ADDENDUM TO "THE KOLMOGOROV–RIESZ COMPACTNESS THEOREM" (EXPO. MATH. 28:385–394 (2010))

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ABSTRACT. We show how to improve on Theorem 10 in [3], describing when subsets in $W^{1,p}(\mathbb{R}^n)$ are totally bounded subsets of $L^q(\mathbb{R}^n)$ for $p < n$ and $p \le q < p^*$. This improvement was first shown in [1] in the context of Morrey–Sobolev spaces.

We show the following improvement of Theorem 10 in [3]:

Theorem. *Assume $1 \le p < n$ and $p \le q < p^*$, where*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n},$$

and let $\mathcal{F}$ be a bounded subset of $W^{1,p}(\mathbb{R}^n)$. Assume that for every $\epsilon > 0$ there exists some R so that, for every $f \in \mathcal{F}$,

$$\int_{|x|>R} |f(x)|^p \, dx < \epsilon^p. \tag{1}$$

Then $\mathcal{F}$ is a totally bounded subset of $L^q(\mathbb{R}^n)$.

Remark 1. The improvement over the original consists of replacing

$$\int_{|x|>R} \big(|f(x)|^p + |\nabla f(x)|_p^p\big) \, dx < \epsilon^p$$

by the weaker inequality (1). (Here $|\cdot|_p$ is the l^p norm on $\mathbb{R}^n$.)

Remark 2. The improvement we prove here was first shown in the more complicated setting of Morrey–Sobolev spaces in [1]. Here we present a direct argument in the setting of [3].

Proof. The Sobolev embedding theorem states that $W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$ and that the inclusion map is bounded. Thus $\mathcal{F}$ is a bounded subset of $L^q(\mathbb{R}^n)$.

We will need the Gagliardo–Nirenberg–Sobolev inequality, which states that there is a constant C (only dependent on p and n) such that

$$\|f\|_{p^*} \le C\|\nabla f\|_p$$

for all $f \in C_c^1(\mathbb{R}^n)$. For a proof, see, e.g., [2, Sect. 5.6.1, p. 263]. This inequality extends to any $f \in W^{1,p}(\mathbb{R})$: Let $f_n \in C_c^1(\mathbb{R})$ converge to f in $W^{1,p}(\mathbb{R})$. Then the inequality implies that (f_n) is Cauchy in $L^{p^*}(\mathbb{R})$, so it has a limit, which must be f itself, since some subsequence converges pointwise. The continuity of the norms finishes the argument.

The above inequality and the interpolation inequality $\|f\|_q \le \|f\|_p^\theta \, \|f\|_{p^*}^{1-\theta}$ where

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^*} \qquad (0 < \theta \le 1)$$

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yield

$$\|f\|_q \le C^{1-\theta}\|f\|_p^\theta\,\|\nabla f\|_p^{1-\theta}.$$

Now let $\epsilon > 0$, and pick R as in the statement of the theorem. Let $\phi \in C_c^\infty(\mathbb{R}^n)$ be a function with $0 \le \phi \le 1$ and $|\nabla\phi|_p \le 1$ satisfying $\phi(x) = 1$ when $|x| \le R$.

Then $\phi\mathcal{F} = \{\phi f : f \in \mathcal{F}\}$ is bounded in $W^{1,p}(\mathbb{R}^n)$, and by [3, Theorem 10] (or the usual Rellich–Kondrachov theorem on a ball of radius $R+2$), $\phi\mathcal{F}$ is totally bounded in $L^q(\mathbb{R}^n)$.

We find that every $f \in \mathcal{F}$ is uniformly approximated in the L^q norm by ϕf:

$$\begin{aligned}
\|f - \phi f\|_q &\le C^{1-\theta}\|(1-\phi)f\|_p^\theta\,\|\nabla((1-\phi)f)\|_p^{1-\theta} \\
&\le C^{1-\theta}\epsilon^\theta\,\|(1-\phi)\nabla f - f\nabla\phi\|_p^{1-\theta} \\
&\le C^{1-\theta}\epsilon^\theta\,\big(\|(1-\phi)\nabla f\|_p + \|f\nabla\phi\|_p\big)^{1-\theta} \\
&\le \big(C2^{1-1/p}\|f\|_{1,p}\big)^{1-\theta}\epsilon^\theta \\
&\le M\epsilon^\theta
\end{aligned}$$

where the constant M depends only on n, p, q, and $\mathcal{F}$. In the next to last line, we used Jensen's inequality on the form $u + v \le 2^{1-1/p}(u^p + v^p)^{1/p}$ when $u, v \ge 0$, together with the definition of the $W^{1,p}$ norm.

Thus every member of $\mathcal{F}$ lies within a distance $M\epsilon^\theta$ of $\phi\mathcal{F}$ in L^q norm. Since $\phi\mathcal{F}$ is totally bounded and $M\epsilon^\theta$ can be made arbitrarily small, it follows that $\mathcal{F}$ is totally bounded. □

References

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