Disturbance Attenuation of $n + 1$ Coupled Hyperbolic PDEs

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***Abstract*— In this paper, we present a state-feedback and a state-observer for disturbance attenuation problems for a class of $n + 1$ coupled linear hyperbolic partial differential equations. The disturbance and the sensing are located at the left boundary of the system while the actuation is located at the right boundary of the system (anti-collocated setup). The designs are based on the backstepping method and rely on boundary measurement only. The feedback control law is found by utilizing the fact that the closed-form solution of the equivalent target system can be obtained. Furthermore, by defining a modified $\mathbb{L}_2$-norm, we show the observer is exponentially stable. A numerical example inspired from an oil well drilling problem is presented to validate the results.**

I. INTRODUCTION

Control design methods for linear and nonlinear distributed parameter systems which are based directly on the PDEs formulation has been established, e.g., [1], [2]. Those approaches can be used as alternative to an early lumping approach where the infinite dimensional systems in PDEs is replaced by a finite dimensional approximation in ODEs by employing finite difference, proper orthogonal decomposition, or using steady state model. Based on where the actuators and sensors are located, the control methods for PDEs can be divided into in-domain control and boundary control. Boundary control is considered to be physically more realistic because actuation and sensing are usually located at the boundary of the systems.

A systematic method for boundary control of PDEs was developed in [3]. The method is called backstepping and is primarily used for nonlinear ODE systems in strict-feedback form. Thus, the method represents a major shift from finite-dimensional to infinite-dimensional systems. The backstepping method has been successfully used as a tool for control design and state estimation of many type of linear and nonlinear PDEs with Volterra nonlinearities [4]. The method uses change of variable by shifting the system state using a Volterra operator. A property of Volterra operator is that the state transformation is triangular which ensures the invertibility of the change of the variable. Furthermore, using method of successive approximation [5] or Marcum Q-functions [6] one may found an explicit expression for the transformation kernel. Thus, the feedback law can be constructed explicitly and the closed-loop solutions can be found in closed-form.

The dual methods for observer design using boundary sensing has been also developed in a similar way to the boundary control [7]. The observer gain can be easily computed numerically and/or, for some cases, analytically. They can be designed in such a way that the observer error system is exponentially stabilized. Furthermore, using separation principle, the exponentially convergent observer can be combined with the backstepping controller to solve the output-feedback problem.

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A. Problem Description

In this paper, we consider the following disturbance attenuation problem for a class of first-order linear hyperbolic systems with $n + 1$ states[1]

$$u_t^i + \lambda_i(x) u_x^i = \sum_{j=1}^{n} \sigma_{i,j}(x) u^j + \omega_i(x) v \tag{1}$$

$$v_t - \mu(x) v_x = \sum_{j=1}^{n} \theta_j(x) u^j \tag{2}$$

$$u^i(0,t) = q_i v(0,t) + C_i X(t) \tag{3}$$

$$v(1,t) = \sum_{j=1}^{n} \rho_j u^j(1,t) + U(t) \tag{4}$$

$$\dot{X}(t) = AX(t) \tag{5}$$

where $x \in [0,1]$, $t \geq 0$, and $q_i \neq 0$, $\forall i = 1, \cdots, n$. X and $C_i^\intercal$ are m-dimensional vectors, and A is an $m \times m$ matrix.

Assumption 1: The transport velocities satisfy the following inequalities

$$-\mu(x) < 0 < \lambda_1(x) < \cdots < \lambda_n(x) \tag{6}$$

which indicates that the u^i states evolve from the left to the right, whereas the v state evolves from the right to the left.

Assumption 2: A is an arbitrary matrix. Furthermore, (A, C_i), $\forall i = 1, \cdots, n$, are observable.

Assumption 3: $\mu, \lambda_i \in C^1([0,1])$, and $\sigma_{i,j}, \omega_i, \theta_i \in C([0,1])$, $\forall i, j \in \{1, \cdots, n\}$.

The objective of this paper is to design a state-feedback control law $U(t)$ at $x = 1$ and a state-observer using only one measurement at $x = 0$ based on the backstepping method.

B. Motivation

Boundary control of hyperbolic systems has attracted a lot of attention lately (e.g., [8], [9]) because they can be used to model many physical processes especially fluid flow either in an open channel [10] or inside the pipeline [11]. In recent papers, (1)-(4) is used to model one-phase [12] and two-phase [13] fluid flow inside the drill pipe in an oil well drilling. Indeed, the present paper is motivated from a practical problem in oil well drilling. A particularly

[1]For the sake of brevity, $u = u(x,t)$ and $v = v(x,t)$.

interesting problem is when drilling from a floating rig. The drill string will move with the heaving motion of the floating rig causing a major pressure variations over the drill bit [14]. If the pressure variation is greater than the acceptable range, it can damage the formation thus reducing productivity. Therefore, there is an incentive to compensate the heave motion (disturbance attenuation) using a control system.

