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Particle Production in Gravitational Fields

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Abstract

In this thesis, we discuss the phenomenon of particle production in gravitational fields. We present the general framework in flat Friedmann-Robertson-Walker spacetimes and apply it to an asymptotically static model in two dimensions. In such cases, the particle number can be obtained analytically in the asymptotic future. We discuss ways to give a physical meaning to the particle number in expanding universes and apply it to the same model. We find that in this case, the method of instantaneous vacuum provide meaningful results for the particle number as long as the particle production is not too big. Finally, a model for production of massless scalar particles during inflation is discussed. In this model, the dominant contribution to the energy density and particle number comes from long-wavelength fluctuations produced during inflation, rather than quantum fluctuations produced after inflation.

Sammendrag

I denne oppgaven diskuteres fenomenet partikkelproduksjon i gravitasjonsfelt. Vi presenterer den generelle teorien i flate Friedmann-Robertson-Walker-tidrom og anvender den på en asymptotisk statisk modell i to dimensjoner. I denne modellen kan partikkelnummeret finnes analytisk. Vi diskuterer måter for å gi fysisk mening til partikkeltallet i ekspanderende univers og anvender det på den samme modellen. Vi finner her at instantant-vakuum-metoden gir et meningfult resultat for partikkeltallet så lenge partikkelproduksjonen ikke er for stor. Til sist diskuterer vi en modell for produksjon av masseløse skalarpartikler under inflasjon. I denne modellen gir fluktuasjoner med lang bølgelengde produsert under inflasjon det største bidraget til partikkelproduksjon, heller enn kvantefluktuasjoner produsert etter inflasjon.

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Conventions and Notation

- We use natural units, i.e we set $\hbar = c = 1$. Sometimes, when explicitly stated we will also take $G_N = 1$.
- The Minkowski metric is denoted $\eta_{\mu\nu}$. We use the signature with $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.
- The metric of a general curved spacetime is denoted by $g_{\mu\nu}(x)$. Moreover, the determinant of the metric is denoted by g.
- For general relativity, we use the same conventions as [4].
- For partial derivatives, we often use the short-hand notations $\frac{\partial}{\partial x^\mu} = \partial_\mu$. When acting on a field, we also use the notation $\partial_\mu \phi \equiv \phi_{,\mu}$
- For covariant derivatives, we use the symbol ∇_μ. When acting on a field, we also use the notation $\nabla_\mu \phi \equiv \phi_{;\mu}$.

Introduction

The purpose of this Master thesis is to investigate quantum effects in a time-varying classical background. In particular we are interested in ways to obtain information about the particle number in an expanding universe. In chapter 1, we give a general introduction to cosmology. We derive the field equation of general relativity and discuss homogeneous and isotropic spacetimes. In chapter 2, we briefly discuss dark matter and some of the possible candidates. This is followed by chapter on inflation. We discuss the problems of the standard scenario of big bang cosmology and how inflationary cosmology may solve these. We also talk briefly about the simplest models for chaotic inflation, where inflation is driven by a single scalar field with power-law potentials as $V \sim \phi^2$ and ϕ^4. In chapter 4, we present the general framework for discussing particle production by a free scalar field in expanding universes. In contrast to Minkowski space, the expansion of the universe causes a friction term in the equation of motion, which leads to a time-dependent effective mass of the scalar field. The energy of the scalar field is not conserved and particles are created. In Chapter 5, we investigate first a solvable model in 2-dimensional spacetime. In this model, one is able to obtain analytically the particle number. Then, we address the question of how to do this in models which cannot be solved analytically. Finally, in chapter 6, we discuss a model for particle production during inflation. We solve the Friedmann equations together with the equation for the scalar-inflaton field in a curved spacetime numerically and see how massless particles minimally coupled to gravity can be produced in large amount towards the end of inflation. We have also included an appendix that contains the details that are necessary, but disturbing to the discussion.

Chapter 1

Cosmology

1.1 A brief introduction to general relativity

Einsteins theory of general relativity is a theory of space, time and gravitation. It is therefore the natural framework in which the universe as a whole is described and a necessary tool when studying cosmology. It is constructed around the *equivalence principle* and is written in the language of differential geometry. The equivalence principle states that *all bodies fall in the same way in a gravitational field.* Mathematically speaking, spacetime is a 4-dimensional manifold on which a metric $g_{\mu\nu}(x)$ is defined. The manifold is (pseudo-) Riemannian which means that the line element relating events in spacetime can be written

$$\mathrm{d}s^2 = g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{1.1}$$

while pseudo means that we allow $\mathrm{d}s^2 < 0$. In this sense, general relativity is very different from the remaining laws of nature since it describes gravity purely as a geometric property rather than in terms of a force. Objects follows geodesics in curved spacetime. In this text, we will not delve into the mathematics of general relativity, but a little knowledge of tensor manipulations are presumed. Instead, we will demonstrate how one can derive the field equations from a variational principle.

1.1.1 Einsteins field equations

The field equations of general relativity relates the curvature to the matter content of the spacetime in consideration. They can be derived from an action principle if we consider the following action

$$S_{EH} = \int_\Omega \mathrm{d}^4x\sqrt{-g}\left(\mathcal{R} - 2\Lambda\right), \tag{1.2}$$

known as the Einstein-Hilbert action. Note the we have already added the contribution from a cosmological constant Λ. A variation of S_{EH} with respect to the metric, leads to variations of the curvature scalar $\mathcal{R} = R^{\mu}_{\mu}$ and $\sqrt{-g}$. We find

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{1.3}$$

$$\delta R = \delta(g^{\mu\nu}R_{\mu\nu}) = \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}. \tag{1.4}$$

From the equivalence principle, we can at a single point P in spacetime choose a coordinate system where

$$\Gamma^{\lambda}_{\mu\nu} = 0 \quad , \partial_{\kappa}g^{\mu\nu} = \nabla_{\kappa}g^{\mu\nu} = 0. \tag{1.5}$$

Physically, this means that we always can find a local frame around a point that can be regarded as freely falling. From this we obtain

$$g^{\mu\nu}\delta\mathcal{R}_{\mu\nu} = g^{\mu\nu}\left(\partial_{\kappa}\delta\Gamma^{\kappa}_{\mu\nu} - \partial_{\nu}\delta\Gamma^{\lambda}_{\mu\lambda}\right) \tag{1.6}$$

$$= \frac{1}{\sqrt{-g}}\partial_k\left(\sqrt{-g}\delta w^k\right). \tag{1.7}$$

We note that δw^k is a vector. The relation holds at the point P in a certain coordinate system, but since both sides are scalars, it should also hold in any other coordinate system. Moreover, the point P is arbitrary, so it holds everywhere. By requiring that the variation vanishes at the boundary, we can now drop this term. Assuming $\delta S = 0$ for an arbitrary variation now implies that

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{EH}}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} \equiv G_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{1.8}$$

Adding all matter fields to the Lagrangian, we can write

$$\mathcal{L} = \frac{1}{2\kappa}\mathcal{L}_{HE} + \mathcal{L}_{\text{matter}}. \tag{1.9}$$

Since we expect that the energy-momentum (em-) tensor should be the source of the gravitational field, we should obtain

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{1.10}$$

The factor κ should be chosen such that we obtain Newtonian dynamics in the weak-field approximation. Comparing with the Poisson equation $\nabla^2\Phi = 4\pi G\rho$, one obtains $\kappa = 8\pi G_N$. We may therefore write Einsteins field equations as

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G_N T_{\mu\nu}.} \tag{1.11}$$

A fundamental feature of the equations is that they directly relate the curvature of the universe with its energy content. It is sometimes useful to write the Einsteins field equations in a slightly different way, using the Ricci tensor as the only geometrical term. By contracting one index with the metric tensor $g^{\mu\nu}$ and putting $\mu = \nu$, we obtain (introducing $T^\mu_\mu \equiv \mathcal{T}$) the relation

$$\mathcal{R} = 8\pi G_N \mathcal{T} + 4\Lambda. \tag{1.12}$$

We can therefore instead express (1.11) as

$$\boxed{R_{\mu\nu} = -8\pi G_N \left(T_{\mu\nu} - \frac{1}{2}\mathcal{T} \right) + \Lambda g_{\mu\nu}.} \tag{1.13}$$

From this, one deduce that an empty universe without a cosmological constant is characterized by

$$R_{\mu\nu} = 0. \tag{1.14}$$

1.1.2 The energy-momentum tensor

Energy-momentum tensor of a perfect fluid. A perfect fluid is an idealized fluid where viscous and heat conductive properties are ignored. It is looking isotropic in its rest frame and we may completely characterize it by its energy density and pressure. This implies that the energy-momentum tensor must be diagonal. Furthermore the nonzero spacelike components should be equal due to isotropy of space. We can therefore write it in the following way in the rest frame

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -P \end{pmatrix}, \tag{1.15}$$

where ρ denotes energy density and P pressure. Also ρ and P should only depend on time in a homogeneous universe. The only relevant tensors are $g^{\mu\nu}$ and the four-velocity u^μ of the fluid. This implies that we can write

$$T^{\mu\nu} = A(\rho, P) u^\mu u^\nu + B(\rho, P) g^{\mu\nu}. \tag{1.16}$$

From comparison with (1.15), we then obtain

$$\boxed{T^{\mu\nu} = (\rho + P) u^\mu u^\nu - P g^{\mu\nu}.} \tag{1.17}$$

In the case where $P = 0$, we obtain the EM-tensor for pressureless (nonrelativistic) matter,

$$T^{\mu\nu} = \rho u^\mu u^\nu. \tag{1.18}$$

Energy-momentum tensor for Maxwell field. The free electromagnetic field is described by the action

$$S_{\mathrm{em}} = -\frac{1}{4}\int \mathrm{d}^4x\sqrt{-g}g_{\mu\alpha}g_{\nu\beta}F^{\mu\nu}F^{\alpha\beta}. \tag{1.19}$$

After a variation with respect to the metric, one obtains

$$T^{\mathrm{em}}_{\mu\nu} = -F_{\mu\lambda}F^{\lambda}_{\nu} + \frac{1}{4}g_{\mu\nu}F_{\gamma\delta}F^{\gamma\delta}. \tag{1.20}$$

Note that the electromagnetic energy-momentum tensor is traceless

$$T^{\mu}_{\mu} = -F_{\mu\lambda}F^{\mu\lambda} + \frac{1}{4}(4)F_{\gamma\delta}F^{\gamma\delta} = 0. \tag{1.21}$$

1.2 Homogeneous and isotropic universes

1.2.1 The cosmological principle

After having briefly discussed general relativity, we may now ask the following question: Which solutions of Einsteins field equations describe the universe we live in? Surprisingly, we can at least to some extent answer this question in the sense that we can construct an idealized model that is consistent with the observational data available to us. At the heart of the model lies the following principle:

> *The Universe is at every time of its evolution both homogeneous and isotropic.*

By homogeneous, we mean that general physical properties are the same everywhere in the universe and isotropic means that the universe looks the same in every direction when observed from a given point. From this we deduce that isotropy around two points implies homogeneity, while homogeneity does not imply isotropy. As an example of the last statement, will a homogeneous universe with everything moving in a single direction not be isotropic. The cosmological principle seems at first to be a much too strong statement. The Solar system does for example not look isotropic at all, neither does the Milky Way. When we apply the cosmological principle, we are however only talking about the largest scales in the universe. The Cosmic Microwave Background radiation (CMB) provides evidence for isotropy at large scales. It would be most remarkable if we occupy a special place in the universe and one therefore feels that isotropy should also hold at other places too, implying homogeneity. While the last argument was previously used for philosophical reasons, there is today also good evidence for homogeneity.

1.2.2 The FRW universe

We now consider a class of universes that are homogeneous and isotropic, known as the Friedmann-Robertson-Walker (FRW) spacetime. It is given in comoving coordinates (r, θ, ϕ) by the line element

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2 \left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\Omega \right], \tag{1.22}$$

where $\mathrm{d}\Omega = \sin^2 \vartheta \mathrm{d}\varphi^2 + \mathrm{d}\vartheta^2$. Here, $k = \pm 1$ corresponds to positive and negative curvature respectively and $k = 0$ to flat 3-space. The terminology *open universe* for $k = -1$ and *closed universe* for $k = +1$ is often also used. This is because geometrically $k = -1$ correspond to a three-dimensional hyperboloid, while $k = 1$ yields a three-dimensional spherical surface. It is convenient to define

$$S(\chi) = \begin{cases} \sin(\chi) & \text{for } k = 1, \\ \chi \equiv r & \text{for } k = 0, \\ \sinh(\chi) & \text{for } k = -1, \end{cases}$$

such that the metric becomes

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2 \left[d\chi^2 + S(\chi)^2 \mathrm{d}\Omega \right]. \tag{1.23}$$

Note that for $k = 0$ introducing conformal time $\mathrm{d}\eta = \frac{\mathrm{d}t}{a(t)}$, the metric becomes conformally flat,

$$\mathrm{d}s^2 = a(t)^2 \left(\mathrm{d}\eta^2 - \mathrm{d}\mathbf{x}^2 \right), \tag{1.24}$$

where $\mathrm{d}\mathbf{x}^2$ is shorthand notation for the spatial part of the line-element.

Conformal symmetry A *conformal* transformation of spacetime is a transformation that shrinks or stretches spacetime. It acts on the metric in the following way

$$g_{\mu\nu}(x) \mapsto \tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x), \tag{1.25}$$

where $\Omega(x)$ is a continuous, nonvanishing, finite, real function. Under a conformal transformation the Christoffel symbols change as

$$\begin{aligned} \Gamma^\sigma_{\mu\nu} \mapsto \tilde{\Gamma}^\sigma_{\mu\nu} &= \frac{1}{2} \tilde{g}^{\sigma\rho} \left(\tilde{g}_{\rho\mu,\nu} + \tilde{g}_{\rho\nu,\mu} - \tilde{g}_{\mu\nu,\rho} \right) \\ &= \Gamma^\sigma_{\mu\nu} + \Omega^{-1} \left[\delta^\sigma_\mu \Omega_{;\nu} + \delta^\sigma_\nu \Omega_{;\mu} - g_{\mu\nu} g^{\sigma\rho} \Omega_{;\rho} \right]. \end{aligned} \tag{1.26}$$

With this in hand we may now also calculate the change in the Riemann tensor, Ricci tensor, etc. We will later need the change of the curvature scalar. It is given by

$$\begin{aligned} \mathcal{R} \mapsto \tilde{\mathcal{R}} &= \tilde{g}^{\mu\lambda} \left[\tilde{\Gamma}^\delta_{\mu\lambda,\delta} - \tilde{\Gamma}^\delta_{\lambda\delta,\mu} + \tilde{\Gamma}^\delta_{\sigma\delta} \tilde{\Gamma}^\sigma_{\mu\lambda} - \tilde{\Gamma}^\delta_{\mu\sigma} \tilde{\Gamma}^\sigma_{\delta\lambda} \right] \\ &= \Omega^{-2} \mathcal{R} - 2(d-1) \Omega^{-3} \Omega_{;\mu\nu} g^{\mu\nu} + (d-1)(d-4) \Omega^{-4} \Omega_{;\mu} \Omega_{;\nu} g^{\mu\nu}, \end{aligned} \tag{1.27}$$

where d is the dimension of spacetime.

1.2.3 The Friedmann equations

Combining the FRW metric (1.22) with the energy-momentum tensor for a perfect fluid results in what are known as the Friedmann equations. The symmetries of the FRW universe reduce the number of independent equations from ten to two. To obtain the equations, we need the relevant components of the Ricci-tensor $R^{\mu\nu}$. From the nonzero component of the metric tensor $g_{\mu\nu}$,

$$g_{00} = 1,\ g_{11} = \frac{a^2}{1-kr^2},\ g_{22} = a^2 r^2,\ g_{33} = a^2 r^2 \sin^2\theta, \tag{1.28}$$

we can derive the Christoffel symbols

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho} \right). \tag{1.29}$$

As an example,

$$\Gamma^{0}_{11} = \frac{1}{2} g^{0\rho} \left(g_{\rho 1,1} + g_{\rho 1,1} - g_{11,\rho} \right) = -\frac{1}{2} \partial_t \left(\frac{-a^2}{1-kr^2} \right) = \frac{a\dot{a}}{1-kr^2}, \tag{1.30}$$

and the others are found similarly. The Ricci tensor is given in terms of the Christoffel symbols by

$$R_{\mu\nu} = \Gamma^{\sigma}_{\mu\sigma,\nu} - \Gamma^{\sigma}_{\mu\nu,\sigma} + \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu} - \Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma} \tag{1.31}$$

This expression looks somewhat ugly, but luckily the only nonzero components are

$$\begin{aligned}
R_{11} &= 3\frac{\ddot{a}}{a}, \\
R_{22} &= -\left(1-kr^2\right)^{-1} \left(a\ddot{a} + 2\dot{a}^2 + 2k \right), \\
R_{33} &= -r^2 \left(a\ddot{a} + 2\dot{a}^2 + 2k \right), \\
R_{44} &= -r^2 \sin^2\theta \left(a\ddot{a} + 2\dot{a}^2 + 2k \right).
\end{aligned} \tag{1.32}$$

We can now also derive the curvature scalar

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu} = \frac{6}{a^2} \left(a\ddot{a} + \dot{a}^2 + k \right). \tag{1.33}$$

Focusing on the RHS of the Einstein equation (1.13), we note that the 4-velocity in our comoving coordinate system (t, r, θ, ϕ) is simply given by $u_\mu = \delta^0_\mu$. The time-time component then becomes

$$3\frac{\ddot{a}}{a} = -4\pi G_N \left(\rho + 3P \right) + \Lambda, \tag{1.34}$$

while the $\mu = \nu = i = 1, 2, 3$ all yield the same equation,

$$a\ddot{a} + 2\dot{a}^2 + 2k = 4\pi G_N \left(\rho - p \right) a^2 + \Lambda a^2. \tag{1.35}$$

Eliminating $\ddot{a}$ from the second equation we get the Friedmann (-Lemaître) equations in their usual form

$$\boxed{\begin{aligned} H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G_N}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \\ \frac{\ddot{a}}{a} &= \frac{\Lambda}{3} - \frac{4\pi G_N}{3}(\rho + 3P). \end{aligned}} \tag{1.36}$$

It is customary to introduce the critical density ρ_c as the the energy density at zero spatial curvature,

$$\rho_c = \rho(k=0) = \frac{3H^2}{8\pi G_N}, \tag{1.37}$$

and the abundance Ω_i of different constituents as

$$\Omega_i = \frac{\rho_i}{\rho_c}. \tag{1.38}$$

We therefore see that Λ act as a constant contribution to the total energy density,

$$\rho_\Lambda = \frac{\Lambda}{8\pi G_N}, \quad \Omega_\Lambda = \frac{\Lambda}{3H^2}. \tag{1.39}$$

1.2.4 Different constituent contributions to energy density

We can derive the scale dependence of different energy forms by means of energy conservation assuming adiabatic expansion

$$\mathrm{d}U + P\mathrm{d}V = 0, \tag{1.40}$$

if we make the assumption that each energy constituent have an equation of state $P(\rho) = w\rho$, with w being constant. We note that pressureless matter has an equation of state with $w = 0$, while radiation obey $w = 1/3$. The last one follows since the electromagnetic energy-momentum tensor is traceless, $T^\mu_\mu = 0$, thus $\rho - 3P = 0$. The cosmological constant has a rather peculiar E.o.S that can be derived from thermodynamics

$$P = -\left(\frac{\partial U}{\partial V}\right)_S = -\frac{\partial(\rho_\Lambda V)}{\partial V} = -\rho_\Lambda. \tag{1.41}$$

Equation (1.40) becomes

$$\mathrm{d}\left(\rho a^3\right) = -3Pa^2\mathrm{d}a. \tag{1.42}$$

Writing out both sides and eliminating P by the E.o.S yields a separable differential equation,

$$-3(1+w)\frac{\mathrm{d}a}{a} = \frac{\mathrm{d}\rho}{\rho}. \tag{1.43}$$

We therefore obtain

$$\boxed{\rho \propto a^{-3(1+w)} = \begin{cases} a^{-3} & \text{for matter } (w=0), \\ a^{-4} & \text{for radiation } (w=1/3), \\ \text{constant} & \text{for } \Lambda \ (w=-1). \end{cases}} \tag{1.44}$$

We can write the solution at arbitrary time in terms of the energy density ρ_0 at present time t_0

$$\rho(t) = \rho_0 \left(\frac{a(t)}{a(t_0)}\right)^{-3(1+w)}. \tag{1.45}$$

The total energy density of the universe at a given time with three different components (matter, radiation, Λ) is thus given by

$$\rho(t) = \rho_{m,0}\left(\frac{a_0}{a(t)}\right)^4 + \rho_{r,0}\left(\frac{a_0}{a(t)}\right)^3 + \rho_\Lambda. \tag{1.46}$$

As one can see from (1.45) different types of energy would have dominated at different times during the evolution of the universe. In a universe expanding from a big bang singularity, radiation will have dominated at early times. Since the radiation term drops of faster than the matter term, there would have been a time when matter becomes more dominating. Finally, as the time goes on, the energy density of matter will be so diluted by expansion that the cosmological constant will dominate the energy density. This is of course depending on the different parameters. If Λ is too large, there may never be enough matter to govern the expansion of the universe. In the end, it is an experimental matter to find the value of Λ. A most intriguing fact is that ρ_m and ρ_Λ both seem to be of the same order just today, in spite of the very different scaling.

