OPERATOR SPLITTING FOR THE BENJAMIN–ONO EQUATION

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Abstract. In this paper we analyze operator splitting for the Benjamin–Ono equation, $u_t = uu_x + Hu_{xx}$, where H denotes the Hilbert transform. If the initial data are sufficiently regular, we show the convergence of both Godunov and Strang splitting.

1. Introduction

In this article, we are concerned with operator splitting for the Benjamin–Ono equation. The Benjamin–Ono equation models the evolution of weakly nonlinear internal long waves. It has been derived by Benjamin [4] and Ono [19] as an approximate model for long unidirectional waves at the interface of a two-layer system of incompressible inviscid fluids, one being infinitely deep. In non-dimensional variables, the initial value problem associated with the Benjamin–Ono equation reads

$$\begin{cases} u_t = uu_x + Hu_{xx}, & x \in \mathbb{R},\ 0 \le t \le T, \\ u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where H denotes the Hilbert transform defined by the principle value integral

$$Hu(x) := \text{P.V.}\,\frac{1}{\pi}\int_{\mathbb{R}} \frac{u(x-y)}{y}\,dy.$$

The Benjamin–Ono equation is, at least formally, completely integrable [3] and thus possesses an infinite number of conservation laws. For example, the momentum and the energy, given by

$$M(u) := \int u^2\,dx, \text{ and } E(u) := \frac{1}{2}\int \left|D_x^{1/2}u\right|^2\,dx + \frac{1}{6}\int u^3\,dx,$$

are conserved for solutions of (1.1).

We also consider the corresponding $2L$-periodic problem

$$\begin{cases} u_t = uu_x + H_{\text{per}}u_{xx}, & 0 \le t \le T, \\ u|_{t=0} = u_0, \end{cases} \qquad x \in \mathbb{T}, \tag{1.2}$$

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where $\mathbb{T} = \mathbb{R}/2L\mathbb{Z}$, u_0 is $2L$ periodic and the periodic Hilbert transform is defined by the principle value integral

$$H_{\mathrm{per}}u(x) = \mathrm{P.V.}\frac{1}{2L}\int_{-L}^{L} \cot(\frac{\pi}{2L}y)u(x-y)\,dy.$$

The initial value problem (1.1) has been extensively studied in recent years. Well-posedness of (1.1) in $H^s(\mathbb{R})$ for $s > 3$ was proved by Iorio [13] by using purely hyperbolic energy methods. Then, Ponce [20] derived a local smoothing effect associated to the dispersive part of the equation, which combined with compactness methods, enabled him to prove well-posedness also for $s = 3$.

By combining a complex version of the Cole–Hopf transform with Strichartz estimates, Tao [21] was able to show well-posedness of the Cauchy problem (1.1) in $H^1(\mathbb{R})$. This well-posedness was extended to $H^s(\mathbb{R})$ for $s > 1$ by Burq and Planchon [5] and for $s \geq 0$ by Ionescu and Kenig [12].

In the periodic setting, Molinet [18] proved global well-posedness in $H^s(\mathbb{T})$ for $s \geq 1/2$. Furthermore, he was able to improve the global well-posedness results to $L^2(\mathbb{T})$ in [17].

We employ operator splitting, i.e., the construction of an approximate solution by concatenating the solutions of the separate problems

$$v_t = Hv_{xx} \tag{1.3}$$

and

$$w_t = ww_x. \tag{1.4}$$

More precisely, the operator splitting method is built up as follows [8]: Consider a general partial differential equation

$$u_t = C(u), \quad u|_{t=0} = u_0, \tag{1.5}$$

where $C(u)$ is a differential operator. Furthermore, assume $C(u)$ can be written as a sum of more elementary operators, say

$$C(u) = A(u) + B(u).$$

For a positive and small time step Δt we discretize the time with n steps such that $t_n = n\Delta t < T$. Instead of solving equation (1.5) directly, we solve the two subequations

$$v_t = A(v), \quad \text{and} \quad w_t = B(w),$$

for each time step, and concatenate the solutions. The simplest form for an operator splitting solution of (1.5) is formed solving the first subequation using the solution from the second subequation as initial data when solving at each time step. Writing out this procedure gives

$$u_{n+1} = \Pi^{\Delta t}(u_n) = \Phi_A^{\Delta t} \circ \Phi_B^{\Delta t}(u_n) = [\Phi_A^{\Delta t} \circ \Phi_B^{\Delta t}]^n(u_0), \tag{1.6}$$

where u_n is the operator splitting solution at time t_n, thus $u_n \approx u(\,\cdot\,, t_n)$, and $\Phi_A^t(v_0)$ and $\Phi_B^t(w_0)$ are the exact solution operators of the above subequations at time t with initial data v_0 and w_0, respectively. This is the well-known *Godunov splitting method.*

Other and more sophisticated methods for forming an operator splitting solution of (1.5) are created by solving the two subequations for different split step sizes, and composing the solution operators in a more complicated way. By solving one of the subequations for half the step size composed with the solution of the other subequation for a full time step, we obtain the *Strang splitting method*, which is given as

$$u_{n+1} = \Psi^{\Delta t}(u_n) = \Phi_A^{\Delta t/2} \circ \Phi_B^{\Delta t} \circ \Phi_A^{\Delta t/2}(u_n) = [\Phi_A^{\Delta t/2} \circ \Phi_B^{\Delta t} \circ \Phi_A^{\Delta t/2}]^n(u_0). \tag{1.7}$$

For $t \in [t_n, t_{n+1})$ define $u_{\Delta t}(t)$ by

$$u_{\Delta t}(t) = \Pi^{t-t_n}(u_n),$$

in case of Godunov splitting and by

$$u_{\Delta t}(t) = \Psi^{t-t_n}(u_n),$$

in case of Strang splitting.

In our case A and B are given by

$$A(u) = H(u_{xx}), \quad \text{and} \quad B(u) = uu_x.$$

Our main results are that the operator splitting schemes converge in L^2 with a rate $\mathcal{O}(\Delta t)$ for the Godunov splitting, and at a rate of $\mathcal{O}(\Delta t^2)$ for the Strang splitting. However, we mention that our method requires a well-posedness theory for the full Benjamin–Ono equation, and cannot be used as a constructive existence theorem. The approach applied here has successfully been applied to a plethora of other equations including the Korteweg–de Vries (KdV) equation, the Schrödinger–Poisson, the cubic nonlinear Schrödinger equation, the viscous Burgers' equation, the Benney–Lin equation, the Kawahara equation, as well as the active scalar equation [9, 16, 10, 11, 6, 7]. However, we stress that each equation requires its own estimates and individual treatment. In the present case both the Hilbert transform and the rather restricted well-posedness of the Benjamin–Ono equation pose new technical challenges.

The rest of the paper is organized as follows: In Section 2, we collect well-posedness results for (1.1) and state results for operator splitting schemes. Sections 3 and 4 present the proof of the main results for Godunov and Strang splitting, respectively.