C. Previous Works

The aim of disturbance attenuation is to design a boundary control input $U(t)$ at $x = 1$ such that

$$\sum_{i=1}^{n} u^i(0,t) = rv(0,t) \tag{7}$$

with $r \neq \sum_{i=1}^{n} q_i$, is achieved at least asymptotically as time approaches infinity. For $n = 1$, this problem has been solved in [15]. It was shown that the disturbance attenuation is achieved if the estimates that converge to the actual values can be obtained. To this end, an exponentially convergent collocated state-observer based on the backstepping method was successfully designed. For $n \geq 2$, however, it was shown in [16] that the collocated state-observer is difficult to be obtained even when considering measurements of all states at the boundary due to the impossibility to add lower triangular integral coupling terms in the target system. Therefore, in this paper we develop an anti-collocated state-observer where only one measurement is taken at one boundary while the actuator is located at the other end.

D. Contribution of the Paper

The contributions of this paper are divided into two parts. The first contribution is to design a controller for (1)-(5) by following the step in [15] (minor contribution). The major contribution is to design an observer for (1)-(5) using an augmented Lyapunov function. To this end, we define the following modified $\mathbb{L}_2$-norm

$$\|f(t)\|_\delta^2 = \int_0^1 e^{\delta x} f(x,t)^\intercal f(x,t)\,\mathrm{d}x \tag{8}$$

for $\delta \in \mathbb{R}$ and $f : [0,1] \times \mathbb{R} \mapsto \mathbb{R}^m$, $\forall m \in \{1, \cdots, n\}$. We write the usual $\mathbb{L}_2$-norm as $\|f(t)\|_{\mathbb{L}_2} = \|f(t)\|_0$.

E. Organization of the Paper

The paper is organized as follows. In section II, we derived a full-state feedback control law for (1)-(5). In section III, we design an anti-collocated state-observer. A numerical example is presented in section IV. Finally, section V contains conclusions and recommendations.

II. FULL-STATE FEEDBACK CONTROL DESIGN

In this section, we develop a full-state feedback control law that can be used to attenuate the disturbance by utilizing the results of [15] and [16]. In [16], for $n \geq 2$ the following Volterra integral transformation of the second kind is used

$$\begin{aligned} \beta &= v - \int_0^x \sum_{i=1}^{n} K^i(x,\xi) u^i(\xi,t)\,\mathrm{d}\xi \\ &\quad - \int_0^x K^{n+1}(x,\xi) v(\xi,t)\,\mathrm{d}\xi \end{aligned} \tag{9}$$

$$\alpha^i = u^i \tag{10}$$

to transform (1)-(4) into an exponentially stable target system. The kernel functions $K^i = K^i(x,\xi)$, $\forall i \in \{1, \cdots, n+1\}$ have to satisfy the following system of first-order hyperbolic PDEs

$$\begin{aligned} \mu(x) K_x^j - \lambda_j(\xi) K_\xi^j &= \lambda_j'(\xi) K^j + \sum_{i=1}^{n} \sigma_{i,j}(\xi) K^i \\ &\quad + \theta_j(\xi) K^{n+1} \end{aligned} \tag{11}$$

$$\begin{aligned} \mu(x) K_x^{n+1} + \mu(\xi) K_\xi^{n+1} &= -\mu'(\xi) K^{n+1} \\ &\quad + \sum_{i=1}^{n} \omega_i(\xi) K^i \end{aligned} \tag{12}$$

$\forall j \in \{1, \cdots, n\}$, with boundary conditions

$$K^j(x,x) = -\frac{\theta_j(x)}{\lambda_j(x) + \mu(x)} \tag{13}$$

$$\mu(0) K^{n+1}(x,0) = \sum_{i=1}^{n} q \lambda_i(0) K^i(x,0) \tag{14}$$

$\forall j \in \{1, \cdots, n\}$, evolving on triangular domain $\Upsilon = \{(x,\xi) \in \mathbb{R}^2 | 0 \leq \xi \leq x \leq 1\}$. The existence of the transformation kernels can be shown by transforming the differential equations into integral forms using the method of characteristic and using successive approximation method to solve the well-posedness problem.

Remark 1: The transformation (9)

$$T : (\mathbb{L}^2([0,1],\mathbb{R}))^{n+1} \rightarrow \mathbb{L}^2([0,1],\mathbb{R}) \tag{15}$$

$$(u^i, v) \mapsto \beta \tag{16}$$

is invertible, i.e., there exists a unique continuous inverse kernel $L^i(x,\xi)$, $\forall i \in \{1, \cdots, n+1\}$ such that

$$\begin{aligned} v(x,t) &= \beta(x,t) + \int_0^x \sum_{i=1}^{n} L^i(x,\xi) \alpha^i(\xi,t)\,\mathrm{d}\xi \\ &\quad + \int_0^x L^{n+1}(x,\xi) \beta(\xi,t)\,\mathrm{d}\xi \end{aligned} \tag{17}$$

and for each $i \in \{1, \cdots, n\}$, the kernels satisfy

$$L^i(x,\xi) = K^i(x,\xi) + \int_\xi^x K^i(x,\xi) L^{n+1}(\xi,s)\,\mathrm{d}s \tag{18}$$

Remark 2: If $n = 1$, the explicit kernel functions can be obtained using Marcum Q-functions [6]. Hence, the feedback laws can be constructed explicitly and the closed-loop solutions can be found in closed-form.

