SOLUTIONS OF THE CAMASSA–HOLM EQUATION WITH ACCUMULATING BREAKING TIMES

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ABSTRACT. We present two initial profiles to the Camassa–Holm equation which yield solutions with accumulating breaking times.

1. INTRODUCTION

The Camassa–Holm (CH) equation [3, 4]

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1.1}$$

which serves as a model for shallow water [11], has been studied intensively over the last twenty years, due to its rich mathematical structure. For example, it is bi-Hamiltonian [13], completely integrable [6] and has infinitely many conserved quantities, see e.g. [23]. Yet another property attracted considerable attention: Even smooth initial data can lead to classical solutions, which only exist locally due to wave breaking, see e.g. [7, 8, 9]. That means the spatial derivative $u_x(t,.)$ of the solution $u(t,.)$ becomes unbounded within finite time, while $\|u(t,.)\|_{H^1}$ remains bounded. In addition, energy concentrates on sets of measure zero when wave breaking takes place. Neglecting this concentration yields to a dissipation and hence to the so-called dissipative solutions [2, 21]. However, taking care of the energy, yields another class of solutions, the conservative ones [1, 20]. Moreover, it is also possible to take out only a fraction of the concentrated energy, giving rise to the recently introduced α-dissipative solutions [18]. A very illustrating example for this phenomenon is given by the so-called peakon-antipeakon solutions, which enjoy wave breaking and therefore can be prolonged thereafter in various ways as presented in detail in [15, 18].

However, as already the study of the peakon-antipeakon solutions shows, there are only very few solutions of the CH equation, which can be computed explicitly. Even in the case of travelling wave solutions, which have been classified by J. Lenells in [22], some of them are only given implicitely. Having a close look at the construction of various types of solutions [1, 2, 17, 20, 21], reveals that they are based on a reformulation of the CH equation as a system of ordinary differential equations in a suitable Banach space by a generalized method of characteristics. Thus computing solutions explicitly, would involve a change of variables from Eulerian to Lagrangian coordinates, solving the equation in Lagrangian coordinates and finally changing back from Lagrangian to Eulerian coordinates, as outlined in Section 2. A task

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which in many cases seems to be impossible. Thus most results concerning the prediction of wave breaking are obtained by following solutions along characteristics, see [5, 7, 14].

However, a good understanding of solutions along characteristics allows the prediction of wave breaking in the nearby future, and in the case of conservative solutions, which one can follow both forward and backward in time, also to find out if wave breaking occurred recently, [14]. Based on this knowledge it is possible to construct some initial data $u_0(x)$, which has $t = 0$ as an accumulation point of breaking times in the conservative case, as we will see in Section 3. We will prove the following result.

Theorem 1.1. *Let $q \in (0, 1)$ and*

$$u_0(x) = \begin{cases} \frac{1}{2}q(1 + \frac{1-q^4}{2}x), & \text{for } x \in [-\frac{2}{1-q^4}, 0], \\ k_j x + d_j, & \text{for } x \in [x_j, x_{j+1}], j \in \mathbb{N} \\ 0, & \text{otherwise}, \end{cases} \tag{1.2}$$

where the endpoints of the intervals $[x_j, x_{j+1}]$ are inductively defined through

$$x_0 = 0, \quad x_{2j+2} - x_{2j+1} = x_{2j+1} - x_{2j} = q^{4j} \tag{1.3}$$

and the constants k_j and d_j satisfy

$$\begin{aligned} k_{2j} &= -\frac{1}{q^{j-1}}, \quad k_{2j+1} = \frac{1}{2}\frac{q + q^4}{q^j}, \\ d_{2j} &= \frac{1}{2}\frac{1}{q^{j-1}(1-q^4)}(4 - 3q^{4j} - q^{4(j+1)}), \\ d_{2j+1} &= -\frac{1}{2}\frac{1}{q^{j-1}(1-q^4)}(2 + 2q^3 - q^{4j+3} - 2q^{4j+4} - q^{4j+7}). \end{aligned}$$

Furthermore, denote by $u(t, x)$ the conservative solution of the CH equation with initial data $u(0, x) = u_0(x)$, then $t = 0$ is an accumulation point of breaking times.

In Section 4 we are going to have a closer look at the cuspons with exponential decay [22], a class of travelling wave solutions with non-vanishing asymptotics, for which wave breaking takes place at any time. However, for any fixed time t, the Radon measure is purely absolutely continuous, which means that no energy is concentrated on sets of measure zero. This means in particular, that the set of points where wave breaking occurs at time t, consists of a single point both in Eulerian and Lagrangian coordinates, as we will see. Additional we are going to show that the breaking point is not traveling along one characteristic with respect to time, but is metaphorically speaking, jumping from one characteristic to the next one. These observations are very interesting since usually wave breaking is linked to the concentration of energy on sets of measure zero in Eulerian coordinates, which corresponds to wave breaking taking place on sets of positive measure in Lagrangian coordinates. Thus it is natural that manipulating the concentrated energy gives rise to different solution concepts. For this example however the question turns up of how the dissipative solution looks like? Does it coincide with the conservative one or not?

Finally, Section 5 concludes this note, by an observation concerning the concentration of energy in the case of accumulating breaking times.

2. Background material

The aim of this section is to outline the construction of conservative solutions of the CH equation, which consists of two main parts. On the one hand the interplay of Eulerian and Lagrangian coordinates and on the other hand the reformulation of the CH equation in Lagrangian coordinates. We will restrict ourselves to presenting those results, which are going to play a key role in what follows. For details we refer the interested reader to [14] and [18].

Let us start with the interplay between Eulerian and Lagrangian coordinates. It is well-known that solutions of the CH equation might enjoy wave breaking within finite time. This means that the solution itself remains bounded while its spatial derivative becomes unbounded from below pointwise. In particular, energy concentrates on sets of measure zero at breaking time, which can be described with the help of positive, finite Radon measures. Thus, the admissible set of Eulerian coordinates, $\mathcal{D}$, is defined as follows.