2. Operator splitting

The upcoming analysis relies heavily on local well-posedness of (1.1) in H^s in the following sense: For a given time T, there exists an $R > 0$ such for all $u_0 \in H^s$ with $\|u_0\|_{H^s} \leq R$, there exists a unique strong solution $u \in C([0,T], H^s)$ of (1.1) with initial data u_0, and the dependence on the initial data is locally Lipschitz continuous; i.e., there is a constant $K = K(R,T) < \infty$ such that for two solutions $\widetilde{u}$ and u corresponding to initial data $\widetilde{u}_0$ and u_0, respectively, in the H^s ball of radius R, we have

$$\|\widetilde{u}(t) - u(t)\|_{H^s} \leq K\|\widetilde{u}_0 - u_0\|_{H^s} \quad \text{for } 0 \leq t \leq T. \tag{2.1}$$

Observe that this requirement says that the map taking initial data to solution is Lipschitz continuous. Unfortunately, for the Benjamin–Ono equation, (2.1) is valid only for $s = 0$. In fact, in [14] it is remarked that the solution map is not uniformly continuous from H^s to H^s for any $s > 0$, because of the derivative in the nonlinearity and the relatively weak smoothing effects of the linear part of the equation. Note, however, that the construction in [14] does not prohibit the solution map from being uniformly continuous or Lipschitz continuous in a weaker topology such as L^2. For the Benjamin–Ono equation we have the following result [21, Thm. 1.1] and [1, Thm. 5.3.1]:

Theorem 2.1. *Let $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Corresponding to the initial data u_0 with $\|u_0\|_{H^s} \le R$, there exists a unique solution u of (1.1) with initial data u_0, that is, $u(0) = u_0$, such that*

$$u \in C^k(\mathbb{R}_+; H^{s-2k}(\mathbb{R})),$$

for all $k \in \mathbb{N}$ with $s - 2k \ge -1$. Furthermore, for another solution $\widetilde{u}(t)$ with initial data $\widetilde{u}_0 \in H^s(\mathbb{R})$ such that $\|\widetilde{u}_0\|_{H^s} \le R$, we find

$$\|\widetilde{u}(t) - u(t)\|_{L^2} \le K(R,T)\|\widetilde{u}_0 - u_0\|_{L^2}, \quad \text{for } \ 0 \le t \le T.$$

A similar result holds for the periodic case: [18, Thm. 1.1]:

Theorem 2.2. *For all $u_0 \in H^s(\mathbb{T})$ with $s \ge \frac{1}{2}$ and for all $T > 0$, there exists a solution u of the Benjamin–Ono equation (1.2) satisfying*

$$u \in C\left([0,T]; H^s(\mathbb{T})\right).$$

Moreover, $u \in C_b(\mathbb{R}, L^2(\mathbb{T}))$ and the map $u_0 \mapsto u$ is continuous from H^s into $C\left([0,T]; H^s(\mathbb{T})\right)$ and Lipschitz on every bounded set from H^s_0 into $C\left([0,T]; H^s_0(\mathbb{T})\right)$. Here $H^s_0(\mathbb{T})$ denotes the closed subset of $H^s(\mathbb{T})$ with mean zero.

For Godunov splitting, we consider solutions bounded by

$$\|u(t)\|_{H^{5/2}} \le \rho < R \quad \text{for } \ 0 \le t \le T, \tag{2.2}$$

in particular, $u_0 \in H^{5/2}$. We show the following result.

Theorem 2.3 (First-order convergence in L^2)**.** *Let u be the unique solution of (1.1), and assume that u satisfies (2.2). Define the Godunov approximation $u_{\Delta t}$ by (1.6). Then for any $T > 0$ there is a $\overline{\Delta t} > 0$ such that for $\Delta t \le \overline{\Delta t}$ and $t \le T$, we have*

$$\|u_{\Delta t}(t) - u(t_n)\|_{L^2} \le C_1 \Delta t.$$

Here, $\overline{\Delta t}$ and C_1 only depend on $\|u_0\|_{H^{5/2}}$, ρ, and T.

Regarding Strang splitting, we consider solutions bounded by

$$\|u(t)\|_{H^{9/2}} \le \rho < R \quad \text{for} \quad 0 \le t \le T. \tag{2.3}$$

In this case, we assume that $u_0 \in H^{9/2}$. Then we show the following result.

Theorem 2.4 (Second-order convergence in L^2)**.** *Let u be the unique solution of (1.1), and assume that u satisfies (2.3). Define the Strang approximation $u_{\Delta t}$ by (1.7). Then there is a $\overline{\Delta t} > 0$ such that for $\Delta t \le \overline{\Delta t}$ and $t \le T$, we have*

$$\|u_{\Delta t}(t) - u(t)\|_{L^2} \le C_2 \Delta t^2.$$

Here, $\overline{\Delta t}$ and C_2 only depend on $\|u_0\|_{H^{9/2}}$, ρ, and T.

Since the exact solution operator for Burgers' equation eventually will produce discontinuities independently of the smoothness of the initial data, the initial value problem for Burgers' equation is not well posed in any Sobolev space with positive exponent. However, if the initial values are smooth, discontinuities will not be created instantaneously, and if you know that the solution is smooth, it is actually smoother than you think. The precise result reads as follows.

Lemma 2.5. *Let $r \geq r_0 > 3/2$. Then following results hold:*
(i) If $\|\Phi_B^t(u_0)\|_{H^{r_0}} \leq \alpha$ for $0 \leq t \leq \Delta t$, then $\|\Phi_B^t(u_0)\|_{H^r} \leq e^{C\alpha t}\|u_0\|_{H^r}$ for $0 \leq t \leq \Delta t$, where the constant C is independent of u_0 and Δt.
(ii) If $\|u_0\|_{H^{r_0}} \leq M$, then there exists a time $\bar{t}(M) > 0$ such that $\|\Phi_B^t(u_0)\|_{H^{r_0}} \leq C(M)$ for $0 \leq t \leq \bar{t}(M)$.

Proof. For any number $r \in \mathbb{R}$, let $H^r(\mathbb{R})$ be the Sobolev space consisting of all tempered distributions f such that

$$\|f\|_r = \left(\int_{\mathbb{R}} \langle \xi \rangle^{2r} |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2} < \infty,$$

with $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$ and $\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} e^{i\xi x} f(x)\, dx$ is the Fourier transform of f. Furthermore, we define an integral operator Λ^r on tempered distributions by

$$\Lambda^r(f) = \mathcal{F}^{-1}(\langle \xi \rangle^r \hat{f}),$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Since the inverse Fourier transformation preserves the L^2 norm, it is evident that $\|\Lambda^r(f)\|_{L^2} = \|f\|_{H^r}$. Moreover, it is easy to see that Λ^r is linear, and commutes with the derivative, i.e., $\Lambda^r(f)_x = \Lambda^r(f_x)$.

Let u be a solution of Burgers' equation. Then $\Lambda^r(u_t) = \Lambda^r(uu_x)$. Taking the standard L^2 inner product, denoted $\langle \cdot\,, \cdot \rangle_{L^2}$, with $\Lambda^r u$ yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \|\Lambda^r u\|_{L^2}^2 &= \langle \Lambda^r u, \Lambda^r u_t \rangle_{L^2} = \langle \Lambda^r u, \Lambda^r u u_x \rangle_{L^2} \\ &= \langle \Lambda^r u, u \Lambda^r u_x \rangle_{L^2} + \langle \Lambda^r u, \Lambda^r u u_x - u\Lambda^r u_x \rangle_{L^2}. \end{aligned}$$

The first term of the above expression can be estimated as follows

$$\begin{aligned} |\langle \Lambda^r u, u\Lambda^r u_x \rangle_{L^2}| &= \left| \int_{\mathbb{R}} u(\Lambda^r u)_x (\Lambda^r u) dx \right| = \left| \frac{1}{2} \int_{\mathbb{R}} u_x (\Lambda^r u)^2 dx \right| \\ &\leq \|u_x\|_{L^\infty} \|\Lambda^r u\|_{L^2}^2 \leq C \|u\|_{H^{r_0}} \|u\|_{H^r}^2 , \end{aligned}$$

where we have used the Sobolev inequality

$$\|u_x\|_{L^\infty} \leq C \|u_x\|_{H^{r_0 - 1}} \leq C \|u\|_{H^{r_0}} ,$$

which holds since $r_0 - 1 > 1/2$. The second term can be estimated by the Cauchy–Schwarz inequality, i.e.,

$$|\langle \Lambda^r u, \Lambda^r u u_x - u\Lambda^r u_x \rangle_{L^2}| \leq \|\Lambda^r u\|_{L^2} \|\Lambda^r u u_x - u\Lambda^r u_x\|_{L^2} .$$

