

ON NONLOCAL QUASILINEAR EQUATIONS AND THEIR LOCAL LIMITS

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ABSTRACT. We introduce a new class of quasilinear nonlocal operators and study equations involving these operators. The operators are degenerate elliptic and may have arbitrary growth in the gradient. Included are new nonlocal versions of p -Laplace, ∞ -Laplace, mean curvature of graph, and even strongly degenerate operators, in addition to some nonlocal quasilinear operators appearing in the existing literature. Our main results are comparison, uniqueness, and existence results for viscosity solutions of linear and fully nonlinear equations involving these operators. Because of the structure of our operators, especially the existence proof is highly non-trivial and non-standard. We also identify the conditions under which the nonlocal operators converge to local quasilinear operators, and show that the solutions of the corresponding nonlocal equations converge to the solutions of the local limit equations. Finally, we give a (formal) stochastic representation formula for the solutions and provide many examples.

1. INTRODUCTION

In this paper we introduce a new class of gradient dependent Lévy type diffusion operators and study the well-posedness, stability, and some asymptotic behavior of equations involving such operators. The operators we will consider are the following,

$$L[u, Du] = (L_1 + L_2)[u, Du]$$

where

$$L_1[u, Du](x) = \int_{\mathbb{R}^P} u(x + j_1(Du, z)) - u(x) - j_1(Du, z) \cdot Du(x) \, d\mu_1(z), \quad (1.1)$$

$$L_2[u, Du](x) = \int_{\mathbb{R}^P} u(x + j_2(Du, z)) - u(x) \, d\mu_2(z), \quad (1.2)$$

and μ_1, μ_2 are non-negative Lévy measures and j_1, j_2 are measurable functions (see Section 2). Here the strength and direction of the diffusion depend on the gradient, and hence as we explain below, these operators are natural generalizations of the local (non-divergence form) quasilinear operators

$$L_0(Du, D^2u) = \frac{1}{2} \text{tr}(\sigma(Du)\sigma(Du)^T D^2u) + b(Du)Du.$$

The operators are allowed to degenerate ($j_1 = 0$ or $j_2 = 0$ in some set) and have arbitrary growth in the gradient, so ∞ -Laplace, p -Laplace, and strongly degenerate operators are included. Included

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are also “explicit” operators of the form (cf. Section 3.2),

$$a(Du) \left[-(-\Delta)^{\frac{\alpha}{2}} u \right] \quad \text{for all } \alpha \in (0, 2) \text{ and } a \in C(\mathbb{R}^N; \mathbb{R}^+). \quad (1.3)$$

We want to study equations involving the operator L , and to simplify and focus on the new issues, the main part of this paper is devoted to the following special problem:

$$F(u, Du, L[u, Du]) = f(x) \quad \text{in } \mathbb{R}^N, \quad (1.4)$$

where we assume F to be (degenerate) elliptic and strictly increasing in u (i.e. $D_u F > 0$). But for this equation, we make an effort to push for very general results. First we obtain comparison, uniqueness, stability, and existence results for bounded solutions of (1.4). These results are highly non-trivial due to the implicit nature of our operators and our weak integrability assumptions. Especially existence is very challenging as we discuss below. We then identify the limit problems where nonlocal operators converge to local ones,

$$L_\varepsilon[\phi, D\phi] \rightarrow L_0(D\phi, D^2\phi) \quad \text{as } \varepsilon \rightarrow 0,$$

for any smooth and bounded function ϕ , and prove that the solutions u_ε of the corresponding nonlocal equations

$$F(u_\varepsilon, Du_\varepsilon, L_\varepsilon[u_\varepsilon, Du_\varepsilon]) = f(x) \quad \text{in } \mathbb{R}^N, \quad (1.5)$$

converge locally uniformly to the solution of the local equation

$$F(u, Du, L_0(Du, D^2u)) = f(x) \quad \text{in } \mathbb{R}^N. \quad (1.6)$$

We refer to Section 2 for the precise assumptions and results, and to Section 6 for extensions to more general problems like parabolic problems and problems with several nonlocal operators. Here we just remark that (i) the weak solution concept we use is bounded viscosity solutions, (ii) generators L of every pure jump Lévy processes are included as linear special cases, and (iii) a typical special case of (1.4) satisfying our assumptions is the quasilinear equation

$$-L[u, Du](x) + u(x) = f(x) \quad \text{in } \mathbb{R}^N, \quad (1.7)$$

with bounded uniformly continuous f .

Let us illustrate our results on ∞ -Laplace type operators. In the local case (e.g. [29]) this operator has “diffusion” (Brownian motion, generator $(-\Delta)$) only in the gradient direction:

$$\Delta_\infty u(x) = \text{tr}[Du(x)Du(x)^T D^2u(x)] = (Du(x) \cdot D)^2 u(x). \quad (1.8)$$

Natural nonlocal generalizations are operators with e.g. α -stable diffusion ($\alpha \in (0, 2)$) along the gradient direction. The generator of the symmetric α -stable process is the fractional Laplacian [2],

$$-(-\Delta)^{\alpha/2} u(x) = \int_{\mathbb{R}^N} u(x+z) - u(x) - (z \cdot Du(x)) \mathbb{1}_{|z|<1} \frac{c_\alpha dz}{|z|^{N+\alpha}},$$

and hence the corresponding nonlocal version of the ∞ -Laplace operator would take the form

$$\mathcal{L}_{\Delta_\infty}^{\alpha/2}[u](x) = \int_{\mathbb{R}^1} u(x + Du(x)z) - u(x) - Du(x) \cdot Du(x)z \mathbb{1}_{|z|<1} \frac{c_\alpha dz}{|z|^{1+\alpha}}. \quad (1.9)$$

This operator is in the form L with $j_1 = Du \cdot z = j_2$, $\mu_1 = \mathbb{1}_{|z|<1}\mu$, and $\mu_2 = \mathbb{1}_{|z|\geq 1}\mu$, where $d\mu = \frac{c_\alpha dz}{|z|^{1+\alpha}}$. By our results, $L = \mathcal{L}_{\Delta_\infty}^{\alpha/2}$ gives rise to well-posed equations (1.4), and since

$$\mathcal{L}_{\Delta_\infty}^{\alpha/2}[\phi](x) \rightarrow \Delta_\infty \phi(x) \quad \text{as } \alpha \rightarrow 2^-$$

for smooth bounded ϕ , it also follows that (possibly non-smooth viscosity) solutions of (1.5) with $L_\varepsilon = \mathcal{L}_{\Delta_\infty}^{1-\varepsilon}$ will converge as $\varepsilon \rightarrow 0$ to the solution of (1.6) with $L_0 = \Delta_\infty$.

A similar construction can be carried out for “any” local (non-divergence form) quasilinear operator and “any” Lévy diffusion, thereby producing a corresponding quasilinear Lévy diffusion. Under our assumptions this new operator is well-posed, and can approximate the original local operator. This will be explained in Remark 2.7. In Section 3 we present a (formal) stochastic interpretation of our equations and give many more examples. Included are several nonlocal versions of the ∞ -Laplace, the p -Laplace, and the mean curvature of graph operators; versions that are modulations of singular integral operators and others based on bounded nonlocal operators. It is interesting to note that the limit operator L_0 will include also a drift term ($b \neq 0$) whenever the measures $\mu_{2,\varepsilon}$ in the L_2 -term has a non-zero mean value near $z = 0$, see assumption (\mathbf{M}_ε) in section 2.2. The reason is that in L_2 this mean is not compensated by a first-order gradient term as in L_1 .

The literature on nonlocal equations is very large, and we will restrict the following discussion to nonlocal quasilinear problems and the questions that we address in this paper: Well-posedness, stability and asymptotic limits. We will not discuss important issues such as regularity of solutions or numerical algorithms. In the literature, typically the nonlocal quasilinear operators either have “coefficients” depending on u or on Du (but see also [13]). In the former case you find e.g. all the equations of porous medium type, see e.g. [11, 16, 7, 18] and references therein. The second case is the case that we consider in this paper. Here the literature seems to be rather recent. In the calculus of variations, such equations can be obtained as Euler-Lagrange equations by minimizing fractional Sobolev norms ($W^{p,\frac{\alpha}{2}}$ -norms) [28, 17, 27] or truncated versions of such norms [1]. In the first three papers, (variational) fractional p and ∞ -Laplace operators are introduced. In [23], a different “variational” type of nonlocal operators is studied by non-variational viscosity solution techniques. In one space dimension, non-variational equations of the type

$$u_t + |u_x|^m (-\Delta)^{\frac{\alpha}{2}} u = 0 \quad \text{in} \quad \mathbb{R}^1 \times (0, T) \quad (1.10)$$

have been studied with viscosity solution techniques in e.g. [22, 33] for different values of $m > 0$ and $\alpha \in (0, 2)$. Such equations are motivated either by dislocation dynamics or porous medium flow, and along with their natural extensions to arbitrary space dimensions, they belong to the class of equations we study here (cf. sections 3.2 and 6). Non-variational nonlocal ∞ -Laplace type operators are introduced in [10, 9], and shown in [10] to be connected to a sequence of Tug of War games. But none of these operators have an implicit form as our operators do. Our operators are not variational, and among existing (multi-dimensional) work they resemble most closely the operators of [10, 9], especially [10]. However, whereas the operators in [10, 9] have bounded dependence on the gradient but are discontinuous where it is zero, our operators are continuous but may have arbitrary growth in the gradient. The operators in [10, 9] correspond to normalized ∞ -Laplacians, which in the local case take the form (see e.g. [30, 29])

$$\frac{1}{|Du(x)|^2} \Delta_\infty u(x) = \left(\frac{Du(x)}{|Du(x)|} \cdot D \right)^2 u(x),$$

while our version (1.9) corresponds to an unnormalized one (i.e. to $\Delta_\infty u$).

In this paper we work with viscosity solutions. This weak solution concept is not distributional and does not involve integration. It is very well adapted to the implicit and degenerate form of our equations. The viscosity solution concept was introduced by Crandall and Lions in the early 1980s to get uniqueness of solutions of first order Hamilton-Jacobi equations. Later it has been extended to wide range of problems, including many nonlocal ones. The standard reference for local problems is [15]. For nonlocal problems, we only refer to [6, 24] for the basic well-posedness theory for problems posed in the whole space. But we mention that there is a large literature on regularity and properties of solutions, asymptotic problems, boundary conditions, approximations and numerics, relation to stochastic processes, applications etc.. The problems we consider here represent a natural

class of nonlocal quasilinear equations where the viscosity solution techniques still apply and give comparison and uniqueness.

In fact we have optimized the assumptions to allow for very general dependence on the gradients in L and F at the cost of no dependence on the variable x ! We have also made an effort to optimize the assumption on j_i and μ_i . In both cases our assumptions are much more general than in [6, 24]. In the doubling of variables argument of the comparison proof, these differences to [6, 24] are e.g. reflected in a different choice of test function and two of the limits being taken in the reverse order. Reversing the limits is contrary to most viscosity solutions proofs, but it is essential in our proof. A side effect is that $\frac{|\bar{x}-\bar{y}|^2}{\varepsilon} \not\rightarrow 0$ and hence that we cannot consider equations with non-trivial x -dependence. Existence, on the other hand, does not follow from clever modifications of commonly used arguments. Because of the implicit form of the equations, with the gradient dependence in j_1 and j_2 , compactness arguments do not work and it seems not possible to adapt Perron's method either. Instead we propose a new argument based on a so-called *Sirtaki method* inspired by [4]. It involves several regularization and approximation arguments, a Schauder fixed point argument, and several limit problems. In each limit problem, we obtain a limit solving the relevant limit equation by the half relaxed limit method combined with strong comparison results. The argument is non-standard and highly non-trivial.

In section 6, we give the extension to the parabolic case (Cauchy problems) and to problems with many nonlocal operators including e.g. Bellman-Isaacs type equations. A natural open question is to study less degenerate equations without the assumption that $D_u F > 0$, like uniformly elliptic or even p and ∞ -Laplace equations. Another one is to consider such equations on domains with boundary conditions. Finally, we mention that in an upcoming paper we will study the local limits of nonlocal equations under assumptions that are optimized w.r.t. the x -dependence. In this case we also give explicit convergence rates.

Outline. We present the main results in Section 2 and give several examples and a stochastic interpretation in Section 3. Then, precise definitions of viscosity solutions appear in Section 4 and the proofs of the comparison, existence and concentration results are given in Section 5. In Section 6 we extend our results to parabolic problems and problems with many nonlocal operators, and in the appendix at the end of the paper, we give the proofs of some technical results we need.

Notation. The notation $UC(\mathbb{R}^N)$ denotes the set of uniformly continuous functions defined on \mathbb{R}^N and $BUC(\mathbb{R}^N)$ is the space of bounded, uniformly continuous functions; *usc* [resp. *lsc*] stands for upper semicontinuous [resp. lower semicontinuous]; the spaces C^1/C^2 are the spaces of functions having continuous first-order / second-order derivatives; $C^{0,\alpha}, C^{1,\alpha}$ stand for the usual Hölder spaces; C_b denotes the space of continuous, bounded functions; \limsup^* and \liminf_* are the half-relaxed limits (more precise definitions in the text where they are used); we denote by $\mathbf{1}_A$ the indicator function of the set A ; a modulus of continuity is a subadditive function $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\lim_{s \rightarrow 0^+} \omega(s) = 0$; the notation $a \wedge b$ stands for the min of a and b , $a \vee b$ is for the max and $s^+ = \max(s, 0)$. Note that in this paper $x \in \mathbb{R}^N$ for $N \geq 1$ while $z \in \mathbb{R}^P$ for $P \geq 1$; finally, $\mathbb{R}^{P \times Q}$ denotes the space of matrices with P rows and Q columns.

2. THE MAIN RESULTS

The results of this section essentially implies that for “any” quasilinear 2nd order local operator L_0 , “any” well-posed local equation (1.6), and “any” nonlocal Lévy type operator, there is a corresponding Lévy type quasilinear operator L and a *well-posed* nonlocal equation (1.4). Moreover,

the solution of any such local equation can be *approximated locally uniformly* by the solutions of a multitude of different nonlocal equations.

2.1. Comparison, uniqueness, and existence. Let us first list the assumptions under which we construct a general existence and uniqueness theory for (1.4):

(M) μ_1 and μ_2 are non-negative Radon measures on $\mathbb{R}^P \setminus \{0\}$ satisfying

$$\int_{|z|>0} |z|^2 d\mu_1(z) + \int_{|z|>0} d\mu_2(z) < \infty.$$

(J1) $j_1(p, z)$ and $j_2(p, z)$ are Borel measurable functions from $\mathbb{R}^N \times \mathbb{R}^P$ into \mathbb{R}^N , continuous in p for a.a. $z \in \mathbb{R}^P$, and for any $r > 0$ there is a $C_{j,r} > 0$ such that for all $|p| < r$,

$$\int_{|z|>0} |j_1(p, z)|^2 d\mu_1(z) \leq C_{j,r}.$$

(J2) For any $r > 0$, there is a modulus of continuity $\omega_{j,r}$ such that for all $|p|, |q| < r$,

$$\int_{|z|>0} |j_1(p, z) - j_1(q, z)|^2 d\mu_1(z) \leq \omega_{j,r}(p - q).$$

(J3) There exists $\delta_0 > 0$ such that for any $r > 0$ and $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\sup_{|p|<r} \int_A |j_1(p, z)|^2 d\mu_1(z) < \varepsilon$$

for every Borel set $A \subset \{0 < |z| < \delta_0\}$ such that $\int_A |z|^2 \mu_1(dz) < \eta$.

(F1) $F : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, and for any $u \in \mathbb{R}$, $p \in \mathbb{R}^N$, $\ell \leq \ell'$, $F(u, p, \ell) \geq F(u, p, \ell')$.

(F2) For any $M > 0$, there exist $\gamma_M > 0$ such that for all $p \in \mathbb{R}^N$, $l \in \mathbb{R}$, and $-M \leq v \leq u \leq M$,

$$F(u, p, l) - F(v, p, l) \geq \gamma_M(u - v).$$

(F3) For any $M, r > 0$, there exists a modulus of continuity $\omega_{M,r}$ such that for any $|u| \leq M$ and $|p|, |q|, |\ell|, |\ell'| \leq r$,

$$\left| F(u, p, \ell) - F(u, q, \ell') \right| \leq \omega_{M,r}(|p - q| + |\ell' - \ell|).$$

(F4) $f \in UC(\mathbb{R}^N)$.

(F5) $f \in BUC(\mathbb{R}^N)$ and all quantities in **(F2)**, **(F3)** are independent of M .

We give now the precise results and refer to Section 5 for the proofs.

Theorem 2.1. (Comparison results)

(a) [Quasilinear case] Assume **(M)**, **(J1)**–**(J2)**, and **(F4)**. If $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is a bounded usc subsolution of (1.7) and $v : \mathbb{R}^N \rightarrow \mathbb{R}$ is a bounded lsc supersolution of (1.7), then $u \leq v$ in \mathbb{R}^N .

(b) [Fully nonlinear case] Assume **(M)**, **(J1)**–**(J2)**, **(F1)**–**(F4)** hold. If $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is a bounded usc viscosity subsolution of (1.4) and $v : \mathbb{R}^N \rightarrow \mathbb{R}$ is a bounded lsc viscosity supersolution of (1.4), then $u \leq v$ in \mathbb{R}^N .

We have the following immediate consequences of this comparison result.

Corollary 2.2. *Under the assumptions of Theorem 2.1:*

(a) [Uniqueness] *There is a most one solution $u \in C_b(\mathbb{R}^N)$ of (1.4) (respectively of (1.7)).*

(b) [Uniform continuity] *Any solution $u \in C_b(\mathbb{R}^N)$ of (1.4) (respectively of (1.7)) belongs to $BUC(\mathbb{R}^N)$ and*

$$\gamma_M \omega_u(h) \leq \omega_f(h) \quad (\text{respectively } \omega_u(h) \leq \omega_f(h)),$$

where $M = \|u\|_\infty$ and $\omega_\phi(r) = \sup_{x \in \mathbb{R}^N, |y| < r} |\phi(x+y) - \phi(x)|$ denotes the modulus of continuity of $\phi(x)$.

(c) [L^∞ -bound] *If also (F5) holds with $\gamma_M = \gamma$ (independent of M), then any solution $u \in C_b(\mathbb{R}^N)$ of (1.4) (respectively of (1.7)) satisfies*

$$\gamma \|u\|_\infty \leq \|f\|_\infty \quad (\text{respectively } \|u\|_\infty \leq \|f\|_\infty).$$

Proof. (a) is immediate from Theorem 2.1, while (c) follows since $\pm \frac{1}{\gamma} \|f\|_\infty$ are super and subsolutions of (1.4). To prove (b), note that $v_\pm(x) = u(x+h) \pm \frac{1}{\gamma_M} \omega_f(|h|)$ is a super and subsolution of (1.4). By Theorem 2.1, $v_-(x) \leq u(x) \leq v_+(x)$, and hence $|u(x) - u(x+h)| \leq \frac{1}{\gamma_M} \omega_f(|h|)$. \square

Theorem 2.3 (Existence). *Under the assumptions of Theorem 2.1, (J3), and (F5), there exists a unique bounded viscosity solution $u \in BUC(\mathbb{R}^N)$ of (1.4) (respectively of (1.7)).*

Let us now briefly comment on the assumptions.

