_i^x\} \quad (5)$$

$$\mathcal{U}_i = \{u_i \in \mathbb{R}^{m_i} \mid C_i^u u_i \leq d_i^u\} \quad (6)$$

and they include the origin in their interior. Here, $C_i^x \in \mathbb{R}^{n_c, x_i \times n_i}$, $C_i^u \in \mathbb{R}^{n_c, u_i \times m_i}$, $d_i^x \in \mathbb{R}^{n_c, x_i}$, $d_i^u \in \mathbb{R}^{n_c, u_i}$ and n_{c, x_i} and n_{c, u_i} are the number of constraints imposed on x_i and u_i , respectively.

It can be seen from (4)-(6) that the constraints imposed on the subsystems are not coupled. In order to consider coupled constraints, the approach described in the next section should be slightly modified.

It is supposed that a full measurement $\bar{x} = [\bar{x}_1, \bar{x}_2, \dots, \bar{x}_M]$ of the overall state is available at the current time t . The optimal regulation problem is considered where the goal is to steer the overall state of the system (3) to the origin. For the current overall state \bar{x} , the regulation NMPC solves the optimization problem:

Problem P1 (Centralized NMPC):

$$V^{\text{opt}}(\bar{x}) = \min_U J(U, \bar{x}) \quad (7)$$

subject to $x_{i|t} = \bar{x}$ and:

$$x_{i,t+k|t} \in \mathcal{X}_i, \quad i = 1, \dots, M, \quad k = 1, \dots, N \quad (8)$$

$$u_{i,t+k} \in \mathcal{U}_i, \quad i = 1, \dots, M, \quad k = 0, 1, \dots, N-1 \quad (9)$$

$$x_{i,t+k+1|t} = f_i(x_{i,t+k|t}, u_{i,t+k}) \quad (10)$$

$$i = 1, \dots, M, \quad k = 0, 1, \dots, N-1$$

$$x_{t+k|t} = [x_{1,t+k|t}, x_{2,t+k|t}, \dots, x_{M,t+k|t}], \quad k = 0, 1, \dots, N \quad (11)$$

$$u_{t+k} = [u_{1,t+k}, u_{2,t+k}, \dots, u_{M,t+k}], \quad k = 0, 1, \dots, N-1 \quad (12)$$

with $U = [u_t, u_{t+1}, \dots, u_{t+N-1}]$ and the cost function given by:

$$J(U, \bar{x}) = \sum_{k=0}^N \sum_{i=1}^M l_i(x_{i,t+k|t}, u_{i,t+k}) \quad (13)$$

Here, $l_i(x_{i,t+k|t}, u_{i,t+k}) = \|x_{i,t+k|t}\|_{Q_i}^2 + \|u_{i,t+k}\|_{R_i}^2$ is the stage cost for the i -th subsystem with symmetric weighting matrices $Q_i, R_i > 0$, and N is a finite horizon.

3. DISTRIBUTED NMPC BY SEQUENTIAL LINEARIZATION AND ACCELERATED GRADIENT METHOD

3.1 Approximation of the NMPC problem by a linear MPC problem

Here, we locally approximate the dynamics of the subsystems (3) by linear models. Let at time t , $U_i^0 = [u_{i,t}^0, u_{i,t+1}^0, \dots, u_{i,t+N-1}^0]$ and $X_i^0 = [x_{i,t|t}^0, x_{i,t+1|t}^0, \dots, x_{i,t+N-1|t}^0]$ be given trajectories of the control input and the state of the i -th subsystem for the prediction horizon N . Taylor series expansion of the right-hand side of the model (3) about the point (U_i^0, X_i^0) leads to the locally linear model:

$$x_{i,t+k+1} = \sum_{j=1}^M (A_{ij,t+k} x_{j,t+k} + B_{ij,t+k} u_{j,t+k}) + g_{i,t+k} \quad (14)$$

$$k = 0, 1, \dots, N-1, \quad i = 1, \dots, M$$

where the matrices $A_{ij,t+k}$, $B_{ij,t+k}$ and the vector $g_{i,t+k}$ are computed as:

$$A_{ij,t+k} = \nabla_{x_j} f_i(x_{t+k|t}^0, u_{t+k}^0), \quad B_{ij,t+k} = \nabla_{u_j} f_i(x_{t+k|t}^0, u_{t+k}^0)$$

$$g_{i,t+k} = -\sum_{j=1}^M (A_{ij,t+k} x_{j,t+k}^0 + B_{ij,t+k} u_{j,t+k}^0) + f_i(x_{t+k|t}^0, u_{t+k}^0) \quad (15)$$

$$k = 0, 1, \dots, N-1, \quad i, j = 1, \dots, M$$

In (15), $u_{t+k}^0 = [u_{1,t+k}^0, u_{2,t+k}^0, \dots, u_{M,t+k}^0]$ and $x_{t+k|t}^0 = [x_{1,t+k|t}^0, x_{2,t+k|t}^0, \dots, x_{M,t+k|t}^0]$. It can be observed that (14)-(15) is a linear time-varying approximation of the model (3).

As in Giselsson and Rantzer (2014), the following tightened constraint sets are introduced:

$$(1-\delta)\mathcal{X}_i = \{x_i \in \mathbb{R}^{n_i} \mid C_i^x x_i \leq (1-\delta)d_i^x\} \quad (16)$$

$$(1-\delta)\mathcal{U}_i = \{u_i \in \mathbb{R}^{m_i} \mid C_i^u u_i \leq (1-\delta)d_i^u\} \quad (17)$$

where $\delta \in (0, 1)$ is the amount of relative constraint tightening. Then, for the locally linear dynamics (14)-(15) with initial state $\bar{x} = [x_{1,t|t}^0, x_{2,t|t}^0, \dots, x_{M,t|t}^0]$, the linear MPC problem is formulated:

Problem P2 (Centralized linear MPC):

$$V^*(\bar{x}) = \min_U J(U, \bar{x}) \quad (18)$$

subject to $x_{i|t} = \bar{x}$, constraints (14)-(15) and:

$$x_{i,t+k|t} \in (1-\delta)\mathcal{X}_i, \quad i = 1, \dots, M, \quad k = 1, \dots, N \quad (19)$$

$$u_{i,t+k} \in (1-\delta)\mathcal{U}_i, \quad i = 1, \dots, M, \quad k = 0, 1, \dots, N-1 \quad (20)$$

where the cost function $J(U, \bar{x})$ is defined by (13).

3.2 Representation and solution of the linear MPC problem as a distributed Quadratic Programming problem

As it is shown in Giselsson et al. (2013), by stacking all decision variables (the control input trajectory and the state trajectory along the horizon) into one vector $Y \in \mathbb{R}^{n_Y}$ with dimension $n_Y = \sum_{i=1}^M N(n_i + m_i)$:

$$\begin{aligned}
Y &= [x_{1,t+1|t}, u_{1,t}, x_{1,t+2|t}, u_{1,t+1}, \dots, x_{1,t+N|t}, u_{1,t+N-1}, \\
&\vdots \\
&x_{M,t+1|t}, u_{M,t}, x_{M,t+2|t}, u_{M,t+1}, \dots, x_{M,t+N|t}, u_{M,t+N-1}]
\end{aligned} \quad (21)$$

the optimization problem P2 can be written as a Quadratic Programming (QP) problem:

Problem P3 (QP problem):

$$V^*(\bar{x}) = \min_Y \frac{1}{2} Y^T \bar{H} Y \quad (22)$$

subject to:

$$\bar{A}Y = \bar{B}\bar{x} - \bar{G} \quad (23)$$

$$\bar{C}Y \leq (1-\delta)\bar{d} \quad (24)$$

Here, $\bar{H} \in \mathbb{R}^{n_Y \times n_Y}$, $\bar{A} \in \mathbb{R}^{n_{\bar{A}} \times n_Y}$, $\bar{B} \in \mathbb{R}^{n_{\bar{A}} \times n_{\bar{x}}}$, $\bar{G} \in \mathbb{R}^{n_{\bar{A}}}$, $\bar{C} \in \mathbb{R}^{n_{\bar{C}} \times n_Y}$, $\bar{d} \in \mathbb{R}^{n_{\bar{C}}}$ ($n_{\bar{A}} = \sum_{i=1}^M (N-1)n_i$, $n_{\bar{x}} = \sum_{i=1}^M n_i$,

