Reliability Analysis with Metamodel Line Sampling

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Abstract

This paper presents an approach for reliability analysis of engineering structures, referred to as Metamodel Line Sampling (MLS). The approach utilizes a metamodel of the performance function, within the framework of the Line Sampling method, to reduce computational demands associated with the reliability analysis of engineering structures. Given a metamodel of the performance function, the failure probability is estimated as a product of a metamodel-based failure probability and a correction coefficient. The correction coefficient accounts for the error in the metamodel estimate of failure probability introduced by the replacement of the performance function with a metamodel. Computational efficiency and accuracy of the MLS approach are evaluated with the Kriging metamodel on analytical reliability problems and a practical reliability problem of a monopile foundation for offshore wind turbine. The MLS approach demonstrated efficient performance in low to medium-dimensional reliability problems.

Keywords: metamodel, reliability, line sampling, kriging, monopile, offshore.

1. Introduction

Reliability analysis is performed to address the inherent randomness of structural parameters and a lack of knowledge about the driving processes defining the behavior of structures. A primary interest in reliability analysis of structures is to evaluate the probability of unsafe or undesired state of the structure, i.e., failure probability, P_F. Given an n-dimensional vector of random variables affecting the performance of a structure, $\mathbf{Z} = [Z_1, ..., Z_n]^T \in \Omega$, in the variable space Ω, associated with the joint probability density function (*pdf*), $f_{\mathbf{Z}}(\mathbf{z})$, P_F is defined as:

$$P_F = P(\mathbf{Z} \in F) = \int_F f_{\mathbf{Z}}(\mathbf{z})d\mathbf{z} = \int_{\mathbb{R}^n} I_F(\mathbf{z}) f_{\mathbf{Z}}(\mathbf{z})d\mathbf{z} \quad (1)$$

where $\mathbf{z} \in \mathbb{R}^n$ denotes a realization of $\mathbf{Z}$, F is the failure domain, I_F is an indicator function such that $I_F(\mathbf{z}) = 1$ if $\mathbf{z} \in F$ and $I_F(\mathbf{z}) = 0$ otherwise. In this study, $\mathbf{Z}$ is defined as a vector of independent standard normal random variables with the joint *pdf* $\phi_{\mathbf{Z}}$, in the standard normal space Ω. In the case of a general random vector $\mathbf{X}$, composed of non-normal and dependent random variables, it is assumed that a probability preserving transformation, $\mathbf{Z} = \Theta_{\mathbf{X},\mathbf{Z}}(\mathbf{X})$ (e.g., Nataf [8]) exists. It is worth noting that the transformations to the standard normal space are often approximate and can introduce additional nonlinearities in the shape of the failure domain.

The state of a structure or an engineering system is commonly evaluated by a so-called performance function, $g(\mathbf{z})$. $g(\mathbf{z})$ plays a central role in the reliability analysis of structures, because it separates the n-dimensional variable space Ω into a safe $g(\mathbf{z}) > 0$, and an unsafe domain $\{\mathbf{z} \in F \subset \mathbb{R}^n : g(\mathbf{z}) \leq 0\}$ by the hypersurface denoted as the failure limit state $\{\mathbf{z} \in L : g(\mathbf{z}) = 0\}$. In the majority of applications $g(\mathbf{z})$ is an implicit function of the random structural parameters, $\mathbf{z}$, (e.g., finite element model). The implicit formulation of the performance function introduces constraints on the applicable mathematical tools for the evaluation of P_F, as often only pointwise evaluations of the performance function and its gradients are obtainable.

Analytical solutions of P_F are achievable only for a limited group of problems with explicit formulations of $g(\mathbf{z})$ and simple definitions of failure domains. In reliability analysis of structures, P_F is often evaluated numerically by employing optimization (e.g., First and Second Order Reliability Method) or sampling methods (e.g., Monte Carlo, Importance Sampling, Subset Simulation) [28]. Among these, the Monte Carlo (MC) method is widely used due to its straightforward implementation and robust performance [28]. The MC method is based on

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drawing N independent identically distributed (i.i.d.) samples $\mathbf{z}_i \sim \phi_{\mathbf{Z}}(\mathbf{z}); i = 1, ..., N$ and evaluating $g(\mathbf{z}_i)$ at these samples. The unbiased estimate of the failure probability, $\hat{P}_F$, is calculated as the ratio of the number of failed samples, N_F, over the total number of samples, N:

$$\hat{P}_F = \frac{1}{N}\sum_{i=1}^{N} I_F(\mathbf{z}_i) = \frac{N_F}{N} \tag{2}$$

where N_F is binomial distributed random variable, which leads to the coefficient of variation of $\hat{P}_F$, $\mathrm{CoV}(\hat{P}_F) \approx \sqrt{(1-\hat{P}_F)/(\hat{P}_F N)}$. Investigation of the $\mathrm{CoV}(\hat{P}_F)$ reveals that the $\hat{P}_F$ is independent of the dimensionality of the problem in the MC method, and that the $\mathrm{CoV}(\hat{P}_F)$ reduces with increasing N. For a small $\hat{P}_F$, a relatively large N is necessary to obtain a reasonably low $\mathrm{CoV}(\hat{P}_F)$. Large numbers of simulations of $g(\mathbf{z})$ are frequently infeasible to execute as the models used to evaluate $g(\mathbf{z})$ can be computationally demanding. Although the MC method is accurate, robust and independent of the dimensionality of the reliability problem, the method is considered to be inefficient when evaluating small P_F and/or when computationally intensive structural models are used to evaluate the performance function.

The previously mentioned inefficiency of the MC method has led to the development of various methods suited for the estimation of small P_F in probabilistic analysis of structures. The Importance Sampling (IS) method, based on the MC approach, introduces an importance *pdf* with a relatively high density over the failure domain of the variable space (e.g., [2]). By sampling the importance *pdf*, the IS method can provide $\hat{P}_F$ with reduced computational expense when compared to the MC method [28]. A series of benchmark tests conducted in [28] showed that the IS approach is applicable in low to medium-dimensional problems ($n < 100$) with efficiency and accuracy dependent on the implementation of the method. The Line Sampling (LS) method, based on the IS approach, evaluates P_F by a number of conditional one-dimensional reliability problems along an important direction, which points to the failure domain nearest to the origin of Ω [28]. Benchmark tests in [28] showed high accuracy and efficient performance of the LS method in high-dimensional problems.

An alternative method for estimating P_F in high-dimensional problems is the Subset Simulation (SS) method [1]. In the SS method, P_F is expressed as a product of a series of conditional failure probabilities corresponding to, prior to the analysis, unknown intermediate failure limits. The conditional failure probabilities can be selected to be relatively high (e.g., $P = 0.1$), requiring consequently a small number of samples to be evaluated accurately.

Reductions in computational demands associated with the reliability analysis of engineering structures can be also achieved by replacing $g(\mathbf{z})$ with a computationally less expensive metamodel $\tilde{g}(\mathbf{z})$. Metamodels are commonly built by implementing statistical learning methods [14] (e.g., Neural Networks [26], Support Vector Machines [3, 16], regression, or Kriging [9]) on a set of observations of $g(\mathbf{z})$ in the variable space. Several metamodel implementations showed high efficiency and accuracy in low to medium-dimensional problems ($n < 100$) (e.g., [9, 3]). An approach which aims at reducing computational cost commonly associated with the reliability analysis, referred to as Metamodel Line Sampling (MLS), is presented in this study. The MLS approach combines the efficiency of the LS method with a relatively low computational cost of $\tilde{g}(\mathbf{z})$ to provide reductions in computational expenses. Given $\tilde{g}(\mathbf{z})$, P_F is evaluated as a product of a metamodel-based failure probability and a correction coefficient. The correction coefficient accounts for the uncertainty in the metamodel-based failure probability, resulting from the replacement of $g(\mathbf{z})$ with $\tilde{g}(\mathbf{z})$. The performance of the MLS approach is evaluated on analytical reliability problems and a practical reliability problem of a monopile foundation for offshore wind turbine.

2. Metamodel Line Sampling

2.1. Line Sampling

LS is a method which formulates a reliability problem as a number of conditional one-dimensional reliability problems in the standard normal space [27]. The formulation of the LS method is based on the assumption that an important direction, $\boldsymbol{\alpha}$, can be approximated. $\boldsymbol{\alpha}$ points to the region of the failure domain nearest to the origin of Ω, as illustrated in Figure 1. An MC estimate of P_F is calculated based on a number of conditional one-dimensional reliability problems along $\boldsymbol{\alpha}$. The one-dimensional reliability problems are conditioned on the MC samples from the $(n-1)$-dimensional standard normal space of random variables orthogonal to $\boldsymbol{\alpha}$. Based on the set of benchmark tests [27], it is reported that the LS method has a wide range of applications in reliability analysis of structures, except for strongly nonlinear performance functions where $\boldsymbol{\alpha}$ cannot be estimated.

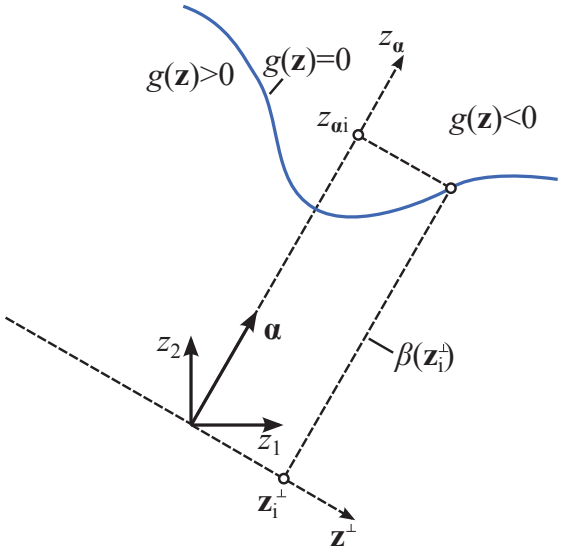

Figure 1: Line sampling method.

Given $\boldsymbol{\alpha}$, the failure domain, F, can be expressed as shown in [27]:

$$F = \left\{\mathbf{z} \in \mathbb{R}^n : z_\alpha \in F_\alpha(z_1^\perp, ..., z_{n-1}^\perp)\right\} \quad (3)$$

where z_α is a realization of the random variable, Z_α, which is defined along $\boldsymbol{\alpha}$, $\mathbf{z}^\perp \in \mathbb{R}^{n-1}$ is a realization of a vector of random variables orthogonal to $\boldsymbol{\alpha}$, denoted as $\mathbf{Z}^\perp$, while F_α is a function representing the failure domain along $\boldsymbol{\alpha}$, defined on $\mathbb{R}^{n-1}$ [27]. Then P_F can be expressed as:

$$P_F = \int_{\mathbb{R}^n} I_F(\mathbf{z})\phi_{\mathbf{Z}}(\mathbf{z})d\mathbf{z} = E_{\mathbf{z}^\perp}\left[\Phi(F_\alpha(\mathbf{z}^\perp))\right] \quad (4)$$

In the case that $F_\alpha(\mathbf{z}^\perp)$ lies within the half open interval $[\beta(\mathbf{z}^\perp), \infty)$, the one-dimensional conditional failure probability can be evaluated as $\Phi(F_\alpha(\mathbf{z}^\perp)) = \Phi(-\beta(\mathbf{z}^\perp))$, where $\beta(\mathbf{z}^\perp)$ is a 'reliability index', as indicated in Figure 1. An unbiased estimate of P_F is calculated on a set of samples $\left\{\mathbf{z}_i^\perp \sim \phi_{\mathbf{Z}^\perp}(\mathbf{z}^\perp) : i = 1, ..., N\right\}$ as:

$$\hat{P}_F = \frac{1}{N}\sum_{i=1}^{N}\Phi(F_\alpha(\mathbf{z}_i^\perp)) = \frac{1}{N}\sum_{i=1}^{N}\Phi(-\beta(\mathbf{z}_i^\perp)) = \frac{1}{N}\sum_{i=1}^{N}P_{Fi} \quad (5)$$

where $P_{Fi} = \Phi(-\beta(\mathbf{z}_i^\perp))$. Variance of the estimator $\hat{P}_F$ is estimated as:

$$\mathrm{Var}(\hat{P}_F) = \frac{1}{N(N-1)}\sum_{i=1}^{N}\left(P_{Fi} - \hat{P}_F\right)^2 \quad (6)$$

Coefficient of variation of $\hat{P}_F$, defined as $\mathrm{CoV}(\hat{P}_F) \approx \sqrt{\mathrm{Var}(\hat{P}_F)}/\hat{P}_F$, is commonly used as a convergence measure of $\hat{P}_F$.

2.2. Metamodel-Based Failure Probability

As discussed in Section 1, reliability analysis of structures can be a computationally intensive and time consuming task. One of the approaches to reduce the computational demands is to approximate $g(\mathbf{z})$ with a computationally less expensive metamodel, $\tilde{g}(\mathbf{z})$. A metamodel is commonly constructed by implementing statistical learning methods on a set of observations of $g(\mathbf{z})$ obtained with an information gathering process known as Design of Experiments (DoE) (e.g., Latin Hypercube Sampling). Some of the early metamodels employed first- and second-order polynomials to approximate the limit state in the proximity of the design point (i.e., the most probable point at the limit state) (e.g., [4]). More recent applications of metamodels are based on Neural Networks (e.g., [25]), Support Vector Machines (e.g., [3]), and Kriging metamodels (e.g., [9]).
An estimate of the metamodel failure probability, $P_{\tilde{F}}$, is obtained by coupling a metamodel with a reliability method (e.g., [3, 9]). An LS formulation of $P_{\tilde{F}}$, based on $\tilde{g}(\mathbf{z})$ in Ω, is defined as:

$$P_{\tilde{F}} = \int_{\mathbb{R}^n} I_{\tilde{F}}(\mathbf{z})\phi_{\mathbf{Z}}(\mathbf{z})d\mathbf{z} = E_{\mathbf{z}^{\tilde{\perp}}}\left[\Phi(F_{\tilde{\alpha}}(\mathbf{z}^{\tilde{\perp}}))\right] \quad (7)$$

where $I_{\tilde{F}}$ is an indicator function defined by $\tilde{g}(\mathbf{z})$, $\tilde{\boldsymbol{\alpha}}$ is an important direction determined by $\tilde{g}(\mathbf{z})$, $\mathbf{z}^{\tilde{\perp}} \in \mathbb{R}^{n-1}$ is a vector of random variables orthogonal to $\tilde{\boldsymbol{\alpha}}$, while $F_{\tilde{\alpha}}$ is the function in Eq. 3 defined with respect to $\tilde{\boldsymbol{\alpha}}$. With $g(\mathbf{z})$ replaced by $\tilde{g}(\mathbf{z})$, as illustrated in Fig. 2, $\hat{P}_{\tilde{F}}$ and $\mathrm{Var}(\hat{P}_{\tilde{F}})$ can be calculated according to Eqs. 5 and 6. In this study $\tilde{\boldsymbol{\alpha}}$ is approximated with $\boldsymbol{\alpha}$ since the goal of a metamodel is to provide a good approximation of the limit state in the proximity of the important direction. However, it is worth noting that although the metamodel might provide a good approximation of the limit state, $\tilde{\boldsymbol{\alpha}}$ does not necessarily need to be close to $\boldsymbol{\alpha}$.