Remark 2.4. (i) μ_1 and μ_2 are Lévy measures [2] by (M). Conversely, any Lévy measure μ can be written as $\mu_1 + \mu_2$ for μ_1 and μ_2 satisfying (M):

$$\mu = \mu \mathbb{1}_{|z| < 1} + \mu \mathbb{1}_{|z| \geq 1} =: \mu_1 + \mu_2.$$

(ii) Assumptions on j are optimized w.r.t. the dependence in p at the cost of no dependence on x ! A typical example is

$$j_i(p, z) = j(p)z,$$

where $z \in \mathbb{R}^P$ and $j : \mathbb{R}^N \rightarrow \mathbb{R}^{N \times P}$ only needs to be continuous. (J1) and (J3) follow from the stronger assumption $|j_i(p, z)| \leq c|z|$ for z near 0. (J3) implies that $\left\{ \frac{|j_i(p, z)|^2}{|z|^2} \right\}_{|p| < r}$ is $|z|^2 \mu_1(dz)$ equi-integrable on $\{0 < |z| < \delta\}$ for any $\delta \leq \delta_0$, cf. Appendix A. We need it to construct solutions under our general assumptions but not for comparison.

(iii) By (M), (J1), and a Taylor expansion, $L[\phi, D\phi](x)$ is well-defined for any $\phi \in C^2(\mathbb{R}^N) \cap C_b(\mathbb{R}^N)$.

(iv) (F2) implies degenerate ellipticity and strict monotonicity in u , while (F3) allows for very general p -dependence at the cost no x -dependence. Compare (F3) to e.g. assumption (3.14) in [15].

(v) The assumptions on integrability and p -dependence of j and the (p, l) -dependence of F of this paper are much more general than e.g. in [24, 6].

2.2. Local limits. We also study the convergence of solutions of the nonlocal equation (1.5) to the local equation (1.6), including separate results for the quasilinear case where (1.5) and (1.6) take the simpler forms

$$-L_\varepsilon[u_\varepsilon, Du_\varepsilon](x) + u_\varepsilon(x) = f(x) \quad \text{in } \mathbb{R}^N, \quad (2.1)$$

$$-L_0(Du, D^2u) + u(x) = f(x) \quad \text{in } \mathbb{R}^N, \quad (2.2)$$

where the local operator L_0 is precisely defined in Definition 2.5 below. Concerning (2.1), we use the decomposition $L_\varepsilon = L_{1,\varepsilon} + L_{2,\varepsilon}$ with

$$L_{i,\varepsilon}[\phi, D\phi](x) = \int_{\mathbb{R}^P} \phi(x + j_i(D\phi(x), z) - \phi(x) - \delta_{i,1} j_i(D\phi(x), z) \cdot D\phi(x)) d\mu_{i,\varepsilon}, \quad i = 1, 2,$$

where $\delta_{i,1} = 1$ if $i = 1$ and 0 otherwise. In order to prove the convergence result as $\varepsilon \rightarrow 0$ we need the following additional assumptions:

(M $_\varepsilon$) $\mu_\varepsilon = (\mu_{1,\varepsilon}, \mu_{2,\varepsilon})$ satisfies **(M)** for every $\varepsilon > 0$, and there exists $A_1, A_2 \in \mathbb{R}^{P \times N}$ and $a \in \mathbb{R}^P$ such that for every $Y \in \mathbb{R}^{P \times P}$, $q \in \mathbb{R}^P$, and $\delta > 0$, as $\varepsilon \rightarrow 0$

$$\begin{aligned} \int_{|z| < \delta} z^T Y z d\mu_{1,\varepsilon} &\rightarrow \text{tr}[A_1^T Y A_1], \\ \int_{|z| < \delta} (z^T Y z + q \cdot z) d\mu_{2,\varepsilon} &\rightarrow \text{tr}[A_2^T Y A_2] + a \cdot q, \\ \int_{|z| > \delta} (d\mu_{1,\varepsilon} + d\mu_{2,\varepsilon}) &\rightarrow 0. \end{aligned}$$

(J4) For $i = 1, 2$, the function $(p, z) \mapsto j_i(p, z)$ is continuous, z -differentiable at $z = 0$ locally uniformly in p , $j_i(p, 0) = 0$, and the function $\sigma_i : \mathbb{R}^N \rightarrow \mathbb{R}^{N \times P}$, defined by

$$\sigma_i(p) := D_z j_i(p, 0), \quad \text{is continuous.}$$

Definition 2.5. For any vector $p \in \mathbb{R}^N$ and matrix $X \in \mathbb{R}^{N \times N}$, we define:

$$L_0(p, X) := \frac{1}{2} \text{tr}[\tilde{\sigma}_1(p) \tilde{\sigma}_1(p)^T X] + \frac{1}{2} \text{tr}[\tilde{\sigma}_2(p) \tilde{\sigma}_2(p)^T X] + b(p) \cdot p,$$

where $\tilde{\sigma}_i(p) := \sigma_i(p) A_i$, $i = 1, 2$, $b(p) := a^T \sigma_2(p)$, for $A_1, A_2, a, \sigma_1, \sigma_2$ given by **(M $_\varepsilon$)** and **(J4)**.

The limit result is the following:

Theorem 2.6. (Local limits) Let L_0 be given by Definition 2.5.

(a) [Quasilinear case] Assume **(M $_\varepsilon$)**, **(J1)**–**(J4)** and **(F5)**. Then any sequence of solutions u_ε of (2.1) converges locally uniformly as $\varepsilon \rightarrow 0$ to the solution u of (2.2).

(b) [Fully nonlinear case] Assume **(M $_\varepsilon$)**, **(J1)**–**(J4)**, and **(F1)**–**(F5)**. Then any sequence of solutions u_ε of (1.5) converges locally uniformly as $\varepsilon \rightarrow 0$ to the solution u of (1.6).

Remark 2.7. (i) **(M $_\varepsilon$)** is a concentration assumption implying e.g. $z^T Y z \mu_{1,\varepsilon}(dz) \rightarrow \text{tr}[A_1^T Y A_1] \delta_0$ in measure. This is a convergence result for measures in \mathbb{R}^P and not in \mathbb{R}^N . Note that a plays a role only for the L_2 -part of L . Illustrative examples are the following singular and truncated $(2 - \varepsilon)$ -stable like Lévy measures:

$$\begin{aligned} \mu_{1,\varepsilon}(dz) &= \varepsilon \frac{g(z)}{|z|^{N+2-\varepsilon}} \mathbf{1}_{|z| < 1} dz \quad \text{where} \quad \lim_{z \rightarrow 0} g(z) = g(0) \neq 0, \\ \mu_{2,\varepsilon}(dz) &= \varepsilon \frac{g(z)}{|z|^{N+2-\varepsilon}} \mathbf{1}_{\varepsilon < |z| < 1} dz \quad \text{where} \quad g \text{ is } C^1 \text{ at } z = 0 \text{ and } g(0) \neq 0. \end{aligned}$$

Both satisfy **(M $_\varepsilon$)**: $\mu_{1,\varepsilon}$ with $A_1 = g(0)I$, $A_2 = 0$, $a = 0$ and $\mu_{2,\varepsilon}$ with $A_1 = 0$, $A_2 = g(0)I$, $a = Dg(0)$.

(ii) By **(J4)** and Definition 2.5,

$$L_0(p, X) = \text{tr}[A_1 Y_1 A_1] + \text{tr}[A_2 Y_2 A_2] + a \cdot q$$

for $Y_i = \sigma_i(p)^T X \sigma_i(p) \in \mathbb{R}^{P \times P}$, $i = 1, 2$, and $q = \sigma_2(p)p \in \mathbb{R}^P$.

(ii) If **(F5)** and **(J4)** (and **(F1)**–**(F3)**) hold, there exists a unique viscosity solution of (2.2) (and of (1.6)) satisfying the strong comparison principle, cf. Theorem 5.1 in [15] and Lemma 5.14 below.

(iii) We may specify (“any”) $\sigma(p)$ first, and then for every Lévy measure $(\mu_{1,\varepsilon}, \mu_{2,\varepsilon})$ satisfying the concentration assumption **(M $_\varepsilon$)**, we get a nonlocal approximation L_ε of the local operator L_0 . Moreover, the corresponding equations, (1.5) and (1.4), are well-posed with solutions that converge to one another under very general assumptions.

3. STOCHASTIC INTERPRETATION AND EXAMPLES

3.1. Stochastic interpretation. Formally equation (1.4) is always the Dynamic Programming Equation of an implicitly defined stochastic control problem or game. E.g. the solution u of (1.7) satisfies formally

$$u(x) = \mathbb{E}^x \left(\int_0^\infty e^{-t} f(X_t) dt \right) \quad (3.1)$$

where X_t is a pure jump Lévy-Ito process satisfying

$$X_t = x + \int_0^t \int_{|z|>0} j_1(Du(X_{s-}), z) \tilde{N}_1(dz, ds) + \int_0^t \int_{|z|>0} j_2(Du(X_{s-}), z) N_2(dz, ds), \quad (3.2)$$

where \tilde{N}_1 is a compensated Poisson random measure, N_2 is a finite intensity Poisson random measure, and \mathbb{E}^x is the expectation w.r.t. the law of X (which starts at x). By a Lévy-Ito process we mean a *Lévy type stochastic integral* defined in Chapter 4.3.3 in [2], and we refer to e.g. [2, 14] for definitions of the other probabilistic terms mentioned above. Formally, the generator A of X_t is given by the formula

$$Au(x) = L[u, Du](x)$$

for u in the domain of A (equation (6.36) in [2]). Moreover, X_t generates a semigroup T_t defined by $T_t \phi(y) = \mathbb{E}^y \phi(X_t^y)$ with the convention that $X_0^y = y$ almost surely, and u is then the 1-resolvent R_1 (chapter 3 in [2]) of this the semi-group applied to f , i.e. $R_1 f(x) = 1u(x)$. By the resolvent identity,

$$u - Au = (I - A)R_1 f(x) = f(x) \quad \text{in} \quad \mathbb{R}^N,$$

i.e. u satisfies equation (1.7) at least formally. To make this discussion rigorous, we need the assumptions of section 2 and some additional ones including smoothness of u . Following chapter 6.7 in [2], it suffices to assume in addition that Assumptions 6.6.1 and 6.7.1 of [2] hold. We do not state them here, we only remark that they are satisfied if e.g.

$$j_1(p, z) = j(p)z \text{ with } j \in W_{\text{loc}}^{1,\infty}, \quad 0 \leq \mu(dz) \leq \frac{C dz}{|z|^{N+\alpha}} \text{ with } \alpha \in (0, 2), \quad u \in C_0^2 \text{ and } f \in C_0.$$

Note that then Du is bounded and Lipschitz. In this case it follows from Theorem 6.7.4 of [2] that T_t is a Feller semi-group with generator A as above and that u is in the domain of A . By Theorem 3.2.9 of [2] the resolvent R_1 exists and satisfies the resolvent identity above for any $f \in C_0(\mathbb{R}^N)$.

We have the following result:

Proposition 3.1. *If u and X_t satisfy the assumptions mentioned above and (3.1) and (3.2) hold (in the strong sense), then u is a classical solution of (1.7).*

3.2. Isotropic operators involving the fractional Laplacian. We will explain why products of the fractional Laplacian and a positive scalar function of the gradient (cf. (1.3)), are operators of the type we consider here in this paper.

By the scaling properties of the Levy measure $\frac{c_{N,\alpha}}{|z|^{N+\alpha}}dz$ and a change of variables,

$$-a^\alpha(-\Delta)^\alpha u(x) = \int_{\mathbb{R}^N} u(x+az) - u(x) - az \cdot Du(x) \frac{c_{N,\alpha} dz}{|z|^{N+\alpha}} \quad \text{for all } a \geq 0, \alpha \in (0, 2),$$

and hence for every x ,

$$-a(Du(x))(-\Delta)^\alpha u(x) = \int_{\mathbb{R}^N} u(x + a^{\frac{1}{\alpha}}(Du(x))z) - u(x) - a^{\frac{1}{\alpha}}(Du(x))z \cdot Du(x) \frac{c_{N,\alpha} dz}{|z|^{N+\alpha}}.$$

It is immediate that assumptions **(M)**, **(J1)**–**(J3)** are all satisfied for this operator when $\alpha \in (0, 2)$ and $a \in C(\mathbb{R}^N; \mathbb{R}^+)$.

In one space dimension and with $a(p) = |p|^{m-1}$, $m > 0$, this operator appear in models of dislocations in crystals ($m = 1$) [21, 22, 8], and in certain nonlocal porous medium models ($m > 1$) [33] as the (integrated) equation for the cumulative distribution function.

3.3. Examples. We introduce now some classes of quasilinear nonlocal operators with special focus on operators of p -Laplacian, ∞ -Laplacian, and mean curvature of graph type. Recall the definitions of the local and fractional ∞ -Laplacian in (1.8) and (1.9). To define other nonlocal operators we need the following Lemma.

Lemma 3.2. *Let $p \geq 1$, $r_p = -1 + \sqrt{p-1}$ and I be $N \times N$ identity matrix.*

$$(a) \quad I + (p-2) \frac{\xi \otimes \xi}{|\xi|^2} = a_p(\xi) a_p^T(\xi) \quad \text{where} \quad a_p(\xi) := I + r_p \frac{\xi \otimes \xi}{|\xi|^2}.$$

$$(b) \quad I - \frac{\xi \otimes \xi}{1 + |\xi|^2} = \tilde{a}(\xi) \tilde{a}^T(\xi) \quad \text{where} \quad \tilde{a}(\xi) := I - \frac{\xi \otimes \xi}{|\xi|^2} \left(1 - \frac{1}{\sqrt{1 + |\xi|^2}} \right).$$

(c) *The functions $\tilde{a}(\xi)$ and $|\xi|^{\frac{p-2}{2}} a_p(\xi)$, $p \geq 2$, are continuous in \mathbb{R}^N .*

The proof is straightforward, using that $r_p^2 + r_p = p - 2$. In view of the lemma,

$$\begin{aligned} \Delta_p u(x) &= \operatorname{div} \left(|Du(x)|^{p-2} Du(x) \right) = |Du(x)|^{p-2} \left(\Delta u(x) + (p-2) \frac{\Delta_\infty u(x)}{|Du(x)|^2} \right) \\ &= \operatorname{tr} [|Du(x)|^{p-2} D^2 u(x)] + (p-2) \operatorname{tr} [|Du(x)|^{p-4} Du(x) Du(x)^T D^2 u(x)] \\ &= \operatorname{tr} [\sigma_p(Du(x)) \sigma_p^T(Du(x)) D^2 u(x)] \quad \text{where} \quad \sigma_p(\xi) = |\xi|^{\frac{p-2}{2}} a_p(\xi), \end{aligned}$$

$$\begin{aligned} H[u](x) &= \operatorname{div} \left(\frac{Du(x)}{\sqrt{1 + |Du(x)|^2}} \right) = \frac{1}{\sqrt{1 + |Du(x)|^2}} \left(\Delta u(x) - \frac{\Delta_\infty u(x)}{1 + |Du(x)|^2} \right) \\ &= \operatorname{tr} \left(\tilde{\sigma}(Du(x)) \tilde{\sigma}^T(Du(x)) D^2 u(x) \right) \quad \text{where} \quad \tilde{\sigma}(\xi) = \frac{\tilde{a}(\xi)}{(1 + |\xi|^2)^{\frac{1}{4}}}, \end{aligned}$$

where $\tilde{\sigma}$ and σ_p are continuous for $p \geq 2$.

First type of examples: Quasilinear versions of every generator of pure jump Lévy processes [2]. E.g. nonlocal fractional Laplace type operators,

$$\begin{aligned} \mathcal{L}_{\Delta_\infty}^{\alpha/2}[u](x) & \text{ already defined in (1.9),} \\ \mathcal{L}_{\Delta_p}^{\alpha/2}[u](x) &= \int_{\mathbb{R}^N} u(x + \sigma_p(Du(x))z) - u(x) - \sigma_p(Du(x))z \cdot Du(x) \mathbb{1}_{|z|<1} \frac{c_\alpha dz}{|z|^{N+\alpha}}, \\ \tilde{\mathcal{L}}_{\Delta_p}^{\alpha/2}[u](x) &= \int_{\mathbb{R}^N} u(x + |Du(x)|^{\frac{p-2}{2}} z) - u(x) - |Du(x)|^{\frac{p-2}{2}} z \cdot Du(x) \mathbb{1}_{|z|<1} \frac{c_\alpha dz}{|z|^{N+\alpha}} \\ & \quad + (p-2) \int_{\mathbb{R}^1} u\left(x + |Du(x)|^{\frac{p-4}{2}} Du(x)z\right) - u(x) - |Du(x)|^{\frac{p-4}{2}} Du(x)z \cdot Du(x) \mathbb{1}_{|z|<1} \frac{c_\alpha dz}{|z|^{1+\alpha}}, \\ \mathcal{L}_H^{\alpha/2}[u](x) &= \int_{\mathbb{R}^N} u\left(x + \bar{\sigma}(Du(x))z\right) - u(x) - \bar{\sigma}(Du(x))z \cdot Du(x) \mathbb{1}_{|z|<1} \frac{c_\alpha dz}{|z|^{N+\alpha}} \end{aligned}$$

where $p \geq 2$ and $c_\alpha = O(2 - \alpha)$ is the constant of $\Delta^{\alpha/2}$. The fractional Laplacian is the generator of the symmetric α -stable process, and the above nonlocal versions can be seen “generators” of gradient dependent modulations of this process. To be more precise, $\mathcal{L}_{\Delta_\infty}^{\alpha/2}$ is a nonlocal version of the infinity Laplace operator; both $\mathcal{L}_{\Delta_p}^{\alpha/2}$ and $\tilde{\mathcal{L}}_{\Delta_p}^{\alpha/2}$ are nonlocal versions of the p -Laplace operator, depending on how we write it; finally $\mathcal{L}_H^{\alpha/2}$ is a nonlocal version of the curvature operator H . Note that these operators are of the form $L = L_1 + L_2$ where both $L_1 \neq 0$ ($|z| < 1$) and $L_2 \neq 0$ ($|z| > 1$).

Second type of examples: Quasilinear versions of the generators of some Lévy-Ito jump-processes defined by stochastic differential equations (SDEs) driven by pure jump Lévy processes [2, 14]. An example is the operator from the CGMY model for the price of a European option in Finance [14],

$$\mathcal{L}u(x) = \int_{\mathbb{R}^1} u(x+z) - u(x) - Du(x)(e^z - 1) \frac{C e^{-Mz^+ - Gz^-} dz}{|z|^{1+Y}},$$

for $C, G, M > 0$, $Y \in (0, 2)$, and the following new nonlocal infinity Laplacian (compare to $\mathcal{L}_{\Delta_\infty}^{\alpha/2}$):

$$\mathcal{J}_{\Delta_\infty}^{\alpha/2}[u](x) = \int_{\mathbb{R}^1} u(x + Du(x)z) - u(x) - Du(x) \cdot Du(x)(e^z - 1) \frac{c_\alpha e^{-Mz^+ - Gz^-} dz}{|z|^{1+\alpha}}.$$

In this case $L_1 \neq 0$ and $L_2 = 0$, and $L = \mathcal{J}_{\Delta_\infty}^{\alpha/2}$ is a gradient dependent modulation of \mathcal{L} . Here \mathcal{L} is not the generator of a Lévy process, but the exponential of a Lévy process [14] (after a transformation). The driving (Lévy) process here is a tempered α -stable process [14]. Other quasilinear versions (p -Laplace etc.) can be easily be constructed as above.