$n_{\bar{C}} = \sum_{i=1}^M N(n_{c,x_i} + n_{c,u_i})$). In (22)-(24) we have:

$$\bar{H} = \text{diag}\{\bar{H}_1, \bar{H}_2, \dots, \bar{H}_M\} \quad (25)$$

$$\bar{A} = [\bar{A}_1 | \bar{A}_2 | \dots | \bar{A}_M]^T, \bar{B} = [\bar{B}_1 | \bar{B}_2 | \dots | \bar{B}_M]^T \quad (26)$$

$$\bar{G} = [\bar{G}_1 | \bar{G}_2 | \dots | \bar{G}_M]^T$$

$$\bar{C} = \text{diag}\{\bar{C}_1 | \bar{C}_2 | \dots | \bar{C}_M\}, \bar{d} = \text{diag}\{\bar{d}_1 | \bar{d}_2 | \dots | \bar{d}_M\} \quad (27)$$

For the i -th subsystem the matrix $\bar{H}_i \in \mathbb{R}^{n_{Y_i} \times n_{Y_i}}$ ($n_{Y_i} = N(n_i + m_i)$) is defined as:

$$\bar{H}_i = \text{diag}\{\underbrace{W_i, W_i, \dots, W_i}_{N \text{ elements}}\}, W_i = \begin{bmatrix} Q_i & 0 \\ 0 & R_i \end{bmatrix} \quad (28)$$

and the matrices $\bar{A}_i \in \mathbb{R}^{n_{\bar{A}_i} \times n_{Y_i}}$, $\bar{B}_i \in \mathbb{R}^{n_{\bar{A}_i} \times n_{\bar{x}_i}}$, $\bar{G}_i \in \mathbb{R}^{n_{\bar{A}_i}}$, $\bar{C}_i \in \mathbb{R}^{n_{\bar{C}_i} \times n_{Y_i}}$ and the vector $\bar{d}_i \in \mathbb{R}^{n_{\bar{C}_i}}$ ($n_{\bar{A}_i} = (N-1)n_i$, $n_{\bar{C}_i} = N(n_{c,x_i} + n_{c,u_i})$), $i = 1, \dots, M$ are:

$$\bar{A}_i = \begin{bmatrix} \bar{A}_{i,1,1} & \bar{A}_{i,1,2} & \dots & \bar{A}_{i,1,M} \\ \bar{A}_{i,2,1} & \bar{A}_{i,2,2} & \dots & \bar{A}_{i,2,M} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{A}_{i,N-1,1} & \bar{A}_{i,N-1,2} & \dots & \bar{A}_{i,N-1,M} \end{bmatrix}, \bar{B}_i = \begin{bmatrix} -A_{i1,t} - A_{i2,t} \dots - A_{iM,t} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad (29)$$

$$\bar{G}_i = [g_{i,t}, g_{i,t+1}, \dots, g_{i,t+N-1}]^T$$

$$\bar{C}_i = \text{diag}\{\underbrace{C_i^{x,u}, C_i^{x,u}, \dots, C_i^{x,u}}_{N \text{ elements}}\}, C_i^{x,u} = \begin{bmatrix} C_i^x & 0 \\ 0 & C_i^u \end{bmatrix} \quad (30)$$

$$\bar{d}_i = \text{diag}\{\underbrace{d_i^{x,u}, d_i^{x,u}, \dots, d_i^{x,u}}_{N \text{ elements}}\}, d_i^{x,u} = [d_i^x, d_i^u]^T \quad (31)$$

In (29), $\bar{A}_{i,k,j}$, $k = 1, 2, \dots, N-1$, $j = 1, 2, \dots, M$ depend on the matrices $A_{ij,t+k}$ and $B_{ij,t+k}$ of the linear model (14)-(15) and $A_{i1,t}, A_{i2,t}, \dots, A_{iM,t}$ are the matrices $A_{ij,t+k}$ for $k = 0$, $j = 1, 2, \dots, M$.