To proceed further, we need the following inequalites which can be readily verified using the mean value theorem: For $r > 1$, and any ξ and η,

$$\left|(1+\xi^2)^{r/2} - (1+\eta^2)^{r/2}\right| \leq C\,|\xi - \eta| \left[(1+(\xi-\eta)^2)^{\frac{r-1}{2}} + (1+\eta^2)^{\frac{r-1}{2}} \right], \tag{2.4a}$$

$$|\eta| \leq \left(1+\eta^2\right)^{1/2}, \tag{2.4b}$$

where C is a constant. At this point, we also recall Young's inequality for convolutions

$$\|u * v\|_{L^2} \leq \|u\|_{L^1} \|v\|_{L^2} .$$

With the above inequalities, we calculate

$$\|\Lambda^r u u_x - u\Lambda^r u_x\|_{L^2}$$

$$
\begin{aligned}
&= \Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\Big((1+\xi^2)^{r/2}-(1+\eta^2)^{r/2}\Big)\,\hat{u}(\xi-\eta)\hat{u}_x(\eta)d\eta\Big)^2 d\xi\Big)^{1/2}\\
&\le C\Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\Big[(1+(\xi-\eta)^2)^{\frac{r-1}{2}}+(1+\eta^2)^{\frac{r-1}{2}}\Big]\,|\xi-\eta|\,|\hat{u}(\xi-\eta)\hat{u}_x(\eta)|\,d\eta\Big)^2 d\xi\Big)^{1/2}\\
&\le C\Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\Big[1+(\xi-\eta)^{\frac{r-1}{2}}\Big]\,|\xi-\eta|\;|\hat{u}(\xi-\eta)\hat{u}_x(\eta)|\;d\eta\Big)^2 d\xi\Big)^{1/2}\\
&\qquad\qquad + C\Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\Big[1+\eta^{\frac{r-1}{2}}\Big]\,|\hat{u}_x(\xi-\eta)|\;|\eta\hat{u}(\eta)|\;d\eta\Big)^2 d\xi\Big)^{1/2}\\
&\le C\Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\big[1+(\xi-\eta)^{\frac{r}{2}}\big]\,|\hat{u}(\xi-\eta)|\;|\hat{u}_x(\eta)|\;d\eta\Big)^2 d\xi\Big)^{1/2}\\
&\qquad\qquad + C\Big(\int_{\mathbb{R}}\Big(\int_{\mathbb{R}}\big[1+\eta^{\frac{r}{2}}\big]\,|\hat{u}(\eta)|\;|\hat{u}_x(\xi-\eta)|\;d\eta\Big)^2 d\xi\Big)^{1/2}\\
&\le C\left\|\hat{u}_x\right\|_{L^1}\,\left\|(1+\xi^2)^r\hat{u}(\xi)\right\|_{L^2}.
\end{aligned}
$$

For the first factor, observe that

$$
\begin{aligned}
\left\|\hat{u}_x\right\|_{L^1} &= \int_{\mathbb{R}}|\hat{u}_x(\xi)|\;d\xi = \int_{\mathbb{R}}|\xi|\;|\hat{u}(\xi)|\;d\xi\\
&\le \Big(\int_{\mathbb{R}}\left(1+\xi^2\right)^{r_0}|\hat{u}(\xi)|^2\;d\xi\Big)^{1/2}\Big(\int_{\mathbb{R}}\frac{\xi^2}{(1+\xi^2)^{r_0}}\,d\xi\Big)^{1/2}\\
&= C_{r_0}\left\|u\right\|_{H^{r_0}}, \quad \text{for } r_0>3/2.
\end{aligned}
$$

Thus, combining the above estimates, we obtain

$$
|\langle\Lambda^r u,\Lambda^r uu_x-u\Lambda^r u_x\rangle_{L^2}|\le C\left\|u\right\|_{H^{r_0}}\left\|u\right\|_{H^r}^2.
$$

Therefore

$$
\frac{d}{dt}\left\|u\right\|_{H^r}^2\le C\left\|u\right\|_{H^{r_0}}\left\|u\right\|_{H^r}^2 \tag{2.5}
$$

which proves the first part (i) of the Lemma 2.5. Observe that, we can also use $r_0=r$ in (2.5), which implies

$$
\frac{d}{dt}\left\|u\right\|_{H^r}\le C\left\|u\right\|_{H^r}^2. \tag{2.6}
$$

The second part (ii) of Lemma 2.5 follows by comparing (2.6) with the majorizing differential equation $y'=Cy^2$. □

3. Godunov splitting

In the previous subsection, we have presented several results which now will prove useful. In what follows, we first estimate the local error for the Godunov splitting, before we use this estimate to find a bound for the global error.

We start by a general perturbation result. We write $e^{tA}v=\Phi_A^t(v)$ to indicate the linearity of the flow of A. We start from the variation-of-constants formula [15, Thm. 4.2.4] for $u(t)=\Phi_{A+B}^t(u_0)$,

$$
u(t)=e^{tA}u_0+\int_0^t e^{(t-s)A}B(u(s))\,ds, \tag{3.1}
$$

which is just the formula $\phi(t) - \phi(0) = \int_0^t \dot{\phi}(s)\, ds$ for $\phi(s) = e^{(t-s)A}u(s)$. Furthermore, we have[1]

$$B(u(s)) = B(e^{sA}u_0) + \int_0^s dB(e^{(s-\sigma)A}u(\sigma))[e^{(s-\sigma)A}B(u(\sigma))]\, d\sigma, \tag{3.2}$$

which is nothing but the formula $B(\varphi(s)) - B(\varphi(0)) = \int_0^s dB(\varphi(\sigma))[\dot{\varphi}(\sigma)]\, d\sigma$ for $\varphi(\sigma) = e^{(s-\sigma)A}u(\sigma)$. We insert (3.2) into (3.1) with $t = \Delta t$ to obtain

$$u(\Delta t) = e^{\Delta tA}u_0 + \int_0^{\Delta t} e^{(\Delta t - s)A}B(e^{sA}u_0)\, ds + e_1 \tag{3.3}$$

with

$$e_1 = \int_0^{\Delta t}\int_0^s e^{(\Delta t-s)A}dB(e^{(s-\sigma)A}u(\sigma))[e^{(s-\sigma)A}B(u(\sigma))]\, d\sigma\, ds. \tag{3.4}$$

We next turn to results specifically for the Godunov splitting. The main tool for proving Theorem 2.3 is a local error estimate.

Lemma 3.1. *Assume that hypothesis (2.2) holds for the solution $u(t) = \Phi^t_{A+B}(u_0)$ of (1.1). If the initial data u_0 is in $H^{5/2}$, then the local error of the Godunov splitting (1.6) is bounded in L^2 by*

$$\|\Pi^{\Delta t}(u_0) - \Phi^{\Delta t}_{A+B}(u_0)\|_{L^2} \le c_1 \Delta t^2,$$

where c_1 only depends on $\|u_0\|_{H^{5/2}}$.

