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MULTIMODAL METHOD IN SLOSHING

I. A. Lukovsky[1], **A. N. Timokha**[1,2]

[1] *Institite of Mathematics, National Academy of Sciences of Ukraine Ukraine, 01601, Kiev, Tereschenkivska, 3 str.*
[2] *Centre for Autonomous Marine Operations and Systems, Department of Marine Technology, Norwegian University of Science and Technology Norway, NO-7491, Trondheim*

The multimodal method reduces the free–surface sloshing problem to a (modal) system of nonlinear ordinary differential equations. The method was originally proposed for non–impulsive hydrodynamic loads but, recently, it was successfully extended to the sloshing–induced slamming. In the 50-60's, the method was employed in the Computational Fluid Dynamics (CFD) but has lost the contest the algorithms of the 90-00's. Nowadays, the method plays the dual role: firstly, as a unique analytical tool for studying nonlinear sloshing regimes, their stability, and chaos as well as for simulations when traditional CFD fails (e.g., containers with a perforated screen) and, secondly, as a source of the modal systems which are analogies of the Kordeweg–de–Vries, Boussinesq, etc. equations but for the contained liquid. The paper surveys the state–of–the–art and existing modal systems, specifies open problems.

1. Introduction. Coupled rigid tank–and–sloshing is a *hybrid mechanical system* in which the tank moves, normally, with six degrees of freedom governed by a system of ordinary differential equations (ODEs) but sloshing is described by a free–surface problem suggesting an infinite number of degrees of freedom. The multimodal method introduces the *hydrodynamic generalised coordinates* (HGCs) and derives a (modal) system (MS) of nonlinear ODEs with respect to them. The MSs facilitate analytical studies on resonant sloshing regimes, their stability, secondary resonance, chaos, etc. that looks questionable with traditional CFD.

The present review runs through the context of a plenary lecture delivered by the authors to experts in differential equations on the Bogolyubov Reading (DIFF–2013, June, 2013, Sevastopol, Ukraine) where, along with historical aspects, ideas and open problems of the multimodal method, the structure, dimension and features of the derived nonlinear MSs (NMSs), their solutions *vs.* container shape, liquid filling (depth), and forcing were discussed. The paper is schematically divided into three sections representing the past, the present, and the future of the multimodal method, respectively. Readers interesting in Faraday waves are referred to [92, 93]. Newbies in sloshing are recommended to have one of the textbooks [58, 91, 144] whose subject indexes should help understanding the terminology.

2. The past. Perhaps, the word *"multimodal"* comes from [50]. However, the multimodal method was first proposed forty years behind, in the 50–60's, when researchers were first real challenged by sloshing in aircraft, spacecraft and marine containers. An enthusiastic atmosphere of these years is described in the memoir article of Abramson [3] who headed the corresponding NASA program. A scientific heritage of the pioneering studies was systematised in [2, 34, 67, 89, 203, 209, 210] (USA) and [1, 66, 103, 110, 154, 155, 165, 166, 168, 190] (Soviet Union). An emphasis was, primary, on *theoretical linear* (small-amplitude) sloshing, *experimental nonlinear* sloshing phenomena, and slosh–suppressing devices. The USA collection of experiments was best reviewed in [2]. Experiments of the SU–scientists were conducted in Moscow, Kiev [110, 144], Dnipropetrovsk [16], and Tomsk [17,19] but they remain, mostly, unpublished.

The ***linear multimodal method*** (see, original [174,181,189], canonical [66,145,154,214] and contemporary [58,170] descriptions) was proposed in the 50's to reduce the linear sloshing problem to an infinite set of linear oscillators (ODEs) called, all together, the *linear* MS (LMS) in which the inhomogeneous terms are functions of the six generalised coordinates (degrees of freedom) of the rigid body motion but the unknowns are HGCs responsible for amplifications (relative to the hydrostatic shape) of the natural sloshing modes. The method interprets sloshing as a conservative mechanical system with an infinite number of degrees of freedom; it needs to know, *a priori*, the *natural sloshing modes*, φ_n, and frequencies, σ_n, as well as the linear *Stokes–Joukowski potentials*, $\mathbf{\Omega}_0 = (\Omega_{01}, \Omega_{02}, \Omega_{03})$. Coupling LMS and dynamic equations of the carrying rigid container is facilitated by the linearised *Lukovsky formulas* which express the hydrodynamic forces and moments in terms of HGCs. The hydrodynamic coefficients in both LMS and the Lukovsky formulas are integrals over $\mathbf{\Omega}_0$, φ_n, and their derivatives. This means that having known $\mathbf{\Omega}_0$ and φ_n, by utilising analytical and/or numerical methods of ODEs makes it possible to find the *semi–analytical solution* of any linear sloshing problem (see, Ch. 5 of [58] and [59,101,102,226]) and, thereby, describe the *linear coupled "rigid tank–sloshing" dynamics*.