Remark 3.3. Since we do not allow for x -dependence in j_1 and j_2 at the level of the PDE (1.4), we can only consider generators of very special SDEs. In the example above the coefficients in the SDE will depend on X_t , but after a change of variables this dependence is lost in the corresponding PDE.

Third type of examples: Versions of the above nonlocal operators with truncated and hence non-singular measures. Simply replace $d\mu(z)$ in the definition of L by $\mathbb{1}_{|z|>r} d\mu(z)$, e.g.

$$\mathcal{L}_{\Delta_\infty}^{\alpha/2,r}[u](x) = \int_{\mathbb{R}^1} \left(u(x + Du(x)z) - u(x) - Du(x) \cdot Du(x)(e^z - 1) \right) \mathbb{1}_{|z|>r} \frac{c_\alpha g(z) dz}{|z|^{1+\alpha}},$$

where $g(0) \neq 0$ and g is C^1 at $z = 0$. Note that here $L = \mathcal{L}_{\Delta_\infty}^{\alpha/2,r}$ with $L_1 = 0$ and $L_2 \neq 0$, and (\mathbf{M}_ε) holds with $A_2 = g(0)I$ and $a = Dg(0)$.

3.4. Remarks.

(a) [Continuity in α, p] All the operators above will be continuous in $(\alpha, p) \in (0, 2) \times [2, \infty)$. For example for any bounded C^2 function ϕ and sequence $(\alpha', p') \rightarrow (\alpha, p) \in (0, 2) \times [2, \infty)$,

$$\mathcal{L}_{\Delta_{p'}}^{\alpha'/2} \phi \longrightarrow \mathcal{L}_{\Delta_p}^{\alpha/2} \phi \quad \text{in } \mathbb{R}^N.$$

(b) [The limit $\alpha \rightarrow 2$] If u is smooth and bounded, then by easy computations,

$$\mathcal{L}_{\Delta_\infty}^{\alpha/2}[u], \mathcal{J}_{\Delta_\infty}^{\alpha/2}[u] \longrightarrow \Delta_\infty u, \quad \mathcal{L}_{\Delta_p}^{\alpha/2}[u], \tilde{\mathcal{L}}_{\Delta_p}^{\alpha/2}[u] \longrightarrow \Delta_p u, \quad \mathcal{L}_H^{\alpha/2}[u] \longrightarrow H[u],$$

$$\text{and } \mathcal{L}_{\Delta_\infty}^{\alpha/2, 2-\alpha}[u] \longrightarrow g(0)\Delta_\infty u + Dg(0)Du$$

point-wise as $\alpha \rightarrow 2$. Hence *all* of these operators converge to their local counterparts including the truncated ones. These latter operators also give rise to a drift term (when μ is non-symmetric!). Note that in these examples assumption (\mathbf{M}_ε) hold with $\alpha = 2 - \varepsilon$, $A = I$ or $A = g(0)I$, and $a = 0$ or $a = Dg(0)$.

(c) [Growth assumptions] Our assumptions allow for extreme growth in the gradient and nonlocal terms. Our results cover the equation

$$u - F(L[u, Du](x)) = f(x)$$

for any continuous nondecreasing function F and any good operator L as above, e.g.

$$u - \left(e^{\mathcal{L}_{\Delta_p}^{\alpha/2}[u]} - 1 \right) = f(x) \quad \text{for any } p \geq 2.$$

4. VISCOSITY SOLUTIONS

In this section, we introduce the good notion of weak solution for equation (1.4). We prove that we have two equivalent definitions and that the solution concept is stable with respect to pointwise limits of uniformly bounded solutions.

We start by splitting L_1 in (1.1) into two parts: $L_1 = L_\delta + L^\delta$ for $\delta > 0$, where

$$L_\delta[\phi, D\phi](x) := \int_{|z| < \delta} \phi(x + j_1(D\phi(x), z)) - \phi(x) - j_1(D\phi(x), z) \cdot D\phi(x) \, d\mu_1(z), \quad (4.1)$$

$$L^\delta[u, p](x) := \int_{|z| \geq \delta} u(x + j_1(p, z)) - u(x) - j_1(p, z) \cdot p \, d\mu_1(z) \quad (p \in \mathbb{R}^N).$$

In view of (\mathbf{M}) , L_δ is well-defined for any C^2 function ϕ and L^δ for any bounded function u . Likewise, the operator $L_2[u, p]$ is also well-defined for any $p \in \mathbb{R}^N$ and bounded measurable function u . Recall that those integrals are taken over \mathbb{R}^P . Now we can introduce the concept of solutions that we will use in this paper.

Definition 4.1.

(a) A bounded usc function u is a viscosity subsolution of (1.4) if for any $\delta > 0$, any C^2 function ϕ , and any global maximum point x of $u - \phi$,

$$F\left(u(x), D\phi(x), L_\delta[\phi, D\phi](x) + L^\delta[u, D\phi](x) + L_2[u, D\phi](x)\right) \leq f(x). \quad (4.2)$$

(b) A bounded lsc function u is a viscosity supersolution of (1.4) if for any $\delta > 0$, any C^2 function ϕ , and any global minimum point x of $u - \phi$,

$$F\left(u(x), D\phi(x), L_\delta[\phi, D\phi](x) + L^\delta[u, D\phi](x) + L_2[u, D\phi](x)\right) \geq f(x). \quad (4.3)$$

(c) A viscosity solution is a bounded continuous function u which is both a subsolution and a supersolution.

Another possible definition is the following:

Definition 4.2.

(a) A bounded usc function u is a viscosity subsolution of (1.4) if for any bounded C^2 function ϕ , and any global maximum point x of $u - \phi$,

$$F\left(u(x), D\phi(x), L[\phi, D\phi](x)\right) \leq f(x). \quad (4.4)$$

(b) A bounded lsc function u is a viscosity subsolution of (1.4) if for any bounded C^2 function ϕ , and any global minimum point x of $u - \phi$,

$$F\left(u(x), D\phi(x), L[\phi, D\phi](x)\right) \geq f(x). \quad (4.5)$$

(c) A viscosity solution is a bounded continuous function u which is both a subsolution and a supersolution.

Remark 4.3. We may assume without loss of generality that the extrema of $u - \phi$ are strict and that $\phi = u$ at the extremal point. The latter comes from shifting the test function by a constant. To make an extremum (say a maximum) point x strict, we replace ϕ by $\phi + \delta\psi$ where $\delta > 0$ and

$$\psi \in C^2(\mathbb{R}^N) \cap W^{2,\infty}(\mathbb{R}^N), \quad \psi = 0 \text{ and } D\psi = 0 \text{ at } x, \quad \text{and} \quad \psi > 0 \text{ elsewhere,}$$

and send $\delta \rightarrow 0$ in the final step of the proof. As opposed to the local case, the δ -terms will now be visible throughout the computations and vanish only in the final step.

Lemma 4.4. *If (M), (J1), (F1), and (F4) hold, then Definitions 4.1 and 4.2 are equivalent.*

Proof. The proof is pretty standard [32, 3, 24, 6]. Since $(u - \phi)$ has a max in x , $L^\delta[u, D\phi](x) \leq L^\delta[\phi, D\phi](x)$ and $L_2[u, D\phi](x) \leq L_2[\phi, D\phi](x)$, and hence by (F1), (F4), and since $L = L_\delta + L^\delta + L_2$, inequality (4.4) follows from (4.2). Conversely, we may assume the max is strict (see Remark 4.3). Then there exists a smooth and uniformly bounded function ϕ_ε such that $u \leq \phi_\varepsilon \leq \phi$ and $\phi_\varepsilon \rightarrow u$ a.e. as $\varepsilon \rightarrow 0$. It immediately follows that also $u - \phi_\varepsilon$ and $\phi_\varepsilon - \phi$ have maximum points at x . Hence, since $D\phi_\varepsilon(x) = D\phi(x)$ and by the definition of L (monotonicity and $L = L_\delta + L^\delta + L_2$),

$$L[\phi_\varepsilon, D\phi_\varepsilon](x) \leq L_\delta[\phi, D\phi](x) + L^\delta[\phi_\varepsilon, D\phi](x) + L_2[\phi_\varepsilon, D\phi](x).$$

Hence, by inequality (4.4) with ϕ_ε replacing ϕ , inequality (4.2) with ϕ_ε replacing u follows. Now we conclude by sending $\varepsilon \rightarrow 0$, using (M), (J1), (F1), (F4), and the dominated convergence theorem. \square

Next, we show that this solution concept is stable with respect to local uniform limits, to so-called half-relaxed limits, and more generally to very general perturbations of the equation. Consider

$$F_\varepsilon(u_\varepsilon, Du_\varepsilon, L_\varepsilon[u_\varepsilon, Du_\varepsilon]) = f_\varepsilon(x) \quad \text{in} \quad \mathbb{R}^N, \quad (4.6)$$

where $(f_\varepsilon, F_\varepsilon, L_\varepsilon := L_{1,\varepsilon} + L_{2,\varepsilon})$ satisfy (F4), (F1), (M), and (J1) for each fixed $\varepsilon > 0$, and where

$$\begin{aligned} L_{1,\varepsilon}[u, Du](x) &= \int_{\mathbb{R}^P} u(x + j_{1,\varepsilon}(Du, z)) - u(x) - j_{1,\varepsilon}(Du, z) \cdot Du(x) \, d\mu_{1,\varepsilon}(z), \\ L_{2,\varepsilon}[u, Du](x) &= \int_{\mathbb{R}^P} u(x + j_{2,\varepsilon}(Du, z)) - u(x) \, d\mu_{2,\varepsilon}(z). \end{aligned}$$

Then we define the “half-relaxed limits”:

$$\bar{u}(x) := \limsup_{y \rightarrow x, \varepsilon \rightarrow 0} u_\varepsilon(y) \quad \text{and} \quad \underline{u}(x) := \liminf_{y \rightarrow x, \varepsilon \rightarrow 0} u_\varepsilon(y). \quad (4.7)$$

$$\bar{f}(x) := \limsup_{y \rightarrow x, \varepsilon \rightarrow 0} f_\varepsilon(y) \quad \text{and} \quad \underline{f}(x) := \liminf_{y \rightarrow x, \varepsilon \rightarrow 0} f_\varepsilon(y). \quad (4.8)$$

$$\bar{F}(u, p, l) := \limsup_{\substack{(v, q, m) \rightarrow (u, p, l) \\ \varepsilon \rightarrow 0}} F_\varepsilon(v, q, m) \quad \text{and} \quad \underline{F}(u, p, l) := \liminf_{\substack{(v, q, m) \rightarrow (u, p, l) \\ \varepsilon \rightarrow 0}} F_\varepsilon(v, q, m). \quad (4.9)$$

Lemma 4.5 (Stability 1). *Assume $\{f_\varepsilon, F_\varepsilon, L_\varepsilon\}_\varepsilon$ satisfy **(M)**, **(J1)**, **(F1)**, **(F4)** for any $\varepsilon > 0$,*

$$\liminf_{\varepsilon \rightarrow 0} L_\varepsilon[\phi, D\phi](x_\varepsilon) \leq L[\phi, D\phi](x) \quad (\text{resp. } \limsup_{\varepsilon \rightarrow 0} L_\varepsilon[\phi, D\phi](x_\varepsilon) \geq L[\phi, D\phi](x)),$$

for all bounded $\phi \in C^2$ and all sequences $x_\varepsilon \rightarrow x$, and that $\{u_\varepsilon\}_{\varepsilon > 0}$ is a sequence of uniformly bounded subsolutions (resp. supersolutions) of (4.6).

Then $\bar{u}(x)$ is a subsolution (resp. $\underline{u}(x)$ supersolution) of (1.4) with (f, F) replaced by (\bar{f}, \underline{F}) (resp. (\underline{f}, \bar{F})).

We also have the following stability result.

Lemma 4.6 (Stability 2). *Assume **(M)**, **(J1)**, **(F1)**, **(F4)** hold, and $\{u_a\}_{a \in A}$, for a set A , is a family of uniformly bounded subsolutions (resp. supersolutions) of (1.4).*

(a) *If for any n , u_{a_n} is continuous and $u_{a_n} \rightarrow u$ locally uniformly as $n \rightarrow \infty$, then u is a continuous bounded subsolution (resp. supersolution) of (1.4).*

(b) *$u := \sup_{a \in A} u_a$ is a subsolution (resp. $v = \inf_{a \in A} u_a$ is a supersolution) of (1.4).*

The proofs follow after the next remark.

Remark 4.7. (i) Similar type of results can be found in [6], but without variation in L .

(ii) Compare Lemma 4.6-(a) to the no stability w.r.t. local uniform convergence result of [10]. In [9] there is stability, but the nonlocal operators are more different from ours than in [10].

Proof of Lemma 4.5. The proof is quite standard, see e.g. [6] (Theorem 2) for a similar proof. We only do the subsolution case since the supersolution case is similar. Assume ϕ is C^2 and bounded and $\bar{u} - \phi$ has a global maximum at x , we will show that inequality (4.4) holds and we are done.

Modifying the test function if necessary (as in Remark 4.3, assuming also $\psi(y) = 1$ for $|y| > 1$), we may assume the maximum is unique, strict, and can not be attained at infinity. In fact, we may assume that

$$(\bar{u} - \phi)(x) > \sup_{|y-x| > r} (\bar{u} - \phi)(y) \quad \text{for any } r > 0. \quad (4.10)$$

Then we take a subsequence such that $\bar{u}(x) = \lim_\varepsilon u_\varepsilon(x_\varepsilon)$, and note that by (4.10) and classical arguments [12, Lemma V.1.6], we may find a sequence $\{y_\varepsilon\}_\varepsilon$ such that

$$u_\varepsilon - \phi \quad \text{has a global maximum at } y_\varepsilon, \quad y_\varepsilon \rightarrow x, \quad \text{and} \quad u_\varepsilon(y_\varepsilon) \rightarrow \bar{u}(x).$$

Since u_ε is a subsolution of (4.6),

$$F_\varepsilon(u_\varepsilon(y_\varepsilon), D\phi(y_\varepsilon), l_\varepsilon) \leq f_\varepsilon(y_\varepsilon) \quad \text{where} \quad l_\varepsilon = L_\varepsilon[\phi, D\phi](y_\varepsilon).$$

By the construction of y_ε and the assumption of the Lemma,

$$\liminf_\varepsilon l_\varepsilon \leq L[\phi, D\phi](x). \quad (4.11)$$

Hence if we take a further subsequence in ε such that $l_\varepsilon \rightarrow \liminf_\varepsilon l_\varepsilon$, then by the definition of (\bar{f}, \underline{F}) and continuity, **(F1)** and **(F4)**,

$$\underline{F}(\bar{u}(x), D\phi(x), \liminf_\varepsilon l_\varepsilon) \leq \bar{f}(x).$$

and inequality (4.4) then follows from (4.11) and monotonicity **(F1)**. \square

Proof of Lemma 4.6. (a) Let ϕ be C^2 and bounded and $x_\varepsilon \rightarrow x$. By assumptions **(M)** and **(J1)**, and the dominated convergence theorem,

$$\lim_{x_\varepsilon \rightarrow x} L[\phi, D\phi](x_\varepsilon) = L[\phi, D\phi](x).$$

Hence by Lemma 4.5, $\bar{u}(x) = \limsup_{y \rightarrow x, n \rightarrow \infty} u_{a_n}(y)$ is a (bounded) subsolution of (1.4). Since u_{a_n} is continuous and $u_{a_n} \rightarrow u$ locally uniformly, it follows that $\bar{u} = u$ and u is continuous.

(b) The proof is similar to the proof of Lemma 4.5 and we only do the subsolution case. Assume $u - \phi$ has a strict global max at x . By the definition of the supremum, there is a sequence $u_{a_k}(x_k) \rightarrow u(x)$ as $k \rightarrow \infty$. As in the previous proof we may find a sequence $\{y_k\}_k$ such that

$$u_{a_k} - \phi \text{ has a global maximum at } y_k, \quad y_k \rightarrow x, \quad \text{and} \quad u_{a_k}(y_k) \rightarrow u(x).$$

Since u_{a_k} is a subsolution of (1.4),

$$F(u_{a_k}(y_k), D\phi(y_k), l_k) \leq f(y_k) \quad \text{where} \quad l_k = L[\phi, D\phi](y_k).$$

By the construction of y_k , assumptions **(M)** and **(J1)**, and the dominated convergence theorem,

$$\lim_k l_k = L[\phi, D\phi](x),$$

and then by the continuity, **(F1)** and **(F4)**, inequality (4.4) holds. \square

5. PROOFS OF THE MAIN RESULTS

5.1. Proof of Theorem 2.1 (comparison).

Proof of Theorem 2.1-(a). We proceed by contradiction, assuming that $M := \sup(u - v) > 0$.

Let $\varepsilon, R > 0$ and define

$$\Phi_{\varepsilon, R}(x, y) := u(x) - v(y) - \phi(x, y), \tag{5.1}$$

where

$$\phi(x, y) = \frac{1}{\varepsilon^2} \varphi(x - y) + \psi\left(\frac{x}{R}\right) + \psi\left(\frac{y}{R}\right), \tag{5.2}$$

and φ, ψ are smooth bounded radially symmetric and radially non-decreasing functions such that

$$\varphi(x) = \begin{cases} |x|^2 & \text{for } |x| < 1 \\ 2 & \text{for } |x| > 4 \end{cases} \quad \text{and} \quad \psi(x) = \begin{cases} 0 & \text{for } |x| < \frac{1}{2} \\ 2(\|u\|_\infty + \|v\|_\infty) + 1 & \text{for } |x| > 1 \end{cases}$$

By penalization (the ψ -terms) the supremum of $\Phi_{\varepsilon, R}$ is attained at a point (\bar{x}, \bar{y}) , and since $M > 0$ this supremum is positive when R is big enough (see **1**) below:

$$M_{\varepsilon, R} := \max \Phi_{\varepsilon, R} = \Phi(\bar{x}, \bar{y}) > 0.$$

For the sake of simplicity we drop the reference to ε, R for the maximum point. By the inequality $\Phi(\bar{x}, \bar{x}) + \Phi(\bar{y}, \bar{y}) \leq 2\Phi(\bar{x}, \bar{y})$, it follows that $\frac{2}{\varepsilon^2} \varphi(\bar{x} - \bar{y}) \leq u(\bar{x}) - u(\bar{y}) + v(\bar{x}) - v(\bar{y})$, and hence

$$\varphi(\bar{x} - \bar{y}) \leq (\|u\|_\infty + \|v\|_\infty) \varepsilon^2. \tag{5.3}$$

By taking $\varepsilon > 0$ small enough, we can always assume that

$$\varphi(\bar{x} - \bar{y}) = |\bar{x} - \bar{y}|^2 \quad \text{and} \quad (D\varphi)(\bar{x} - \bar{y}) = 2(\bar{x} - \bar{y}).$$

In particular, $|\bar{x} - \bar{y}| \leq (\|u\|_\infty + \|v\|_\infty)\varepsilon$ and this estimate is independent of R .