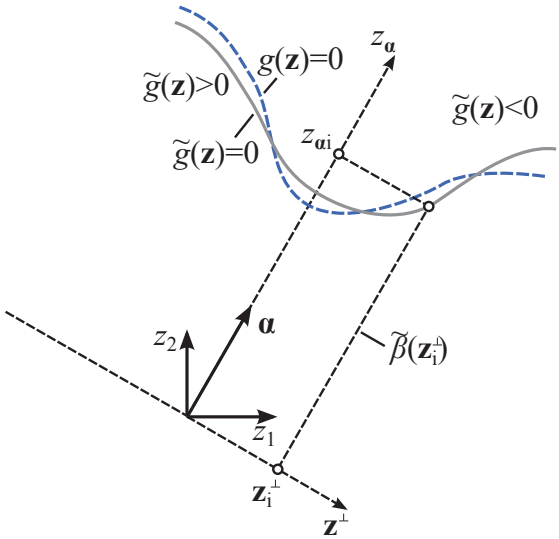

Figure 2: Metamodel-based Line Sampling.

2.3. Correction Coefficient

Although some metamodels can provide an error estimate to quantify the epistemic uncertainty associated with the metamodel (e.g., Kriging prediction variance), this measure cannot quantify the overall error resulting from replacing $g(\mathbf{z})$ with $\tilde{g}(\mathbf{z})$. This problem was recognized in [9], where a correction coefficient, defined as a ratio of P_F and $P_{\tilde{F}}$, is implemented in the Kriging-based importance sampling. The correction coefficient in [9] is estimated by sampling a ratio of the indicator function and a Kriging-based probabilistic classification function on samples from an importance sampling distribution.
In this study, a metamodel independent correction coefficient, κ, is implemented to quantify the error resulting from substituting $g(\mathbf{z})$ with $\tilde{g}(\mathbf{z})$. κ is formulated by considering the probability of a union of two events:

$$P(F \cup \tilde{F}) = P(F) + P(\tilde{F}) - P(F \cap \tilde{F}) \quad (8)$$

where F represents a failure event (domain) defined by $g(\mathbf{z})$, while $\tilde{F}$ is a failure event (domain) defined by $\tilde{g}(\mathbf{z})$. After rearranging Eq. 8, κ is formulated as a ratio of P_F and $P_{\tilde{F}}$:

$$\kappa = \frac{P(F)}{P(\tilde{F})} = \frac{P(F \cup \tilde{F})}{P(\tilde{F})} + \frac{P(F \cap \tilde{F})}{P(\tilde{F})} - 1 \quad (9)$$

κ is dependent on two terms denoted as κ_U and κ_I:

$$\kappa = \frac{1}{\kappa_U} + \kappa_I - 1 \quad (10)$$

where

$$\kappa_U = \frac{P(\tilde{F})}{P(F \cup \tilde{F})}; \quad 0 \leq \kappa_U \leq 1 \tag{11}$$

$$\kappa_I = \frac{P(F \cap \tilde{F})}{P(\tilde{F})}; \quad 0 \leq \kappa_I \leq 1 \tag{12}$$

Given a relatively accurate metamodel approximation of the limit state function it follows that $\kappa_U \approx 1$ and $\kappa_I \approx 1$, which leads to $\kappa \approx 1$. The definition of κ is not metamodel dependent which means that it can be integrated in various metamodel-based approaches for reliability analysis to quantify the error introduced by substituting $g(\mathbf{z})$ with $\tilde{g}(\mathbf{z})$.

An expression for κ_U is formulated by introducing an importance distribution $h_{F\cup\tilde{F}}(\mathbf{z})$ in the integral which defines $P(\tilde{F})$:

$$P(\tilde{F}) = \int_{\mathbb{R}^n} I_{\tilde{F}}(\mathbf{z})\phi_{\mathbf{Z}}(\mathbf{z})d\mathbf{z} = \int_{\mathbb{R}^n} \frac{I_{\tilde{F}}(\mathbf{z})\phi_{\mathbf{Z}}(\mathbf{z})}{h_{F\cup\tilde{F}}(\mathbf{z})} h_{F\cup\tilde{F}}(\mathbf{z})d\mathbf{z} \tag{13a}$$

where $h_{F\cup\tilde{F}}(\mathbf{z})$ is defined as a product of the indicator function denoting union of events F and $\tilde{F}$, $I_{F\cup\tilde{F}}(\mathbf{z})$, and the joint *pdf* $\phi_{\mathbf{Z}}(\mathbf{z})$:

$$h_{F\cup\tilde{F}}(\mathbf{z}) = \frac{I_{F\cup\tilde{F}}(\mathbf{z})\phi_{\mathbf{Z}}(\mathbf{z})}{\int_{\mathbb{R}^n} I_{F\cup\tilde{F}}(\mathbf{z})\phi_{\mathbf{Z}}(\mathbf{z})d\mathbf{z}} = \frac{I_{F\cup\tilde{F}}(\mathbf{z})\phi_{\mathbf{Z}}(\mathbf{z})}{P(F\cup\tilde{F})} \tag{13b}$$

With the $h_{F\cup\tilde{F}}(\mathbf{z})$ in Eq. 13b inserted in Eq. 13a, κ_U is defined as:

$$\kappa_U = \frac{P(\tilde{F})}{P(F\cup\tilde{F})} = \int_{\mathbb{R}^n} \frac{I_{\tilde{F}}(\mathbf{z})}{I_{F\cup\tilde{F}}(\mathbf{z})} h_{F\cup\tilde{F}}(\mathbf{z})d\mathbf{z} \tag{13c}$$

An MC estimator of κ_U can be constructed by drawing samples from $h_{F\cup\tilde{F}}(\mathbf{z})$ and evaluating the ratio of the indicator functions $I_{\tilde{F}}(\mathbf{z})$ and $I_{F\cup\tilde{F}}(\mathbf{z})$. Due to an unknown value of the normalizing constant $P(F\cup\tilde{F})$ prior to a reliability analysis, a Markov Chain Monte Carlo or a resampling method (e.g., [12]) can be used to draw samples from $h_{F\cup\tilde{F}}(\mathbf{z})$.

At the fundamental level, the problem of estimating the failure probability, $P_F = E[I_F(\mathbf{z})]$, is substituted with the problems of estimating the correction coefficients κ_U and κ_I (the estimate of κ_I will be discussed shortly). In the case of $\kappa_U = E_{h_{F\cup\tilde{F}}}[I_{\tilde{F}}(\mathbf{z})/I_{F\cup\tilde{F}}(\mathbf{z})]$, the substitution will require a smaller sample size to be evaluated with comparable accuracy if the event corresponding to κ_U is less rare than the failure event, $E_{h_{F\cup\tilde{F}}}[I_{\tilde{F}}(\mathbf{z})/I_{F\cup\tilde{F}}(\mathbf{z})] > E[I_F(\mathbf{z})]$. In the case of a relatively accurate metamodel approximation of the limit state $\kappa_U \approx 1$, which provides a potential to reduce the computational demands commonly associated with the estimation of rare failure events.

Due to the focus on the LS method, an LS estimator of κ_U is constructed in this study. Assuming that $\boldsymbol{\alpha}$ can be estimated, a marginal distribution of $h_{F\cup\tilde{F}}(\mathbf{z})$ in the $(n-1)$-dimensional standard normal space orthogonal to $\boldsymbol{\alpha}$, denoted as $h^{\perp}_{F\cup\tilde{F}}(\mathbf{z}^{\perp})$, can be defined as presented in the example of $h^{\perp}_{F}(\mathbf{z}^{\perp})$ in Appendix A. To implement an LS estimator of κ_U, Eq. 13c is formulated as:

$$\kappa_U = \int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} \frac{I_{\tilde{F}}(\mathbf{z})}{I_{F\cup\tilde{F}}(\mathbf{z})} h_{F_\alpha\cup\tilde{F}_\alpha}(z_\alpha)dz_\alpha \right] h^{\perp}_{F\cup\tilde{F}}(\mathbf{z}^{\perp})d\mathbf{z}^{\perp} \tag{13d}$$

where $h_{F_\alpha\cup\tilde{F}_\alpha}(z_\alpha)$ is a distribution of random variables along $\boldsymbol{\alpha}$ conditioned on $F\cup\tilde{F}$. For a given $\left\{\mathbf{z}^{\perp}_j \sim h^{\perp}_{F\cup\tilde{F}}(\mathbf{z}^{\perp}) : j = 1, ..., N_U\right\}$, the one-dimensional integral in Eq. 13d is rewritten as:

$$\kappa_U(\mathbf{z}^{\perp}_j) = \kappa_{Uj} = \int_{\mathbb{R}} \frac{I_{\tilde{F}}(z_\alpha, \mathbf{z}^{\perp}_j)}{I_{F\cup\tilde{F}}(z_\alpha, \mathbf{z}^{\perp}_j)} h_{F_\alpha\cup\tilde{F}_\alpha}(z_\alpha)dz_\alpha \tag{13e}$$

Based on a set of samples $\left\{z_{\alpha k} \sim h^{\perp}_{F_\alpha\cup\tilde{F}_\alpha}(z_\alpha) : k = 1, ..., N_\alpha\right\}$ an MC estimator of κ_{Uj} can be defined as:

$$\hat{\kappa}_{Uj} = \frac{1}{N_\alpha} \sum_{k=1}^{N_\alpha} \frac{I_{\tilde{F}}(z_{\alpha k}, \mathbf{z}^{\perp}_j)}{I_{F\cup\tilde{F}}(z_{\alpha k}, \mathbf{z}^{\perp}_j)} \tag{13f}$$

In the case that the failure domains, $F_\alpha(\mathbf{z}^{\perp})$ and $\tilde{F}_\alpha(\mathbf{z}^{\perp})$, are specified with the intervals $[\beta(\mathbf{z}^{\perp}), \infty)$ and $\left[\tilde{\beta}(\mathbf{z}^{\perp}), \infty\right)$ respectively, an LS estimator of $\kappa_U(\mathbf{z}^{\perp}_j)$ is defined:

$$\hat{\kappa}_{Uj} = \frac{\Phi(-\tilde{\beta}(\mathbf{z}^{\perp}_j))}{\Phi\left(-\min\left[\tilde{\beta}(\mathbf{z}^{\perp}_j), \beta(\mathbf{z}^{\perp}_j)\right]\right)} \tag{13g}$$

An estimate of κ_U is then calculated by solving the following $(n-1)$-dimensional integral:

$$\kappa_U = \int_{\mathbb{R}^{n-1}} \kappa_U(\mathbf{z}^{\perp}) h^{\perp}_{F\cup\tilde{F}}(\mathbf{z}^{\perp})d\mathbf{z}^{\perp} \tag{13h}$$

A self-weighted importance sampling estimate of κ_U is implemented to utilize the relation between $h^{\perp}_{F\cup\tilde{F}}(\mathbf{z}^{\perp})$ and the $(n-1)$-dimensional standard normal distribution orthogonal to $\boldsymbol{\alpha}$, $\phi_{\mathbf{Z}^{\perp}}(\mathbf{z}^{\perp})$ as shown in the example of $h^{\perp}_{F}(\mathbf{z}^{\perp})$ in Appendix A. Given a set of samples $\left\{\mathbf{z}^{\perp}_j \sim \phi_{\mathbf{Z}^{\perp}}(\mathbf{z}^{\perp}) : j = 1, ..., N_U\right\}$, a set of weights, $w_{F\cup\tilde{F}}(\mathbf{z}^{\perp})$ can be calculated as presented in the example of $h^{\perp}_{F}(\mathbf{z}^{\perp})$ in Appendix A. A self-weighted importance sampling estimate of κ_U is calculated as a weighted average:

$$\hat{\kappa}_U = \sum_{j=1}^{N_U} w_{F\cup\tilde{F}}(\mathbf{z}^{\perp}_j)\hat{\kappa}_{Uj} \tag{13i}$$

Variance of $\hat{\kappa}_U$ is estimated as:

$$\mathrm{Var}(\hat{\kappa}_U) = \sum_{j=1}^{N_U} w^2_{F\cup\tilde{F}}(\mathbf{z}^{\perp}_j) \cdot \left(\hat{\kappa}_{Uj} - \hat{\kappa}_U\right)^2 \tag{13j}$$

Similar to κ_U, an expression for κ_I is formulated by introducing an importance distribution $h_{\tilde{F}}(\mathbf{z})$ into the integral defining $P(F\cap\tilde{F})$:

$$P(F\cap\tilde{F}) = \int_{\mathbb{R}^n} I_{F\cap\tilde{F}}(\mathbf{z})\phi_{\mathbf{Z}}(\mathbf{z})d\mathbf{z} = \int_{\mathbb{R}^n} \frac{I_{F\cap\tilde{F}}(\mathbf{z})\phi_{\mathbf{Z}}(\mathbf{z})}{h_{\tilde{F}}(\mathbf{z})} h_{\tilde{F}}(\mathbf{z})d\mathbf{z} \tag{14a}$$

where $h_{\tilde{F}}(\mathbf{z})$ is defined as a product of the indicator function denoting the event $\tilde{F}$, $I_{\tilde{F}}$, and the joint *pdf* $\phi_{\mathbf{Z}}(\mathbf{z})$:

$$h_{\tilde{F}}(\mathbf{z}) = \frac{I_{\tilde{F}}\phi_{\mathbf{Z}}(\mathbf{z})}{\int_{\mathbb{R}^n} I_{\tilde{F}}(\mathbf{z})\phi_{\mathbf{Z}}(\mathbf{z})d\mathbf{z}} = \frac{I_{\tilde{F}}\phi_{\mathbf{Z}}(\mathbf{z})}{P(\tilde{F})} \tag{14b}$$

With the $h_{\tilde{F}}(\mathbf{z})$ in Eq. 14b inserted in Eq. 14a, κ_I is defined as:

$$\kappa_I = \frac{P(F \cap \tilde{F})}{P(\tilde{F})} = \int_{\mathbb{R}^n} \frac{I_{F\cap\tilde{F}}(\mathbf{z})}{I_{\tilde{F}}(\mathbf{z})} h_{\tilde{F}}(\mathbf{z})d\mathbf{z} \tag{14c}$$