The three Stokes–Joukowski velocity potentials, $\Omega_{0i}, i = 1, 2, 3$, deduce the inhomogeneous forcing terms associated with three angular degrees of freedom of the rigid tank. They are solutions of the Neumann boundary problems in the mean (hydrostatic) liquid domain Q_0 (the linear Stokes–Joukowski potential problems, LSJPPs) which were first derived by Nikolay Joukowski (1885) [99] when examining a spatially–moving rigid body with a cavity completely filled by an ideal incompressible liquid. Exact analytical Ω_{0i} are a rare exception (Ch. 5 of [58]).

The natural sloshing modes are eigenfunctions of a spectral boundary problem (the natural sloshing problem, NSP) in Q_0. The spectral parameter κ appears in the mean free-surface (Σ_0) boundary condition and $\sigma_n = \sqrt{\kappa_n g}$ (g is the gravity acceleration). The traces $\varphi_n|_{\Sigma_0}$ define the standing wave patterns which were first described for an upright circular basin by Mikhail Ostrogradsky. His manuscript [182] was submitted to the Paris Academy of Sciences in 1826 and, later on, revisited by Poisson and Rayleigh [35] for other tank shapes. Rigorous mathematical theory of NSP was created in the 60's (Ch. VI in [66] and [39,170]). It states, in particular, that (i) the spectrum consists uniquely of positive eigenvalues, κ_n, with the only limited point at the infinity (in contrast to the external water waves problem which yields a continuous spectrum) and (ii) $\varphi_n|_{\Sigma_0}$ constitute, together with a nonzero constant, a Fourier basis on Σ_0. The fact (i) is important for understanding why the Kordeweg–de–Vries, Boussinesq, etc. equations (sea waves) and MSs (contained liquid) follow from the same free–surface problem but are of the different mathematical nature. The fact (ii) is a foundation stone for introducing the HGCs. In the 60–70's, S. Krein [106,109] (see, also [28,29]) generalised these results to a *viscous incompressible liquid*, but Kopachevskii — to a *capillary one* (Part II in [172] and [106,107]).

In the 50-80's, the research focus was on constructing *analytically approximate solutions of NSP and LSJPP*. Brilliant ideas were proposed (see, a collection of them in [145] and an amazing solution for circular/spherical tank in [23]). Because of the volume conservation condition, these solutions were, normally, obtained by the Trefftz method suggesting an analytical harmonic functional basis exemplified by the harmonic polynomials [145] whose completeness in the star–shaped domains was proved in [216,217]. The Trefftz solutions become especially efficient and provide a uniform convergence when the corner–point singularity [66, 104, 105, 223, 224] is accounted for [59, 65]. Nowadays, the harmonic polynomials are extensively used in numerical methods, e.g., in the Harmonic Polynomial Cell (HPC) method [197, 198]. Growing computer facilities of the 90–00's made finding φ_n and $\mathbf{\Omega}_0$ *no problem*. An exception may be non-smooth tanks equipped with baffles, screens, etc. that leads to the strongly singular behaviour of $\nabla\varphi_n$ [59–62, 65, 79].