Proof. Set

$$u_1 = \Pi^{\Delta t}(u_0) = e^{\Delta tA}\left(\Phi^{\Delta t}_B(u_0)\right).$$

The first-order Taylor expansion with integral remainder term

$$\Phi^{\Delta t}_B(v) = v + \Delta t B(v) + \underbrace{\Delta t^2 \int_0^1 (1-\theta) dB(\Phi^{\theta\Delta t}_B(v))[B(\Phi^{\theta\Delta t}_B(v))]\, d\theta}_{e_2} \tag{3.5}$$

is justified for any $v \in H^{5/2}$ and for sufficiently small Δt by Lemma 2.5. We therefore obtain

$$u_1 = e^{\Delta tA}u_0 + \Delta t e^{\Delta tA}B(u_0) + e_2.$$

Thus the error can be written

$$u_1 - u(\Delta t) = \Delta t\, e^{\Delta tA}B(u_0) - \int_0^{\Delta t} e^{(\Delta t-s)A}B(e^{sA}u_0)\, ds + (e_2 - e_1), \tag{3.6}$$

and therefore the principal error term is just the quadrature error of the rectangle rule applied to the integral over $[0, \Delta t]$ of the function

$$f(s) = e^{(\Delta t-s)A}B(e^{sA}u_0). \tag{3.7}$$

We express the quadrature error in first-order Peano form,

$$\Delta t\, f(0) - \int_0^{\Delta t} f(s)\, ds = \Delta t^2 \int_0^1 \kappa_1(\theta)\, f'(\theta\Delta t)\, d\theta,$$

[1]Here we introduce the second-order Taylor expansion $\Psi(f+g) = \Psi(f) + d\Psi(f)[g] + \int_0^1 (1-\alpha)d^{(2)}\Psi(f+\alpha g)[g]^2 d\alpha$ for an operator Ψ, see [2, p. 29] for notation and proofs.

where κ_1 is the real-valued, bounded Peano kernel of the rectangle rule. Thus, the L^2-error after one step is bounded as

$$\|u_1 - u(\Delta t)\|_{L^2} \leq (\Delta t)^2 \int_0^1 \|\kappa_1(\theta)\, f'(\theta\Delta t)\|_{L^2}\, d\theta + \|(e_2 - e_1)\|_{L^2}. \tag{3.8}$$

Next, we find that

$$\begin{aligned} f'(s) &= -e^{(\Delta t - s)A}[A,B](e^{sA}u_0) \\ [A,B](v) &= dA(v)[B(v)] - dB(v)[Av] \\ &= H[(vv_x)_{xx}] - v_x H(v_{xx}) - vH(v_{xxx}) \\ &= H(vv_{xxx}) + 3H(v_x v_{xx}) - v_x H(v_{xx}) - vH(v_{xxx}). \end{aligned} \tag{3.9}$$

By Lemma 3.2 below, we obtain the commutator bound,

$$\|[A,B](v)\|_{L^2} \leq C\,\|v\|^2_{H^{5/2}}.$$

Since e^{tA} preserves the Sobolev norms, we have

$$\langle u, Hu_{xx}\rangle_{L^2} = -\langle Hu, u_{xx}\rangle_{L^2} = -\langle (Hu)_{xx}, u\rangle_{L^2} = -\langle Hu_{xx}, u\rangle_{L^2},$$

implying that $\langle u, Hu_{xx}\rangle_{L^2} = 0$. It follows

$$\|f'(s)\|_{L^2} \leq C\,\|u_0\|^2_{H^{5/2}},$$

and hence the quadrature error is $O(\Delta t^2)$ in the L^2-norm for $u_0 \in H^{5/2}$. The L^2-norm of the remainder term $e_2 - e_1$ is bounded by $C\Delta t^2$ for $u_0 \in H^{5/2}$ for sufficiently small Δt (by using Lemma 2.5 (ii)). Specifically,

$$\begin{aligned} \|e_1\|_{L^2} &\leq \int_0^{\Delta t}\int_0^s \left\| e^{(\Delta t - s)A} dB(e^{(s-\sigma)A}u(\sigma))[e^{(s-\sigma)A}B(u(\sigma))] \right\|_{L^2} d\sigma\, ds \\ &\leq \int_0^{\Delta t}\int_0^s \left\| \Big(\big(e^{(s-\sigma)A}u(\sigma)\big)\big(e^{(s-\sigma)A}B(u(\sigma))\big)\Big)_x \right\|_{L^2} d\sigma\, ds \\ &\leq C\int_0^{\Delta t}\int_0^s \|u(\sigma)\|_{H^1}\, \|B(u(\sigma))\|_{H^1}\, d\sigma\, ds \\ &\leq C\int_0^{\Delta t}\int_0^s \|u(\sigma)\|_{H^1}\, \|u(\sigma)\|_{H^1}\, \|u(\sigma)\|_{H^2}\, d\sigma\, ds \\ &\leq C\int_0^{\Delta t}\int_0^s \|u(\sigma)\|^3_{H^2}\, d\sigma\, ds \leq C\Delta t^2 R^3, \end{aligned}$$

and

$$\begin{aligned} \|e_2\|_{L^2} &\leq \Delta t^2 \int_0^1 (1-\theta) \left\| e^{\Delta t A} dB(\Phi_B^{\theta\Delta t}(u_0))[B(\Phi_B^{\theta\Delta t}(u_0))] \right\|_{L^2} d\theta \\ &\leq \Delta t^2 \int_0^1 \left\| \Big((\Phi_B^{\theta\Delta t}(u_0))(B(\Phi_B^{\theta\Delta t}(u_0)))\Big)_x \right\|_{L^2} d\theta \\ &\leq \Delta t^2 C \int_0^1 \left\|\Phi_B^{\theta\Delta t}(u_0)\right\|_{H^1} \left\|B(\Phi_B^{\theta\Delta t}(u_0))\right\|_{H^1} d\theta \\ &\leq \Delta t^2 C \int_0^1 \left\|\Phi_B^{\theta\Delta t}(u_0)\right\|^2_{H^1} \left\|\Phi_B^{\theta\Delta t}(u_0)\right\|_{H^2} d\theta \\ &\leq \Delta t^2 C \int_0^1 \left\|\Phi_B^{\theta\Delta t}(u_0)\right\|^3_{H^2} d\theta \leq C\Delta t^2 R^3. \end{aligned}$$

This completes the proof. □

Lemma 3.2. *If $v \in H^{5/2}(\mathbb{R})$ and $[A,B](v)$ is given by (3.9), then*

$$\|[A,B](v)\|_{L^2} \le C\,\|v\|_{H^2}\,\|v\|_{H^{5/2}}\,, \tag{3.10}$$

for some constant C.

Proof. From (3.9) we have

$$[A,B](v) = g(v)_x + 2H(v_x v_{xx}),$$

with

$$g(v) := H(vv_{xx}) - vH(v_{xx}).$$

First we show

$$\|g(v)_x\|_2 \le \|v_x\|_2\,\|v\|_{H^{5/2}}\,.$$

Recall that $\mathcal{F}$ denotes the Fourier transform. Thus

$$\mathcal{F}(g(v))(\xi) = I(\xi) - II(\xi)$$

where

$$I(\xi) := \mathcal{F}\left(vH(v_{xx})\right)(\xi) = -i\int \hat{v}(\xi-\eta)\,\hat{v}(\eta)\,\mathrm{sign}\,(\eta)\;|\eta|^2 d\eta,$$

$$II(\xi) := \mathcal{F}(H(vv_{xx}))(\xi) = -i\,\mathrm{sign}\,(\xi)\int \hat{v}(\xi-\eta)\,\hat{v}\eta\,|\eta|^2 d\eta.$$