The idea is to transform (1)-(2) into an equivalent system (Lemma 1) for which a closed-loop solution can be obtained (Lemma 2). The closed-form solution allows us to derive a stabilizing state feedback control law that attenuates the disturbance (Theorem 1). Lemmas and theorem presented in this section are obtained by generalized the results of disturbance attenuation in [15].

Lemma 1: Let $K^i,\ i = 1, \cdots, n+1$ be the solution to (11)-(14). If the control law is given by

$$\begin{aligned} U(t) &= V(t) - \sum_{j=1}^{n} \rho_j u^j(1,t) \\ &\quad + \int_0^1 \sum_{i=1}^{n} K^i(1,\xi) u^i(\xi,t)\,\mathrm{d}\xi \\ &\quad + \int_0^1 K^{n+1}(1,\xi) v(\xi,t)\,\mathrm{d}\xi \end{aligned} \tag{19}$$

then the transformations (9)-(10) and their inverses maps (1)-(5) into the following equivalent system

$$\begin{aligned} \alpha_t^i &= -\lambda_i(x)\alpha_x^i + \sum_{j=1}^{n} \sigma_{i,j}(x)\alpha^j + \omega_i(x)\beta \\ &\quad + \sum_{j=1}^{n} \int_0^x c_{i,j}(x,\xi)\alpha^j(\xi,t)\,\mathrm{d}\xi \\ &\quad + \int_0^x \kappa_i(x,\xi)\beta(\xi,t)\,\mathrm{d}\xi \end{aligned} \tag{20}$$

$$\beta_t = \mu(x)\beta_x - \sum_{i=1}^{n} \lambda_i(0) K^i(x,0) C_i X(t) \tag{21}$$

$$\alpha^i(0,t) = q_i \beta(0,t) + C_i X(t) \tag{22}$$

$$\beta(1,t) = V(t) \tag{23}$$

$$\dot{X} = AX \tag{24}$$

where

$$c_{i,j} = \omega_i(x) K^j + \int_\xi^x \kappa_i(x,s) K^j(s,\xi)\,\mathrm{d}s \tag{25}$$

$$\kappa_i = \omega_i(x) K^{n+1} + \int_\xi^x \kappa_i(x,s) K^{n+1}(s,\xi)\,\mathrm{d}s \tag{26}$$

$\forall i, j = 1, \cdots, n$.

Remark 3: The term $V(t)$ in the control law (19) is used to compensate the disturbance. Intuitively, $V(t)$ should depend on $X(t)$.

Remark 4: In the absence of disturbance, the transformed system (20)-(23) is exactly the same with the target system used in [16].

Remark 5: Equation (21) is a nonhomogeneous hyperbolic equation whose explicit solution can be obtained using the method of characteristic. By utilizing a semigroup property of (24), the explicit solution can be used to derive the following Lemma.

Lemma 2: For $t \geq d$ solution of (20)-(24) satisfy

$$\begin{aligned} \beta(0,t) - \beta(1,t-d) &= -\int_0^d \sum_{i=1}^{n} \lambda_i(0) K^i(1 - h^{-1}(\tau), 0) \\ &\quad \times C_i e^{-A\tau}\,\mathrm{d}\tau X(t) \end{aligned} \tag{27}$$

Theorem 1: Suppose $\sum_{i=1}^n q_i \neq 0$, $r \neq \sum_{i=1}^n q_i$, and

$$\begin{aligned} U(t) &= KX(t) - \sum_{j=1}^{n} \rho_j u^j(1,t) \\ &\quad + \int_0^1 \sum_{i=1}^{n} K^i(1,\xi) u^i(\xi,t)\,\mathrm{d}\xi \\ &\quad + \int_0^1 K^{n+1}(1,\xi) v(\xi,t)\,\mathrm{d}\xi \end{aligned} \tag{28}$$

where

$$\begin{aligned} K &= \frac{1}{r - \sum_{i=1}^n q_i} \sum_{i=1}^{n} C_i e^{Ad} \\ &\quad + \int_0^d \sum_{i=1}^{n} \lambda_i(0) K^i(1 - h^{-1}(\tau), 0) \\ &\quad \times C_i e^{A(d-\tau)}\,\mathrm{d}\tau \end{aligned} \tag{29}$$

Then $\sum_{i=1}^n u^i(0,t) = rv(0,t)$ for all $t \geq d$.

To evaluate the control law (28), all states need to be measured throughout the domain. Furthermore, the disturbance should also be measured. In the following section, we show how to obtain an observer using measurement only at the boundary. For time being, let us assume that estimates are available for u^i, v, and X and denote them $\hat{u}^i$, $\hat{v}$, and $\hat{X}$. Let $\tilde{u}^i = u - \hat{u}^i$, $\tilde{v} = v - \hat{v}$, and $\tilde{X} = X - \hat{X}$. If we replace, u^i, v, and X in (28) with their estimates, we obtain the following result.