The ***theoretical nonlinear sloshing*** was founded by Penny & Price [184], Moiseev [167], Narimanov [175], and Perko [169, 185] in the 50–60's. By adopting the perturbation theory technique [184], ***Moiseev*** [167] showed how to construct, analytically, an *asymptotic steady–state* (*periodic*, frequency–domain) solution of the nonlinear sloshing problem in a rigid tank performing a prescribed horizontal and/or angular small–amplitude harmonic motion with the forcing frequency σ close to the lowest natural sloshing frequency σ_1. He assumed a finite liquid depth of an ideal incompressible liquid with irrotational flows and proved that, if the nondimensional forcing amplitude (scaled by the tank breadth) is a small parameter $\epsilon \ll 1$, the primary excited mode(s) is of the order $O(\epsilon^{1/3})$ and the matching resonant asymptotics (the so-called Moiseev detuning) is $|\sigma^2 - \sigma_1^2|/\sigma_1^2 = O(\epsilon^{2/3})$. The Moiseev technique implicitly assumed that there is no the so-called *secondary resonance* [22, 56–58, 225]. It was originally realised for a two-dimensional rectangular tank [44, 178]. Other tank shapes were considered in [11, 89, 151, 202]. Obtaining the Moiseev solution yields huge and tedious derivations. In the 80's, looking for an *almost-periodic* sloshing, ***Miles*** [157, 158] generalised Moiseev's results by deriving the so-called *Miles equations* which govern a slow-time variation of dominant, $O(\epsilon^{1/3})$, amplitudes of the primary excited HGC(s). He considered a horizontal harmonic excitation of an upright circular cylindrical tank, adopted the Moiseev asymptotic ordering and detuning; separation of the fast and slow time scales which was done, directly, in the *Bateman–Luke variational* principle [9, 84, 126]. The Miles equations were later derived for an upright rectangular tank. Using these equations is a rather popular approach in applied mathematical studies on almost-periodic resonant sloshing, in detecting periodic orbits and clarifying the chaos [88, 157, 158]. Both horizontal and vertical (Faraday waves) harmonic excitations were in focus [70, 71, 85, 88, 158–163]; [108, 199] used the Miles equations for studying the "rigid tank–contained liquid" system with a limited power supply forcing.

By using the perturbation theory, ***Narimanov*** [175] derived a *historically–first* version of *weakly–nonlinear modal systems* WNMSs. Narimanov did not know Mosieev's results, but he postulated asymptotic relationships between the HGCs and *hydrodynamic generalised velocities* (HGVs) as if these may follow from the Moiseev solution. The original derivations in [175] (the same in [204–207]) have had algebraic errors which were corrected by Lukovsky [136, 140, 144, 176]. Narimanov's technique leads to huge and tedious derivations increasing, dramatically, with increasing number of HGCs. As a consequence, all existing Narimanov's modal systems are of a low dimension; they couple from two to five HGCs. These systems were derived for upright tanks of circular, annular and rectangular cross-sections, conical and spherical tanks as well as for an upright circular cylindrical tank with a rigid–ring baffle [80, 136, 140, 176]. Nowadays, this *method is rarely employed* being replaced by variational versions of the multimodal method [127–129] based, normally, on the Bateman–Luke variational principle [9, 58, 84, 126, 144] which deduces, in a natural way, both dynamic and kinematic relations of the sloshing problem [58, 140, 195, 221, 222]. Most probably, such a variational multimodal method was first proposed in 1976 by ***Miles*** and ***Lukovsky*** [140, 156] who derived, independently, a *fully nonlinear modal system* (Miles–Lukovsky system, MLS) with respect to HGCs and HGVs for sloshing in an upright tank performing a prescribed translatory motion. Later on, Lukovsky derived the MLS for an arbitrary rigid tank motion [50,137], proposed the so-called *nonconformal mapping technique* to get the MLS for tanks with non-vertical walls [63,75,133,136,143], and derived the so-called *Lukovsky formulas* for the hydrodynamic force and moment [140,144] (Ch. 7 of [58] gives an alternative derivation). He also showed how to use the Bateman–Luke formalism for deriving the dynamic equations of the coupled "rigid tank–contained liquid" system [139,140] and to incorporate the damping into the variational formulation [141,144]. Taking asymptotic relationships between HGCs and HGVs reduces MLS to a weakly–nonlinear form yielding what is today called the *weakly–nonlinear multimodal systems* (WNMSs, [58,86,130,134,144]). Both MLS and WNMs require that the *analytically approximate natural sloshing modes*, φ_n, and the *nonlinear Stokes–Joukowski potentials*, $\mathbf{\Omega}$, are analytically continuable over Σ_0. This and other limitations are discussed in Ch. 7 of [58] and in [63,86,134].