1.2.5 Cosmological models with a single energy component

Let us for a moment consider a $k=0$ universe dominated by a single source of energy with equation of state $P = w\rho$. We can insert (1.45) into the first Friedmann equation and find

$$\dot{a}^2 = \frac{8\pi G_N}{3} a^2 \rho_c(t_0)\left(\frac{a}{a_0}\right)^{-3(1+w)} = H_0^2 a_0^{3+3w} a^{-(1+3w)}. \tag{1.47}$$

The differential equation is separable and integrating from 0 to t_0 yields

$$t_0 H_0 = \frac{2}{3+3w} = \begin{cases} \frac{2}{3} & \text{for matter,} \\ \frac{1}{2} & \text{for radiation,} \\ \to \infty & \text{for } \Lambda. \end{cases} \tag{1.48}$$

The time-scale H_0^{-1} is known as the Hubble age of the universe. Integrating instead up to an arbitrary time, we obtain the time-dependence of the scale factor,

$$\boxed{a(t) \propto t^{2/(3+3w)} = \begin{cases} t^{\frac{2}{3}} & \text{for matter,} \\ t^{\frac{1}{2}} & \text{for radiation,} \\ \exp(t) & \text{for } \Lambda. \end{cases}} \tag{1.49}$$

1.3 Observational status

1.3.1 The expanding universe

The standard model of cosmology, famously known as the big bang-theory has its origin from the discovery that the universe expands. By studying spectral lines of large number of galaxies, one measured most of the galaxies to be redshifted

$$z \equiv \frac{\Delta\lambda}{\lambda} > 0. \tag{1.50}$$

Moreover, the amount of redshift of the galaxies was found to be proportional to its distance d. By interpreting this redshift as a Doppler shift, i.e that $z = v_r$ (assuming $v \ll c \equiv 1$), one obtains the following formula for the recession velocity known as *Hubble's law*

$$v_r = H_0 d. \tag{1.51}$$

The proportionality factor is the Hubble constant H_0. The law is indeed not universal, rather an approximation for $z \ll 1$ and the Hubble constant is not really a constant but a time-dependent parameter $H(t)$. To see this, let's Taylor expand the scale factor around $t = t_0$,

$$a(t) = a(t_0) + (t - t_0)\dot{a}(t_0) + \frac{1}{2}(t - t_0)^2\ddot{a}(t_0) + ... \tag{1.52}$$

$$= a(t_0)\left[1 + (t - t_0)H_0 - \frac{1}{2}(t - t_0)^2 q_0 H_0^2\right]. \tag{1.53}$$

where

$$\boxed{H_0 \equiv \frac{\dot{a}(t_0)}{a(t_0)}, \quad q_0 \equiv -\frac{\ddot{a}(t_0)a(t_0)}{\dot{a}(t_0)^2}.} \tag{1.54}$$

We see that we can identify the Hubble constant H_0 with the parameter in the Friedmann equation taken at $t = t_0$. We have also introduced the deceleration parameter q which can be determined from studying deviations from Hubble's law. For small redshifts

$$1 - z \approx \frac{1}{1+z} = \frac{a(t)}{a(t_0)} \approx 1 + (t - t_0)H_0, \tag{1.55}$$

and Hubbles law follows as long as $H_0(t_0 - t) \ll 1$. The Hubble constant is usually given in the rescaled form,

$$H_0 = 100\ h\ \mathrm{km\ s^{-1}\ Mpc^{-1}}, \tag{1.56}$$

to separate out its uncertainty from derived parameters.

1.3.2 The cosmic microwave background radiation

The cosmic microwave background (CMB) is the name given to the almost isotropic radiation that fills the universe. It was first discovered in 1965 by Arno Penzias and Robert Wilson, although predicted much earlier by George Gamow and others. The spectrum of radiation, which lies in the microwave range, can be described very well as a black-body spectrum of temperature $T = 2.7255$K. This discovery is one of the cornerstone predictions of the big bang model. As the early hot universe cooled and after a few hundred thousand years reached the temperature of a few thousand kelvin, neutral atoms could form and the universe became transparent. This is referred to as the *surface of last scattering*. The CMB is the relic from the photon decoupling. Due to the expansion of the universe, this radiation is highly redshifted, explaining why we observe the radiation in the microwave range. Another important feature is that CMB is almost perfectly isotropic and provides therefore evidence for the cosmological principle, with

$$\frac{\delta T}{T} \sim \frac{\delta \rho}{\rho} \sim 10^{-5}, \tag{1.57}$$

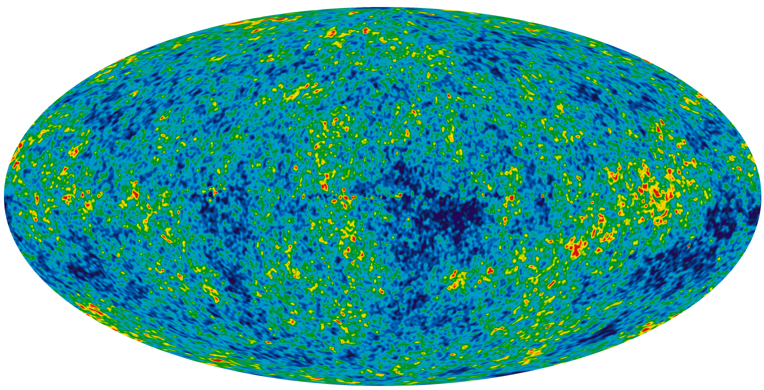

Figure 1.1: Temperature fluctuations of the CMB. This image shows a temperature range around average of $\pm 200\mu$K, illustrated in colors with blue being the coldest. Credit: NASA/WMAP Science Team.

1.3.3 The ΛCDM-model

Using as foundation the cosmological principle, we saw that one could construct a general class of spacetimes known as Friedmann-Walker-Robertson spacetimes. Including energy components from different species, we could construct different cosmological models. One of the goals of observational cosmology is then to specify the values of the different parameters. That involves specifying the present day parameters

$$H_0, \quad \Omega_{m,0}, \quad \Omega_{r,0}, \quad \Omega_{\Lambda,0}. \tag{1.58}$$

As late as in 1998 an additional important discovery was made. Not only is the universe expanding, it is expanding at an ever increasing rate. This lead to the re-introduction of Einsteins *biggest blunder*, a cosmological constant Λ. Putting cosmological data together one has today came up with a model known as the ΛCDM-model which is consistent with all cosmological observations. In this model, we live in a universe with $\Omega_m + \Omega_\Lambda = 1$ and zero curvature $k = 0$. Below, we have listed a table of cosmological parameters.

Parameter	Symbol	Value
Hubble parameter	h	0.704 ± 0.025
Cold dark matter density	Ω_{cdm}	$\Omega_{\text{cdm}} h^2 = 0.112 \pm 0.006$
Baryon density	Ω_b	$\Omega_b h^2 = 0.00225 \pm 0.0006$
Cosmological constant	Ω_Λ	$\Omega_\Lambda = 0.73 \pm 0.03$
Radiation density	Ω_r	$\Omega_r h^2 = 2.47 \times 10^{-5}$

Table 1.1: Table of cosmological parameters. Data taken from [6].

Chapter 2

Dark Matter

2.1 Motivation

The best evidence for the existence dark matter in galaxies comes from observations of *rotation curves.* A rotation curve is the graph of orbital velocities of objects in a galaxy as function of the distance to its center. Studying the rotation curve of a galaxy enables us to compute its mass distribution. The formula for the rotation curve of a spiral galaxy can easily be derived if we assume that the objects in it follows circular orbits. Using Newtons law of gravitation, the orbital velocity $v(r)$ is given by

$$v(r) = \sqrt{\frac{G_N M(r)}{r}}, \tag{2.1}$$

where $M(r)$ is the mass inside a sphere of radius r. By studying the visible matter in galaxies one observes matter moving faster than one would expect from gravitational attraction alone. If visible matter provided all the mass in the galaxy, then the rotation curve would fall off at $r > r_{\text{disc}}$. Instead, one observes that $v(r)$ is more or less constant out to much larger radii indicating $M(r) \propto r$ for $r_{\text{disc}} \lesssim r$ out to $r \gg r_{\text{disc}}$.

2.2 Production Mechanisms and Candidates

One distinguishes between hot (HDM) and cold dark matter (CDM) from whether it is composed of particles being relativistic or not at chemical decoupling. HDM are typically neutrinos. The neutrinos, which are interacting only very weakly, will not collapse into tightly bound object and should remain less condensed. The study of structure formation of the Universe indicate that most of the dark matter should be cold since HDM will in general not permit galaxies to form in the way we observe. CDM, has in contrast been very successful explaining structure formation. It is both very well motivated and is able to reproduce most features of the observed

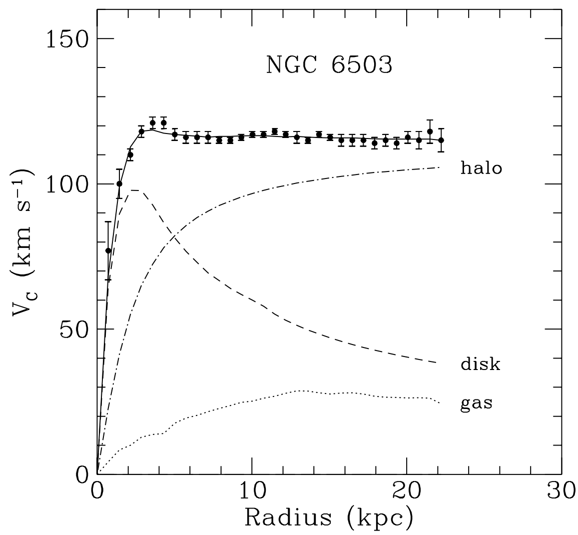

Figure 2.1: Rotation curve of the spiral galaxy NGC 6503. Data point show observed rotation curves compared to the rotation curves of a disk, an intergalactic gas and a dark matter halo. Picture taken from [10]

Universe. One can divide dark matter candidates into two categories according to whether it is thermal or not. Thermal dark matter are particles that once was in thermal equilibrium with radiation and ordinary matter. A general class of candidates for cold dark matter are weakly interacting massive particles (WIMPs). WIMPs in equilibrium in the early Universe has naturally the right abundance to be CDM. In the early hot Universe, WIMPs interacted with ordinary matter through reactions like

$$\chi\bar{\chi} \longleftrightarrow e^+e^-, \mu^+\mu^-, ... \tag{2.2}$$

At temperatures $T \gg m_\chi$, colliding pairs of particles and antiparticles had sufficient energy to produce pair of WIMPs efficiently. Initially, the annihilation process were in equilibrium with the production process with rate given by

$$\Gamma_{\text{ann}} = \langle \sigma_{\text{ann}} v \rangle n_{\text{eq}}. \tag{2.3}$$

Here, the WIMP annihilation cross-section σ_{ann} times the relative velocity of the annihilating WIMPs v is averaged of the thermal distribution, and n_{eq} is denoting the number density of WIMPs in chemical equilibrium. As the Universe cooled, the temperature of the plasma eventually fall below the WIMP mass and the number of WIMPs produced decreased exponentially due to the Boltzmann factor in

$$n_{\text{eq}} = \left(\frac{m_\chi T}{2\pi} \right) \mathrm{e}^{-m_\chi/T}. \tag{2.4}$$

As a consequence, the annihilation rate went down. When the annihilation rate became smaller than the Hubble parameter H, the production of WIMPs became frozen (chemical decoupling). From this point, the energy density of WIMPs decrease as a^{-3}. The rate of change can be found from

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{\mathrm{d}n}{\mathrm{d}a}\dot{a} = -3Hn. \tag{2.5}$$

Adding contributions from creation and annihilation processes also, one obtains

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -3Hn - \langle \sigma_{\mathrm{ann}} v \rangle (n^2 - n_{\mathrm{eq}}). \tag{2.6}$$

Assuming constant entropy $S = sV$ of the Universe, one get another equation

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -3Hs. \tag{2.7}$$

One usually introduces the dimensionless variables $Y = n/s$ and $x = m/T$ and use instead of t, the temperature T as the independent variable. From this one obtains

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = \frac{1}{3H}\frac{\mathrm{d}s}{\mathrm{d}x}\langle \sigma_{\mathrm{ann}} v \rangle \left(Y^2 - Y_{\mathrm{ann}}^2\right). \tag{2.8}$$

Solving this equations can be used to obtain the present WIMP abundance

$$\Omega_\chi h^2 \simeq \frac{3 \times 10^{-27}\mathrm{cm}^3\mathrm{s}^{-1}}{\langle \sigma_{\mathrm{ann}} v \rangle}. \tag{2.9}$$

Since evidence of dark matter first came, there have been a lot of work associated with the development of suitable candidates for dark matter. Here, we briefly discuss some of them. In general, there are several conditions that must be satisfied for something to be considered a good dark matter candidate. For example, they must be stable on cosmological time scales to still be around. They must also, naturally, interact very weakly with electromagnetic radiation. Finally, they must have the right relic density.

Supersymmetric dark matter Among the best motivated extensions of the standard Model of particle physics is supersymmetry, the main observation being its ability to stabilize the mass scale of electroweak symmetry breaking. A motivation that is more interesting in this context was the realization that the lightest supersymmetric particle in models conserving R-parity would be a very good dark matter candidate. To meet this requirement, the lightest supersymmetric particle should have a mass $\lesssim 50$ TeV if it was once in thermal equilibrium. This holds for the neutralino χ or the sneutrino $\tilde{\nu}$ and can also be extended for a gravitino.

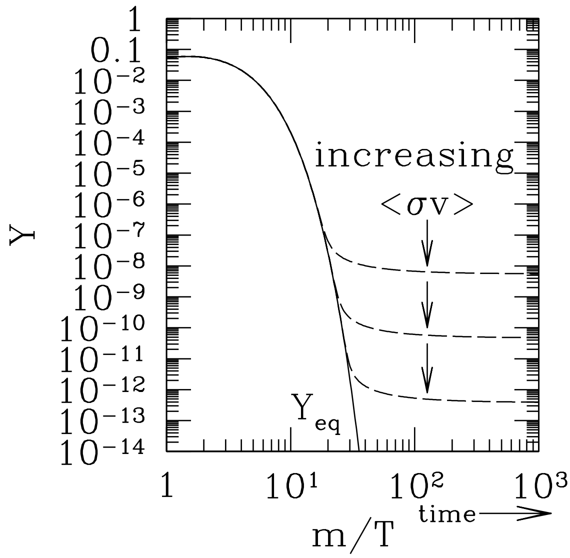

Figure 2.2: Evolution of WIMP number density in the early Universe during chemical decoupling. The figure shows $Y = n/s$ as a function of $x = m/T$. Solid line shows the evolution for dark matter staying in thermal equilibrium, while the dashed lines show decoupling for different annihilation cross-sections. Picture taken from [14]

Axions and the strong CP problem Axions are introduced to solve the 'strong CP problem' of the standard model. Consider the QCD-Lagrangian

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \sum_{j=1}^{n}\left[\bar{q}_j\gamma^\mu \mathrm{i}D_\mu q_j - (m_j q^\dagger_{\mathrm{L}_j} q_{\mathrm{R}_j} + \text{h.c.})\right] + \frac{\theta g^2}{32\pi^2}G^a_{\mu\nu}\tilde{G}^{a\mu\nu}. \tag{2.10}$$

The last term is a 4-divergence and therefore does not contribute to perturbation theory. It contributes however through non-perturbative effects. One can show that the θ dependence must be present if none of the current quark masses vanish and that the dependence can be parametrized by

$$\bar{\theta} = \theta - \arg(m_1, m_2, ..., m_n). \tag{2.11}$$

If $\bar{\theta} \neq 0$, then QCD violates P (parity) and CP (charge conjugation + parity). Since we don't see CP violation in strong interactions, an upper limit of $\bar{\theta} < 10^{-9}$ can be set from experiments. The smallness of $\bar{\theta}$ is strange since the quark masses originates from the electroweak sector (violating P and CP). A proposal to solve the CP problem was put forward by Peccei and Quinn postulating the existence of a global U(1) quasi-symmetry broken by nonperturbative instanton effects of QCD.

The axion is the (quasi-) Nambu-Goldstone boson associated with spontaneous symmetry breaking of $U_{PQ}(1)$. The presence of this symmetry implies that

$$\bar{\theta} \to \bar{\theta} - \frac{a(x)}{f_a}, \tag{2.12}$$

solving the strong CP problem. Here, f_a is called the axion decay constant. The axion mass is then given by

$$m_a \sim 0.6\mathrm{eV}\frac{10^7\mathrm{GeV}}{f_a}, \tag{2.13}$$

and all axion couplings are inversely proportional to f_a. The abundance of axions depends on their production mechanism, but there exist models that give it the necessary and efficient abundance to be dark matter.