As discussed earlier, the problem of estimating the failure probability, $P_F = E[I_F(\mathbf{z})]$, is substituted with the problems of estimating the correction coefficients κ_U and κ_I. In the case of $\kappa_I = E_{h_{\tilde{F}}}[I_{F\cap\tilde{F}}(\mathbf{z})/I_{\tilde{F}}(\mathbf{z})]$, the substitution will require a lower smaller sample size to be evaluated with comparable accuracy if the event corresponding to κ_I is less rare than the failure event, $E_{h_{\tilde{F}}}[I_{F\cap\tilde{F}}(\mathbf{z})/I_{\tilde{F}}(\mathbf{z})] \geq E[I_F(\mathbf{z})]$. In the case of a relatively accurate metamodel approximation of the limit state $\kappa_I \approx 1$, which provides a potential to reduce the computational demands commonly associated with the estimation of rare failure events.
To implement an LS estimator of κ_I, Eq. 14c is formulated as:

$$\kappa_I = \int_{\mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} \frac{I_{F\cap\tilde{F}}(\mathbf{z})}{I_{\tilde{F}}(\mathbf{z})} h_{\tilde{F}_{\boldsymbol{\alpha}}}(z_{\boldsymbol{\alpha}})dz_{\boldsymbol{\alpha}} \right] h^{\perp}_{\tilde{F}}(\mathbf{z}^{\perp})d\mathbf{z}^{\perp} \tag{14d}$$

where $h_{\tilde{F}_{\boldsymbol{\alpha}}}(z_{\boldsymbol{\alpha}})$ is a distribution of random variables along $\boldsymbol{\alpha}$ conditioned on $\tilde{F}$, while $h^{\perp}_{\tilde{F}}(\mathbf{z}^{\perp})$ is a marginal distribution of $h_{\tilde{F}}(\mathbf{z})$ in the $(n-1)$-dimensional space orthogonal to $\boldsymbol{\alpha}$. For a given $\left\{\mathbf{z}^{\perp}_j \sim h^{\perp}_{\tilde{F}}(\mathbf{z}^{\perp}) : j = 1, ..., N_I\right\}$, the one-dimensional integral in Eq. 14d is rewritten as:

$$\kappa_I(\mathbf{z}^{\perp}_j) = \kappa_{Ij} = \int_{\mathbb{R}} \frac{I_{F\cap\tilde{F}}(z_{\boldsymbol{\alpha}}, \mathbf{z}^{\perp}_j)}{I_{\tilde{F}}(z_{\boldsymbol{\alpha}}, \mathbf{z}^{\perp}_j)} h_{\tilde{F}_{\boldsymbol{\alpha}}}(z_{\boldsymbol{\alpha}})dz_{\boldsymbol{\alpha}} \tag{14e}$$

Based on a set of samples $\left\{z_{\boldsymbol{\alpha}k} \sim h^{\perp}_{\tilde{F}_{\boldsymbol{\alpha}}}(z_{\boldsymbol{\alpha}}) : k = 1, ..., N_{\boldsymbol{\alpha}}\right\}$ an MC estimator of κ_{Ij} can be defined as:

$$\hat{\kappa}_{Ij} = \frac{1}{N_{\boldsymbol{\alpha}}} \sum_{k=1}^{N_{\boldsymbol{\alpha}}} \frac{I_{F\cap\tilde{F}}(z_{\boldsymbol{\alpha}k}, \mathbf{z}^{\perp}_j)}{I_{\tilde{F}}(z_{\boldsymbol{\alpha}k}, \mathbf{z}^{\perp}_j)} \tag{14f}$$

In the case that the failure domains, $F_{\boldsymbol{\alpha}}(\mathbf{z}^{\perp})$ and $\tilde{F}_{\boldsymbol{\alpha}}(\mathbf{z}^{\perp})$, are specified with the intervals $[\beta(\mathbf{z}^{\perp}), \infty)$ and $\left[\tilde{\beta}(\mathbf{z}^{\perp}), \infty\right)$ respectively, an LS estimator of $\kappa_I(\mathbf{z}^{\perp}_j)$ can be defined as:

$$\hat{\kappa}_{Ij} = \frac{\Phi\left(-\max\left[\tilde{\beta}(\mathbf{z}^{\perp}_j), \beta(\mathbf{z}^{\perp}_j)\right]\right)}{\Phi(-\tilde{\beta}(\mathbf{z}^{\perp}_j))} \tag{14g}$$

An estimate of κ_I is then calculated by solving an $(n-1)$-dimensional integral:

$$\kappa_I = \int_{\mathbb{R}^{n-1}} \kappa_I(\mathbf{z}^{\perp}) h^{\perp}_{\tilde{F}}(\mathbf{z}^{\perp})d\mathbf{z}^{\perp} \tag{14h}$$

A self-weighted importance sampling estimate of κ_I is developed to utilize the relation between $h^{\perp}_{\tilde{F}}(\mathbf{z}^{\perp})$ and the $(n-1)$-dimensional standard normal distribution orthogonal to $\boldsymbol{\alpha}$, $\phi_{\mathbf{Z}^{\perp}}(\mathbf{z}^{\perp})$. Given a set of samples $\left\{\mathbf{z}^{\perp}_j \sim \phi_{\mathbf{Z}^{\perp}}(\mathbf{z}^{\perp}) : j = 1, ..., N_I\right\}$, a set of weights $w_{\tilde{F}}(\mathbf{z}^{\perp})$ can be calculated as presented in the example of $h^{\perp}_{\tilde{F}}(\mathbf{z}^{\perp})$ in Appendix A. A self-weighted importance sampling estimate of κ_I is calculated as a weighted average:

$$\hat{\kappa}_I = \sum_{j=1}^{N_I} w_{\tilde{F}}(\mathbf{z}^{\perp}_j)\hat{\kappa}_{Ij} \tag{14i}$$

Variance of $\hat{\kappa}_I$ is estimated as:

$$\mathrm{Var}(\hat{\kappa}_I) = \sum_{j=1}^{N_I} w^2_{\tilde{F}}(\mathbf{z}^{\perp}_j) \cdot \left(\hat{\kappa}_{Ij} - \hat{\kappa}_I\right)^2 \tag{14j}$$

Given $\hat{\kappa}_U$ and $\hat{\kappa}_I$ and their respective variances, an approximate value of $\hat{\kappa}$ and $\mathrm{Var}(\hat{\kappa})$ can be obtained as shown in Appendix B. A first-order approximation of $\hat{\kappa}$, which corresponds to substituting κ_U and κ_I in Eq. 10 with their estimates, $\hat{\kappa}_U$ and $\hat{\kappa}_I$, is shown in Appendix B to be asymptotically unbiased and consistent estimator. Provided with the estimate of the bias in Eq. B.4, a bias-correction term can be added to the first-order approximation of $\hat{\kappa}$ as follows:

$$\hat{\kappa} \approx \frac{1}{\hat{\kappa}_U} + \hat{\kappa}_I - 1 - \frac{\mathrm{Var}(\hat{\kappa}_U)}{\hat{\kappa}^3_U} \tag{15}$$

The estimator in Eq. 15 remains approximate since the bias-correction term, $\mathrm{Var}(\hat{\kappa}_U)/\hat{\kappa}^3_U$, is an approximation of the bias. Additionally, $\hat{\kappa}$ in Eq. 15 remains to be an asymptotically unbiased and consistent estimator. As shown in Appendix B, the variance of the estimator in Eq. 15 is evaluated as follows:

$$\mathrm{Var}(\hat{\kappa}) \approx \frac{\mathrm{Var}(\hat{\kappa}_U)}{\hat{\kappa}^4_U} + \mathrm{Var}(\hat{\kappa}_I) \tag{16}$$

The coefficient of variation of $\hat{\kappa}$, $\mathrm{CoV}(\hat{\kappa}) \approx \sqrt{\mathrm{Var}(\hat{\kappa})}/\hat{\kappa}$ is used as a measure of convergence of $\hat{\kappa}$ in this study.

2.4. MLS Failure Probability

The estimate of P_F with the corresponding variance is derived based on an unbiased estimate of the metamodel-based failure probability, $\hat{P}_{\tilde{F}}$ and asymptotically unbiased estimate of the correction coefficient $\hat{\kappa}$. The estimates $\hat{P}_{\tilde{F}}$ and $\hat{\kappa}$ are independent as they are evaluated on samples from different distributions. The estimate of P_F then becomes:

$$\hat{P}_F = \hat{P}_{\tilde{F}} \cdot \hat{\kappa} \tag{17}$$

Given that $\hat{P}_{\tilde{F}}$ is an unbiased estimator and $\hat{\kappa}$ is asymptotically unbiased estimator, $\hat{P}_F$ is an asymptotically unbiased estimator. Expressions for the corresponding variance, $\mathrm{Var}(\hat{P}_F)$, and coefficient of variation, $\mathrm{CoV}(\hat{P}_F)$ are derived in Appendix C. Since the values of $\hat{P}_{\tilde{F}}$ and $\hat{\kappa}$ are usually calculated with relatively low CoVs (e.g., < 0.1), the $\mathrm{CoV}(\hat{P}_F)$ can be relatively

accurately approximated as shown in Appendix C:

$$\mathrm{CoV}(\hat{P}_F) \approx \sqrt{\mathrm{CoV}^2(\hat{P}_{\tilde{F}}) + \mathrm{CoV}^2(\hat{\kappa})} \tag{18}$$

3. Metamodel

3.1. Design of Experiments

A metamodel, $\tilde{g}(\mathbf{z})$, is constructed to approximate the functional relationship between $\mathbf{z}$ and $g(\mathbf{z})$. In reliability analysis of structures $\tilde{g}(\mathbf{z})$ is implemented to classify a combination of random structural parameters as a safe or failure state of the structure with lower computational expenses than $g(\mathbf{z})$. A metamodel can be built by implementing interpolation, regression or classification methods from the group of statistical learning methods [14] on a set of pointwise evaluations of $g(\mathbf{z})$:

$$\Gamma = \left\{\left(\mathbf{z}_j, g(\mathbf{z}_j)\right), j = 1, .., S\right\} \tag{19}$$

In reliability analysis of structures Γ is obtained through a Design of Experiments (DoE), commonly initiated by a space-filling DoE and updated by a model-based or an adaptive DoE. The initial space-filling DoE attempts to gather the largest amount of information to construct a metamodel by filling Ω. Common approaches for the space-filling DoE include Monte Carlo sampling, Latin Hypercube Sampling [21], Full factorial design [22], or K-means clustering [9].

On the other hand, model-based or adaptive DoE utilizes information provided by a model (e.g., structural model, metamodel) to construct an optimal DoE. For example, several adaptive Kriging-based DoEs were formulated on the information provided by the Kriging variance (e.g., [9, 17]).

A model-based DoE, adapted to the LS method, is implemented in this study. Motivation for an LS based DoE (LS-DoE) stems from the requirement that an efficient implementation of a metamodel for reliability analysis requires accurate approximation of $g(\mathbf{z})$ around the limit state with a relatively low number of observations. The limitation in the number of observations is commonly encountered due to computationally demanding structural models, employed to generate observations, and a decrease in efficiency of certain metamodels (e.g., Kriging) with an increase in the number of observations. For these reasons, the LS-DoE is formulated as a sampling strategy to select realizations of random parameters from the variable space in the proximity of the limit state. The sampling strategy for the LS-DoE is derived by expanding the concept of the limit state with 'artificial uncertainty'. In the expanded formulation, the position of the limit state is assumed to be uncertain and modeled with a pseudo random variable $\epsilon \sim f_\epsilon(\epsilon)$:

$$g'(\mathbf{z}) = g(\mathbf{z}) + \epsilon \tag{20}$$

Similar to the LS method, the LS-DoE conducts a series of line searches along $\boldsymbol{\alpha}$ to locate the limit state defined by $g'(\mathbf{z})$ as illustrated in Figure 3. The set of observations obtained with the LS-DoE is specified as:

$$\Gamma = \left\{\left(\mathbf{z}_j, g(\mathbf{z}_j)\right) : g'(\mathbf{z}_j) = 0, g(\mathbf{z}_j) = -\epsilon_j, j = 1, .., S\right\} \tag{21}$$

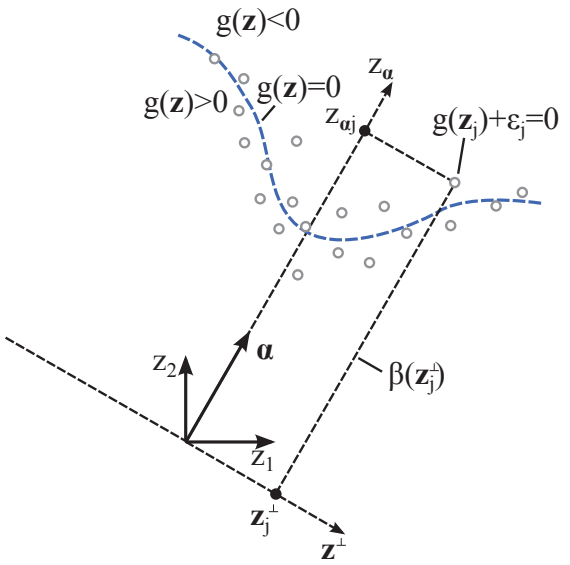

Figure 3: Illustration of the LS-DoE.

ϵ can be selected as a zero-mean random variable with the standard deviation on the magnitude of $g(\mathbf{z})$ at the mean of Ω. This enables the LS-DoE to focus observations of $g(\mathbf{z})$ on the region extending from around the limit state to the mean of Ω. The LS-DoE can be adaptively updated during the metamodel training by modifying the standard deviation of ϵ to provide a varying resolution of observations around the limit state. For example, one can stepwise reduce the standard deviation of ϵ to adaptively increase the number of observations in the proximity of the limit state. It is expected that the LS-DoE can provide a comparable information for an approximation of the limit state with a lower number of observations than a space-filling DoE due to its focus on observations in the proximity of the limit state.

3.2. Kriging Metamodel

In this study, a Kriging predictor is implemented as a metamodel due to several efficient applications in the field of reliability analysis (e.g., [9]). A relatively robust and numerically efficient DACE Matlab^{TM} Kriging toolbox [5] is employed to estimate Kriging parameters and to predict values at unobserved locations.