The rare nonlinear sloshing simulations of the 60-80's were based on the Galerkin scheme, finite difference (maker-and-cell, etc.) and finite element methods [6,40,41,45,171,173,180,191, 200,208]. By expanding the velocity potential in a Fourier series by the natural sloshing modes, ***Perko*** [169,185] proposed a *numerical multimodal method* for simulating the short–term transients. In the 70-80's, Perko's method was used, with modifications, by *Chakhlov* [18,37,38] and *Limarchenko* [114–117,119,121]. Because Limarchenko also utilised the Lagrange variational principle and the perturbation theory, he has proposed, in fact, a *weakly–nonlinear variational numerical multimodal method* which is, generally, not applicable for analytical studies but rather for *ad hoc* simulations.

The ***CFD simulations*** of the nonlinear sloshing gave rise *en masse* only in the 90's. The success was initially associated with FLOW–3D which is a famous Navier–Stokes commercial solver of the 90-00's (numerous examples were given by Solaas [201] who discussed advantages and drawbacks). Volume of Fluid (VoF), Smoothed Partitions Hydromechanics (SPH), and their modifications provided, using a parallel computation, rather accurate, efficient and robust simulations. Readers are referred to [25] which reviews the numerical sloshing of the 80–90's.

3. The present. In the 00's, the *theoretical nonlinear sloshing* split into almost–independent *numerical* and *analytical* directions. Recent advances of the ***numerical sloshing*** are reported in [192] (see, also introductions in [24,42,82,211,219,227,228]). The best CFD solvers are based on a viscous and fully-nonlinear statement and allow, by conducting simulations with different initial scenarios (conditions), for modelling the special free–surface phenomena: fragmentation, wave breaking, overturning, roof and wall impacts, and flip-through. These phenomena cannot be accurately described within the framework of physical (mathematical) models adopted by the ***analytical sloshing*** that are, typically, of the weakly–nonlinear nature, assume ideal potential flows, and smooth instant free-surface patterns. In the XIX century, the same splitting has occurred in the sea waves theory which is nowadays investigated, almost independently, with CFD and approximate analytical models. The latter models are exemplified by the Kordeweg–de–Vries, Boussinesq, etc. partial differential equations [183]. The approximate analytical models are typically not used in practical computations but rather focus on quantifying and classifying surface waves, their stability, chaos, and others *vs.* the initial scenarios and input parameters, the cases, when the CFD is not very efficient.

Because the *Perko type method* pursues direct numerical simulations but employs simplified mathematical models of the analytical sloshing, it lost the context both CFD (in an accurate computing of the hydrodynamic loads and descriptions of the aforementioned free–surface phenomena) and the *analytical sloshing* (in the input/initial parameter studies). This explains why the method is rarely used, today, restricted to a few fully–nonlinear simulations by MLS for a rectangular tank [111, 196] and weakly-nonlinear ones in [47, 55, 118–120, 122–124]. Another drawback of the Perko type simulations is that these are unrealistically *stiff* so that artificial damping terms must be incorporated to damp the rising parasitic higher harmonics [47,55] and, thereby, to facilitate computations on a relatively long time scale. Conducting the simulations may be justified for the screen–equipped tanks when the screen–induced damping is very high and there are no reasons in the artificial damping terms [122–124].

Reincarnation of the multimodal method as a powerful tool of the *analytical sloshing* is, to some extent, explained by growing practical interest to the smooth (without internal structures) Liquefied Natural Gas (LNG) tanks. The revolutionary paper [50] re-derived MLS and initiated a series of publications on WNMSs, mostly, for *upright tanks of rectangular and circular (annular) cross–sections*, when exact analytical φ_n and $\mathbf{\Omega}_0$ exist, and liquid depths are finite. The WNMSs were applied for description of steady–state and transient resonant waves; they were validated by model tests.