Definition 2.1 (Eulerian coordinates)**.** *The set $\mathcal{D}$ is composed of all pairs (u,μ) such that $u\in H^1(\mathbb{R})$ and μ is a positive, finite Radon measure whose absolutely continuous part, μ_{ac}, satisfies*

$$\mu_{\mathrm{ac}} = u_x^2\,dx. \tag{2.1}$$

Rewriting the CH equation in the weak formulation yields

$$u_t + uu_x + P_x = 0 \tag{2.2}$$

where

$$P(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}u^2(t,z)dz + \frac{1}{4}\int_{\mathbb{R}} e^{-|x-z|}d\mu(t,z). \tag{2.3}$$

A close inspection of (2.2) reveals that one can try to compute solutions of the CH equation using the method of characteristics. Indeed, this is possible but only under the assumption that μ is a purely absolutely continuous Radon measure and that we are given some initial characteristic $y_0(\xi)=y(0,\xi)$. Moreover, due to wave breaking we can only expect to obtain local solutions so far. Thus $y(t,\xi)$ is the solution to

$$y_t(t,\xi) = u(t,y(t,\xi)), \tag{2.4}$$

for some given initial data $y_0(\xi)=y(0,\xi)$, and additional we introduce the function

$$U(t,\xi) = u(t,y(t,\xi)), \tag{2.5}$$

whose time evolution is given through (2.2).

In the general case where μ is a positive finite Radon measure, we use the following mapping from Eulerian coordinates $\mathcal{D}$ to Lagrangian coordinates $\mathcal{F}$ to obtain an admissible initial characteristic $y_0(\xi)$ for any initial data $(u_0,\mu_0)\in\mathcal{D}$.

Definition 2.2. *For any (u,μ) in $\mathcal{D}$, let*

$$y(\xi) = \sup\left\{y \mid \mu((-\infty,y)) + y < \xi\right\}, \tag{2.6a}$$

$$h(\xi) = 1 - y_\xi(\xi), \tag{2.6b}$$

$$U(\xi) = u\circ y(\xi). \tag{2.6c}$$

Then $(y,U,h)\in\mathcal{F}$. We denote by $L\colon\mathcal{D}\to\mathcal{F}$ the mapping which to any element $(u,\mu)\in\mathcal{D}$ associates $X=(y,U,h)\in\mathcal{F}$ given by (2.6).

The big advantage of this change of variables is due to the fact that y, U, and h are all functions, and they will remain functions for all times. However there are a lot of properties these functions have to satisfy to enable us to construct global (conservative) solutions to the CH equation. All of them are collected in the following definition.

Definition 2.3 (Lagrangian coordinates)**.** *The set $\mathcal{F}$ is composed of all $X = (\zeta, U, h)$, such that*

$$(\zeta, U, h, \zeta_\xi, U_\xi) \in L^\infty(\mathbb{R}) \times [L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})]^4, \tag{2.7a}$$

$$y_\xi \geq 0, h \geq 0, y_\xi + h > 0 \textit{ almost everywhere}, \tag{2.7b}$$

$$y_\xi h = U_\xi^2 \textit{ almost everywhere}, \tag{2.7c}$$

$$y + H \in G, \tag{2.7d}$$

where we denote $y(\xi) = \zeta(\xi) + \xi$ and $H(\xi) = \int_{-\infty}^{\xi} h(\eta) d\eta$.

Here we denote by G the subgroup of the group of homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$ such that

$$f - \text{Id} \text{ and } f^{-1} - \text{Id both belong to } W^{1,\infty}(\mathbb{R}), \tag{2.8a}$$

$$f_\xi - 1 \text{ belongs to } L^2(\mathbb{R}), \tag{2.8b}$$

where Id denotes the identity function. In particular, G coincides with the set of relabelling functions, which enable us to identify equivalence classes in $\mathcal{F}$. This is necessary since we have 3 Lagrangian coordinates in contrast to 2 Eulerian coordinates.

In the case of conservative solutions, the reformulation of the CH equation in Lagrangian coordinates is given through

$$\zeta_t = U, \tag{2.9a}$$

$$\zeta_{\xi,t} = U_\xi, \tag{2.9b}$$

$$U_t = -Q, \tag{2.9c}$$

$$U_{\xi,t} = \frac{1}{2}h + (U^2 - P)y_\xi, \tag{2.9d}$$

$$h_t = 2(U^2 - P)U_\xi, \tag{2.9e}$$

where

$$P(t, \xi) = \frac{1}{4}\int_{\mathbb{R}} e^{-|y(t,\xi)-y(t,\eta)|}(2U^2 y_\xi + h)(t, \eta)\, d\eta, \tag{2.10}$$

and

$$Q(t, \xi) = -\frac{1}{4}\int_{\mathbb{R}} \text{sign}(\xi - \eta) e^{-|y(t,\xi)-y(t,\eta)|}(2U^2 y_\xi + h)(t, \eta)\, d\eta. \tag{2.11}$$

One can show that both $P(t, .)$ and $Q(t, .)$ belong to $H^1(\mathbb{R})$ and that this system of ordinary differential equations admits global unique solutions in $\mathcal{F}$.

Hence it remains to get back from Lagrangian to Eulerian coordinates

Definition 2.4. *Given any element $X = (y, U, h) \in \mathcal{F}$, we define (u, μ) as follows*

$$u(x) = U(\xi) \textit{ for any } \xi \textit{ such that } x = y(\xi), \tag{2.12a}$$

$$\mu = y_\#(h(\xi)\, d\xi), \tag{2.12b}$$

We have that (u,μ) belongs to $\mathcal{D}$. We denote by $M\colon \mathcal{F}\to\mathcal{D}$ the mapping which to any X in $\mathcal{F}$ associates the element $(u,\mu)\in\mathcal{D}$ as given by (2.12)*.*

In particular, the mapping M maps elements $X\in\mathcal{F}$ belonging to one and the same equivalence class in $\mathcal{F}$ to one and the same element in $\mathcal{D}$, see [20].