Theorem 2: Suppose $\sum_{i=1}^n q_i \neq 0$, $r \neq \sum_{i=1}^n q_i$ and

$$\begin{aligned} U(t) &= K\hat{X}(t) - \sum_{j=1}^{n} \rho_j \hat{u}^j(1,t) \\ &\quad + \int_0^1 \sum_{i=1}^{n} K^i(1,\xi) \hat{u}^i(\xi,t)\,\mathrm{d}\xi \\ &\quad + \int_0^1 K^{n+1}(1,\xi) \hat{v}(\xi,t)\,\mathrm{d}\xi \end{aligned} \tag{30}$$

Then, there exists a constant c such that $|\sum_{i=1}^n u^i(0,t) - rv(0,t)| \leq c\|(\tilde{X}, \tilde{u}, \tilde{v})\|$ for all $t \geq d$, where

$$\begin{aligned} \|(\tilde{X}, \tilde{u}, \tilde{v})\|^2 &= \Bigg(|\tilde{X}(t)|^2 + \sum_{j=1}^{n} \tilde{u}^i(1,t)^2 \\ &\quad + \int_0^1 \left(\sum_{j=1}^{n} \tilde{u}^i(x,t)^2 + \tilde{v}(x,t)^2 \right) \mathrm{d}x \Bigg) \end{aligned} \tag{31}$$

Remark 6: The theorem stated that the disturbance attenuation $\sum_{i=1}^n u^i(0,t) = rv(0,t)$ with $r \neq \sum_{i=1}^n q_i > 0$, is only achieved if the estimates that converge to the actual value can be obtained.

III. ANTI-COLLOCATED OBSERVER DESIGN

To obtain measurements across the domain in distributed parameter systems are often difficult. The common setting is to place the sensors at the boundaries. For example, in oil well drilling, the measurement can be obtained either from the well head sensor (located at the top of the well) or from the drill bit sensor (located at the bottom of the well). Suppose the only available measurement is $v(0,t)$. The observer is designed as follow

$$\begin{aligned}
\hat{u}^i_t + \lambda_i(x)\hat{u}^i_x &= \sum_{j=1}^n \sigma_{i,j}(x)\hat{u}^j + \omega_i(x)\hat{v}(0,t) \\
&\quad + p_i(x)\tilde{v}(0,t) && (32)\\
\hat{v}_t - \mu(x)\hat{v}_x &= \sum_{j=1}^n \theta_j(x)\hat{u}^j + p_{n+1}(x)\tilde{v}(0,t) && (33)\\
\hat{u}^i(0,t) &= q_i v(0,t) + C\hat{X}(t) && (34)\\
\hat{v}(1,t) &= \sum_{j=1}^n \rho_j \hat{u}^j(1,t) + U(t) && (35)\\
\dot{\hat{X}}(t) &= A\hat{X}(t) - L\tilde{v}(0,t) && (36)
\end{aligned}$$

where the functions $p_i(x)$ and the vector $L \in \mathbb{R}^m$ are output injection gains to be determined. Subtracting (32)-(36) by (1)-(5), we get

$$\begin{aligned}
\tilde{u}^i_t + \lambda_i(x)\tilde{u}^i_x &= \sum_{j=1}^n \sigma_{i,j}(x)\tilde{u}^j + \omega_i(x)\tilde{v} \\
&\quad + p_i(x)\tilde{v}(0,t) && (37)\\
\tilde{v}_t - \mu(x)\tilde{v}_x &= \sum_{j=1}^n \theta_j(x)\tilde{u}^j + p_{n+1}(x)\tilde{v}(0,t) && (38)\\
\tilde{u}^i(0,t) &= C\tilde{X}(t) && (39)\\
\tilde{v}(1,t) &= \sum_{j=1}^n \rho_j \tilde{u}^j(1,t) && (40)\\
\dot{\tilde{X}}(t) &= A\tilde{X}(t) - L\tilde{v}(0,t) && (41)
\end{aligned}$$

In [16], the following transformations were used

$$\begin{aligned}
\tilde{u}^i &= \tilde{\alpha}^i + \int_0^x m^i(x,\xi)\tilde{\beta}(\xi,t)\,\mathrm{d}\xi && (42)\\
\tilde{v} &= \tilde{\beta} + \int_0^x m^{n+1}(x,\xi)\tilde{\beta}(\xi,t)\,\mathrm{d}\xi && (43)
\end{aligned}$$

where the kernels satisfy a first-order hyperbolic system

$$\begin{aligned}
\lambda_i(x)m^i_x - \mu(\xi)m^i_\xi &= \mu'(\xi)m^i + \sum_{j=1}^n \sigma_{i,j}(x)m^j \\
&\quad + \omega_i(x)m^{n+1} && (44)\\
\mu(x)m^{n+1}_x + \mu(\xi)m^{n+1}_\xi &= -\mu'(\xi)m^{n+1} \\
&\quad - \sum_{j=1}^n \theta_j(x)m^j && (45)
\end{aligned}$$

$\forall i \in \{1, \cdots, n\}$, with boundary conditions

$$\begin{aligned}
m^i(x,x) &= \frac{\omega_i(x)}{\lambda_i(x)+\mu(x)} && (46)\\
m^{n+1}(1,\xi) &= \sum_{j=1}^n \rho_j m^j(1,\xi) && (47)
\end{aligned}$$

$\forall i \in \{1, \cdots, n\}$.

The idea is to transform (37)-(41) into an equivalent system (Lemma 3) for which the (exponential) stability of the system is proved by using an augmented Lyapunov function (Lemma 4). Since the transformations (42)-(43) are also invertible, the error system (37)-(41) is exponentially stable. Hence, the estimates converge to the actual values.