In the Kriging predictor, the set of pointwise observations of $g(\mathbf{z})$ in Eq. 21 is interpreted as a realization of a Gaussian process (e.g., [9]):

$$g(\mathbf{z}) = \mathbf{q}(\mathbf{z})^T\boldsymbol{\eta} + U(\mathbf{z}) \tag{22}$$

which can be decomposed into a regression model defined by a set of known functions $\mathbf{q} = \{q_r, r = 1, ..., R\}$ and unknown coefficients $\boldsymbol{\eta}$, and a zero-mean stationary Gaussian process $U(\mathbf{z})$. The zero-mean stationary Gaussian process, U, specifies the covariance structure of the Gaussian process in Eq. 22, with variance σ_g^2, and a correlation function ρ:

$$Cov\left\{g(\mathbf{z}'), g(\mathbf{z}'')\right\} = \sigma_g^2\rho(\mathbf{z}', \mathbf{z}''); \quad (\mathbf{z}', \mathbf{z}'') \in \Omega \tag{23}$$

A Gaussian correlation model is selected for ρ:

$$\rho(\mathbf{z}', \mathbf{z}'') = \exp\left(-\sum_{i=1}^{n} \frac{(z_i' - z_i'')^2}{\theta_i^2}\right) \tag{24}$$

where $\{\theta_i : i = 1, ..., n\}$ is a set of parameters.
Given a realization, $\{\mathbf{z}_0 \in \Omega\}$, the Kriging predictor of $g(\mathbf{z}_0)$ based on the set of observations, Γ, is defined as a linear unbiased predictor:

$$\tilde{g}(\mathbf{z}_0) = \sum_{j=1}^{S} \zeta_j g(\mathbf{z}_j) \tag{25}$$

where $\boldsymbol{\zeta} = \left\{\zeta_j, j = 1, ..., S\right\}$ is a set of unknown weights. The set of unknown weights $\boldsymbol{\zeta}$ is determined by minimizing the prediction error variance:

$$\hat{\boldsymbol{\zeta}} = \arg\min_{\boldsymbol{\zeta}} \mathrm{Var}\left\{g(\mathbf{z}_0) - \tilde{g}(\mathbf{z}_0)\right\} \tag{26}$$

with respect to unbiasedness constraint:

$$E\left\{\tilde{g}(\mathbf{z}_0)\right\} = E\left\{g(\mathbf{z}_0)\right\} \tag{27}$$

After estimating $\hat{\boldsymbol{\zeta}}$, the prediction is calculated as:

$$\tilde{g}(\mathbf{z}_0) = \sum_{j=1}^{S} \hat{\zeta}_j g(\mathbf{z}_j) \tag{28}$$

with associated error variance:

$$\sigma_{\tilde{g}}^2 = \mathrm{Var}\left\{g(\mathbf{z}_0) - \tilde{g}(\mathbf{z}_0)\right\} =$$

$$\sigma_g^2\left(1 - 2\sum_{j=1}^{S} \hat{\zeta}_j \rho(\mathbf{z}_0, \mathbf{z}_j) + \sum_{j=1}^{S}\sum_{k=1}^{S} \hat{\zeta}_j \hat{\zeta}_k \rho(\mathbf{z}_j, \mathbf{z}_k)\right) \tag{29}$$

Details on the implementation of the Kriging predictor can be found in various sources (e.g., [5]).
The accuracy of the predictor in Eq. 28 depends on the selection of $\mathbf{q}$, $\boldsymbol{\eta}$, σ_g^2 and θ. The set of functions, $\mathbf{q}$, can be selected as an optimal set of functions when performing regression analysis on observations in Eq. 19. After determining the optimal regression fit, $\boldsymbol{\eta}$, σ_g^2, and $\{\theta_i : i = 1, ..., n\}$ are calculated as maximum likelihood estimates (MLEs) [5] on the set of observations in Eq. 19. The error variance, $\sigma_{\tilde{g}}^2$, is dependent on the number S and the location of samples in Eq. 19. The value of $\sigma_{\tilde{g}}^2$ can be reduced by generating new pointwise observations of $g(\mathbf{z})$ from the region of the variable space where predictions are to be made.

4. MLS Implementation

This section summarizes the implementation of the MLS method. The flowchart in Figure 4 illustrates the three main steps of the MLS method. In the first step, a metamodel of $g(\mathbf{z})$ is trained according to the Algorithm 1. Once the metamodel is trained, the metamodel-based failure probability $P_{\tilde{F}}$ and the correction coefficient, κ, are evaluated by two independent steps defined according to Algorithms 2 and 3 respectively. Finally, P_F is calculated as a product of the estimates of $P_{\tilde{F}}$ and κ.

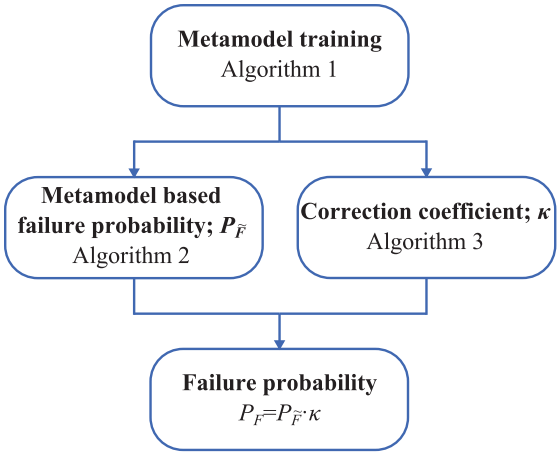

Figure 4: Flowchart of the MLS method.

The implementation of the MLS method is initiated with the metamodel training based on the LS-DoE, as presented in Algorithm 1. The implementation of the LS-DoE requires the important direction, $\boldsymbol{\alpha}$, to be specified. For example, $\boldsymbol{\alpha}$ can be approximated by a gradient vector of $g(\mathbf{z})$ pointing in the direction of steepest descent. In [19] $\boldsymbol{\alpha}$ was determined as a unit vector which points to a set of samples generated with the Markov Chain Monte Carlo method from the distribution of random variables conditioned on the failure event, $h(\mathbf{z}) = I_F(\mathbf{z}) f_{\mathbf{Z}}(\mathbf{z}))/P_F$. Alternatively, Adaptive Line Sampling method [6] can be implemented, where an initial crude estimate of $\boldsymbol{\alpha}$ is updated during the reliability analysis to converge to a near optimal $\boldsymbol{\alpha}$.
As presented in Algorithm 1, a metamodel is stepwise refined with S observations, generated by the LS-DoE, until maximum number of refinement steps, N_S, is achieved or convergence of κ within the bounds $\kappa_{\min}$ and $\kappa_{\max}$ occurs. A relatively accurate estimate of κ, according to Algorithm 3, can present a computationally demanding task at early stages of the metamodel refinement. For this reason, a relatively low number of samples, $N_\kappa^t > 50$, is used during the metamodel refinement to obtain a relatively crude estimate of κ. The refinement procedure is conducted until P_F and $P_{\tilde{F}}$ are within less than one order of magnitude, which corresponds to $\kappa_{\min} \geq 0.1$ and $\kappa_{\max} \leq 10$. As discussed in Section 3.1, ϵ can be selected as a zero-mean random variable with standard deviation on the magnitude of $g(\mathbf{z})$ at the mean.

Once $\tilde{g}(\mathbf{z})$ is trained, $P_{\tilde{F}}$ and κ can be estimated according to Algorithms 2 and 3 respectively. Since Algorithms 2 and 3 are independent, the estimates of $P_{\tilde{F}}$ and κ can be calculated in parallel to optimize computational efforts.
Assuming that $\tilde{g}(\mathbf{z})$ approximates the limit state reasonably well, $\boldsymbol{\alpha}$ specified for Algorithm 1 can be used for the LS estimate of $P_{\tilde{F}}$. Additionally, the implementation of Algorithm 2 requires the specification of the limitations in the number of line searches for the LS estimate of $P_{\tilde{F}}$, $N_{P_{\tilde{F}}}^{\min}$ and $N_{P_{\tilde{F}}}^{\max}$, together with the target coefficient of variation, $\tau_{\tilde{F}}$. Since $\mathrm{CoV}(\hat{P}_{\tilde{F}})$ and

Algorithm 1 Metamodel training

Define $\boldsymbol{\alpha}, N_S, N_\kappa^t, S, \kappa_{\min}, \kappa_{\max}, f_\epsilon(\epsilon)$
Initiate $i \leftarrow 0$
while $\{i < N_S \quad \& \quad \kappa_{\min} \leq \hat{\kappa} \leq \kappa_{\max}\}$ **do**
  $i \leftarrow i + 1$
  $j \leftarrow (i-1) \cdot S$
  while $(i-1) \cdot S \leq j < i \cdot S$ **do**
    $j \leftarrow j + 1$
    $\mathbf{z}_j^\perp \sim N(0, \mathbb{I}_{n-1})$ ▷ $\mathbb{I}_{n-1}$ is an identity matrix of size $(n-1)$
    $\epsilon_j \sim f_\epsilon(\epsilon)$
    Solve $g(\boldsymbol{\alpha}\beta_j + \mathbf{z}_j^\perp) + \epsilon_j = 0$ for β_j ▷ Performance function line search
    $\mathbf{z}_j \leftarrow \boldsymbol{\alpha}\beta_j + \mathbf{z}_j^\perp$, $g(\mathbf{z}_j) \leftarrow -\epsilon_j$ ▷ Update DoE
  end while
  Train $\tilde{g}(\mathbf{z})$ on $\Gamma = \{(\mathbf{z}_k, g(\mathbf{z}_k)), k = 1, .., j\}$
  Evaluate $\hat{\kappa}$ according to Algorithm 3 with $N_\kappa^{\max} = N_\kappa^t$
end while
return $\tilde{g}(\mathbf{z})$

Algorithm 2 Metamodel-based failure probability, $P_{\tilde{F}}$

Define $\boldsymbol{\alpha}, N_{P_F}^{\min}, N_{P_F}^{\max}, \tau_{\tilde{F}}$
Initiate $i \leftarrow 0$
while $\left\{i < N_{P_F}^{\max} \quad \& \quad \mathrm{CoV}(\hat{P}_{\tilde{F}}) > \tau_{\tilde{F}}\right\}$ **do**
  $i \leftarrow i + 1$
  $\mathbf{z}_i^\perp \sim N(0, \mathbb{I}_{n-1})$ ▷ $\mathbb{I}_{n-1}$ is an identity matrix of size $(n-1)$
  Solve $\tilde{g}(\boldsymbol{\alpha}\tilde{\beta}_i + \mathbf{z}_i^\perp) = 0$ for $\tilde{\beta}_i$ ▷ Metamodel line search
  if $i \geq N_{P_F}^{\min}$ **then**
    $\hat{P}_{\tilde{F}} \leftarrow \frac{1}{i} \sum_{j=1}^{i} \Phi(-\tilde{\beta}_j)$
    $\mathrm{Var}(\hat{P}_{\tilde{F}}) \leftarrow \frac{1}{i(i-1)} \sum_{j=1}^{i} \left(\Phi(-\tilde{\beta}_j) - \hat{P}_{\tilde{F}}\right)^2$
    $\mathrm{CoV}\,(\hat{P}_{\tilde{F}}) \leftarrow \sqrt{\mathrm{Var}(\hat{P}_{\tilde{F}})}/\hat{P}_{\tilde{F}}$
  end if
end while
return $\hat{P}_{\tilde{F}}, \mathrm{CoV}(\hat{P}_{\tilde{F}})$

$\mathrm{CoV}(\hat{\kappa})$ contribute equally to $\mathrm{CoV}(\hat{P}_F)$ in Eq. 18, the following target values can be selected; $\tau_{\tilde{F}} = \tau_\kappa = \tau_F/\sqrt{2}$, where τ_κ and τ_F are target coefficients of variation for $\hat{\kappa}$ and $\hat{P}_F$ respectively.

κ is calculated according to Algorithm 3 with the target coefficient of variation τ_κ and minimum and maximum number of line searches, $N_\kappa^{\max}$ and $N_\kappa^{\min}$.

Finally, $\hat{P}_F$ and the corresponding $\mathrm{CoV}(\hat{P}_F)$ are calculated according to Eqs. 17 and 18.

5. Application Example with Parabolic Failure Limit

In the following section, the performance of the MLS approach will be illustrated with a reliability problem defined by a parabolic failure limit. The problem is studied for a range of dimensions to evaluate the effect of dimensionality on the performance of the MLS method.

5.1. Problem Definition

The reliability problem is defined by an n-dimensional paraboloid studied in [18]:

$$g(\mathbf{Z}) = a \cdot \sum_{i=2}^{n} Z_i^2 - Z_1 - b \tag{30}$$

where $\mathbf{Z} = [Z_1, ..., Z_n]^T \in \Omega$ is an n-dimensional vector of independent random variables distributed by the joint multivariate normal *pdf*, $\phi_{\mathbf{Z}}(\mathbf{z}) = N_n(\mu_{\mathbf{Z}}, \boldsymbol{\Sigma})$, where $\mu_{\mathbf{Z}} = 0$ and $\boldsymbol{\Sigma} = \mathbb{I}_n$ is an identity matrix of size n. The parameters a and b in Eq. 30 are constants defining the shape of the paraboloid. High-dimensional applications of the reliability problem (Eq. 30) can be assessed by implementing the following substitution:

$$Q = \sum_{i=2}^{n} Z_i^2 \tag{31}$$

where $Q \sim \chi_{n-1}^2$ is a Chi-square distributed random variable with $n-1$ degrees of freedom. With the substitution implemented, the n-dimensional reliability problem can be transformed to an equivalent 2-dimensional problem with a linear performance function:

$$g(Z_1, Q) = a \cdot Q - Z_1 - b \tag{32}$$

The joint *pdf* of the equivalent 2-dimensional reliability problem is defined by a product: $f(z_1, q) = \phi_{Z_1}(z_1) \cdot f_Q(q)$, where $\phi_{Z_1}(z_1) = N(0, 1)$, and $f_Q(q) = \chi_{n-1}^2$. The transformation of the n-dimensional problem (Eq. 30) to an equivalent 2-dimensional problem (Eq. 32) is applied to illustrate the performance of the LS-DoE.

The reliability problem in Eq. 30 is evaluated for a range of dimensions with the paraboloid parameters a and b defined in Table 1.

Table 1: Parameters for the reliability problem in Eq. 30.

n	a	b	σ_ϵ	S
2	1	-3	1	10
10	1	0	1	50
100	0.1	4.5	1.5	200

5.2. Kriging Metamodel

The implementation of the MLS method is initiated with the Kriging metamodel training according to Algorithm 1. The effect of a regression model on the performance of the Kriging predictor is evaluated by implementing a linear, $\mathbf{q}_1 = \{1, z_1, ..., z_n\}$, and a quadratic regression model, $\mathbf{q}_2 = \left\{1, z_1, ..., z_n, z_1^2, ..., z_n^2\right\}$. For the paraboloid parameters a and b in Table 1, the problem features a single design point along the z_1 axis. Based on these observations, $\boldsymbol{\alpha}$ is select as a unit vector parallel to the z_1 axis with the sign dependent on the value of b.