Before focusing on results related to our further investigations, we would like to emphasize that there are several ways to prolong solutions past wave breaking. Most of them are related to how the energy is manipulated at breaking time. In the case of conservative solutions we mean that the total amount of energy remains unchanged with respect to time, that is

$$\|u(t,.)\|_{L^2}^2+\mu(t,\mathbb{R})=\|u_0\|_{L^2}^2+\mu_0(\mathbb{R}).$$

For a detailed discussion of this topic we refer to [18] and for more information on how wave breaking is described in Eulerian and Lagrangian coordinates we refer to [14].

Of great importance for us, will be the prediction of wave breaking. Although we cannot determine exactly at which points (t,x) wave breaking takes place, we can at least determine if wave breaking occurs in the nearby future or not. The result is contained in the following theorem, which is a slight reformulation of [14, Theorem 1.1].

Theorem 2.5. *Given $(u_0,\mu_0)\in\mathcal{D}$ and denote by $(u(t),\mu(t))\in\mathcal{D}$ the global conservative solution of the CH equation at time t with initial data (u_0,μ_0). Moreover, let $C=2(\|u_0\|_{L^2}^2+\mu_0(\mathbb{R}))$, then the following holds:*

(i) If $u_{0,x}(x)<-\sqrt{2C}$ for some $x\in\mathbb{R}$, then wave breaking will occur within the time interval $[0,T]$, where T denotes the solution of

$$\frac{u_{0,x}(x)+\sqrt{2C}}{u_{0,x}(x)-\sqrt{2C}}=\exp(-\sqrt{2C}T). \tag{2.13}$$

(ii) If $u_{0,x}(x)>\sqrt{2C}$ for some $x\in\mathbb{R}$, then wave breaking occured within the time interval $[T,0]$, where T denotes the solution of

$$\frac{u_{0,x}(x)+\sqrt{2C}}{u_{0,x}(x)-\sqrt{2C}}=\exp(-\sqrt{2C}T). \tag{2.14}$$

As far as the situation in Lagrangian coordinates is concerned much more is known, due to the fact that the prediction of wave breaking is based on following solutions along characteristics. A key role, in that context, plays the following set,

$$\kappa_{1-\gamma}=\{\xi\in\mathbb{R}\mid \frac{h_0}{y_{0,\xi}+h_0}(\xi)\geq 1-\gamma,\, U_{0,\xi}(\xi)\leq 0\},\quad \gamma\in[0,\frac{1}{2}]. \tag{2.15}$$

Every condition imposed on points $\xi\in\kappa_{1-\gamma}$ is motivated by what is known about wave breaking. Indeed, if wave breaking occurs at some time t_b, then energy is concentrated on sets of measure zero in Eulerian coordinates, which correspond to the sets where $\frac{h}{y_\xi+h}(t_b,\xi)=1$ in Lagrangian coordinates. Furthermore, at time t_b the solution u is bounded while its spatial derivative u_x becomes unbounded from below pointwise, see [10, 19]. In Lagrangian coordinates this means that $U_\xi(t_b,\xi)=0$ and $U_\xi(t,\xi)$ changes sign from negative to positive at breaking time, [14].

The next lemma, which is a reformulation of [18, Corollary 18], enables us to predict wave breaking in the nearby future along characteristics.

Lemma 2.6. *Let $X_0 \in \mathcal{F}$. Denote by $X = (\zeta, U, \zeta_\xi, U_\xi, h) \in C(\mathbb{R}_+, \mathcal{F})$ the global solution of (2.9) with initial data X_0 and by $\tau_1(\xi) \geq 0$ the first time wave breaking occurs at the point ξ. Moreover, let $M = \left\| U_0^2 y_{0,\xi} + h_0 \right\|_{L^1}$. Then the following statements hold:*

(i) We have

$$\left\| \frac{1}{y_\xi + h}(t, \cdot) \right\|_{L^\infty} \leq 2e^{C(M)T} \left\| \frac{1}{y_{0,\xi} + h_0} \right\|_{L^\infty}, \tag{2.16}$$

and

$$\|(y_\xi + h)(t, \cdot)\|_{L^\infty} \leq 2e^{C(M)T} \|y_{0,\xi} + h_0\|_{L^\infty} \tag{2.17}$$

for all $t \in [0, T]$ and a constant $C(M)$ which depends on M.

(ii) There exists a $\gamma \in (0, \frac{1}{2})$ depending only on M such that if $\xi \in \kappa_{1-\gamma}$, then $\frac{y_\xi}{y_\xi + h}(t, \xi)$ is a decreasing function and $\frac{U_\xi}{y_\xi + h}(t, \xi)$ is an increasing function, both with respect to time for $t \in [0, \min(\tau_1(\xi), T)]$. Therefore we have

$$\frac{U_{0,\xi}}{y_{0,\xi} + h_0}(\xi) \leq \frac{U_\xi}{y_\xi + h}(t, \xi) \leq 0 \quad \text{and} \quad 0 \leq \frac{y_\xi}{y_\xi + h}(t, \xi) \leq \frac{y_{0,\xi}}{y_{0,\xi} + h_0}(\xi), \tag{2.18}$$

for $t \in [0, \min(\tau_1(\xi), T)]$. In addition, for γ sufficiently small, depending only on M and T, we have

$$\kappa_{1-\gamma} \subset \{\xi \in \mathbb{R} \mid 0 \leq \tau_1(\xi) < T\}. \tag{2.19}$$

(iii) For any given $\gamma > 0$, there exists $\hat{T} > 0$ such that

$$\{\xi \in \mathbb{R} \mid 0 < \tau_1(\xi) < \hat{T}\} \subset \kappa_{1-\gamma}. \tag{2.20}$$

(i) ensures that the function $\frac{h}{y_\xi + h}(t, \xi)$ is well-defined. (ii) gives us a possibility to fix at first some time interval $[0, T]$ and thereafter by finding a suitable γ to identify points which enjoy wave breaking for sure. (iii) on the other hand gives us a possibility to choose γ first and identifying a time interval $[0, T]$ thereafter, such that $\kappa_{1-\gamma}$ contains all points enjoying wave breaking within $[0, T]$.