Superheavy dark matter As a last example, we discuss a possible dark matter candidate at the other end of the mass scale, namely superheavy dark matter. Typically, dark matter candidates cannot be too heavy since they would then overclose the Universe. It is however possible to consider dark matter candidates produced out of thermal equilibrium. These particles, often dubbed WIMPZILLAS, are particularly interesting since they are created through the process of gravitational particle production. They are usually believed to be produced at the last stages of inflation or during reheating after inflation. For them to be considered a dark matter candidate they need to have a mass about $\sim 10^{13}$GeV. Also, they need to be stable or have an expected lifetime of order the age of the Universe. In the latter case, their decay products may give rise to ultra high energy cosmic rays (above GZK limit). The concept om gravitational particle production will be discussed in great detail in chapter 4-6. The important realization here is that particles produced with masses comparable to the Hubble parameter at the end of inflation $H_{\mathrm{e}} \sim 10^{-6} M_{\mathrm{p}}$ may give the right abundance to be dark matter. This alone, is a big motivation for studying gravitational particle production.

Chapter 3

Inflation

3.1 Shortcomings of the big bang-model

In spite of the successes of the standard big bang scenario, it also suffers from some problems. These problems, which we will discuss in the following, are not problems in the sense that they lead to contradictions with the theory itself, rather they are shortcomings that cannot be explained by it. Most notably are the horizon (and homogeneity-) and the flatness problem. They have in common that they require an incredible fine-tuning of the initial conditions of the universe (or at least of the classical era below the Planck density[1]). It would be more satisfying with a theory that could explain these issues as a necessity rather than merely a historical accident. Inflation is such a theory. In inflationary cosmology one adds to the standard model a period of rapid (exponential) expansion in the very early universe. In this period, the universe expands for a short period of time at least 60 e-folds (i.e. grows by a factor $e^{60} \approx 0.1$ billion$\times$billion$\times$billion) and dilutes the energy density of matter to nearly zero. As we will see, this does to some extent improve the above mentioned problems

The homogeneity and the horizon problem Observations of the large-scale structure of the universe provides good evidence for homogeneity and isotropy. Since looking out in the universe means looking back in time, the early universe must have been very homogeneous. This homogeneity could of course have been driven by some unknown physical mechanism. What is more intriguing is that the mechanism cannot have been perfect since it's exactly the small amount of inhomogeneity in the early universe that has allowed for structure formation. Homogeneity also leads to another unsatisfactory feature of the big bang-model. Relativity dictates that information cannot propagate faster than the speed of light. Why are

[1] General Relativity is generally believed to break down at the Planck scale corresponding to about 10^{-43}s after the big bang. Prior to that era, quantum fluctuations would have been so large that a theory of Quantum Gravity is needed to describe it.

even causally disconnected regions of the universe homogeneous? In an expanding universe, the particle horizon is defined as the proper distance outwards it remains possible to observe a particle by exchange of light signals. It is given by ($c \equiv 1$)

$$l_H(t_0) = a_0 \int_0^{t_0} \frac{\mathrm{d}t}{a(t)}. \tag{3.1}$$

If we live in a universe that has been dominated by matter (or radiation) since the big bang, the particle horizon is given by

$$l_H(t_0) = \int_0^{t_0} \left(\frac{t_0}{t}\right)^{\alpha} = \frac{t_0}{1-\alpha} = \begin{cases} 3t_0 & \text{for matter,} \\ 2t_0 & \text{for radiation,} \end{cases} \tag{3.2}$$

while the scale factor grows like $t^{3/2}$ ($t^{1/2}$) according to (1.49). This means that any length scale l contained completely inside the horizon today was at some time t, with $0 < t < t_0$ outside the horizon. In other words, we observe homogeneous structures in the universe that has supposedly never been in causal contact. To solve the horizon problem, we need the scale factor to grow faster than the horizon.

The Flatness Problem In Chapter 1, we defined the critical density as the energy density at zero spatial curvature

$$\rho_c = \frac{3H^2}{8\pi G_N}. \tag{3.3}$$

We also introduced the relative abundance Ω_i of different energy species as the ratio of their corresponding energy density and the critical density ρ_c. This means that we can write the first Friedmann equation as

$$\frac{k}{a^2} = H^2 \left[\sum_i \Omega_i - 1\right] \equiv H^2(\Omega_{\text{tot}} - 1), \tag{3.4}$$

which implies that $\Omega_{\text{tot}} = 1$ corresponds to a flat universe. As already mentioned, a universe governed by a single energy component with an equation of state $P = w\rho$ has an energy-density given by

$$\Omega_i \propto \frac{1}{H^2} a^{-3(1+w_i)}. \tag{3.5}$$

This means that both the matter- ($w = 0$) and the radiation term ($w = 1/3$) decrease faster than the curvature term in the Friedmann equation. Taking the time-derivative of (3.4), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Omega_{\text{tot}} - 1| = -2|k|\frac{\ddot{a}}{\dot{a}^3}. \tag{3.6}$$

Therefore, in a decelerating universe (e.g. matter/radiation dominated) the curvature tend to increase. Since observations today indicate $\Omega_{\text{tot}} = 1.002 \pm 0.011$ we would need an incredible fine tuning of the early universe. To see how much, (3.4) can be converted into redshift. The left hand side scales as $(1+z^2)$, while the Hubble parameter scales as $(1+z)^3$ or $(1+z)^4$ for matter and radiation, respectively. During most of the time since the Planck ages $t_p \sim 10^{-43}$, the universe has been radiation dominated. We can therefore obtain an order of magnitude estimate

$$|\Omega_{\mathrm{tot,pl}} - 1| \sim |\Omega_{\mathrm{tot,0}} - 1|(1 + z_{\mathrm{p}})^{-2} \sim 10^{-2} t_{\mathrm{p}}/t_0 \sim 10^{-62}. \tag{3.7}$$

The relic problems Although we do not yet have a good understanding of the physics at high energy scales, most of the ideas developed to explain it involves production of particles not being part of the ordinary standard model. Among these ideas, we find most notably supersymmetry, grand unified theories and supergravity, having in common that they include particles that can only be produced in the early universe before nucleosynthesis. By claiming that the big bang theory is valid up to Planck scale, the energy must once have been high enough for these particles to be produced. Moreover, some of these particles come out with long lifetimes and would be expected to dominate the current energy density. This happens for example if particles only interacts via gravitation. These particles include among others monopoles, gravitons and modulies, and are often referred to as relic particles since they are relics of the early universe.

Solution by inflation From equation (3.6), we see that an early phase of accelerated expansion called inflation[2], would drive the curvature term to zero. Moreover, inflation may be so efficient that the curvature produced by a later radiation or matter dominated phase could be negligible, thus solving the flatness problem. It could also solve the horizon problem, since the expansion of the universe during inflation grows faster than the horizon. Thus causally connected regions would be driven to superhorizon scales. From the second Friedmann equation, we have

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3P) = -\frac{4\pi G_N}{3}(1 + 3w)\rho, \tag{3.8}$$

so we need $w < -1/3$ for acceleration to occur. Consider a universe governed by an E.o.S with $w = -1$, i.e. $P = -\rho$. In such a universe, the energy density wouldn't change and the curvature term would soon become negligible leading to

$$H^2 = \frac{8\pi G_N}{3}\rho. \tag{3.9}$$

A constant ρ would therefore lead to a constant Hubble parameter and as a consequence to $a = \exp(Ht)$. Therefore, all relic particles produced before inflation would be exponentially suppressed. When inflation ends, the universe may be below the temperature necessary to produce these relic particles.

[2]The term *inflation* was introduced by Alan Guth.

3.2 Models for inflation

What mechanism could produce an equation of state with $w < -1/3$? A possibility we have already seen, is a universe dominated by a cosmological constant Λ. It would yield an E.o.S with $w = -1$, leading to exponential expansion. However, the effect of a cosmological constant increases with time and hence could not be identified with the small one observed today. In a universe with such a large cosmological constant, inflation would never end. A possible, and certainly more interesting way to produce the effect of a cosmological constant is to introduce a scalar field rolling down a potential $V(\phi)$. This scalar field is often referred to as the *inflaton field.* We will discuss the behavior of scalar fields in curved spacetime in more detail in the next chapter, but it suffices to say that the energy density and pressure of a scalar field in a FRW universe is given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{1}{2}(\nabla\phi)^2, \tag{3.10}$$

$$P = \frac{1}{2}\dot{\phi}^2 - V(\phi) - \frac{1}{6}(\nabla\phi)^2. \tag{3.11}$$

If we assume the scalar field is spatially constant (or study a patch of the universe where it is homogeneous), it will produce an equation of state given by

$$w = \frac{P}{\rho} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} \in [-1, 1]. \tag{3.12}$$

As long as the potential $V(\phi)$ dominate the energy density, we have $w \approx -1$ and inflation will occur.

3.2.1 Early models for inflation

One of the first models for inflation that had a clear physical motivation was put forward by Alan Guth in 1981. In this model, based on cosmological phase transitions a scalar field was trapped in a local minimum of its potential with $V(\phi) > 0$. As long as the field remained there, the energy density would be dominated by the potential leading to inflation according to (3.12). This state is known as a *false vacuum*, since it acts as if it was the lowest possible energy state. Classically it would be stable, since there is no energy available to push the field over the potential barrier. It would therefore not be possible to distinguish it from the true vacuum. However, due to quantum effects the field may tunnel through the potential barrier to the true vacuum thus ending inflation. Unfortunately, this model had problems of its own. The bubbles produced when the false vacuum decayed produced way too much inhomogeneities. Also, the idea of a field trapped in a local minimum has fine-tuning problems of its own.

3.2.2 Chaotic inflation

Today, most models are based around an idea suggested by Andrei Linde in 1983. The evolution a scalar field in an expanding universe is given by

$$\boxed{\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0.} \tag{3.13}$$

This equation is analogous to the one of a damped harmonic oscillator, with the Hubble parameter H acting as friction. If the field ϕ finds itself at a large value of the potential, then according to (3.9), H will be large and ϕ will roll slowly. We will have a situation with $w \approx -1$ and the universe will expand quasi-exponentially. As the field rolls down the potential, there will be some point when the system is no longer heavily damped and the kinetic term of the energy density will become important, causing inflation to end. An advantage of the chaotic inflation scenario is that inflation occurs even in theories with $V \sim \phi^2$ or $V \sim \phi^4$, but is not limited to these. It happens in any theory with a region that allows for a slow-roll regime. In particular, it can happen for every power-law potential for $\phi \gtrsim M_{\mathrm{p}}$

$$V(\phi) = \frac{\lambda_n \phi^4}{4M_{\mathrm{p}}^{n-4}}, \quad \lambda_n \ll 1. \tag{3.14}$$

Initial conditions To study the evolution of a universe filled with a scalar field ϕ, we need a way to set the initial conditions. Let us consider the universe as it emerges from the Planck-era, with energy density $\rho \sim M_{\mathrm{p}}^4$. Only from this point, with $\rho \lesssim M_{\mathrm{p}}^4$ may we describe the universe in terms of the laws of classical physics. Thus, we should at this point require that the energy density of the scalar field is given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \sim M_{\mathrm{p}}^4. \tag{3.15}$$

If in some patch of the universe, we have

$$\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 \lesssim V(\phi), \tag{3.16}$$

inflation will start. Moreover, within a Planck time $t_{\mathrm{p}} \sim 10^{-43}$s, $\frac{1}{2}\dot{\phi}^2$ and $\frac{1}{2}(\nabla\phi)^2$ will be driven to values much smaller than $V(\phi)$ ensuring that inflation continue.

Slow-roll approximation The equation for the scalar field together with the Friedmann equation can be simplified if we assume that the field is slowly rolling, i.e. $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$. We then obtain

$$\dot{\phi} = \frac{\ddot{\phi} + V'(\phi)}{3H} \approx -\frac{V'(\phi)}{3H}, \tag{3.17}$$

and

$$H^2 = \frac{8\pi G_N}{3}\rho \approx \frac{8\pi G_N}{3}V(\phi). \tag{3.18}$$

Using the two equations, we find that

$$\epsilon \equiv \frac{1}{2}\left(\frac{V'}{8\pi G_N V}\right)^2 \ll 1, \tag{3.19}$$

is valid in the slow-roll regime. Another slow-roll parameter can be derived from $|\ddot{\phi}| \ll |V'|$ after differentiation of (3.17)

$$\eta \equiv \frac{V''}{8\pi G_N V} \ll 1. \tag{3.20}$$

Assuming (3.19) and (3.20) are fulfilled, it is easy to calculate the number of e-folds, N, produced by inflation as the scalar field rolls from ϕ_1 to ϕ_2

$$N \equiv \ln\left(a_2/a_1\right) = \int_{t_1}^{t_2} H\mathrm{d}t = \int_{\phi_1}^{\phi_2} \frac{\mathrm{d}\phi}{\dot{\phi}} H = 8\pi G_N \int_{\phi_2}^{\phi_1} \mathrm{d}\phi \frac{V(\phi)}{V'(\phi)}. \tag{3.21}$$

Inflation in $\lambda\phi^4$-theory. Let's consider a theory of inflation with the potential

$$V(\phi) = \frac{1}{4}\lambda\phi^4, \quad \text{with } \lambda \ll 1 \tag{3.22}$$

as in figure 3.1. When $\phi > \lambda^{-1/4}M_\mathrm{p}$, the energy density of the scalar field is above the Planck density $\rho > \rho_\mathrm{p} = M_\mathrm{p}^4$ and we can no longer neglect the strong quantum fluctuations of spacetime in our description. This state is referred to as a spacetime foam and a theory of quantum gravity is a necessity. As $\frac{M_\mathrm{p}}{3} \lesssim \phi \lesssim \lambda^{-1/4}M_\mathrm{p}$, the field is rolling slowly down the potential and the universe expands quasi-exponentially with $w \approx -1$. When $\phi < M_\mathrm{p}/3$, the field oscillates around the minimum of the potential transferring its energy into particles. In this model, the equations in the slow-roll regime becomes

$$\dot{\phi} = -\frac{\lambda\phi^3}{3H} = -\frac{\lambda\phi^3}{3}\sqrt{\frac{3}{2\pi G_N \lambda\phi^4}} = -M_\mathrm{p}\sqrt{\frac{\lambda}{6\pi}}\phi, \tag{3.23}$$

written in terms of the Planck mass $M_p = G_N^{-1/2}$. The solution in the slow-roll regime therefore becomes

$$\phi(t) = \phi_0 \exp\left(-\sqrt{\frac{\lambda}{6\pi}}M_\mathrm{p}t\right), \tag{3.24}$$

where ϕ_0 is the initial value of the inflaton field. Moreover, the number of e-folds becomes

$$N = \frac{2\pi}{M_\mathrm{p}^2}\int_{\phi_\mathrm{e}}^{\phi} \phi\, \mathrm{d}\phi = \frac{\pi}{M_\mathrm{p}^2}\left(\phi^2 - \phi_\mathrm{e}^2\right). \tag{3.25}$$

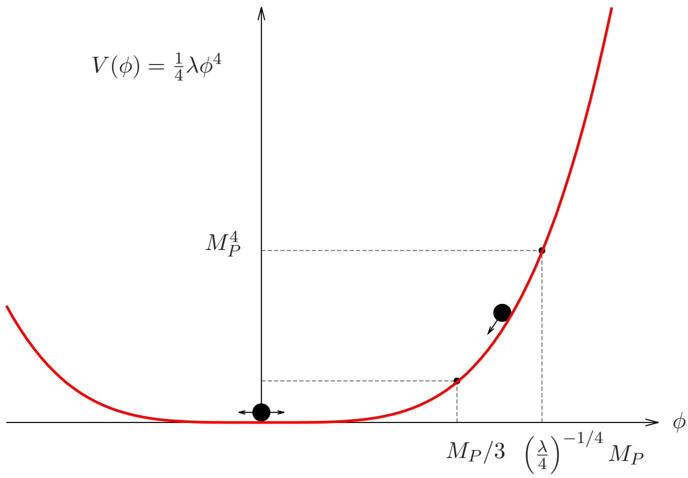

Figure 3.1: A scalar field rolling in a potential with $V(\phi) = \frac{1}{4}\lambda\phi^4$.

In order to achieve 60 efoldings, assuming inflation ends at $\phi_{\rm e} \sim M_{\rm p}/3$ we see that we need at least

$$\phi_i = \sqrt{\left(\frac{60}{\pi} + \frac{1}{9}\right)} M_{\rm p} \approx 4.4 M_{\rm p}. \tag{3.26}$$

Massive inflation. We can also consider a theory of inflation with the inflaton potential as a mass-term

$$V(\phi) = \frac{1}{2} m^2 \phi^2. \tag{3.27}$$

In this model, inflation occurs for $M_{\rm p}/6 \lesssim \phi \lesssim m^{-1}$. During inflation, the field ϕ is to a good approximation given by

$$\phi(t) = \phi_i - \frac{M_{\rm p} m}{2\sqrt{3\pi}} t. \tag{3.28}$$

Moreover, the number of efoldings becomes according to (3.21)

$$N = \frac{2\pi}{M_{\rm p}^2} \left(\phi^2 - \phi_e^2\right). \tag{3.29}$$

Hence $N \simeq 60$ require

$$\phi_i \sim 3.1 M_{\rm P}. \tag{3.30}$$

3.3 Reheating after inflation

As we have seen, inflation in the early universe is able to solve the mentioned problems. This however, comes at a price. Inflation leaves the universe very cold and empty. Not only does it get rid of all relic particles, but all other kinds of particles as well. During inflation all the energy density of the universe is contained in the inflaton field. Therefore, when inflation is over there must be a mechanism where the inflaton decays into other forms of energy and *reheats* the universe. Reheating is the theory of particle production by the inflaton field leading subsequently to thermalization. The details of the theory depend often on the model of inflation one considers. If the decay of the inflaton happens slowly, the decay products may have time to interact with each other and come to a state of thermal equilibrium at the reheating temperature T_r. This is typically the way it happens if the inflaton field couples directly to other fields and one can study the process of the inflaton field decaying to different forms of energy perturbatively. There exists however a much more effective way to for the inflaton field to decay, namely via parametric resonance. This mechanism, which occurs in the nonperturbatiev regime is in general much more effective. Consider a case where the inflaton field ϕ is coupled to a scalar field χ via the interaction term

$$V_{\text{int}} = \frac{1}{2} g^2 \phi^2 \chi^2. \tag{3.31}$$

In such a model the equation of motion for χ will have a frequency depending on ϕ [11]. The oscillations of the inflaton will then cause the resonant modes of χ to be amplified. This effect was named *preheating* because it transfers the energy density of the inflaton field rapidly to fluctuations of the field χ, and to distinguish it from the slower perturbative mechanisms of reheating.

Chapter 4

Particle Production in an Expanding Universe.

Combining quantum mechanics and the special theory of relativity leads to the possibility of creation and annihilation of particles. In curved spacetime things can be said to be even more interesting. Upon quantizing a field, particles can be created from the vacuum by the expansion of the universe itself with no other external field present. In this chapter we develop the necessary tools for studying gravitational particle production of scalar fields in a spatially flat FRW universe. In chapter 5, we apply the formalism developed here on a case which can be solved analytically. Note that while we quantize the scalar field, the gravitational field is treated purely as a classical field by means of general relativity. It is clear that this semi-classical picture will break down for strong gravitational fields. On the other hand, many of the results of semi-classical radiation theory developed prior to the theory of quantum electrodynamics is in compliance with the full machinery itself, and one may hope that quantum fields in classical gravitational backgrounds shows a similar feature. Much of the discussion in this chapter follow [2].

