

Measurability of Intersections of Measurable Multifunctions

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We prove universal compact-measurability of the intersection of a compact-measurable Souslin family of closed-valued multifunctions. This generalizes previous results on intersections of measurable multifunctions. We introduce the unique maximal part of a multifunction which is defined on the quotient given by an equivalence relation. Measurability of this part of a multifunction is proven in a special case. We show how these results apply to the spectral theory of measurable families of closed linear operators.^{1 2 3}

1 Introduction.

If $\omega \mapsto F(\omega)$ and $\omega \mapsto G(\omega)$ are set-valued functions (\equiv multifunctions), then it is reasonable to consider properties of the multifunctions $F \cap G$ and $F \cup G$. F is compact-measurable if $\{\omega \mid F(\omega) \cap K = \emptyset\}$ is measurable for all compacts K . Borel-, closed-, and open-measurability are defined likewise. It is trivial to verify measurability of finite or countable unions of measurable multifunctions. Himmelberg *et al* [3, p.166,ex.2] give an example of two closed-valued open-measurable multifunctions into a Polish space for which the intersection is not open-measurable. On the other side they prove compact-measurability of the intersection of a countable family of closed-valued compact-measurable multifunctions into a metric space. A metric space has second countable compacts. We prove compact-measurability of the intersection of a countable family of closed-valued compact-measurable multifunctions into a Hausdorff space with second countable compacts. The idea of our proof is different from the proof of Himmelberg *et al* and was motivated by the separation ideas in the work of Spakowski *et al* [6]. Spakowski *et al* prove open-measurability of the intersection of a countable family of compact-valued open-measurable multifunctions into a countably C -separated Hausdorff space. We generalize this to the case of a countably C -separated T_1 -space, with a simplification of their proof. We remark that a space is C -separated and T_1 if and only if every compact set is closed.

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The counterexample of Himmelberg *et al* disappears when the σ -algebra is completed. Castaing *et al* [2] prove universal Borel-measurability of the intersection of a countable family of closed-valued universally open-measurable multifunctions into a Souslin space. Our main result on intersections is the Souslin intersection theorem which gives universal compact-measurability of the intersection of a compact-measurable Souslin family of closed-valued multifunctions into a Hausdorff space with second countable compacts.

In the final section we consider multifunctions of the form

$$H(\omega) := \bigcup_{b \in G(\omega)} F(\omega, b)$$

$$K(\omega) := \bigcap_{b \in G(\omega)} F(\omega, b).$$

Universal open-measurability of H is proven, but the corresponding proof for K fails. Of particular interest is the case

$$F_{\sim}(\omega) := \bigcap_{b \sim \omega} F(b),$$

where \sim is an equivalence relation. F_{\sim} is the unique maximal part of F which is defined on the quotient given by \sim . We apply the Souslin intersection theorem to prove measurability of F_{\sim} in a special case.

To motivate the reader further we give here an example of one of the many possible applications of the theory of measurable multifunctions, and in particular the Souslin intersection theorem. Consider the Schrödinger operator

$$H_x = -\Delta + W_0 + \sum_{i=1}^{\infty} x_i W_i,$$

where Δ is the Laplace differential operator, the W_i 's are functions acting by multiplication, and x is a sequence of numbers. Under suitable assumptions H_x is realized as a closed linear operator acting in the Hilbert space of square integrable functions [5]. H_x is closed iff the graph $G(H_x) := \{(f, H_x f) \mid f \in D(H_x)\}$ is a closed set. If the multifunction $x \mapsto G(H_x)$ is open-measurable, we say that $\{H_x\}$ is a measurable family of closed operators. In this case it turns out that the spectrum of the operator gives a closed-valued universally open-measurable multifunction $x \mapsto \Sigma(x) := \sigma(H_x)$. Σ is typically even lower semicontinuous in the special case of the above Schrödinger operator. It is of particular importance to understand the dependence of the spectrum on the behavior of the sequence x at infinity. One possibility is to consider two sequences x, y to be equal at infinity $x \sim y$ if there exists an N such that for all $n \geq N$, $x_n = y_n$. The interesting part of the spectrum is then

the tail part Σ_{\sim} , which is the unique maximal part of the spectrum defined on the quotient given by \sim . The Souslin intersection theorem gives open-measurability of the tail part of the spectrum. Equip the set of sequences with the product measure $\mu = \otimes \mu_i$, where the μ_i 's are probability measures. We prove existence of a closed set A such that $\mu\{x \mid \Sigma_{\sim}(x) = A\} = 1$.