Lemma 3: The transformation (42)-(43) maps the system

$$\begin{aligned}
\tilde{\alpha}^i_t + \lambda_i(x)\tilde{\alpha}^i_x &= \sum_{j=1}^n \sigma_{i,j}(x)\tilde{\alpha}^j \\
&\quad + \sum_{j=1}^n \int_0^x g_{i,j}(x,\xi)\tilde{\alpha}^j(\xi,t)\,\mathrm{d}\xi && (48)\\
\tilde{\beta}_t - \mu(x)\tilde{\beta}_x &= \sum_{j=1}^n \theta_j(x)\tilde{\alpha}^j \\
&\quad + \sum_{j=1}^n \int_0^x h_j(x,\xi)\tilde{\alpha}^j(\xi,t)\,\mathrm{d}\xi && (49)\\
\tilde{\alpha}^i(0,t) &= C\tilde{X}(t) && (50)\\
\tilde{\beta}(1,t) &= \sum_{j=1}^n \rho_j \tilde{\alpha}^j(1,t) && (51)\\
\dot{\tilde{X}} &= A\tilde{X} - L\tilde{\beta}(0,t) && (52)
\end{aligned}$$

into (37)-(41) with

$$\begin{aligned}
p_i(x) &= -\mu(0)m^i(x,0) && (53)\\
p_{n+1}(x) &= -\mu(0)m^{n+1}(x,0) && (54)
\end{aligned}$$

where the integral coupling coefficients are defined by the following equations

$$\begin{aligned}
g_{i,j} &= -\theta_j(\xi)m^i - \int_\xi^x m^i(x,s)h_j(s,\xi)\,\mathrm{d}s && (55)\\
h_i &= -\theta_i(\xi)m^{n+1} - \int_\xi^x m^{n+1}(x,s)h_i(s,\xi)\,\mathrm{d}s && (56)
\end{aligned}$$

Proof: Computing the first derivative of (42) with respect to t, integration by parts, and using (55), yields

$$\begin{aligned}
\tilde{u}^i_t(x,t) &= -\lambda_i(x)\tilde{\alpha}^i_x(x,t) + \sum_{j=1}^n \sigma_{i,j}(x)\tilde{\alpha}^j(x,t) \\
&\quad + \mu(x)m^i(x,x)\tilde{\beta}(x,t) - \mu(0)m^i(x,0)\tilde{\beta}(0,t) \\
&\quad - \int_0^x \mu(\xi)m^i_\xi(x,\xi)\tilde{\beta}(\xi,t)\,\mathrm{d}\xi \\
&\quad - \int_0^x \mu'(\xi)m^i(x,\xi)\tilde{\beta}(\xi,t)\,\mathrm{d}\xi && (57)
\end{aligned}$$

Computing the first derivation of (42) with respect to x and substituting into (57), yields

$$\begin{aligned}\tilde{u}_t^i(x,t) &= -\lambda_i(x)\tilde{u}_x^i(x,t) + \lambda_i(x)m^i(x,x)\tilde{\beta}(x,t)\\ &+\mu(x)m^i(x,x)\tilde{\beta}(x,t)\\ &+\sum_{j=1}^n \sigma_{i,j}(x)\tilde{\alpha}^j(x,t) - \mu(0)m^i(x,0)\tilde{\beta}(0,t)\\ &+\int_0^x \lambda_i(x)m_x^i(x,\xi)\tilde{\beta}(\xi,t)\,\mathrm{d}\xi\\ &-\int_0^x \mu(\xi)m_\xi^i(x,\xi)\tilde{\beta}(\xi,t)\,\mathrm{d}\xi\\ &-\int_0^x \mu'(\xi)m^i(x,\xi)\tilde{\beta}(\xi,t)\,\mathrm{d}\xi \qquad (58)\end{aligned}$$

Remark that

$$\begin{aligned}&\omega_i(x)\tilde{\beta}(x,t) - \omega_i(x)\tilde{\beta}(x,t) = \omega_i(x)\tilde{v}(x,t)\\ &-\omega_i(x)\tilde{\beta}(x,t) - \int_0^x \omega_i(x)m^{n+1}(x,\xi)\tilde{\beta}(\xi,t)\,\mathrm{d}\xi \qquad (59)\end{aligned}$$

while from (42), we have

$$\begin{aligned}\sum_{j=1}^n \sigma_{i,j}(x)\tilde{\alpha}^j(x,t) &= \sum_{j=1}^n \sigma_{i,j}(x)\tilde{u}^i(x,t) \qquad (60)\\ &-\int_0^x \sum_{j=1}^n \sigma_{i,j}(x)m^i(x,\xi)\tilde{\beta}(\xi,t)\,\mathrm{d}\xi\end{aligned}$$