4.1 Models in flat FRW-universes

4.1.1 Field equation for a scalar field

The simplest Lagrangian that we can write down for a scalar field in Minkowski space (i.e in flat spacetime) is given by

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\phi_{,\mu}\,\phi_{,\nu} - V(\phi), \tag{4.1}$$

where the potential $V(\phi)$ may be

$$V(\phi) = \frac{1}{2}m^2\phi^2, \tag{4.2}$$

and the Minkowski metric is given by $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. The Lagrangian with the potential (4.2) describes a free (noninteracting) scalar field of mass m. To see this, note that

$$\left.\frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\right|_{\phi=\phi_{\min}} \equiv m^2. \tag{4.3}$$

Hence a term quadratic in the field act as a mass term, while higher order terms generate self-interactions of the field. We want to promote this Lagrangian to a general curved spacetime with the metric $g_{\mu\nu}(x)$. We must then make the following replacements when constructing the action.

- We must replace the Minkowski metric by an arbitrary metric

$$\eta_{\mu\nu} \longrightarrow g_{\mu\nu}(x).$$

- Promote ordinary derivatives to covariant derivatives

$$\partial_\mu \longrightarrow \nabla_\mu.$$

- Use a covariant volume element

$$\mathrm{d}^4 x = \mathrm{d}^3\mathbf{x}\ \mathrm{d}t \longrightarrow \mathrm{d}^4 x\sqrt{-g}.$$

The action resulting from this procedure describes a scalar field minimally coupled scalar field to the gravitational field

$$S = \int \sqrt{-g}\ \mathrm{d}^4 x \left[\frac{1}{2} g^{\mu\nu}\phi_{;\mu}\,\phi_{;\nu} - V(\phi)\right]. \tag{4.4}$$

By considering a flat FRW-universe given by the line element (Chapter 1)

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2 \mathrm{d}\mathbf{x}^2, \tag{4.5}$$

this action simplifies to[1]

$$S = \int \mathrm{d}^4 x\ a^3 \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2a}(\nabla\phi)^2 - V(\phi)\right]. \tag{4.6}$$

It is now straightforward to find the resulting equation of motion for the scalar field. Varying the action yields

$$\begin{aligned}
\delta S &= \int \mathrm{d}^4 x\ a^3 \left[\dot{\phi}\delta\dot{\phi} - \frac{1}{a^2}(\nabla\phi)\cdot\delta(\nabla\phi) - V'\delta\phi\right] \\
&= \int \mathrm{d}^4 x\ \left[-\frac{\mathrm{d}}{\mathrm{d}t}(a^3\dot{\phi}) + a\nabla^2\phi - a^3 V'\right]\delta\phi \\
&= \int \mathrm{d}^4 x\ a^3 \left[-\ddot{\phi} - 3H\dot{\phi} + \frac{1}{a^2}\nabla^2\phi - V'\right]\delta\phi = 0.
\end{aligned}$$

[1]For a scalar field, covariant derivatives correspond to ordinary derivatives.

We have here integrated by parts using Gauß' theorem and assumed vanishing boundary terms. Note that we also have introduced the Hubble parameter $H \equiv \frac{\dot{a}}{a}$. Now, since the variation $\delta\phi$ is arbitrary, $\delta S = 0$ is only possible if what is inside the brackets vanish. We therefore obtain the following field equation for a scalar field a in flat FRW background

$$\boxed{\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + V'(\phi) = 0.} \tag{4.7}$$

We now make some important observations. We see that the term $3H\dot{\phi}$ acts as a friction term for the oscillating field. This is due to the expansion of the universe considered. As we will see later, this friction will lead to particle production. Moreover, the gradient of ϕ becomes suppressed as a increases and can therefore often be neglected in the expanding universe. It is now very convenient to introduce conformal time, given by $\mathrm{d}\eta = \frac{1}{a}\mathrm{d}t$. This yields e.g.[2]

$$\ddot{\phi} = \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1}{a}\phi'\right) = \frac{1}{a^2}\phi'' - \frac{a'}{a^3}\phi', \tag{4.8}$$

$$H = \frac{\dot{a}}{a} = \frac{a'}{a^2} \equiv \frac{\mathcal{H}}{a}. \tag{4.9}$$

We substitute these expressions into (4.7) and obtain

$$\boxed{\phi'' + 2\mathcal{H}\phi' - \nabla^2\phi + a^2\frac{\partial V}{\partial\phi} = 0.} \tag{4.10}$$

For convenience, we also introduce the auxiliary field given by $\chi = a\phi$, i.e.

$$\phi' = \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1}{a}\chi\right) = \frac{1}{a}\chi' - \frac{a'}{a^2}\chi, \tag{4.11}$$

$$\phi'' = \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1}{a}\chi' - \frac{a'}{a^2}\chi\right) = \frac{1}{a}\chi'' - 2\frac{a'}{a^2}\chi' - \left(\frac{a''}{a^2} - 2\frac{a'^2}{a^3}\right)\chi. \tag{4.12}$$

We substitute these into (4.10) and let the potential be a mass term to obtain

$$\boxed{\chi'' - \nabla^2\chi + \left(m^2a^2 - \frac{a''}{a}\right)\chi = 0.} \tag{4.13}$$

[2] We now adapt the notation $'$ for derivatives with respect to conformal time, e.g. $\phi' = \frac{\mathrm{d}\phi}{\mathrm{d}\eta}$.

This is the usual Klein Gordon equation in Minkowski space, but now with a time-dependent effective mass

$$m_{\text{eff}}^2(\eta) = m^2 a^2 - \frac{a''}{a}. \tag{4.14}$$

The explicit time-dependence of the effective mass is a signature of the lack of energy conservation. As we see, the influence made by the gravitational field is now contained in the effective mass.

Conformal coupling and conformal symmetry The coupling to gravity given in the action (4.4) is called minimal since it is the minimally required interaction between a scalar field and the gravitational field which still remains compatible with General Relativity. More generally we could add a coupling to the Ricci Scalar Curvature $\mathcal{R}$ (or even to $R_{\mu\nu}$ or $R_{\mu\nu\rho\sigma}$) by introducing the term $-\frac{\xi}{2}\mathcal{R}\phi^2$ in the Lagrangian. This coupling is known as *conformal coupling* if ξ has the form

$$\xi(d) = \frac{1}{4}\frac{d-2}{d-1}, \tag{4.15}$$

in d dimensions. The addition of this term results in a correction of the mass of the scalar field proportional to the Ricci-scalar

$$\chi'' - \nabla^2\chi + \left[(m^2 + \mathcal{R}\xi)a^2 - \frac{a''}{a}\right]\chi = 0. \tag{4.16}$$

In Chapter 1, we found the scalar curvature in flat FRW spacetime. For $k = 0$, it becomes

$$\mathcal{R} = \frac{6}{a^2}(a\ddot{a} - \dot{a}^2) = \frac{6}{a^2}\frac{a''}{a}. \tag{4.17}$$

Inserting this yields

$$\chi'' - \nabla^2\chi + \left[m^2a^2 - 6\frac{a''}{a}\left(\frac{1}{6} - \xi\right)\right]\chi = 0. \tag{4.18}$$

Thus the conformal choice $\xi = \frac{1}{6}$ in 4 dimensions simplify the equation to

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} - \nabla^2 + m^2a^2\right]\chi = 0. \tag{4.19}$$

Let us see what happens if we apply the conformal transformation (1.25) to a massless scalar field in FRW spacetime conformally coupled to gravity. Consider the equation of motion changing as

$$\left[\Box + \frac{1}{4}\frac{d-2}{d-1}\mathcal{R}\right]\phi \mapsto \left[\tilde{\Box} + \frac{1}{4}\frac{d-2}{d-1}\tilde{\mathcal{R}}\right]\tilde{\phi}, \tag{4.20}$$

with $\tilde{\phi}(x) \equiv \Omega^{(2-d)/2}\phi(x)$. We need to know how the d'Alembertian changes under the transformation

$$\Box \mapsto \tilde{\Box} = \tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu = \frac{1}{\sqrt{-\tilde{g}}}\partial_\mu \left[\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu\right] \tag{4.21}$$

$$= \Omega^{-2}\Box + (d-2)\Omega^{-3}g^{\mu\nu}\Omega_{,\mu}\,\partial_\nu. \tag{4.22}$$

Combining this and the change of the Ricci scalar (1.27), we find

$$\left[\tilde{\Box} + \frac{1}{4}\frac{d-2}{d-1}\tilde{\mathcal{R}}\right]\tilde{\phi} = \Omega^{-(d+2)/2}\left[\Box + \frac{1}{4}\frac{d-2}{d-1}\mathcal{R}\right]\phi. \tag{4.23}$$

We therefore obtain the important result that the field equations are for a massless scalar field invariant under conformal transformations.

Energy-momentum tensor Using definition (1.10), with the help of (1.3), we find that for a scalar field in curved spacetime, the (stress-) energy-momentum tensor becomes

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}\mathcal{L}, \tag{4.24}$$

hence it coincides with the canonical energy-momentum tensor. The energy density and pressure become

$$\rho = T^{00} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi), \tag{4.25}$$

$$P = \frac{1}{3}\sum_{i=1}^{3}T^{ii} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{6}(\nabla\phi)^2 - V(\phi). \tag{4.26}$$

4.1.2 Mode expansion

We now wish to expand the auxiliary field χ into Fourier modes, i.e. we set

$$\chi(\mathbf{x},\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\chi_{\mathbf{k}}(\eta)\mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{x}}. \tag{4.27}$$

Inserting this expansion into (4.13), we see that we get a set of decoupled differential equations, one for each Fourier mode $\chi_{\mathbf{k}}(\eta)$. Introducing the time-dependent oscillation frequency

$$\omega_k(\eta) = k^2 + m^2_{\mathrm{eff}}(\eta), \tag{4.28}$$

they look like a set of time-dependent oscillators.

$$\chi''_{\mathbf{k}} + \omega_k^2(\eta)\chi_{\mathbf{k}} = 0. \tag{4.29}$$

By taking the complex conjugate of equation (4.27), using that χ is real, we obtain the following requirement for the modes

$$\chi^*_{\mathbf{k}}(\eta) = \chi_{-\mathbf{k}}(\eta). \tag{4.30}$$

The general solution can now be written in terms of isotropic mode functions

$$\chi_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{2}}\left(a_{\mathbf{k}}v^*_k(\eta) + a^\dagger_{-\mathbf{k}}v_k(\eta)\right). \tag{4.31}$$

where $\{v_k, v^*_k\}$ is a basis for the solution space of (4.29). The isotropy of the modes is a great simplification which can be made as a consequence of the isotropy of the FRW-universe. For now, the integration constants $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$ are c-numbers, so by $\dagger$ we really mean complex conjugated. Now, since the set $\{v_k, v^*_k\}$ satisfy (4.29) it has the property that the quantity

$$W(v_k, v^*_k) = v'_k v^*_k - v_k v^{*\prime}_k = 2\mathrm{i}\ \mathrm{Im}(v'v^*), \tag{4.32}$$

known as the Wronskian of $\{v_k, v^*_k\}$, is time-independent and nonzero (see Appendix). We can therefore use it to normalize the set by a proper scaling to enforce that

$$\mathrm{Im}(v'v^*) = \frac{W(v,v^*)}{2\mathrm{i}} = 1. \tag{4.33}$$

Finally, let us insert the expansion (4.31) into (4.27). The result can be written as

$$\chi_{\mathbf{k}}(\mathbf{x},\eta) = \frac{1}{\sqrt{2}}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\left(a_{\mathbf{k}}v^*_k(\eta)\mathrm{e}^{\mathrm{i}\mathbf{kx}} + a^\dagger_{\mathbf{k}}v_k(\eta)\mathrm{e}^{-\mathrm{i}\mathbf{kx}}\right). \tag{4.34}$$

4.2 Canonical Quantization of Scalar Fields

We now want to recast the classical field $\chi(\mathbf{x},\eta)$ into a quantum field-operator $\hat\chi(\mathbf{x},\eta)$. The quantization proceeds as in flat spacetime by introducing the canonically conjugated momentum $\hat\pi \equiv \hat\chi'$ and by imposing the usual equal-time commutation relations. The momentum density operator is written explicitly as

$$\hat\pi(\mathbf{x},\eta) = \frac{1}{\sqrt{2}}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\left(a_{\mathbf{k}}v^{*\prime}_k(\eta)\mathrm{e}^{\mathrm{i}\mathbf{kx}} + a^\dagger_{\mathbf{k}}v_k{}'(\eta)\mathrm{e}^{-\mathrm{i}\mathbf{kx}}\right). \tag{4.35}$$

The Hamiltonian for the quantum field is given by

$$\hat H(\eta) = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\left[\hat\pi^2 + (\nabla\hat\chi)^2 + m^2_{\mathrm{eff}}(\eta)\hat\chi^2\right], \tag{4.36}$$

while the equal-time commutation relations become

$$\begin{aligned}
&[\hat\chi(\mathbf{x},\eta), \hat\pi(\mathbf{y},\eta)] = \mathrm{i}\delta^3(\mathbf{x}-\mathbf{y}),\\
&[\hat\chi(\mathbf{x},\eta), \hat\chi(\mathbf{y},\eta)] = [\hat\pi(\mathbf{x},\eta), \hat\pi(\mathbf{y},\eta)] = 0.
\end{aligned} \tag{4.37}$$

Promoting $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{\dagger}$ to operators, it is easy to show that they must satisfy

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^{\dagger}] = \delta^3(\mathbf{k} - \mathbf{k}'), \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}_{\mathbf{k}}^{\dagger}, \hat{a}_{\mathbf{k}'}^{\dagger}] = 0. \tag{4.38}$$

We should therefore interpret them as creation and annihilation operators. Now, given a set of creation and annihilation operators $\{a_{\mathbf{k}}, a_{\mathbf{k}}^{\dagger}\}$, we can define the corresponding vacuum as

$$\hat{a}_{\mathbf{k}} |0\rangle = 0 \ \ \forall \ \mathbf{k}. \tag{4.39}$$

In the same manner, we can use creation operators to make many-particle states as

$$|_{(a)} m_{\mathbf{k_1}}, n_{\mathbf{k_2}}, \cdots\rangle = \frac{1}{\sqrt{m!n!...}} \left[\left(\hat{a}_{\mathbf{k_1}}^{\dagger} \right)^m \left(\hat{a}_{\mathbf{k_n}}^{\dagger} \right)^n \cdots \right] |_{(a)} 0\rangle \, . \tag{4.40}$$

4.3 Bogolyubov Transformations

Given a set of operators $\{\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}}^{\dagger}\}$ we can construct a basis of quantum states in a Hilbert space with an unambiguous physical meaning only if we have selected a particular set of mode functions $\{v_k, v_k^*\}$. However, when the set solves the equation

$$v_k'' + \omega_k^2(\eta) v_k = 0, \tag{4.41}$$

then so will the set given by

$$u_k(\eta) = \alpha_k v_k(\eta) + \beta_k v_k^*(\eta), \tag{4.42}$$

where α_k and β_k are time-independent complex coefficients. If we let

$$|\alpha_k|^2 - |\beta_k|^2 = 1, \tag{4.43}$$

then

$$\mathrm{Im}(u' u^*) = u_k' u_k^* - u_k u_k^{*\prime} = 1, \tag{4.44}$$

also. The mode functions are thus a priori on equal footing. We may therefore instead choose another set of creation and annihilation operators $\{\hat{b}_k, \hat{b}_k^{\dagger}\}$ corresponding to the mode functions u_k with the same commutation relations. Since the Fourier modes $\chi_{\mathbf{k}}$ should be independent of choice of basis, we have

$$a_{\mathbf{k}} v_k^*(\eta) + a_{-\mathbf{k}}^{\dagger} v_k(\eta) = b_{\mathbf{k}} u_k^*(\eta) + b_{-\mathbf{k}}^{\dagger} u_k(\eta). \tag{4.45}$$

Inserting the relation (4.42) for u_k on the right hand side yields

$$\boxed{\hat{a}_{\mathbf{k}} = \alpha_k^* \hat{b}_{\mathbf{k}} + \beta_k \hat{b}_{\mathbf{k}}^{\dagger}, \quad \hat{a}_{\mathbf{k}}^{\dagger} = \alpha_k \hat{b}_{\mathbf{k}}^{\dagger} + \beta_k^* \hat{b}_{\mathbf{k}}.} \tag{4.46}$$

The relations above are known as *Bogolyubov transformations* and α_k and β_k as the corresponding *Bogolyubov coefficients.* Different Fock spaces will have different vacua

$$\hat{a}_{\mathbf{k}} \left|_{(a)}\ 0\right\rangle = 0 \ \ \forall\ \mathbf{k}, \ \ \hat{b}_{\mathbf{k}} \left|_{(b)}\ 0\right\rangle = 0 \ \ \forall\ \mathbf{k}, \tag{4.47}$$

and in the same manner we can construct many-particle states of a- and b-particles respectively. Note that $|_{(b)}\ 0\rangle$ is a squeezed state in terms of $|_{(a)}\ 0\rangle$. To see this, we expand the b-vacuum in terms of the a-vacuum. After some algebra, one finds

$$|_{(b)}\ 0\rangle = \prod_k \frac{1}{|\alpha_k|^{1/2}} \left(\sum_{n=0}^{\infty} \left(\frac{\beta_k}{2\alpha_k} \right)^n |_{(a)}\ n_{\mathbf{k}}, n_{-\mathbf{k}}\rangle \right). \tag{4.48}$$

Hence, in general, the b-vacuum could contain a-particles and vice versa. Another way to see this is to calculate the expected number of a-particles in the b-vacuum. The number operator of b-particles is given by $\hat{N}_k^{(a)} = \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}$. Hence we find

$$\begin{aligned} \langle_{(b)}0|\hat{N}_k^{(a)}|_{(b)}0\rangle &= \langle_{(b)}0|\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}|_{(b)}0\rangle \\ &= \langle_{(b)}0| \left(\alpha_k \hat{b}_{\mathbf{k}}^\dagger + \beta_k^* \hat{b}_{\mathbf{k}} \right) \left(\alpha_k^* \hat{b}_{\mathbf{k}} + \beta_k \hat{b}_{\mathbf{k}}^\dagger \right) |_{(b)}0\rangle \\ &= |\beta_k|^2 \langle_{(b)}0|\hat{b}_{\mathbf{k}} \hat{b}_{\mathbf{k}}^\dagger|_{(b)}0\rangle \\ &= |\beta_k|^2 \delta^{(3)}(0). \end{aligned} \tag{4.49}$$

The divergent factor $\delta^{(3)}(0)$ is of the harmless type relating to the infinite volume of space. The meaningful quantities are anyway the mean number density in the $\mathbf{k}$ mode

$$n_{\mathbf{k}} = |\beta_k|^2, \tag{4.50}$$

and the mean density of all particles

$$n = \int \mathrm{d}^3\mathbf{k}\ |\beta_k|^2. \tag{4.51}$$

The total energy density becomes

$$\rho = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} |\beta_k|^2 \omega_k. \tag{4.52}$$

We see that finiteness of the energy density require $|\beta_k|^2$ to decay faster than k^{-4} for large k.