2 Preliminaries.

Let \mathcal{E} be a σ -algebra of sets in Ω , so \mathcal{E} is a family of subsets of Ω - closed under complements and countable unions. The empty set is countable, so $\emptyset \in \mathcal{E}$. Ω equipped with \mathcal{E} is a measurable space. Let $\mu : \mathcal{E} \rightarrow [0, \infty)$ be a finite measure, which means that the measure of the union of a disjoint countable family of measurable sets equals the sum of the measure of each set. In particular $\mu(\emptyset) = 0$, since an empty sum is zero. A measure space is a set equipped with a measure. \mathcal{E}_{μ} is the family of sets B such that there exists $A, C \in \mathcal{E}$ with $A \subset B \subset C$ and $\mu(C \setminus A) = 0$. The completion \mathcal{E}_{μ} of \mathcal{E} is a σ -algebra, and μ extends to \mathcal{E}_{μ} . The universal completion is $\hat{\mathcal{E}} := \bigcap_{\nu} \mathcal{E}_{\nu}$, for all positive finite measures ν . The sets in \mathcal{E} , $\hat{\mathcal{E}}$, and \mathcal{E}_{μ} are denoted measurable, universally measurable, and μ -measurable respectively. Observe $\mathcal{E} \subset \hat{\mathcal{E}} \subset \mathcal{E}_{\mu}$, and in particular $\hat{\mathcal{E}}_{\mu} = \mathcal{E}_{\mu}$. The σ -algebra generated by a family of subsets in Ω is the intersection of all σ -algebras containing the family. The σ -algebra $\otimes_{\alpha \in \Lambda} \mathcal{E}_{\alpha}$ is the σ -algebra generated by $\{\{\omega \in \prod_{\alpha} \Omega_{\alpha} \mid \omega(\beta) \in A_{\beta}\} \mid \beta \in \Lambda, A_{\beta} \in \mathcal{E}_{\beta}\}$. $\otimes_{\alpha} \mathcal{E}_{\alpha}$ is the product σ -algebra in $\prod_{\alpha} \Omega_{\alpha}$.

Let X be a topological space. This means that there is a family τ of subsets of X - closed under arbitrary unions and finite intersections. The empty union and intersection imply in particular $\emptyset, X \in \tau$. The sets in τ are called open and the complement of an open set is called closed. X is Hausdorff (T_2) if for any given $x \neq y$ there exist open disjoint sets U, V with $x \in U, y \in V$. X is a T_1 -space if every finite set is a closed set. A family $\{A_{\alpha}\}$ of sets covers a set K if $K \subset \bigcup_{\alpha} A_{\alpha}$. A set K is compact if every family $\{U_{\iota}\}$ of open sets which covers K contains a finite family $\{U_{\iota_1}, \dots, U_{\iota_N}\}$ which covers K . A finite set is compact, and in a Hausdorff space a compact set is closed. τ is C -separating [6] if for any given compact set K and a disjoint closed set F there exists an open set containing F , but not intersecting K . A space is T_1 and C -separated if and only if every compact set is closed. X is separable if there exists a countable set $D \subset X$ such that $D \cap U \neq \emptyset$ for all open sets $D \neq \emptyset$. A topology τ is second countable if there exists a countable $\tau_0 \subset \tau$ such that any open set is a union of sets from τ_0 . X is countably C -separated if there

exists a countable C -separating family of open sets. In this case every open set is a countable union of closed sets. If K is a subset of X , then $\tau_K := \{K \cap U \mid U \in \tau\}$ is the relative topology in K . X has second countable compacts if τ_K is second countable whenever K is compact. The family \mathcal{B} of Borel sets is the σ -algebra generated by the topology.