Substituting (59) and (60) into (58), yield

$$\begin{aligned}\tilde{u}_t^i &= -\lambda_i(x)\tilde{u}_x^i + \omega_i(x)\tilde{v} + \sum_{j=1}^n \sigma_{i,j}(x)\tilde{u}^i\\ &+\left(\lambda_i(x)m^i(x,x) + \mu(x)m^i(x,x) - \omega_i(x)\right)\tilde{\beta}\\ &-\mu(0)m^i(x,0)\tilde{\beta}(0,t)\\ &-\int_0^x \sum_{j=1}^n \sigma_{i,j}(x)m^i(x,\xi)\tilde{\beta}(\xi,t)\,\mathrm{d}\xi\\ &-\int_o^x \omega_i(x)m^{n+1}(x,\xi)\tilde{\beta}(\xi,t)\,\mathrm{d}\xi\\ &+\int_0^x \lambda_i(x)m_x^i(x,\xi)\tilde{\beta}(\xi,t)\,\mathrm{d}\xi\\ &-\int_0^x \mu(\xi)m_\xi^i(x,\xi)\tilde{\beta}(\xi,t)\,\mathrm{d}\xi\\ &-\int_0^x \mu'(\xi)m^i(x,\xi)\tilde{\beta}(\xi,t)\,\mathrm{d}\xi \qquad (61)\end{aligned}$$

Substituting (44), (46), and (53) into (61), yield (37). (38) is obtained using the similar steps. ■

Lemma 4: If there exists a positive definite matrix P and an observer gain L such that

$$\begin{pmatrix} A^TP + PA + p\sum_{i=1}^n \lambda_i(0)C_i^TC_i & -PL\\ -L^TP & 0\end{pmatrix} \preceq 0 \qquad (62)$$

where p is a sufficiently large number, then the origin of (48)-(49) is exponentially stable.

Proof: We define a candidate Lyapunov function as $V(t) = V_1(t) + V_2(t)$, where

$$\begin{aligned}V_1(t) &= \tilde{X}^TP\tilde{X} \qquad (63)\\ V_2(t) &= \int_0^1 pe^{-\delta x}\sum_{i=1}^n \tilde{\alpha}^i(x,t)^2 + e^{\delta x}\tilde{\beta}(x,t)^2\,\mathrm{d}x \qquad (64)\end{aligned}$$

Computing its first derivative with respect to t, yield

$$\dot{V}_1(t) = \begin{pmatrix}\tilde{X}\\ \tilde{\beta}(0,t)\end{pmatrix}^T \begin{pmatrix} A^TP + PA & -PL\\ -L^TP & 0\end{pmatrix}\begin{pmatrix}\tilde{X}\\ \tilde{\beta}(0,t)\end{pmatrix} \qquad (65)$$

$$\begin{aligned}\dot{V}_2(t) &= p\sum_{i=1}^n \lambda_i(0)\tilde{\alpha}^i(0,t)^2 - pe^{-\delta}\sum_{i=1}^n \lambda_i(1)\tilde{\alpha}^i(1,t)^2\\ &-p\delta\int_0^1 e^{-\delta x}\sum_{i=1}^n \lambda_i(x)\tilde{\alpha}^i(x,t)^2\,\mathrm{d}x\\ &+p\int_0^1 e^{-\delta x}\sum_{i=1}^n \lambda_i(x)'\tilde{\alpha}^i(x,t)^2\,\mathrm{d}x\\ &+2p\int_0^1 e^{-\delta x}\sum_{i=1}^n \tilde{\alpha}^i(x,t)\Gamma_{\tilde{\alpha}}^i(\tilde{\alpha})\,\mathrm{d}x\\ &+e^\delta\mu(1)\tilde{\beta}(1,t)^2 - \mu(0)\tilde{\beta}(0,t)^2\\ &-\delta\int_0^1 e^{\delta x}\mu(x)\tilde{\beta}(x,t)^2\,\mathrm{d}x\\ &-\int_0^1 e^{\delta x}\mu(x)'\tilde{\beta}(x,t)^2\,\mathrm{d}x\\ &+2\int_0^1 e^{\delta x}\tilde{\beta}(x,t)\Gamma_{\tilde{\beta}}(\tilde{\alpha})\,\mathrm{d}x \qquad (66)\end{aligned}$$

where

$$\begin{aligned}\Gamma_{\tilde{\alpha}}^i(\tilde{\alpha}) &= \sum_{j=1}^n\left[\sigma_{i,j}(x)\tilde{\alpha}^j + \int_0^x g_{i,j}(x,\xi)\tilde{\alpha}^j(\xi,t)\,\mathrm{d}\xi\right] \qquad (67)\\ \Gamma_{\tilde{\beta}}(\tilde{\alpha}) &= \sum_{j=1}^n\left[\theta_j(x)\tilde{\alpha}^j + \int_0^x h_j(x,\xi)\tilde{\alpha}^j(\xi,t)\,\mathrm{d}\xi\right] \qquad (68)\end{aligned}$$

Using Cauchy-Schwartz's and Young's inequalities there exists M such that

$$\begin{aligned}&2p\left|\int_0^1 e^{-\delta x}\sum_{i=1}^n \tilde{\alpha}^i(x,t)\Gamma_{\tilde{\alpha}}^i(\tilde{\alpha})\,\mathrm{d}x\right| \le\\ &pM\int_0^1 e^{-\delta x}\sum_{i=1}^n \tilde{\alpha}^i(x,t)^2\,\mathrm{d}x \qquad (69)\\ &2\left|\int_0^1 e^{\delta x}\tilde{\beta}(x,t)\Gamma_{\tilde{\beta}}(\tilde{\alpha})\,\mathrm{d}x\right| \le \qquad (70)\\ &M\int_0^1 e^{\delta x}\tilde{\beta}(x,t)^2\,\mathrm{d}x + Me^{2\delta}\int_0^1 e^{-\delta x}\sum_{i=1}^n \tilde{\alpha}^i(x,t)^2\,\mathrm{d}x\\ &e^\delta\mu(1)\tilde{\beta}(1,t)^2 \le\\ &e^\delta\mu(1)n\max_j|\rho_j|^2\sum_{i=1}^n \tilde{\alpha}^i(1,t)^2 \qquad (71)\end{aligned}$$