Computing the Bogolyubov coefficients To compute the Bogolyuov coefficients α_k and β_k, we need to know the mode functions $v_k(\eta)$,$u_k(\eta)$ and their first derivatives at some time η, say η_0. From equation (4.42) and its derivative evaluated at η_0, we have

$$u_k(\eta_0) = \alpha_k v_k(\eta_0) + \beta_k v_k^*(\eta_0), \tag{4.53}$$

$$u_k{}'(\eta_0) = \alpha_k v_k{}'(\eta_0) + \beta_k v_k^{*\prime}(\eta_0). \tag{4.54}$$

These two expressions can be combined to solve for α_k and β_k. We obtain

$$\alpha_k = \frac{v_k^* {u_k}' - u_k {v_k^*}'}{{v_k}' v_k^* - v_k {v_k^*}'} = \frac{v_k^* {u_k}' - u_k {v_k^*}'}{2\mathrm{i}}, \tag{4.55}$$

where we also have used the normalization condition. Similarly, we find

$$\beta_k^* = \frac{{u_k}' v_k - u_k {v_k}'}{2\mathrm{i}}. \tag{4.56}$$

We can write the expression in a more compact way by making use of the Wronskian.

$$\begin{aligned} \alpha_k &= \left.\frac{W(u_k, v_k^*)}{2\mathrm{i}}\right|_{\eta=\eta_0}, \\ \beta_k &= \left.\frac{W(v_k^*, u_k^*)}{2\mathrm{i}}\right|_{\eta=\eta_0}. \end{aligned} \tag{4.57}$$

These relations hold at any time η_0.

4.4 Ambiguity of the Vacuum

In the last section, we saw that the notion of particle number and vacuum in a curved spacetime depends on which set of mode functions we work with. All mode functions related by the transformation (4.42) stand a priori on equal footing. A natural question to ask is then which set of modes correspond best to the physical vacuum. Albeit a good question, the answer cannot be provided without specifying the quantum measurement process. As an example will an inertial (freely falling) and an accelerated observer not in general agree upon the present particle density, even in Minkowski space. However, in Minkowski space, all inertial observers will always measure the same vacuum. Even this feature is lost in a generally curved spacetime since there is no preferred coordinate system. The particle concept in Minkowski space depends on the ability to decompose the field ϕ into plane waves

$$\phi_k \sim \exp(i\mathbf{kx} - \mathrm{i}\omega_k t). \tag{4.58}$$

A localized particle with momentum k is described by a wave packet of uncertainty Δk. The momentum of the particle is well-defined as long as $\Delta k \ll k$. The natural length scale associated with the wave packet is $\lambda \sim 1/\Delta k \gg 1/k$. However, if the geometry of spacetime varies to significantly across a region $l \sim \lambda$, the plane wave-picture does not make sense anymore since they no longer represent a valid approximation to the solution of the wave equation. In this section, we will address ways to give a definition of the particle concept in curved spacetimes.

4.4.1 The instantaneous vacuum

In Minkowski space, the vacuum is defined as the lowest energy-eigenstate of the Hamiltonian. This enables us to pick out a set of mode functions $\{v_k, v_k^*\}$ that defines the vacuum unambiguously. The Hamiltonian

$$\hat{H}(\eta) = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\left[\hat{\pi}^2 + (\nabla\hat{\chi})^2 + m_{\mathrm{eff}}^2(\eta)\hat{\chi}^2\right], \tag{4.59}$$

is however explicitly time-dependent due to the effective mass and does not have time-independent eigenvectors that can serve as vacuum. Still, we can define the *instantaneous vacuum* at η_0, $|_{\eta_0}0\rangle$ as the lowest energy-state of the instantaneous Hamiltonian $\hat{H}(\eta_0)$. The procedure is to compute the expectation value $\langle_v 0|\hat{H}(\eta_0)|_v 0\rangle$ for the vacuum-state $|_v 0\rangle$ determined by arbitrarily chosen mode functions $v_k(\eta)$ and then minimize the expectation value with respect to these, or what amounts to the same, finding the lowest eigenvalue of these. We therefore insert the mode-expansion of the auxiliary field

$$\chi_{\mathbf{k}}(\mathbf{x},\eta) = \frac{1}{\sqrt{2}}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\left(a_{\mathbf{k}} v_k^*(\eta)\mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{x}} + a_{\mathbf{k}}^\dagger v_k(\eta)\mathrm{e}^{-\mathrm{i}\mathbf{k}\mathbf{x}}\right), \tag{4.60}$$

into the Hamiltonian (4.59). We can calculate the integrals over configuration space. They are of the type

$$\int \frac{\mathrm{d}^3\mathbf{x}}{(2\pi)^3}\ \mathrm{e}^{\mathrm{i}(\mathbf{k}-\mathbf{p})\mathbf{x}} = \delta^3(\mathbf{k}-\mathbf{p}). \tag{4.61}$$

The first term in (4.59) for example, becomes

$$\begin{aligned}
\frac{1}{2}\int \mathrm{d}^3\mathbf{x}\ \hat{\pi}^2 &= \frac{1}{4}\int \mathrm{d}^3\mathbf{k}\ \mathrm{d}^3\mathbf{p}\ \Big[\delta^3(\mathbf{p}+\mathbf{k})\left(v_k^{*\prime} v_p^{*\prime} a_{\mathbf{k}} a_{\mathbf{p}} + v_k{}' v_p{}' a_{\mathbf{k}}^\dagger a_{\mathbf{p}}^\dagger\right) \\
&\qquad\qquad + \delta^3(\mathbf{p}-\mathbf{k})\left(v_k^{*\prime} v_p{}' a_{\mathbf{k}} a_{\mathbf{p}}^\dagger + v_k{}' v_p^{*\prime} a_{\mathbf{k}}^\dagger a_{\mathbf{p}}\right)\Big] \\
&= \frac{1}{4}\int \mathrm{d}^3\mathbf{k}\ \left[v_k^{*2} a_{\mathbf{k}} a_{-\mathbf{k}} + v_k{}^2 a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + |v_k|^2 (a_{\mathbf{k}} a_{\mathbf{k}}^\dagger + a_{\mathbf{k}}^\dagger a_{\mathbf{k}})\right].
\end{aligned}$$

The other terms are found in the same manner. Combining all together, using $[a_{\mathbf{k}}, a_{\mathbf{p}}^\dagger] = \delta^3(\mathbf{p}-\mathbf{k})$, we obtain

$$H(\eta) = \frac{1}{4}\int \mathrm{d}^3\mathbf{k}\Big[E_k(2a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \delta^3(0)) + F_k a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + F_k{}^* a_{\mathbf{k}} a_{-\mathbf{k}}\Big], \tag{4.62}$$

where

$$E_k(\eta) = |v_k{}'|^2 + \omega_k^2(\eta)|v_k|^2, \tag{4.63}$$

$$F_k(\eta) = v_k{}'^2 + \omega_k^2(\eta) v_k{}^2. \tag{4.64}$$

Computing the energy-expectation value $\langle_v 0|\hat{H}(\eta_0)|_v 0\rangle$, we see that only one term survives,

$$\langle_v 0|\hat{H}(\eta_0)|_v 0\rangle = \frac{1}{4}\delta^3(0)\int \mathrm{d}^3\mathbf{k}\ E_k(\eta_0). \tag{4.65}$$

Again, one should not worry too much about the divergence. Instead we should talk about the energy density ϵ given by

$$\epsilon(\eta_0) = \frac{1}{4} \int \mathrm{d}^3\mathbf{k} \Big(|v_k{}'(\eta_0)|^2 + \omega_k^2(\eta_0)|v_k(\eta_0)|^2 \Big). \tag{4.66}$$

Minimizing $\epsilon(\eta_0)$ is equivalent to minimizing $E_k(\eta_0)$, while obeying the normalization condition $\mathrm{Im}(v'v^*) = 1$. We therefore make the ansatz

$$v_k = r_k \exp(\mathrm{i}\alpha_k). \tag{4.67}$$

Inserting the ansatz into the normalization condition, we obtain

$$r_k^2 \alpha_k{}' = 1. \tag{4.68}$$

Hence we find

$$E_k(\eta_0) = |v_k{}'|^2 + \omega_k^2 |v_k|^2 = {r'_k}^2 + \frac{1}{r_k^2} + \omega_k^2 r_k^2. \tag{4.69}$$

This expression is extremized if

$$\frac{\partial E_k}{\partial r_k} = \frac{\partial E_k}{\partial r_k{}'} = 0. \tag{4.70}$$

Thus we obtain

$$r_k{}'(\eta_0) = 0, \quad \text{and} \quad r_k(\eta_0) = \frac{1}{\sqrt{\omega(\eta_0)}}. \tag{4.71}$$

This is clearly a minimum if and only if $\omega_k^2 > 0$. Hence the existence of the instantaneous vacuum relies on the condition $\omega_k^2 > 0$. By inserting these expressions into the ansatz, we find the initial conditions that determine the mode functions that define the instantaneous vacuum

$$\boxed{v_k(\eta_0) = \frac{1}{\sqrt{\omega_k(\eta_0)}} \mathrm{e}^{\mathrm{i}\alpha_k(\eta_0)}, \quad v_k{}' = \mathrm{i}\omega_k(\eta_0) v_k(\eta_0)} \tag{4.72}$$

Note that we are free to choose the the phase α_k since any choice will minimize the expression.

Bogolyubov Coefficients in the Instantaneous vacuum We now set out to find the Bogolyubov coefficients relating two instantaneous vacua $|_{\eta_1}0\rangle$ and $|_{\eta_2}0\rangle$. Note that instantaneous vacuum $|_{\eta}0\rangle$ at intermediate time $\eta > \eta_0$ is a squeezed quantum state with respect to $|_{\eta_1}0\rangle$. We let $\alpha_k(\eta)$ and $\beta_k(\eta)$ denote the instantaneous Bogolyubov coefficients relating the initial vacuum $|_{\eta_1}0\rangle$ and the state $|_\eta 0\rangle$. We can express these coefficients in terms of the modes $v_k(\eta)$ using (4.57) with the condition (4.72) on u_k

$$\alpha_k(\eta) = \frac{\mathrm{i}\omega_k v_k^* - v_k^{*\prime}}{2\mathrm{i}\sqrt{\omega_k}}, \quad \beta_k(\eta) = \frac{\mathrm{i}\omega_k v_k^* + v_k^{*\prime}}{2\mathrm{i}\sqrt{\omega_k}}. \tag{4.73}$$

It is useful introduce the function $\zeta_k(\eta)$ given by

$$\zeta_k(\eta) = \frac{\beta_k^*(\eta)}{\alpha_k^*(\eta)} = -\frac{v_k{}'(\eta) - \mathrm{i}\omega_k(\eta)v_k(\eta)}{v_k{}'(\eta) + \mathrm{i}\omega_k(\eta)v_k(\eta)}. \tag{4.74}$$

Taking the derivative of $\zeta_k(\eta)$, we obtain after some algebra

$$\frac{\mathrm{d}\zeta_k}{\mathrm{d}\eta} = -2\mathrm{i}\omega_k\zeta_k + \left(1 - \zeta_k^2\right)\frac{\omega_k{}'}{2\omega_k}, \tag{4.75}$$

where we have used that the modes $v_k(\eta)$ satisfy the mode equation (4.41). From the condition that $v_k{}'(\eta_0) = \mathrm{i}\omega_k v_k(\eta_0)$, we obtain $\zeta_k(\eta_0) = 0$. It can be shown that ζ_k will be small when

$$\zeta_k(\eta) = \mathcal{O}\left(\omega'(\eta)/\omega^2(\eta)\right). \tag{4.76}$$

This is however a strong condition that is sufficient, but often not necessary. As long as ζ_k is small, we may now use time-dependent perturbation theory. As a first approximation, we set $1 - \zeta_{(1)} = 1$ on the RHS of equation (4.75). We then obtain the initial value problem

$$\frac{\mathrm{d}\zeta_{(1)}}{\mathrm{d}\eta} = -2\mathrm{i}\omega_k\zeta_{(1)} + \frac{\omega_k{}'}{2\omega_k}, \quad \zeta_{(1)}(\eta_0) = 0. \tag{4.77}$$

The solution is given in terms of the integral

$$\zeta_{(1)}(\eta) = \int_{\eta_1}^{\eta} \mathrm{d}\eta' \frac{1}{2\omega(\eta')}\frac{\mathrm{d}\omega(\eta')}{\mathrm{d}\eta'} \exp\left[-2\mathrm{i}\int_{\eta'}^{\eta} \mathrm{d}\eta''\omega(\eta'')\right]. \tag{4.78}$$

We can improve the approximation by iteration as follows

$$\frac{\mathrm{d}\zeta_{(n+1)}}{\mathrm{d}\eta} = -2\mathrm{i}\omega_k\zeta_{(n+1)} + \left(1 - \zeta_{(n)}\right)\frac{\omega_k{}'}{2\omega_k}, \quad \zeta_{(n+1)}(\eta_0) = 0, \tag{4.79}$$

which gives us the recurrence relation

$$\zeta_{(n+1)}(\eta) = \int_{\eta_1}^{\eta} \mathrm{d}\eta' \frac{\omega'(\eta')}{2\omega(\eta')}\left(1 - \zeta_{(n)}^2\right) \exp\left[-2\mathrm{i}\int_{\eta'}^{\eta} \mathrm{d}\eta''\omega(\eta'')\right]. \tag{4.80}$$

We can now write the Bogolyubov coefficients in terms of the solution as

$$\begin{aligned} \alpha_k(\eta) &= \frac{1}{\sqrt{1 - |\zeta_k(\eta)|^2}}, \\ \beta_k(\eta) &= \frac{\zeta_k^*(\eta)}{\sqrt{1 - |\zeta_k(\eta)|^2}}. \end{aligned} \tag{4.81}$$

From the earlier discussion we find that the mean number density of particles in a mode $\mathbf{k}$ is given by

$$\boxed{n_{\mathbf{k}} = |\beta_{\mathbf{k}}|^2 = \frac{|\zeta_k|^2}{1-|\zeta_k|^2}.} \tag{4.82}$$

4.4.2 The Adiabatic Vacuum

In a slowly changing spacetime, what is called the adiabatic vacuum gives sometimes a more meaningful notion of particle number than the instantaneous vacuum. The procedure is based on the WKB approximation for the solution of

$$\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\chi_k(\eta) + \omega_k^2(\eta)\chi_k(\eta) = 0. \tag{4.83}$$

For the ansatz

$$\chi_k(\eta) = \frac{1}{\sqrt{W_k(\eta)}} \exp\left[\mathrm{i}\int_{\eta_0}^{\eta} W_k(\eta)\mathrm{d}\eta\right], \tag{4.84}$$

one finds that the function $W_k(\eta)$ have to satisfy the nonlinear equation

$$W_k^2 = \omega_k^2 - \frac{1}{2}\left[\frac{W_k{}''}{W_k} - \frac{3}{2}\left(\frac{W_k{}'}{W_k}\right)^2\right]. \tag{4.85}$$

For a slowly changing spacetime, the derivative terms of this equation will be small compared to ω_k^2, so as a zeroth approximation one has

$$W_k^{(0)}(\eta) = \omega_k(\eta). \tag{4.86}$$

One can now obtain higher order estimate by iteration. To second order, one has

$$W_k^{(2)} = \omega_k\left(1 - \frac{1}{4}\frac{\omega_k{}''}{\omega_k^3} + \frac{3}{8}\frac{\omega_k{}'^2}{\omega_k^4}\right). \tag{4.87}$$

Similarly, one can obtain higher order estimates. The series is however, as in the case of the instantaneous vacuum, asymptotic and the approximation reaches a best value at some order N.

Chapter 5

Analytically Solvable Model

We have yet to see an explicit example where production of particles by a gravitational field occurs. In this chapter, we consider particle production in an exactly solvable model in 1+1 dimensional spacetime. It was first studied in [1]. In this model, the scale factor changes as

$$C(\eta) \equiv a^2(\eta) = A + B\tanh(\rho\eta), \quad A > B \geq 0, \ \rho > 0. \tag{5.1}$$

$C(\eta)$ is called the conformal scale factor and is shown in figure 5.1. A feature of

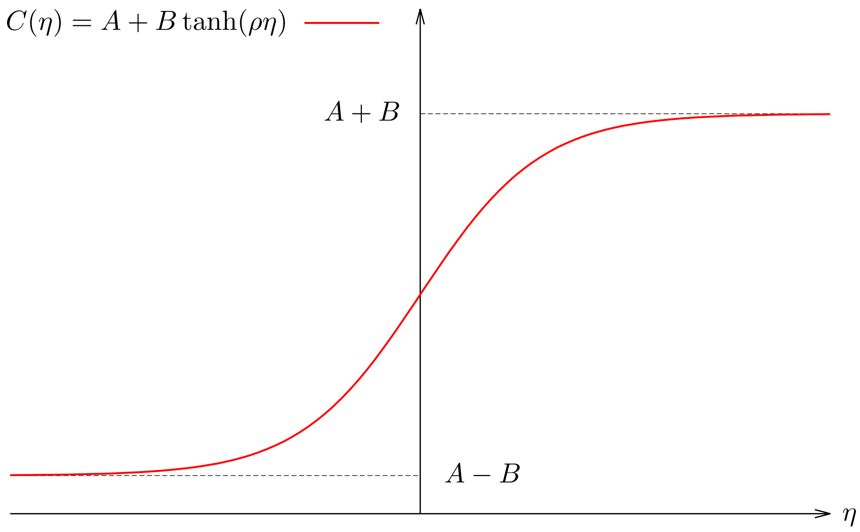

Figure 5.1: The conformal scale factor $C(\eta)$ of an asymptotically static universe undergoing a period of smooth expansion.

this model is that it approaches Minkowski space in the infinite past and future.

Observe that the amount of expansion is governed by the parameter B and the rate at which it occur by ρ. Restricting our attention to the case of a spatially flat FRW universe, the line-element becomes

$$\mathrm{d}s^2 = a^2(\eta)(\mathrm{d}\eta^2 - \mathrm{d}x^2). \tag{5.2}$$

Note that the line element written in this form is manifestly conformally equivalent to Minkowski space through the transformation $g_{\mu\nu}(x) \mapsto a^{-2}(\eta) g_{\mu\nu}(x)$. As we discussed in Chapter 4, solving the equation of motion for the scalar field is equivalent to solving the mode equations

$$\boxed{u_k^{''} + \omega_k^2(\eta) u_k = 0.} \tag{5.3}$$

We will consider the minimally coupled case, which in two dimensions correspond to $\xi = 0$, i.e.[1]

$$\omega_k^2(\eta) = k^2 + m_{\text{eff}}^2(\eta) = k^2 + m^2\left(A + B\tanh(\rho\eta)\right). \tag{5.4}$$

Before trying to solve equation (5.3), we make the following observation. As η tend to $-\infty$ the equation approaches the one of a harmonic oscillator with energy

$$\omega_{\text{in}} = \sqrt{k^2 + m^2(A - B)}. \tag{5.5}$$

Similarly as $\eta \to \infty$ the equation approaches a harmonic oscillator with energy

$$\omega_{\text{out}} = \sqrt{k^2 + m^2(A + B)}. \tag{5.6}$$

Therefore we should look for solutions of (5.3) with asymptotic modes behaving purely as positive frequency exponentials

$$u_k^{\text{in/out}}(\eta) \overset{\eta\to\mp\infty}{\longrightarrow} C_{\text{in/out}} \exp\left(-\mathrm{i}\omega_{\text{in/out}}\eta\right). \tag{5.7}$$

Moreover, by use of the normalization constant (4.33) we can already read off the integration constants $C_{\text{in/out}} = \left(2\omega_{\text{in/out}}\right)^{-\frac{1}{2}}$.