A metric d on a set S is a function $d : S^2 \rightarrow [0, \infty)$ such that for all x, y, z : $d(x, y) = 0 \Leftrightarrow x = y$, $d(x, y) = d(y, x)$, and $d(x, z) \leq d(x, y) + d(y, z)$. The set $\{y \mid d(x, y) < r\}$ is the open ball with center at x and radius r . The family of sets which are unions of open balls is a topology. A sequence x_1, x_2, \dots is convergent if there exists an x such that $d(x, x_1), d(x, x_2), \dots$ converges to 0. The sequence is Cauchy if $d(x_n, x_m)$ converges to 0 as $n, m \rightarrow \infty$. A metric space is complete if every Cauchy sequence is convergent. A Polish space is a separable, complete metric space. A Souslin space S is a metric space which is the range of a continuous function with a Polish domain [1]. Let $A \in \mathcal{E} \otimes \mathcal{B}$, where \mathcal{E} is a σ -algebra in Ω and \mathcal{B} is the family of Borel sets in a Souslin space S . The projection

$$\Pr_{\Omega} A := \{\omega \mid (\omega, x) \in A\}$$

need not be measurable, but it is universally measurable by the Souslin projection theorem [2, p.75, Thm. III.23].

A multifunction $F : \Omega \rightarrow X$ is a function $F : \Omega \rightarrow \mathcal{P}(X)$. This means that for each $\omega \in \Omega$ there is assigned a subset $F(\omega)$ of X . Let U be a subset of X . The set of ω such that $F(\omega)$ intersects U is denoted by

$$F^{-1}(U) := \{\omega \mid F(\omega) \cap U \neq \emptyset\}.$$

In particular F^{-1} is a multifunction $F^{-1} : \mathcal{P}(X) \rightarrow \Omega$. If $f : \Omega \rightarrow X$ is a function, then $\{f\} : \Omega \rightarrow X$ is the singleton-valued multifunction $\omega \mapsto \{f(\omega)\}$. We observe $f^{-1} = \{f\}^{-1}$, so statements about a function f which only involves f^{-1} may have natural generalizations to multifunctions. F^{-1} preserves unions

$$F^{-1}\left(\bigcup_{\lambda} A_{\lambda}\right) = \bigcup_{\lambda} F^{-1}(A_{\lambda}),$$

but not intersections, since

$$F^{-1}(A \cap B) \subset F^{-1}(A) \cap F^{-1}(B),$$

and the opposite inclusion is false. f^{-1} preserves unions, the empty set, the full set, intersections, complements, and disjoint unions. Only the first two of these six statements generalise to F^{-1} . The graph of F is $Gr F := \{(\omega, x) \mid x \in F(\omega)\}$. Gr is

a bijection between multifunctions from Ω to X and subsets of $\Omega \times X$. Gr preserves unions, the empty set, the full set, intersections, complements, and disjoint unions.

Let Ω be a measurable space, let X be a topological space, and let $F : \Omega \rightarrow X$ be a multifunction. F is (universally) open-measurable if $F^{-1}(U)$ is (universally) measurable for all open sets U . Borel-, closed-, compact-measurable is defined likewise. If F is singleton-valued, then Borel-, closed-, and open-measurability are equivalent. If X is a metric space and F is closed-valued, then Borel- implies closed- implies open- implies compact-measurability, and furthermore

$$Gr F = \{(\omega, x) \mid d(x, F(\omega)) = 0\},$$

so F has a measurable graph if F is open-measurable. If X is Hausdorff and a countable union of compacts, then compact- implies closed-measurability. If F is open-measurable and non-empty closed-valued into a Polish space, then

$$F(\omega) = \{x_1(\omega), x_2(\omega), \dots\}^-$$

for suitable measurable x_i 's. This is the selection theorem [2, p.65, Theorem III.6]. If F has a measurable graph, then F is universally Borel-measurable. This follows from the identity

$$F^{-1}(A) = \Pr_{\Omega}[\Omega \times A \cap Gr F],$$

and the Souslin projection theorem. A family $\{F_s\}_{s \in S}$ of multifunctions is compact-measurable if the multifunction $(\omega, s) \mapsto F_s(\omega)$ is compact-measurable.

3 Intersection of Multifunctions.

In this section we will prove the compact-measurability results on intersection of compact-measurable multifunctions. Some examples and consequences are discussed in the final section.

Since the idea of countable separation as found in [6] is essential in the following we include a proof of:

Lemma 3.1 *In a second countable Hausdorff space there exists a countable basis which Hausdorff separates the compacts.*

Proof. The family \mathcal{B} of finite unions of sets from a countable basis is a countable separant for the compacts: Let M and N be disjoint compact sets. The Hausdorff property gives disjoint open sets U' and V' with $M \subset U'$ and $N \subset V'$. U' is a union of sets from the basis. This gives a covering of M and the compactness of M gives

a finite subcovering. The union U of the sets in the finite subcovering belongs to \mathcal{B} . A set V in \mathcal{B} with the property $N \subset V \subset V'$ is found likewise. The conclusion is the existence of two disjoint open sets U and V in the basis \mathcal{B} with $M \subset U$ and $N \subset V$. \square

The proof of the following Lemma is a modification of the separation ideas found in [6] where open-measurability of the intersection of two compact-valued open-measurable multifunctions into a countably C -separated space is proven. A separable metric space is countably C -separated, but a non-separable metric space is not countably C -separated since a countably C -separated space is separable. The following holds in particular for multifunctions into a metric space.