Algorithm 3 Correction coefficient κ

Define $\boldsymbol{\alpha}, N_\kappa^{\max}, N_\kappa^{\min}, \tau_\kappa$
Initiate $i \leftarrow 0$
while $\{i < N_\kappa^{\max}$ & $\mathrm{CoV}(\hat{\kappa}) > \tau_\kappa\}$ **do**
  $i \leftarrow i + 1$
  $\mathbf{z}_i^{\perp} \sim N(0, \mathbb{I}_{n-1})$ ▷ $\mathbb{I}_{n-1}$ is an identity matrix of size $(n-1)$
  Solve $g(\boldsymbol{\alpha}\beta_i + \mathbf{z}_i^{\perp}) = 0$ for β_i ▷ Performance function line search
  Solve $\tilde{g}(\boldsymbol{\alpha}\tilde{\beta}_i + \mathbf{z}_i^{\perp}) = 0$ for $\tilde{\beta}_i$ ▷ Metamodel line search
  $\kappa_{Ui} \leftarrow \Phi(-\tilde{\beta}_i)/\Phi\left(-\min\left[\tilde{\beta}_i, \beta_i\right]\right)$ ▷ Indicator ratio
  $w_{Ui} \leftarrow \Phi\left(-\min\left[\tilde{\beta}_i, \beta_i\right]\right)$ ▷ Weight
  $\kappa_{Ii} \leftarrow \Phi\left(-\max\left[\tilde{\beta}_i, \beta_i\right]\right)/\Phi(-\tilde{\beta}_i)$ ▷ Indicator ratio
  $w_{Ii} \leftarrow \Phi(-\tilde{\beta}_i)$ ▷ Weight
  if $i \geq N_\kappa^{\min}$ **then**
    $w_{Uj} \leftarrow w_{Uj} / \sum_{j=1}^{i} w_{Uj}; j = 1, ..., i$ ▷ Normalize weights
    $\hat{\kappa}_U \leftarrow \sum_{j=1}^{i} w_{Uj}\kappa_{Uj}$
    $\mathrm{Var}(\hat{\kappa}_U) \leftarrow \sum_{j=1}^{i} w_{Uj}^2(\kappa_{Uj} - \hat{\kappa}_U)^2$
    $w_{Ij} \leftarrow w_{Ij} / \sum_{j=1}^{i} w_{Ij}; j = 1, ..., i$ ▷ Normalize weights
    $\hat{\kappa}_I \leftarrow \sum_{j=1}^{i} w_{Ij}\kappa_{Ik}$
    $\mathrm{Var}(\hat{\kappa}_I) \leftarrow \sum_{j=1}^{i} w_{Ij}^2(\kappa_{Ij} - \hat{\kappa}_I)^2$
    $\hat{\kappa} \leftarrow 1/\hat{\kappa}_U + \hat{\kappa}_I - 1 - \mathrm{Var}(\hat{\kappa}_U)/\hat{\kappa}_U^3$ ▷ Correction coefficient estimate
    $\mathrm{Var}(\hat{\kappa}) \leftarrow \mathrm{Var}(\hat{\kappa}_U)/\hat{\kappa}_U^4 + \mathrm{Var}(\hat{\kappa}_I)$
    $\mathrm{CoV}(\hat{\kappa}) \leftarrow \sqrt{\mathrm{Var}(\hat{\kappa})}/\hat{\kappa}$
  end if
end while
return $\hat{\kappa}$, $\mathrm{CoV}(\hat{\kappa})$

To generate the LS-DoE, ϵ is defined as a zero-mean normal random variable with the standard deviation σ_ϵ specified in Table 1. The performance of the LS-DoE on the reliability problem in Eq. 30 can be examined on realizations in Figures 5 (a) and (b) for $n = 2$ and $n = 100$ respectively. Figures 5 (a) and (b) display a set of 100 observations generated by the LS-DoE. From Figures 5 (a) and (b) it can be detected that the LS-DoE is able to provide a set of independent observations in the proximity of the limit state for the reliability problem in Eq. 30.
The metamodel is stepwise refined with S observations, as specified in Table 1, until $0.2 \leq \hat{\kappa} \leq 5$ or $N_S \leq 5$. During the metamodel refinement an estimate of κ is evaluated according to Algorithm 3 with $N_\kappa^t = 50$ line searches.

In addition to the computational demands associated with the performance function evaluations during the metamodel training, a significant computational expense can be required to evaluate MLEs for the Kriging parameters. A parametric study was conducted to investigate the effect of n in the problem in Eq. 30 and the LS-DoE size on the computational time required by the pattern search algorithm in the DACE library to locate MLEs. Simulations in Table 2 were performed with the DACE library [5] in Matlab^{TM} on an Intel®Core^{TM} i7-3770 CPU @ 3.40 GHz running on Windows®7.

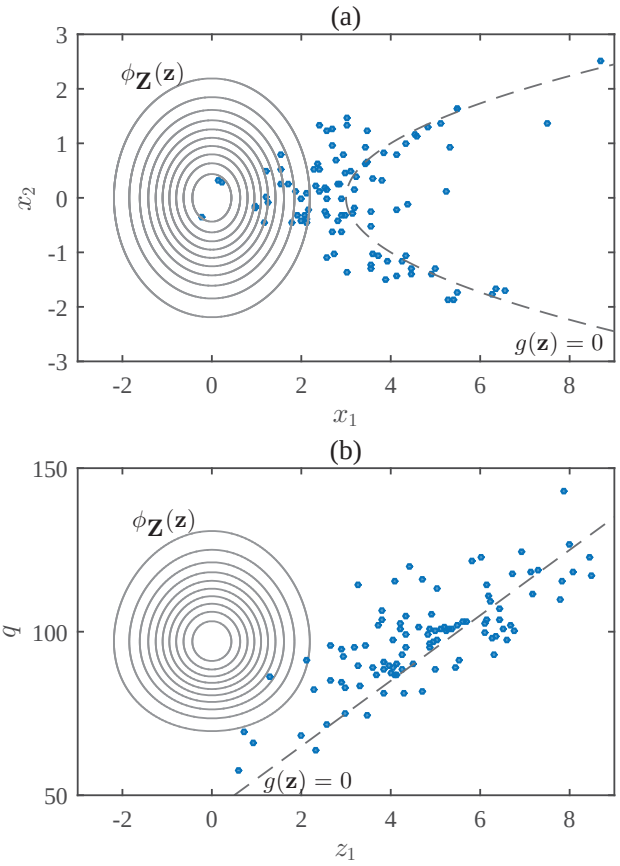

Figure 5: Realizations of the LS-DoE with $\epsilon \sim N(0, 1)$ for the reliability problem in Eq. 30; (a) $n = 2$, (b) $n = 100$.

Table 2: Computational time required to calculate MLEs for the Kriging parameters in seconds.

n	S				
	20	200	500	1000	2000
2	0.01	0.10	0.48	2.71	11.88
10	0.03	0.46	3.35	16.72	76.03
100	-	12.59	70.94	302.89	1629.80

The results in Table 2 indicate that potential computational savings of the MLS approach can be affected by the computational expenses employed to locate MLEs for the Kriging parameters in high-dimensional reliability problems with large LS-DoE sizes.

5.3. Results

An estimate of P_F, defined as a product of $\hat{P}_{\tilde{F}}$ and $\hat{\kappa}$, is obtained by evaluating Algorithms 2 and 3 respectively. Following the procedure in Algorithm 2, $\hat{P}_{\tilde{F}}$ is evaluated with the following parameters $N_{P_{\tilde{F}}}^{\min} = 50$, $N_{P_{\tilde{F}}}^{\max} = 10^5$, and $\tau_{\tilde{F}} = 0.05/\sqrt{2}$. $\hat{\kappa}$ is calculated with the following convergence criteria $N_\kappa^{\min} = 50$, $N_\kappa^{\max} = 10^5$, and $\tau_\kappa = 0.05/\sqrt{2}$ according to Algorithm 3. In total, 30 reliability analyses were conducted to evaluate the

performance of the MLS approach with the target coefficient of variation $\tau_F = \sqrt{\tau_{\tilde{F}}^2 + \tau_\kappa^2} = 0.05$. The results in Table 3 are average values over 30 MLS reliability analyses.

The performance of the MLS method is compared to several reliability methods; LS, SS, and MC. The results in Table 3 are averages over 30 reliability analyses conducted with each of the reliability methods. The implementation of the LS method is based on $\boldsymbol{\alpha}$ defined for the MLS method in Section 5.2. The SS method is carried out with intermediate probability levels of 0.1, estimated with 20000 samples generated with the Modified Metropolis algorithm. Upper and lower bounds for the $\mathrm{CoV}(\hat{P}_F)$ of the SS estimate of P_F were calculated as defined in [1].

Table 3 summarizes the comparison of the performance of several reliability methods with the MLS method on the problem in Eq. 30. The reliability methods are compared with respect to the average number of calls to the performance function, N, and average estimates of P_F and $\mathrm{CoV}(\hat{P}_F)$. In the case of the MLS approach, N represents the total number of calls to the performance function for the metamodel training and evaluation of κ. The computational cost associated with $\hat{P}_{\tilde{F}}$ is not included in N due to a relatively low computational cost of a metamodel evaluation (i.e., on the magnitude of a milisecond or lower for a single evaluation) when compared to a performance function evaluation in a general reliability problem. Due to a relatively simple formulation of the performance function in Eq. 30 no additional computational costs were required for the determination of $\boldsymbol{\alpha}$ for the MLS and LS methods. In a general reliability problem the computational cost related to the determination of $\boldsymbol{\alpha}$ can be on the magnitude of n performance function evaluations. Additionally, as a result of the formulation of the performance function in Eq. 30, line search along $\boldsymbol{\alpha}$ can be solved analytically with a single evaluation. Depending on the nonlinearity of a performance function in a general reliability problem, line search can require ≥ 2 performance function evaluations for a single line search.

Table 3: Results for the reliability problem in Eq. 30.

Method	MLS		LS	SS	MC
	$\mathbf{q}_1$	$\mathbf{q}_2$			
$n = 2$					
N	135	55	433	101800	843190
$\hat{P}_F \cdot 10^4$	4.80	4.81	4.80	4.62	4.76
CoV (%)	3.97	3.54	4.99	≥ 5.39 ≤ 11.78	5.00
$n = 10$					
N	1201	71	41893	100000	518360
$\hat{P}_F \cdot 10^4$	7.94	7.77	7.67	7.58	7.74
CoV (%)	3.53	3.54	5.00	≥ 4.93 ≤ 10.54	5.00
$n = 100$					
N	100930	251	45066	104200	1057890
$\hat{P}_F \cdot 10^4$	3.82	3.77	3.76	3.90	3.79
CoV (%)	6.63	3.54	5.00	≥ 5.43 ≤ 11.95	5.00

It can be observed from Table 3 that the MLS approach can provide consistent estimates of P_F for the reliability problem in Eq. 30, comparable to several commonly used reliability methods (i.e., LS, SS, and MC). The MLS approach demonstrates a potential to reduce computational expenses associated with reliability problems by estimating P_F with similar $\mathrm{CoV}(\hat{P}_F)$ for a smaller or comparable N with respect to the reliability methods considered in Table 3.

The comparison between the computational expenses of the MLS method with regression models $\mathbf{q}_1$ and $\mathbf{q}_2$ reveals that the quality of the metamodel approximation considerably affects the performance of the MLS method. For example, due to the inability of the Kriging predictor with $\mathbf{q}_1$ to accurately approximate the performance function for $n = 100$, the MLS method does not provide reductions in computational expenses when compared to the other reliability methods in Table 3. Computational expenses for the MLS implementation with $\mathbf{q}_1$ are dominated by the performance function evaluations employed to estimate κ as presented in Table 4. On the other hand, the calculation expenses for the MLS implementation with $\mathbf{q}_2$ are composed of the minimal number of samples for a multivariate regression analysis with $\mathbf{q}_2$ (i.e., $2n + 1$), and the specified minimum number of samples for the estimate of κ, $N_\kappa^{\min} = 50$.

Table 4: Average ratio (%) of the computational expenses employed to estimate κ over N.

n	2	10	100
$\mathbf{q}_1$	80.5	80.1	99.0
$\mathbf{q}_2$	90.9	70.4	19.9

To investigate if the estimator $\mathrm{CoV}(\hat{P}_F)$ in Eq. 18 is a reasonable estimate of the accuracy of $\hat{P}_F$, the empirical coefficient of variation, $\mathrm{CoV}_E(\hat{P}_F)$, calculated from 30 MLS analyses is compared to the predicted values. The consistency between $\mathrm{CoV}(\hat{P}_F)$ and its estimate, $\mathrm{CoV}_E(\hat{P}_F)$, in Table 5 confirms that $\mathrm{CoV}(\hat{P}_F)$ is a sound estimate of accuracy of $\hat{P}_F$.

Table 5: Comparison between the empirical and the predicted CoV for the MLS method.

	n	2	10	100
$\mathbf{q}_1$	CoV (%)	3.97	3.53	6.63
	CoV_E (%)	3.44	4.80	6.70
$\mathbf{q}_2$	CoV (%)	3.54	3.54	3.54
	CoV_E (%)	4.20	3.19	3.31

6. Parallel System Analysis

A two component parallel system reliability problem, studied in [13], is evaluated to investigate the performance of the MLS approach on a problem with a highly nonlinear performance function. The reliability problem is defined as:

$$g(z_1, z_2) = \max \begin{cases} 2 - z_2 + \exp(-0.1 z_1^2) + (0.2 z_1)^4 \\ 4.5 - z_1 z_2 \end{cases} \quad (33)$$

where z_1 and z_2 are realizations of two independent standard normally distributed random variables, Z_1 and Z_2. The problem features a single design point, $\mathbf{z}_P = (1.6148, 2.7806)^T$ as displayed in Figure 6.

The implementation of the MLS approach is initiated with the Kriging metamodel training according to Algorithm 1 with $\mathbf{q}_1$ and $S = 50$ observations per refinement step until $0.2 \leq \hat{\kappa} \leq 5$ or $N_S \leq 5$. The Kriging metamodel is trained on observations of the performance function generated by the LS-DoE. The LS-DoE is constructed with $\boldsymbol{\alpha} = (0.5030, 0.8643)^T$, pointing in the direction of $\mathbf{z}_P$, and a zero-mean normally distributed pseudo random variable $\epsilon \sim N(0, 2)$. To demonstrate the performance of the LS-DoE on the problem in Eq. 33, a set of 100 realizations is presented in Figure 6. During the metamodel refinement $\hat{\kappa}$ is evaluated according to Algorithm 3 with $N_\kappa^t = 50$ line searches.

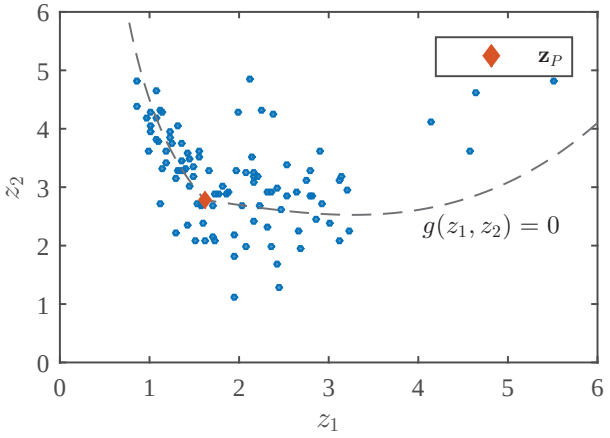

Figure 6: Realization of the LS-DoE with $\epsilon \sim N(0, 1)$ for the reliability problem in Eq. 6.