Therefore for $\xi > 0$, we have

$$I(\xi) = -i\int_{\eta>0} \hat{v}(\xi-\eta)\,\hat{v}(\eta)\,|\eta|^2 d\eta + i\int_{\eta<0} \hat{v}(\xi-\eta)\,\hat{v}(\eta)\,|\eta|^2 d\eta,$$

$$II(\xi) = -i\int_{\eta>0} \hat{v}(\xi-\eta)\,\hat{v}(\eta)\,|\eta|^2 d\eta \;-\; i\int_{\eta<0} \hat{v}(\xi-\eta)\,\hat{v}(\eta)\,|\eta|^2 d\eta,$$

and for $\xi < 0$,

$$I(\xi) = -i\int_{\eta>0} \hat{v}(\xi-\eta)\,\hat{v}(\eta)\,|\eta|^2 d\eta + i\int_{\eta<0} \hat{v}(\xi-\eta)\,\hat{v}(\eta)\,|\eta|^2 d\eta,$$

$$II(\xi) = i\int_{\eta>0} \hat{v}(\xi-\eta)\,\hat{v}(\eta)\,|\eta|^2 d\eta \;+\; i\int_{\eta<0} \hat{v}(\xi-\eta)\,\hat{v}(\eta)\,|\eta|^2 d\eta.$$

This implies for $\xi > 0$,

$$\mathcal{F}(g(v))(\xi) = I(\xi) - II(\xi) = 2i\int_{\eta<0} \hat{v}(\xi-\eta)\,\hat{v}(\eta)\,|\eta|^2 d\eta,$$

and for $\xi < 0$,

$$\mathcal{F}(g(v))(\xi) = I(\xi) - II(\xi) = -2i\int_{\eta>0} \hat{v}(\xi-\eta)\,\hat{v}(\eta)\,|\eta|^2 d\eta.$$

Using Parseval's relation, we obtain

$$\begin{aligned}
\|g(v)_x\|_2^2 = \|\mathcal{F}(g(v)_x)\|_2^2 &= \int_0^\infty |\xi|^2 |\mathcal{F}(g(v))(\xi)|^2 d\xi + \int_{-\infty}^0 |\xi|^2 |\mathcal{F}(g(v))(\xi)|^2 d\xi \\
&\le 4\underbrace{\int_0^\infty |\xi|^2 \left|\int_{\eta<0} \hat{v}(\xi-\eta)\hat{v}(\eta)|\eta|^2 d\eta\right|^2 d\xi}_{\mathcal{A}}
\end{aligned}$$

$$+ \underbrace{4\int_{-\infty}^{0} |\xi|^2 \left| \int_{\eta>0} \hat{v}(\xi-\eta)\hat{v}(\eta)|\eta|^2 d\eta \right|^2 d\xi}_{\mathcal{B}} .$$

Next we estimate $\mathcal{A}$. Note that, for $\eta < 0$ and $\xi > 0$, we have

$$|\eta| \le |\xi - \eta|.$$

Using the above inequality we obtain

$$\mathcal{A} \le \int_0^\infty |\xi|^2 \left(\int_{\eta<0} |\xi-\eta||\hat{v}(\xi-\eta)| \;\; |\eta| \; |\hat{v}(\eta)| d\eta \right)^2 d\xi.$$

Using the Cauchy–Schwarz inequality we find

$$\begin{aligned} \mathcal{A} &\le \int_0^\infty |\xi|^2 \left(\int_{\eta<0} |\xi-\eta|^2 |\hat{v}(\xi-\eta)|^2 d\eta \right) \left(\int_{\eta<0} |\eta|^2 \; |\hat{v}(\eta)|^2 d\eta \right) d\xi \\ &\le \|v_x\|_2^2 \underbrace{\int_0^\infty |\xi|^2 \int_{\eta<0} |\xi-\eta|^2 |\hat{v}(\xi-\eta)|^2 d\eta d\xi}_{\mathcal{C}} . \end{aligned}$$

Next we estimate $\mathcal{C}$. Using change of variables and integration by parts, we have

$$\begin{aligned} \mathcal{C} &= \int_0^\infty \xi^2 \left(\int_\xi^\infty y^2 |\hat{v}(y)|^2 dy \right) d\xi \\ &= \frac{1}{3} \int_0^\infty \frac{d}{d\xi}(\xi^3) \left(\int_\xi^\infty y^2 |\hat{v}(y)|^2 dy \right) d\xi \\ &= -\frac{1}{3} \int_0^\infty \xi^3 \; \frac{d}{d\xi} \left(\int_\xi^\infty y^2 |\hat{v}(y)|^2 dy \right) d\xi \\ &= \frac{1}{3} \int_0^\infty |\xi|^5 |\hat{v}(\xi)|^2 d\xi \le \|v\|_{H^{5/2}}^2 . \end{aligned}$$

Therefore we have

$$\mathcal{A} \le \|v_x\|_2^2 \, \|v\|_{H^{5/2}}^2 .$$

A similar argument shows that

$$\mathcal{B} \le \|v_x\|_2^2 \, \|v\|_{H^{5/2}}^2 .$$

Finally, we consider

$$\begin{aligned} \|H(v_x v_{xx})\|_2 &= \|v_x v_{xx}\|_2 \le \|v_x\|_\infty \|v_{xx}\|_2 \\ &\le C \|v_x\|_{H^1} \|v\|_{H^2} \le C \|v\|_{H^2} \|v\|_{H^{5/2}} . \end{aligned}$$

This completes the proof of the lemma. □

Proof of Theorem 2.3. The proof compares the error propagation with the exact flow. In our approach the necessary regularity for estimating local errors by Lemma 3.1 is ensured by Lemma 2.5 (i), via the following induction argument.

We make the induction hypothesis that for $k \le n-1$,

$$\begin{aligned} \|u_k\|_{L^2} &\le R, \\ \|u_k\|_{H^{5/2}} &\le e^{2cRk\Delta t} \|u_0\|_{H^{5/2}} \le C_0, \\ \|u_k - u(t_k)\|_{L^2} &\le \gamma \Delta t, \end{aligned}$$

where $C_0 = e^{2cRT}\|u_0\|_{H^{5/2}}$ with c from Lemma 2.5 (i), and $\gamma = K(R,T)c_1(C_0)T$ with $K(R,T)$ from the local Lipschitz bound (2.1) and $c_1(C_0)$ is the constant of Lemma 3.1 for starting values bounded by C_0 in $H^{5/2}$. We then show that the above bounds also hold for $k = n$ as long as $n\Delta t \le T$ and Δt is sufficiently small.

We denote, with $\Phi^t = \Phi^t_{A+B}$ for brevity,

$$u_n^k = \Phi^{(n-k)\Delta t}(u_k),$$

which is the value at time t_n of the exact solution of (1.1) starting with initial data u_k at time t_k. Note that

$$u_n = u_n^n, \quad u(t_n) = u_n^0.$$

We estimate

$$\begin{aligned}\|u_n - u(t_n)\|_{L^2} &\le \sum_{k=0}^{n-1} \|u_n^{k+1} - u_n^k\|_{L^2} \\ &= \sum_{k=0}^{n-1} \|\Phi^{(n-k-1)\Delta t}(\Pi^{\Delta t}(u_k)) - \Phi^{(n-k-1)\Delta t}(\Phi^{\Delta t}(u_k))\|_{L^2}.\end{aligned}$$