From the maximum of $\Phi_{\varepsilon,R}$ it follows that $u(x) - \phi(x, \bar{y})$ has a global maximum point at \bar{x} and $v(y) - (-\phi)(\bar{x}, y)$ has a global minimum point at \bar{y} . Subtracting the corresponding viscosity inequalities for u and v (cf. Definition 4.1) gives for any $\delta > 0$ that

$$\begin{aligned} 0 &\geq - \left(L_\delta[\phi(\cdot, \bar{y}), D_x\phi](\bar{x}) - L_\delta[(-\phi)(\bar{x}, \cdot), D_y(-\phi)](\bar{y}) \right) \\ &\quad - \left(L^\delta[u, D_x\phi](\bar{x}) - L^\delta[v, D_y(-\phi)](\bar{y}) \right) \\ &\quad - \left(L_2[u, D_x\phi](\bar{x}) - L_2[v, D_y(-\phi)](\bar{y}) \right) \\ &\quad - (f(\bar{x}) - f(\bar{y})) + (u(\bar{x}) - v(\bar{y})) \\ &\geq -I_\delta - I^\delta - I_2 - \omega_f(\bar{x} - \bar{y}) + (u(\bar{x}) - v(\bar{y})). \end{aligned} \tag{5.4}$$

The strategy is now to estimate I_δ , I^δ , and I_2 , and prove that when sending first $\delta \rightarrow 0$, then $R \rightarrow \infty$, and finally $\varepsilon \rightarrow 0$,

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{R \rightarrow \infty} \limsup_{\delta \rightarrow 0} (I_\delta + I^\delta + I_2) \leq 0.$$

We will also show that

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{R \rightarrow \infty} \limsup_{\delta \rightarrow 0} (u(\bar{x}) - v(\bar{y})) \geq M, \tag{5.5}$$

and hence by the viscosity inequality (5.4) we get the contradiction that concludes the proof:

$$0 \geq M.$$

We proceed in 4 steps:

1) We show that (5.5) holds. First note that $u(\bar{x}) - v(\bar{y}) = M_{\varepsilon,R} + \phi(\bar{x}, \bar{y})$ does not depend on δ . Then by the maximum point property, it follows that

$$M_{\varepsilon,R} \rightarrow M_\varepsilon := \sup \left(u(x) - v(y) - \frac{1}{\varepsilon^2} \varphi(x - y) \right) \quad \text{and} \quad \psi\left(\frac{\bar{x}}{R}\right) + \psi\left(\frac{\bar{y}}{R}\right) \rightarrow 0$$

as $R \rightarrow \infty$ (see Lemma 2.3 in [25]). Observe now that $M \leq M_\varepsilon \leq M_{\varepsilon'}$ for $\varepsilon \leq \varepsilon'$, and hence by monotone convergence, $M_\varepsilon \searrow \tilde{M}$ for some $\tilde{M} \geq M$. Since $\tilde{M} = \limsup_{\varepsilon \rightarrow 0} \limsup_{R \rightarrow \infty} \limsup_{\delta \rightarrow 0} (u(\bar{x}) - v(\bar{y}))$, we are done.

2) To estimate the I_δ -term, we Taylor expand to find that

$$\begin{aligned} &\int_{|z| < \delta} \phi(\bar{x} + j_1(D_x\phi(\bar{x}, \bar{y}), z), \bar{y}) - \phi(\bar{x}, \bar{y}) - j_1(D_x\phi(\bar{x}, \bar{y}), z) \cdot D_x\phi(\bar{x}, \bar{y}) \, d\mu_1(z) \\ &\quad \leq \|D^2\phi\|_\infty \int_{|z| < \delta} |j_1(D_x\phi(\bar{x}, \bar{y}), z)|^2 \, d\mu_1(z) = o_\delta(1) \end{aligned}$$

for fixed $\varepsilon, R > 0$. Here the $o_\delta(1)$ comes from assumption **(J1)** and dominated convergence as $\delta \rightarrow 0$. After a similar estimate for $L_\delta[-\phi, D_y(-\phi)]$, we conclude that $I_\delta \rightarrow 0$ as $\delta \rightarrow 0$ and $\varepsilon, R > 0$ are fixed.

3) We estimate I^δ . Using the notation $j_{\bar{x}}(z) := j_1(D_x\phi(\bar{x}, \bar{y}), z)$ and $j_{\bar{y}}(z) := j_1(D_y(-\phi)(\bar{x}, \bar{y}), z)$, and the maximum point property of $\Phi_{\varepsilon, R}$,

$$\Phi_{\varepsilon, R}(\bar{x} + j_{\bar{x}}, \bar{y} + j_{\bar{y}}) \leq \Phi_{\varepsilon, R}(\bar{x}, \bar{y}),$$

we see that

$$\begin{aligned} I^\delta &= \int_{\delta \leq |z|} \left(\left(u(\bar{x} + j_{\bar{x}}(z)) - u(\bar{x}) \right) - \left(v(\bar{y} + j_{\bar{y}}(z)) - v(\bar{y}) \right) \right. \\ &\quad \left. + j_{\bar{x}}(z) \cdot D_x\phi(\bar{x}, \bar{y}) + j_{\bar{y}}(z) \cdot D_y\phi(\bar{x}, \bar{y}) \right) d\mu_1(z) \\ &\leq \int_{\delta \leq |z|} \left(\phi(\bar{x} + j_{\bar{x}}(z), \bar{y} + j_{\bar{y}}(z)) - \phi(\bar{x}, \bar{y}) - D_x\phi(\bar{x}, \bar{y}) \cdot j_{\bar{x}}(z) - D_y\phi(\bar{x}, \bar{y}) \cdot j_{\bar{y}}(z) \right) d\mu_1(z). \end{aligned}$$

Since $D^2\varphi$ is bounded and $|D^2\psi(\frac{x}{R})| \leq \frac{1}{R^2}\|D^2\psi\|_\infty < \infty$, a short computation using Taylor expansions shows that

$$I^\delta \leq \int_{\delta \leq |z|} \left(\frac{1}{2\varepsilon^2}\|D^2\varphi\|_\infty |j_{\bar{x}}(z) - j_{\bar{y}}(z)|^2 + \frac{1}{2R^2}\|D^2\psi\|_\infty (|j_{\bar{x}}(z)|^2 + |j_{\bar{y}}(z)|^2) \right) d\mu_1(z).$$

To proceed we compute the gradients,

$$D_x\phi(\bar{x}, \bar{y}) = p_\varepsilon + \frac{1}{R}D\psi\left(\frac{\bar{x}}{R}\right), \quad D_y(-\phi)(\bar{x}, \bar{y}) = p_\varepsilon - \frac{1}{R}D\psi\left(\frac{\bar{y}}{R}\right), \quad p_\varepsilon = \frac{2(\bar{x} - \bar{y})}{\varepsilon^2},$$

and note that for fixed $\varepsilon > 0$, they are uniformly bounded for $R > 1$ by estimate (5.3). Hence, there is $r_\varepsilon > 0$ such that $|D\phi(\bar{x}, \bar{y})| \leq r_\varepsilon$ for all $\delta > 0$ and $R > 1$, and then by assumptions **(J1)** and **(J2)**,

$$\begin{aligned} I^\delta &\leq O\left(\frac{1}{\varepsilon^2}\right)\omega_{j, r_\varepsilon}(D_x\phi(\bar{x}, \bar{y}) - D_y(-\phi)(\bar{x}, \bar{y})) + O\left(\frac{1}{R^2}\right)C_{j, r_\varepsilon} \\ &\leq O\left(\frac{1}{\varepsilon^2}\right)\omega_{j, r_\varepsilon}\left(O\left(\frac{1}{R}\right)\right) + O\left(\frac{1}{R^2}\right)C_{j, r_\varepsilon}. \end{aligned}$$

We first send $\delta \rightarrow 0$ since nothing depends on δ on the right-hand side, and then we send $R \rightarrow \infty$ and find that

$$\limsup_{R \rightarrow \infty} \limsup_{\delta \rightarrow 0} I^\delta \leq 0.$$

4) Finally, we estimate I_2 . First note that by the maximum point property, the positivity of ϕ , the calculations of gradients in (c), and estimate (5.3),

$$\begin{aligned} I_2 &\leq \int_{|z|>0} \phi\left(\bar{x} + j_2\left(p_\varepsilon + \frac{1}{R}D\psi\left(\frac{\bar{x}}{R}\right), z\right), \bar{y} + j_2\left(p_\varepsilon - \frac{1}{R}D\psi\left(\frac{\bar{y}}{R}\right), z\right)\right) - \phi(\bar{x}, \bar{y}) \, d\mu_2(z) \\ &\leq \frac{1}{\varepsilon^2} \int_{|z|>0} \sup_{x, y \in \mathbb{R}^N} \left\{ \varphi\left(x + j_2\left(p_\varepsilon + \frac{1}{R}D\psi\left(\frac{x}{R}\right), z\right) - \left(y + j_2\left(p_\varepsilon - \frac{1}{R}D\psi\left(\frac{y}{R}\right), z\right)\right) \right. \right. \\ &\quad \left. \left. - \varphi(x - y) \right\} d\mu_2(z) \\ &\quad + \int_{|z|>0} \psi\left(\frac{\bar{x} + j_2\left(p_\varepsilon + \frac{1}{R}D\psi\left(\frac{\bar{x}}{R}\right), z\right)}{R}\right) + \psi\left(\frac{\bar{y} + j_2\left(p_\varepsilon - \frac{1}{R}D\psi\left(\frac{\bar{y}}{R}\right), z\right)}{R}\right) \, d\mu_2(z) \\ &:= J_1 + J_2. \end{aligned}$$

Now we send $R \rightarrow \infty$ in J_1 . Then by compactness, p_ε will up to a subsequence converge to a limit that we also call p_ε . By the boundedness of $D\psi$ and p -continuity of $j_2(p, z)$ for a.e. z in **(J1)**,

$$\lim_{R \rightarrow \infty} \sup_{x \in \mathbb{R}^N} \left| j_2\left(p_\varepsilon \pm \frac{1}{R}D\psi\left(\frac{x}{R}\right), z\right) - j_2(p_\varepsilon, z) \right| = 0 \quad \text{for a.e. } z.$$

Hence since φ is a Lipschitz continuous function,

$$\begin{aligned} & \sup_{x,y \in \mathbb{R}^N} \left| \varphi\left(x - y - j_2\left(p_\varepsilon + \frac{1}{R}D\psi\left(\frac{x}{R}\right), z\right) + j_2\left(p_\varepsilon - \frac{1}{R}D\psi\left(\frac{y}{R}\right), z\right)\right) - \varphi(x - y) \right| \\ & \leq \|D\varphi\|_\infty \sup_{x,y \in \mathbb{R}^N} \left| j_2\left(p_\varepsilon + \frac{1}{R}D\psi\left(\frac{x}{R}\right), z\right) - j_2\left(p_\varepsilon - \frac{1}{R}D\psi\left(\frac{y}{R}\right), z\right) \right| \rightarrow 0 \end{aligned}$$

as $R \rightarrow \infty$ for a.e. z . Hence, the J_1 -integrand is a uniformly bounded function converging to 0 as $R \rightarrow \infty$ for a.e. z . Hence by the dominated convergence theorem (for fixed ε),

$$\limsup_{R \rightarrow \infty} J_1 \leq 0.$$

Now we send $R \rightarrow \infty$ in J_2 . Here we use the fact that

$$\psi\left(\frac{\bar{x}}{R}\right) + \psi\left(\frac{\bar{y}}{R}\right) \rightarrow 0 \quad \text{as} \quad R \rightarrow \infty,$$

which is a simple consequence of the maximum point property (see Lemma 2.3 in [25]). Since j_2 is locally bounded for a.e. fixed z by **(J1)** and ψ is continuous,

$$\frac{j_2\left(p_\varepsilon + \frac{1}{R}D\psi\left(\frac{\bar{y}}{R}\right), z\right)}{R} \rightarrow 0 \quad \text{for a.a. } z$$

as $R \rightarrow \infty$, and hence

$$\psi\left(\frac{\bar{x} + j_2\left(p_\varepsilon + \frac{1}{R}D\psi\left(\frac{\bar{x}}{R}\right), z\right)}{R}\right) \rightarrow \psi(0) = 0 \quad \text{for a.a. } z$$

as $R \rightarrow \infty$. Since ψ is bounded, we can use the dominated convergence theorem to conclude that

$$\limsup_{R \rightarrow \infty} J_2 = 0.$$

Since I_2 is independent of δ , we can now conclude that

$$\limsup_{R \rightarrow \infty} \limsup_{\delta \rightarrow 0} I_2 \leq 0,$$

and the proof is complete. \square

Proof of Theorem 2.1-(b). Part of the proof is similar to the previous proof. We start by assuming that $M := \sup\{u(x) - v(x)\} > 0$ and consider the maximum $M_{\varepsilon,R}$ of

$$\Phi_{\varepsilon,R}(x, y) := u(x) - v(y) - \phi(x, y),$$

where ϕ was defined in the proof of Theorem 2.1-(a). Since $M_{\varepsilon,R} \rightarrow M_\varepsilon \geq M$ as $R \rightarrow \infty$, we may assume that $M_{\varepsilon,R} \geq M/2$ and $u(\bar{x}) > v(\bar{y})$. Since $u - \phi(\cdot, \bar{y})$ has a global max in \bar{x} and $v - (-\phi)(\bar{x}, \cdot)$ has a global min in \bar{y} , we subtract the corresponding viscosity inequalities and find that

$$\begin{aligned} & F\left(u(\bar{x}), p_\varepsilon + O\left(\frac{1}{R}\right), \underbrace{L_\delta[\phi(\cdot, \bar{y}), D_x\phi](\bar{x}) + L^\delta[u, D_x\phi](\bar{x}) + L_2[u, D_x\phi](\bar{x})}_{I_x}\right) \\ & - F\left(v(\bar{y}), p_\varepsilon + O\left(\frac{1}{R}\right), \underbrace{L_\delta[-\phi(\bar{x}, \cdot), D_y(-\phi)](\bar{y}) + L^\delta[v, D_y(-\phi)](\bar{y}) + L_2[v, D_y(-\phi)](\bar{y})}_{I_y}\right) \quad (5.6) \\ & \leq f(\bar{x}) - f(\bar{y}), \end{aligned}$$

where $p_\varepsilon = 2(\bar{x} - \bar{y})/\varepsilon$.

We now estimate the different terms. By the estimates in the proof of Theorem 2.1-(a),

$$\begin{aligned} \left| L_\delta[\phi(\cdot, \bar{y}), D_x\phi](\bar{x}) - L_\delta[\phi(\bar{x}, \cdot), D_y(-\phi)](\bar{y}) \right| &= \frac{1}{\varepsilon^2} o_\delta(1), \\ \left| L^\delta[u, D_x\phi](\bar{x}) - L^\delta[v, D_y(-\phi)](\bar{y}) \right| + \left| L_2[u, D_x\phi](\bar{x}) - L_2[v, D_y(-\phi)](\bar{y}) \right| &= \left(1 + \frac{1}{\varepsilon^2}\right) o_R(1), \end{aligned}$$

and hence

$$|I_x - I_y| = \left(1 + \frac{1}{\varepsilon^2}\right) (o_R(1) + o_\delta(1)).$$

By the order we will take the limits, we may and will always assume that terms on the right hand sides are bounded (by 1 for example). Moreover, by the estimates in the proof of Theorem 2.1-(a),

$$\begin{aligned} |D\phi(\bar{x}, \bar{y})| &\leq C\left(\frac{1}{\varepsilon} + \frac{1}{R}\right), \\ |D^2\phi(\bar{x}, \bar{y})| &\leq C\left(\frac{1}{\varepsilon^2} + \frac{1}{R^2}\right), \\ |L_\delta[\phi(\cdot, \bar{y}), D_x\phi](\bar{x})| &\leq \|D^2\phi\|_\infty \int_{|z|<\delta} |j_1(D_x\phi(\bar{x}, \bar{y}), z)|^2 d\mu_1(z), \\ |L_\delta[-\phi(\bar{x}, \cdot), D_y(-\phi)](\bar{y})| &\leq \|D^2\phi\|_\infty \int_{|z|<\delta} |j_1(D_y(-\phi)(\bar{x}, \bar{y}), z)|^2 d\mu_1(z), \\ |L_2[u, D_x\phi](\bar{x})| + |L_2[v, D_y(-\phi)](\bar{y})| &\leq 2(\|u\|_\infty \vee \|v\|_\infty) \mu_2(\mathbb{R}^N), \end{aligned}$$

and by the maximum point property,

$$\begin{aligned} L^\delta[u, D_x\phi](\bar{x}) &\leq L^\delta[\phi(\cdot, \bar{y}), D_x\phi(\bar{x}, \bar{y})](\bar{x}) \\ &\leq \|D_x^2\phi\|_\infty \int_{|z|>0} |j(D_x\phi(\bar{x}, \bar{y}), z)|^2 d\mu_1(z) - \|D_y^2\phi\|_\infty \int_{|z|>0} |j(D_y(-\phi)(\bar{x}, \bar{y}), z)|^2 d\mu_1(z) \\ &\leq L^\delta[v, D_y(-\phi)](\bar{y}). \end{aligned}$$

If $\varepsilon > 0$ is fixed, then by **(M)** and **(J1)**, these terms are uniformly bounded for $R > 1$ and $\delta > 0$. These and the previous bounds then implies that there is a $C_\varepsilon > 0$ such that

$$-C_\varepsilon \leq L^\delta[v, D_y(-\phi)](\bar{y}) \leq L^\delta[u, D_x\phi](\bar{x}) + |L^\delta[v, D_y(-\phi)](\bar{y}) - L^\delta[u, D_x\phi](\bar{x})| \leq C_\varepsilon,$$

and similarly we can show that $|L^\delta[u, D_x\phi](\bar{x})| \leq C_\varepsilon$. Hence there is $r_\varepsilon > 0$ such that

$$|D\phi(\bar{x}, \bar{y})| + |I_x| + |I_y| \leq r_\varepsilon \quad \text{for all } R > 1, \delta > 0.$$

By (5.6) and the previous estimates, **(F2)**, **(F3)**, **(F4)**, we see

$$\begin{aligned} \frac{\gamma M}{2} &\leq F(u(\bar{x}), p_\varepsilon - O(\frac{1}{R}), I_y) - F(u(\bar{x}), p_\varepsilon + O(\frac{1}{R}), I_x) + f(\bar{x}) - f(\bar{y}) \\ &\leq \omega_{\|u\|_\infty \vee \|v\|_\infty, r_\varepsilon} \left(O(\frac{1}{R}) + \left(1 + \frac{1}{\varepsilon^2}\right) (o_R(1) + o_\delta(1)) \right) + o_\varepsilon(1). \end{aligned}$$

Sending first $\delta \rightarrow 0$, then $R \rightarrow \infty$, and finally $\varepsilon \rightarrow 0$, we get again $M \leq 0$. This is a contradiction and the result follows. \square

5.2. Proof of Theorem 2.3 (existence). A major challenge we face when we want to prove existence, is the implicit nature of equation (1.4) with a gradient dependence inside the j functions. It seems non-trivial to use Perron's method for such equations, and since fixed point iterations require convergence of the full sequence to get the equation in the limit, compactness argument (yielding subsequences) can not work. We have been able to overcome the problem by a nontrivial approximation procedure, which is inspired by the "Sirtaki method" of [4], along with a fixed point argument using Schauder's fixed point theorem. We start by proving existence for an approximate problem in a bounded set, and then pass to the limit using the method of half-relaxed limits and strong comparison of the limit equation.