Following the MLS algorithm, an estimate of P_F is calculated as a product of $\hat{P}_{\tilde{F}}$ and $\hat{\kappa}$. $\hat{P}_{\tilde{F}}$ is evaluated as specified in Algorithm 2 with $N_{P_{\tilde{F}}}^{\min} = 50$, $N_{P_{\tilde{F}}}^{\max} = 10^5$, and $\tau_{\tilde{F}} = 0.01$. $\hat{\kappa}$ is calculated according to Algorithm 3 to satisfy the following convergence criteria $N_\kappa^{\min} = 50$, $N_\kappa^{\max} = 10^5$, and $\tau_\kappa = 0.05/\sqrt{2}$. In total, 30 reliability analyses were conducted to evaluate the performance of the MLS approach with the target coefficient of variation $\tau_F = \sqrt{\tau_{\tilde{F}}^2 + \tau_\kappa^2} = 0.0367$. The MLS estimates are validated with 30 reliability analyses conducted with the LS, IS, SS, and MC method. The LS method is implemented with $\boldsymbol{\alpha}$ pointing in the direction of $\mathbf{z}_P$. An importance sampling distribution for the IS approach is defined as a bivariate normal distribution $N(\mathbf{z}_P, \mathbb{I}_2)$ with the mean at $\mathbf{z}_P$ and a two-dimensional identity matrix $\mathbb{I}_2$ as the covariance matrix. The SS method is carried out with intermediate probability levels of 0.1, estimated with 20000 samples generated with the Modified Metropolis algorithm. Upper and lower bounds for the $\mathrm{CoV}(\hat{P}_F)$ of the SS estimate of P_F were calculated as defined in [1].

Table 6 summarizes the comparison of the performance of several reliability methods with the MLS method on the problem in Eq. 6. The reliability methods are compared with respect to the average N, $\hat{P}_F$, and $\mathrm{CoV}(\hat{P}_F)$. In the case of the MLS approach, N represents the total number of calls to the performance function used to determine $\boldsymbol{\alpha}$, train the metamodel and evaluate κ. After comparing the results in Table 6 it can be observed that the MLS approach can provide estimates of P_F for the reliability problem in Eq. 6, comparable to several commonly used reliability methods. The MLS approach provided estimates P_F with similar $\mathrm{CoV}(\hat{P}_F)$ for a smaller or comparable N with respect to the reliability methods considered in Table 6. On average, computational expenses employed to train the metamodel dominate in this example with 53.3 % of the total performance function evaluations.

To investigate if the estimator $\mathrm{CoV}(\hat{P}_F)$ in Eq. 18 is a reasonable estimate of the accuracy of $\hat{P}_F$, the empirical coefficient of variation, $\mathrm{CoV}_E(\hat{P}_F)$ is compared to $\mathrm{CoV}(\hat{P}_F)$. Based on the results in Table 6, $\mathrm{CoV}_E(\hat{P}_F) = 5.19\%$ is slightly higher than the estimated value $\mathrm{CoV}(\hat{P}_F) = 3.46\%$.

Table 6: Results for the reliability problem in Eq. 33.

Method	MLS	LS	IS	SS	MC
$n = 2$					
N	762	2905	2883	114000	1650190
$\hat{P}_F \cdot 10^4$	2.42	2.45	2.43	2.80	2.43
CoV (%)	3.52	4.00	4.95	≥ 5.70 ≤ 12.87	5.00

7. Reliability Analysis of a Laterally Loaded Pile

The development of an efficient approach for reliability analysis is important for a wide range of engineering problems, including natural hazards (e.g., water triggered landslides) and engineering structures (e.g., offshore wind turbines). In this study, a reliability analysis of a monopile foundation for offshore wind turbines is performed with the MLS method to examine the performance of the approach on a practical engineering problem. The reliability analysis is conducted to quantify the effects of uncertainties in lateral load and soil properties on the ultimate limit state of a monopile foundation. The response of a monopile is simulated by a numerical finite element pile-soil model, which is an implicit function of the lateral load and soil parameters. The performance of the MLS approach is validated with the LS method.

7.1. Numerical Pile-soil Model.

The response of a pile to lateral load can be simulated by a finite element model, known as the API or p-y model (e.g., [10]). This model is based on Winkler's beam on elastic foundation theory, where the response of soil is simulated by a series of elastic springs. The original Winkler model assumes elastic behavior of soil, while in the API formulation it incorporates nonlinearities. The nonlinearities in soil response are modeled by p-y curves, where p is the soil reaction per unit length of a pile, and y is the lateral displacement of a pile. P-y curves were

developed by backcalculating a series of field test on laterally loaded piles performed in different soil types. The API model is currently recommended in several design codes for offshore wind turbine foundations (e.g., [10]).

The monopile, in this study, is a hollow tube with length L_P=30 m, diameter of D=5.0 m, and pile wall thickness of t=0.05 m. The pile material is steel with Young's modulus of E=2.1 · 10^5 MPa, and a Poisson's ratio of ν=0.3. The material behavior of the pile is assumed to be linearly elastic. Soil response is simulated by a series of springs with material behavior defined by p-y curves for medium stiff clay. Basic elements of the monopile model are presented in Figure 7.

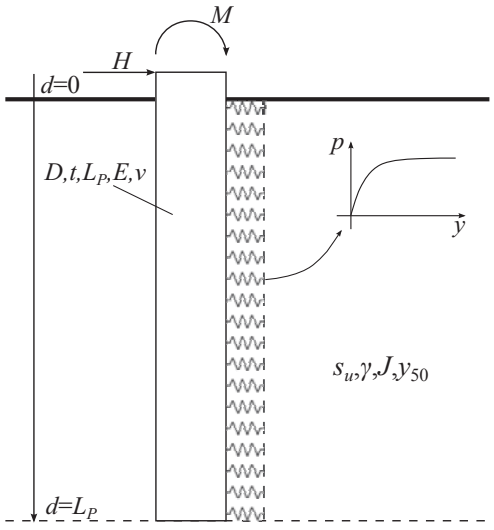

Figure 7: Laterally loaded monopile foundation.

Reliability analysis of the pile-soil system is conducted by considering the lateral load, H, and the undrained shear strength of the clay, s_u, as random variables. The pile is laterally loaded with a horizontal force H and a moment $M = H \cdot 30$ m at $d = 0$ m. Other parameters of the p-y curves are assumed to be deterministic with the following values; unit weight γ=18.0 kN/m^3, empirical model parameter J=0.25, strain corresponding to one half of the maximum principal stress difference y_{50}=0.005.

7.2. Random Load

Uncertainties associated with H are modeled by a Gumbel distributed random variable:

$$H \sim f_H(\mu_H, \mu_H \cdot \mathrm{CoV}(H)) \tag{34}$$

where μ_H = 2500 kN is the mean, and CoV(H) = 0.1 is the coefficient of variation, with the corresponding *pdf* presented in Figure 8.

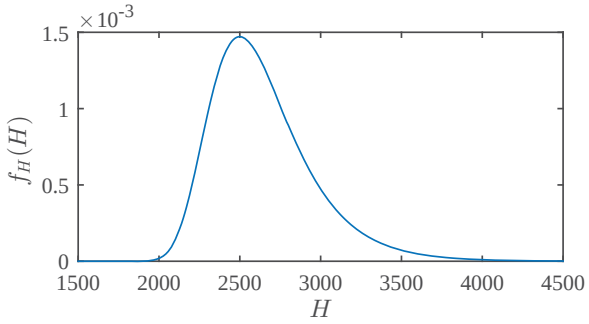

Figure 8: Probability density function of H.

7.3. Soil Variability

Variability of s_u is expected to significantly influence the pile-soil response since s_u is directly related to the peak value of soil resistance. The variability of s_u is modeled by means of an one-dimensional random field:

$$\left\{s_u(d); d \in G \subset \mathbb{R}^1\right\} \sim f_{s_u}(s_u) \tag{35}$$

where d is soil depth or the reference variable $\{d \in G : 0 \le d \le L_P\}$, G is the studied domain, and $f_{s_u}(s_u)$ is a *pdf* specifying the random field. $f_{s_u}(s_u)$ is a multivariate lognormal *pdf*, with mean linearly increasing with depth:

$$\mu_{s_u} = \alpha_{s_u} + \beta_{s_u} \cdot d \tag{36}$$

where α_{s_u} and β_{s_u} are parameters of the mean function. The covariance structure of the random field is determined by a given coefficient of variation CoV(s_u) and a Markov correlation function:

$$\rho_{\ln s_u}(\tau) = \exp\left(-2 \cdot \frac{|d' - d''|}{\theta_d}\right) \tag{37}$$

where $\{(d', d'') \in G\}$, and θ_d is the correlation length of $\ln s_u$. Realizations of the random field in Eq. 35 are generated by discretizing the domain $\{d \in G : 0 \le d \le L_P\}$ into P = 40 intervals, with interval length of $d_L = L_P/P$. The generated random field is fully described by a P-dimensional *joint pdf*, $f_{s_u}(s_u)$. The intervals are selected to correspond to the discretization of the finite element mesh of the numerical pile-soil model. In this study, the random field, $\{s_u(d); d \in G\}$, is specified with the following parameters; α_{s_u} = 50 kPa, β_{s_u} = 3 kPa/m, CoV(s_u) = 0.4, θ_d = 2 m, and P = 40. A realization of $\{s_u(d); d \in G\}$ is presented in Figure 9.

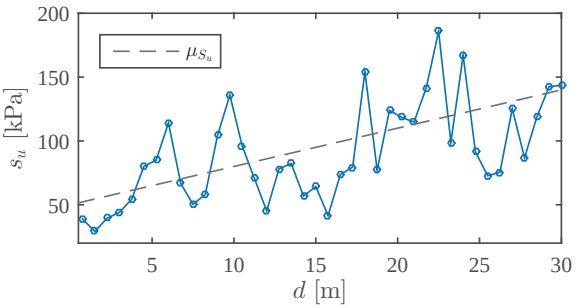

Figure 9: A random field realization of s_u.

7.4. Reliability Analysis

The reliability analysis of the monopile foundation is performed to quantify the effects of uncertainties in the set of random variables $\mathbf{X} = [H, s_u]^T$ on the ultimate limit state. In this study, the ultimate limit state is defined by the monopile steel

yield strength, $\sigma_{\text{lim}} = 235$ MPa. A transformation to a vector of independent standard normal distributed random variables, $\mathbf{Z} \in \Omega$, is applied to implement the MLS approach. The performance function is thus defined as:

$$g(\mathbf{z}) = \sigma_{\text{lim}} - \sigma(\mathbf{z}) \tag{38}$$

where $\sigma = \sigma(\mathbf{z})$ is the maximum stress along the monopile.
The implementation of the MLS method for the reliability problem in Eq. 38 is initiated with the Kriging metamodel training based on an LS-DoE, as defined in Algorithm 1. To generate an LS-DoE it is necessary to specify $\boldsymbol{\alpha}$ and the pseudo-random variable, ϵ. $\boldsymbol{\alpha}$ is determined by evaluating gradients of the performance function at the mean point of the standard normal space, $\mu_{\mathbf{Z}}$. It is expected that a gradient vector pointing in the direction of the steepest descent of the performance function is a reasonable estimate of $\boldsymbol{\alpha}$. A normalized gradient with respect to the i-th random variable is defined as:

$$\omega_i = \frac{\left.\frac{\partial g(\mathbf{z})}{\partial z_i}\right|_{\mu_{\mathbf{Z}}}}{\sqrt{\sum_{j=1}^{P+1}\left(\left.\frac{\partial g(\mathbf{z})}{\partial z_j}\right|_{\mu_{\mathbf{Z}}}\right)^2}} \tag{39}$$

where $\partial g(\mathbf{z})/\partial z_i$ is the derivative of $g(\mathbf{z})$ with respect to the i-th random variable. Estimates of ω are evaluated by the central difference scheme with two evaluations of $g(\mathbf{z})$ per random variable. Estimated ω are presented in Figure 10 with the gradient with respect to H on the first position, followed by gradients with respect to $P = 40$ random variables associated with the s_u random field discretization. Based on the estimates of ω in Figure 10, $\boldsymbol{\alpha}$ is selected as a unit vector in the standard normal space pointing in the positive direction, parallel to the axis assigned to H.

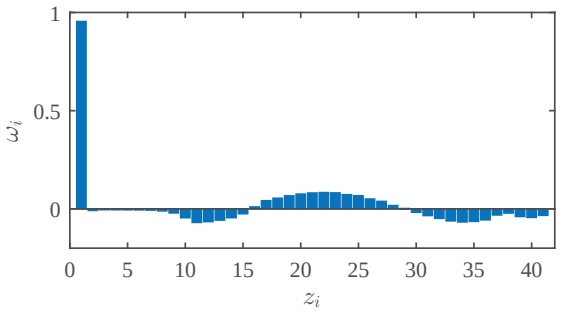

Figure 10: Performance function gradients at $\mu_{\mathbf{Z}}$.

ϵ is modeled as a zero-mean normally distributed random variable with $\sigma_\epsilon = 30$ kPa, selected to be on the magnitude $g(\mu_{\mathbf{Z}}) \approx 170$. The metamodel is stepwise refined with $S = 100$ observations from the LS-DoE until $0.2 \leq \hat{\kappa} \leq 5$ or $N_S \leq 10$. During the metamodel training a crude estimate of κ is calculated with $N_\kappa^t = 100$ simulations according to Algorithm 3. To minimize the number of evaluations of the performance function, the N_κ^t samples used to evaluate $\hat{\kappa}$ in Algorithm 1 are integrated within the set of samples in Algorithm 3 for an accurate estimate of κ once the metamodel is trained.