Substituting these inequalities into (66), yields

$$\begin{aligned}\dot{V}_2(t) \leq\ & \tilde{X}^T\left(p\sum_{i=1}^{n}\lambda_i(0)C_i^TC_i\right)\tilde{X}\\ &-\left(pe^{-\delta}n\min_i\lambda_i(1)-e^{\delta}\mu(1)n\max_j|\rho_j|^2\right)\\ &\times\sum_{i=1}^{n}\tilde{\alpha}^i(1,t)\\ &-\left(p\delta\min_i\inf_{x\in[0,1]}\lambda_i(x)-pn\kappa_1-pM-Me^{2\delta}\right)\\ &\times\int_0^1 e^{-\delta x}\sum_{i=1}^{n}\tilde{\alpha}^i(x,t)^2\,\mathrm{d}x\\ &-\left(\delta\inf_{x\in[0,1]}\mu(x)+\kappa_2-M\right)\\ &\times\int_0^1 e^{\delta x}\tilde{\beta}(x,t)^2\,\mathrm{d}x \end{aligned} \tag{72}$$

where $\kappa_1 = \max_i\sup_{x\in[0,1]}\lambda_i(x)'$ and $\kappa_2 = \inf_{x\in[0,1]}\mu(x)'$. Therefore, we have

$$\begin{aligned}&\dot{V}(t)=\dot{V}_1(t)+\dot{V}_2(t)\\ \leq\ & \begin{pmatrix}\tilde{X}\\ \tilde{\beta}(0,t)\end{pmatrix}^T\\ &\times\begin{pmatrix}A^TP+PA+p\sum_{i=1}^{n}\lambda_i(0)C_i^TC_i & -PL\\ -L^TP & 0\end{pmatrix}\\ &\begin{pmatrix}\tilde{X}\\ \tilde{\beta}(0,t)\end{pmatrix}\\ &-\left(pe^{-\delta}n\min_i\lambda_i(1)-e^{\delta}\mu(1)n\max_j|\rho_j|^2\right)\\ &\times\sum_{i=1}^{n}\tilde{\alpha}^i(1,t)\\ &-\left(p\delta\min_i\inf_{x\in[0,1]}\lambda_i(x)-pn\kappa_1-pM-Me^{2\delta}\right)\\ &\times\int_0^1 e^{-\delta x}\sum_{i=1}^{n}\tilde{\alpha}^i(x,t)^2\,\mathrm{d}x\\ &-\left(\delta\inf_{x\in[0,1]}\mu(x)+\kappa_2-M\right)\int_0^1 e^{\delta x}\tilde{\beta}(x,t)^2\,\mathrm{d}x \end{aligned} \tag{73}$$

If we choose

$$p \geq \max\left(\frac{e^{\delta}\mu(1)\max_j|\rho_j|^2}{e^{-\delta}\min_i\lambda_i(1)},\ \frac{Me^{2\delta}}{\delta\min_i\inf_{x\in[0,1]}\lambda_i(x)-n\kappa_1-M}\right) \tag{74}$$

$$\delta \geq \frac{M-\kappa_2}{\inf_{x\in[0,1]}\mu(x)} \tag{75}$$

Then, there exists a positive constant ν such that $\dot{V} \leq -\nu V(t)$, which completes the proof. ■

Remark 7: Since (1)-(5) is a linear system, the separation principle holds; i.e., the combination of a separately designed stable state-feedback and stable observer results in a stabilizing output-feedback controller.

Remark 8: From the backstepping transformations (42)-(43) and their inverses [16], there exists positive constants a_1, a_2, a_3, b_1, b_2, and b_3, such that

$$\|\tilde{\alpha}(t)\|_{-\delta}^2 \leq a_1\|\tilde{u}(t)\|_{\mathbb{L}_2}^2+a_2\|\tilde{v}(t)\|_{\mathbb{L}_2}^2 \tag{76}$$

$$\|\tilde{\beta}(t)\|_{\delta}^2 \leq a_3\|\tilde{v}(t)\|_{\mathbb{L}_2}^2 \tag{77}$$

$$\|\tilde{u}(t)\|_{\mathbb{L}_2}^2 \leq b_1\|\tilde{\alpha}(t)\|_{-\delta}^2+b_2\|\tilde{\beta}(t)\|_{\delta}^2 \tag{78}$$

$$\|\tilde{\beta}(t)\|_{\mathbb{L}_2}^2 \leq b_3\|\tilde{v}(t)\|_{\delta}^2 \tag{79}$$

Corollary 1: Under the assumption of Theorem 2, Lemma 3, and Lemma 4, the error system (37)-(41) is exponentially stable in the norm (31).