5.1 Analytical Solution

We now set out to solve equation (5.3), i.e.

$$u_k^{''}(\eta) + \left[k^2 + m^2\left(A + B\tanh(\rho\eta)\right)\right] u_k(\eta) = 0, \tag{5.8}$$

and make the substitution

$$\xi = \frac{1 + \tanh(\rho\eta)}{2}. \tag{5.9}$$

[1] In two dimensions the minimal and the conformal coupling coincides.

The new differential operators then become

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\eta} &= \frac{\mathrm{d}\xi}{\mathrm{d}\eta}\frac{\mathrm{d}}{\mathrm{d}\xi} = \frac{\rho}{2}\left(1-\tanh^2(\rho\eta)\right)\frac{\mathrm{d}}{\mathrm{d}\xi} = 2\rho\xi(1-\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}, \\
\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} &= 4\rho^2\xi^2(1-\xi)^2\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + 4\rho^2\xi(1-\xi)(1-2\xi)\frac{\mathrm{d}}{\mathrm{d}\xi},
\end{aligned} \tag{5.10}$$

while the remaining terms can be written as

$$\begin{aligned}
k^2 + m^2(A + B\tanh(\rho\eta)) &= k^2 + m^2(A + B(2\xi - 1)) \\
&= \omega_{\text{in}}^2 + 2Bm^2\xi \\
&= \omega_{\text{in}}^2 + (\omega_{\text{out}}^2 - \omega_{\text{in}}^2)\xi \\
&= \omega_{\text{in}}^2(1-\xi) + \omega_{\text{out}}^2\xi
\end{aligned} \quad . \tag{5.11}$$

Substituting the above expressions into the equation and dividing through it by the overall factor $4\rho^2\xi^2(1-\xi)^2$, we obtain a Riemann type differential equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \left(\frac{1}{\xi-1} + \frac{1}{\xi} +\right)\frac{\mathrm{d}}{\mathrm{d}\xi} + \left(\frac{\omega_{\text{out}}^2}{4\rho^2}\frac{1}{\xi-1} - \frac{\omega_{\text{in}}^2}{4\rho^2}\frac{1}{\xi}\right)\frac{1}{\xi(\xi-1)}\right]\tilde{u}(\xi) = 0. \tag{5.12}$$

The solutions are given in terms of hypergeometric functions[2]. The solution behaving as a pure frequency exponential as $\eta \to -\infty$ is given by

$$\begin{aligned}
u_k^{\text{in}}(\eta) =& \frac{1}{\sqrt{2\omega_{\text{in}}}}\exp\mathrm{i}\left(-\omega_+\eta - \frac{\omega_-}{\rho}\ln[2\cosh\rho\eta]\right) \\
\times & \quad {}_2F_1\left[\frac{\mathrm{i}\omega_-}{\rho}, 1 + \frac{\mathrm{i}\omega_-}{\rho}; 1 - \frac{\mathrm{i}\omega_{\text{in}}}{\rho}; \xi\right].
\end{aligned} \tag{5.13}$$

Similarly, the solution behaving as a pure positive frequency exponential in the asymptotic future is given by

$$\begin{aligned}
u_k^{\text{out}}(\eta) =& \frac{1}{\sqrt{2\omega_{\text{out}}}}\exp\mathrm{i}\left(-\omega_+\eta - \frac{\omega_-}{\rho}\ln[2\cosh\rho\eta]\right) \\
\times & \quad {}_2F_1\left[\frac{\mathrm{i}\omega_-}{\rho}, 1 + \frac{\mathrm{i}\omega_-}{\rho}; 1 + \frac{\mathrm{i}\omega_{\text{out}}}{\rho}; 1 - \xi\right].
\end{aligned} \tag{5.14}$$

In the solutions above, we have for simplicity also introduced

$$\omega_\pm = \frac{1}{2}\left(\omega_{\text{out}} \pm \omega_{\text{in}}\right). \tag{5.15}$$

By making use of the transformation formulas of hypergeometric functions (see Appendix), we may express the in and out solutions in terms of each other by linear transformations

$$u_k^{\text{in}}(\eta) = \alpha_k u_k^{\text{out}}(\eta) + \beta_k u_{-k}^{\text{out}*}(\eta), \tag{5.16}$$

[2]The full details can be found in the appendix.

with the coefficients

$$\alpha_k = \left(\frac{\omega_{\text{out}}}{\omega_{\text{in}}}\right)^{\frac{1}{2}} \frac{\Gamma(1 - \frac{\mathrm{i}\omega_{\text{in}}}{\rho})\Gamma(-\frac{\mathrm{i}\omega_{\text{out}}}{\rho})}{\Gamma(1 - \frac{\mathrm{i}\omega_+}{\rho})\Gamma(-\frac{\mathrm{i}\omega_+}{\rho})}, \tag{5.17}$$

$$\beta_k = \left(\frac{\omega_{\text{out}}}{\omega_{\text{in}}}\right)^{\frac{1}{2}} \frac{\Gamma(1 - \frac{\mathrm{i}\omega_{\text{in}}}{\rho})\Gamma(\frac{\mathrm{i}\omega_{\text{out}}}{\rho})}{\Gamma(1 + \frac{\mathrm{i}\omega_+}{\rho})\Gamma(\frac{\mathrm{i}\omega_+}{\rho})}. \tag{5.18}$$

5.2 Physical Interpretation

A fortunate feature of spacetimes that are asymptotically Minkowskian in the remote past and future is that there is a natural choice of the vacuum in the asymptotic limits, namely the state that is empty to all inertial observers. We denote the vacuum in the asymptotic past as $|0_{\text{in}}\rangle$. It is defined through

$$a_{\mathbf{k}}^{\text{in}} |0_{\text{in}}\rangle = 0, \quad \forall \mathbf{k}. \tag{5.19}$$

Similarly, the unambiguous vacuum state in the asymptotic future, denoted $|0_{\text{out}}\rangle$ is given by

$$a_{\mathbf{k}}^{\text{out}} |0_{\text{out}}\rangle = 0, \quad \forall \mathbf{k}. \tag{5.20}$$

The in- and out operators are as discussed in the previous chapter related by the transformations

$$a_{\mathbf{k}}^{\text{out}} = \alpha_k a_{\mathbf{k}}^{\text{in}} + \beta_k^* a_{-\mathbf{k}}^{\text{in}\dagger}, \tag{5.21}$$

$$a_{\mathbf{k}}^{\text{in}} = \alpha_k^* a_{\mathbf{k}}^{\text{out}} - \beta_k^* a_{-\mathbf{k}}^{\text{out}\dagger}. \tag{5.22}$$

Consider the quantum field sitting in the vacuum-state $|0_{\text{in}}\rangle$, defined by (5.19) in terms of the in-modes u_k^{in}. In the infinite past, the spacetime is Minkowskian. All inertial observers will see the state as empty of particles and therefore identify it as the true (physical) vacuum. In the infinite future, the quantum field remains in the state $|0_{\text{in}}\rangle$ in the Heisenberg picture, but inertial observers will no longer see it as the physical vacuum. The physical vacuum is now the state $|0_{\text{out}}\rangle$ defined in terms of the out-mode $u_{\mathbf{k}}^{\text{out}}$. Since $|0_{\text{in}}\rangle \neq |0_{\text{out}}\rangle$, all unaccelerated observers will see the state $|0_{\text{in}}\rangle$ as nonempty. In the kth mode, the expected particle number is given through the Bogoliubov coefficients relating the in- and out modes

$$n_k = |\beta_k|^2 = \frac{\sinh^2(\frac{\pi\omega_-}{\rho})}{\sinh(\frac{\pi\omega_{\text{in}}}{\rho})\sinh(\frac{\pi\omega_{\text{out}}}{\rho})}. \tag{5.23}$$

Note that in the massless limit, we have $\omega_- \to 0$ and the particle production vanish. This is an example of a more general result. No particle creation can occur from conformally invariant fields propagating in spacetimes conformal to

Minkowski space. The presence of a mass term breaks this symmetry and causes particle production. This is easy enough to understand. With $m = 0$ and the transformation $g_{\mu\nu} \mapsto a^{-2}(t)g_{\mu\nu}$, the field becomes equivalent to a free field in Minkowski space. Therefore all inertial observers will measure a state without particles. This result also generalizes to higher spin fields. As the rate of expansion is determined by the parameter ρ, we find for $\rho \to 0$

$$|\beta_k|^2 \to \mathrm{e}^{-2\pi\omega_{\mathrm{in}}/\rho}. \tag{5.24}$$

The particle number falls exponentially with a slower rate of expansion. A more precise statement is that the particle number vanish for small $\rho/\omega_{\mathrm{in}}$, corresponding to $\rho \ll k, m$. This means that production of modes with energy $\omega \gtrsim \rho$ are strongly suppressed. This is reasonable since the gravitational field must supply more energy to provide for a large rest mass or a large k.

5.3 Validity of the instantaneous vacuum

In this section, we want to test the applicability of the instantaneous vacuum prescription. The existence of the instantaneous vacuum relied on the assumption $\omega_k^2 > 0$, which is always true in our case. The relevant formulas for obtaining the particle number in the instantaneous vacuum was derived in chapter 4. We found that

$$n_{\mathbf{k}} = |\beta_{\mathbf{k}}|^2 = \frac{|\zeta_k|^2}{1 - |\zeta_k|^2}, \tag{5.25}$$

where the function ζ_k could be estimated perturbatively. To a first approximation

$$\zeta_{(1)}(\eta) = \int_{\eta_1}^{\eta} \mathrm{d}\eta' \frac{1}{2\omega(\eta')} \frac{\mathrm{d}\omega(\eta')}{\mathrm{d}\eta'} \exp\left[-2\mathrm{i}\int_{\eta'}^{\eta} \mathrm{d}\eta'' \omega(\eta'')\right]. \tag{5.26}$$

Recall that in deriving this expression, the validity of perturbation theory relied on $|\zeta_1|^2 \ll 1$. Coensequently, the approximation should be good only for

$$|\beta_{\mathbf{k}}^{(1)}|^2 = \frac{|\zeta_{(1)}|^2}{1 - |\zeta_{(1)}|^2} \ll 1. \tag{5.27}$$

Given this, we can apply the method if the function $\omega_k(\eta)$ is known as a function of η. A sufficient, but not necessary condition for $|\zeta|^2 \ll 1$, was the adiabatic condition

$$\frac{\mathrm{d}\omega_k}{\mathrm{d}\eta} \frac{1}{\omega_k^2(\eta)} \ll 1. \tag{5.28}$$

For the model we consider, we have

$$\omega_k^2(\eta) = k^2 + m^2 \left(A + B \tanh\left(\rho\eta\right)\right), \tag{5.29}$$

and the integral (5.26) must be solved numerically. We can however do the integral in the exponent analytically. We obtain

$$\int \omega(\eta)\, \mathrm{d}\eta = \frac{1}{p}\left[\omega_{\text{out}} \cdot \operatorname{arctanh}\left(\frac{\omega(\eta)}{\omega_{\text{out}}}\right) - \omega_{\text{in}} \cdot \operatorname{arctanh}\left(\frac{\omega(\eta)}{\omega_{\text{in}}}\right)\right] + C. \tag{5.30}$$

The condition (5.28) becomes

$$\frac{\mathrm{d}\omega_k}{\mathrm{d}\eta}\frac{1}{\omega_k^2(\eta)} = \frac{m^2 B\rho(1-\tanh^2(\rho\eta))}{\left[k^2+m^2\left(A+B\tanh\left(\rho\eta\right)\right)\right]^{\frac{3}{2}}} \ll 1. \tag{5.31}$$

This criterion is, as it turns out too strict. Below, we show some plots that illustrate the validity of the method. We have plotted the solution obtained with formula (5.26) and compared them to the exact expression

$$n_k = |\beta_k|^2 = \frac{\sinh^2(\frac{\pi\omega_-}{\rho})}{\sinh(\frac{\pi\omega_{\text{in}}}{\rho})\sinh(\frac{\pi\omega_{\text{out}}}{\rho})}. \tag{5.32}$$

From these plots, we see as we expected that validity of the approximation breaks down for $|\beta_{\mathbf{k}}|^2$ close to unity. We have also, naively plotted the particle number as a function of time.

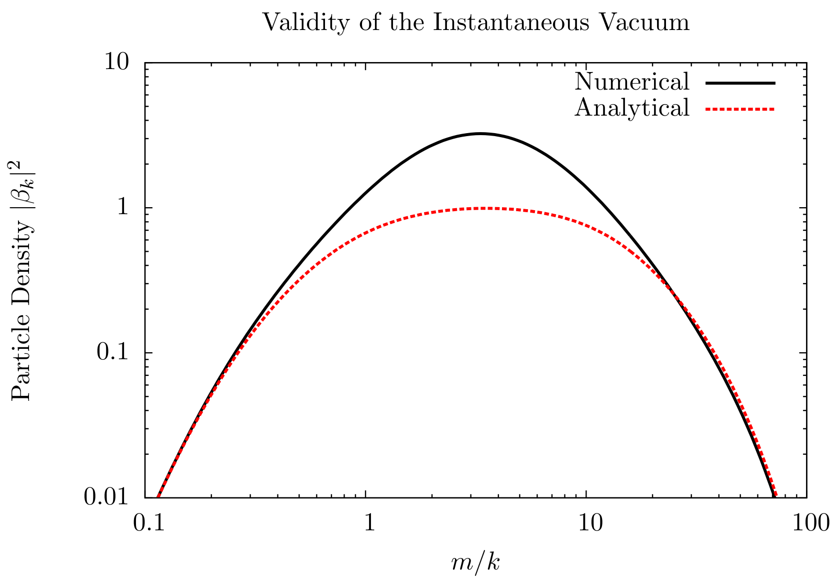

Figure 5.2: Particle density obtained with the method instantaneous vacuum as a function of mass for $k = 0.01, A = 20, B = 19, \rho = 1$.

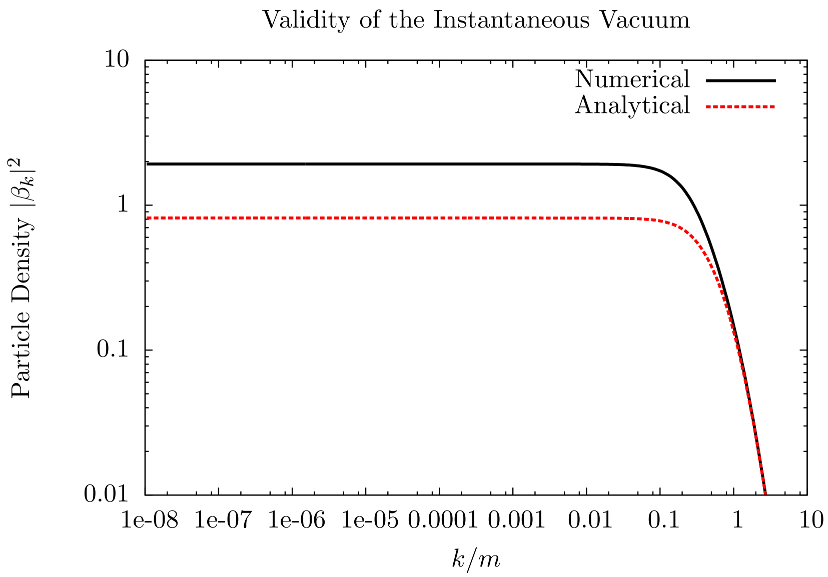

Figure 5.3: Particle density as a function of momentum for $A = 2, B = 1.85, m = 0.01, \rho = 1$.

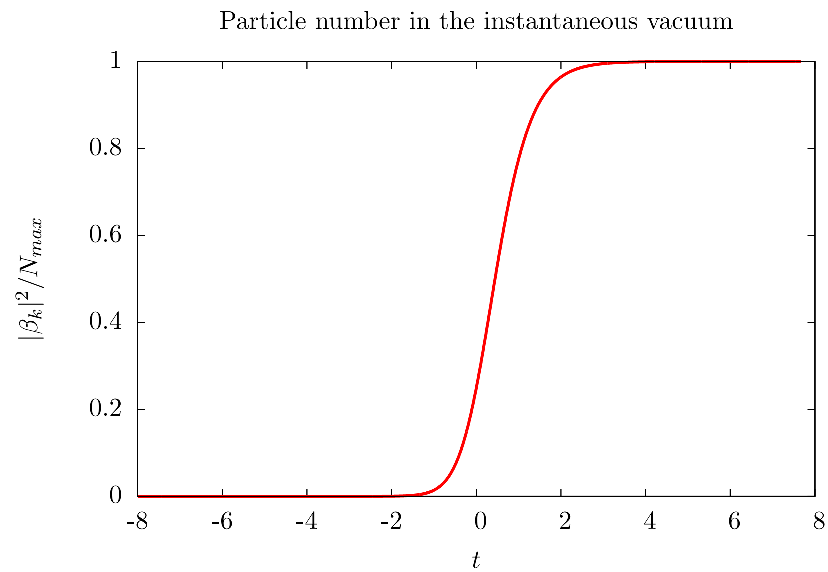

Figure 5.4: Particle density as a function of time in a k-mode. Here, $A = 20, B = 19.9, k = 0.01, p = 1, m = 0.0001$

Chapter 6

Particle Production by Inflation

We have in the last chapter seen an explicit example of particle production by gravitational fields. In this chapter, we study the phenomenon in the context of inflationary cosmology. The invention of inflationary cosmology led to a more relaxed attitude to the effects of gravitational particle production in the early universe. Although gravitational effects are very important near the Planck age $t_{\rm P} \sim 10^{-43}s$, inflation will in general dilute the energy density of relics produced exponentially small and gravitational particle production is therefore usually a very inefficient mechanism. There are however some exceptions to this rule. In many of the models extending the standard model to higher energy scales, particles with weak scale masses and Planck suppressed couplings are predicted. Most notably are the gravitino and the scalar moduli. These particles, will, as it turns out often be copiously produced in the early universe during the end of inflation. They can have catastrophic consequences destroying the predictions of nucleosynthesis. One of the goals of this chapter is to produce the results of [9].

6.1 The evolution of an inflationary universe

As discussed earlier, the basic tools for studying the dynamics of inflationary cosmology are the Friedmann equations together with the equation for a scalar field ϕ moving in a potential with its energy density being the dominating form of energy in the universe. They are for completeness repeated here in terms of the Planck mass[1] $M_p = G_N^{-1}$

$$H^2 = \frac{8\pi}{3M_{\rm p}^2}\rho = \frac{8\pi}{3M_{\rm p}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{6.1}$$

[1]The convention $8\pi G_N \equiv M_{\rm p}^2 \equiv 1$ is often used to get rid of the factor $8\pi G_N$ in the equations.