The linear MPC problem P2 can be solved distributedly by applying the dual accelerated gradient algorithm in Giselsson et al. (2013). The distribution is enabled by solving the dual problem to problem P3, which is created by introducing dual variables $\lambda \in \mathbb{R}^{n_{\bar{A}}}$ for the equality constraints (23) and dual

variables $\mu \in \mathbb{R}^{n_{\bar{C}}}$ for the inequality constraints (24). It is shown in Giselsson et al. (2013) that the dual problem can be written as:

$$\max_{\lambda, \mu \geq 0} D(\bar{x}, \lambda, \mu) \quad (32)$$

where $D(\bar{x}, \lambda, \mu)$ is the dual cost function:

$$\begin{aligned}
D(\bar{x}, \lambda, \mu) &= -\frac{1}{2} (\bar{A}^T \lambda + \bar{C}^T \mu)^T \bar{H}^{-1} (\bar{A}^T \lambda + \bar{C}^T \mu) \\
&\quad - \lambda^T (\bar{B}\bar{x} - \bar{G}) - \mu^T \bar{d} (1 - \delta)
\end{aligned} \quad (33)$$

In order to perform distributedly the iterations of the dual accelerated gradient method, the following vector $Y_i \in \mathbb{R}^{n_{Y_i}}$ ($n_{Y_i} = N(n_i + m_i)$) of decision variables, associated to the i -th subsystem, is introduced:

$$Y_i = [x_{i,t+1|t}, u_{i,t}, x_{i,t+2|t}, u_{i,t+1}, \dots, x_{i,t+N|t}, u_{i,t+N-1}] \quad (34)$$

Also, let $\lambda_i \in \mathbb{R}^{n_{\bar{A}_i}}$ and $\mu_i \in \mathbb{R}^{n_{\bar{C}_i}}$ be the dual variables for the equality and the inequality constraints, related to the i -th subsystem. Then, the distributed iterations of the dual accelerated gradient method are:

$$Y_i^r = -\bar{H}_i^{-1} \left(\sum_{j=1}^M \bar{A}_j^T \lambda_j^r + \bar{C}_i^T \mu_i^r \right) \quad (35)$$

$$\bar{Y}_i^r = Y_i^r + \frac{r-1}{r+2} (Y_i^r - Y_i^{r-1}) \quad (36)$$

$$\lambda_i^{r+1} = \lambda_i^r + \frac{r-1}{r+2} (\lambda_i^r - \lambda_i^{r-1}) + \frac{1}{L} (\bar{A}_i \bar{Y}_i^r - (\bar{B}_i \bar{x}_i - \bar{G}_i)) \quad (37)$$

$$\begin{aligned}
\mu_i^{r+1} &= \\
&\max(0, \mu_i^r + \frac{r-1}{r+2} (\mu_i^r - \mu_i^{r-1}) + \frac{1}{L} (\bar{C}_i \bar{Y}_i^r - \bar{d}_i (1 - \delta)))
\end{aligned} \quad (38)$$

$i = 1, 2, \dots, M$

where \bar{A}_j^i are the columns of the matrix \bar{A}_j corresponding to the decision vector Y_j . Here, $L = \|\llbracket \bar{A}^T, \bar{C}^T \rrbracket^T \bar{H}^{-1} \llbracket \bar{A}^T, \bar{C}^T \rrbracket\|$ is the Lipschitz constant to the gradient of the dual function (33) and r is the iteration number. Note that because of the couplings in the dynamics models of the subsystems, in (35) the computation of the decision variables Y_i^r for the i -th subsystem requires to have information about the dual variables λ^r for the whole system. For the same reason, in (37) the update of the dual variables λ_i associated to the i -th subsystem uses the information about the decision variables \bar{Y}^r for the entire system. Since there are no couplings in the control input and state constraints of the subsystems (cf. (4)-(6)), in (38) the update of the dual variables μ_i for the i -th subsystem requires information only about the decision variables \bar{Y}_i^r for this subsystem.