Hence, we can state the following disturbance attenuation result of the $n+1$ coupled linear hyperbolic PDEs.

Theorem 3: Consider the observer (32)-(36) and the control law (28) in closed loop with the system (1)-(5). There exists positive constants c_1, c_2, and c_3 such that

$$\begin{aligned}&\left|\sum_{i=1}^{n}u^i(0,t)-rv(0,t)\right|\\ \leq\ & c_1\|(\tilde{X},\tilde{u},\tilde{v})\|\\ \leq\ & c_2\|(\tilde{X}(0),\tilde{u}(x,0),\tilde{v}(x,0))\|e^{-c_3t}\end{aligned} \tag{80}$$

IV. NUMERICAL SIMULATIONS

The example is inspired from the heave problem in oil well drilling [14]. An oil well is drilled from a floating rig causing the drill pipe to move with the heaving motion of the floating rig and induce major pressure variations over the drill bit at the bottom of the well. During drilling, an active heave compensation system is used. However, during connection (plugging a new segment of drill pipe), the active compensation system is shut down. Instead, a back-pressure pump is installed near the well head and the pressure along the annulus can be controlled by the choke manifold.

During connection period, which can last for 20 to 40 minutes, the drilling fluid (usually mud) and formation fluids (usually gas) are mixed at the bottom of the well creating a multi-phase flow along the annulus. To model the flow, a two-phase flow model known as the drift-flux model [17] is employed. The model is a quasi-linear hyperbolic equations but can be linearized in its equilibrium [13] and simplified into the following form[2]

$$u_t^1(x,t)+1.1u_x^1(x,t) = 2u^2(x,t)+4v(x,t) \tag{81}$$

$$u_t^2(x,t)+1.2u_x^2(x,t) = 0 \tag{82}$$

$$v_t(x,t)-v_x(x,t) = u^2(x,t) \tag{83}$$

Furthermore, one should notice that u^2 is a Riemann invariant for the system, i.e., satisfy a pure transport equation, which characterized mass fraction of gas in the two-phase model. The boundary conditions are given by

$$u^1(0,t) = v(0,t)+CX(t) \tag{84}$$

$$u^2(0,t) = v(0,t)+CX(t) \tag{85}$$

$$v(1,t) = -0.8u^2(1,t)+U(t) \tag{86}$$

$$\dot{X}(t) = AX(t) \tag{87}$$

[2]For simplicity all coefficients are assumed constants.

In this example, the disturbance matrices are given as follow

$$C = \begin{pmatrix} -2 & 1 \end{pmatrix} \quad A = \begin{pmatrix} -10 & -5 \\ -6 & -7 \end{pmatrix} \tag{88}$$

The vector gain L and the respective diagonal positive definite matrix P associated with the Lyapunov function are chosen as follow

$$L = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad P = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \tag{89}$$

To stabilize the states u^1, u^2, and v under the disturbance, the backstepping control law (30) is employed. The results of the state and the estimation error can be shown in the following figures

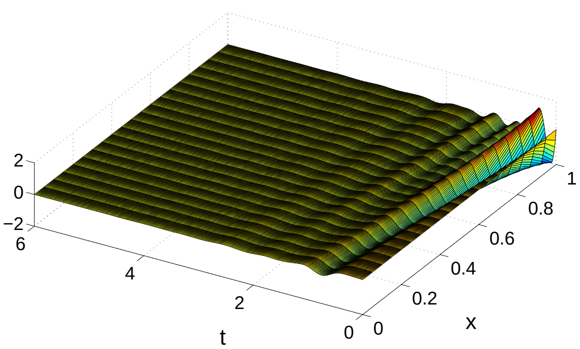

Fig. 1: Profile of $v(x, t)$ for output-feedback control.

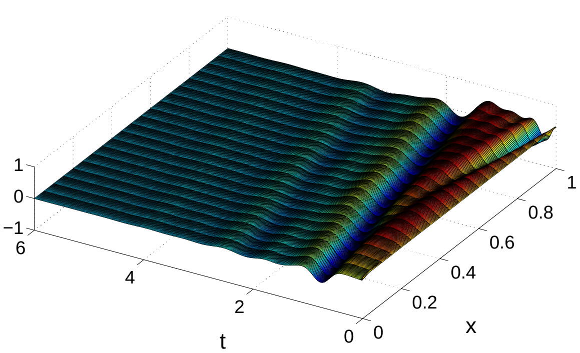

Fig. 2: Estimation error $v(x, t)$.

Clearly, the control law (30) successfully stabilizes the states and the observer (32)-(36) converges to the actual value.

V. CONCLUSIONS AND RECOMMENDATIONS

We have solved disturbance attenuation problem for a class of $n+1$ coupled first-order linear hyperbolic PDEs with a single boundary input. The control law requires measurement of the states and the disturbance. The control gain for the state is obtained by solving a first-order hyperbolic system, while the control gain for the disturbance is computed explicitly. The state-observer is constructed in an anti-collocated setup. Since the system is linear, from the separation principle, the combined observer and feedback is stable. The potential application of the proposed method is to solve the heave problem during drilling connection in oil well drilling from a floating rig. There is an interest to develop a collocated state-observer since a reliable measurement usually available at the well head ($x = 1$), thus this can be a subject for the future research.

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