Given a Kriging metamodel, $\hat{P}_F$ is calculated as a product of $\hat{P}_{\tilde{F}}$ and $\hat{\kappa}$, according Algorithms 2 and 3 respectively. $\hat{P}_{\tilde{F}}$ is evaluated with the target coefficient of variation $\tau_{\tilde{F}} = 0.05/\sqrt{2}$ with $N_{P_{\tilde{F}}}^{\min} = 50$ and $N_{P_{\tilde{F}}}^{\max} = 10^5$. $\hat{\kappa}$ is calculated with the following convergence criteria $N_\kappa^{\min} = 50$, $N_\kappa^{\max} = 10^5$, and $\tau_\kappa = 0.05/\sqrt{2}$. In total, 30 reliability analyses were conducted to investigate the performance of the MLS method on the reliability problem in Eq. 38 with $\tau_F = \sqrt{\tau_{\tilde{F}}^2 + \tau_\kappa^2} = 0.05$. Average MLS estimates of P_F and $\text{CoV}(P_F)$ among 30 reliability analyses are presented with the average number of evaluations of the performance function, N, in Table 7. N includes the number of performance function evaluations conducted to determine $\boldsymbol{\alpha}$, train the Kriging metamodel, and evaluate $\hat{\kappa}$.
The performance of the MLS method is validated by conducting 10 reliability analyses with the LS method. The implementation of the LS method is based on the $\boldsymbol{\alpha}$, estimated for the MLS approach. Total number of performance function evaluations, N, for the LS approach includes the performance function evaluations used to determine $\boldsymbol{\alpha}$ and evaluate P_F. Average values of $\hat{P}_F$, $\text{CoV}(\hat{P}_F)$, and N among 10 reliability analyses conducted with the LS method are presented in Table 7.

Table 7: Results for the reliability problem in Eq. 38.

Method	MLS	LS
$n = 41$		
N	6889	33983
$\hat{P}_F \cdot 10^{10}$	1.74	1.78
CoV (%)	4.90	5.00

From Table 7 it can be observed that the MLS method provided an estimate of P_F comparable to the LS estimate with reduced computational expense. The MLS approach provided estimates with on average 20% of the performance function evaluations required by the LS method. The computational expenses employed to evaluate κ dominate on average in this example with 80.0% of the total performance function evaluations. To verify that the estimator in Eq. 18 is a sound estimate of the accuracy of $\hat{P}_F$ the empirical coefficient of variation, $\text{CoV}_E(\hat{P}_F)$ is compared to $\text{CoV}(\hat{P}_F)$. Based on 30 MLS estimates $\text{CoV}_E(\hat{P}_F) = 6.86\%$ is slightly higher than the average MLS estimate $\text{CoV}(\hat{P}_F) = 4.90\%$. These results indicate that $\text{CoV}(\hat{P}_F)$ in Eq. 18 provides a reasonable estimate of accuracy of $\hat{P}_F$.

8. Conclusion

Reliability analysis of structures can be a computationally challenging task if the failure probability to be estimated is low and/or if the structural model is computationally complex. The Metamodel Line Sampling method shows considerable potential to reduce the computational efforts for such problems by utilizing the efficiency of the Line Sampling method with a relatively low computational cost of a metamodel of the performance function.

A metamodel of the performance function is trained on a set of observations generated with a Line Sampling-based Design of Experiments. The Line Sampling Design of Experiments provides a set of observations in the proximity of the limit state by expanding the definition of the limit state with artificial uncertainty.

Once a metamodel is trained, the estimate of failure probability is calculated as a product of a metamodel-based failure probability and a correction coefficient. The correction coefficient accounts for the error in the metamodel estimate of failure probability resulting from the replacement the performance function with a metamodel. The estimate of failure probability is asymptotically unbiased and consistent.

The performance of the Metamodel Line Sampling method was examined on academic reliability problems and a practical reliability problem of a monopile foundation for offshore wind turbines. The implementation of the Metamodel Line Sampling with the Kriging predictor provided accurate estimates of failure probability under lower or comparable computational expense when compared to several commonly used reliability methods. The Metamodel Line Sampling approach performs optimally in reliability problems with no strong nonlinearities in the performance function, such that the importance direction can be estimated. Based on the studied examples, efficient performance is observed in low to medium-dimensional reliability problems ($n < 100$). The limitation in the dimensionality of a reliability problem is mainly due to the decrease in efficiency and accuracy of the Kriging metamodel with increasing dimensionality of a reliability problem.

Appendix A. Distribution of Random Variables Conditioned on the Failure Event

The distribution of random variables in Ω, conditioned on the failure event, is defined as:

$$h_F(\mathbf{z}) = \frac{I_F(\mathbf{z})\phi_{\mathbf{Z}}(\mathbf{z})}{\int_{\mathbb{R}^n} I_F(\mathbf{z})\phi_{\mathbf{Z}}(\mathbf{z})d\mathbf{z}} = \frac{I_F(\mathbf{z})\phi_{\mathbf{Z}}(\mathbf{z})}{P_F} \tag{A.1}$$

Assuming that an important direction, $\boldsymbol{\alpha}$, can be approximately estimated, the marginal distribution of $h_F(\mathbf{z})$ in the $(n-1)$-dimensional standard normal space of random variables orthogonal to $\boldsymbol{\alpha}$, can be obtained by integrating out the random variable along $\boldsymbol{\alpha}$, denoted as Z_α.

$$h_F^\perp(\mathbf{z}^\perp) = \int_{\mathbb{R}} h_F(\mathbf{z})dz_\alpha \tag{A.2}$$

With the formulation of $h_F(\mathbf{z})$ from Eq. A.1 incorporated into Eq. A.2, the following expression is obtained:

$$h_F^\perp(\mathbf{z}^\perp) = \int_{\mathbb{R}} \frac{I_F(\mathbf{z})\phi_{\mathbf{Z}}(\mathbf{z})}{P_F}dz_\alpha \tag{A.3}$$

Due to the rotational symmetry and independence of Ω:

$$h_F^\perp(\mathbf{z}^\perp) = \int_{\mathbb{R}} \frac{I_F(\mathbf{z})\phi_{Z_\alpha}(z_\alpha)\phi_{\mathbf{Z}^\perp}(\mathbf{z}^\perp)}{P_F}dz_\alpha \tag{A.4}$$

$$h_F^\perp(\mathbf{z}^\perp) = \frac{\phi_{\mathbf{Z}^\perp}(\mathbf{z}^\perp)}{P_F}\int_{\mathbb{R}} I_F(\mathbf{z})\phi_{Z_\alpha}(z_\alpha)dz_\alpha \tag{A.5}$$

Following the implementation of the LS method, the integral in Eq. A.5 can be expressed as $\Phi\left(-\beta(\mathbf{z}^\perp)\right)$, where $\beta(\mathbf{z}^\perp)$ is the solution of a line search along $\boldsymbol{\alpha}$ defined as $g\left(\boldsymbol{\alpha}\beta(\mathbf{z}^\perp) + \mathbf{z}^\perp\right) = 0$. The marginal distribution is defined as:

$$h_F^\perp(\mathbf{z}^\perp) = \frac{\Phi\left(-\beta(\mathbf{z}^\perp)\right)\phi_{\mathbf{Z}^\perp}(\mathbf{z}^\perp)}{P_F} \tag{A.6}$$

The proportionality operator can be used to define the relation between $\phi_{\mathbf{Z}^\perp}(\mathbf{z}^\perp)$ and $h_F^\perp(\mathbf{z}^\perp)$ because P_F is usually unknown prior to a reliability analysis;

$$h_F^\perp(\mathbf{z}^\perp) \propto \Phi\left(-\beta(\mathbf{z}^\perp)\right)\phi_{\mathbf{Z}^\perp}(\mathbf{z}^\perp) \tag{A.7}$$

Due to an unknown value of the normalizing constant, P_F, the distribution in Eq. A.7 can be sampled by Markov Chain Monte Carlo (e.g., Metropolis-Hastings algorithm [12]) or resampling methods (e.g., rejection method [12]).

The relation between $\phi_{\mathbf{Z}^\perp}(\mathbf{z}^\perp)$ and $h_F^\perp(\mathbf{z}^\perp)$ is used to define self-normalized importance sampling estimates (e.g., mean, variance). Self-normalized importance sampling estimates are evaluated on $\left\{\mathbf{z}_i^\perp \sim \phi_{\mathbf{Z}^\perp}(\mathbf{z}^\perp); i = 1, ..., K\right\}$ with the following weights:

$$w_i = \frac{h_F^\perp(\mathbf{z}_i^\perp)/\phi_{\mathbf{Z}^\perp}(\mathbf{z}_i^\perp)}{\sum_{j=1}^{K} h_F^\perp(\mathbf{z}_j^\perp)/\phi_{\mathbf{Z}^\perp}(\mathbf{z}_j^\perp)} = \frac{\Phi\left(-\beta(\mathbf{z}_i^\perp)\right)}{\sum_{j=1}^{K}\Phi\left(-\beta(\mathbf{z}_j^\perp)\right)} \tag{A.8}$$

Consider a function $l(\mathbf{z}^\perp)$ such that $\mu_l = \int l(\mathbf{z}^\perp)h_F^\perp(\mathbf{z}^\perp)d\mathbf{z}^\perp$ and $\sigma_l^2 = \int(l(\mathbf{z}^\perp) - \mu_l)^2 h_F^\perp(\mathbf{z}^\perp)d\mathbf{z}^\perp$ exist with $\sigma_l^2 > 0$. A self-normalized importance sampling mean estimate is calculated as:

$$\hat{\mu}_l = \sum_{i=1}^{K} w_i \cdot l(\mathbf{z}_i^\perp) \tag{A.9}$$

The variance of the self-normalized importance sampling estimator in Eq. A.9 is defined as:

$$\mathrm{Var}(\hat{\mu}_l) = \sum_{i=1}^{K} w_i^2 \mathrm{Var}(l(\mathbf{z}_i^\perp)) = \sum_{i=1}^{K} w_i^2\left(l(\mathbf{z}_i^\perp) - \hat{\mu}_l\right)^2 \tag{A.10}$$

Provided that all the observations have the same variance, $\mathrm{Var}(l(\mathbf{z}_i^\perp)) = \sigma_l^2$, the variance can be expressed as $\mathrm{Var}(\hat{\mu}_l) = \sigma_l^2 \sum_{i=1}^{K} w_i^2$. To investigate the effect of unequally weighted samples on the mean estimate, consider the unweighted mean based on K^e independent observations. The unweighted mean has variance σ_l^2/K^e. After setting $\mathrm{Var}(\hat{\mu}_l) = \sigma_l^2/K^e$ and solving for K^e, a

so-called effective sample size is obtained as follows:

$$K^e = 1/\sum_{i=1}^{K} w_i^2 \tag{A.11}$$

Appendix B. Correction Coefficient Estimate

A relatively simple approximation of the estimator for κ can be obtained by inserting the estimators $\hat{\kappa}_U$ and $\hat{\kappa}_I$ in Eq. 10:

$$\kappa(\kappa_U, \kappa_I) \approx \kappa(\hat{\kappa}_U, \hat{\kappa}_I) = \frac{1}{\hat{\kappa}_U} + \hat{\kappa}_I - 1 \tag{B.1}$$

Due to the ratio $1/\hat{\kappa}_U$, the estimator in Eq. B.1 is biased. The bias in the estimator is evaluated with the delta method by examining the expectation of the Taylor's series expansion of $\kappa(\hat{\kappa}_U, \hat{\kappa}_I)$ at κ_U and κ_I. The Taylor's series expansion of $\kappa(\hat{\kappa}_U, \hat{\kappa}_I)$ up to the second-order terms is evaluated as follows:

$$\begin{aligned}\kappa(\hat{\kappa}_U, \hat{\kappa}_I) \approx\ & \kappa(\hat{\kappa}_U, \hat{\kappa}_I)|_{\kappa_U,\kappa_I} + \\ & (\hat{\kappa}_U - \kappa_U)\frac{\partial\kappa(\hat{\kappa}_U, \hat{\kappa}_I)}{\partial\hat{\kappa}_U}\bigg|_{\kappa_U,\kappa_I} + (\hat{\kappa}_I - \kappa_I)\frac{\partial\kappa(\hat{\kappa}_U, \hat{\kappa}_I)}{\partial\hat{\kappa}_I}\bigg|_{\kappa_U,\kappa_I} \\ & + \frac{1}{2}(\hat{\kappa}_U - \kappa_U)^2\frac{\partial^2\kappa(\hat{\kappa}_U, \hat{\kappa}_I)}{\partial\hat{\kappa}_U^2}\bigg|_{\kappa_U,\kappa_I} + \frac{1}{2}(\hat{\kappa}_I - \kappa_I)^2\frac{\partial^2\kappa(\hat{\kappa}_U, \hat{\kappa}_I)}{\partial\hat{\kappa}_I^2}\bigg|_{\kappa_U,\kappa_I} \\ & + (\hat{\kappa}_U - \kappa_U)(\hat{\kappa}_I - \kappa_I)\frac{\partial^2\kappa(\hat{\kappa}_U, \hat{\kappa}_I)}{\partial\hat{\kappa}_U\partial\hat{\kappa}_I}\bigg|_{\kappa_U,\kappa_I}\end{aligned} \tag{B.2}$$

After applying the expectation operator to the terms of the Taylor's series expansion in Eq. B.2 the following expression is obtained:

$$\begin{aligned}E\left[\kappa(\hat{\kappa}_U, \hat{\kappa}_I)\right] &\approx \frac{1}{\kappa_U} + \kappa_I - 1 + \frac{\mathrm{Var}(\hat{\kappa}_U)}{\kappa_U^3} \\ &= E\left[\kappa(\kappa_U, \kappa_I)\right] + \frac{\mathrm{Var}(\hat{\kappa}_U)}{\kappa_U^3}\end{aligned} \tag{B.3}$$

Based on Eq. B.3 the bias in $\kappa(\hat{\kappa}_U, \hat{\kappa}_I)$ can be approximated as follows:

$$E\left[\kappa(\hat{\kappa}_U, \hat{\kappa}_I)\right] - E\left[\kappa(\kappa_U, \kappa_I)\right] \approx \frac{\mathrm{Var}(\hat{\kappa}_U)}{\kappa_U^3} \tag{B.4}$$

