Properness conditions for actions and coactions

S. Kaliszewski, Magnus B. Landstad, and John Quigg

Dedicated to R. V. Kadison — teacher and inspirator

ABSTRACT. Three properness conditions for actions of locally compact groups on C^*-algebras are studied, as well as their dual analogues for coactions. To motivate the properness conditions for actions, the commutative cases (actions on spaces) are surveyed; here the conditions are known: proper, locally proper, and pointwise properness, although the latter property has not been so well studied in the literature. The basic theory of these properness conditions is summarized, with somewhat more attention paid to pointwise properness. C^*-characterizations of the properties are proved, and applications to C^*-dynamical systems are examined. This paper is partially expository, but some of the results are believed to be new.

1. Introduction

In our recent study of C^*-covariant systems (A, G, α) and crossed product algebras between the full crossed product $A \rtimes_\alpha G$ and the regular crossed product $A \rtimes_{\alpha,r} G$, it turns out that various generalizations of the concept of proper actions of G play an important role. We therefore start by taking a closer look at this concept, and it turns out that even for a classical action of G on a space X we made what we believe to be new discoveries.

Classically (going back to Bourbaki [**Bou60**]), a G-space X is called *proper* if the map from $G \times X$ to $X \times X$ given by

$$(s, x) \mapsto (x, sx)$$

is proper, i.e., inverse images of compact sets are compact.

We call the action *pointwise proper* if the map from G to X given by

$$s \mapsto sx$$

is proper for each $x \in X$.

There is also an intermediate property: X is *locally proper* if each point of X has a G-invariant neighbourhood on which G acts properly.

Apparently the above terminology is not completely standard. For a discrete group, [**DV97**] uses the terms *discontinuous*, *properly discontinuous*, and *strongly properly discontinuous* instead of pointwise proper, locally proper, and proper, respectively. Palais uses *Cartan* instead of locally proper. And [**Kos65**] uses the terms P_2, P_1, and P, respectively. A characteristic property of properness (see Lemma 2.3 below) is sometimes referred to as "compact sets are wandering".

It is folklore that for proper G-spaces X the full crossed product $C_0(X) \rtimes_\alpha G$ is isomorphic to the reduced crossed product $C_0(X) \rtimes_{\alpha,r} G$ (see [**Phi89**] for the second countable case). In Proposition 6.12 (perhaps also folklore) we show that this carries over to locally proper actions. We will show in Theorem 6.2 (believed to be new) that this is true also if X is first countable, but the action is only assumed to be pointwise proper.

We propose the following as natural generalizations of properness to a general C^*-covariant system (A, G, α):

DEFINITION.
- (A, G, α) is *s-proper* if for all $a, b \in A$ the map
$$g \mapsto \alpha_g(a)b \text{ is in } C_0(G, A).$$
- (A, G, α) is *w-proper* if for all $a \in A$, $\phi \in A^*$ the map
$$g \mapsto \phi(\alpha_g(a)) \text{ is in } C_0(G).$$

This is consistent with the classical case, for $A = C_0(X)$ we have
$$(X, G) \text{ is proper} \iff (C_0(X), G) \text{ is s-proper}$$
$$(X, G) \text{ is pointwise proper} \iff (C_0(X), G) \text{ is w-proper}.$$

One indication that w-properness is an interesting property is the following

PROPOSITION. *Suppose (A, G, α) is w-proper, π a representation of A, and $s \mapsto U_s$ a continuous map into the unitaries (but not necessarily a homomorphism) such that $\pi(\alpha_s(a)) = U_s \pi(a) U_s^*$. Then for all ξ, η in the Hilbert space the coefficient function $s \mapsto \langle U_s \xi, \eta \rangle$ is in $C_0(G)$.*

We treat the classical situation of a G-space X in Sections 2 and 3, and discuss general C^*-covariant systems in Section 4.

For a C^*-covariant system (A, G, α), there are various definitions of properness (by Rieffel and others) involving some integrability properties. We show in Section 5 that they imply s- or w-properness. The main purpose of these integrability properties is to define a suitable fixed point algebra in $M(A)$, so our properness definitions are too general for this purpose.

The natural dual concept of a C^*-covariant system is that of a *coaction*. As we briefly describe in Section 7, it turns out that s- and w-properness can be defined in a similar way for coactions, and we describe some of the relevant results.

In Section 8 we describe a general construction of crossed product algebras between $A \rtimes_\alpha G$ and $A \rtimes_{\alpha,r} G$. We claim that the interesting ones are obtained by first taking as our group C^*-algebra $C^*(G)/I$ where I is a *small* ideal of $C^*(G)$ (*i.e.* I is δ_G-invariant and contained in the kernel of the regular representation λ of $C^*(G)$). We showed in [KLQ13] that I is a small ideal of $C^*(G)$ if and only if the annihilator $E = I^\perp$ in $B(G)$ is a *large* ideal, in the sense that it is a nonzero, weak* closed, and G-invariant ideal of the Fourier-Stieltjes algebra $B(G)$. There are various interesting examples, see [BG] and [KLQ13].

Now to a C^*-covariant system (B, G, α) and E as above one can define an E-crossed product $B \rtimes_{\alpha, E} G$ between the full and the reduced crossed product. In [KLQ13] we show that if the coaction is w-proper then there is a Galois theory describing these crossed products.

Finally we mention the work by Kirchberg, Baum, Guentner, and Willet [**BGW**] on the Baum-Connes conjecture. They have shown that there is a unique minimal exact and Morita compatible functor that assigns to a C^*-covariant system (A, G, α) a C^*-algebra between $A \rtimes_\alpha G$ and $A \rtimes_{\alpha, r} G$. At least one of the authors doubts that this minimal functor is an E-crossed product for some large ideal E, although this remains an open problem.

In Sections 2–6 we give a fairly detailed exposition, in particular proofs of results we believe to be new. Sections 7–8 will be more descriptive, referring to the literature for details and proofs.

2. Actions on spaces

Throughout, G will be a locally compact group, A will be a C^*-algebra, and X will be a locally compact Hausdorff space. We will be concerned with actions α of G on A, and we just say (A, α) is an action since the group G will typically be fixed. If G acts on X then we sometimes call X a *G-space*, and the *associated action* $(C_0(X), \alpha)$ is defined by

$$\alpha_s(f)(x) = f(s^{-1}x) \quad \text{for } s \in G, f \in C_0(X), x \in X.$$

Recall that, since the map $(s, x) \mapsto sx$ from $G \times X$ to X is continuous, the associated action α is *strongly continuous* in the sense that for all $f \in C_0(X)$ the map $s \mapsto \alpha_s(f)$ from G to $C_0(X)$ is continuous for the uniform norm.

The following notation is borrowed from Palais [**Pal61**]:

NOTATION 2.1. If G acts on X, then for two subsets $U, V \subset X$ we define

$$((U, V)) = \{s \in G : sU \cap V \neq \varnothing\}.$$

Note that if U and V are compact then $((U, V))$ is closed in G.

Much of the following discussion of actions on spaces is well-known; we present it in a formal way for convenience. We make no attempt at completeness, but at the same time we include many proofs to make this exposition self-contained. When a result can be explicitly found in [**Pal61**], we give a precise reference, but lack of such a reference should not be taken as any claim of originality. In much of the literature on proper actions the spaces are only required to be Hausdorff, or completely regular; in the proofs we will take full advantage of our assumption that our spaces are locally compact Hausdorff.

DEFINITION 2.2. A G-space X is *proper* if the map $\phi : X \times G \to X \times X$ defined by $\phi(x, s) = (x, sx)$ is proper, i.e., inverse images of compact sets are compact.

The following is routine, and explains why properness is sometimes referred to as "compact sets are wandering" (e.g., [**Rie82**, Situation 2]):

LEMMA 2.3. *A G-space X is proper if and only if for every compact $K \subset X$ the set $((K, K))$ is compact, equivalently for every compact $K, L \subset X$ the set $((K, L))$ is compact.*

EXAMPLE 2.4. If H is a closed subgroup of G, then it is an easy exercise that the action of G on the homogeneous space G/H by translation is proper if and only if H is compact.

The following result is contained in [**Pal61**, Theorem 1.2.9].

PROPOSITION 2.5. *A G-space X is proper if and only if for all $x, y \in X$ there are neighborhoods U of x and V of y such that $((U, V))$ is relatively compact.*

PROOF. One direction is obvious, since if the action is proper we only need to choose the neighborhoods U and V to be compact.

Conversely, assume the condition involving pairs of points x, y, and let $K \subset X$ be compact. To show that $((K, K))$ is compact, we will prove that any net $\{s_i\}$ in $((K, K))$ has a convergent subnet. For every i we can choose $x_i \in K$ such that $s_i x_i \in K$. Passing to subnets and relabeling, we can assume that $x_i \to x$ and $s_i x_i \to y$ for some $x, y \in K$. By assumption we can choose compact neighborhoods U of x and V of y such that $((U, V))$ is compact. Without loss of generality, for all i we have $x_i \in U$ and $s_i x_i \in V$, and hence $s_i \in ((U, V))$. Thus $\{s_i\}$ has a convergent subnet by compactness. $\square$

DEFINITION 2.6. A G-space X is *locally proper* if it is a union of open G-invariant sets on which G acts properly.

Palais uses the term *Cartan* instead of locally proper. The forward direction of the following result is [**Pal61**, Proposition 1.2.4].

LEMMA 2.7. *A G-space X is locally proper if and only if every $x \in X$ has a neighborhood U such that $((U, U))$ is compact.*

PROOF. First assume that the action is locally proper, and let $x \in X$. Choose an open G-invariant set V containing x on which G acts properly. Then choose a compact neighborhood U of x contained in V. Then $((U, U))$ is compact by properness.

Conversely, assume the condition involving compact sets $((U, U))$. Choose an open neighborhood V of x such that $((V, V))$ is relatively compact, and let $U = GV$. We will show that the action of G on U is proper. Let $y, z \in U$. Choose $s, t \in G$ such that $y \in sV$ and $z \in tV$. Then we have neighborhoods sV of y and tV of z, and

$$((sV, tV)) = t((V, V))s^{-1}$$

is relatively compact. $\square$

The following result displays a kind of semicontinuity of the sets $((V, V))$, and in also of the stability subgroups. The forward direction is [**Pal61**, Proposition 1.1.6].

PROPOSITION 2.8. *A G-space X is locally proper if and only if for all $x \in X$, the isotropy subgroup G_x is compact and for every neighborhood U of G_x there is a neighborhood V of x such that $((V, V)) \subset U$.*

PROOF. First assume that the action is locally proper. We argue by contradiction. Suppose we have $x \in X$ and a neighborhood U of G_x such that for every neighborhood V of x there exists $s \in ((V, V))$ such that $s \notin U$. Fix a neighborhood R of x such that $((R, R))$ is compact. Restricting to neighborhoods V of x with $V \subset R$, we see that we can find nets $\{s_i\}$ in the complement U^c and $\{y_i\}$ in R such that

- $s_i y_i \in R$ for all i,
- $y_i \to x$, and
- $s_i y_i \to x$.