We begin with the linear case (1.7). The simple adaptations for the general case are given at the end of this section. Consider now the following approximate problem: find $u \in C^2(\overline{B}_R)$ such that

$$\begin{cases} u - T_M \left[L_k^R[u, Du] \right] - \varepsilon \Delta u = f(x), & x \in B_R(0), \\ u = 0, & x \in \partial B_R(0), \end{cases} \quad (5.7)$$

where $L_k^R[v, Dv] = L_{1,k}^R[v, Dv] + L_{2,k}^R[v, Dv]$ for

$$\begin{aligned} L_{1,k}^R[v, Dv](x) &= \int_{\mathbb{R}^P} v \left(P_R(x + j_1(Dv(x), z)) \right) - v(x) - Dv(x) \cdot j_1(Dv(x), z) \mathbb{1}_{|z| < 1} \, d\mu_{1,k}(z), \\ L_{2,k}^R[v, Dv](x) &= \int_{\mathbb{R}^P} v \left(P_R(x + j_2(Dv(x), z)) \right) - v(x) \, d\mu_{2,k}(z), \end{aligned}$$

T_M is a truncation and P_R the orthogonal projection onto \overline{B}_R ,

$$T_M[f] := \min(\max(f, -M), M), \quad P_R(x) := \begin{cases} x & \text{if } |x| \leq R, \\ \frac{R}{|x|}x & \text{if } |x| > R, \end{cases}$$

and the measures

$$\mu_{1,k} := \rho_k * (\mu_1 \cdot \mathbb{1}_{1/k < |z| < k}) \quad \text{and} \quad \mu_{2,k} := \rho_k * \mu_2,$$

for a mollifier $\rho_k(z) = k^P \rho(kz)$, $0 \leq \rho \in C^\infty(\mathbb{R}^P)$ is symmetric with support in B_1 and $\int \rho = 1$.

Remark 5.1. (i) The truncated mollified measures $\mu_{1,k}$ and $\mu_{2,k}$ are absolutely continuous with respect to the Lebesgue measure with bounded densities, see Lemma 5.3 below.

(ii) From the definition it follows that

$$|P_R(x) - P_R(y)| \leq |x - y| \quad \text{for all } x, y \in \mathbb{R}^N.$$

The projection allows us to look for solutions that are defined only in \overline{B}_R and not in all of \mathbb{R}^N as in equation (1.4). The new nonlocal term is of Neumann-type, corresponding to jump processes that are projected back to the boundary of the domain immediately upon leaving it (cf.[4, 5]).

To prove existence we first strengthen the assumptions on j and f , later we do the general case.

(J1') $j_1(p, z)$ and $j_2(p, z)$ are Borel measurable, locally bounded, continuous in p for a.e. z , and for every $r > 0$ there is C_r such that for all $|p| \leq r$ and $|z| < 1$,

$$|j_1(p, z)| \leq C_r |z|$$

(J2') for any $K > 0$, there exists $C = C(K)$ such that for any $|p|, |q|, |z| \leq K$,

$$|j_i(p, z) - j_i(q, z)| \leq C |p - q|.$$

(F5') $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is bounded and Lipschitz continuous.

Remark 5.2. For any $x \in B_R$ and $u \in C^2(\overline{B}_R)$, all terms in (5.7) are well-defined and the equation holds in the classical sense (since $\mu_{i,k}$ are bounded and P_R is continuous, the integral terms are well-defined because of **(J1')**). Note that **(J1')** and **(M)** implies **(J1)**, while μ_1 bounded and compactly supported and **(J2')** implies **(J2)**.

We state the properties of $\mu_{i,k}$ that we will need later. The proof is given in Appendix B.

Lemma 5.3. *Assume **(M)** holds and $\delta > 0$.*

(a) *The measures $\mu_{1,k}$ and $\mu_{2,k}$ have densities $\bar{\mu}_{1,k}$ and $\bar{\mu}_{2,k}$ with respect to the Lebesgue measure on \mathbb{R}^N such that*

$$\|\bar{\mu}_{1,k}\|_\infty, \|\bar{\mu}_{2,k}\|_\infty < \infty \quad \text{and} \quad \int_{|z|<\delta} |z|^2 \bar{\mu}_{1,k}(z) dz \leq 4 \int_{|z|<\delta} |z|^2 \mu_1(dz).$$

From now on, let (ψ_k) be a sequence of functions and $C > 0$ constants independent of p, z that differ from line to line.

(b) *If $\|\psi_k\|_\infty \leq C$ for all k , and $\sup_{\delta \leq |z| < K} |\psi_k(z) - \psi(z)| \xrightarrow{k \rightarrow \infty} 0$ for any $K > \delta$, then*

$$\int_{|z| \geq \delta} \psi_k(z) d\mu_{1,k}(z) \xrightarrow{k \rightarrow \infty} \int_{|z| \geq \delta} \psi(z) \mu_1(dz).$$

(c) *If $|\psi_k(z)| \leq C|z|^2$ for all k, z , and $\sup_{|z| < \delta} |\psi_k(z) - \psi(z)| \xrightarrow{k \rightarrow \infty} 0$, then*

$$\int_{0 < |z| < \delta} \psi_k(z) d\mu_{1,k}(z) \xrightarrow{k \rightarrow \infty} \int_{0 < |z| < \delta} \psi(z) \mu_1(dz).$$

(d) *If $\|\psi_k\|_\infty \leq C$ for all k , and $\sup_{|z| < K} |\psi_k(z) - \psi(z)| \xrightarrow{k \rightarrow \infty} 0$ for any $K > 0$, then*

$$\int_{|z| > 0} \psi_k(z) d\mu_{2,k}(z) \xrightarrow{k \rightarrow \infty} \int_{|z| > 0} \psi(z) \mu_2(dz).$$

We also need the following results.

Lemma 5.4. *Assume **(M)**, **(J1')**, **(J2')** and let $v \in C^{1,\theta}(\overline{B}_R)$ for some $\theta \in (0, 1)$. Then*

(a) *the function $x \mapsto L_k^R[v, Dv](x)$ belongs to $C^{0,\theta}(\overline{B}_R)$;*

(b) *if $v_n \rightarrow v$ in $C^{1,\theta}(\overline{B}_R)$, then $L_k^R[v_n, Dv_n] \rightarrow L_k^R[v, Dv]$ in $C^{0,\theta}(\overline{B}_R)$.*

Proof. (a) We only do the proof for the L_1 -term since the L_2 -case is similar but easier. Below $x, y \in \overline{B}_R$, $\frac{1}{k} < |z| < k$ (the support of $\mu_{1,k}$), and C_* will denote all constants (that may vary from line to line) depending only on $R, k, \varepsilon, v, j_i, \mu_i, N$. Since v is $C^{1,\theta}$,

$$|v(x)| + |Dv(x)| + \frac{|Dv(x) - Dv(y)|}{|x - y|^\theta} \leq \|v\|_{C^{1,\theta}}.$$

Then, $|v(x) - v(y)| \leq C_* |x - y|$ and by Lipschitz continuity of P_R and assumption **(J2')**,

$$\begin{aligned} \left| v \left(P_R(x + j_1(Dv(x), z)) \right) - v \left(P_R(y + j_1(Dv(y), z)) \right) \right| &\leq C_* (|x - y| + |Dv(x) - Dv(y)|), \\ \left| Dv(x) \cdot j_1(Dv(x), z) - Dv(y) \cdot j_1(Dv(y), z) \right| &\leq C_* |Dv(x) - Dv(y)|. \end{aligned}$$

Thus, all these quantities are controlled by $C_*|x - y|^\theta$. Since the measure $\mu_{1,k}$ is bounded and supported in $\frac{1}{k} < |z| < k$, it then follows that

$$|L_{1,k}^R[v, Dv](x) - L_{1,k}^R[v, Dv](y)| \leq C_*|x - y|^\theta \mu_{1,k}(\mathbb{R}^N),$$

and the proof of (a) is complete.

(b) By assumption

$$|v_n(x) - v(y)| + \frac{|v_n(x) - v(y)|}{|x - y|} + \frac{|Dv_n(x) - Dv(y)|}{|x - y|^\theta} \leq \|v_n - v\|_{C^{1,\theta}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

By similar computations as above, we end up with

$$|L_k^R[v_n, Dv_n](x) - L_k^R[v, Dv](y)| \leq C_* \|v_n - v\|_{C^{1,\theta}} \mu_{1,k}(\mathbb{R}^N) |x - y|^\theta,$$

for a C_* independent of x, y, n . It follows that $L_k^R[v_n, Dv_n] \rightarrow L_k^R[v, Dv]$ in $C^{0,\theta}(\overline{B}_R)$ as $n \rightarrow \infty$. \square

We can now prove an existence result for the approximate problem (5.7).

Proposition 5.5. *Assume (M), (J1'), (J2'), (F5'), and let $\varepsilon, R, M > 0, k \in \mathbb{N}$. Then there exists a classical solution $u \in C^2(\overline{B}_R)$ of (5.7).*

Proof. The proof is based on Schauder's fixed point theorem (cf. e.g. [20, Corollary 11.2]).

1) Let $X := C^{1,\theta_0}(\overline{B}_R)$ for a fixed $\theta_0 \in (0, 1)$, and define

$$\mathcal{C} = \{w \in X : \|w\|_\infty \leq M + \|f\|_\infty\}.$$

Note that \mathcal{C} is a convex and closed subset of X . On \mathcal{C} we now define a map $\mathcal{T} = \mathcal{T}_{\varepsilon,R,M,k}$ in the following way: for every $v \in \mathcal{C}$, $u = \mathcal{T}(v)$ is the classical solution of the Dirichlet problem

$$\begin{cases} u - \varepsilon \Delta u = T_M \left[L_k^R[v, Dv] \right] + f(x) & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R. \end{cases} \quad (5.8)$$

When $v \in X$, $L_k^R[v, Dv] \in C^{0,\theta_0}$ by Lemma 5.4, and then by the definition of T_M and (F5'),

$$w := T_M \left[L_k^R[v, Dv] \right] + f(x) \in C^{0,\theta_0}(\overline{B}_R).$$

Since B_R is a smooth domain, classical results ([20, Corollary 6.9]) then tell us that there exists a unique classical solution $u \in C^{2,\theta_0}(\overline{B}_R) (\subset X)$ of (5.8). Moreover, by the maximum principle and the definition of T_M , $\|u\|_\infty \leq M + \|f\|_\infty$. We conclude that \mathcal{T} is a well-defined map from \mathcal{C} into \mathcal{C} .

2) We show that $\mathcal{T} : \mathcal{C} \rightarrow \mathcal{C}$ is continuous with respect to norm of $X = C^{1,\theta_0}(\overline{B}_R)$. Take a sequence $\{v_n\} \subset \mathcal{C}$ such that $v_n \rightarrow v$ in X . By subtracting the equations for u_n and u_p , we see that $w := u_n - u_p$ is a classical solution of

$$\begin{cases} w - \varepsilon \Delta w = T_M \left[L_k^R[v_n, Dv_n] \right] - T_M \left[L_k^R[v_p, Dv_p] \right] =: g_{n,p}, & \text{in } B_R, \\ w = 0, & \text{in } \partial B_R. \end{cases}$$

By the maximum principle, we then find that

$$\|w\|_{C^0(\overline{B}_R)} \leq \|g_{n,p}\|_{C^0(\overline{B}_R)},$$

and by standard C^{2,θ_0} -theory (e.g. [20, Thm 6.6]),

$$\|u_n - u_p\|_{C^{2,\theta_0}(\overline{B}_R)} = \|w\|_{C^{2,\theta_0}(\overline{B}_R)} \leq C \left(\|w\|_{C^0(\overline{B}_R)} + \|g_{n,p}\|_{C^{0,\theta_0}(\overline{B}_R)} \right).$$

Hence, since $T_M[L_k^R[v_n, Dv_n]] \rightarrow T_M[L_k^R[v, Dv]]$ in $C^{0,\theta_0}(\overline{B}_R)$ by Lemma 5.4 (b), it follows that (u_n) is a Cauchy sequence in $C^{2,\theta_0}(\overline{B}_R)$ and hence also in X . By completeness, the limit u exists and belongs to $C^{2,\theta_0}(\overline{B}_R) \cap \mathcal{C}$ since \mathcal{C} is closed in X .

By the C^{2,θ_0} convergence of u_n , the C^{1,θ_0} convergence of v_n , and Lemma 5.4, we can pass to the limit in the equation to see that $u = \mathcal{T}(v)$. It follows that $\mathcal{T}(v_n) \rightarrow \mathcal{T}(v)$ in X , and we conclude that \mathcal{T} is continuous in \mathcal{C} .

3) We now show that $\mathcal{T}(\mathcal{C})$ is relatively compact. Take any sequence $\{u_n\} \subset \mathcal{T}(\mathcal{C})$. Then there exists a sequence $\{v_n\} \subset \mathcal{C}$ such that $u_n = \mathcal{T}(v_n)$. It follows that

$$u_n - \varepsilon \Delta u_n = g_n,$$

where $|g_n| \leq M + \|f\|_\infty$. By the maximum principle and $W^{2,p}$ -theory (e.g. [20, Thm 9.11]), we have the following two a priori estimates for any $1 < p < \infty$,

$$\|u_n\|_{L^\infty(B_R)} \leq M + \|f\|_\infty,$$

$$\|u_n\|_{W^{2,p}(B_R)} \leq C \left(\|u_n\|_{L^p(B_R)} + \|g_n\|_{L^p(B_R)} \right) \leq 2C|B_R|^{1/p} (M + \|f\|_\infty).$$

By compact embeddings of Sobolev spaces [20, Thm 7.26], we can extract a subsequence $u_{\varphi(n)}$ converging in $C^{1,\theta}(\overline{B}_R)$ for any $\theta \in (0, 1)$, in particular for $\theta = \theta_0$. This provides a subsequence which converges in X , and proves the claim.

4) By Schauder's fixed point theorem, there exists a function $u \in \mathcal{C}$ such that $u = \mathcal{T}(u)$, which means that we have a $C^2(\overline{B}_R)$ -solution of (5.7) and the proof is complete. \square

We proceed to prove existence under the restrictive assumptions **(J1')** and **(J2')**. We will need the following result.

Lemma 5.6. *Assume **(M)**, **(J1')**, $f, g \in C^0(\overline{B}_R)$. Let u and v be $C^2(\overline{B}_R)$ solutions of*

$$u - T_M[L_k^R[u, Du]] - \varepsilon \Delta u \leq f(x) \quad \text{and} \quad v - T_M[L_k^R[v, Dv]] - \varepsilon \Delta v \geq g(x) \quad \text{in } B_R.$$

If $u \leq v$ on ∂B_R and $f \leq g$ in B_R , then $u \leq v$ in \overline{B}_R .

Remark 5.7. The result is a comparison result for smooth sub and supersolutions of (5.7). It implies uniqueness of classical solutions of (5.7).

Proof. Let $w := u - v$ and $\bar{m} := \max_{\overline{B}_R} w$. If this max is attained at $x_0 \in \partial B_R$, since $u \leq v$ there, we have $\bar{m} \leq 0$. Otherwise, there is an interior point $x_0 \in B_R$ such that $m = w(x_0)$. By assumption,

$$w(x_0) - \left\{ T_M[L_k^R[u, Du]](x_0) - T_M[L_k^R[v, Dv]](x_0) \right\} - \varepsilon \Delta w(x_0) \leq (f - g)(x_0).$$

Since x_0 is a maximum point, and u, v are smooth, $Dw(x_0) = (Du - Dv)(x_0) = 0$, $\Delta w(x_0) \geq 0$, and $(u - v)(x_0) \geq (u - v)(y)$ for all $y \in \overline{B}_R$. By the latter inequality,

$$u\left(P_R(x_0 + j(Du(x_0), z))\right) - u(x_0) \leq v\left(P_R(x_0 + j(Dv(x_0), z))\right) - v(x_0),$$

and hence

$$L_k^R[u, Du](x_0) - L_k^R[v, Dv](x_0) \leq 0.$$

Since T_M is a non-decreasing function and $f \leq g$, we can conclude that

$$\bar{m} = w(x_0) \leq 0,$$

and the proof is complete because in either case, we get $\bar{m} \leq 0$. \square

Corollary 5.8. *If $u_{R,M,k,\varepsilon}$ is the solution of (5.7), then $\|u_{R,M,k,\varepsilon}\|_{L^\infty(B_R)} \leq \|f\|_{L^\infty(\mathbb{R}^N)}$.*

Proof. Follows from Lemma 5.6 with $-\|f\|_\infty/u/\|f\|_\infty$ as subsolution/solution/supersolution. \square

Proposition 5.9. *Assume **(M)**, **(J1')**, **(J2')**, **(J2)**, and **(F5')**. Then there exists a unique viscosity solution $u \in C_b(\mathbb{R}^N)$ of (1.7).*

Proof. Let $R > 0$, $k = M = 1/\varepsilon = n \in \mathbb{N}$, and $u_{R,n}$ be the corresponding solution of (5.7) given by Proposition 5.5. Using the ‘‘half relaxed limit’’ method, we first we send $R \rightarrow \infty$ and then $n \rightarrow \infty$ and show that we can obtain from $u_{R,n}$ a function u which is the viscosity solution of (1.7).