3.3 Algorithm for distributed NMPC by sequential linearization and distributed Quadratic Programming

In Grancharova and Johansen (2014), an approach to distributed NMPC for interconnected nonlinear systems with separable coupled dynamics has been proposed, which is based on an exact on-line solution of the resulting NLP and QP sub-problems. Here, a suboptimal algorithm is proposed that differs from the one in Grancharova and Johansen (2014)

in two aspects: 1) it does not involve finding the exact solution of the optimization problem, instead a suboptimal solution is obtained by linearizing the system dynamics and performing distributed iterations of the dual accelerated gradient method for linear MPC (Giselsson et al. (2013)), 2) a more general class of interconnected nonlinear systems is considered, since it does not assume separability of the couplings.

The suggested algorithm for distributed NMPC includes two loops. In the outer loop, the dynamics of the nonlinear system (3) is locally approximated with a linear model (14)-(15) about the current guess for the control input trajectory and the corresponding state trajectory of the system (3). Then, in the inner loop, a suboptimal solution to the resulting QP problem P3 is found by applying the distributed iterations (35)-(38) of the dual accelerated gradient method.

Before describing the algorithm, some notation is needed. Let $U(t)=[u_t, u_{t+1}, \dots, u_{t+N-1}]$ be the current update of the control input trajectory and denote with $X(t)=[x_{t+1|t}, x_{t+2|t}, \dots, x_{t+N|t}]$ the corresponding trajectory of the nonlinear system (3) obtained for initial state $x_{t|t} = \bar{x}$, i.e.:

$$\begin{aligned} x_{i,t+k+1|t} &= f_i(x_{t+k|t}, u_{t+k}) \\ i &= 1, \dots, M, k = 0, 1, \dots, N-1 \end{aligned} \quad (39)$$

Then, the current update $Y(t)$ of the decision variables can be easily constructed according to (21). Respectively, if updates Y^r are obtained by performing the iterations (35)-(38), the corresponding update of the U^r of the control input trajectory can be extracted from it. Further, assume that a relative tolerance $\varepsilon > 0$ of achieving the extremum of the cost function is specified, i.e. the iterations in the outer loop will terminate if the following condition is satisfied:

$$\frac{|J(U_2, \bar{x}) - J(U_1, \bar{x})|}{J(U_1, \bar{x})} \leq \varepsilon \quad (40)$$

Here, U_1, U_2 and $J(U_1, \bar{x}), J(U_2, \bar{x})$ are the control input trajectories and cost function values obtained in two sequential iterations in the outer loop of the algorithm.

Suppose that the relative constraint tightening δ , the relative tolerance ε and the number R of iterations (35)-(38) are specified. Then, the algorithm for distributed NMPC is described as follows.

Algorithm 1 (Distributed NMPC by sequential linearization and accelerated gradient iterations):

1. Given δ , ε and R . Let $t = 0$, $U(t) = [0, 0, \dots, 0]$.
2. Let the state at time t be $x(t) = \bar{x} = [\bar{x}_1, \dots, \bar{x}_M]$.
3. Compute the state trajectory $X(t)$ of the nonlinear system (3) corresponding to initial state \bar{x} and control input trajectory $U(t)$ and the associated cost function value $J_2 := J(U, \bar{x})$ by using (13). Form the vector $Y(t)$ of decision variables.
4. **Do**
5. $J_1 := J_2$