The ***two-dimensional*** weakly–nonlinear resonant sloshing in a ***smooth rectangular tank*** with a ***finite liquid depth*** was studied by using diverse WNMSs [50, 56, 68, 86, 87, 94, 95, 98]. The forcing was small, $O(\epsilon)$, and the forcing frequency σ was close to the lowest natural sloshing frequency σ_1. The system [50] was based on the Narimanov–Moiseev asymptotics. Transient and steady–state predictions were validated for both a prescribed harmonic excitation [50] and a coupling with external surface waves (floating tank) [112,113,193]. The steady–state resonant sloshing was characterised by the Duffing like response curves with the theoretically soft spring as the *depth-to-breadth ratio* $>$ 0.3368... and the hard spring is for $<$ 0.3368... so that the *theoretical nondimensional critical depth* $=$ 0.3368... [58, 220]. A purely mathematical analysis of the WNMS was reported in [68, 86, 87].

The Narimanov–Moiseev WNMS [50] becomes physically inapplicable with *increasing the forcing amplitude* and at the *critical and small (shallow) liquid depths* due to the ***secondary resonance***, $n\sigma \approx \sigma_n$ for some n, that leads to a nonlinearity–driven amplification of the $n\sigma$ harmonics and an energy transfer from primary (σ_1) to secondary (σ_n) excited HGCs. Handling the secondary resonance for a finite liquid depth requires the so-called *adaptive modal systems* (AMSs) [56,87] suggesting a few extra dominant (secondary excited) HGCs $\sim O(\epsilon^{1/3})$ for which a dominant higher harmonics contribution is theoretically predicted ($n\sigma \approx \sigma_n$). The adaptive multimodal method concept was extensively validated by experiments [55–57]. Whereas the secondary resonance occurs, the response curves are characterised by double peaks at the primary resonance zone; the peaks grow with increasing the forcing amplitude. The AMSs and their structure are intensively discussed in Ch. 8 of [58]. Based on the AMS modelling, [87] showed that the *critical depth* is a function of the *forcing amplitude* (0.3368... is the limit case as $\epsilon \to 0$) and explains the experimental value 0.28 in [69]. The AMSs [94, 95, 98] use a *sophisticated asymptotic ordering* which is not based on the the secondary resonance concept but rather on experimental observations of the surface wave patterns.

The commensurate/almost–commensurate ***shallow/small depth*** liquid sloshing spectrum leads to a hydrodynamic jump [218]/multiple secondary resonances [57]. Deriving the WNMS [57] for small liquid depths in a rectangular tank needed a Boussinesq fourth–order asymptotic ordering (combining Moiseev's and Kordeweg–de–Vries' asymptotics [177–179]) where all the HGCs and the nondimensional liquid depth were the same order $O(\epsilon^{1/4})$. By truncating this infinite–dimensional system and incorporating the ***linear damping*** terms (due to the laminar boundary layer and the bulk viscosity; Ch. 6 in [58] and [27,100,152,164,215]) provided a good agreement with experiments [32, 33, 57], for both steady-state and transient sloshing. As in Chester's experiments [32,33,57], the theoretical response curves [5,57] had then a fingers–like shape with many peaks at the primary resonance zone ($\sigma \approx \sigma_1$). Increasing the excitation amplitude and/or decreasing the liquid depth made the small–depth WNMS [57] physically inapplicable due to the breaking and overturning waves, bores as well as the free-surface fragmentation which yield an enormously large damping. A detailed experimental classification of the shallow water sloshing (four different wave types were described), in general, and these phenomena, in particular, was reported in Ch. 8 of [58]. Damping due to the aforementioned free–surface phenomena is similar to that for the ***roof impact*** whose effect was included into the WNMSs of [50,56] in [49] by using the Wagner theory.s Accounting for the damping in the shallow water case is a challenge.