Lemma 3.2 *The intersection $F \cap G$ of two closed-valued compact-measurable multifunctions into a Hausdorff space with second countable compacts is a compact-measurable multifunction.*

Proof. Let K be compact. The previous Lemma gives a countable family \mathcal{B} of open sets which Hausdorff separates the compacts in K . If $(U, V) \in \mathcal{B}' \Leftrightarrow [U, V \in \mathcal{B}, U \cap V = \emptyset]$, then

$$\{\omega \mid F(\omega) \cap G(\omega) \cap K = \emptyset\} = \bigcup_{(U, V) \in \mathcal{B}'} \{\omega \mid F(\omega) \cap K \cap U^c = \emptyset \text{ and } G(\omega) \cap K \cap V^c = \emptyset\}.$$

This proves compact-measurability of the multifunction $F \cap G$ since $K \cap U^c$ and $K \cap V^c$ are compact sets. \square

The next Theorem generalizes from finite intersections of multifunctions to countable intersections. Theorem 3.4 in [3] gives the same result in the case of multifunctions into a metric space X . Our proof is different and gives a more general result since a metric space has second countable compacts, but the converse is false.

Theorem 3.3 *The intersection of a sequence of closed-valued compact-measurable multifunctions into a Hausdorff space with second countable compacts is a compact-measurable multifunction.*

Proof. When K is compact, the set

$$\{\omega \mid [\bigcap_{n \geq 1} F_n(\omega)] \cap K = \emptyset\} = \bigcup_{m \geq 1} \{\omega \mid [\bigcap_{1 \leq n \leq m} F_n(\omega)] \cap K = \emptyset\}$$

is measurable since a finite intersection of closed-valued compact-measurable multifunctions is compact-measurable. \square

Our main result on intersections is the Souslin intersection Theorem:

Theorem 3.4 *Let S be a Souslin space. The intersection $\bigcap_{s \in S} F_s$ of a compact-measurable family $\{F_s\}_{s \in S}$ of closed-valued multifunctions into a Hausdorff space with second countable compacts is a universally compact-measurable multifunction.*

Proof. If K is compact and $F_s(\omega)$ is closed, then

$$\{\omega \mid [\bigcap_{s \in S} F_s(\omega)] \cap K = \emptyset\} = \bigcup_{n=1}^{\infty} \bigcup_{\vec{s} \in S^n} \{\omega \mid [\bigcap_{i=1}^n F_{s_i}(\omega)] \cap K = \emptyset\}.$$

S^n is a Souslin space, so the above set is universally measurable from the Souslin projection theorem and the compact-measurability of $(\omega, \vec{s}) \mapsto [\bigcap_{i=1}^n F_{s_i}(\omega)]$ which we proceed to prove. Let $G_i(\omega, \vec{s}) := F_{s_i}(\omega)$, so $G_i^{-1}(A) = \{(\omega, \vec{s}) \mid F_{s_i}(\omega) \cap A \neq \emptyset\}$. The multifunction G_i is compact-measurable from the compact measurability of the family $\{F_s\}_{s \in S}$. The multifunction $\bigcap_{i=1}^n G_i$ is then also compact-measurable. \square

The following Theorem is a kind of dual to the previous.

Theorem 3.5 *A countable intersection of closed-measurable compact-valued multifunctions into a second countable Hausdorff space is closed-measurable. Let S be a Souslin space. The intersection $\bigcap_{s \in S} F_s$ of a closed-measurable family $\{F_s\}_{s \in S}$ of compact-valued multifunctions into a second countable Hausdorff space is a universally closed-measurable multifunction.*

Proof. This is similar to the above. \square

The following Theorem generalizes Theorem 3.7 in [6] since a compact-valued multifunction into a countably C -separated space is open-measurable if and only if it is closed-measurable. We remove the Hausdorff assumption, and go beyond countable intersections.