6. Obtain a linear model (14)-(15) of the system (3) around the trajectories $(U(t), X(t))$.
7. **For** $r = 0, 1, \dots, R$ **do**
8. **If** $r = 0$ **then**
9. Initialize iterations (35)-(38) with $Y^{-1} = Y(t)$, $\lambda^0 = \lambda^{-1} = 0$, $\mu^0 = \mu^{-1} = 0$.
10. **else**
11. Let $Y^{r-1} := Y^r$, $\lambda^{r-1} := \lambda^r$, $\lambda^r := \lambda^{r+1}$, $\mu^{r-1} := \mu^r$, $\mu^r := \mu^{r+1}$.
12. **end**
13. Run iterations (35)-(38) *distributedly* and obtain Y^r , λ^{r+1} , μ^{r+1} . Extract U^r from Y^r .
14. **end**
15. Let $U(t) = U^R$.
16. Compute the state trajectory $X(t)$ of the nonlinear system (3) corresponding to initial state \bar{x} and control input trajectory $U(t)$ and the associated cost function value $J_2 := J(U, \bar{x})$ by using (13). Form the vector $Y(t)$ of decision variables.
17. **while** $|J_2 - J_1|/J_1 > \varepsilon$
18. Apply to the overall system the control input $u(t) = [I \ 0 \ 0 \ \dots \ 0]U(t)$.
19. Let $t = t+1$ and go to step 2.

The steps 4 to 17 in Algorithm 1 include an iterative linearization of the nonlinear system dynamics about the current updates of the control input and state trajectories, distributed solution of the QP problem P3, which results from the linear MPC problem P2, and update of the control input and state trajectories of the nonlinear system¹.

In general, no guarantees for feasibility, stability and desired performance of the suboptimal NMPC computed with Algorithm 1 in closed-loop with the nonlinear system (3) can be given. However, appropriate values for the prediction horizon and the parameters δ , ε and R may be determined by an offline study of the performance of the algorithm with different values of these parameters for various initial conditions of the system. The purpose is to obtain the values of the parameters, for which the closed-loop system will have the desired properties with minimum number of iterations in Algorithm 1.

4. EXAMPLE

4.1 System description

As an example, the quadruple-tank system in Johansson (2000) is considered, which is schematically shown in Fig. 1. The objective is to control the level in the lower two tanks with two pumps. The control inputs are v_1 and v_2 (input voltages to the pumps) and the outputs are y_1 and y_2

¹ Alternatively, an approach similar to (Oliveira and Biegler (1995)) can be applied, where the idea would be to obtain the exact solution of the approximating QP problem P3 and to use it when formulating the approximating QP problem in the next time instant.

(voltages from level measurement devices). The first-principles model, describing the dynamics of the system, is (Johansson (2000)):

$$\dot{h}_1 = -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1}v_1 \quad (41)$$

$$\dot{h}_2 = -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2}v_2 \quad (42)$$

$$\dot{h}_3 = -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2)k_2}{A_3}v_2 \quad (43)$$

$$\dot{h}_4 = -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1)k_1}{A_4}v_1 \quad (44)$$

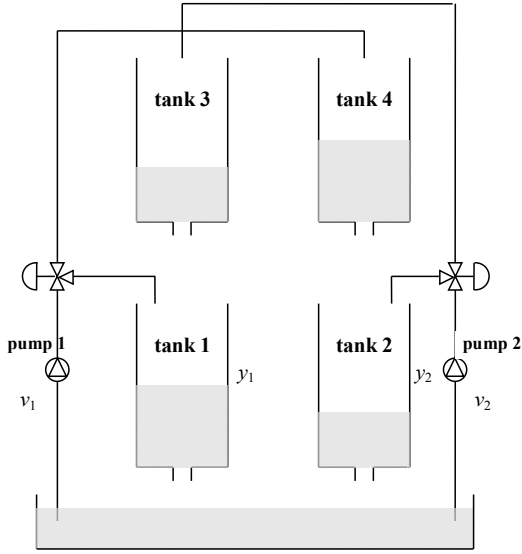


Fig. 1. Quadruple-tank system (Johansson (2000)).

In (41)-(44), A_i is the cross-sectional area of tank i , a_i is the cross-sectional area of the outlet hole of tank i , h_i is the water level in tank i (Johansson (2000)). The voltage applied to pump i is v_i and the corresponding flow is $k_i v_i$. The parameters $\gamma_1, \gamma_2 \in (0, 1)$ are determined from the positions of the two valves. In the simulation experiments, it is chosen that $\gamma_1 = 0.7$, $\gamma_2 = 0.6$, which lead to a minimum-phase behavior of the plant (Johansson (2000)). The flow to tank 1 is $\gamma_1 k_1 v_1$ and the flow to tank 4 is $(1-\gamma_1)k_1 v_1$. The flows to tanks 2 and 3 are $\gamma_2 k_2 v_2$ and $(1-\gamma_2)k_2 v_2$, respectively. The acceleration of gravity is denoted g . The measured level signals are $y_1 = k_c h_1$ and $y_2 = k_c h_2$, where k_c is a constant. The parameter values of the quadruple-tank system are given in (Johansson (2000)).