Obviously the question occurs if a point ξ in Lagrangian coordinates can enjoy wave breaking infinitely many times within a fixed time interval $[0, T]$. According to [18, Corollary 19], which we state here for the sake of completeness, this is not possible.

Lemma 2.7 ([18] Corollary 19)**.** *Denote by $X(t) = (y, U, y_\xi, U_\xi, h)(t)$ the global solution of (2.9) with $X(0) = X_0 \in \mathcal{F}$ in $C(\mathbb{R}_+, \mathcal{F})$. Let $M = \left\| U_0^2 y_{0,\xi} + h_0 \right\|_{L^1}$ and denote by $\tau_j(\xi) \geq 0$ the j'th time wave breaking occurs at a point $\xi \in \mathbb{R}$. Then for any $\xi \in \mathbb{R}$ the possibly infinite sequence $\tau_j(\xi)$ cannot accumulate.*

In particular, there exists a time $\hat{T}$ depending on M such that any point ξ can experience wave breaking at most once within the time interval $[T_0, T_0 + \hat{T}]$ for any $T_0 \geq 0$. More precisely, given $\xi \in \mathbb{R}$, we have

$$\tau_{j+1}(\xi) - \tau_j(\xi) > \hat{T} \text{ for all } j. \tag{2.21}$$

In addition, for $\hat{T}$ sufficiently small, we get that in this case $U_\xi(t, \xi) \geq 0$ for all $t \in [\tau_j(\xi), \tau_j(\xi) + \hat{T}]$.

Here it is important to note that we can only say that the sequence $\tau_j(\xi)$ for $\xi \in \mathbb{R}$ does not accumulate in Lagrangian coordinates. We have, however, no possibility to conclude that the same result holds in Eulerian coordinates for $x \in \mathbb{R}$, since we follow the solution in Lagrangian coordinates along characteristics.

Remark 2.8. *In Section* 4 *we are going to look at the case of a traveling wave solution* $u(t,x)$ *with nonvanishing, but equal asymptotics as* $x \to \pm\infty$. *That is, there exists* $c \in \mathbb{R}$ *such that* $u(t,x) - c \in H^1(\mathbb{R})$ *for all* $t \in \mathbb{R}$. *Also in this case the description of solutions in Lagrangian coordinates is possible, by slightly changing the definition of* $\mathcal{D}$ *and* $\mathcal{F}$, *while leaving Definition* 2.2, *Definition* 2.4 *and* (2.9)–(2.11) *unchanged. To be more explicit we have to replace* $U(\xi) \in H^1(\mathbb{R})$ *by* $U(\xi) - c \in H^1(\mathbb{R})$ *in Definition* 2.3, *while the remaining assumptions remain unchanged. Moreover,* $P(t,\xi) - c^2$ *and* $Q(t,\xi)$ *belong to* $H^1(\mathbb{R})$ *for all* $t \in \mathbb{R}$.

The set of points in Lagrangian coordinates where wave breaking occurs at time t *still coincides with*

$$\{\xi \in \mathbb{R} \mid y_\xi(t,\xi) = 0\}.$$

For details we refer the interested reader to [16] *and* [17].

3. Can breaking times accumulate for solutions of the CH equation?

The aim of this section is to identify some solutions of the CH equation with an accumulating sequence of breaking times. Since it is nearly impossible, beside of some special cases, to compute solutions explicitly, we constructed some initial data, whose corresponding conservative solution has an accumulating sequence of breaking times. The construction is based on Lemma 2.6 and Lemma 2.7, which make it unnecessary to compute the actual solution.

To be more specific, we aimed at finding some initial data $(u_0, \mu_0) \in \mathcal{D}$, with $\mu_0 = \mu_{0,ac}$ such that

- u_0 has compact support, that is $supp(u_0) = [-c, c]$ for some $c > 0$,
- u_0 is a piecewise linear and continuous function, that is there exists an increasing sequence x_j and two sequences k_j and d_j such that
  - (i) $x_{-1} = -c$ and $x_0 = 0$
  - (ii) $(x_{j+1} - x_j) \to 0$ as $j \to \infty$ (nonincreasing)
  - (iii) $k_{2j} < 0$ and $k_{2j+1} > 0$ for $j \in \mathbb{N}$
  - (iv) $-k_{2j} \to \infty$ and $k_{2j+1} \to \infty$ as $j \to \infty$ (strictly increasing)
  - (v) $u_0(x) = k_j x + d_j$ for $x \in [x_j, x_{j+1}]$, $j = -1, 0, 1, 2, \ldots$
- $u_0 \in H^1(\mathbb{R})$

Each of these assumptions is motivated by what is known about the prediction of wave breaking. Hence we want to have a close look at them, before turning to the proof of Theorem 1.1. Although u_0 has compact support, $u_{0,x}$ is not going to be bounded on $[0, c]$. Thus in order for u_0 to be in $H^1(\mathbb{R})$ there must be a balance between the increasing sequences $-k_{2j}$ and k_{2j+1} and the decreasing sequence $x_{j+1} - x_j$. Moreover, the condition $|k_j| \to \infty$ is necessary to impose since Lemma 2.6 (*ii*) points out that given some time interval $[0, T]$, we can be sure that wave breaking occurs for all points which lie inside $\kappa_{1-\gamma}$ for γ small enough. Thus if we can find for each $\gamma > 0$ infinitely many nonintersecting intervals in Lagrangian coordinates which lie inside $\kappa_{1-\gamma}$, the claim follows. So what does $\xi \in \kappa_{1-\gamma}$ mean? If μ_0 is absolutely continuous, which we assume, then

$$y_{0,\xi}(\xi) = \frac{1}{1 + u_{0,x}^2(y_0(\xi))} = \frac{1}{1 + k_j^2}, \quad \text{for } y_0(\xi) \in [x_j, x_{j+1}], \tag{3.1}$$