Then $s_i \in ((R, R))$ for all i, so passing to subnets and relabeling we can assume that $s_i \to s$ for some $s \in G$. Then $s_i y_i \to sx$, so $sx = x$. Thus $s \in G_x$. But then eventually $s_i \in U$, which is a contradiction.

Conversely, assume the condition regarding isotropy groups and neighborhoods thereof, and let $x \in X$. Since G_x is compact, we can choose a compact neighborhood U of G_x, and then we can choose a neighborhood V of x such that $((V, V)) \subset U$. Then $((V, V))$ is relatively compact, and we have shown that the action is locally proper. □

The following result is contained in [**Pal61**, Theorem 1.2.9].

PROPOSITION 2.9. *A G-space X is proper if and only if it is locally proper and $G \backslash X$ is Hausdorff.*

PROOF. First assume that the action is proper. Then it is locally proper, and to show that $G \backslash X$ is Hausdorff, we will prove that if a net $\{Gx_i\}$ in $G \backslash X$ converges to both Gx and Gy then $Gx = Gy$. Since the quotient map $X \to G \backslash X$ is open, we can pass to a subnet and relabel so that without loss of generality $x_i \to x$. Then again passing to a subnet and relabeling we can find $s_i \in G$ such that $s_i x_i \to y$. Choose compact neighborhoods U of x and V of y, so that $((U, V))$ is compact by properness. Without loss of generality $x_i \in U$ and $s_i x_i \in V$ for all i. Then $s_i \in ((U, V))$ for all i, so by compactness we can pass to subnets and relabel so that $\{s_i\}$ converges to some $s \in G$. Then $s_i x_i \to sx$, so $sx = y$, and hence $Gx = Gy$.

Conversely, assume that the action is locally proper and $G \backslash X$ is Hausdorff. Let $x, y \in X$. By assumption we can choose a compact neighborhood U of x such that $((U, U))$ is compact. Now choose any compact neighborhood V of y. To show that the action is proper, we will prove that $((U, V))$ is compact. Let $\{s_i\}$ be any net in $((U, V))$. For each i choose $x_i \in U$ such that $s_i x_i \in V$. By compactness we can pass to subnets and relabel so that $x_i \to z$ and $s_i x_i \to w$ for some $z \in U$ and $w \in V$. Then by Hausdorffness we can write

$$Gz = \lim Gx_i = \lim Gs_i x_i = Gw,$$

so we can choose $s \in G$ such that $w = sz$. Then $s_i x_i \to sz$, so

$$s^{-1} s_i x_i \to z.$$

Without loss of generality, for all i we can assume that $s^{-1} s_i x_i \in U$, so that $s^{-1} s_i \in ((U, U))$. By compactness we can pass to subnets and relabel so that $s^{-1} s_i \to t$ for some $t \in G$. Thus $s_i \to st$, and we have found a convergent subnet of $\{s_i\}$. Thus $((U, V))$ is compact. □

EXAMPLE 2.10. It is a well-known fact in topological dynamics that there are actions that are locally proper but not proper, e.g., the action of $\mathbb{Z}$ on

$$[0, \infty) \times [0, \infty) \setminus \{(0, 0)\}$$

generated by the homeomorphism $(x, y) \mapsto (2x, y/2)$, where any compact neighborhood of $\{(1, 0), (0, 1)\}$ meets itself infinitely often. This action is locally proper because its restriction to each of the open sets $[0, \infty) \times (0, \infty)$ and $(0, \infty) \times [0, \infty)$,

which cover the space, are proper. A closely related example is given by letting $\mathbb{R}$ act on the same space by $s(x, y) = (e^s x, e^{-s} y)$.

DEFINITION 2.11. A G-space X is *pointwise proper* if for all $x \in X$ and compact $K \subset X$, the set $((x, K))$ is compact.

The above properness condition does not seem to be very often studied in the dynamics literature, and the term we use is not standard, as far as we have been able to determine.

It is obvious that the above definition can be reformulated as follows:

LEMMA 2.12. *A G-space X is pointwise proper if and only if for every $x \in X$ the map $s \mapsto sx$ from G to X is proper.*

PROPOSITION 2.13. *If a G-space X is pointwise proper then orbits are closed, and hence $G\backslash X$ is T_1.*

PROOF. Let $x \in X$, and suppose we have a net $\{s_i x\}$ in the orbit Gx converging to $y \in X$. Choose a compact neighborhood U of y. Without loss of generality, for all i we have $s_i x \in U$, and hence $s_i \in ((x, U))$. This set is compact by pointwise properness, so passing to a subnet and relabeling we can assume that $s_i \to s$ for some $s \in G$. Then $s_i x \to sx$, so $y = sx \in Gx$. $\square$

NOTATION 2.14. For $x \in X$ let G_x denote the isotropy subgroup.

PROPOSITION 2.15. *A G-space X is pointwise proper if and only if for all $x \in X$ the isotropy subgroup G_x is compact and the map $s \mapsto sx$ from G to Gx is relatively open, equivalently, the action of G on the orbit Gx is conjugate to the action on the homogeneous space G/G_x.*

PROOF. First assume that the action is pointwise proper, and let $x \in X$. Then G_x is trivially compact. By homogeneity it suffices to show that the map $s \mapsto sx$ from G to Gx is relatively open at e. Let W be a neighborhood of e. Suppose that Wx is not a relative neighborhood of x in the orbit Gx. Then we can choose a net $\{s_i\}$ in G such that $s_i x \notin Wx$ and $s_i x \to x$. Choose a neighborhood U of x such that $((U, U))$ is compact. Without loss of generality, for all i we have $s_i x \in U$, and so $s_i \in ((x, U))$. By compactness we can pass to a subnet and relabel so that $s_i \to s$ for some $s \in G$. Then $s_i x \to sx$. Thus $sx = x$, and so $s \in G_x$. But then eventually $s_i \in WG_x$, which is a contradiction because $WG_x x = Wx$.

The converse is obvious, since if G_x is compact the action of G on G/G_x is proper. $\square$

We will show that pointwise properness is weaker than local properness, but for this we need a version of Proposition 2.13 for local properness. The following result is contained in [**Pal61**, Proposition 1.1.4].

LEMMA 2.16. *If a G-space X is locally proper then orbits are closed.*

PROOF. Let $x \in X$, and suppose we have a net $\{s_i\}$ in G such that $s_i x \to y$. Choose an open G-invariant subset U containing y on which G acts properly. Then the action of G on U is pointwise proper, so the orbit Gx is closed in U, and hence $y \in Gx$. $\square$

COROLLARY 2.17. *If a G-space X is locally proper then it is pointwise proper.*

PROOF. Let $x \in X$. Choose an open G-invariant set $U \subset X$ such that the action of G on U is proper. Let $K \subset X$ be compact, and put $L = K \cap Gx$. Then L is compact because Gx is closed, and $L \subset U$. Thus $((x, K)) = ((x, L))$ is compact because $\{x\}$ and L are compact subsets of U and G acts properly on U. $\qquad \square$

EXAMPLE 2.18. This example is taken from [**DV97**, Example 5 in Section 2]. Recall that in Example 2.10 we had an action of $\mathbb{Z}$ on the space

$$X = \big([0, \infty) \times [0, \infty)\big) \setminus \{(0,0\}$$

generated by the homeomorphism $(x, y) \mapsto (2x, y/2)$. We form the quotient of X by identifying $\{0\} \times (0, \infty)$ with $(0, \infty) \times \{0\}$ via

$$(0, y) \sim (1/y, 0).$$

Then the action descends to the identification space, and the quotient action is pointwise proper but not locally proper.

With suitable countability assumptions, there is a surprise:

COROLLARY 2.19 (Glimm). *Let G act on X, and assume that G and X are second countable, and that every isotropy subgroup is compact. Then the following are equivalent:*

  (1) *the action is pointwise proper;*
  (2) *for all $x \in X$ the map $sG_x \mapsto sx$ from G/G_x to Gx is a homeomorphism;*
  (3) *$G \backslash X$ is T_0;*
  (4) *$G \backslash X$ is T_1;*
  (5) *every orbit is locally compact in the relative topology from X;*
  (6) *every orbit is closed in X.*

PROOF. Because we assume that the isotropy groups are compact, we know (1) $\iff$ (2). Glimm [**Gli61**, Theorem 1] proves that, in the second countable case, (2) $\iff$ (3) $\iff$ (5). We also know (1) $\Rightarrow$ (6) $\Rightarrow$ (4). Finally, (4) $\Rightarrow$ (3) trivially. $\qquad \square$

3. C^*-ramifications

Let X be a G-space, and let α be the associated action of G on $C_0(X)$. In this section we examine the ramifications for the action α of the various properness conditions covered in Section 2. For the state of the art in the case of proper actions, see [**EE11**].

NOTATION 3.1. If $\psi : X \to Y$ is a continuous map between locally compact Hausdorff spaces, define $\psi^* : C_0(Y) \to C_b(X)$ by $\psi^*(f) = f \circ \psi$.

It is an easy exercise to show:

LEMMA 3.2. *For a continuous map $\psi : X \to Y$ between locally compact Hausdorff spaces, the following are equivalent:*

  (1) *ψ is proper*
  (2) *ψ^* maps $C_0(Y)$ into $C_0(X)$*
  (3) *ψ^* maps $C_c(Y)$ into $C_c(X)$.*

PROPOSITION 3.3. *The G-space X is proper if and only if for all $f, g \in C_0(X)$ the map $s \mapsto \alpha_s(f)g$ from G to $C_0(X)$ vanishes at infinity.*

PROOF. First assume that the action is proper. Since $C_c(X)$ is dense in $C_0(X)$, by continuity it suffices to show that for all $f, g \in C_c(X)$ the map continuous $s \mapsto \alpha_s(f)g$ from $G \to C_0(X)$ has compact support. Define $f \times g \in C_c(X \times X)$ by

$$f \times g(x, y) = f(x)g(y).$$

Since the map $\phi : G \times X \to X \times X$ given by $g(s, x) = (sx, x)$ is proper, we have $\phi^*(f \times g) \in C_c(G \times X)$, so there exist compact sets $K \subset G$ and $L \subset X$ such that for all $(s, x) \notin K \times L$ we have

$$0 = \phi^*(f \times g)(s, x) = f \times g(sx, x) = f(sx)g(x) = \big(\alpha_{s^{-1}}(f)g\big)(x).$$

Since $s \notin K$ implies $(s, x) \notin K \times L$, we see that the map $s \mapsto \alpha_s(f)g$ is supported in the compact set K^{-1}.

Conversely, assume the condition regarding $\alpha_s(f)g$. To show that the action is proper, we will show that the map ϕ is proper, and by Lemma 3.2 it suffices to show that if $h \in C_c(X \times X)$ then $\phi^*(h) \in C_c(G \times X)$. The support of h is contained in a product $M \times N$ for some compact sets $M, N \subset X$, and we can choose $f, g \in C_c(X)$ with $f = 1$ on M and $g = 1$ on N. Then $h(f \times g) = h$, so it suffices to show that $\phi^*(f \times g)$ has compact support. By assumption the support K of $s \mapsto \alpha_s(f)g$ is compact, and letting L be the support of g we see that for all (s, x) not in the compact set $K^{-1} \times L$ we have

$$\phi^*(f \times g)(s, x) = \big(\alpha_{s^{-1}}(f)g\big)(x) = 0. \qquad \square$$

PROPOSITION 3.4. *The G-space X is pointwise proper if and only if for all $f \in C_0(X)$ and $\mu \in M(X) = C_0(X)^*$ the map*

$$g(s) = \int_X f(sx) \, d\mu(x)$$

is in $C_0(G)$.