The control objective is to keep the water levels h_1 and h_2 at the set-points:

$$h_1^* = 12.4 \text{ cm}, h_2^* = 12.7 \text{ cm} \quad (45)$$

The steady-state values of h_3, h_4, v_1, v_2 , corresponding to these set-points are:

$$h_3^* = 1.6 \text{ cm}, h_4^* = 1.45 \text{ cm}, v_1^* = 3.04 \text{ V}, v_2^* = 2.97 \text{ V} \quad (46)$$

The following variables are introduced:

$$x_{1,1} = h_1 - h_1^*, x_{1,2} = h_4 - h_4^*, x_{2,1} = h_2 - h_2^*, x_{2,2} = h_3 - h_3^* \quad (47)$$

$$u_i = v_i - v_i^*, i = 1, 2 \quad (48)$$

Then, the quadruple-tank system can be considered as consisting of two interconnected sub-systems, which are described by:

Subsystem S₁:

$$\begin{aligned} \dot{x}_{1,1} = & -\frac{a_1}{A_1}\sqrt{2g(x_{1,1} + h_1^*)} + \frac{a_3}{A_1}\sqrt{2g(x_{2,2} + h_3^*)} \\ & + \frac{\gamma_1 k_1}{A_1}(u_1 + v_1^*) \end{aligned} \quad (49)$$

$$\dot{x}_{1,2} = -\frac{a_4}{A_4}\sqrt{2g(x_{1,2} + h_4^*)} + \frac{(1-\gamma_1)k_1}{A_4}(u_1 + v_1^*) \quad (50)$$

Subsystem S₂:

$$\begin{aligned} \dot{x}_{2,1} = & -\frac{a_2}{A_2}\sqrt{2g(x_{2,1} + h_2^*)} + \frac{a_4}{A_2}\sqrt{2g(x_{1,2} + h_4^*)} \\ & + \frac{\gamma_2 k_2}{A_2}(u_2 + v_2^*) \end{aligned} \quad (51)$$

$$\dot{x}_{2,2} = -\frac{a_3}{A_3}\sqrt{2g(x_{2,2} + h_3^*)} + \frac{(1-\gamma_2)k_2}{A_3}(u_2 + v_2^*) \quad (52)$$

The subsystem S₁ influences the dynamics of the subsystem S₂ with the expression $\frac{a_4}{A_2}\sqrt{2g(x_{1,2} + h_4^*)}$, while the subsystem S₂ influences the dynamics of the subsystem S₁ with the expression $\frac{a_3}{A_1}\sqrt{2g(x_{2,2} + h_3^*)}$.

4.2 Simulation results

The performance of the proposed distributed NMPC approach and algorithm is studied by simulations for the quadruple-tank system described above. The ordinary differential equations (49)-(52) are discretized with sampling time of 1 s by applying the Euler's method with step 0.1 s. The constraints imposed on the system (41)-(44) are:

$$0 \leq v_i(t) \leq 6 \text{ V}, i = 1, 2 \quad (53)$$

$$0 \leq h_i(t) \leq 20 \text{ cm}, i = 1, 2, 0 \leq h_i(t) \leq 3 \text{ cm}, i = 3, 4 \quad (54)$$

which by taking into account (45)-(48) become:

$$-3.04 \leq u_1(t) \leq 2.96 \text{ V}, -2.97 \leq u_2(t) \leq 3.03 \text{ V} \quad (55)$$

$$-12.4 \leq x_{1,1}(t) \leq 7.6 \text{ cm}, -1.45 \leq x_{1,2}(t) \leq 1.55 \text{ cm} \quad (56)$$

$$-12.7 \leq x_{2,1}(t) \leq 7.3 \text{ cm}, -1.60 \leq x_{2,2}(t) \leq 1.40 \text{ cm} \quad (57)$$