Theorem 3.6 *A countable intersection of closed-measurable compact-valued multifunctions into a countably C -separated T_1 -space is closed-measurable. Let S be a Souslin space. The intersection $\bigcap_{s \in S} F_s$ of a closed-measurable family $\{F_s\}_{s \in S}$ of compact-valued multifunctions into a countably C -separated T_1 -space is a universally closed-measurable multifunction.*

Proof. Let K be compact and let $\{U_1, U_2, \dots\}$ be the C -separating family of open sets. For each $x \in K^c$ there is an i with $x \in U_i \subset K^c$, since $\{x\}$ is closed by the T_1 assumption. This proves that K is closed. Let F_i be closed-measurable compact-valued multifunctions, and let G be a closed set. $F_1(\omega)$ is closed and $F_2(\omega) \cap G$ is compact, so

$$\{\omega \mid F_1(\omega) \cap F_2(\omega) \cap G = \emptyset\} = \bigcup_i [\{\omega \mid F_1(\omega) \cap U_i^c = \emptyset\} \cap \{\omega \mid F_2(\omega) \cap G \cap U_i = \emptyset\}].$$

This set is measurable, because $F_2 \cap G$ is closed-measurable and then also open-measurable. $F_1(\omega)$ is compact, so

$$\{\omega \mid [\bigcap_{n \geq 1} F_n(\omega)] \cap G = \emptyset\} = \bigcup_{m \geq 1} \{\omega \mid [\bigcap_{1 \leq n \leq m} F_n(\omega)] \cap G = \emptyset\},$$

and the countable intersection result is proven. The Souslin intersection result follows as before. \square

4 Examples.

Let X be the set of complex numbers equipped with the usual topology τ and family \mathcal{B} of Borel sets. Equip $E := \prod_{i=1}^{\infty} X$ with the σ -algebra $\otimes_i \mathcal{B}$ and the measure $\mu = \otimes_i \mu_i$ where $\mu_i(X) = 1$.

4.1 Spectrum of a sequence.

The spectrum is the multifunction $Sp : E \rightarrow X$ defined by

$$Sp(x) := \{x_1, x_2, \dots\}^-.$$

Sp is closed-valued and open-measurable from

$$Sp_i(x) := \{x_i\},$$

$$Sp = [\bigcup_i Sp_i]^-,$$

$$Sp^{-1}(U^{\text{open}}) \stackrel{(1)}{=} [\bigcup_i Sp_i]^{-1}(U) \stackrel{(2)}{=} \bigcup_i Sp_i^{-1}(U),$$

$$Sp_i^{-1}(U) = \{x \mid x_i \in U\},$$

since the sets $\{x \mid x_i \in U\}$ generate the product σ -algebra. The equality $\stackrel{(1)}{=}$ proves more generally that a multifunction is open-measurable if and only if the closure of the multifunction is open-measurable. The equality $\stackrel{(2)}{=}$ does not depend on the countability of the index set and gives more generally the measurability of the union of a measurable family of multifunctions. The corresponding statement for intersections is false and this is why measurability of the intersection of multifunctions is a more delicate question. The intersection relation corresponding to $\stackrel{(2)}{=}$ is

$$[\bigcap_i F_i]^{-1}(A) \subset \bigcap_i F_i^{-1}(A).$$

4.2 The selection theorem.

Let $x = (x_1, x_2, \dots)$ be a sequence of complex valued measurable functions. The above gives open-measurability of the multifunction $\Sigma : \Omega \rightarrow X$

$$\omega \mapsto \Sigma(\omega) := Sp(x(\omega)).$$

Each x_i is a selection for Σ since $x_i(\omega) \in \Sigma(\omega)$. The sequence x is a dense selection for Σ .

Let $\omega \mapsto F(\omega)$ be a closed non-empty valued open-measurable multifunction into X . $X = \mathbf{C}$ is Polish so the selection theorem gives a dense measurable selection x and $F = Sp(x)$. The selection is not unique.

4.3 The essential spectrum.

The essential spectrum of the sequence x is

$$Sp_{ess}(x) := \bigcap_{n \geq 1} Sp(x_n, x_{n+1}, \dots).$$

It follows from the countable intersection theorem that Sp_{ess} is compact-measurable, since each $x \mapsto Sp(x_n, x_{n+1}, \dots)$ is compact-measurable. $X = \mathbf{C}$ is a countable union of compacts, and Sp_{ess} is then also open-measurable.