PROOF. First assume that the action is pointwise proper. Let $f \in C_0(X)$ and $\mu \in M(X)$, and define g as above. Note that g is continuous since the associated action $(C_0(X), \alpha)$ is strongly continuous. Suppose that g does not vanish at ∞, and pick $\varepsilon > 0$ such that the closed set

$$S := \{s \in G : |g(s)| \geq \varepsilon\}$$

is not compact. It is a routine exercise to verify that we can find a sequence $\{s_n\}$ in S and a compact neighborhood V of e such that the sets $\{s_n V\}$ are pairwise disjoint. Then for each $x \in X$ we have $\lim_{n \to \infty} f(s_n x) = 0$, because for fixed x and any $\delta > 0$ it is an easy exercise to see that the compact set $\{s \in G : |f(sx)| \geq \delta\}$ can only intersect finitely many of the sets $\{s_n V\}$. Thus by the Dominated Convergence theorem $\lim_{n \to \infty} g(x_n) = 0$, contradicting $s_n \in S$ for all n.

The converse follows immediately by taking μ to be a Dirac measure and applying Lemma 2.12. $\qquad \square$

Proposition 3.5 below is the first time we need vector-valued integration. There are numerous references dealing with this topic. We are interested in integrating functions $f : \Omega \to B$, where Ω is a locally compact Hausdorff space equipped with a Radon measure μ (sometimes complex, but other times positive, and then frequently infinite), and B is a Banach space. Rieffel [**Rie04**, Section 1] handles continuous bounded functions to a C^*-algebra using C^*-valued weights. Exel [**Exe99**, Section 2] develops a theory of *unconditionally integrable* functions with values

in a Banach space, involving convergence of the integrals over relatively compact subsets of G. Williams [**Wil07**, Appendix B.1] gives an exposition of the general theory of $L^1(\Omega, B)$, that in some sense unifies the treatments in [**DS88**, Chapter 3], [**Bou63**], [**FD88**, Chapter II], and [**HP74**, part I, Section III.1]. However, Williams uses a positive measure throughout, and we occasionally need complex measures; this poses no problem, since the theory of [**Wil07**] can be applied to the positive and negative variations of the real and imaginary parts of a complex measure. We prefer to use [**Wil07**] as our reference for vector-valued integration, mainly because it entails *absolute integrability* rather than *unconditional integrability* (see the first item in the following list). Here are the main properties of $L^1(\Omega, B)$ that we need:

- The map $f \mapsto \int_\Omega f \, d\mu$ from $L^1(\Omega, B)$ to B is bounded and linear, where $\|f\|_1 = \int_\Omega \|f(x)\| \, d|\mu|(x)$.
- If $f \in L^1(\Omega, B)$ and ω is a bounded linear functional on B, then $\omega \circ f \in L^1(\Omega)$ and
$$\omega \left(\int_\Omega f(x) \, d\mu(x) \right) = \int_\Omega \omega(f(x)) \, d\mu(x).$$
- If $f \in L^1(\Omega)$ and $b \in B$ then
$$\int_\Omega (f \otimes b) \, d\mu = \left(\int_\Omega f \, d\mu \right) b,$$
where $(f \otimes b)(x) = f(x)b$.
- Every continuous bounded function from Ω to B is measurable, and is also essentially-separably valued on compact sets, and so is integrable with respect to any complex measure.

Of course, we refer to the elements of $L^1(\Omega, B)$ as the *integrable* functions from Ω to B.

If X is a G-space, then $C_0(X)$ gets a Banach-module structure over $M(G) = C_0(G)^*$ by
$$\mu * f(x) = \int_G f(sx) \, d\mu(s) \quad \text{for } \mu \in M(G), f \in C_0(X), x \in X.$$

Here we are integrating the continuous bounded function $s \mapsto \alpha_s(f)$ with respect to the complex measure μ.

The following is a special case of Proposition 4.6 below.

PROPOSITION 3.5. *The action on X is pointwise proper if and only if for each f the map $\mu \mapsto \mu * f$ is weak*-to-weakly continuous.*

4. Properness conditions for actions on C^*-algebras

Propositions 3.3 and 3.4 motivate the following:

DEFINITION 4.1. An action (A, α) is *s-proper* if for all $a, b \in A$ the map $s \mapsto \alpha_s(a)b$ from G to A vanishes at infinity.

Taking adjoints, we see that the above map could equally well be replaced by $s \mapsto a\alpha_s(b)$.

DEFINITION 4.2. An action (A, α) is *w-proper* if for all $a \in A$ and all $\omega \in A^*$ the map
$$g(s) = \omega\big(\alpha_s(a)\big)$$

is in $C_0(G)$.

We use the admittedly nondescriptive terminology s-proper and w-proper to avoid confusion with the myriad other uses of the word "proper" for actions on C^*-algebras.

REMARK 4.3. It is almost obvious that a G-space X is locally proper if and only if there is a family of α-invariant closed ideals of $C_0(X)$ that densely span $C_0(X)$ and on each of which α has the property in Proposition 3.3. In fact, we will use this in the proof of Proposition 6.12. This could be generalized in various ways to actions on arbitrary C^*-algebras, but since we have no applications of this we will not pursue it here.

Propositions 3.3 and 3.4 can be rephrased as follows:

COROLLARY 4.4. A G-space X is proper if and only if the associated action $(C_0(X), \alpha)$ is s-proper, and is pointwise proper if and only if α is w-proper.

REMARK 4.5. If an action (A, α) is s-proper then it is w-proper, since by the Cohen-Hewitt factorization theorem every functional in A^* can be expressed in the form $\omega \cdot a$, where

$$\omega \cdot a(b) = \omega(ab) \quad \text{for } \omega \in A^*, a, b \in A.$$

On the other hand, Example 2.10 implies that α can be w-proper but not s-proper.

If (A, α) is an action then A gets a Banach module structure over $M(G)$ by

$$\mu * a = \int_G \alpha_s(a) \, d\mu(s) \quad \text{for } \mu \in M(G), a \in A.$$

Proposition 3.5 is the commutative version of the following:

PROPOSITION 4.6. An action (A, α) is w-proper if and only if for each $a \in A$ the map $\mu \mapsto \mu * a$ is weak*-to-weakly continuous.

PROOF. First assume that α is w-proper, and let $a \in A$. Let $\mu_i \to 0$ weak* in $M(G)$, and let $\omega \in A^*$. Then

$$\omega(\mu_i * a) = \omega \left(\int_G \alpha_s(a) \, d\mu_i(s) \right) = \int \omega(\alpha_s(a)) \, d\mu_i(s) \to 0,$$

because the map $s \mapsto \omega(\alpha_s(a))$ is in $C_0(G)$.

Conversely, assume the weak*-weak continuity, and let $a \in A$ and $\omega \in A^*$. If $\mu_i \to 0$ weak* in $M(G)$, then

$$\int_G \omega(\alpha_s(a)) \, d\mu_i(s) = \omega(\mu_i * a) \to 0$$

by continuity. By the well-known Lemma 4.7 below, the element $s \mapsto \omega(\alpha_s(a))$ of $C_b(G)$ lies in $C_0(G)$.   □

In the above proof we appealed to the following well-known fact:

LEMMA 4.7. Let $f \in C_b(G)$. Then $f \in C_0(G)$ if and only if for every net $\{\mu_i\}$ in $M(G)$ converging weak* to 0 we have

$$\int f \, d\mu_i \to 0.$$

The properties of s-properness and w-properness are both preserved by morphisms:

PROPOSITION 4.8. *Let $\phi : A \to M(B)$ be a nondegenerate homomorphism that is equivariant for actions α and β, respectively. If α is s-proper or w-proper, then β has the same property.*

PROOF. First assume that α is s-proper. Let $c, d \in B$. By the Cohen-Hewitt Factorization theorem, $c = c'\phi(a)$ and $d = \phi(b)d'$ for some $a, b \in A$ and $c', d' \in B$. Then

$$\beta_s(c)d = \beta_s(c'\phi(a))\phi(b)d'$$
$$= \beta_s(c')\phi(\alpha_s(a)b)d',$$

which vanishes at infinity because $s \mapsto \alpha_s(a)b$ does and $s \mapsto \beta_s(c')$ is bounded.

Now assume that α is w-proper. Let $b \in B$ and $\omega \in B^*$. We must show that the function $s \mapsto \omega \circ \beta_s(b)$ vanishes at ∞, and it suffices to do this for ω positive. By the Cohen-Hewitt Factorization theorem we can assume that $b = \phi(a^*)c$ with $a \in A$ and $c \in B$. By the Cauchy-Schwarz inequality for positive functionals on C^*-algebras, we have

$$\left|\omega \circ \beta_s(b)\right|^2 = \left|\omega\big(\phi(\alpha_s(a^*))\beta_s(c)\big)\right|^2$$
$$\leq \omega \circ \phi(\alpha_s(a^*a))\omega(\beta_s(c^*c)),$$

which vanishes at ∞ since $s \mapsto \omega \circ \phi(\alpha_s(a^*a))$ does and $s \mapsto \omega(\beta_s(c^*c))$ is bounded. □

In Section 7 we will discuss properness for coactions, the dualization of actions. Here we record an easy corollary of Proposition 4.8 that involves coactions, because it gives a rich supply of s-proper actions. For now we just need to recall that if (A, δ) is a coaction of G, with crossed product C^*-algebra $A \rtimes_\delta G$, then there is a pair of nondegenerate homomorphisms

$$A \xrightarrow{\ j_A\ } M(A \rtimes_\delta G) \xleftarrow{\ j_G\ } C_0(G)$$

such that (j_A, j_G) is a universal covariant homomorphism. The *dual action* $\widehat{\delta}$ of G on $A \rtimes_\delta G$ is characterized by

$$\widehat{\delta}_s \circ j_A = j_A$$
$$\widehat{\delta}_s \circ j_G = j_G \circ \mathrm{rt}_s,$$

where rt is the action of G on $C_0(G)$ by right translation.

COROLLARY 4.9. *Every dual action is s-proper.*

PROOF. If δ is a coaction of G on A, then the canonical nondegenerate homomorphism $j_G : C_0(G) \to M(A \rtimes_\delta G)$ is $\mathrm{rt} - \widehat{\delta}$ equivariant. Thus $\widehat{\delta}$ is s-proper since rt is. □

[**BG12**, Corollary 5.9] says that if an action of a discrete group G on a compact Hausdorff space X is a-T-menable in the sense of [**BG12**, Definition 5.5], then every covariant representation of the associated action $(C(X), \alpha)$ is weakly contained in a representation (π, U), on a Hilbert space H, such that for all ξ, η in a dense subspace

of H the function $s \mapsto (U_s\xi, \eta)$ is in $c_0(G)$. The following proposition shows that w-proper actions on arbitrary C^*-algebras have a quite similar property:

PROPOSITION 4.10. *Let (A, α) be a w-proper action, let π be a representation of A on a Hilbert space H, and for each $s \in G$ suppose we have a unitary operator U_s on H such that $\operatorname{Ad} U_s \circ \pi = \pi \circ \alpha_s$. Then for all $\xi, \eta \in H$ the function*

$$s \mapsto \langle U_s\xi, \eta \rangle$$

vanishes at infinity.