For $k \le n-2$, we have $\|\Pi^{\Delta t}(u_k)\|_{L^2} = \|u_{k+1}\|_{L^2} \le R$, and

$$\begin{aligned}\|\Phi^{\Delta t}(u_k)\|_{L^2} &\le \|\Phi^{\Delta t}(u_k) - \Phi^{\Delta t}(u(t_k))\|_{L^2} + \|\Phi^{\Delta t}(u(t_k))\|_{L^2} \\ &\le K(R,\Delta t)\|u_k - u(t_k)\|_{L^2} + \|u(t_{k+1})\|_{L^2} \\ &\le K(R,\Delta t)\gamma\Delta t + \rho,\end{aligned}$$

cf. (2.2), which is bounded by R if Δt is so small that

$$K(R,\Delta t)\gamma\Delta t \le R - \rho.$$

Using Theorem 2.1 and Lemma 3.1 we therefore have, for $k \le n-1$ and $n\Delta t \le T$,

$$\begin{aligned}&\|\Phi^{(n-k-1)\Delta t}(\Pi^{\Delta t}(u_k)) - \Phi^{(n-k-1)\Delta t}(\Phi^{\Delta t}(u_k))\|_{L^2} \\ &\qquad\le K(R,T)\|\Pi^{\Delta t}(u_k) - \Phi^{\Delta t}(u_k)\|_{L^2} \\ &\qquad\le K(R,T)c_1(C_0)\Delta t^2,\end{aligned}$$

where Π is the Godunov step operator defined in (1.6). With this estimate we obtain, again noting $n\Delta t \le T$,

$$\|u_n - u(t_n)\|_{L^2} \le nK(R,T)c_1(C_0)\Delta t^2 \le \gamma\Delta t.$$

To prove the boundedness of u_n, we choose $\gamma\Delta t \le R - \rho$. Then we have

$$\|u_n\|_{L^2} \le R.$$

Since $\|\Phi^t_A(v)\|_{H^{5/2}} \le \|v\|_{H^{5/2}}$, the Lemma 2.5, for $\Delta t \le \bar{t}(R)$,

$$\|u_n\|_{H^{5/2}} = \|\Phi_A^{\Delta t} \circ \Phi_B^{\Delta t}(u_{n-1})\|_{H^{5/2}} \le e^{2cR\Delta t}\|u_{n-1}\|_{H^{5/2}} \le e^{2cRn\Delta t}\|u_0\|_{H^{5/2}}.$$

Thus, the three necessary results hold by the induction argument, and this completes the proof of the theorem. □

Remark 3.3. *To keep the presentation fairly short we have only provided details in the full line case. However, we note that the same proofs apply mutatis mutandis also in the periodic case.*

4. Strang splitting

To prove the correct convergence rate for Strang splitting, we use the same framework as in the proof of the convergence rate for Godunov splitting. The major difference between the proofs is that for Strang splitting we need to use the higher-order midpoint rule, rather than the rectangle rule applied for the Godunov splitting. In addition, a higher-order series expansion of the involved terms is also necessary to obtain the results.

We only present the results in the full line case, and, as before, the same proofs apply also in the periodic case. Note that our aim is to find the error between the operator splitting solution and the exact (Taylor expanded) solution, and bound it using numerical quadratures. The proof is longer due to the extra order in the Taylor expansion.

Also for Strang splitting the proof is based on a local error estimate.

Lemma 4.1. *Assume that the hypothesis (2.3) holds for the solution $u(t) = \Phi^t_{A+B}(u_0)$ of (1.1). If the initial data u_0 is in $H^{9/2}$, then the local error of the Strang splitting (1.7) is bounded in L^2 by*

$$\left\|\Psi^{\Delta t}(u_0) - \Phi^{\Delta t}_{A+B}(u_0)\right\|_{L^2} \le c_2 \Delta t^3,$$

where c_2 only depends on $\|u_0\|_{H^{9/2}}$.

Proof. We follow [11] and use the second-order Taylor expansion

$$\begin{aligned}\Phi^{\Delta t}_B(v) &= v + \Delta t B(v) + \tfrac12 \Delta t^2 dB(v)[B(v)] \\ &\quad + \Delta t^3 \int_0^1 \tfrac12 (1-\theta)^2 \Big(d^2B(\Phi^{\theta\Delta t}_B(v))[B(\Phi^{\theta\Delta t}_B(v)), B(\Phi^{\theta\Delta t}_B(v))] \\ &\qquad + dB(\Phi^{\theta\Delta t}_B(v))\big[dB(\Phi^{\theta\Delta t}_B(v))[B(\Phi^{\theta\Delta t}_B(v))]\big]\Big)\, d\theta.\end{aligned}$$

Henceforth we abbreviate the integral remainder term as

$$\Delta t^3 \int_0^1 \tfrac12 (1-\theta)^2 \Big(d^2B(B,B) + dB\, dB\, B\Big)\big(\Phi^{\theta\Delta t}_B(v)\big)\, d\theta.$$

Hence,

$$\begin{aligned}u_1 &= e^{\Delta tA}u_0 + \Delta t e^{\Delta tA/2} B\big(e^{\Delta tA/2}u_0\big) + \tfrac12 \Delta t^2 e^{\Delta tA/2} dB\big(e^{\Delta tA/2}u_0\big)[B\big(e^{\Delta tA/2}u_0\big)] \\ &\quad + \Delta t^3 \int_0^1 \tfrac12 (1-\theta)^2 e^{\Delta tA/2}\Big(d^2B(B,B) + dB\, dB\, B\Big)\big(\Phi^{\theta\Delta t}_B(e^{\Delta tA/2}u_0)\big)\, d\theta \\ &= e^{\Delta tA}u_0 + \Delta t e^{\Delta tA/2} B\big(e^{\Delta tA/2}u_0\big) + e_2,\end{aligned}$$

where e_2 is given by

(4.1)

$$\begin{aligned}e_2 &:= \tfrac12 \Delta t^2 e^{\Delta tA/2} dB\big(e^{\Delta tA/2}u_0\big)[B\big(e^{\Delta tA/2}u_0\big)] \\ &\quad + \Delta t^3 \int_0^1 \tfrac12 (1-\theta)^2 e^{\Delta tA/2}\Big(d^2B(B,B) + dB\, dB\, B\Big)\big(\Phi^{\theta\Delta t}_B(e^{\Delta tA/2}u_0)\big)\, d\theta.\end{aligned}$$

Recall (3.3) and (3.4), viz.

$$u(\Delta t) = e^{\Delta tA}u_0 + \int_0^{\Delta t} e^{(\Delta t - s)A} B(e^{sA}u_0)\, ds + e_1$$

where

$$e_1 = \int_0^{\Delta t} \int_0^s e^{(\Delta t - s)A} dB(e^{(s-\sigma)A} u(\sigma))[e^{(s-\sigma)A} B(u(\sigma))]\, d\sigma\, ds.$$

We express the integrand in e_1 by a formula of the type (3.2) by using

$$G(u(\sigma)) = G(e^{\sigma A} u_0) + \int_0^{\sigma} dG(e^{(\sigma-\tau)A} u(\tau))[e^{(\sigma-\tau)A} B(u(\tau))]\, d\tau$$

with

$$G(v) = G_{s,\sigma}(v) = dB(e^{(s-\sigma)A} v)[e^{(s-\sigma)A}\, B(v)],$$

and

$$\begin{aligned} dG(v)[w] = {} & d^2 B\big(e^{(s-\sigma)A} v\big)\big[e^{(s-\sigma)A} w, e^{(s-\sigma)A} B(v)\big] \\ & + dB\big(e^{(s-\sigma)A} v\big)\big[e^{(s-\sigma)A} dB(v)[w]\big]. \end{aligned}$$

This implies

$$\begin{aligned} e_1 = {} & \int_0^{\Delta t} \int_0^s e^{(\Delta t - s)A} dB\left(e^{sA} u_0\right) \left[e^{(s-\sigma)A} B\left(e^{sA} u_0\right)\right] d\sigma ds \\ & + \int_0^{\Delta t} \int_0^s \int_0^{\sigma} dG_{s,\sigma}\big(e^{(\sigma-\tau)A} u(\tau)\big)\big[e^{(\sigma-\tau)A} B(u(\tau))\big]\, d\tau d\sigma ds. \end{aligned} \tag{4.2}$$