1) *Claim:* For every $n \in \mathbb{N}$, the functions

$$\bar{u}_n(x) := \limsup_{y \rightarrow x, R \rightarrow \infty} u_{R,n}(y) \quad \text{and} \quad \underline{u}_n(x) := \liminf_{y \rightarrow x, R \rightarrow \infty} u_{R,n}(y)$$

are bounded viscosity sub- and supersolutions respectively of

$$u - \frac{1}{n} \Delta u - T_n \left[L_n[u, Du] \right] = f \quad \text{in} \quad \mathbb{R}^N. \quad (5.9)$$

where L_k is defined as L in (1.1), but with the measure $\mu_{i,k}$ replacing μ_i for $i = 1, 2$. Moreover,

$$-\|f\|_\infty \leq \underline{u}_n(x) \leq \bar{u}_n(x) \leq \|f\|_\infty \quad \text{in} \quad \mathbb{R}^N.$$

Proof of Claim: First note that \bar{u}_n and \underline{u}_n are defined for every $x \in \mathbb{R}^N$ since $R \rightarrow \infty$, they are semicontinuous and $\underline{u}_n \leq \bar{u}_n$ by definition and bounded by $\|f\|_\infty$ by Corollary 5.8. We show that \bar{u} is a subsolution of (5.9) according to Definition 4.4. Take any bounded test function ϕ and any point $x \in \mathbb{R}^N$ such that $\bar{u} - \phi$ has a global maximum at x . We may as usual assume the maximum is strict. Then there exists a sequence $y_R \rightarrow x$ of maximum points of $u_{R,n} - \phi$ in \bar{B}_R , such that $u_{R,n}(y_R) \rightarrow \bar{u}(x)$. Take R big enough such that $R > |x|$ and $y_R \in B_R$ (since $y_R \rightarrow x$, y_R cannot be located on ∂B_R for R big enough). Since y_R is a maximum point and $u_{R,n}$ is smooth, $Du_{R,n}(y_R) = D\phi(y_R)$, $\Delta u_{R,n}(y_R) \leq \Delta\phi(y_R)$, and (cf. the proof of Lemma 5.6)

$$L_n^R[u_{R,n}, Du_{R,n}](y_R) \leq L_n^R[\phi, D\phi](y_R).$$

Since $u_{R,n}$ satisfies equation (5.7) at the point y_R , it then follows that

$$u_{R,n}(y_R) - \frac{1}{n} \Delta\phi(y_R) - T_n \left[L_n^R[\phi, D\phi](y_R) \right] \leq f(y_R). \quad (5.10)$$

By the boundedness of y_R , the regularity of ϕ , **(J1')**, and the definition of P_R ,

$$\phi \left(P_R(y_R + j_i(D\phi(y_R), z)) \right) \rightarrow \phi(x + j_i(D\phi(x), z)) \quad \text{as} \quad R \rightarrow \infty, \quad \text{for} \quad a.e. \quad z \quad \text{and} \quad i = 1, 2.$$

Hence, since this term is uniformly bounded, $\mu_{i,k}$ is bounded, and **(M)** holds, we can use the dominated convergence theorem to conclude that

$$L_n^R[\phi, D\phi](y_R) \rightarrow L_n[\phi, D\phi](x) \quad \text{as} \quad R \rightarrow \infty.$$

By the regularity of ϕ and the continuity of T_n , we can then pass to the limit as $R \rightarrow \infty$ in (5.10) and find that

$$\bar{u}(x) - \frac{1}{n} \Delta\phi(x) - T_n \left[L_n[\phi, D\phi](x) \right] \leq f(x).$$

We conclude that \bar{u} is a viscosity subsolution of (5.9), and in a similar way we can show that \underline{u} is viscosity supersolution of (5.9). The claim is proved.

2) We now pass to the limit as $n \rightarrow \infty$. We proceed as before, defining

$$\bar{u}(x) := \limsup_{y \rightarrow x, n \rightarrow \infty} \bar{u}_n(y) \quad \text{and} \quad \underline{u}(x) := \liminf_{y \rightarrow x, n \rightarrow \infty} \underline{u}_n(y),$$

where \bar{u}_n and \underline{u}_n are the uniformly bounded sub and supersolutions of (5.9) given by part 1). It immediately follows that $-\|f\|_\infty \leq \underline{u} \leq \bar{u} \leq \|f\|_\infty$.

We prove that \bar{u} is viscosity subsolution of (1.7). Again we take any smooth bounded test function ϕ and global strict maximum point x of $\bar{u} - \phi$ in \mathbb{R}^N . We have a sequence $y_n \rightarrow x$ of maximum points of $u_n - \phi$ such that $u_n(y_n) \rightarrow \bar{u}(x)$. Fix $\delta > 0$ and choose n big enough so that $1/n < \delta$. We split $L_{1,n}$ into $L_{1,n} = L_{\delta,n} + L_n^\delta$, where L_δ and L^δ are defined in the beginning of section 4. Since \bar{u} is a subsolution of (5.9), it follows that

$$u_n(y_n) - \frac{1}{n}\Delta\phi(y_n) - T_n[L_n[\phi, D\phi](y_n)] \leq f(y_n). \quad (5.11)$$

We pass to the limit in the different terms of this inequality. First, let

$$\psi_n(z) = \phi(y_n + j_1(D\phi(y_n), z)) - \phi(y_n) - D\phi(y_n) \cdot j_1(D\phi(y_n), z)\mathbb{1}_{|z|<1}.$$

Since $D^2\phi$ is continuous, $y_n \rightarrow x$ and $D\phi(y_n) \rightarrow D\phi(x)$ are bounded, a Taylor expansion and **(J1')** reveals that there is a C independent of n and z such that

$$|\psi_n(z)| \leq \sup_{t \in (0,1)} \frac{1}{2} \left| D^2\phi\left(y_n + tj_1(D\phi(y_n), z)\right) \right| |j_1(D\phi(y_n), z)|^2 \leq C|z|^2 \quad \text{for } |z| < 1.$$

By regularity of ϕ and **(J2')**, the ψ_n converge uniformly for $|z| \leq \delta$ to

$$\psi(z) := \phi(x + j_1(D\phi(x), z)) - \phi(x) - D\phi(x) \cdot j_1(D\phi(x), z)\mathbb{1}_{|z|<1},$$

and hence we use Lemma 5.3 (c) to conclude that

$$L_{\delta,n}[\phi, D\phi](y_n) \rightarrow L_\delta[\phi, D\phi](x).$$

Simpler but similar arguments, this time using Lemma 5.3 (b) and (d), show that

$$L_n^\delta[\phi, D\phi](y_n) \rightarrow L^\delta[\phi, D\phi](x) \quad \text{and} \quad L_{2,n}[\phi, D\phi](y_n) \rightarrow L_2[\phi, D\phi](x).$$

We conclude that $L_n[\phi, D\phi](y_n) \rightarrow L[\phi, D\phi](x)$, and since this limit is bounded, for n big enough

$$T_n[L_n[\phi, D\phi](y_n)] = L_n[\phi, D\phi](y_n) \rightarrow L[\phi, D\phi](x) \quad \text{as } n \rightarrow \infty.$$

In view of the regularity of ϕ and f , we can then pass to the limit as $n \rightarrow \infty$ in (5.11) to find that

$$\bar{u}(x) - L[\phi, D\phi](x) \leq f(x).$$

Hence \bar{u} is a viscosity subsolution of (1.7). Similar arguments show that \underline{u} is viscosity supersolution.

3) By the comparison result for semicontinuous viscosity solutions, Theorem 2.1 (a), it follows that $\bar{u} \leq \underline{u}$. Note that this result requires **(J2)**, see also Remark 5.2. Since the opposite inequality holds by part 2), $\bar{u} = \underline{u} =: u$, and this function is continuous, bounded by $\|f\|_\infty$, and a viscosity solution of (1.7). The proof is complete. \square

Now we show how to remove assumptions **(J1')**, **(J2')**, and **(F5')** and obtain an existence result in the general case. We do it in two steps, starting by removing **(J2')** and **(F5')** by a regularization argument (mollification), and then we remove **(J1')** by a truncation argument. We will need the following lemma, whose proof is given in Appendix C.

Lemma 5.10. *Assume **(M)**, **(J1')**, and **(J2)** and define $j_{1,k}$ and $j_{2,k}$ by*

$$j_{i,k}(p, z) := (\rho_k * j_i(\cdot, z))(p) = \int_{\mathbb{R}^N} \rho_k(q-p) j_i(q, z) dq \quad \text{for } i = 1, 2,$$

for a mollifier $\rho_k(p) = k^N \rho(kp)$, $0 \leq \rho \in C^\infty(\mathbb{R}^N)$ is symmetric with support in B_1 and $\int \rho = 1$. Then the following holds:

(a) For any $k \in \mathbb{N}$, $j_{1,k}$ and $j_{2,k}$ are locally bounded, and for any $|p| \leq r$, $|z| < 1$, and $k \in \mathbb{N}$,

$$|j_{1,k}(p, z)| \leq C_{r+1/k}|z|, \quad \text{where } C_r \text{ is the constant from (J1')}.$$

(b) For every $k \in \mathbb{N}$ and $r > 0$, there is $C_{k,r}$ such that for all $|p|, |q|, |z| \leq r$,

$$|j_{i,k}(p, z) - j_{i,k}(q, z)| \leq C_{k,r}|p - q| \quad \text{for } i = 1, 2.$$

(c) For any $r > 0$, $|p|, |q| < r$ and $k \in \mathbb{N}$,

$$\int_{|z|>0} |j_{1,k}(p, z) - j_{1,k}(q, z)|^2 d\mu_1(z) \leq \omega_{j,r+\frac{1}{k}}(p - q) \quad \text{where } \omega_{j,r} \text{ is the modulus from (J2)}.$$

(d) There exists $\delta_0 > 0$ such that for any $r > 0$ and $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\sup_{|p|<r, k \in \mathbb{N}} \int_A |j_{1,k}(p, z)|^2 d\mu_1(z) < \varepsilon$$

for every Borel set $A \subset \{0 < |z| < \delta_0\}$ such that $\int_A |z|^2 d\mu_1(z) < \eta$.

(e) Let $L_{1,k}$ and $L_{2,k}$ be defined as in (1.1) and (1.2) with j_1 and j_2 replaced by $j_{1,k}$ and $j_{2,k}$. For all $\phi \in C_b \cap C^2$ and $x_k \rightarrow x$,

$$L_{i,k}[\phi, D\phi(x_k)](x_k) \rightarrow L_i[\phi, D\phi(x)](x) \quad \text{for } i = 1, 2.$$

Here is the general existence result for equation (1.7).

Proposition 5.11. *Assume (M), (J1)–(J3) and (F1)–(F5). Then there exists a viscosity solution of (1.7).*

Proof. We proceed in two steps.

1) Assume in addition (J1') holds. We approximate j_i and f by

$$j_{i,k} \text{ as defined in Lemma 5.10} \quad \text{and} \quad f_k = \rho_k * f,$$

where $k \in \mathbb{N}$ and ρ_k is the mollifier defined in Lemma 5.10. By Lemma 5.10, $j_{i,k}$ satisfy (J1)–(J3) and (J1')–(J2'), while assumption (F5) and properties of mollifiers imply that f_k is bounded and Lipschitz continuous in \mathbb{R}^N (so (F5') holds) and converges uniformly to f in \mathbb{R}^N as $k \rightarrow \infty$.

Let L_k be defined as L with $j_{i,k}$ replacing j_i , and consider the problem

$$u - L_k[u, Du] = f_k \quad \text{in } \mathbb{R}^N.$$

By the above discussion, this problem satisfies all the assumptions of Proposition 5.9 for every $k \in \mathbb{N}$, and hence there exist solutions u_k of this problem for every $k \in \mathbb{N}$. By the comparison principle (Theorem 2.1 (a)),

$$\|u_k\|_\infty \leq \|f_k\|_\infty \leq \|f\|_\infty.$$

As in the proof of Proposition 5.9, we then define the half-relaxed limits

$$\bar{u}(x) = \limsup_{y \rightarrow x, k \rightarrow \infty} u_k(y) \quad \text{and} \quad \underline{u}(x) = \liminf_{y \rightarrow x, k \rightarrow \infty} u_k(y),$$

prove that they are viscosity sub and supersolutions of (1.7), and conclude by the comparison result Theorem 2.1 (a) that $\bar{u} = \underline{u} = u$ is a viscosity solution of (1.7). We just sketch the argument for \bar{u} being a subsolution. Take a smooth test-function ϕ such that $\bar{u} - \phi$ has a strict maximum at $x \in \mathbb{R}^N$. Then there exists a sequence $y_k \rightarrow x$ such that $u_k - \phi$ has a maximum at x_k . Using the

subsolution property for u_k , the fact that $L_k[\phi, D\phi](x_k) \rightarrow L[\phi, D\phi](x)$ (Lemma 5.10 (e)) and that $f_k(x_k) \rightarrow f(x)$, we arrive at

$$\bar{u}(x) - L[\phi, D\phi](x) \leq f(x),$$

hence \bar{u} is a subsolution of (1.7). This completes the existence proof under the assumptions **(M)**, **(J1')**, **(J2)**, **(J3)**, and **(F5)**.

2) We remove the **(J1')** condition through a truncation procedure for the j_1 -term:

$$j_1^M(p, z) := |z| T_M \left(\frac{j_1(p, z)}{|z|} \right),$$

where $T_M(x) = x$ if $|x| \leq M$ and $T_M(x) = M \frac{x}{|x|}$ if $|x| > M$. In this case is easy to see that j_1^M satisfies **(J1)**–**(J3)** and in addition **(J1')** with $C_r = M$. Furthermore, for almost any z ,

$$j_1^M(p, z) \rightarrow j_1(p, z) \quad \text{locally uniformly in } p.$$

Hence by part 1), for any $M > 0$ there exists a solution u_M of

$$u - L^M[u, Du] = f \quad \text{in } \mathbb{R}^N,$$

where L^M is defined as L (cf. (1.1)) but with (j_1^M, j_2) replacing (j_1, j_2) . As in part 1), we define the half-relaxed limits

$$\bar{u}(x) = \limsup_{y \rightarrow x, M \rightarrow \infty} u_M(y) \quad \text{and} \quad \underline{u}(x) = \liminf_{y \rightarrow x, M \rightarrow \infty} u_M(y),$$

prove that they are viscosity sub and supersolutions of (1.7), and conclude by the comparison result Theorem 2.1 (a) that $\bar{u} = \underline{u} = u$ is a viscosity solution of (1.7). In view of previous arguments, the only thing we need to check is that

$$L_1^M[\phi, D\phi](x_M) \rightarrow L_1[\phi, D\phi](x) \quad \text{when } x_M \rightarrow x.$$

To do so, we use note that the integrand converges pointwise a.e. by the regularity of ϕ and the local uniform convergence of j_1^M in the p -variable. Since $|j_1^M(p, z)| \leq |j_1(p, z)|$, j_1^M satisfies the equi-integrability condition **(J3)** uniformly in M . Hence we may use Vitali's convergence theorem (cf. Appendix A and Remark 2.4-(b)) to pass the limit $M \rightarrow \infty$ inside the integral over $|z| < \delta$. Passage to the limit in the integral over $|z| \geq \delta$ is done using the dominated convergence theorem since ϕ is bounded and μ_1 is not singular on this domain. The proof is complete. \square

Proof of Theorem 2.3. Uniqueness and uniform continuity follow from Corollary 2.2. Existence of solutions of (1.7) follows from Proposition 5.11. Existence for (1.4) follows in a similar way as for (1.7) where all difficulties are already present. We only give a very brief sketch of the proof. We start by an approximate problem (5.7). In the general case it is to find $u \in C^2(\bar{B}_R)$ such that

$$\begin{cases} \gamma u + T_M \left[-\gamma u + F(u, Du, L_k^R[u, Du]) \right] - \varepsilon \Delta u = f(x), & x \in B_R(0), \\ u = 0, & x \in \partial B_R(0), \end{cases}$$

where L_k^R is defined below (5.7). Under the same assumptions as the linear case in addition to the assumption that F is also locally Lipschitz, we obtain existence of solutions of this problem following step by step the fixed point argument of the proof of Proposition 5.5. To get an existence result for equation (1.4), we follow the approximation and limit procedures given in the proofs of Proposition 5.9 and 5.11. The only slight difference is that we also need an approximation argument for F (e.g. by mollification again), and when we pass to the limit, we use this time the strong comparison result Theorem 2.1-(b) for the limit equation. It is straight forward to check that this will work out and produce a bounded viscosity solution of (1.4). \square

5.3. Proof of Theorem 2.6 (local limits). Theorem 2.6 is a consequence of the half-relaxed limit method and comparison. Define

$$\bar{u}(x) := \limsup_{\varepsilon \rightarrow 0, y \rightarrow x} u_\varepsilon(y) \quad \text{and} \quad \underline{u}(x) := \liminf_{\varepsilon \rightarrow 0, y \rightarrow x} u_\varepsilon(y).$$

In the quasilinear case we have the following result.

Lemma 5.12. *Assume the assumptions of Theorem 2.6-(a) hold. Then \bar{u} is a viscosity subsolution of (2.2) and \underline{u} is a viscosity supersolution of (2.2).*

Proof. Let ϕ be a smooth bounded function such that $\bar{u} - \phi$ has a global maximum at x . We may assume the maximum is strict and that there is a sequence $\{y_\varepsilon\}_\varepsilon$ of global maximum points of $u_\varepsilon - \phi$ such that $y_\varepsilon \rightarrow x$ and $u_\varepsilon(y_\varepsilon) \rightarrow \bar{u}(x)$ as $\varepsilon \rightarrow 0$. We let $L_{1,\varepsilon} = L_{\varepsilon,\delta} + L_\varepsilon^\delta$ as in (4.1), and to see the localization effect, we also decompose $L_{2,\varepsilon} = \tilde{L}_{\varepsilon,\delta} + \tilde{L}_\varepsilon^\delta$. By the maximum point property $\tilde{L}_{\varepsilon,\delta}[u_\varepsilon, D\phi](y_\varepsilon) \leq \tilde{L}_{\varepsilon,\delta}[\phi, D\phi](y_\varepsilon)$, and then since u_ε is a subsolution of (2.1),

$$-L_{\varepsilon,\delta}[\phi, D\phi](y_\varepsilon) - L_\varepsilon^\delta[u_\varepsilon, D\phi](y_\varepsilon) - \tilde{L}_{\varepsilon,\delta}[\phi, D\phi](y_\varepsilon) - \tilde{L}_\varepsilon^\delta[u_\varepsilon, Du_\varepsilon](y_\varepsilon) + u_\varepsilon(y_\varepsilon) \leq f(y_\varepsilon). \quad (5.12)$$

Then by a Taylor expansion and (J4), for $i = 1, 2$,

$$j_i(D\phi(y_\varepsilon), z) = \sigma_i(D\phi(y_\varepsilon))z + o(z) \quad \text{as} \quad z \rightarrow 0,$$

where $o(z)$ is independent of $\varepsilon > 0$. Another Taylor expansion and hypotheses (M $_\varepsilon$) applied to $Y = \sigma_1(D\phi(x))^T D^2\phi(x) \sigma_1(D\phi(x)) \in \mathbb{R}^{P \times P}$ then gives that

$$\begin{aligned} L_{\varepsilon,\delta}[\phi, D\phi](y_\varepsilon) &= \frac{1}{2} \int_{|z| < \delta} z^T \left[\sigma_1(D\phi(y_\varepsilon))^T D^2\phi(y_\varepsilon) \sigma_1(D\phi(y_\varepsilon)) \right] z \, d\mu_{1,\varepsilon}(z) + o_\delta(1) \\ &= \frac{1}{2} \text{tr} \left[A_1^T \left[\sigma_1(D\phi(x))^T D^2\phi(x) \sigma_1(D\phi(x)) \right] A_1 \right] + o_\varepsilon(1) + o_\delta(1) \\ &= \frac{1}{2} \text{tr} \left[\tilde{\sigma}_1(D\phi(x)) \sigma_1(D\phi(x))^T D^2\phi(x) \right] + o_\varepsilon(1) + o_\delta(1), \end{aligned}$$

where for the last line, we use that $\tilde{\sigma}_1(x, p) = \sigma_1(x, p)A_1$ is a $N \times N$ matrix as is $D^2\phi(x)$, so that we can use the property $\text{tr}(M_1M_2) = \text{tr}(M_2M_1)$. Here the $o_\delta(1)$ -term is independent of ε since the measure $|z|^2 \mu_{1,\varepsilon}(dz)$ has a uniformly bounded mass. Similarly, by (J4) and (M $_\varepsilon$),

$$\begin{aligned} \tilde{L}_{\varepsilon,\delta}[\phi, D\phi](x) &= \frac{1}{2} \int_{|z| < \delta} z^T \left[\sigma_2(D\phi(x))^T D^2\phi(x) \sigma_2(D\phi(x)) \right] z \, d\mu_{2,\varepsilon}(z) \\ &\quad + \int_{|z| < \delta} D\phi(x) \sigma_2(D\phi(x)) z \, d\mu_{2,\varepsilon}(z) + o_\varepsilon(1) + o_\delta(1) \\ &= \frac{1}{2} \text{tr} \left[A_2^T \left[\sigma_2(D\phi(x))^T D^2\phi(x) \sigma_2(D\phi(x)) \right] A_2 \right] \\ &\quad + \sigma_2(D\phi(x)) D\phi(x) \cdot a + o_\varepsilon(1) + o_\delta(1) \\ &= \frac{1}{2} \text{tr} \left[\tilde{\sigma}_2(D\phi(x)) \tilde{\sigma}_2(D\phi(x)) D^2\phi(x) \right] + b(D\phi(x)) \cdot D\phi(x) + o_\varepsilon(1) + o_\delta(1) \end{aligned}$$

By (J1) and continuity, $D\phi(y_\varepsilon)$ and $j(D\phi(y_\varepsilon), z)$ are uniformly bounded in $\varepsilon > 0$ for $|z| < 1$, and by (F5) and Corollary 2.2-(c), u_ε is also uniformly bounded:

$$\|u_\varepsilon\|_\infty \leq \|f\|_\infty.$$

By (\mathbf{M}_ε) it then follows that

$$L_\varepsilon^\delta[u_\varepsilon, D\phi](y_\varepsilon) + \tilde{L}_\varepsilon^\delta[u_\varepsilon, D\phi](y_\varepsilon) = O\left(\int_{|z|\geq\delta} d\mu_\varepsilon(z)\right) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

and $\delta > 0$ is fixed.