PROOF. We can assume that π is nondegenerate. Then we can factor $\xi = \pi(a)\xi'$ for some $a \in A, \xi' \in H$, and we have

$$|\langle U_s\pi(a)\xi', \eta \rangle| = |\langle U_s\xi', \pi(\alpha_s(a^*))\eta \rangle|$$
$$\leq \|\xi'\| \langle \pi(\alpha_s(aa^*))\eta, \eta \rangle^{1/2},$$

so we can appeal to w-properness with $\omega \in A^*$ defined by

$$\omega(b) = \langle \pi(b)\eta, \eta \rangle. \qquad \square$$

REMARK 4.11. Note that in the above proposition we do not require U to be a homomorphism; it could be a projective representation.

REMARK 4.12. Thus it would be interesting to study the relation between a-T-menable actions in the sense of [**BG12**] and pointwise proper actions. As it stands, the connection would be subtle, because an infinite discrete group cannot act pointwise properly on a compact space.

Action on the compacts. The following gives a strengthening of a special case of Proposition 4.10:

PROPOSITION 4.13. *Let H be a Hilbert space, and let α be an action of G on $\mathcal{K}(H)$. For each $s \in G$ choose a unitary operator U_s such that $\alpha_s = \operatorname{Ad} U_s$. The following are equivalent:*

   (1) *α is s-proper;*
   (2) *α is w-proper;*
   (3) *$s \mapsto \langle U_s\xi, \xi \rangle$ vanishes at infinity for all $\xi \in H$.*
   (4) *$s \mapsto \langle U_s\xi, \eta \rangle$ vanishes at infinity for all $\xi, \eta \in H$.*

PROOF. We know $(1) \Rightarrow (2) \Rightarrow (3)$ by Remark 4.5 and Proposition 4.10, and $(3) \Rightarrow (4)$ by polarization.

Assume (4). Let $E(\xi, \eta)$ be the rank-1 operator given by $\zeta \mapsto \langle \zeta, \eta \rangle \xi$. For $\xi, \eta, \gamma, \kappa \in H$, A routine computation shows

$$E(\xi, \eta)\alpha_s(E(\gamma, \kappa)) = \langle U_s\gamma, \eta \rangle E(\xi, \kappa)U_s^*,$$

so

$$\big\| E(\xi, \eta)\alpha_s(E(\gamma, \kappa)) \big\| \leq \big| \langle U_s\gamma, \eta \rangle \big| \| E(\xi, \kappa) \|,$$

which vanishes at infinity. Thus $s \mapsto a\alpha_s(b)$ is in $C_0(G, \mathcal{K}(H))$ whenever a and b are rank-1, and by linearity and density it follows that α ia s-proper. $\qquad \square$

In Proposition 4.13, when U can be chosen to be a representation of G, we have the following:

COROLLARY 4.14. *Let U be a representation of G on a Hilbert space H, and let $\alpha = \operatorname{Ad} U$ be the associated action of G on $\mathcal{K}(H)$. Suppose that ξ is a cyclic vector for the representation U. If $s \mapsto \langle U_s \xi, \xi \rangle$ vanishes at infinity, then α is s-proper.*

PROOF. As in [**BG12**, Remark 2.7], it is easy to see that for all η, κ in the dense subspace of H spanned by $\{U_s \xi : s \in G\}$ the function $s \mapsto \langle U_s \eta, \kappa \rangle$ vanishes at infinity. Then for all $\eta, \kappa \in H$ we can find sequences $\{\eta_n\}, \{\kappa_n\}$ such that $\|\eta_n - \eta\| \to 0$, $\|\kappa_n - \kappa\| \to 0$, and for all n the function $s \mapsto \langle U_s \eta_n, \kappa_n \rangle$ vanishes at infinity. Then a routine estimation shows that the functions $s \mapsto \langle U_s \eta_n, \kappa_n \rangle$ converge uniformly to the function $s \mapsto \langle U_s \eta, \kappa \rangle$, and hence this latter function vanishes at infinity. The result now follows from Proposition 4.13. $\square$

5. Rieffel properness

We will show that if an action (A, α) is proper in Rieffel's sense [**Rie90**, Definition 1.2] (see also [**Rie04**, Definition 4.5] then it is s-proper. Rieffel's definitions of proper action in both of the above papers involve integration of A-valued functions on G, and we have recorded our conventions regarding vector-valued integration in the discussion preceding Proposition 3.5. In [**Rie90**], Rieffel defined an action (A, α) to be *proper* (and we follow [**BE**] in using the term *Rieffel proper*) if $s \mapsto \alpha_s(a)b$ is integrable for all a, b in some dense subalgebra, plus other conditions that we will not need.

COROLLARY 5.1. *Let (A, α) be an action.*

(1) *Suppose that there is a dense α-invariant subset A_0 of A such that for all $a, b \in A_0$ the function*

(5.1)
$$s \mapsto \alpha_s(a)b$$

is integrable. Then α is s-proper in the sense of Definition 4.1.

(2) *Suppose that there is a dense α-invariant subset A_0 of A such that for all $a \in A_0$ and all $\omega \in A^*$ the function*

$$s \mapsto \omega(\alpha_s(a))$$

is integrable. Then α is w-proper in the sense of Definition 4.2.

PROOF. (1) Since the functions (5.1) are uniformly continuous in norm, it follows immediately from the elementary lemma Lemma 5.2 below that $s \mapsto \alpha_s(a)b$ is in $C_0(G, A)$ for all $a, b \in A_0$, and then (1) follows by density.

(2) This can be proved similarly to (1), except now the functions are scalar-valued. $\square$

In the above proof we referred to the following:

LEMMA 5.2. *Let B be a Banach space, and let $f : G \to B$ be uniformly continuous and integrable. Then f vanishes at infinity.*

PROOF. Since the composition of f with the norm on B is uniformly continuous, and $\|f\|_1 = \int_G \|f(s)\|\, ds < \infty$ by hypothesis, so this follows immediately from the scalar-valued case (for which, see [**Car96**, Theorem 1]), and which itself is a routine adaptation of a classical result about scalar-valued functions on $\mathbb{R}$, sometimes referred to as Barbalat's Lemma. $\square$

In the commutative case, Corollary 5.1 (1) has a converse. First, following [**BE**], we will call an action (A, α) *Rieffel proper* if it satisfies the conditions of [**Rie90**, Definition 1.2].

PROPOSITION 5.3. *If $A = C_0(X)$ is commutative, then an action (A, α) is s-proper if and only if it is Rieffel proper.*

PROOF. First assume that α is s-proper. Then by Theorem 4.4 the G-space X is proper, and then it follows from [**Rie04**, Theorem 4.7 and and its proof] that α is Rieffel proper.

Conversely, if α is Rieffel proper, then in particular it satisfies the hypothesis of Corollary 5.1 (1), so α is s-proper.   □

REMARK 5.4. Thus, if the G-space X is proper, then by [**Rie90**, Theorem 1.5] (for the case of free action, see also [**Rie82**, Situation 2], which refers to [**Gre77**]) there is an ideal of $C_0(X) \rtimes_r G$ (which is known to equal $C_0(X) \rtimes G$ in this case — see Proposition 6.12 below) that is Morita equivalent to $C_0(G \backslash X)$. This uses the following: for $f \in C_c(X)$ the integral

$$\widehat{f}(Gx) := \int_G f(sx) \, ds$$

defines $\widehat{f} \in C_c(G \backslash X)$. If the action on X is just pointwise proper, the integral $\int_G f(sx) \, ds$ still makes sense for $f \in C_c(X)$. It would be interesting to know what properties persist in this case.

EXAMPLE 5.5. Proposition 5.3 is not true for arbitrary actions (A, α). For example, let G be the free group $\mathbb{F}_n$ with $n > 1$, and let l be the length function. Haagerup proves in [**Haa79**] that for any $a > 0$ the function $s \mapsto e^{-al(s)}$ is positive definite.

For $k \in \mathbb{N}$ define $h_k(s) = e^{-l(s)/k}$, and let U_k be the associated cyclic representation on a Hilbert space H_k, so that we have a cyclic vector ξ_k for U_k with

$$\langle U_k(s)\xi_k, \xi_k \rangle = h_k(s).$$

For each k, since h_k vanishes at infinity the associated inner action $\alpha_k = \operatorname{Ad} U_k$ of G on $\mathcal{K}(H_k)$ is s-proper, by Corollary 4.14.

We claim that not all these actions α_k can be Rieffel proper. Rieffel shows in [**Rie04**, Theorem 7.9] that the action α is proper in the sense of [**Rie04**, Definition 4.5] if and only if the representation U is square-integrable in the sense of [**Rie04**, Definition 7.8]. This latter definition is somewhat nonstandard, in that it uses concepts from the theory of left Hilbert algebras. Also, Rieffel's definition of proper action in [**Rie04**] is somewhat complicated in that it involves C^*-valued weights. In this paper we prefer to deal with the more accessible definition of Rieffel-proper action in [**Rie90**, Definition 1.2], which Rieffel shows implies the properness condition [**Rie04**, Definition 4.5]. Actually, we need not concern ourselves here with Rieffel's definition of square-integrable representations, rather all we need is his reassurance (see [**Rie04**, Corollary 7.12 and Theorem 7.14] that a cyclic representation of G is square-integrable in his sense if and only if it is contained in the regular representation of G — so his notion of square integrability is equivalent to the more usual one (as he assures us in his comment following [**Rie04**, Definition 7.8]).

Suppose that for every $k \in \mathbb{N}$ the action α_k of G on $\mathcal{K}(H_k)$ is Rieffel proper. Then, as noted above, α_k is also proper in the sense of [**Rie04**, Definition 4.5], and so the representation U_k is contained in the regular presentation λ. Now we argue exactly as in [**BG12**, proof of Proposition 4.4]: since the functions h_k converge to 1 pointwise on the discrete group G, for all $s \in G$ we have

$$\langle U_k(s)\xi_k, \xi_k \rangle \to 1,$$

and hence

$$\|U_k(s)\xi_k - \xi_k\| \to 0.$$

Thus the direct sum representation $\bigoplus_k U_k$ weakly contains the trivial representation. But since each U_k is contained in λ, the direct sum is weakly contained in λ. This gives a contradiction, since $G = \mathbb{F}_n$ is nonamenable.

6. Full equals reduced

DEFINITION 6.1. Let (A, α) be an action. We say the *full and reduced crossed products of (A, α) are equal* if the regular representation

$$\Lambda : A \rtimes_\alpha G \to A \rtimes_{\alpha,r} G$$

is an isomorphism.