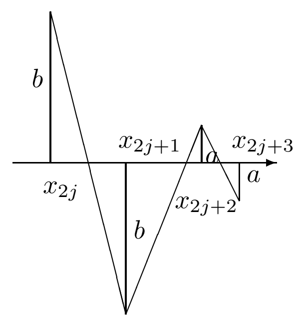

FIGURE 1. An illustration of the piecewise linear function u

which means that $y_0(\xi)$ is an increasing function. Thus

$$\frac{h_0(\xi)}{y_{0,\xi}(\xi)+h_0(\xi)} = 1 - \frac{y_{0,\xi}(\xi)}{y_{0,\xi}(\xi)+h_0(\xi)} = 1 - y_{0,\xi}(\xi) = 1 - \frac{1}{1+k_j^2}, \tag{3.2}$$

for $y_0(\xi) \in [x_j, x_{j+1}]$, where we used $y_{0,\xi}(\xi) + h_0(\xi) = 1$. Furthermore,

$$U_{0,\xi}(\xi) = u_{0,x}(y_0(\xi))y_{0,\xi}(\xi) = \frac{k_j}{1+k_j^2}, \quad \text{for } y_0(\xi) \in [x_j, x_{j+1}]. \tag{3.3}$$

Combining (3.2) and (3.3) yields $\xi \in \kappa_{1-\gamma}$ if and only if $y_0(\xi) \in [x_j, x_{j+1}]$ with $k_j < 0$ and $\gamma \geq \frac{1}{1+k_j^2}$. Thus our assumptions on k_j are chosen in such a way that $\kappa_{1-\gamma}$ consists of infinitely many nonintersecting intervals.

Proof of Theorem 1.1. That $u_0 \in H^1(\mathbb{R})$ relies on the fact that the geometric series $\sum_{n=0}^\infty q^n$ converges for $q \in (0,1)$ and the proof is left to the reader. We rather focus on showing the existence of a convergent sequence of breaking times.

Let $M = \|u_0\|_{H^1}^2 = \left\|U_0^2 y_{0,\xi} + h_0\right\|_{L^1}$, then according to Lemma 2.7 there exists a time $\hat{T}$ depending on M such that any point $\xi \in \mathbb{R}$ in Lagrangian coordinates can experience wave breaking at most once within the time interval $[0, \hat{T}]$. Moreover, Lemma 2.6 (ii) implies that there exists $\hat{\gamma} \in (0, \frac{1}{2})$ depending only on M and $\hat{T}$ such that

$$\kappa_{1-\hat{\gamma}} \subset \{\xi \in \mathbb{R} \mid 0 \leq \tau_1(\xi) < \hat{T}\}, \tag{3.4}$$

and such that (2.18) holds for all $\xi \in \kappa_{1-\hat{\gamma}}$.

Let $\gamma \in (0, \hat{\gamma})$ and $\xi \in \kappa_{1-\gamma}$, then we have

$$\left(\frac{U_\xi}{y_\xi + h}\right)_t = \frac{1}{2} + \left(U^2 - P - \frac{1}{2}\right)\frac{y_\xi}{y_\xi + h} - (2U^2 - 2P + 1)\frac{U_\xi^2}{(y_\xi + h)^2}, \tag{3.5}$$

and in particular $\frac{U_\xi}{y_\xi + h}(t,\xi)$ changes sign from negative to positive at breaking time $\tau_1(\xi) < \hat{T}$. Since $\left\|(U^2 - P)(t,.)\right\|_{L^\infty}$ can be bounded uniformly by a constant depending on M, we have that

$$\frac{1}{2} - C(M)\gamma \leq \left(\frac{U_\xi}{y_\xi + h}\right)_t \leq \frac{1}{2} + C(M)\gamma, \tag{3.6}$$

for some constant $C(M)$ only dependent on M. Assume additional that γ is so small that $\frac{1}{2}-C(M)\gamma>0$, then (3.6) allows us to derive an upper and a lower bound on $\tau_1(\xi)$ for $\xi\in\kappa_{1-\gamma}$ such that $\frac{h_0}{y_{0,\xi}+h_0}(\xi)=1-\gamma$. Indeed, on the one hand (3.6) implies that

$$\left(\frac{1}{2}-C(M)\gamma\right)t-\sqrt{\gamma(1-\gamma)}\leq\frac{U_\xi}{y_\xi+h}(t,\xi), \tag{3.7}$$

and hence $\tau_1(\xi)\leq T_{1,\gamma}=\frac{\sqrt{\gamma(1-\gamma)}}{\frac{1}{2}-C(M)\gamma}$. On the other hand, we have

$$\frac{U_\xi}{y_\xi+h}(t,\xi)\leq\left(\frac{1}{2}+C(M)\gamma\right)t-\sqrt{\gamma(1-\gamma)}, \tag{3.8}$$

which implies $T_{2,\gamma}=\frac{\sqrt{\gamma(1-\gamma)}}{\frac{1}{2}+C(M)\gamma}\leq\tau_1(\xi)$. Thus $T_{2,\gamma}\leq\tau_1(\xi)\leq T_{1,\gamma}$ for all $\xi\in\kappa_{1-\gamma}$ such that $\frac{h_0}{y_{0,\xi}+h_0}(\xi)=1-\gamma$. Seen as a function of γ, both $T_{1,\gamma}$ and $T_{2,\gamma}$ are strictly decreasing or equivalently

$$\lim_{\gamma\to 0}T_{1,\gamma}=0 \quad \text{and} \quad \lim_{\gamma\to 0}T_{2,\gamma}=0. \tag{3.9}$$

By definition the sequence $k_{2j}^2=\frac{1}{q^{2(j-1)}}\to\infty$ as $j\to\infty$. Thus the corresponding sequence $\gamma_j=\frac{1}{1+\kappa_{2j}^2}\to 0$ as $j\to\infty$ and each $\kappa_{1-\gamma_j}$ consists of infinitely many non-intersecting intervals. Thus choosing to every $j\in\mathbb{N}$ a point $z_j\in[x_{2j},x_{2j+1}]$ with breaking time t_j, the above argument shows that $t_j\to 0$. □