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3M_{\mathrm{p}}^2}(\rho + 3P) = -\frac{8\pi}{3M_{\mathrm{p}}^2}\left(\dot{\phi}^2 - V(\phi)\right), \tag{6.2}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0. \tag{6.3}$$

We consider in this section the evolution of the universe in the simplest model of inflation, where the inflaton potential is a power law $V \sim \phi^4$ or $V \sim \phi^2$. We first consider the model with potential $V(\phi) = \lambda\phi^4$. Realistic values for the inflaton self-coupling are in this model $\lambda \sim 10^{-13} - 10^{-14}$. Below we show the evolution of the inflaton field and of the scale factor in this model. We have taken $\lambda = 9 \cdot 10^{-14}$. The plots are shown in Planck units and we have for comparison also included the analytical solution obtained in chapter 3 from assuming that the energy density is dominated by the potential

$$\phi(t) = \phi_0 \exp\left(-\sqrt{\frac{\lambda}{6\pi}} M_{\mathrm{p}} t\right). \tag{6.4}$$

As we can see from the plots an inflationary epoch of 60 e-foldings in this model last for about 10^{-34}s to 10^{-35}s.

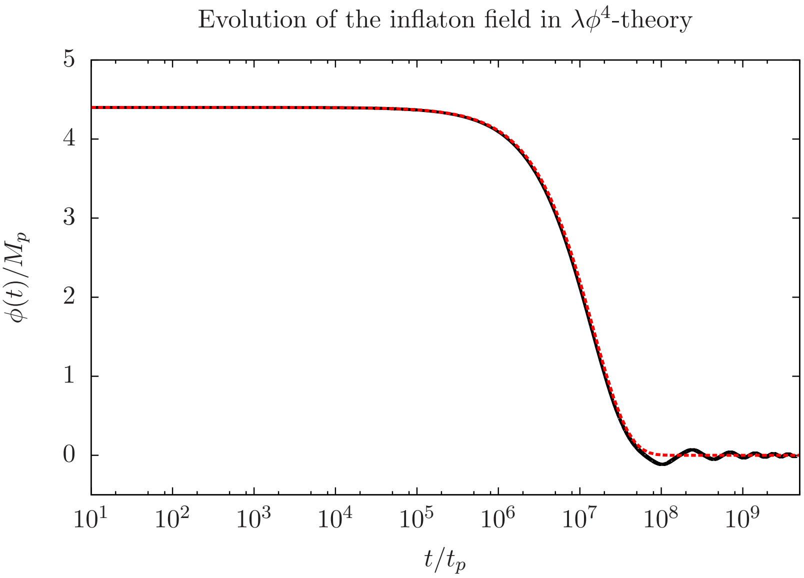

Figure 6.1: Inflaton field during inflation in a $\lambda\phi^4$-theory with 60 e-foldings of expansion. The red dashed curved shows the slow-roll solution

We can also consider a theory of inflation with the inflaton potential as a mass-term $V(\phi) = \frac{1}{2}m^2\phi^2$. A typical value for the inflaton mass is in this model $m \sim 10^{-6} M_{\mathrm{p}}$.

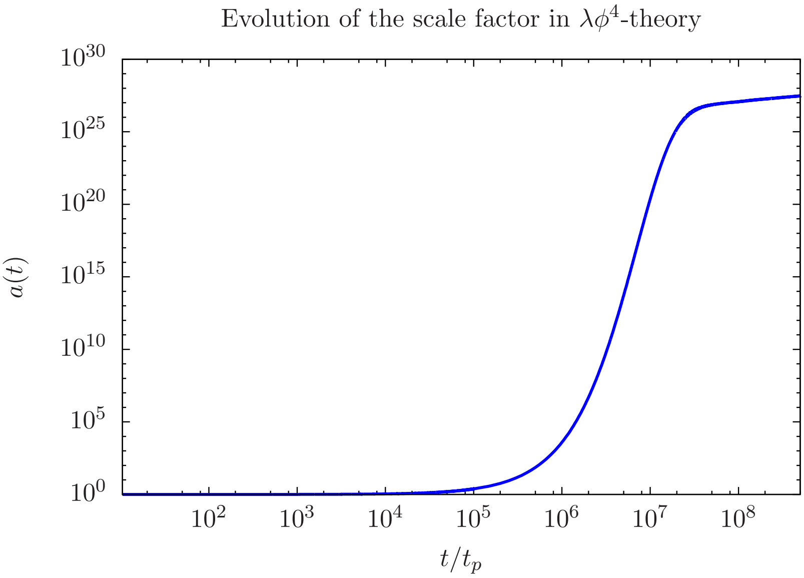

Figure 6.2: Scale factor during inflation in a $\lambda\phi^4$-theory with 60 e-foldings of expansion.

In order to achieve 60 e-folds we need to start the inflation at values $3.1M_p$ or more. As long as the energy density is dominated by the potential, the scalar field should evolve according to

$$\phi(t) = \phi_0 - \frac{M_p m}{2\sqrt{3\pi}} t. \tag{6.5}$$

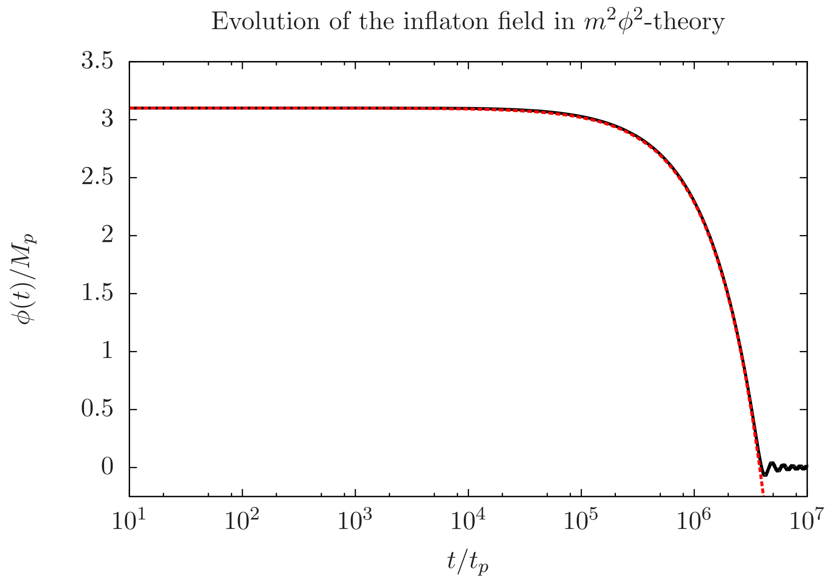

Figure 6.3: Inflaton field during inflation in a theory with $V(\phi) = \frac{1}{2}m^2\phi^2$ with 60 e-foldings of expansion. The red dashed curved shows the slow-roll solution

6.2 Generation of scalar particles

We now consider a scalar field χ with the potential

$$V(\chi) = \frac{1}{2}\left(m^2 - \xi\mathcal{R}\right)\chi^2. \tag{6.6}$$

We have earlier derived the Ricci scalar for the FRW-universe (1.33). Studying particle production during the end of a long period of expansion justifies setting $k = 0$,

$$\mathcal{R} = -\frac{6}{a^2}\left(\ddot{a}a + \dot{a}^2\right). \tag{6.7}$$

We proceed from here as before by performing a Fourier transform and introducing conformal time and field variables defined as $\mathrm{d}\eta \equiv \frac{\mathrm{d}t}{a}$,$f_k \equiv a\chi_k$. The equation for the modes then becomes

$$f_k{}'' + \omega_k^2 f_k = 0, \tag{6.8}$$

where

$$\omega_k^2 = k^2 + m^2a^2 - \frac{a''}{a}\left(1 - 6\xi\right). \tag{6.9}$$

The scale factor is determined by the evolution of the inflaton field ϕ with potential $V(\phi)$. In conformal time the motion of the inflaton is given by

$$\phi'' + 2\frac{a'}{a}\phi' + a^2\frac{\partial V(\phi)}{\partial \phi}. \tag{6.10}$$

Similarly, we can write down the Friedmann equations in conformal time. They are given by

$$a'' = \frac{{a'}^2}{a} - \frac{8\pi}{3M_\mathrm{p}^2}\left(a{\phi'}^2 - a^3V(\phi)\right), \tag{6.11}$$

$$\frac{{a'}^2}{a} = \frac{\rho a^3}{3M_\mathrm{p}^2} = \frac{1}{3M_\mathrm{p}^2}\left(\frac{a}{2}{\phi'}^2 + a^3V(\phi)\right). \tag{6.12}$$

Equation (6.10), (6.11) and (6.12) are not independent of each other, so we are free to choose our favorite two of the three equations to solve. The remaining one can be used to check energy conservation during numerical simulations.

Initial conditions for the scale factor and inflaton In our numerical simulations we normalize the scale factor as $a(t_0) = 1$. This determines the initial value of the Hubble parameter

$$H_0 = \dot{a}(t_0) = \sqrt{\frac{8\pi V(\phi_0)}{3M_\mathrm{p}^2}}. \tag{6.13}$$

Initial conditions for the modes As a first approximation, one can use as initial conditions for the mode positive frequency vacuum fluctuations

$$f_k = \frac{1}{\sqrt{2k}}\mathrm{e}^{-ikt}. \tag{6.14}$$

However, for fluctuations produced at the last stages of a long period of inflation, one should begin with the fluctuations generated from the previous stage of inflation. For a massless scalar field minimally coupled to gravity on should use Hankel functions [8]:

$$f_k(t) = \frac{\mathrm{i}a(t)H}{\sqrt{2k^3}}\left(1 + \frac{k}{\mathrm{i}H}\mathrm{e}^{-Ht}\right)\exp\left(\frac{\mathrm{i}k}{H}\mathrm{e}^{-Ht}\right), \tag{6.15}$$

where H is the Hubble parameter at the beginning of the calculation. One should also take into account that long-wavelength perturbations are produced at earlier stages of inflation when H is greater than at the beginning of the calculation. If the duration of inflation is very long then one can use $f_k = \frac{1}{\sqrt{2k}}\mathrm{e}^{-ikt}$ to a good approximation. If the stage is short, doing so will (as we will see) underestimate the amplitude $\langle\chi^2\rangle$. For de Sitter space the Hankel function solution gives

$$|f_k|^2 = \frac{a^2 H^2}{2k^3}. \tag{6.16}$$

This expression can be corrected for by using the value of the Hubble constant H_k when a mode of momentum k crossed the horizon. For the $\lambda\phi^4-$theory it can be approximated as

$$H_k = \sqrt{\frac{2\pi\lambda}{3}}\left(\phi_e^2 - \frac{1}{\pi}\ln\left(\frac{k}{H_e}\right)\right), \tag{6.17}$$

where ϕ_e and H_e are values for the inflaton and the Hubble parameter at the end of inflation.

6.2.1 Numerical results in the $\lambda\phi^4$-theory.

We first study particle production in the model with inflaton potential $V(\phi) = \frac{1}{4}\lambda\phi^4$ with $\lambda = 9\cdot 10^{-14}$. The plot (6.4) shows $k^2|f_k|^2$ as a function k. They show results from runs taken at the last stages of inflation with initial values ranging from $\phi_i = 1.5M_\mathrm{p}$ (the lowest curve) to $2M_\mathrm{p}$. This corresponds roughly to the last 10-15 efoldings of inflation. The data are taken after ten oscillations of the inflaton field. The upper curve shows the runs taken with the Hankel function solutions. The numerical simulation show that runs taken closer to then end of inflation are suppressed in the infrared part of the spectrum while they all coincide in the UV. This suppression is due to starting inflation at late time and using (6.14) as initial conditions. The plots show that long wavelength fluctuations are primarily produced during inflation. They give the main contribution to the number density. The left part of the plot show long wavelength modes produced during inflation. These modes crossed the horizon first and were therefore frozen in at a large value of the Hubble parameter and therefore with highest amplitude. The lowest modes did not however have time to cross the horizon and are therefore suppressed. The right side of the plot show modes created near or during the oscillatory stages which were not amplified a lot. The numerical simulation also show that a long period of inflation can be mimicked by a proper set of initial conditions. Instead of using the fluctuations (6.14) one could use the de Sitter space solution (6.16) with the Hubble parameter corrected to the value it had as the relevant frequency mode crossed the horizon. The simulation shows that this is equivalent to running a simulation with a long period of inflation.
The late time solution of (6.8) can be represented in terms of Bogolyubov coefficients,

$$f_k(\eta) = \frac{\alpha_k(\eta)}{\sqrt{2\omega_k}}\mathrm{e}^{-\mathrm{i}\int\omega_k \mathrm{d}\eta} + \frac{\beta_k(\eta)}{\sqrt{2\omega_k}}\mathrm{e}^{\mathrm{i}\int\omega_k \mathrm{d}\eta}, \tag{6.18}$$

where $n_k = |\beta_k|^2$ is interpreted as the particle number and $\omega_k n_k$ the energy density of a given mode. In this context however, the dominant contribution to particle

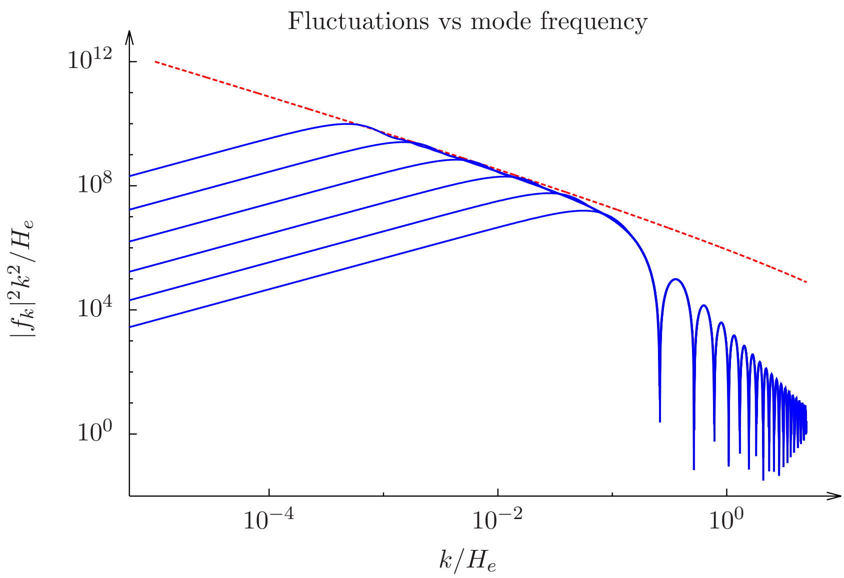

Figure 6.4: Fluctuations vs mode frequency in the $\lambda\phi^4$-theory. The plot shows run taken with initial values of the inflaton ranging from $1.5M_\mathrm{p}$ to $2.0M_\mathrm{p}$. The dashed curved show the corresponding Hankel function solutions.

production from the field χ comes from long-wavelength fluctuations that crossed the horizon before the end of inflation. These modes do not oscillate and the Bogolyubov coefficients have no clear physical meaning any longer. In turns however out that the mode amplitudes $|f_k|^2$ contain the relevant information for obtaining the particle number.

6.3 Analytical theory for particle production

The numerical simulation shows that the long-wavelength modes give the dominant contribution to particle production. One is for this reason able to study the problem analytically. The long-wavelength modes will at late times become nonrelativistic and their number density is given by

$$n_\chi = \frac{\rho_\chi}{m} = \frac{1}{2}m\chi^2. \tag{6.19}$$

The modes that remain outside horizon acts like a classical homogeneous field with amplitude

$$\langle\chi^2\rangle = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\ |\chi_k|^2 = \frac{1}{2\pi^2a^2}\int \mathrm{d}k\ k^2|f_k|^2. \tag{6.20}$$

Using the Hankel function solutions (6.15), we obtain

$$\langle\chi^2\rangle = \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3k}{k}\left(\frac{\mathrm{e}^{-2Ht}}{2}+\frac{H^2}{2k^2}\right). \tag{6.21}$$

Setting $H=0$, we see that the first term corresponds to the vacuum fluctuation contribution in Minkowski space. The second term arises because of inflation. It is shown in [8] that long-wavelength fluctuations with $m^2 \ll H^2$ in a de Sitter space lasting for a finite amount of time behave as

$$\langle\chi^2\rangle = \frac{3H^4}{8\pi^2 m^2}\left[1-\exp\left(-\frac{2m^2}{3H}t\right)\right]. \tag{6.22}$$

In the massless limit, this expression becomes

$$\langle\chi^2\rangle = \frac{H^3 t}{4\pi^2}. \tag{6.23}$$

The expression becomes more complicated in realistic models of inflation, where the Hubble parameter varies with time. In such cases the (massless) fluctuations satisfy [9]

$$\frac{\mathrm{d}\langle\chi^2\rangle}{\mathrm{d}t} = \frac{H(t)^3}{4\pi^2} \tag{6.24}$$

In the case of the $\lambda\phi^4$-theory, we have

$$H^2 = \frac{2\pi\lambda}{3M_\mathrm{p}^2}\phi^4, \tag{6.25}$$

and the differential equation reads

$$\frac{\mathrm{d}\langle\chi^2\rangle}{\mathrm{d}t} = \frac{1}{4\pi^2}\left(\frac{2\pi\lambda}{3}\right)^{3/2}\frac{\phi(t)^6}{M_\mathrm{p}^3} = \frac{1}{4\pi^2}\left(\frac{2\pi\lambda}{3}\right)^{3/2}\frac{\phi_0^6}{M_\mathrm{p}^3}\exp\left(-\sqrt{\frac{6\lambda}{\pi}}M_\mathrm{p}t\right), \tag{6.26}$$

where we have inserted the solution (6.4). Integrating up to large t, we obtain

$$\boxed{\langle\chi^2\rangle = \frac{\lambda}{18}\left(\frac{\phi_0^3}{M_\mathrm{p}^2}\right)^2.} \tag{6.27}$$

The typical amplitude is then given by

$$\chi_0 = \sqrt{\langle\chi^2\rangle} = \sqrt{\frac{\lambda}{18}}\frac{\phi_0^3}{M_\mathrm{p}^2}. \tag{6.28}$$

This shows that the amplitude depend strongly on ϕ_0.

Closing Remarks

The primary goal of this thesis was to investigate the phenomenon of particle production by gravitational fields. This is indeed an intriguing subject since it is based on using the methods of quantum field theory in curved spacetime. We only considered the most simple example, where a minimally coupled scalar field propagated in a flat FRW spacetime. We studied this analytically for an asymptotically static model. Then we investigated the validity of the instantaneous vacuum prescription, where one minimizes the Hamiltonian at a given moment of time. The particle number could then be obtained by means of perturbation theory. We found that such a prescription could be meaningful for some momentum modes, as long as the particle production in that mode was not too big. However, it turns out that the most interesting realistic cases of particle production happens during inflation, or shortly after inflation during reheating. If the particle production is very efficient, such as with light moduli fields, this leads to unacceptable cosmological consequences [9]. If the production is very inefficient, such as with superheavy particles, they can be a possible dark matter candidate [10]. In the last chapter we discussed particle production during inflation, and saw that the dominant contribution to the energy density and particle number came from long-wavelength modes produced during inflation and frozen in with high amplitudes. In such cases, the formalism of Bogolyubov coefficients was less useful since these modes were not exhibiting particle-like behavior. One could however study these in terms of the mode amplitude which contains the relevant information.

Due to the time it took to develop a feel for the subject, there was only so much we could do. In this we thesis, we have only discussed scalar fields in expanding FRW-universes. An extension to higher spin field could therefore be interesting to look at. As an example, one could study the production of gravitinos in the early universe. Moreover, we could have done the numerical calculations we did for other models of inflation.
On a personal level, one main goal of this work was to acquire the knowledge required to follow some of the ongoing research in the field. Particle production from inflation represents a fascinating idea that should be taken serious in all inflationary models. It has also given me a broader overview of cosmology in general.