It is an old theorem [**Phi89**] that if X is a second countable proper G-space then the associated action $(C_0(X), \alpha)$ has full and reduced crossed products equal. It is folklore that the second-countability hypothesis can be removed — see the proof of Proposition 6.12 and Remark 6.14. We extend this to pointwise proper actions and weaken the countability hypothesis:

THEOREM 6.2. *If X is a first countable pointwise proper G-space, then the full and reduced crossed products of the associated action $(C_0(X), \alpha)$ are equal.*

We need some properties of the "full = reduced" phenomenon for actions. First, it is frequently inherited by invariant subalgebras:

LEMMA 6.3. *Let (A, α) and (B, β) be actions, and let $\phi : A \to M(B)$ be an injective $\alpha - \beta$ equivariant homomorphism. Suppose that the crossed-product homomorphism*

$$\phi \rtimes G : A \rtimes_\alpha G \to M(B \rtimes_\beta G)$$

is faithful. If the full and reduced crossed products of β are equal, then the full and reduced crossed products of α are equal.

PROOF. We have a commutative diagram

$$
\begin{array}{ccc}
A \rtimes_\alpha G & \xrightarrow{\phi \rtimes G} & M(B \rtimes_\beta G) \\
\Lambda_\alpha \downarrow & & \downarrow \Lambda_\beta \\
A \rtimes_{\alpha,r} G & \xrightarrow{\phi \rtimes_r G} & M(B \rtimes_{\beta,r} G),
\end{array}
$$

and the composition $\Lambda_\beta \circ (\phi \rtimes G)$ is faithful, and therefore Λ_α is faithful. $\square$

Next, "full = reduced" is preserved by extensions:

LEMMA 6.4. *Let (A, α) be an action, and let J be a closed invariant ideal of A. If the actions of G on J and on A/J both have full and reduced crossed products equal, then the full and reduced crossed products of α are equal.*

PROOF. Let $\phi : J \hookrightarrow A$ be the inclusion map, and let $\psi : A \to A/J$ be the quotient map. We have a commutative diagram

$$
\begin{array}{ccccc}
J \rtimes G & \xrightarrow{\phi \rtimes G} & A \rtimes G & \xrightarrow{\psi \rtimes G} & A/J \rtimes G \\
\Lambda_J \downarrow & & \Lambda_A \downarrow & & \downarrow \Lambda_{A/J} \\
J \rtimes_r G & \xrightarrow[\phi \rtimes_r G]{} & A \rtimes_r G & \xrightarrow[\psi \rtimes_r G]{} & A/J \rtimes_r G.
\end{array}
$$

The argument is a routine diagram-chase. The vertical maps are the regular representations, which are surjective, and moreover Λ_J and $\Lambda_{A/J}$ are injective by hypothesis. Since J is an ideal, the map $\phi \rtimes G$ is an isomorphism onto the kernel of $\psi \rtimes G$ [**Gre78**, Proposition 12]. Further, since J is an invariant subalgebra, $\phi \rtimes_r G$ is injective. Let x be in the kernel of Λ_A. Then

$$
0 = (\psi \rtimes_r G) \circ \Lambda_A(x) = \Lambda_{A/J} \circ (\psi \rtimes G)(x),
$$

so x is in the kernel of $\psi \rtimes G$. Thus $x \in J \rtimes G$, and

$$
0 = \Lambda_A \circ (\phi \rtimes G)(x) = (\phi \rtimes_r G) \circ \Lambda_J(x),
$$

so $x = 0$. $\qquad\square$

Next we show that "full = reduced" is preserved by direct sums:

LEMMA 6.5. *Let $\{(A_i, \alpha_i)\}_{i \in I}$ be a family of actions, and assume that the full and reduced crossed products are equal for every α_i. Then the direct sum action*

$$
\left(\bigoplus_{i \in I} A_i, \bigoplus_{i \in I} \alpha_i \right)
$$

also has full and reduced crossed products equal.

PROOF. By Lemma 6.4, the conclusion holds if I has cardinality 2, and by induction it holds if I is finite. By [**Gre78**, Proposition 12], we can regard $(\bigoplus_{i \in I} A_i) \rtimes G$ as the inductive limit of the ideals $(\bigoplus_{i \in F} A_i) \rtimes G$ for finite $F \subset I$. Similarly (but not requiring the reference to [**Gre78**]), we can regard $(\bigoplus_{i \in I} A_i) \rtimes_r G$ as the inductive limit of the ideals $(\bigoplus_{i \in F} A_i) \rtimes_r G$. For every finite $F \subset I$ we have a commutative diagram

$$
\begin{array}{ccc}
(\bigoplus_{i \in F} A_i) \rtimes G & \hookrightarrow & (\bigoplus_{i \in I} A_i) \rtimes G \\
\Lambda_F \downarrow \simeq & & \downarrow \Lambda_I \\
(\bigoplus_{i \in F} A_i) \rtimes_r G & \hookrightarrow & (\bigoplus_{i \in I} A_i) \rtimes_r G,
\end{array}
$$

where the vertical arrows are the regular representations. Thus Λ_I must be an isomorphism, by properties of inductive limits. $\qquad\square$

COROLLARY 6.6. *Let (A, α) be an action, let $\{(A_i, \alpha_i)\}_{i \in I}$ be a family of actions for which the full and reduced crossed products are equal, and for each i let $\phi_i : A \to M(A_i)$ be an $\alpha - \alpha_i$ equivariant homomorphism. Let*

$$\phi : A \to M\left(\bigoplus_{i \in I} A_i\right)$$

be the associated equivariant homomorphism. Suppose that $\bigcap_{i \in I} \ker \phi_i = \{0\}$, and that the crossed-product homomorphism

$$A \rtimes_\alpha G \to M\left(\left(\bigoplus_{i \in I} A_i\right) \rtimes_{\alpha_i} G\right)$$

is faithful. Then α also has full and reduced crossed products equal.

PROOF. This follows immediately from Lemmas 6.3 and 6.5.   □

We are almost ready for the proof of Theorem 6.2, but first we need to recall the notion of quasi-regularity, and we only need this in the special case of closed orbits:

DEFINITION 6.7 (special case of [**Gre78**, Page 221]). Let G act on X, and assume that all orbits are closed. Then the associated action of G on $C_0(X)$ is *quasi-regular* if for every irreducible covariant representation (π, U) of $(C_0(X), G)$ there is an orbit $G \cdot x$ such that

$$\ker \pi = \{f \in C_0(X) : f|_{G \cdot x} = 0\}.$$

In this case, π factors through a faithful representation ρ of $C_0(G \cdot x)$ such that the covariant pair (ρ, U) is an irreducible representation of the restricted action $(C_0(G \cdot x), \alpha)$. By [**Gre78**, Corollary 19], the action is quasi-regular if the orbit space $G \backslash X$ is second countable or *almost Hausdorff* in the sense that every closed subset contains a dense relative open Hausdorff subset. Here we will prove a variant of this result:

PROPOSITION 6.8. *If a G-space X is pointwise proper and first countable, then the associated action of G on $C_0(X)$ is quasi-regular.*

We first need a topological property of pointwise proper actions on first countable spaces:

LEMMA 6.9. *If a G-space X is pointwise proper and first countable, then each orbit is a countable decreasing intersection of open G-invariant sets.*

PROOF. Since orbits are closed, the quotient space $G \backslash X$ is T_1. Since the quotient map is continuous and open, $G \backslash X$ is first countable. In particular, every point is a countable decreasing intersection of open sets, and the result follows.   □

REMARK 6.10. In Lemma 6.9 the first countability assumption could be weakened to: every point in X is a G_δ.

It seems to us that the proof of Proposition 6.8 is clearer if we separate out a special case:

LEMMA 6.11. *If a G-space X is pointwise proper and first countable, and if there is an irreducible covariant representation (π, U) of $(C_0(X), G)$ such that π is faithful, then X consists of a single orbit.*

PROOF. We can extend π to a representation of the algebra of bounded Borel functions on X, and we let P be the associated spectral measure (see, e.g., [**Mur90**, Theorem 2.5.5] for a version of the relevant theorem in the nonsecond-countable case; Murphy states the theorem for compact Hausdorff spaces, but it applies equally well to locally compact spaces by passing to the one-point compactification). Since (π, U) is irreducible, for every G-invariant Borel set E we have $P(E) = 0$ or 1. In particular each orbit has spectral measure 0 or 1, and there can be at most one orbit with measure 1.

Claim: every nonempty G-invariant open subset O of X has spectral measure 1. It suffices to show that $P(O) \neq 0$. Since $O \neq \varnothing$, we can choose a nonzero $f \in C_0(X)$ supported in O. Then

$$0 \neq \pi(f) = \pi(f\chi_O) = \pi(f)P(O),$$

so $P(O) \neq 0$.

Let $x \in X$. We will show that $X = G \cdot x$. By Lemma 6.9 we can choose a decreasing sequence $\{O_n\}$ of open G-invariant sets with $\bigcap_1^\infty O_n = G \cdot x$. By the properties of spectral measures, we have

$$P(G \cdot x) = \lim_n P(O_n) = 1.$$

Thus every orbit has spectral measure 1, so there can be only one orbit. $\square$

PROOF OF PROPOSITION 6.8. Let (π, U) be an irreducible covariant representation of $(C_0(X), G)$ on a Hilbert space H. Then $\ker \pi$ is a G-invariant ideal of $C_0(X)$, so there is a closed G-invariant subset Y of X such that

$$\ker \pi = \{f \in C_0(X) : f|_Y = 0\}.$$

We will show that Y consists of a single orbit. The restriction map $f \mapsto f|_Y$ is a G-equivariant homomorphism of $C_0(X)$ to $C_0(Y)$, and $\ker \pi = C_0(X \setminus Y)$, so π factors through a faithful representation ρ of $C_0(Y)$ such that (ρ, U) is an irreducible covariant representation of $(C_0(Y), G)$. Then Y is a single orbit, by Lemma 6.11. $\square$

PROOF OF THEOREM 6.2. For each $x \in X$, the orbit $G \cdot x$ is closed, the isotropy subgroup G_x is compact, and the canonical bijection $G/G_x \to G \cdot x$ is an equivariant homeomorphism. Thus G_x is in particular amenable, so it follows from the above and [**QS92**, Corollary 4.3] (see also [**Kas88**, Theorem 3.15]) the associated action of G on $C_0(G \cdot x)$ has full and reduced crossed products equal. The restriction map $\phi_x : C_0(X) \to C_0(G \cdot x)$ is equivariant, and we get an equivariant injective homomorphism

$$\phi : C_0(X) \to M\left(\bigoplus_{x \in X} C_0(G \cdot x)\right).$$

By Proposition 6.8 the action of G on $C_0(X)$ is quasi-regular, so every irreducible covariant representation of $(C_0(X), G)$ factors through a representation of $(C_0(G \cdot x), G)$ for some orbit $G \cdot x$. It follows that the crossed-product homomorphism

$$\phi \rtimes G : C_0(X) \rtimes G \to M\left(\left(\bigoplus_{x \in X} C_0(G \cdot x)\right) \rtimes G\right)$$

is faithful. Therefore the theorem follows from Corollary 6.6. $\square$

The above strategy can also be used to prove the following folklore result, which is a mild extension of Phillips' full-equals-reduced theorem. Actually, we could not find the following result explicitly recorded in the literature, but it seems to us that it must have been noticed before.