Remark 3.1. *Before continuing with another interesting example, we would like to point out another interesting fact about our constructed initial data u_0. According to Theorem* 2.5 *it is not only possible to predict if wave breaking occurs in the nearby future or not, but also if wave breaking occurred recently or not. Moreover, it is also possible to adapt in the conservative case Lemma* 2.6 *and Lemma* 2.7 *to going backward in time. Thus following a similar argument to the one presented above one can also show that there exists an increasing sequence of breaking times with limiting value zero.*

4. Cuspons with exponential decay - from a wave breaking point of view

In [22, 24] J. Lenells classified all travelling wave solutions of the Camassa–Holm equation. One of them, namely the cuspons with exponential decay, serve as an interesting example of solutions enjoying wave breaking. On the one hand wave breaking occurs at any time and on the other hand the energy is not concentrated on sets with positive measure in Lagrangian coordinates, but in a single point which does not correspond to the singular continuous part of the positive, finite Radon measure.

Definition 4.1 (Cuspon with exponential decay)**.** *Given m, s, and M such that $m<s<M$, let $\kappa=\frac{1}{2}(s-2m-M)$, then the cuspon with exponential decay and speed $s+\kappa$ is defined through*

$$u(t,x)=\phi(x-(s+\kappa)t)+\kappa$$

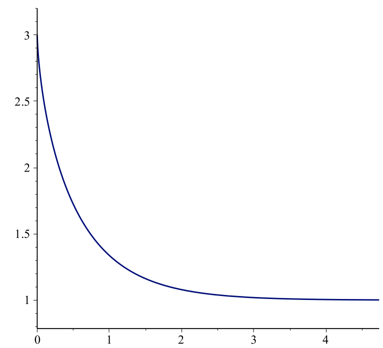

FIGURE 2. Plot of the function $\phi(x)$ for $m = 1$, $s = 3$, and $M = 5$.

where $\phi(x)$ is implicitly given through

$$\phi_x^2 = \frac{(M-\phi)(\phi-m)^2}{(s-\phi)} \tag{4.1}$$

and satisfies

$$\phi(-x) = \phi(x) \tag{4.2a}$$
$$\phi(0) = s \tag{4.2b}$$
$$\phi_x(x) < 0 \quad \textit{for} \quad x > 0 \tag{4.2c}$$
$$\lim_{x\to\infty} \phi(x) = m. \tag{4.2d}$$

Note that for any $\phi(x)$ defined through (4.1) and (4.2), we have that $m \leq \phi(x) \leq s$ for all $x \in \mathbb{R}$. Thus $\phi_x(x)$ is finite for all $x \in \mathbb{R}\backslash\{0\}$,

$$\lim_{x\to 0-} \phi_x(x) = \infty \quad \text{and} \quad \lim_{x\to 0+} \phi_x(x) = -\infty$$

and in particular

$$\lim_{x\to 0} \phi_x^2(x) = \infty.$$

This implies for $u(t,x) = \phi(x-(s+\kappa)t) + \kappa$, that $u_x(t,x)$ is well-defined for all $x \in \mathbb{R}\backslash\{(s+\kappa)t\}$ and

$$\lim_{x\to(s+\kappa)t} u_x^2(t,x) = \infty \qquad \text{for all } t.$$

Or in order words wave breaking occurs for the solution $u(t,x)$ defined in Definition 4.1 for all points (t,x) such that $x = (s+\kappa)t$.

Breaking points in Lagrangian coordinates. Let $u(0,x) = \phi(x) + \kappa$ and denote by $y_0(\xi) = y(0,\xi)$ the initial characteristics in Lagrangian coordinates given through (2.6a). Then we claim that the point $x = 0$ in Eulerian coordinates is mapped to the single point $\bar\xi$ in Lagrangian coordinates which satisfies $y_0(\bar\xi) = 0$ and

$$\{\bar\xi\} = \{\xi \in \mathbb{R} \mid y_{0,\xi}(\xi) = 0\}. \tag{4.3}$$

Let us start by considering the function $g(x) = x + \int_{-\infty}^x \phi_x^2(y)dy$, which has as a pseudo inverse the function $y(\xi)$ (cf. (2.6a)). If $\int_{-\infty}^x \phi_x^2(y)dy$ exists for all $x \in (-\infty, 0]$, then it follows from the symmetry of $\phi(x)$ that the function $g(x)$ is well-defined, strictly increasing and bijektiv. This means in particular to any $x \in \mathbb{R}$ there exists a unique $\xi \in \mathbb{R}$ such that $y(\xi) = x$. Indeed, we have for $x \leq 0$,

$$\begin{aligned}\int_{-\infty}^{x} \phi_x^2(y)dy &= \int_{-\infty}^{x} \frac{\sqrt{M-\phi(y)}(\phi(y)-m)}{\sqrt{s-\phi(y)}}\phi_x(y)dy \\ &= \int_{m}^{\phi(x)} \frac{\sqrt{M-z}(z-m)}{\sqrt{s-z}}dz \\ &\leq \sqrt{M-m}(s-m)\int_{m}^{s}\frac{1}{\sqrt{s-z}}dz \\ &= 2\sqrt{(M-m)(s-m)}(s-m) < \infty\end{aligned}$$

and hence $y(\xi)$ is bijective.