4.4 The tail part of a multifunction

Let $\sigma : E \rightarrow X$ be a closed-valued open-measurable multifunction. The tail dependent part of σ is given by

$$\sigma_{\sim}(x) := \bigcap_{n \geq 1} \bigcap_{\vec{y} \in \mathbf{C}^{n-1}} \sigma(\vec{y}, x_n, x_{n+1}, \dots).$$

σ_{\sim} is then universally open-measurable from the Souslin intersection theorem. This conclusion holds also if we only assume that $x \mapsto \sigma(x)$ is a closed-valued universally open-measurable multifunction into X : The set Y of all closed sets in X is equipped with the σ -algebra \mathcal{F} generated by sets $\{A^{closed} \mid A \cap U^{open} \neq \emptyset\}$. This σ -algebra is countably generated. Then there exists a closed-valued open-measurable ς such that $\mu\{x \mid \varsigma(x) \neq \sigma(x)\} = 0$ [4, p.103], and $\mu\{x \mid \varsigma_{\sim}(x) \neq \sigma_{\sim}(x)\} = 0$ which gives the claim.

The σ -algebra \mathcal{F} contains sets Z_1, Z_2, \dots which separates the points in Y . Kolmogorov's zero-one law [4, p.104] gives $\mu\{x \mid \sigma_{\sim}(x) \in Z_i\} \in \{0, 1\}$. We conclude existence of a closed set $A \subset X$ such that $\mu\{x \mid \sigma_{\sim}(x) = A\} = 1$.

5 Multifunctions restricted to equivalence classes.

Define the relation \sim on E by

$$x \sim y \Leftrightarrow [\exists N \forall n \geq N x_n = y_n].$$

The relation is an equivalence relation since it is transitive ($x \sim y, y \sim z \Rightarrow x \sim z$), reflexive ($x \sim x$), and symmetric ($x \sim y \Rightarrow y \sim x$). In the previous subsection we established the measurability of the tail dependent part

$$\sigma_{\sim}(x) = \bigcap_{y \sim x} \sigma(y)$$

of σ . In the following we will consider a generalization of this.

Let \sim be an equivalence relation on a set E and consider a multifunction $F : E \rightarrow X$. The \sim -part F_{\sim} of F is $F_{\sim}(x) = \bigcap_{y \sim x} F(y)$. It follows that F_{\sim} is

(i) Included $F_{\sim}(x) \subset F(x)$.

(ii) Invariant $x \sim y \Rightarrow F_{\sim}(x) = F_{\sim}(y)$.

(iii) Maximal If G is included in F and is \sim -invariant, then $G \subset F_{\sim}$.

Conversly it follows that a multifunction H which is included, invariant, and maximal is unique, so $H = F_{\sim}$.

The measurability of F_{\sim} is a fundamental question. As shown above the Souslin intersection theorem gives a positive result for compact-measurability in a special case. A negative result for open-measurability follows from the next subsection.

5.1 A general union result.

Let $G : \Omega \rightarrow E$ and $F : \Omega \times E \rightarrow X$ be multifunctions. Define the multifunction $H : \Omega \rightarrow X$ by

$$H(\omega) := \bigcup_{b \in G(\omega)} F(\omega, b).$$

The following identity holds

$$H^{-1}(U) = P_{\Omega} [\Omega \times E \times U \cap Gr G \times X \cap Gr F].$$

Assume open-measurability of F and G and that E and X are Polish. It follows that H is universally open-measurable.

Consider next the multifunction $K : \Omega \rightarrow X$ defined by

$$K(\omega) := \bigcap_{b \in G(\omega)} F(\omega, b).$$

A special case is given by $\Omega = E$, $b \in G(\omega) \Leftrightarrow b \sim \omega$, and $F(\omega, b) = F(\omega)$, in which case $K = F_{\sim}$. Measurability properties of K follows from the identity

$$Gr K^c = Pr_{\Omega \times X} [Gr G \times X \cap Gr F^c].$$

Assume in addition to the above that Ω is Polish and is equipped with the corresponding Borel field. It follows that $Gr K$ is Borel if and only if

$$Pr_{\Omega \times X} Gr G \times X \cap Gr F^c$$

is Borel, since the complement of a Souslin set is Souslin if and only if it is Borel [1, p.205]. This implies that open-measurability of K fails without further assumptions, since there exist Borel sets which have non-Borel projections. The question of compact-measurability of K is left open.

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