PROPOSITION 6.12. *If a G-space X is locally proper then the associated action $(C_0(X), \alpha)$ has full and reduced crossed products equal.*

Note that there is no countability hypothesis on X.

We need the following, which will play a role similar to that of Corollary 6.6 in the pointwise proper case:

COROLLARY 6.13. *Let (A, α) be an action, and let $\{J_i\}_{i \in I}$ be a family of G-invariant ideals that densely span A. If for every i the restriction of the action to J_i has full and reduced crossed products equal, then the action on A has the same property.*

PROOF. For each i let $\alpha_i = \alpha|_{J_i}$, let $\phi_i : A \to M(J_i)$ be the $\alpha - \alpha_i$ equivariant homomorphism induced by the A-bimodule structure on J_i, and let $\phi : A \to M(\bigoplus_{i \in I} J_i)$ be the associated equivariant homomorphism, Since $A = \overline{\text{span}}_{i \in I} J_i$, we have $\bigcap_{i \in I} \ker \phi_i = \{0\}$. Thus, by Corollary 6.6 we only need to show that

$$\phi \rtimes G : A \rtimes_\alpha G \to M\left(\left(\bigoplus_{i \in I} J_i\right) \rtimes_{\alpha_i} G\right)$$

is faithful. Suppose that $\ker(\phi \rtimes G) \neq \{0\}$. The ideals $J_i \rtimes_{\alpha_i} G$ densely span $A \rtimes_\alpha G$, since the J_i's densely span A. Thus we can find $i \in J$ such that

$$\{0\} \neq \ker(\phi \rtimes G) \cap (J_i \rtimes_{\alpha_i} G) = \ker(\phi|_{J_i} \rtimes G).$$

But $\phi|_{J_i} \rtimes G$ is faithful since $\phi|_{J_i}$ is faithful and J_i is a G-invariant ideal, so we have a contradiction. $\square$

PROOF OF PROPOSITION 6.12. First, if the G-space X is actually proper, then $G\backslash X$ is Hausdorff, so by [**Gre78**, Corollary 19] the action of G on $C_0(X)$ is quasi-regular, so the conclusion follows as in the proof of Proposition 6.2. In the general case, X is a union of open G-invariant subsets U_i, on each of which G acts properly. Then $C_0(X)$ is densely spanned by the ideals $C_0(U_i)$, so by properness the associated actions α_i have full and reduced crossed products equal, and hence the conclusion follows from Lemma 6.13. $\square$

REMARK 6.14. In the above proof we appealed to [**Gre78**, Corollary 19], whose proof involved dense points in irreducible closed sets. In the spirit of the techniques of the current paper, we offer an alternative argument: assume that X is a proper G-space. To see that the action is quasi-regular, as in the proof of Proposition 6.8 we can assume without loss of generality that there is an irreducible covariant representation (π, U) of $(C_0(X), G)$ such that π is faithful. We must show that X consists of a single G-orbit. Suppose $G \cdot x$ and $G \cdot y$ are distinct orbits in X. By properness, the quotient space $G\backslash X$ is Hausdorff, so we can find disjoint open neighborhoods of $G \cdot x$ and $G \cdot y$ in $G\backslash X$, and hence nonempty disjoint open G-invariant sets U and V in X. But, as in the proof of Lemma 6.11, letting P denote the spectral measure associated to the representation π of $C_0(X)$, every nonempty G-invariant open subset O of X has $P(O) = 1$. Since we cannot have two disjoint open sets with spectral measure 1, we have a contradiction.

The above methods quickly lead to another property of the crossed product. Recall that a C^*-algebra is called *CCR*, or *liminal* [**Dix77**, Definition 4.2.1], if every irreducible representation is by compacts. In the second countable case, the following result is contained in [**Wil07**, Proposition 7.31].

PROPOSITION 6.15. *Let X be a G-space. In either of the following two situations, the crossed product $C_0(X) \rtimes G$ is CCR:*

(1) *the action of G is locally proper;*
(2) *the action is pointwise proper and X is first countable.*

PROOF. (1) If the G-space X is actually proper, then this is well-known. To illustrate how the above methods apply, we give the following argument. We have seen above that the action is quasi-regular, and hence for every irreducible covariant representation (π, U) of $(C_0(X), G)$ factors through an irreducible representation of the restriction of the action to $(C_0(G \cdot x), G)$ for some $x \in X$. The G-spaces $G \cdot x$ and G/G_x are isomorphic, and $C_0(G/G_x) \rtimes G$ is Morita equivalent to $C^*(G_x)$ by Rieffel's version of Mackey's Imprimitivity Theorem [**Rie74**, Section 7]. Since the isotropy subgroup G_x is compact, $C^*(G_x)$ is CCR, and hence the image of the integrated form $\rho \times U$, which equals the image of $\pi \times U$, is the algebra of compact operators.

In the general case, X is a union of open G-invariant proper G-spaces U_i, so $C_0(X) \rtimes G$ is the closed span of the CCR ideals $C_0(U_i) \rtimes G$. Since every C^*-algebra has a largest CCR ideal [**Dix77**, Proposition 4.2.6], $C_0(X) \rtimes G$ must be CCR.

(2) By Proposition 6.8 the action is quasi-regular, and it follows as in part (1) that $C_0(X) \rtimes G$ is CCR.   □

REMARK 6.16. As remarked in [**AD02**, Example 2.7 (3)], it follows from [**ADR00**, Corollary 2.1.17] that if an action of G on X is proper then the action is amenable (a condition involving approximation by positive-definite functions). By [**AD02**, Theorem 5.3], if a G-space X is amenable then the associated action α on $C_0(X)$ has full and reduced crossed products equal. This raises a question: is every pointwise proper action amenable? It seems that amenability of the G-space is closely related to equality of full and reduced crossed products: by [**Mat14**, Theorem 3.3], for an action of a discrete exact group G on a compact space X, if α has full and reduced crossed products equal then the action is amenable. Unfortunately, this is of no help for our question, because a noncompact group cannot act pointwise properly on a compact space.

7. Properness conditions for coactions

We will now dualize the properness properties of Definitions 4.1 and 4.2.

To motivate how this will go, we pause to recall some basic facts regarding C^*-tensor products, commutative C^*-algebras, and actions.

For locally compact Hausdorff spaces X, Y we have the standard identifications

$$C_0(X \times Y) = C_0(X) \otimes C_0(Y)$$

and

$$C_b(X) = M(C_0(X)).$$

For a C^*-algebra A we have

$$A \otimes C_0(G) = C_0(G, A)$$

and
$$M(A \otimes C_0(G)) = C_b(G, M^\beta(A)),$$
where $M^\beta(A)$ denotes the multiplier algebra $M(A)$ with the strict topology.

For an action (A, α) we have a homomorphism
$$\widetilde{\alpha} : A \to M(A \otimes C_0(G))$$
given by
$$\widetilde{\alpha}(f)(s, x) = \widetilde{\alpha}(f)(s)(x) = f(sx) = \alpha_{s^{-1}}(f)(x).$$
In fact, the image of $\widetilde{\alpha}$ lies in the C^*-subalgebra $\widetilde{M}(A \otimes C_0(G))$, where for any C^*-algebras A and D
$$\widetilde{M}(A \otimes D) := \{m \in M(A \otimes D) : m(1 \otimes D) \cup (1 \otimes D)m \subset A \otimes D\}.$$

Using the above facts, Corollary 4.4 can be restated as follows:

LEMMA 7.1. *An action (A, α) is s-proper if and only if*
$$\widetilde{\alpha}(A)(A \otimes 1_{M(C_0(G))}) \subset A \otimes C_0(G),$$
and is w-proper if and only if for all $\omega \in A^*$,
$$(\omega \otimes \mathrm{id}) \circ \widetilde{\alpha}(A) \subset C_0(G).$$

Now consider a coaction (A, δ) of G. The main difference from actions is that the commutative C^*-algebra $C_0(G)$ is replaced by $C^*(G)$. Here we will use the standard conventions for tensor products and coactions (see, e.g., [**EKQR06**, Appendix A], in particular, the coaction is a homomorphism
$$\delta : A \to \widetilde{M}(A \otimes C^*(G)).$$

DEFINITION 7.2. A coaction (A, δ) is *s-proper* if
$$\delta(A)\big(A \otimes 1_{M(C^*(G))}\big) \subset A \otimes C^*(G),$$
and is *w-proper* if for all $\omega \in A^*$ we have
$$(\omega \otimes \mathrm{id}) \circ \delta(A) \subset C^*(G).$$

REMARK 7.3. In [**KLQ**, Definition 5.1] we introduced the above properness conditions, but in that paper we used the term *proper coaction* for the above s-proper coaction, and *slice proper coaction* for the above w-proper coaction (because it involves the slice map $\omega \otimes \mathrm{id}$). After we submitted [**KLQ**], we learned that Ellwood had defined properness more generally for coactions of Hopf C^*-algebras [**Ell00**, Definition 2.4]. Indeed, Proposition 3.3 is essentially [**Ell00**, Theorem 2.9(b)]. Definition 7.2 should also be compared with Condition (A1) in [**GK03**, Section 4.1], which concerns discrete quantum groups and involves the algebraic tensor product.

REMARK 7.4. An action on $C_0(X)$ can be w-proper without being s-proper, and a fortiori a coaction can be w-proper without being s-proper, even for G abelian.

REMARK 7.5. (1) Just as every action of a compact group is s-proper, every coaction of a discrete group is s-proper, because then we in fact have $\delta(A) \subset A \otimes C^*(G)$.

(2) For any locally compact group G the canonical coaction δ_G on $C^*(G)$ given by the comultiplication is s-proper, because it is symmetric in the sense that
$$\delta_G = \Sigma \circ \delta_G,$$

where Σ is the flip automorphism on $C^*(G) \otimes C^*(G)$.