We return to the error formula (3.6) and write the principal error term

$$\Delta t\, e^{\Delta t A/2} B\big(e^{\Delta t A/2} u_0\big) - \int_0^{\Delta t} e^{(\Delta t - s)A} B(e^{sA} u_0)\, ds$$

in second-order Peano form

$$\Delta t f(\tfrac{1}{2}\Delta t) - \int_0^{\Delta t} f(s)\, ds = \Delta t^3 \int_0^1 \kappa_2(\theta)\, f''(\theta \Delta t)\, d\theta$$

with the second-order Peano kernel κ_2 of the midpoint rule and f is defined by (3.7) with

$$f''(s) = e^{(\Delta t - s)A} [A, [A, B]](e^{sA} u_0).$$

By Lemma 4.2, proven below, we obtain the double commutator bound

$$\|[A, [A, B]](v)\|_{L^2} \le C \|v\|_{H^{9/2}}^2 .$$

Thus, it follows that

$$\|f''(s)\|_{L^2} \le C \|u_0\|_{H^{9/2}}^2 .$$

Lemma 4.2. *For $v \in H^{9/2}$, we have*

$$\|[A, [A, B]](v)\|_{L^2} \le 2 \|v_x\|_{L^2} \|v\|_{H^{9/2}} + C \|v\|_{H^4}^2 ,$$

for some constant C.

Proof of Lemma 4.2. The Lie double commutator is given by

$$[A, [A, B]](v) := [A, L](v) = dA(v)[L(v)] - dL(v)[Av],$$

where L is defined by

$$L(v) = [A, B](v) = g(v)_x + 2H(v_x v_{xx}),$$

with

$$g(v) := H(v v_{xx}) - v H(v_{xx}).$$

A direct computation shows that

$$dL(v)[w] = \Big(H(vw_{xx} + wv_{xx}) - \big(vH(w_{xx}) + wH(v_{xx})\big)\Big)_x + 2H(v_x w_{xx} + w_x v_{xx}).$$

Using the Leibniz rule, we have the following identity

$$\begin{aligned}\big(H(vw) - vH(w)\big)_{xx} &= H(v_{xx}w) + H(w_{xx}v) + 2H(v_x w_x)\\ &\quad - vH(w_{xx}) - v_{xx}H(w) - 2v_x H(w_x).\end{aligned}$$

Thus

$$dL(v)[w] = \big(H(vw) - vH(w)\big)_{xxx} + \mathcal{E}(v,w),$$

where

$$\mathcal{E}(v,w) = \big(v_{xx}H(w) - wH(v_{xx}) + 2v_x H(w_x)\big)_x.$$

For $w = A(v) = H(v_{xx})$, using the property $H^2 = -I$, we obtain

$$\begin{aligned}H(vw) - vH(w) &= H(vH(v_{xx})) - vH^2(v_{xx})\\ &= H(vH(v_{xx})) + vv_{xx}\\ &= H(vH(v_{xx}) - H(vv_{xx})) = -H(g(v)).\end{aligned}$$

Thus,

$$dL(v)[A(v)] = -H\big(g(v)_{xxx}\big) + \mathcal{E}(v, A(v)).$$

Again,

$$dA(v)[L(v)] = H\big(L(v)_{xx}\big) = H\big(g(v)_{xxx}\big) - 2\big(v_x v_{xx}\big)_{xx}.$$

Therefore,

$$[A,[A,B]](v) = 2H\big(g(v)_{xxx}\big) + \mathcal{D}(v)$$

where

$$\mathcal{D}(v) = -\mathcal{E}(v, A(v)) - 2\big(v_x v_{xx}\big)_{xx}.$$

Repeatedly using that $\|v\|_{L^\infty} \le \|v\|_{H^1}$, we see that

$$\|\mathcal{D}(v)\|_2 \le C\,\|v\|_{H^4}^2,$$

for some numerical constant C. Next we claim that

$$\|g(v)_{xxx}\|_2 \le 2\,\|v_x\|_2\,\|v\|_{H^{9/2}}. \tag{4.3}$$

Using the Parseval relation, we obtain

$$\begin{aligned}\|g(v)_{xxx}\|_2^2 &= \|\mathcal{F}(g(v)_{xxx})\|_2^2\\ &= \int_0^\infty |\xi|^6 |\mathcal{F}(g(v))(\xi)|^2\,d\xi + \int_{-\infty}^0 |\xi|^6 |\mathcal{F}(g(v))(\xi)|^2\,d\xi\\ &\le 4\underbrace{\int_0^\infty |\xi|^6 \left|\int_{\eta<0} \hat{v}(\xi-\eta)\hat{v}(\eta)|\eta|^2\,d\eta\right|^2 d\xi}_{\mathcal{A}}\\ &\quad + 4\underbrace{\int_{-\infty}^0 |\xi|^6 \left|\int_{\eta>0} \hat{v}(\xi-\eta)\hat{v}(\eta)|\eta|^2\,d\eta\right|^2 d\xi}_{\mathcal{B}}.\end{aligned}$$

Next we estimate $\mathcal{A}$. Note that, for $\eta < 0$ and $\xi > 0$, we have

$$|\eta| \le |\xi - \eta|.$$

Using the above inequality we obtain

$$\mathcal{A} \le \int_0^\infty |\xi|^6 \left(\int_{\eta<0} |\xi-\eta||\hat{v}(\xi-\eta)| \quad |\eta| \; |\hat{v}(\eta)| d\eta \right)^2 d\xi.$$

Using the Cauchy–Schwarz inequality we infer

$$\begin{aligned} \mathcal{A} &\le \int_0^\infty |\xi|^6 \left(\int_{\eta<0} |\xi-\eta|^2 |\hat{v}(\xi-\eta)|^2 d\eta \right) \left(\int_{\eta<0} |\eta|^2 \; |\hat{v}(\eta)|^2 d\eta \right) d\xi \\ &\le \|v_x\|_2^2 \underbrace{\int_0^\infty |\xi|^6 \int_{\eta<0} |\xi-\eta|^2 |\hat{v}(\xi-\eta)|^2 d\eta d\xi}_{\mathcal{C}} . \end{aligned}$$

Next we estimate $\mathcal{C}$. Using a change of variables and integration by parts, we have

$$\begin{aligned} \mathcal{C} &= \int_0^\infty \xi^6 \left(\int_\xi^\infty y^2 |\hat{v}(y)|^2 dy \right) d\xi \\ &= \frac{1}{7} \int_0^\infty \frac{d}{d\xi}(\xi^7) \left(\int_\xi^\infty y^2 |\hat{v}(y)|^2 dy \right) d\xi \\ &= -\frac{1}{7} \int_0^\infty \xi^7 \; \frac{d}{d\xi} \left(\int_\xi^\infty y^2 |\hat{v}(y)|^2 dy \right) d\xi \\ &= \frac{1}{7} \int_0^\infty |\xi|^9 |\hat{v}(\xi)|^2 d\xi \le \|v\|_{H^{9/2}}^2 . \end{aligned}$$