Denote by $\bar{\xi}$ the unique $\xi \in \mathbb{R}$ such that $y(\bar{\xi}) = 0$. Then we have that $\phi_x(x)$ is well-defined for all $x \in \mathbb{R}\backslash\{0\}$ and hence

$$y_{0,\xi}(\xi) = \frac{1}{1+\phi_x^2(y_0(\xi))} = \frac{(s-\phi(y_0(\xi)))}{(s-\phi(y_0(\xi))) + (M-\phi(y_0(\xi)))(\phi(y_0(\xi))-m)^2}$$

for all $\xi \in \mathbb{R}\backslash\{\bar{\xi}\}$. In particular, $y_{0,\xi}(\xi) > 0$ for all $\xi \in \mathbb{R}\backslash\{\bar{\xi}\}$. Thus if we can show that $y_{0,\xi}(\bar{\xi}) = 0$, we obtain as an immediate consequence (4.3). Therefore, let $x > 0$, then it follows by Definition 4.1 that

$$\begin{aligned}x = \int_0^x dy &= -\int_0^x \frac{\sqrt{s-\phi(y)}}{\sqrt{M-\phi(y)}(\phi(y)-m)}\phi_x(y)dy \\ &= \int_{\phi(x)}^{s} \frac{\sqrt{s-z}}{\sqrt{M-z}(z-m)}dz \\ &\geq \frac{1}{(M-m)^{3/2}}\int_{\phi(x)}^{s}\sqrt{s-z}dz \\ &= \frac{2}{3}\left(\frac{s-\phi(x)}{M-m}\right)^{3/2}\end{aligned}$$

or equivalently

$$0 \leq s-\phi(x) = \phi(0)-\phi(x) \leq (M-m)\left(\frac{3}{2}x\right)^{2/3} \qquad \text{for all } x \in \mathbb{R}$$

since $\phi(x)$ is symmetric. Hence, given some $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|\phi(0)-\phi(x)| < \varepsilon \quad \text{ for all } |x| < \delta.$$

Moreover, it is well-known that $y_0(\xi)$ is Lipschitz continuous with Lipschitz constant at most one. Thus

$$|y_0(\xi) - y_0(\bar{\xi})| < \delta \quad \text{ for all } |\xi - \bar{\xi}| < \delta$$

and the definition of $y_0(\xi)$ yields

$$\begin{aligned}|\bar{\xi}-\xi| &= |y_0(\bar{\xi})-y_0(\xi)| + |\int_{y_0(\xi)}^{y_0(\bar{\xi})}\phi_x^2(z)dz| \\ &\geq |y_0(\bar{\xi})-y_0(\xi)| + |\int_{y_0(\xi)}^{y_0(\bar{\xi})}\frac{(M-s)(s-\varepsilon-m)^2}{\varepsilon}dz|\end{aligned}$$

$$\geq |y_0(\bar{\xi}) - y_0(\xi)| \left(\frac{\varepsilon + (M-s)(s-\varepsilon-m)^2}{\varepsilon} \right)$$

Hence

$$\frac{y_0(\bar{\xi}) - y_0(\xi)}{\bar{\xi} - \xi} \leq \frac{\varepsilon}{\varepsilon + (M-s)(s-\varepsilon-m)^2} \quad \text{for } |\bar{\xi} - \xi| < \delta.$$

Since we can choose ε to be any positive real number, we have

$$y_{0,\xi}(\bar{\xi}) = \lim_{\xi \to \bar{\xi}} \frac{y_0(\bar{\xi}) - y_0(\xi)}{\bar{\xi} - \xi} = 0.$$

The cusp is not traveling along a single characteristic. Finally, we want to show that the cusp is not travelling along a characteristic in Lagrangian coordinates. In particular, we are going to show that the peak, metaphorically speaking, jumps from one characteristic to the next. Recall therefore that the cuspon with exponential decay and speed $s + \kappa$ is given by

$$u(t,x) = \phi(x - (s+\kappa)t) + \kappa.$$

Thus by definition we have

$$\begin{aligned} P(t,x) &= \frac{1}{4} \int_{\mathbb{R}} e^{-|x-z|} (2u^2 + u_x^2)(t,z) dz \\ &= \frac{1}{4} \int_{\mathbb{R}} e^{-|(x-(s+\kappa)t)-(z-(s+\kappa)t)|} (2(\phi(z-(s+\kappa)t)+\kappa)^2 + \phi_x^2(z-(s+\kappa)t)) dz \\ &= \frac{1}{4} \int_{\mathbb{R}} e^{-|(x-(s+\kappa)t)-z|} (2(\phi(z)+\kappa)^2 + \phi_x^2(z)) dz \\ &= P(0, x-(s+\kappa)t). \end{aligned}$$

Similar considerations yield

$$P_x(t,x) = P_x(0, x-(s+\kappa)t).$$

Moreover, $P(0,x)$ is an even function, and hence $P_x(0,x)$ is an odd function and satisfies

$$P_x(t,(s+\kappa)t) = P_x(0,0) = 0.$$

Denote by $y(t,\xi)$ the characteristics at time t, where $y(0,\xi)$ is defined through (2.6a) and satisfies

$$y_t(t,\xi) = U(t,\xi).$$

Then we have

$$U(t,\xi) = u(t,y(t,\xi)) = \phi(y(t,\xi) - (s+\kappa)t) + \kappa, \tag{4.4a}$$

$$P(t,\xi) = P(t,y(t,\xi)) = P(0, y(t,\xi) - (s+\kappa)t) \tag{4.4b}$$

$$Q(t,\xi) = P_x(t,y(t,\xi)) = P_x(0, y(t,\xi) - (s+\kappa)t) \tag{4.4c}$$

and hence

$$(y(t,\xi) - (s+\kappa)t)_t = \phi(y(t,\xi) - (s+\kappa)t) - s \leq 0. \tag{4.5}$$

Thus the characteristics to the left and to the right of the cusp travel at a speed slower than the one of the cusp. In particular, one can show using subsolutions, that to any $\xi > \bar{\xi}$ there exists a time $T > 0$ such that $y(t,\xi) > (s+\kappa)t$ for all $t < T$, $y(T,\xi) = (s+\kappa)T$ and $U(t,\xi) = (s+\kappa)$. Obviously we have for $\xi > \bar{\xi}$ that $y(t,\xi) < (s+\kappa)t$ for all $t > 0$.