The prediction horizon in the centralized NMPC problem is $N = 10$ and the weighting matrices in the cost function (13) are $Q_1 = Q_2 = \text{diag}(10, 1)$, $R_1 = R_2 = 0.1$. The Algorithm 1 is used to generate the two control inputs for initial states of the subsystems S₁ and S₂:

$$[x_{1,1}(0) \ x_{1,2}(0) \ x_{2,1}(0) \ x_{2,2}(0)] = [-2.4 \ -0.45 \ -2.7 \ -0.60] \quad (58)$$

The trajectories of the control inputs and the states associated to the two subsystems are depicted in Fig. 2 to Fig. 4.

The trajectories obtained with the suboptimal distributed NMPC approach are compared to those corresponding to the centralized NMPC approach, which solves problem P1 at each time instant. The solution of problem P1 is obtained by using the sequential QP method, where the QP subproblem is solved

by applying an active set strategy. The values of the parameters in Algorithm 1 are $\delta = 0.1$, $\varepsilon = 0.05$ and $R = 80$. It can be seen that the distributed suboptimal NMPC approach leads to feasible trajectories and the level of suboptimality is acceptable.

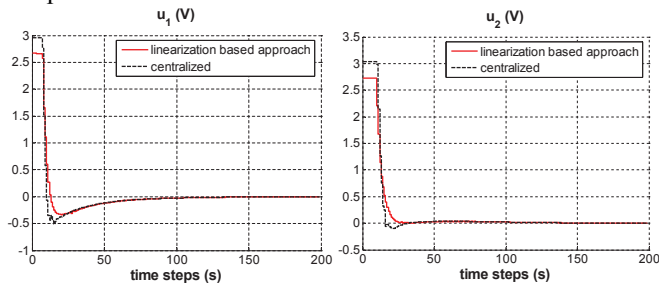


Fig. 2. The control inputs for the two subsystems.

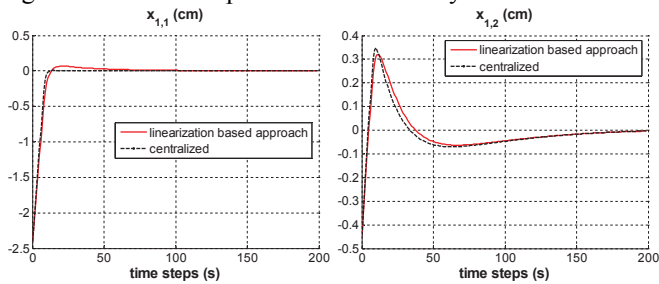


Fig. 3. The states of subsystem S_1 .

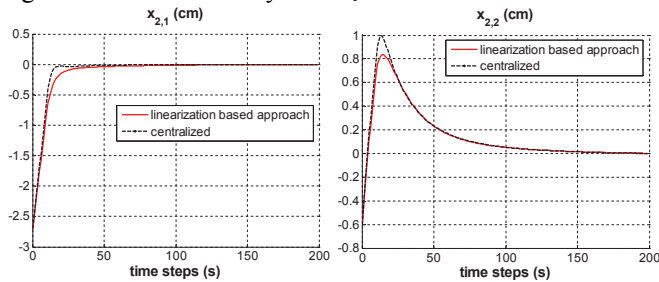


Fig. 4. The states of subsystem S_2 .

The computational cost of both approaches is compared and the results are presented in Table 1. The computations are performed on a 1.73 GHz Intel Pentium processor. It can be seen that the computational costs of both approaches are close, but the advantage of the distributed method is that it allows the computation of the suboptimal control inputs to be done autonomously by the subsystems without the need for centralized optimization.

Table 1. Comparison of approaches.

Method	Average CPU time, s	Maximal CPU time, s
Suboptimal distributed NMPC with linearization of system dynamics	0.16	0.30
Centralized NMPC	0.05	1.14

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