Therefore we have

$$4\mathcal{A} \le \|v_x\|_2^2 \, \|v\|_{H^{9/2}}^2 .$$

A similar argument shows that

$$4\mathcal{B} \le \|v_x\|_2^2 \, \|v\|_{H^{9/2}}^2 .$$

This completes the proof of (4.3) and thereby of the lemma. □

Now for the difference of (4.1) and (4.2),

$$e_2 - e_1 = \tfrac{1}{2}\Delta t^2 g(\tfrac{1}{2}\Delta t, \tfrac{1}{2}\Delta t) - \int_0^{\Delta t} \int_0^s g(s,\sigma)\, d\sigma\, ds + \tilde{e}_2 - \tilde{e}_1,$$

where

$$\begin{aligned} g(s,\sigma) &= e^{(\Delta t - s)A} dB(e^{sA} u_0)\, [e^{(s-\sigma)A} B(e^{\sigma A} u_0)], \\ \tilde{e}_1 &= \int_0^{\Delta t} \int_0^s \int_0^\sigma dG_{s,\sigma} \left(e^{(\sigma-\tau)A} u(\tau) \right) e^{(\sigma-\tau)A} B(u(\tau))\, d\tau d\sigma ds, \\ \tilde{e}_2 &= \Delta t^3 \int_0^1 \tfrac{1}{2}(1-\theta)^2 e^{\Delta t A/2} \Big(d^2 B(B,B) + dB\, dB\, B \Big) (\Phi_B^{\theta \Delta t}(u_0))\, d\theta. \end{aligned}$$

To estimate the remainder terms $\tilde{e}_i$, for $i = 1, 2$, we calculate

$$\begin{aligned} \|dG_{s,\sigma}(v) w\|_{L^2} &\le \left\| d^2 B \left(e^{(s-\sigma)A} v \right) [e^{(s-\sigma)A} B(v), e^{(s-\sigma)A} w] \right\|_{L^2} \\ &\quad + \left\| dB \left(e^{(s-\sigma)A} v \right) [e^{(s-\sigma)A} dB(v)[w]] \right\|_{L^2} \\ &\le C \left(\|B(v)\|_{H^1} \|w\|_{H^1} + \|v\|_{H^1} \|dB(v)[w]\|_{H^1} \right) \\ &\le C \left(\|v\|_{H^2}^2 \|w\|_{H^1} + \|v\|_{H^1} \|v\|_{H^2} \|w\|_{H^2} \right) \end{aligned}$$

$$\leq C \|v\|_{H^2}^2 \|w\|_{H^2},$$

and

$$\begin{aligned}\left\|\left(d^2B(B,B) + dB\,dB\,B\right)(v)\right\|_{L^2} &\leq \left\|d^2B(B(v),B(v))\right\|_{L^2} + \|dB(v)[dB(v)[B(v)]]\|_{L^2} \\ &\leq C\left(\|B(v)\|_{H^1}^2 + \|v\|_{H^1}\|dB(v)[B(v)]\|_{H^1}\right) \\ &\leq C\left(\|v\|_{H^2}^4 + \|v\|_{H^2}^2\|B(v)\|_{H^2}\right) \\ &\leq C\|v\|_{H^3}^4.\end{aligned}$$

Then, using Lemma 3.1, the remainder terms are bounded by

$$\|\tilde{e}_1\|_{L^2} + \|\tilde{e}_2\|_{L^2} \leq C\Delta t^3 \|u_0\|_{H^3}^4. \tag{4.4}$$

The first two terms in $e_2 - e_1$ are the quadrature error of a first-order two-dimensional quadrature formula, which is bounded by

$$\begin{aligned}&\left\|\tfrac{1}{2}\Delta t^2 g(\tfrac{1}{2}\Delta t, \tfrac{1}{2}\Delta t) - \int_0^{\Delta t}\int_0^s g(s,\sigma)\,d\sigma\,ds\right\|_{L^2} \\ &\qquad\qquad \leq C\Delta t^3\big(\max\|\partial g/\partial s\|_{L^2} + \max\|\partial g/\partial\sigma\|_{L^2}\big),\end{aligned}$$

where the maxima are taken over the triangle $\{(s,\sigma) : 0 \leq \sigma \leq s \leq \Delta t\}$. In order to estimate the partial derivatives we write

$$g(s,\sigma) = e^{(\Delta t - s)A} dB(v(s)) w(s,\sigma),$$

where

$$v(s) = e^{sA}u_0 \quad\text{and}\quad w(s,\sigma) = e^{(s-\sigma)A}B(v(\sigma)).$$

With this notation

$$\begin{aligned}\frac{\partial g}{\partial s} &= e^{(\Delta t - s)A}\left(-AdB(v(s))w(s,\sigma)) + d^2B(Av(s), w(s,\sigma)) + dB(v(s))Aw(s,\sigma)\right) \\ &= e^{(\Delta t - s)A}\left(-A(v(s)w(s,\sigma)) + Av(s)w(s,\sigma) + v(s)Aw(s,\sigma)\right)_x.\end{aligned}$$

Now

$$\begin{aligned}\left\|\frac{\partial g}{\partial s}\right\|_{L^2} &\leq \|-A(v(s)w(s,\sigma)) + (Av(s))w(s,\sigma) + v(s)Aw(s,\sigma)\|_{H^1} \\ &\leq \|A(vw)\|_{H^1} + \|(Av)w\|_{H^1} + \|vAw\|_{H^1} \\ &\leq C\|v\|_{H^3}\|w\|_{H^3} \leq C\|v\|_{H^4}^3 \leq C\|u_0\|_{H^4}^3.\end{aligned}$$

For the other partial derivative, we get

$$\frac{\partial g}{\partial \sigma} = e^{(\Delta t - s)A} dB(v(s))\left(e^{(s-\sigma)A}\left(-AB(v(s)) + dB(v(\sigma))Av(\sigma)\right)\right),$$

so that

$$\begin{aligned}\left\|\frac{\partial g}{\partial \sigma}\right\|_{L^2} &\leq C\|v(s)\|_{H^1}\|-AB(v(s)) + dB(v(\sigma))Av(\sigma)\|_{H^1} \\ &\leq \|v\|_{H^1}\|AB(v)\|_{H^1} + \|v\|_{H^1}\|dB(v)[Av]\|_{H^1} \\ &\leq \|v_{H^1}\|\left(\|(vv_x)_{xx}\|_{H^1} + \|v\|_{H^2}\|A(v)\|_{H^2}\right) \\ &\leq C\|v\|_{H^4}^3.\end{aligned}$$

Therefore

$$\left\|\frac{\partial g}{\partial \sigma}\right\|_{L^2} \leq C \left\|u_0\right\|_{H^4}^3 .$$

Thus

$$\left\|e_2 - e_1\right\|_{L^2} \leq C\Delta t^3 \big(\left\|u_0\right\|_{H^4}^3 + \left\|u_0\right\|_{H^3}^4 \big) \leq C\Delta t^3, \tag{4.5}$$

which together with the bound for the quadrature error of the midpoint rule for f yields the stated result. □

Proof of Theorem 2.4. We argue as in the proof of Theorem 2.3, but now assume inductively that $\|u_k - u(t_k)\|_{L^2} \leq \gamma\Delta t^2$. With Ψ denoting the Strang step operator defined in (1.7), we have

$$\begin{aligned}
\|u_n - u(t_n)\|_{L^2} &\leq \sum_{k=0}^{n-1} \left\|\Phi^{(n-k-1)\Delta t}(\Psi^{\Delta t}(u_k)) - \Phi^{(n-k-1)\Delta t}(\Phi^{\Delta t}(u_k))\right\|_{L^2} \\
&\leq \sum_{k=0}^{n-1} K(R,T)\|\Psi^{\Delta t}(u_k) - \Phi^{\Delta t}(u_k)\|_{L^2} \\
&\leq \sum_{k=0}^{n-1} K(R,T)c_2(C_0)\Delta t^3 \leq K(R,T)c_2(C_0)T\Delta t^2.
\end{aligned}$$

This completes the proof of Theorem 2.4. □

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