If (A, δ) is a coaction, then A gets a Banach module structure over the Fourier-Stieltjes algebra $B(G) = C^*(G)^*$ by

$$f \cdot a = (\mathrm{id} \otimes f) \circ \delta(a) \quad \text{for } f \in B(G), a \in A.$$

In [**KLQ**, Lemma 5.2] we proved the following dual analogue of Lemma 4.6

LEMMA 7.6. *A coaction* (A, δ) *is w-proper if and only if for all* $a \in A$ *the map* $f \mapsto f \cdot a$ *is weak*-to-weakly continuous.*

PROOF. See [**KLQ**, Lemma 5.2]. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

s-properness and w-properness are both preserved by morphisms. For w-properness this is proved in [**KLQ**, Proposition 5.3], and here it is for s-properness:

PROPOSITION 7.7. *Let* $\phi : A \to M(B)$ *be a nondegenerate homomorphism that is equivariant for coactions* δ *and* ε, *respectively. If* δ *is s-proper then* ε *has the same property.*

PROOF. We have

$$
\begin{aligned}
(B \otimes 1)\varepsilon(B) &= (B\phi(A) \otimes 1)(\phi \otimes \mathrm{id})(\delta(A))\varepsilon(B) \\
&= (B \otimes 1)(\phi(A) \otimes 1)(\phi \otimes \mathrm{id})(\delta(A))\varepsilon(B) \\
&= (B \otimes 1)(\phi \otimes \mathrm{id})\big((A \otimes 1)\delta(A)\big)\varepsilon(B) \\
&\subset (B \otimes 1)(\phi \otimes \mathrm{id})(A \otimes C^*(G))\varepsilon(B) \\
&= (B \otimes C^*(G))\varepsilon(B) \\
&\subset B \otimes C^*(G)).
\end{aligned}
$$
$\qquad\qquad\qquad\qquad\qquad\qquad\square$

COROLLARY 7.8. *Every dual coaction is s-proper.*

PROOF. If (A, α) is an action, then the canonical nondegenerate homomorphism $i_G : C^*(G) \to M(A \rtimes_\alpha G)$ is $\delta_G - \widehat{\alpha}$ equivariant, where δ_G is the canonical coaction on $C^*(G)$ given by the comultiplication. Thus $\widehat{\alpha}$ is s-proper since δ_G is. $\quad\square$

Recall that if (A, δ) is a coaction then the *spectral subspaces* $\{A_s\}_{s \in G}$ are given by

$$A_s = \{a \in M(A) : \delta(a) = a \otimes s\},$$

and the *fixed-point algebra* is $A^\delta = A_e$.

PROPOSITION 7.9. *Suppose* $A \cap A^\delta \neq \{0\}$. *Then the following are equivalent:*
(1) δ *is s-proper;*
(2) δ *is w-proper;*
(3) G *is discrete.*

PROOF. We know (1) implies (2) and (3) implies (1). Assume (2), and let $a_e \in A \cap A^\delta$ be nonzero. Then

$$f \mapsto f \cdot a_e = (\mathrm{id} \otimes f) \circ \delta(a_e) = (\mathrm{id} \otimes f)(a_e \otimes 1) = f(e)a_e$$

is weak*-weak continuous from $B(G)$ to A, so $f \mapsto f(e)$ is a weak* continuous linear functional on $B(G)$, which implies $e \in C^*(G)$, and hence G is discrete. $\quad\square$

REMARK 7.10. Of course, the above proposition applies if A is unital. Also note that when G is nondiscrete a coaction (A, δ) can be s-proper and still have nonzero spectral subspaces A_s (and hence nontrivial fixed-point algebra A^δ, but these will be subspaces in $M(A)$ that intersect A trivially.

For the next lemma, recall that if (A, δ) is a coaction, then a projection $p \in M(A)$ is called δ-invariant if $p \in A^\delta$, and in this case δ restricts to a coaction δ_p on the corner pAp:

$$\delta_p(pap) = (p \otimes 1)\delta(a)(p \otimes 1) \in M(pAp \otimes C^*(G)) \quad \text{for } a \in A.$$

LEMMA 7.11. *Let (A, δ) be a coaction, and let p be a δ-invariant projection in $M(A)$. If (A, δ) is s-proper, then so is the corner coaction (pAp, δ_p) defined above.*

PROOF. This is a routine computation:

$$\begin{aligned}
\delta_p(pAp)(pAp \otimes 1) &\subset (p \otimes 1)\delta(A)(A \otimes 1)(p \otimes 1) \\
&\subset (p \otimes 1)(A \otimes C^*(G))(p \otimes 1) \\
&= pAp \otimes C^*(G).
\end{aligned}$$
$\square$

For the definitions of normalization and maximalization, we refer to [**EKQR06**, Appendix A.7] and [**EKQ04**]. Normalizations and maximalizations always exist, and are unique up to equivariant isomorphism.

PROPOSITION 7.12. *For any coaction (A, δ), the following are equivalent:*
1. *(A, δ) is s-proper;*
2. *The normalization (A^n, δ^n) is s-proper;*
3. *The maximalization (A^m, δ^m) is s-proper.*

PROOF. It follows from Proposition 4.8 that (1) implies (2) and (3) implies (1), and a careful examination of the construction of the maximalization in [**EKQ04**] (particularly Lemma 3.6 and the proof of Theorem 3.3 in that paper) shows that (2) implies (3). $\square$

REMARK 7.13. In case the above proof seems overly fussy, note that it would not be enough to observe that the double-dual coaction $\widehat{\widehat{\delta}}$ is automatically s-proper and the maximalization δ^m is Morita equivalent to $\widehat{\widehat{\delta}}$, because s-properness is *not* preserved by Morita equivalence — otherwise every coaction of an amenable group would be s-proper!

Recall from [**KMQW10**, Proposition 3.1] that if $\mathcal{A} \to G$ is a Fell bundle then there is a coaction $\delta_\mathcal{A}$ of G on the (full) bundle algebra $C^*(\mathcal{A})$.

PROPOSITION 7.14. *Let $\mathcal{A} \to G$ be a Fell bundle. Then the coaction $(C^*(\mathcal{A}), \delta_\mathcal{A})$ is s-proper.*

PROOF. We must show that for all $a, b \in C^*(\mathcal{A})$ we have $\delta(a)(b \otimes 1) \in C^*(\mathcal{A}) \otimes C^*(G)$, and by density and nondegeneracy it suffices to take $a \in \Gamma_c(\mathcal{A})$ and b of the form $f \cdot b$ for $f \in A(G) \cap C_c(G)$:

$$\begin{aligned}
\delta(a)(f \cdot b \otimes 1) &= \int_G \left(a(t)f \cdot b \otimes t\right) dt \\
&= \int_G \left(a(t)b \otimes tf\right) dt \quad \text{(justified below)} \\
&\in C^*(\mathcal{A}) \otimes C^*(G),
\end{aligned}$$

because the integrand

$$t \mapsto a(t)b \otimes tf$$

is in $C_c(G, C^*(\mathcal{A}) \otimes C^*(G))$. In the above computation we used the equality

$$a(t)f \cdot b \otimes t = a(t)b \otimes tf \quad \text{for all } t \in G,$$

which we justify as follows: computing inside $M(C^*(\mathcal{A}) \otimes C^*(G))$, we have

$$
\begin{aligned}
a(t)f \cdot b \otimes t &= \big(a(t) \otimes t\big)\big(f \cdot b \otimes 1\big) \\
&= \big(a(t) \otimes t\big)\big(b \otimes f\big) \quad \text{(justified below)} \\
&= a(t)b \otimes tf,
\end{aligned}
$$

where we must now justify the equality $f \cdot b \otimes 1 = b \otimes f$: both sides can be regarded as compactly supported strictly continuous functions from G to $M(C^*(\mathcal{A}) \otimes C^*(G))$, and for all $s \in G$ we have

$$
\begin{aligned}
(f \cdot b \otimes 1)(s) &= (f \cdot b)(s) \otimes 1 \\
&= f(s)b(s) \otimes 1 \\
&= b(s) \otimes f(s) \quad \text{(since } f(s) \in \mathbb{C}\text{)} \\
&= (b \otimes f)(s). \qquad\qquad\qquad \square
\end{aligned}
$$

REMARK 7.15. Let $\mathcal{A}$ be a Fell bundle over G, and let

$$\delta_{\mathcal{A}}^r = (\mathrm{id} \otimes \lambda) \circ \delta_{\mathcal{A}} : C^*(\mathcal{A}) \to M(C^*(\mathcal{A}) \otimes C_r^*(G))$$

be the reduction of the coaction $\delta_{\mathcal{A}}$. [**Bus10**, Theorem 3.10] shows that $\delta_{\mathcal{A}}^r$ is *integrable* in the sense that the set of positive elements a in A for which $\delta_{\mathcal{A}}^r(a)$ is in the domain of the operator-valued weight $\mathrm{id} \otimes \varphi$ is dense in A^+, where φ is the Plancherel weight on $C_r^*(G)$.

Corollary 7.16 below is a dual analogue of Corollary 5.1 (1). To explain the terminology, we recall a few things from Buss' thesis [**Bus07**]. Buss worked with reduced coactions, but as he points out in [**Bus07**, Remark 2.6.1 (4)], the theory carries over to full coactions by considering the reductions of the coactions. Throughout, (A, δ) is a coaction of G.

Let φ be the Plancherel weight on $C^*(G)$, let $\mathcal{M}_\varphi^+ = \{c \in C^*(G)^+ : \varphi(c) < \infty\}$, $\mathcal{N}_\varphi = \{c \in C^*(G) : c^*c \in \mathcal{M}_\varphi\}$, and $\mathcal{M}_\varphi = \mathrm{span}\, \mathcal{M}_\varphi^+$, so that $\mathcal{M}_\varphi^+$ is a hereditary cone in $C^*(G)$, and coincides with both $\mathcal{M}_\varphi \cap C^*(G)^+$ and $\mathrm{span}\, \mathcal{N}_\varphi^* \mathcal{N}_\varphi$, and φ extends uniquely to a linear functional on $\mathcal{M}_\varphi$.

Let $\mathrm{id} \otimes \varphi$ denote the associated $M(A)$-valued weight on $A \otimes C^*(G)$, with associated objects $\mathcal{M}_{\mathrm{id} \otimes \varphi}^+$, $\mathcal{N}_{\mathrm{id} \otimes \varphi}$, and $\mathcal{M}_{\mathrm{id} \otimes \varphi}$, and characterized as follows: for $x \in (A \otimes C^*(G))^+$ we have $x \in \mathcal{M}_{\mathrm{id} \otimes \varphi}^+$ if and only if there exists $a \in M(A)^+$ such that

$$\theta(a) = (\mathrm{id} \otimes \varphi)\big((\theta \otimes \mathrm{id})(x)\big) \quad \text{for all } \theta \in A^{*+},$$

in which case $(\mathrm{id} \otimes \varphi)(x) = a$. We have $(\mathrm{id} \otimes \varphi)(a \otimes c) = \varphi(c)a$ for all $a \in A$ and $c \in \mathcal{M}_\varphi$.

Let $\Lambda : \mathcal{N}_\varphi \to L^2(G)$ be the canonical embedding associated to the GNS construction for φ, so that $\Lambda(bc) = \lambda(b)\Lambda(c)$ for all $b \in C^*(G)$ and $c \in \mathcal{N}_\varphi$.

Let $\mathrm{id} \otimes \Lambda : \mathcal{N}_{\mathrm{id} \otimes \varphi} \to M(A \otimes L^2(G)) = \mathcal{L}(A, A \otimes L^2(G))$ be the map associated to the KSGNS construction for $\mathrm{id} \otimes \varphi$, characterized by

$$(\mathrm{id} \otimes \Lambda)(x)^*\big(a \otimes \Lambda(c)\big) = (\mathrm{id} \otimes \varphi)\big(x^*(a \otimes c)\big)$$

for all $x \in \mathcal{N}_{\mathrm{id}\otimes\varphi}$, $a \in A$, and $c \in \mathcal{N}_\varphi$. We have $(\mathrm{id}\otimes\Lambda)(a\otimes c) = a\otimes\Lambda(c)$ for all $a \in A$ and $c \in \mathcal{N}_\varphi$, and

$$(\mathrm{id}\otimes\Lambda)(xy) = (\mathrm{id}\otimes\lambda)(x)(\mathrm{id}\otimes\Lambda)(y)$$

for all $x \in M(A\otimes C^*(G))$ and $y \in \mathcal{N}_{\mathrm{id}\otimes\Lambda}$.