Sending first $\varepsilon \rightarrow 0$ and then $\delta \rightarrow 0$ in (5.12) then leads to

$$-L_0(D\phi(x), D^2\phi(x)) + \bar{u}(x) \leq f(x),$$

and hence \bar{u} is a viscosity subsolution of (2.2). In a similar way we can show that \underline{u} is a viscosity supersolution of (2.2). We omit the proof. \square

Then it is enough to invoke the comparison principle for the limit (local) equation to conclude that Theorem 2.6-(a) holds:

Proof of Theorem 2.6-(a). By Lemma 5.12 \bar{u} is a subsolution and \underline{u} is a supersolution of (2.2), hence $\bar{u} \leq \underline{u}$ by the comparison principle for (2.2), see [15] or Lemma 5.14 below. Since $\underline{u} \leq \bar{u}$ by definition, $\bar{u} = \underline{u} =: u$, and hence $u_\varepsilon \rightarrow u$ point-wise, u is continuous and a viscosity solution of (2.2). Now we show that the convergence is locally uniform. Fix any $R > 0$ and take $x_\varepsilon \in B_R(0)$ such that

$$\max_{|x|\leq R} (u_\varepsilon(x) - u(x)) = u_\varepsilon(x_\varepsilon) - u(x_\varepsilon)$$

for any $\varepsilon > 0$. Since $|x_\varepsilon| \leq R$, there exists a convergent subsequence $x_\varepsilon \rightarrow \bar{x}$ for some $|\bar{x}| \leq R$. By the continuity of u , the definition of \bar{u} , and the fact that $u = \bar{u}$, it follows that

$$\limsup_{\varepsilon \rightarrow 0} \max_{|x|\leq R} (u_\varepsilon(x) - u(x)) = \limsup_{\varepsilon \rightarrow 0} (u_\varepsilon(x_\varepsilon) - u(x_\varepsilon)) \leq \bar{u}(\bar{x}) - u(\bar{x}) = 0.$$

A similar argument shows that $\liminf_{\varepsilon \rightarrow 0} \max_{|x|\leq R} (u_\varepsilon(x) - u(x)) \geq 0$. Combined with similar arguments for $-(u_\varepsilon - u)$, this shows that

$$\lim_{\varepsilon \rightarrow 0} \max_{|x|\leq R} |u_\varepsilon(x) - u(x)| = 0 \quad \text{for all } R > 0,$$

and we are done. \square

In the fully nonlinear case, the strategy is the same and we begin with a half-relaxed limit result.

Lemma 5.13. *Assume the assumptions of Theorem 2.6-(b) hold. Then \bar{u} is a viscosity subsolution of (1.6) and \underline{u} is a viscosity supersolution of (1.6).*

Proof. Take a test-function ϕ such that $\bar{u} - \phi$ has a maximum at \bar{x} that we can assume to be strict. Hence $u_\varepsilon - \phi$ also has a maximum at some x_ε , and $x_\varepsilon \rightarrow \bar{x}$ and $u_\varepsilon(x_\varepsilon) \rightarrow \bar{u}(\bar{x})$ as $\varepsilon \rightarrow 0$. Hence $\tilde{L}_{\varepsilon,\delta}[u_\varepsilon, D\phi](x_\varepsilon) \leq \tilde{L}_{\varepsilon,\delta}[\phi, D\phi](x_\varepsilon)$, and since u_ε is a subsolution and $(\mathbf{F1})$ holds,

$$F(u_\varepsilon(x_\varepsilon), D\phi(x_\varepsilon), L_{\varepsilon,\delta}[\phi, D\phi](x_\varepsilon) + L_\varepsilon^\delta[u_\varepsilon, D\phi](x_\varepsilon) + \tilde{L}_{\varepsilon,\delta}[\phi, D\phi](x_\varepsilon) + \tilde{L}_\varepsilon^\delta[u_\varepsilon, D\phi](x_\varepsilon)) \leq f(x_\varepsilon).$$

As in the proof of Lemma 5.12,

$$L_\varepsilon\text{-terms} \rightarrow L_0(D\phi(\bar{x}), D^2\phi(\bar{x})) \quad \text{as } \varepsilon \rightarrow 0.$$

By the continuity of F , we then send $\varepsilon \rightarrow 0$ to find that

$$F(\bar{u}(\bar{x}), D\phi(\bar{x}), L_0(D\phi(\bar{x}), D^2\phi(\bar{x}))) \leq f(\bar{x}),$$

which means that \bar{u} is a subsolution of (1.6). In a similar way we can show that \underline{u} is a supersolution of (1.6). \square

Then we prove a comparison result for the limit equation.

Lemma 5.14. *Under the assumptions of Theorem 2.6-(b), if u is a bounded usc subsolution of (1.6) and v is a bounded lsc supersolution of (1.6), then $u \leq v$ in \mathbb{R}^N .*

To prove this result, first note that by **(F1)** and **(F2)**, the nonlinearity in (1.6),

$$H(x, u, p, X) := F\left(u, p, L_0(p, X)\right) - f(x), \quad (5.13)$$

is strictly increasing in u and nonincreasing in X . Moreover, it is straightforward to check that we also have the following result.

Lemma 5.15. *Assume **(J4)** and **(F3)** hold, H is defined in (5.13), and $M, r, \hat{r}, R, \varepsilon > 0$. Let $x, y \in \mathbb{R}^N$, $|u| \leq M$, $p = 2(x - y)/\varepsilon^2 + o_R(1)$, $|p| < r$, $X, Y \in \mathcal{S}_N$, and $-\hat{r} \leq X \leq Y \leq \hat{r}$ such that*

$$\left(-\frac{8}{\varepsilon^2} + o_R(1)\right) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \frac{1}{\varepsilon^2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + o_R(1) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Then there are moduli of continuity $\omega_M, \omega_{M,r,\hat{r}}$ such that

$$H\left(y, u, p - O\left(\frac{1}{R}\right), Y\right) - H\left(x, u, p + O\left(\frac{1}{R}\right), X\right) \leq \omega_M(x - y) + \frac{1}{\varepsilon^2} \omega_{M,r,\hat{r}}(o_R(1)) + o_R(1).$$

Proof of Lemma 5.14. In view of our assumptions and Lemma 5.15, the proof is standard. It can be obtained by following the line of reasoning of the proof of Theorem 5.1 in [15]. We omit the details. \square

Proof of Lemma 2.6-(b). In view of Lemma 5.13 and 5.14, we can conclude the proof exactly as for Theorem 2.6-(a) above. \square

6. EXTENSIONS

6.1. Parabolic equations. In this section we extend the results to the case of quasilinear and fully nonlinear parabolic equations,

$$u_t - L[u, Du] = f(x, t) \quad \text{in} \quad Q_T := \mathbb{R}^N \times (0, T), \quad (6.1)$$

$$u_t + F(u, Du, L[u, Du]) = f(x, t) \quad \text{in} \quad Q_T, \quad (6.2)$$

with initial data

$$u(x, 0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N.$$

These extensions are straightforward since the time variable does not play an important role here. We keep the same notation, definitions, and most of the assumptions as in the previous sections. However, we take the following parabolic versions of **(F2)**, **(F4)** and **(F5)**:

(F2') assumption **(F2)** holds with $\gamma_M = 0$.

(F6) $u_0 \in UC(\mathbb{R}^N)$ and $f \in UC(Q_T)$.

(F6') u_0 and f satisfy **(F6)**, are bounded, and the quantities in **(F2)**, **(F3)** are independent of M .

As usual for parabolic problems, we do no longer need strict monotonicity in u , see **(F2')**. We have the following parabolic version of the existence and comparison results.

Theorem 6.1. (Comparison results)

(a) [Quasilinear case] Assume **(M)**, **(J1)**–**(J2)** and **(F6)**. If u is a bounded usc subsolution of (6.1), v a bounded lsc supersolution of (6.1), and $u(x, 0) \leq v(x, 0)$ in \mathbb{R}^N , then $u \leq v$ in \overline{Q}_T .

(b) [Fully nonlinear case] Assume **(M)**, **(J1)**–**(J2)**, **(F1)**, **(F2')**, **(F3)**, and **(F6)**. If u is a bounded usc viscosity subsolution of (6.2), v a bounded lsc viscosity supersolution of (6.2), and $u(x, 0) \leq v(x, 0)$ in \mathbb{R}^N , then $u \leq v$ in \overline{Q}_T .

Sketch of proof. We assume by contradiction that

$$m = \sup_{\mathbb{R}^N \times [0, T]} (u - v) > 0.$$

We need to double the variables in time as well as in space and consider

$$\Phi_{\varepsilon, \beta, R, c, \delta}(x, y, s, t) := u(x, t) - v(y, s) - \phi(x, t, y, s)$$

where

$$\phi(x, t, y, s) = \frac{1}{\varepsilon^2} \varphi(x - y) + \frac{|t - s|^2}{\beta^2} + \psi\left(\frac{x}{R}\right) + \psi\left(\frac{y}{R}\right) + \frac{c}{T - t} + \delta m \frac{t}{T},$$

$\delta \in (0, 1)$, and φ and ψ are defined in the proof of Theorem 2.1.

A standard argument shows that

$$\sup_{\mathbb{R}^{2N} \times [0, T]^2} \Phi_{\varepsilon, \beta, R, c, \delta}(x, y, t, s) \geq \sup_{\mathbb{R}^N \times [0, T]} \Phi_{\varepsilon, \beta, R, c, \delta}(x, x, t, t) \geq m - o_R(1) - c - \delta m > 0$$

for c small enough and R big enough. By definition, $\Phi_{\varepsilon, \beta, R, c}$ will attain its supremum at some point $(\bar{x}, \bar{t}, \bar{y}, \bar{s})$, and since $u(x, 0) \leq v(x, 0)$, we may assume that both $\bar{t}, \bar{s} > 0$ by taking ε and β small enough. Because of the term $c/(T - t)$, we may also assume that $\bar{t}, \bar{s} < T$. Hence we may use the viscosity inequalities for u and v . After we have subtracted these inequalities and observed that $\delta m \frac{1}{T} \leq \phi_t - \phi_s$, we use the continuity of the equation to send $\beta \rightarrow 0$ and obtain $\bar{s} = \bar{t} \in (0, T)$. The inequality corresponding to (5.6) then takes the form

$$\delta m \frac{1}{T} \leq F(v(\bar{y}, \bar{t}), \dots) - F(u(\bar{x}, \bar{t}), \dots) + f(\bar{x}, \bar{t}) - f(\bar{y}, \bar{t}).$$

Since $m > 0$, we can use **(F2')** to see that

$$\delta m \frac{1}{T} \leq F(u(\bar{x}, \bar{t}), \dots) - F(u(\bar{x}, \bar{t}), \dots) + f(\bar{x}, \bar{t}) - f(\bar{y}, \bar{t}) = R.H.S.$$

At this stage we proceed as in the proof of Theorem 2.1 (but omitting **(F2)**), and show that

$$\delta m \frac{1}{T} \leq \lim_{c \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \lim_{R \rightarrow \infty} \lim_{\beta \rightarrow 0} R.H.S. \leq 0.$$

This is a contradiction to our original assumption $m > 0$ and the proof is complete. \square

Theorem 6.2 (Existence). *Under the assumptions of Theorem 6.1, **(F6')**, and **(J3)**, there exists a bounded viscosity solution of (6.2).*

Sketch of proof. We first assume **(J1')**, **(J2')** and that $(x, t) \mapsto f(x, t)$, $x \mapsto u_0(x)$ are bounded and Lipschitz continuous. The general result will follow from passage to the limit as in Proposition 5.11. Then we consider the parabolic version of the approximation we used in Section 5.2:

$$\begin{cases} u_t + T_M \left[F(u, Du, L_k^R[u, Du]) \right] - \varepsilon \Delta u = f(x, t), & (x, t) \in B_R(0) \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial B_R(0) \times (0, T), \\ u(x, 0) = u_0(x), & x \in B_R(0). \end{cases}$$

Assuming this problem has a solution $u_{M,R,k,\varepsilon}$, we pass to the limit as $R \rightarrow \infty$, $k \rightarrow \infty$, $M \rightarrow \infty$, $\varepsilon \rightarrow 0$ as in the elliptic case. Using half-relaxed limits and comparison for the limit equation, we show that the sequence of solutions has a limit which is a solution of (6.2). Existence is then proved.

To prove that there exists a solution of the approximate problem we use Schauder's fixed point theorem and the argument given in Proposition 5.5 with some small modifications:

1) We use Schauder's fixed point theorem in the Banach space

$$X := H_{1+\theta_0}([0, T] \times \overline{B}_R)$$

for some $\theta_0 \in (0, 1)$. The space $H_{1+\theta_0}$ ($= C^{1+\theta_0, \frac{1+\theta_0}{2}}$) is a standard parabolic Hölder space where $u \in H_{1+\theta_0}$ e.g. implies that $Du \in H_{\theta_0}$. See page 46 in [26] or Section 1.2.3 in [34] for the definition.

2) The time-dependent version of Lemma 5.4 remains valid: if $v \in X$, then $(x, t) \mapsto L_k^R[v, Dv](x, t)$ belongs to $H_{\theta_0}(\overline{B}_R \times [0, T])$, and if $v_n \rightarrow v$ in X , then $L_k^R[v_n, Dv_n] \rightarrow L_k^R[v, Dv]$ in $H_{\theta_0}(\overline{B}_R \times [0, T])$.

3) The C^{2,θ_0} regularity result that we use in Lemma 5.5 (step 2) is replaced by the parabolic $H_{2+\theta_0}$ version in Theorem 4.28 in [26].

4) In Lemma 5.5 (step 3), instead of the $W^{2,p}$ -theory we use the parabolic $W_p^{2,1}$ -theory of Theorem 7.17 in [26]. We also use the compact embedding of $W_p^{2,1}$ into X for p big enough, see Theorem 1.4.1 in [34]. \square

For the local limit result, we introduce the local parabolic equations

$$u_t - L_0(Du, D^2u) = f(x, t) \quad \text{in} \quad Q_T, \quad (6.3)$$

$$u_t + F(u, Du, L_0(Du, D^2u)) = f(x, t) \quad \text{in} \quad Q_T, \quad (6.4)$$

with an initial data $u(x, 0) = u_0(x)$. The result is the following:

Theorem 6.3. (Localization)

(a) [Quasilinear case] *Under the assumptions of Theorem 6.1-(a), **(F6')**, **(M $_{\varepsilon}$)**, and **(J4)**, any sequence of solutions u_{ε} of (2.1) converge locally uniformly in \overline{Q}_T as $\varepsilon \rightarrow 0$ to the solution u of (6.3).*

(b) [Fully nonlinear case] *Under the assumptions of Theorem 6.1-(b), **(F6')**, **(M $_{\varepsilon}$)**, **(J4)**, any sequence of solutions u_{ε} of (1.5) converge locally uniformly in \overline{Q}_T as $\varepsilon \rightarrow 0$ to the solution u of (6.4).*

Sketch of Proof. We use uniform boundedness of u_{ε} and the half-relaxed limits

$$\overline{u}(x) := \limsup_{\varepsilon \rightarrow 0, y \rightarrow x, s \rightarrow t} u_{\varepsilon}(y, s) \quad \text{and} \quad \underline{u}(x) := \liminf_{\varepsilon \rightarrow 0, y \rightarrow x, s \rightarrow t} u_{\varepsilon}(y, s),$$

and prove that \overline{u} and \underline{u} are respectively sub and supersolutions of the local limit problem. Local uniform convergence is then obtained after proving that the limit problem satisfies the comparison

principle. The proofs of the comparison principles for (6.3) and (6.4) are similar to the proofs in stationary case with standard modification such as doubling also the time variables. \square

6.2. More general nonlocal operators and equations. As is common in viscosity solution theory, our results and proofs extend easily to equations involving many different operators L and equations of Bellman-Isaacs type involving infima and/or suprema of indexed operators and equations of the type we have studied before. An example is the following equation:

$$\sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \left\{ F^{\alpha, \beta} \left(u(x), Du(x), L_{1, \alpha, \beta}[u, Du](x), \dots, L_{m, \alpha, \beta}[u, Du](x) \right) - f_{\alpha, \beta}(x) \right\} = 0$$

where

$$L_{i, \alpha, \beta}[u, Du](x) := \int_{\mathbb{R}^{P_i}} \left[(u(x + j_{i, \alpha, \beta}(Du, z)) - u(x) - j_{i, \alpha, \beta}(Du, z) \cdot Du(x) \mathbf{1}_{|z| < 1}) \right] d\mu_{i, \alpha, \beta}(z),$$

and for fixed (i, α, β) , $\mu_{i, \alpha, \beta}$ is a measure on \mathbb{R}^{P_i} and $j_{i, \alpha, \beta} : \mathbb{R}^N \times \mathbb{R}^{P_i} \rightarrow \mathbb{R}^N$.