Appendix A

Relevant Mathematics.

A.1 Time dependent Oscillators

We now discuss some features of the time-dependent one-dimensional oscillator

$$\ddot{x} + \omega(t)x = 0. \tag{A.1}$$

It is a second order equation and therefore has a two-dimensional solution space, that is, if $x_1(t)$ and $x_2(t)$ are two linearly independent solutions of (A.1) then every solution $x_3(t)$ can be written as a linear combination of $x_1(t)$ and $x_2(t)$. The **Wronskian** of two functions x_1 and x_2 is defined as

$$W(x_1, x_2) \equiv \dot{x}_1 x_2 - x_1 \dot{x}_2. \tag{A.2}$$

A solution $x(t)$ of equation (A.1) have a time independent wronskian. To see this note that

$$\frac{\mathrm{d}}{\mathrm{d}t} W(x, x^*) = \ddot{x}x^* - x\ddot{x}^* = -\omega(t)(xx^* - xx^*) = 0. \tag{A.3}$$

We may therefore normalize $x(t)$ such that $\mathrm{Im}(x'x^*) = \frac{W[x,x^*]}{2\mathrm{i}} = 1$.

A.2 Eulers Gamma function

The gamma-function obey the following relations

$$\Gamma(1 + x) = x\Gamma(x), \tag{A.4}$$

$$|\Gamma(\mathrm{i}y)|^2 = \frac{\pi}{y \sinh(\pi y)}, \qquad \text{when } y \text{ i real.} \tag{A.5}$$

A.3 Riemann's differential equation and Hypergeometric functions

A *hypergeometric series* is a series of the form

$$ {}_2F_1(\alpha,\beta,\gamma,z) = 1 + \frac{\alpha\cdot\beta}{\gamma\cdot 1}z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\cdot 1\cdot 2}z^2 + \cdots \tag{A.6} $$

Hypergeometric functions satisfy the following important transformation formulas

$$ F(\alpha,\beta;\gamma;z) = (1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma;z), \tag{A.7} $$

$$ \begin{aligned} F(\alpha,\beta;\gamma,z) &= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta;\alpha+\beta-\gamma;1-z) \\ &+ (1-z)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-z). \end{aligned} \tag{A.8} $$

Riemann's differential equation is given by

$$ \begin{aligned} &\frac{\mathrm{d}^2u}{\mathrm{d}z^2} + \left[\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c}\right]\frac{\mathrm{d}u}{\mathrm{d}z} \\ &\quad + \left[\frac{\alpha\alpha'(a-b)(a-c)}{z-a} + \frac{\beta\beta'(b-c)(b-a)}{z-b} \right. \\ &\quad \left. + \frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right]\frac{u}{(z-a)(z-b)(z-c)} = 0, \end{aligned} \tag{A.9} $$

and satisfies $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'-1=0$. It may be written schematically as follows

$$ u = P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}. \tag{A.10} $$

and the following transformation formulas are then valid

$$ \begin{aligned} &\left(\frac{z-a}{z-b}\right)^k \left(\frac{z-c}{z-b}\right)^l P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} \\ &= P\begin{Bmatrix} a & b & c & \\ \alpha+k & \beta+l & \gamma-k-l & z \\ \alpha'+k & \beta'+l & \gamma'-k-l & \end{Bmatrix}. \end{aligned} \tag{A.11} $$

The hypergeometric differential equation is a special case of (A.9) with

$$ u = P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{Bmatrix}. \tag{A.12} $$

A special case of (A.11) is

$$z^k(1-z)^l P\begin{Bmatrix} 0 & 1 & \infty & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P\begin{Bmatrix} 0 & 1 & \infty & \\ \alpha+k & \beta+l & \gamma-k-l & z \\ \alpha'+k & \beta'+l & \gamma'-k-l & \end{Bmatrix}. \quad \text{(A.13)}$$

Appendix B

Analytical Solution of Riemanns differential equation

We want to solve the differential equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\eta^2} + \left[k^2 + m^2\left(A + B\tanh(\rho\eta)\right)\right] u(\eta) = 0. \tag{B.1}$$

It is convenient to first make the substitution

$$\xi = \frac{1 + \tanh\left(\rho\eta\right)}{2}. \tag{B.2}$$

Under the substitution, the new differential operators then become

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\eta} &= \frac{\mathrm{d}\xi}{\mathrm{d}\eta}\frac{\mathrm{d}}{\mathrm{d}\xi} = \frac{\rho}{2}\left(1 - \tanh^2(\rho\eta)\right)\frac{\mathrm{d}}{\mathrm{d}\xi} = 2\rho\xi(1-\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}, \\ \frac{\mathrm{d}^2}{\mathrm{d}\eta^2} &= 4\rho^2\xi^2(1-\xi)^2\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + 4\rho^2\xi(1-\xi)(1-2\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}. \end{aligned} \tag{B.3}$$

Recall the definitions

$$\begin{aligned} \omega_{\mathrm{in}} &= k^2 + m^2(A - B), \\ \omega_{\mathrm{out}} &= k^2 + m^2(A + B), \\ \omega_{\pm} &= \frac{1}{2}\left(\omega_{\mathrm{out}} \pm \omega_{\mathrm{in}}\right). \end{aligned} \tag{B.4}$$

We can write the remaining terms as

$$\begin{aligned} k^2 + m^2(A + B\tanh(\rho\eta)) &= k^2 + m^2(A + B(2\xi - 1)) \\ &= \omega_{\mathrm{in}}^2 + 2Bm^2\xi \\ &= \omega_{\mathrm{in}}^2 + (\omega_{\mathrm{out}}^2 - \omega_{\mathrm{in}}^2)\xi \\ &= \omega_{\mathrm{in}}^2(1-\xi) + \omega_{\mathrm{out}}^2\xi. \end{aligned} \tag{B.5}$$

Substituting the above expressions into the equation and dividing through it by the overall factor $4\rho^2\xi^2(1-\xi)^2$, we obtain a Riemann type differential equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}+\left(\frac{1}{\xi-1}+\frac{1}{\xi}\right)\frac{\mathrm{d}}{\mathrm{d}\xi}+\left(\frac{\omega_{\mathrm{out}}^2}{4\rho^2}\frac{1}{\xi-1}-\frac{\omega_{\mathrm{in}}^2}{4\rho^2}\frac{1}{\xi}\right)\frac{1}{\xi(\xi-1)}\right]\tilde{u}(\xi)=0. \tag{B.6}$$

With

$$u=P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & \xi \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}, \tag{B.7}$$

we see that we can take e.g. $a=0$, $b=1$ and $c=\infty$. To determine the other coefficients, we have

$$1-\alpha-\alpha'=1,\quad \alpha\alpha'=\frac{\omega_{\mathrm{in}}^2}{4\rho^2}\Rightarrow\alpha=-\alpha'=\frac{\mathrm{i}\omega_{\mathrm{in}}}{2\rho}, \tag{B.8}$$

$$1-\beta-\beta'=1,\quad \beta\beta'=\frac{\omega_{\mathrm{out}}^2}{4\rho^2}\Rightarrow\beta=-\beta'=\frac{\mathrm{i}\omega_{\mathrm{out}}}{2\rho}, \tag{B.9}$$

$$1-\gamma-\gamma'=1,\quad \gamma\gamma'=0\Rightarrow\gamma=1-\gamma'=0. \tag{B.10}$$

Thus we obtain

$$u=P\begin{Bmatrix} 0 & 1 & \infty & \\ \frac{\mathrm{i}\omega_{\mathrm{in}}}{2\rho} & \frac{\mathrm{i}\omega_{\mathrm{out}}}{2\rho} & 1 & \xi \\ -\frac{\mathrm{i}\omega_{\mathrm{in}}}{2\rho} & -\frac{\mathrm{i}\omega_{\mathrm{out}}}{2\rho} & 0 & \end{Bmatrix}. \tag{B.11}$$

We want to write the solution in terms of hypergeometric functions and therefore use the transformation formula (A.13) to write it as (A.12). The result is

$$u=\xi^{\frac{\mathrm{i}\omega_{\mathrm{in}}}{2\rho}}(1-\xi)^{\frac{\mathrm{i}\omega_{\mathrm{out}}}{2\rho}}\,{}_2F_1\left[\frac{\mathrm{i}\omega_+}{\rho},1+\frac{\mathrm{i}\omega_+}{\rho};1+\frac{\mathrm{i}\omega_{\mathrm{in}}}{\rho};\xi\right]. \tag{B.12}$$

Note that

$$\xi^{\frac{\mathrm{i}\omega_{\mathrm{in}}}{2\rho}}=\left(\frac{1+\tanh(\rho\eta)}{2}\right)^{\frac{\mathrm{i}\omega_{\mathrm{in}}}{2\rho}}=\left(\frac{\mathrm{e}^{\rho\eta}}{2\cosh(\rho\eta)}\right)^{\frac{\mathrm{i}\omega_{\mathrm{in}}}{2\rho}}=e^{\frac{\mathrm{i}\omega_{\mathrm{in}}\eta}{2}}e^{-\frac{\mathrm{i}\omega_{\mathrm{in}}}{2\rho}\ln 2\cosh(\rho\eta)}, \tag{B.13}$$

and similarly

$$(1-\xi)^{\frac{\mathrm{i}\omega_{\mathrm{out}}}{2\rho}}=\left(\frac{1-\tanh(\rho\eta)}{2}\right)^{\frac{\mathrm{i}\omega_{\mathrm{in}}}{2\rho}}=e^{-\frac{\mathrm{i}\omega_{\mathrm{out}}\eta}{2}}e^{-\frac{\mathrm{i}\omega_{\mathrm{out}}}{2\rho}\ln 2\cosh(\rho\eta)}. \tag{B.14}$$

Putting this together, we can write the solution as

$$u(\eta)=\exp\mathrm{i}\left(-\omega_-\eta-\frac{\omega_+}{\rho}\ln 2\cosh(\rho\eta)\right) \tag{B.15}$$

$$\times\,{}_2F_1\left[\frac{\mathrm{i}\omega_+}{\rho},1+\frac{\mathrm{i}\omega_+}{\rho};1+\frac{\mathrm{i}\omega_{\mathrm{in}}}{\rho};\xi\right]. \tag{B.16}$$

Now, we want to write the solution in terms of two sets of independent solutions, one set behaving purely as a positiv and negative frequence exponentials as $\eta \to \infty$. We start by transforming $u(\eta)$ according to (A.7)

$$u(\eta) = \exp \mathrm{i} \left(\omega_+ \eta + \frac{\omega_-}{\rho} \ln 2 \cosh(\rho\eta) \right) \tag{B.17}$$

$$\times\, {}_2F_1 \left[-\frac{\mathrm{i}\omega_-}{\rho}, 1 - \frac{\mathrm{i}\omega_-}{\rho}; 1 + \frac{\mathrm{i}\omega_{\mathrm{in}}}{\rho}; \xi \right] \equiv u^*_{\mathrm{in}}(\eta). \tag{B.18}$$

To see that this is indeed the in-solution, we let look at $\lim_{\eta \to -\infty} u_{\mathrm{in}}(\eta)$. First, we note that

$${}_2F_1 \left[\alpha, \beta; \gamma; \frac{1 + \tanh(\rho\eta)}{2} \right] \to 1, \tag{B.19}$$

$$\ln 2 \cosh(\rho\eta) \to -\rho\eta. \tag{B.20}$$

for large negative η. Combining this we get

$$u_{\mathrm{in}} \to e^{-\mathrm{i}\omega_{\mathrm{in}}\eta} \quad \text{as } \eta \to -\infty. \tag{B.21}$$

We use the Wronskian to normalize the solution and set (as $\eta \to \infty$)

$$W[u_{\mathrm{in}}, u^*_{\mathrm{in}}] = 2\mathrm{i}. \tag{B.22}$$

From this we find

$$\begin{aligned} u_{\mathrm{in}}(\eta) = \frac{1}{\sqrt{2\omega_{\mathrm{in}}}} \exp \mathrm{i} \left(-\omega_+ \eta - \frac{\omega_-}{\rho} \ln 2 \cosh(\rho\eta) \right) \\ \times\, {}_2F_1 \left[\frac{\mathrm{i}\omega_-}{\rho}, 1 + \frac{\mathrm{i}\omega_-}{\rho}; 1 - \frac{\mathrm{i}\omega_{\mathrm{in}}}{\rho}; \xi \right]. \end{aligned} \tag{B.23}$$

The out-solution is the one that approaches $\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{out}}}$ as $\eta \to \infty$. In terms of hypergeometric functions, it is the one expanded around the regular singular point $\xi - 1$. Using formula (A.8), we find

$$\begin{aligned} {}_2F_1 \left[\frac{\mathrm{i}\omega_-}{\rho}, 1 + \frac{\mathrm{i}\omega_-}{\rho}; 1 - \frac{\mathrm{i}\omega_{\mathrm{in}}}{\rho}; \xi \right] &= A_k \, {}_2F_1 \left[\frac{\mathrm{i}\omega_-}{\rho}, 1 + \frac{\mathrm{i}\omega_-}{\rho}; 1 + \frac{\mathrm{i}\omega_{\mathrm{out}}}{\rho}; 1 - \xi \right] \\ &+ B_k (1 - \xi)^{-\frac{\mathrm{i}\omega_{\mathrm{out}}}{\rho}} \, {}_2F_1 \left[1 - \frac{\mathrm{i}\omega_+}{\rho}, -\frac{\mathrm{i}\omega_+}{\rho}; 1 - \frac{\mathrm{i}\omega_{\mathrm{out}}}{\rho}; 1 - \xi \right], \end{aligned} \tag{B.24}$$

with

$$A_k = \frac{\Gamma(1 - \frac{\mathrm{i}\omega_{\mathrm{in}}}{\rho}) \Gamma(-\frac{\mathrm{i}\omega_{\mathrm{out}}}{\rho})}{\Gamma(1 - \frac{\mathrm{i}\omega_+}{\rho}) \Gamma(-\frac{\mathrm{i}\omega_+}{\rho})}, \tag{B.25}$$

$$B_k = \frac{\Gamma(1-\frac{\mathrm{i}\omega_{\mathrm{in}}}{\rho})\Gamma(\frac{\mathrm{i}\omega_{\mathrm{out}}}{\rho})}{\Gamma(\frac{\mathrm{i}\omega_-}{\rho})\Gamma(1+\frac{\mathrm{i}\omega_-}{\rho})}. \tag{B.26}$$

Hence, we obtain

$$u_k^{\mathrm{in}}(\eta) = \alpha_k u_k^{\mathrm{out}}(\eta) + \beta_k u_{-k}^{\mathrm{out}*}(\eta), \tag{B.27}$$

with the out-solution

$$\begin{aligned} u_k^{\mathrm{out}}(\eta) =& \frac{1}{\sqrt{2\omega_{\mathrm{out}}}} \exp \mathrm{i} \left(-\omega_+ \eta - \frac{\omega_-}{\rho} \ln[2\cosh \rho\eta] \right) \\ \times & \quad {}_2F_1 \left[\frac{\mathrm{i}\omega_-}{\rho}, 1 + \frac{\mathrm{i}\omega_-}{\rho}; 1 + \frac{\mathrm{i}\omega_{\mathrm{out}}}{\rho}; 1 - \xi \right]. \end{aligned} \tag{B.28}$$

and the Bogolyubov coefficients

$$\alpha_k = \sqrt{\frac{\omega_{\mathrm{out}}}{\omega_{\mathrm{in}}}} A_k, \tag{B.29}$$

$$\beta_k = \sqrt{\frac{\omega_{\mathrm{out}}}{\omega_{\mathrm{in}}}} B_k. \tag{B.30}$$

The particle number is then found from

$$|\beta_k|^2 = \frac{\omega_{\mathrm{out}}}{\omega_{\mathrm{in}}} \frac{|\Gamma(1-\frac{\mathrm{i}\omega_{\mathrm{in}}}{\rho})|^2 |\Gamma(\frac{\mathrm{i}\omega_{\mathrm{out}}}{\rho})|^2}{|\Gamma(\frac{\mathrm{i}\omega_-}{\rho})|^2 |\Gamma(1+\frac{\mathrm{i}\omega_-}{\rho})|^2}. \tag{B.31}$$

The properties of the Γ-function then yields

$$\left| \Gamma\left(1 - \frac{\mathrm{i}\omega_{\mathrm{in}}}{\rho}\right) \right|^2 = \frac{\pi\omega_{\mathrm{in}}}{\rho \sinh(\frac{\pi\omega_{\mathrm{in}}}{\rho})}, \tag{B.32}$$

$$\left| \Gamma\left(1 + \frac{\mathrm{i}\omega_-}{\rho}\right) \right|^2 = \frac{\pi\omega_-}{\rho \sinh(\frac{\pi\omega_-}{\rho})}, \tag{B.33}$$

$$\left| \Gamma\left(\frac{\mathrm{i}\omega_{\mathrm{out}}}{\rho}\right) \right|^2 = \frac{\pi\rho}{\omega_{\mathrm{out}} \sinh(\frac{\pi\omega_{\mathrm{out}}}{\rho})}, \tag{B.34}$$

$$\left| \Gamma\left(\frac{\mathrm{i}\omega_-}{\rho}\right) \right|^2 = \frac{\pi\rho}{\omega_- \sinh(\frac{\pi\omega_-}{\rho})}. \tag{B.35}$$

Combining these yields

$$\boxed{|\beta_k|^2 = \frac{\sinh^2(\frac{\pi\omega_-}{\rho})}{\sinh(\frac{\pi\omega_{\mathrm{in}}}{\rho})\sinh(\frac{\pi\omega_{\mathrm{out}}}{\rho})}.} \tag{B.36}$$

References

[1] C. Bernard & A. Duncan: *Regularization and Renormalization of Quantum Field Theory in Curved Space-time.* Annals of physics 107, 201-221, 1977.

[2] V. F. Mukhanov & S. Winitzki: *Introduction to Quantum Fields in Classical Backgrounds*, Cambridge University Press, 2007.

[3] I. S. Gradshteyn, I. M. , A. Jeffrey: *Table of Integrals, Series and Products.* 5th ed.

[4] M.P Hobson, G. Efstathiou, A.N. Lasenby:*General Relativity*, Cambridge University Press, 2006.

[5] N.D. Birrell, P. C. W. Davies:*Quantum Fields in Curved Space*, Cambridge University Press, 1982.

[6] J. Beringer et al. :*Particle Data Group*, Phys. Rev. D86, 010001 (2012)

[7] A. Linde: *Inflationary Cosmology*, arXiv:0705.0164

[8] A. Linde: *Particle Physics and Inflationary Cosmology* , arXiv:hep-th/0503203

[9] G. Felder, L. Kofman, A. Linde : arXiv:hep-ph/9909508

[10] G. Bertone, D. Hooper, J. Silk: *Particle Dark Matter*, arXiv:hep-ph/0404175

[11] L. Kofman, A. Linde, A. A. Starobinsky: *Towards the Theory of Reheating After inflation*, arXiv:hep-ph/9704452

[12] A. Guth : arXiv:astro-ph/0404546

[13] E. W. Kolb, M. S. Turner:*The early universe*, Addison Wesley, 1990.

[14] G. Gelmini, P. Gondolo: *DM Production Mechanisms*, arXiv:1009.3690, 2010