Let $\tilde{\xi} \in \mathbb{R}$ such that $y(T, \tilde{\xi}) = (s+\kappa)T$, then we aim at showing that $y(t, \tilde{\xi}) < (s+\kappa)t$ for all $t > T$. Or equivalently, we can show for $z(t, \tilde{\xi}) = y(t, \tilde{\xi}) - (s+\kappa)t$, that $z(t, \tilde{\xi}) < 0$ for all $t > T$. Therefore note that

$$z_t(T, \tilde{\xi}) = U(T, \tilde{\xi}) - (s+\kappa) = \phi(z(T, \tilde{\xi})) - s = 0, \tag{4.6}$$

$$z_{tt}(T, \tilde{\xi}) = -Q(T, \tilde{\xi}) = -P_x(0, z(T, \tilde{\xi})) = 0 \tag{4.7}$$

$$z_{ttt}(T, \tilde{\xi}) = -Q_t(T, \tilde{\xi}) = ? \tag{4.8}$$

Hence, if we can show that $Q_t(T, \tilde{\xi})$ exists and is positive, it follows that the function $U(t, \tilde{\xi})$, seen as a function of t attains a maximum at $t = T$. Thus $z_t(t, \tilde{\xi}) < 0$ for all $t > T$ and, in particular, $z(t, \tilde{\xi}) < 0$ for all $t > T$.

Recall that $z(t, \xi)$ is bijective and continuous and that $Q(t, \xi) = P_x(0, z(t, \xi))$. Thus for all $\xi \in \mathbb{R} \backslash \{\tilde{\xi}\}$, we can apply the chain rule to $Q(t, \xi)$ and obtain

$$Q_t(T, \xi) = P_{xx}(0, z(T, \xi)) z_t(T, \xi) \tag{4.9}$$

$$= (P(0, z(T, \xi)) - (\phi(z(T, \xi)) - \kappa)^2)(\phi(z(T, \xi)) - s) \tag{4.10}$$

$$+ (M - \phi(z(T, \xi)))(\phi(z(T, \xi)) - m)^2. \tag{4.11}$$

If $Q_t(T, \xi)$ is continuous as a function of ξ, we get that

$$Q_t(T, \tilde{\xi}) = (M - s)(s - m)^2 > 0, \tag{4.12}$$

since $P(0, x)$ and $\phi(x)$ are uniformly bounded and $z_t(T, \tilde{\xi}) = 0$.

Thus establishing the existence and continuity of $Q_t(t, \xi)$ will finish the proof of the claim. By definition,

$$Q(t, \xi) = -\frac{1}{4} \int_{-\infty}^{\xi} e^{-(\xi - \eta)} e^{-(\zeta(t, \xi) - \zeta(t, \eta))} (2U^2 y_\xi(t, \eta) + h(t, \eta)) d\eta \tag{4.13}$$

$$+ \frac{1}{4} \int_{\xi}^{\infty} e^{-(\eta - \xi)} e^{-(\zeta(t, \eta) - \zeta(t, \xi))} (2U^2 y_\xi(t, \eta) + h(t, \eta)) d\eta, \tag{4.14}$$

where we used $y(t, \xi) = \xi + \zeta(t, \xi)$. Since the functions

$$f_1(t, \xi, \eta) = e^{-(\zeta(t, \xi) - \zeta(t, \eta))} (2U^2(t, \eta) y_\xi(t, \eta) + h(t, \eta)) \tag{4.15a}$$

$$f_2(t, \xi, \eta) = e^{-(\zeta(t, \eta) - \zeta(t, \xi))} (2U^2(t, \eta) y_\xi(t, \eta) + h(t, \eta)) \tag{4.15b}$$

are both differentiable with respect to t and $f_{1,t}(t, \xi, \eta)$ and $f_{2,t}(t, \xi, \eta)$ can be uniformly bounded for ξ, $\eta \in \mathbb{R}$ and t on a finite time interval, [12, Theorem 2.27] implies the existence of $Q_t(t, \xi)$ and that

$$\begin{aligned} Q_t(t, \xi) = \frac{1}{4} \int_{\mathbb{R}} & e^{-|y(t, \xi) - y(t, \eta)|} (U(t, \xi) - U(t, \eta))(2U^2 y_\xi(t, \eta) + h(t, \eta)) d\eta \\ - \frac{1}{4} \int_{\mathbb{R}} & \operatorname{sign}(\xi - \eta) e^{-|y(t, \xi) - y(t, \eta)|} \\ & \times (-4QU(t, \eta) y_\xi(t, \eta) + 4U^2 U_\xi(t, \eta) - 2PU_\xi(t, \eta)) d\eta. \end{aligned}$$

Finally, following the same lines as the proof of [16, Lemma 3.1], the continuity of $Q_t(t, \xi)$ with respect to ξ can be established.

5. Observation

We had now a closer look at two particular initial data for the CH equation, which yield on the one hand a solution, with accumulating breaking times, see Section 3 and on the other hand a solution which enjoys wave breaking at any time, but the associated measure has neither a discrete nor a singular continuous part, see Section 4. Thus naturally the question arises, can we find to some initial data $(u_0, \mu_0) \in \mathcal{D}$ a global conservative solution $(u(t,x), \mu(t,x)) \in \mathcal{D}$ such that there exist $0 < t_1 < t_2$ such that $\mathrm{supp}(\mu_d(t,x)) \neq \emptyset$ for all $t \in [t_1, t_2]$?

Reformulating this question in Lagrangian coordinates yields. Can we find some initial data $X_0 = (y_0, U_0, h_0) \in \mathcal{F}$ with solution $X(t) = (y(t), U(t), h(t))$ such that there exists $0 < t_1 < t_2$ such that $y_\xi(t,\xi) = 0$ on an interval of positive length for all $t \in [t_1, t_2]$?

Due to [20, Lemma 2.7 (ii)], which we state in a moment for the sake of completeness, the answer is no.

Lemma 5.1 ([20, Lemma 2.7 (ii)]). *Given initial data $X_0 = (\zeta_0, U_0, h_0)$ in $\mathcal{F}$, let $X(t) = (\zeta(t), U(t), h(t))$ be the short-time solution of (2.9) in $C([0,T], \mathcal{F})$ for some $T > 0$ with initial data $X_0 = (\zeta_0, U_0, h_0)$. Then for almost every $t \in [0,T]$, $y_\xi(t,\xi) > 0$ for almost every $\xi \in \mathbb{R}$.*

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