(i) *Comparison.* We can extend our comparison results easily to this equation if we require $\{j_{i, \alpha, \beta}\}$ and $\{\mu_{i, \alpha, \beta}\}$ to satisfy assumptions **(M)** and **(J1)–(J3)** uniformly with respect to i , α , and β . However, we cannot mix gradient dependence with x -dependence in $j_{i, \alpha, \beta}$ for reasons explained in the introduction. This extension is essentially based on the classical inequality

$$\begin{aligned} & \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \left\{ F^{\alpha, \beta}(u, p, \ell_{\alpha, \beta}) - f_{\alpha, \beta}(x) \right\} - \sup_{\alpha \in \mathcal{A}} \inf_{\beta \in \mathcal{B}} \left\{ F^{\alpha, \beta}(v, q, \ell'_{\alpha, \beta}) - f_{\alpha, \beta}(y) \right\} \\ & \leq \sup_{(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}} \left\{ F^{\alpha, \beta}(u, p, \ell_{\alpha, \beta}) - F^{\alpha, \beta}(v, q, \ell'_{\alpha, \beta}) - f_{\alpha, \beta}(x) + f_{\alpha, \beta}(y) \right\}, \end{aligned}$$

where for each (α, β) , $\ell_{\alpha, \beta}, \ell'_{\alpha, \beta} \in \mathbb{R}^m$. In order to use this inequality in the various passages to the limit, we have of course to use the uniformity with respect to i , α , and β of the constants appearing in our hypotheses on j and μ . Notice that for the F -hypotheses, we have to reformulate them with a vector $\ell \in \mathbb{R}^m$. For instance, ellipticity condition **(F1)** becomes

(F1') $F : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^m$ is continuous and for any $u \in \mathbb{R}$, $p \in \mathbb{R}^N$, $\ell, \ell' \in \mathbb{R}^m$ s.t. $\ell_i \leq \ell'_i$ ($i = 1 \dots m$),

$$F(u, p, \ell) \leq F(u, p, \ell').$$

(ii) *Existence* is obtained for these more general equations as we did in Section 5.2, by a series of approximations including truncations (of the measures and operators), vanishing viscosity and so on. Again, uniformity of the hypotheses with respect to i , α , and β are needed in order to pass to the limit in the various approximations.

(iii) *The local limit results* follow again the same lines as in Section 5.3, though the local operators involve also a sup/inf of operators $L_0^{i, \alpha, \beta}(Du, D^2u)$ defined in Definition 2.5.

Finally, these extensions are also valid for the parabolic versions discussed in Section 6.1.

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APPENDIX A. EQUI-INTEGRABILITY AND CONVERGENCE

We give the definition equi-integrability (also called uniform integrability) and a result that we have used many times in this paper, a generalization of the dominated convergence theorem due to Vitali. Our presentation follow [31] page 133.

Definition A.1. Let $(\Omega, \mathcal{E}, \mu)$ be a positive measure space. A family $(f_i)_{i \in I} \subset L^1(\Omega, \mu)$ is equi-integrable if for any $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\sup_{i \in I} \int_A |f_i(z)| d\mu(z) < \varepsilon$$

for every $A \in \mathcal{E}$ such that $\mu(A) < \eta$.

The Vitali convergence theorem is the following result:

Proposition A.2. Assume $(\Omega, \mathcal{E}, \mu)$ is a positive finite measure space and $(f_n)_{n \in \mathbb{N}} \subset L^1(\Omega, \mu)$ an equi-integrable family such that $f_n \rightarrow f$ μ -a.e.. Then $f_n \rightarrow f$ in $L^1(\Omega, \mu)$.

Note that the dominated convergence theorem is a consequence of this result (on finite measure spaces!) since domination by a fixed, integrable function implies equi-integrability.

APPENDIX B. PROOF OF LEMMA 5.3

(a) Note that $\mu_1 \mathbb{1}_{\frac{1}{k} < |z| < k}$ has a finite mass by **(M)**. An application of Fubini's theorem then shows that the convolution $\mu_{1,k} = (\mu_1 \mathbb{1}_{\frac{1}{k} < |z| < k}) * \rho_k$ is a measure which has density $\bar{\mu}_{1,k}$ with respect to the Lebesgue measure given by

$$\bar{\mu}_{1,k}(x) = \int_{\mathbb{R}^N} \rho_k(x-z) \mathbb{1}_{\frac{1}{k} < |z| < k} \mu_1(dz).$$

This function is bounded for each k :

$$\|\bar{\mu}_{1,k}\|_\infty \leq \|\rho_k\|_\infty \mu_1\left(\left\{\frac{1}{k} < |z| < k\right\}\right) < \infty,$$

A similar argument shows the existence and boundedness of $\bar{\mu}_{2,k}$.

Let $g(z) := |z|^2 \mathbb{1}_{|z| < \delta}$, then by Fubini's theorem and symmetry of ρ_k ,

$$\int_{|z| < \delta} |z|^2 \bar{\mu}_{1,k}(z) dz = \int_{\mathbb{R}^N} g(z) \int_{\mathbb{R}^N} \rho_k(z-y) \mathbb{1}_{\frac{1}{k} < |y| < k} \mu_1(dy) dz = \int_{\mathbb{R}^N} (g * \rho_k)(y) \mathbb{1}_{\frac{1}{k} < |y| < k} \mu_1(dy).$$

Note that $\rho_k * g$ is continuous with support in $\{|z| < \delta + 1/k\}$. By Hölder's inequality,

$$|\rho_k * g|(z) \leq \max_{y \in z + \bar{B}_{1/k}} |g(y)| \cdot 1 \leq (|z| + 1/k)^2 \leq 4|z|^2 \quad \text{for } |z| > 1/k,$$

and hence

$$\int_{|z| < \delta} |z|^2 \bar{\mu}_{1,k}(z) dz \leq 4 \int_{|z| < \delta} |z|^2 \mu_1(dz).$$

The proof of (a) is complete.

(b) Let $\varepsilon > 0$ be given, and split the integral in two using a $K > \delta$ to separate the domains:

$$\begin{aligned} I &:= \int_{|z| \geq \delta} \psi_k(z) d\mu_{1,k}(z) = \int_{\delta \leq |z| \leq K} \psi_k(z) d\mu_{1,k}(z) + \int_{|z| \geq K} \psi_k(z) d\mu_{1,k}(z) = I_1 + I_2, \\ J &:= \int_{|z| \geq \delta} \psi(z) d\mu_1(z) = \int_{\delta \leq |z| \leq K} \psi(z) d\mu_1(z) + \int_{|z| \geq K} \psi(z) d\mu_1(z) = J_1 + J_2. \end{aligned}$$

We will show that if we take K big enough, then $|I_2| + |J_2| < \varepsilon$ for all k , and then if k is big enough, $|I_1 - I_2| < \varepsilon$. The conclusion is that $|I - J| \leq 2\varepsilon$ and the proof is complete.

Consider first I_2 and J_2 . By the definition of $\mu_{1,k}$ and Fubini's theorem, $\mu_{1,k}(\{|z| > K\}) \leq \mu_1(\{|z| > K - 1/k\})$, and then by the dominated convergence theorem and $\mu_1(\{|z| > 1\}) < \infty$,

$$0 \leq \mu_{1,k}(\{|z| > K\}) \leq \mu_1(\{|z| > K - 1/k\}) \rightarrow 0 \quad \text{as } K \rightarrow \infty.$$

Note that this convergence is uniform in k . Hence since ψ_k and ψ are uniformly bounded in k , it follows that $I_2, J_2 \rightarrow 0$ as $K \rightarrow \infty$ uniformly in k .

We complete the proof by showing that $|I_1 - J_1| \rightarrow 0$ as $k \rightarrow \infty$ for any fixed K . Note that

$$|I_1 - J_1| \leq \int_{\delta \leq |z| \leq K} |\psi_k - \psi| \mu_{1,k}(dz) + \left| \int_{\delta \leq |z| \leq K} \psi \mu_{1,k}(dz) - \int_{\delta \leq |z| \leq K} \psi \mu_1(dz) \right|.$$

Consider the first term on the right hand side. Since $\sup_{\delta \leq |z| \leq K} |\psi_k - \psi| \rightarrow 0$ by assumption, and $\mu_{1,k}(\{\delta \leq |z| \leq K\}) \leq \mu_1(\{|z| \geq \delta - \frac{1}{k}\})$ as in the $|z| > K$ case, for $k > \frac{1}{\delta}$ we get

$$\int_{\delta \leq |z| \leq K} |\psi_k - \psi| \mu_{1,k}(dz) \leq \sup_{\delta \leq |z| \leq K} |\psi_k - \psi| \mu_1(\{|z| \geq \delta - \frac{1}{k}\}) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

For the second term, let $g(z) = \psi(z) \mathbf{1}_{\delta \leq |z| \leq K}$ and use Fubini's theorem to see that

$$\int_{\delta \leq |z| \leq K} \psi \mu_{1,k}(dz) = \int_{\mathbb{R}^N} (\rho_k * g)(z) \mathbf{1}_{\frac{1}{k} < |z| < k} \mu_1(dz).$$

In the last integral, the support of the convolution is $\{\delta - \frac{1}{k} \leq |z| \leq K + \frac{1}{k}\}$ so we need $k > \frac{1}{\delta}$. Since $\rho_k * g \rightarrow g$ and $\mathbf{1}_{\frac{1}{k} < |z| < k} \rightarrow 1$ pointwise (almost everywhere) both functions are uniformly bounded, we can pass to the limit using dominated convergence and find that

$$\left| \int_{\delta \leq |z| \leq K} \psi \mu_{1,k}(dz) - \int_{\delta \leq |z| \leq K} \psi \mu_1(dz) \right| \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (\text{B.1})$$

The proof of (b) is complete.

(c) Let $\varepsilon > 0$ be given, and split the integral in two using a $0 < r < \min(\delta, \delta_0)$ to separate the domains:

$$\begin{aligned} I &:= \int_{0 < |z| \leq \delta} \psi_k(z) d\mu_{1,k}(z) = \int_{|z| < r} \psi_k(z) d\mu_{1,k}(z) + \int_{r < |z| \leq \delta} \psi_k(z) d\mu_{1,k}(z) = I_1 + I_2, \\ J &:= \int_{0 < |z| < \delta} \psi(z) d\mu_1(z) = \int_{|z| < r} \psi(z) d\mu_1(z) + \int_{r < |z| \leq \delta} \psi(z) d\mu_1(z) = J_1 + J_2. \end{aligned}$$

We will show that if we take r small enough, then $|I_2| + |J_2| < \varepsilon$ for all k , and then if k is big enough, $|I_1 - I_2| < \varepsilon$. The conclusion is that $|I - J| \leq 2\varepsilon$ and the proof is complete.

The estimate $|I_2| + |J_2| < \varepsilon$ for r small, follows by the assumptions on ψ , part (a), and dominated convergence and **(M)**. For example,

$$\int_{0 < |z| < r} |\psi_k(z)| \mu_{1,k}(dz) \leq C \int_{0 < |z| < r} |z|^2 \mu_{1,k}(dz) \leq 4C \int_{0 < |z| < r} |z|^2 \mu_1(dz) \rightarrow 0 \quad \text{as } r \rightarrow 0.$$

Consider now $|I_1 - I_2|$. We first introduce the functions

$$\tilde{\psi}_k(z) := \frac{\psi_k(z)}{|z|^2} \quad \text{and} \quad \tilde{\psi}(z) := \frac{\psi(z)}{|z|^2},$$

and measures

$$\tilde{\mu}_{1,k}(dz) := |z|^2 \mu_{1,k}(dz) \quad \text{and} \quad \tilde{\mu}_1(dz) := |z|^2 \mu_1(dz).$$

By the assumptions and **(M)**, $\tilde{\psi}_k$ and $\tilde{\psi}$ are uniformly bounded, $\tilde{\psi}_k \rightarrow \tilde{\psi}$ uniformly on $r < |z| < \delta$, and $\tilde{\mu}_{1,k}$ and $\tilde{\mu}_1$ are bounded measures on $0 < |z| < \delta$. It follows that

$$|I_1 - I_2| \leq \int_{r < |z| < \delta} |\tilde{\psi}_k - \tilde{\psi}| \tilde{\mu}_{1,k}(dz) + \left| \int_{r < |z| < \delta} \tilde{\psi} \tilde{\mu}_{1,k}(dz) - \int_{r < |z| < \delta} \tilde{\psi} \tilde{\mu}_1(dz) \right|.$$

The first term converges by uniform convergence of $\tilde{\psi}_k$ and uniform boundedness of $\mu_{1,k}(\{r < |z| < \delta\})$. For the second term, we note that (see part (a))

$$\int_{r < |z| < \delta} \tilde{\psi} \tilde{\mu}_{1,k}(dz) = \int_{r < |z| < \delta} \psi \mu_{1,k}(dz) = \int_{\mathbb{R}^N} (\rho_k * (\psi(\cdot) \mathbb{1}_{r < |\cdot| < \delta}))(z) \mathbb{1}_{\frac{1}{k} < |z| < k} \mu_1(dz).$$

The integrand is uniformly bounded (by $\|\psi(\cdot) \mathbb{1}_{r < |\cdot| < \delta}\|_\infty$) and converges pointwise to $\psi(z) \mathbb{1}_{r < |z| < \delta}$ for a.a. z , so by **(M)** and the dominated convergence theorem, we can conclude that $|I_1 - J_1| \rightarrow 0$ as $k \rightarrow \infty$. The proof of (c) is complete.

(d) This proof is similar to the proof of (b), we omit it.

APPENDIX C. THE PROOF OF LEMMA 5.10

(a) Let $K \subset \mathbb{R}^N \times \mathbb{R}^P$ be any bounded set, then by the definition of ρ_k , Hölder's inequality, and $\int \rho_k(p) dp = 1$, $\|j_{i,k}\|_{L^\infty(K)} \leq \|j_i\|_{L^\infty(K_{1/k})}$ where

$$K_{1/k} = \{(q, z) \in \mathbb{R}^N \times \mathbb{R}^P : \exists p \in \mathbb{R}^N \text{ such that } (p, z) \in K \text{ and } |q - p| < 1/k\}.$$

By Hölder's inequality and **(J1')**, we also find that for $|p| \leq r$ and $|z| < 1$,

$$|j_{1,k}(p, z)| \leq \max_{|q-p| < 1/k} |j(q, z)| \int \rho_k(p) dp \leq C_{r+1/k} |z|,$$

and the proof of (a) is complete.

(b) By similar arguments, for $|p|, |q|, |z| < r$,

$$|j_{i,k}(p, z) - j_{i,k}(q, z)| \leq \int_{|s| < r+1/k} |j(s, z)| |\rho_k(p-s) - \rho_k(q-s)| ds \leq \sup_{\substack{|s| \leq r+1/k \\ |z| \leq r}} |j(s, z)| \|D\rho_k\|_{L^1} |p-q|,$$

and the proof of (b) is complete by **(J1')** and the standard estimate $\|D\rho_k\|_{L^1} \leq k\|D\rho\|_{L^1}$.

(c) By the definition of $j_{1,k}$, properties of mollifiers and Jensen's inequality, Fubini and **(J2)**,

$$\begin{aligned} \int_{|z| > 0} |j_{1,k}(p, z) - j_{1,k}(q, z)|^2 d\mu_1(z) &\leq \int_{|z| > 0} \int_{y \in \mathbb{R}^N} \rho_k(y) |j_1(p-y, z) - j_1(q-y, z)|^2 dy d\mu_1(z) \\ &\leq \int_{y \in \mathbb{R}^N} \rho_k(y) \left(\int_{|z| > 0} |j_1(p-y, z) - j_1(q-y, z)|^2 d\mu_1(z) \right) dy \\ &\leq \int_{y \in \mathbb{R}^N} \rho_k(y) \omega_{j, r+\frac{1}{k}}(p-q) dy \leq \omega_{j, r+\frac{1}{k}}(p-q). \end{aligned}$$

(d) Let $A \subset \{0 < |z| < \delta_0\}$ be a Borel set. Then as in part (c), we use properties of mollifiers and Jensen's inequality, Fubini and **(J2)**, to see that

$$\int_A |j_{1,k}(p, z)|^2 d\mu(z) \leq \int_A \int_{|q| < \frac{1}{k}} \rho_k(q) |j_1(p - q, z)|^2 dq d\mu_1(z) \leq \max_{|q| < \frac{1}{k}} \int_A |j_1(p - q, z)|^2 d\mu_1(z).$$

Now (d) follows from **(J3)** applied with $r + 1$, which is bigger than $r + 1/k$ for any $k \geq 1$.

(e) First fix a z such that $j_i(p, z)$ is continuous in p , cf. **(J1)**. Hence $j_{i,k}(\cdot, z)$ converges locally uniformly to $j_i(\cdot, z)$ as $k \rightarrow \infty$ (see for instance Appendix C, Theorem 6 in [19]). Moreover, $j_i(\cdot, z)$ is locally uniformly continuous in p , say with a modulus ω_r for $|p|, |q| \leq r$. Then

$$|j_{i,k}(p, z) - j_{i,k}(q, z)| \leq \int_{y \in \mathbb{R}^N} |j(p - y, z) - j(q - y, z)| \rho_k(y) dy \leq \omega_r(|p - q|) \cdot 1,$$

and $j_{i,k}(\cdot, z)$ is equicontinuous in p . Combining these two results, it follows that for every $p_k \rightarrow p$,

$$j_{i,k}(p_k, z) \rightarrow j_i(p, z) \quad \text{as } k \rightarrow \infty \quad \text{for a.e. } z. \quad (\text{C.1})$$

Then we let $p_k = D\phi(x_k)$ and $p = D\phi(x)$, and consider $L_{2,k}[\phi, p_k](x_k) = \int \phi(x_k + j_{2,k}(p_k, z)) - \phi(x_k) d\mu_2(dz)$. By (C.1) and continuity of ϕ and $D\phi$,

$$\phi(x_k + j_{2,k}(p_k, z)) - \phi(x_k) \rightarrow \phi(x + j_2(p, z)) - \phi(x) \quad \text{for a.e. } z,$$

and since the integrand is uniformly bounded by the μ_2 -integrable function $2\|\phi\|_\infty$ (cf. **(M)**), the dominated convergence theorem implies that

$$L_{2,k}[\phi, p_k](x_k) \rightarrow L_2[\phi, p](x) \quad \text{as } k \rightarrow \infty.$$

Note that $L_{1,k} = L_{1,\delta,k} + L_{1,k}^\delta$. For the $L_{1,k}^\delta$ -term the proof is more or less the same as for the $L_{2,k}$ -term (see above). It only remains to consider the $L_{1,\delta}$ -term. As above, we see that the integrand converges to $\phi(x + j_1(D\phi(x), z)) - \phi(x) - j_1(D\phi(x), z) \cdot D\phi(x)$ for a.e. z . An application of Taylor's theorem and **(J1')**, show that the integrand uniformly bounded by the μ_1 -integrable function $\|D^2\phi\|_{L^\infty(B_{R_2})} C_{R_1} |z|^2$ where $R_1 = \max(\delta, \max_k |D\phi(x_k)|)$ and

$$R_2 = \max_{k \in \mathbb{N}} |x_k| + \max_{|s|, |z| \leq R_1} |j(s, z)|.$$

Hence we conclude by the dominated convergence theorem that

$$L_{1,\delta,k}[\phi, D\phi(x_k)](x_k) \rightarrow L_{1,\delta}[\phi, D\phi(x)](x) \quad \text{as } k \rightarrow \infty.$$

The proof of (e) is complete.

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