Provided that the samples in Eq. 13i, $\kappa_{Uj}; j = 1, ..., N_S$ have the same variance, $\mathrm{Var}(\kappa_{Uj}) = \mathrm{Var}(\kappa_U)$ the bias becomes:

$$E\left[\kappa(\hat{\kappa}_U, \hat{\kappa}_I)\right] - E\left[\kappa(\kappa_U, \kappa_I)\right] \approx \frac{1}{N_U^e}\frac{\mathrm{Var}(\kappa_U)}{\kappa_U^3} \tag{B.5}$$

where $N_U^e = 1/\sum_{j=1}^{N_U} w_{F\cup\tilde{F}}^2(\mathbf{z}_j^{\perp})$ is the effective sample size, obtained as shown in Eq. A.11. From Eq. B.4 it follows that the estimator in Eq. B.1 is asymptotically unbiased as $\{E\left[\kappa(\hat{\kappa}_U, \hat{\kappa}_I)\right] - E\left[\kappa(\kappa_U, \kappa_I)\right]\} \to 0$ as $N_U^e \to \infty$.
The application of the variance operator to Eq. B.1 leads the following expression:

$$\begin{aligned}\mathrm{Var}\left[\kappa(\hat{\kappa}_U, \hat{\kappa}_I)\right] &= E\left[\{\kappa(\hat{\kappa}_U, \hat{\kappa}_I) - E\left[\kappa(\hat{\kappa}_U, \hat{\kappa}_I)\right]\}^2\right] \\ &\approx E\left[\left\{\frac{1}{\hat{\kappa}_U} + \hat{\kappa}_I - \frac{1}{\kappa_U} - \kappa_I\right\}^2\right] \approx \frac{\mathrm{Var}(\hat{\kappa}_U)}{\kappa_U^4} + \mathrm{Var}(\hat{\kappa}_I) \\ &= \frac{1}{N_U^e}\frac{\mathrm{Var}(\kappa_U)}{\kappa_U^4} + \frac{1}{N_I^e}\mathrm{Var}(\kappa_I)\end{aligned} \tag{B.6}$$

where $N_U^e = 1/\sum_{j=1}^{N_U} w_{F\cup\tilde{F}}^2(\mathbf{z}_j^{\perp})$ and $N_I^e = 1/\sum_{j=1}^{N_U} w_{\tilde{F}}^2(\mathbf{z}_j^{\perp})$ are the effective sample sizes, obtained as shown in Eq. A.11. From Eq. B.6 it follows that $\kappa(\hat{\kappa}_U, \hat{\kappa}_I)$ in Eq. B.1 is a consistent estimator as the variance diminishes as $N_U^e \to \infty$ and $N_I^e \to \infty$.
Provided with the approximation of the bias in Eq. B.4, a modified estimator for κ can be derived as follows:

$$\begin{aligned}\kappa(\kappa_U, \kappa_I) \approx \kappa_b(\hat{\kappa}_U, \hat{\kappa}_I) &= \kappa(\hat{\kappa}_U, \hat{\kappa}_I) - \frac{1}{N_U^e}\frac{\mathrm{Var}(\kappa_U)}{\kappa_U^3} \\ &= \frac{1}{\hat{\kappa}_U} + \hat{\kappa}_I - 1 - \frac{1}{N_U^e}\frac{\mathrm{Var}(\kappa_U)}{\kappa_U^3}\end{aligned} \tag{B.7}$$

The expectation of the estimator in Eq. B.7 is:

$$E\left[\kappa_b(\hat{\kappa}_U, \hat{\kappa}_I)\right] = E\left[\kappa(\hat{\kappa}_U, \hat{\kappa}_I)\right] - \frac{1}{N_U^e}\frac{\mathrm{Var}(\kappa_U)}{\kappa_U^3} \approx \frac{1}{\kappa_U} + \kappa_I - 1 \tag{B.8}$$

The variance of the estimator in Eq. B.7 remains the same as in Eq. B.6:

$$\begin{aligned}\mathrm{Var}\left[\kappa_b(\hat{\kappa}_U, \hat{\kappa}_I)\right] &= E\left[(\kappa_b(\hat{\kappa}_U, \hat{\kappa}_I) - E\left[\kappa_b(\hat{\kappa}_U, \hat{\kappa}_I)\right])^2\right] \\ &= E\left[(\kappa(\hat{\kappa}_U, \hat{\kappa}_I) - E\left[\kappa(\hat{\kappa}_U, \hat{\kappa}_I)\right])^2\right] = \mathrm{Var}\left[\kappa(\hat{\kappa}_U, \hat{\kappa}_I)\right]\end{aligned} \tag{B.9}$$

Appendix C. Estimate of Failure Probability

The estimate of P_F with the corresponding variance is derived based on an unbiased estimate of the metamodel-based failure probability, $\hat{P}_{\tilde{F}}$, with the variance $\mathrm{Var}(\hat{P}_{\tilde{F}})$, and asymptotically unbiased estimate of the correction coefficient $\hat{\kappa}$ with the variance $\mathrm{Var}(\hat{\kappa})$. The estimates $\hat{P}_{\tilde{F}}$ and $\hat{\kappa}$ are independent as they are evaluated on samples from different distributions. The estimate of P_F then becomes:

$$\hat{P}_F = \hat{P}_{\tilde{F}} \cdot \hat{\kappa} \tag{C.1}$$

The variance of $\hat{P}_F$ is obtained as follows:

$$\begin{aligned}\mathrm{Var}(\hat{P}_F) &= \mathrm{Var}(\hat{P}_{\tilde{F}} \cdot \hat{\kappa}) \\ &= E(\hat{P}_{\tilde{F}}^2 \cdot \hat{\kappa}^2) - E(\hat{P}_{\tilde{F}} \cdot \hat{\kappa})^2 \\ &= E(\hat{P}_{\tilde{F}}^2) \cdot E(\hat{\kappa}^2) - E(\hat{P}_{\tilde{F}} \cdot \hat{\kappa})^2\end{aligned} \tag{C.2}$$

Given that $\mathrm{Var}(\hat{P}_{\tilde{F}}) = E(\hat{P}_{\tilde{F}}^2) - E(\hat{P}_{\tilde{F}})^2$ and $\mathrm{Var}(\hat{\kappa}) = E(\hat{\kappa}^2) - E(\hat{\kappa})^2$ the variance becomes:

$$\begin{aligned}\mathrm{Var}(\hat{P}_F) &= \left[\mathrm{Var}(\hat{P}_{\tilde{F}}) + E(\hat{P}_{\tilde{F}}^2)\right] \cdot \left[\mathrm{Var}(\hat{\kappa}) + E(\hat{\kappa}^2)\right] - E(\hat{P}_{\tilde{F}} \cdot \hat{\kappa})^2 \\ &= \mathrm{Var}(\hat{P}_{\tilde{F}})E(\hat{\kappa}^2) + E(\hat{P}_{\tilde{F}}^2)\mathrm{Var}(\hat{\kappa}) + \mathrm{Var}(\hat{P}_{\tilde{F}})\mathrm{Var}(\hat{\kappa})\end{aligned} \tag{C.3}$$

Since $\hat{P}_{\tilde{F}}$ is an unbiased estimate $E(\hat{P}_{\tilde{F}}) = P_{\tilde{F}}$. Similarly, since $\hat{\kappa}$ is asymptotically unbiased $E(\hat{\kappa}) = \kappa$. The variance becomes:

$$\mathrm{Var}(\hat{P}_F) = \mathrm{Var}(\hat{P}_{\tilde{F}})\kappa^2 + \mathrm{Var}(\hat{\kappa})P_{\tilde{F}}^2 + \mathrm{Var}(\hat{P}_{\tilde{F}})\mathrm{Var}(\hat{\kappa}) \quad \text{(C.4)}$$

In order to achieve a relatively accurate estimate of $\hat{P}_F$ with a low variance, it is necessary to achieve correspondingly low values of $\mathrm{Var}(\hat{P}_{\tilde{F}})$ and $\mathrm{Var}(\hat{\kappa})$. Given that $\hat{P}_{\tilde{F}}$ and $\mathrm{Var}(\hat{P}_{\tilde{F}})$ are evaluated with a computationally inexpensive metamodel, it is expected the majority of computational expenses will be utilized to achieve a relatively low value of the second term on the right side of the expression in Eq. C.4, $\mathrm{Var}(\hat{\kappa})P_{\tilde{F}}^2$. After including the expression from Eq. B.6 in $\mathrm{Var}(\hat{\kappa})P_{\tilde{F}}^2$, the following expression is obtained:

$$\mathrm{Var}(\hat{\kappa})P_{\tilde{F}}^2 \approx \left(\frac{1}{N_U^e}\frac{\mathrm{Var}(\kappa_U)}{\kappa_U^4} + \frac{1}{N_I^e}\mathrm{Var}(\kappa_I)\right)P_{\tilde{F}}^2 \quad \text{(C.5)}$$

From Eq. C.5 it can be observed that the decay of $\mathrm{Var}(\hat{\kappa})P_{\tilde{F}}^2$ is advanced by the increase in the effective sample sizes, and low values of $\mathrm{Var}(\kappa_U)$, $\mathrm{Var}(\kappa_I)$, and $P_{\tilde{F}}^2$. On the other hand, the convergence rate of $\mathrm{Var}(\hat{\kappa})P_{\tilde{F}}^2$ is decreased with relatively low values of κ_U. Given a relatively accurate metamodel approximation of the limit state, the events corresponding to κ_U and κ_I become less rare when compared to the failure event. Consequently, this provides a potential to achieve relatively low values of $\mathrm{Var}(\hat{\kappa}_U)$ and $\mathrm{Var}(\hat{\kappa}_I)$ with smaller effective sample sizes, relative to the sample size that would be required to achieve comparable $\mathrm{Var}(\hat{P}_F)$ in the direct estimation of P_F. Given the influence of κ on $\mathrm{Var}(\hat{P}_F)$ in Eqs. C.4 and C.5, it is advised to train a metamodel such that P_F and $P_{\tilde{F}}$ are on the same magnitude, with $0.1 \leq \kappa \leq 10$.

The coefficient of variation of the estimate, $\mathrm{CoV}(\hat{P}_F)$, is derived by dividing Eq. C.4 with $P_{\tilde{F}}^2 \cdot \kappa^2$.

$$\mathrm{CoV}(\hat{P}_F) = \sqrt{\mathrm{CoV}^2(\hat{P}_{\tilde{F}}) + \mathrm{CoV}^2(\hat{\kappa}) + \mathrm{CoV}^2(\hat{P}_{\tilde{F}})\mathrm{CoV}^2(\hat{\kappa})} \quad \text{(C.6)}$$

Given that the estimates $\hat{P}_{\tilde{F}}$ and $\hat{\kappa}$ are usually evaluated with $\mathrm{CoV}(\hat{P}_{\tilde{F}})$ and $\mathrm{CoV}(\hat{\kappa})$ between 1% and 10%, $\mathrm{CoV}(\hat{P}_F)$ can be approximated relatively accurately as follows:

$$\mathrm{CoV}(\hat{P}_F) \approx \sqrt{\mathrm{CoV}^2(\hat{P}_{\tilde{F}}) + \mathrm{CoV}^2(\hat{\kappa})} \quad \text{(C.7)}$$

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References

[1] Au, S.-K., & Beck, J. (2001). Estimation of small failure probabilities in high dimensions by subset simulation. *Probabilistic Engineering Mechanics*, 16.4, 263-277.

[2] Au, S., & Beck, J. (2003). Important sampling in high dimensions. *Structural Safety*, 25(2), 139-163.

[3] Bourinet, J. M., Deheeger, F., & Lemaire, M. (2011). Assessing small failure probabilities by combined subset simulation and support vector machines. *Structural Safety*, 33(6), 343-353.

[4] Bucher, C. G., & Bourgund, U. (1990). A fast and efficient response surface approach for structural reliability problems. *Structural safety*, 7(1), 57-66.

[5] Lophaven S., Nielsen H., Sondergaard J. (2002). Aspects of the Matlab Toolbox DACE. *Report IMM-REP-2002-13, Informatics and Mathematical Modelling, DTU*.

[6] de Angelis, M., Patelli, E., & Beer, M. (2015). Advanced Line Sampling for efficient robust reliability analysis. *Structural Safety*, 52, 170-182.

[7] Depina, I., Le, T. M. H., Eiksund, G. & Benz, T. (2015). Behavior of cyclically loaded monopile foundations for offshore wind turbines in heterogeneous sands. *Computers and Geotechnics*, 65, 266-277.

[8] Der Kiureghian, A., & Liu, P. L. (1986). Structural reliability under incomplete probability information. *Journal of Engineering Mechanics*, 112(1), 85-104.

[9] Dubourg, V., & Sudret, B. (2014). Metamodel-based importance sampling for reliability sensitivity analysis. *Structural Safety*, 49, 27-36.

[10] DNV. (2010). Design of Offshore Wind Turbine Structures, Det Norske Veritas.

[11] Fenton, G. A., & Griffiths, D. V. (2008). Risk assessment in geotechnical engineering, John Wiley & Sons, New York.

[12] Gamerman, D., & Lopes, H. F. (2006). Markov Chain Monte Carlo: Stochastic Simulation for Bayesian Inference: Taylor & Francis.

[13] Grooteman, F. (2008). Adaptive radial-based importance sampling method for structural reliability. *Structural Safety*, 30(6), 533-542.

[14] Hastie, T., Tibshirani, R., & Friedman, J. J. H. (2001). The elements of statistical learning (Vol. 1): Springer New York.

[15] Hastings, W. K. (1970). Monte Carlo sampling methods using Markov chains and their applications. *Biometrika*, 57(1), 97-109.

[16] Hurtado, J. E., & Alvarez, D. A. (2003). Classification approach for reliability analysis with stochastic finite-element modeling. *Journal of Structural Engineering*, 129(8), 1141-1149.

[17] Jones, D. R., Schonlau, M., & Welch, W. J. (1998). Efficient global optimization of expensive black-box functions. *Journal of Global optimization*, 13(4), 455-492.

[18] Katafygiotis, L. S., & Zuev, K. M. (2008). Geometric insight into the challenges of solving high-dimensional reliability problems. *Probabilistic Engineering Mechanics*, 23(2), 208-218.

[19] Koutsourelakis, P., Pradlwarter, H., & Schuëller, G. (2004). Reliability of structures in high dimensions, part I: algorithms and applications. *Probabilistic Engineering Mechanics*, 19(4), 409-417.

[20] Liu, P. L.,& Der Kiureghian, A. (1986). Multivariate distribution models with prescribed marginals and covariances. *Probabilistic Engineering Mechanics*, 1(2), 105-112.

[21] McKay, M. D., Beckman, R. J., & Conover, W. J. (1979). Comparison of three methods for selecting values of input variables in the analysis of output from a computer code. *Technometrics*, 21(2), 239-245.

[22] Montgomery, D. C. (2004). Design and analysis of experiments. John Wiley & Sons, New York.

[23] Montgomery D. C., & Runger G. C. (2010). Applied statistics and probability for engineers, John Wiley & Sons, New York.

[24] Nadim F. (2007). Tools and strategies for dealing with uncertainty in geotechnics, *Probabilistic methods in geotechnical engineering*, Springer Vienna.

[25] Papadopoulos, V., Giovanis, D. G., Lagaros, N. D., & Papadrakakis, M. (2012). Accelerated subset simulation with neural networks for reliability analysis. *Computer Methods in Applied Mechanics and Engineering*, 223, 70-80.

[26] Papadrakakis, M., Papadopoulos, V., & Lagaros, N. D. (1996). Structural reliability analyis of elastic-plastic structures using neural networks and Monte Carlo simulation. *Computer methods in applied mechanics and engineering*, 136(1), 145-163.

[27] Pradlwarter, H. J., Schueller, G. I., Koutsourelakis, P. S., & Charmpis, D. C. (2007). Application of line sampling simulation method to reliability benchmark problems. *Structural Safety*, 29(3), 208-221.

[28] Schuëller G., Pradlwarter H., & Koutsourelakis P. (2004). A critical appraisal of reliability estimation procedures for high dimensions. *Probabilistic Engineering Mechanics*, 19.4, 463-474.