The weight φ extends canonically to $M(C^*(G))$, and the associated objects are denoted by $\overline{\mathcal{M}}_\varphi^+$, $\overline{\mathcal{N}}_\varphi$, and $\overline{\mathcal{M}}_\varphi$. Similarly for the canonical extension of $\mathrm{id}\otimes\varphi$ to $M(A\otimes C^*(G))$, $\overline{\mathcal{M}}_{\mathrm{id}\otimes\varphi}^+$, etc.

Let

$$A_{\mathrm{si}} = \{a \in A : \delta(aa^*) \in \overline{\mathcal{M}}_{\mathrm{id}\otimes\varphi}^+\}.$$

Then the coaction δ is *square-integrable* if A_{si} is dense in A. For $a \in A_{\mathrm{si}}$ define

$$\langle\!\langle a| \in M(A\otimes L^2(G)) = \mathcal{L}(A, A\otimes L^2(G))$$

by

$$\langle\!\langle a|(b) = (\mathrm{id}\otimes\Lambda)\big(\delta(a)^*(b\otimes 1)\big),$$

then define $|a\rangle\!\rangle = \langle\!\langle a|^* \in \mathcal{L}(A\otimes L^2(G), A)$, and for $a, b \in A_{\mathrm{si}}$ define

$$\langle\!\langle a|b\rangle\!\rangle = \langle\!\langle a| \circ |b\rangle\!\rangle \in \mathcal{L}(A\otimes L^2(G)).$$

Then (A, δ) is *continuously square-integrable* if there is a dense subspace $\mathcal{R} \subset A_{\mathrm{si}}$ such that

$$\langle\!\langle a|b\rangle\!\rangle \in A \rtimes_\delta G \subset \mathcal{L}(A\otimes L^2(G)) \quad \text{for all } a, b \in A_{\mathrm{si}}.$$

COROLLARY 7.16. *Every continuously square-integrable coaction is s-proper.*

PROOF. Let (A, δ) be a continuously square-integrable coaction. [**Bus07**, Section 6.8 and Proposition 6.9.4] gives a Fell bundle $\mathcal{A}$ over G and a $\delta_{\mathcal{A}} - \delta$ equivariant surjective homomorphism $\kappa : C^*(\mathcal{A}) \to A$. By Proposition 4.8, every quotient of an s-proper coaction is s-proper, so the corollary follows from Proposition 7.14. □

8. E-crossed products

To every action (B, α) one can associate the full crossed product $B \rtimes_\alpha G$ and the reduced crossed product $B \rtimes_{\alpha,r} G$. But there are frequently many "exotic" crossed products in between, i.e., quotients $(B \rtimes_\alpha G)/J$ where J is a nonzero ideal properly contained in the kernel of the regular representation Λ. In [**KLQ13**], inspired by work of Brown and Guentner [**BG12**], we introduced a tool that produces many (but not all) of these exotica. Our strategy is to base everything on "interesting" C^*-algebras $C^*(G)/I$ between $C^*(G)$ and $C_r^*(G)$. We call a closed ideal I of $C^*(G)$ *small* if it is contained in the kernel of the regular representation λ and is δ_G-*invariant*, i.e., the coaction δ_G descends to a coaction on $C^*(G)/I$. In [**KLQ13**, Corollary 3.13] we proved that I is small if and only if the annihilator $E = I^\perp$ in $B(G)$ is an ideal, which will then be *large* in the sense that it is nonzero, weak* closed, and G-invariant, where $B(G)$ is given the G-bimodule structure

$$(s \cdot f \cdot t)(u) = f(tus) \quad \text{for } f \in B(G), s, t, u \in G.$$

Large ideals automatically contain the reduced Fourier-Stieltjes algebra $B_r(G) = C_r^*(G)^*$ [**KLQ13**, Lemma 3.14], and the map $E \mapsto {}^\perp E$ gives a bijection between the large ideals of $B(G)$ and the small ideals I of $C^*(G)$. For a large ideal E the quotient map

$$q_E : C^*(G) \to C_E^*(G) := C^*(G)/{}^\perp E$$

is equivariant for δ_G and a coaction δ_G^E.

EXAMPLE 8.1. $E = \overline{B(G) \cap C_0(G)}$ is a large ideal, and if G is discrete then G has the Haagerup property if and only if $E = B(G)$ [**BG12**, Corollary 3.5].

EXAMPLE 8.2. For $1 \le p \le \infty$, $E^p := \overline{B(G) \cap L^p(G)}$ is a large ideal. Of course $E^\infty = B(G)$. For $p \le 2$ we have $E^p = B_r(G)$ [**KLQ13**, Proposition 4.2] (and [**BG12**, Proposition 2.11] for discrete G). If $G = \mathbb{F}_n$ for $n > 1$, it has been attributed to Okayasu [**Oka**] and (independently) to Higson and Ozawa (see [**BG12**, Remark 4.5]) that for $2 \le p < \infty$ the ideals E^p are all distinct.

Given an action (B, α), we use large ideals to produce exotic crossed products by involving the dual coaction $\widehat{\alpha}$ on $B \rtimes_\alpha G$. As in [**KLQ**], the process is most cleanly expressed in terms of an abstract coaction (A, δ). An ideal J of A is called δ-*invariant* if δ descends to a coaction on the quotient A/J. We call an ideal J *small* if it is invariant and contained in the kernel of j_A, where (j_A, j_G) is the canonical covariant homomorphism of $(A, C_0(G))$ into the multiplier algebra of the crossed product $A \rtimes_\delta G$. For the coaction $(C^*(G), \delta_G)$, this is consistent with the above notion of small ideals of $C^*(G)$.

Recall that A gets a $B(G)$-module structure by

$$f \cdot a = (\mathrm{id} \otimes f) \circ \delta(a) \quad \text{for } f \in B(G), a \in A.$$

For any large ideal E of $B(G)$,

$$\mathcal{J}(E) = \{a \in A : f \cdot a = 0 \text{ for all } f \in E\}$$

is a small ideal of A [**KLQ**, Observation 3.10]. For a dual coaction $(B \rtimes_\alpha G, \widehat{\alpha})$, we call the quotient

$$B \rtimes_{\alpha, E} G := (B \rtimes_\alpha G)/\mathcal{J}(E)$$

an E-*crossed product*.

In the other direction, for any small ideal J of A,

$$\mathcal{E}(J) = \{f \in B(G) : (s \cdot f \cdot t) \cdot a = 0 \text{ for all } a \in J, s, t \in G\}$$

is an ideal of $B(G)$, which is G-invariant by construction, and which will be weak*-closed if the coaction is w-proper. The following is [**KLQ**, Lemma 6.4]:

LEMMA 8.3. *For any w-proper coaction* (A, δ), *the above maps* $\mathcal{J}$ *and* $\mathcal{E}$ *form a Galois correspondence between the large ideals of* $B(G)$ *and the small ideals of* A.

By *Galois correspondence* we mean that $\mathcal{J}$ and $\mathcal{E}$ reverse inclusions, $E \subset \mathcal{E}(\mathcal{J}(E))$ for every large ideal E of $B(G)$, and $J \subset \mathcal{J}(\mathcal{E}(J))$ for every small ideal J of A.

Since every dual coaction is s-proper, and hence w-proper, Lemma 8.3 is applicable to $(B \rtimes_\alpha G, \widehat{\alpha})$ for any action (B, α). In [**KLQ**, Theorem 6.10] we used this Galois correspondence to exhibit examples of small ideals J that are not of the form $\mathcal{J}(E)$ for any large ideal E. Buss and Echterhoff [**BE13**, Example 5.3] have given examples that are better in the sense that the coaction (A, δ) is of the form $(B \rtimes_\alpha G, \widehat{\alpha})$. Consequently, there are exotic crossed products that are not E-crossed products for any large ideal E.

However, the real goal is not to look at exotic crossed products one at a time, but rather all at once: In [**BGW**], Baum, Guentner, and Willett define a *crossed-product* as a functor $(B, \alpha) \mapsto B \rtimes_{\alpha, \tau} G$, from the category of actions to the category

of C^*-algebras, equipped with natural transformations

$$B \rtimes_\alpha G \longrightarrow B \rtimes_{\alpha,\tau} G$$
$$B \rtimes_{\alpha,r} G,$$

where the vertical arrow is the regular representation, such that the horizontal arrow is surjective.

For a large ideal E of $B(G)$, the E-crossed product

$$(B, \alpha) \mapsto B \rtimes_{\alpha,E} G$$

gives a crossed-product functor in the sense of [**BGW**].

[**BGW**] defines a crossed-product functor τ to be *exact* if for every short exact sequence

$$0 \to (B_1, \alpha_1) \to (B_2, \alpha_2) \to (B_3, \alpha_3) \to 0$$

of actions the corresponding sequence of C^*-algebras

$$0 \to B_1 \rtimes_{\alpha_1,\tau} G \to B_2 \rtimes_{\alpha_2,\tau} G \to B_3 \rtimes_{\alpha_3,\tau} G \to 0$$

is exact, and *Morita compatible* if for every action (B, α) the canonical *untwisting isomorphism*

$$(B \otimes \mathcal{K}_G) \rtimes G \simeq (A \rtimes G) \otimes \mathcal{K}_G,$$

where $\mathcal{K}_G$ denotes the compact operators on $\bigoplus_{n=1}^\infty L^2(G)$, descends to an isomorphism

$$(B \otimes \mathcal{K}_G) \rtimes_\tau G \simeq (A \rtimes_\tau G) \otimes \mathcal{K}_G$$

of τ-crossed products. [**BGW**, Theorem 3.8] (with an assist from Kirchberg) shows that there is a unique minimal exact and Morita compatible crossed product, and [**BGW**] uses this to give a promising reformulation of the Baum-Connes conjecture.

If E is any large ideal of $B(G)$, the E-crossed product

$$(B, \alpha) \mapsto B \rtimes_{\alpha,E} G$$

is a crossed-product functor in the sense of [**BGW**], and it is automatically Morita compatible [**BGW**, Lemma A.5].

It is an open problem whether the minimal functor of [**BGW**] is an E-crossed product for some large ideal E. The counterexamples of [**BE13**] do not necessarily give a negative answer, because it is unknown whether they fit into a crossed-product functor. The state of the art regarding E-crossed products is depressingly meager at this early stage — we do not even know any examples other than $B(G)$ itself of large ideals E for which the E-crossed-product functor is exact for all G! Of course, by definition the $B_r(G)$-crossed product is exact for an *exact* group G (where $B_r(G) = C_r^*(G)^*$ denotes the reduced Fourier-Stieltjes algebra). But nonexact groups are quite mysterious.

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School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona 85287

E-mail address: kaliszewski@asu.edu

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway

E-mail address: magnusla@math.ntnu.no

School of Mathematical and Statistical Sciences, Arizona State University, Tempe, Arizona 85287

E-mail address: quigg@asu.edu