The damping due to the flow separation through a ***perforated screen*** was included into the WNMSs [46–48,122] for sloshing in a rectangular two–dimensional tank with a finite depth. For smaller solidity ratios of the screen, $0 < Sn < 0.5$, and a relatively small forcing, utilising the pressure drop condition [15] yielded an integral term into the existing modal systems [46,122]; the modified WNMSs showed a satisfactory agreement with experiments. A higher solidity ratio, $0.5 < Sn < 1$, also modified the natural sloshing modes and frequencies [60] and, thereby, both linear [48] and nonlinear (increasing excitation amplitudes) [47] WNMSs changed their analytical structure; the secondary resonance became then evident so that the secondary resonant peaks in the primary resonant zone differed from those for the smooth rectangular tank. Numerous WNMSs for various tanks with perforated screens were derived, studied and validated in [83,122–125].

Generalisation of the two-dimensional results [50] to the case of a ***three–dimensional rectangular*** tank was done in [52]. A focus was on the *nearly–square cross–section* leading to the degenerating Stokes natural sloshing modes including those two for the lowest natural frequency σ_1. The corresponding Narimanov–Moiseev WNMS in [52, 54] has nine degrees of freedom with the two dominant $O(\epsilon^{1/3})$ HGCs. The system provided an accurate *classification* of steady–state regimes (planar, diagonal, nearly–diagonal, and swirling) for both longitudinal and diagonal harmonic tank excitations [51, 52]. The asymptotically periodic solution of the WNMS implies an amazing 3D bifurcation diagram, especially, when the cross–section aspect ratio changes from one [20, 21, 54]; a good qualitative agreement with experiments was shown, including for estimating the chaos region. On the other hand, the theoretical transient and steady-state wave response was not quantitatively supported by experiments due to the secondary resonance effect which becomes especially evident for swirling, even if the excitation amplitude was small enough. Accounting for the secondary resonance effect led to the multidimensional AMSs [53, 55] which well predicted swirling and its stability ranges. The numerical stability also showed that the two stable periodic dominant HGCs could co-exist with the unstable higher–order HGCs demonstrating an irregular character. This was an extra source of discrepancy. Another source is that the damping is satisfactory predicted for the lowest HGCs (Ch. 6 of [58] and [52, 55, 100, 152, 164]) but not for the higher ones. The latter damping was strongly affected by the wave breaking and overturning observed in [51, 52, 55]. The adaptive WNMs can also be based on a sophisticated asymptotic ordering deduced from experiments. An example is [97] in which the resonant sloshing subject to obliquely horizontal harmonic excitation was studied.

In the 80's, Lukovsky [131, 139, 140] derived the five–dimensional WNMS for sloshing in an ***upright circular cylindrical tank***, constructed its asymptotic periodic solutions (planar and swirling), studied their stability by the first Lyapunov method, established the chaos in a certain frequency range, and validated these classification results by experiments [4]. The paper [96] re-derived this system, again, classified the steady–state regimes, and conducted the Runge–Kutta simulations (as in [139,144]). This WNMS and the constructed periodic solutions were also revisited in [77] and Ch. 9 of [58]. The paper [77] examined the nodal curves patterns theoretically justifying that these are not a moving straight line. Ch. 9 of [58] showed that the theoretical classification of the periodic regimes in [140, 144] is, generally, supported by the model tests in [194]. The fifth–order asymptotic ordering ($O(\epsilon^{1/5})$ for the two dominant HGCs) gave a negligible contribution to this classification [148] .

Theoretically, the Narimanov–Moiseev asymptotics requires an infinite set of the second and third order HGCs included into WNMSs for ***axisymmetric tanks*** [63, 134]. For upright circular cylindrical tank, such a ‘complete’ WNMS was derived and analysed in [130]. The infinite set of the higher–order HGCs did not influence the Lukovsky classification results on the periodic steady–state solutions except for isolated liquid depths and forcing frequencies when the secondary resonance was expected (see, the depths listed in [22, 80]). Unfortunately, the modal systems from [140] and [130] were not able to quantify the measured steady–state wave amplitudes in [194]. The discrepancy could be clarified by the aforementioned secondary resonance (requiring the AMSs which were not derived for this tank shape, yet) and by the surface tension which was neglected in the multimodal analysis, yet.

Lukovsky also derived and analysed the corresponding five–dimensional WNMS [140, 186, 187] for an ***upright annular cylindrical tank***. It was re-derived and modified by adding some extra third–order HGCs in [213], where new model tests were conducted as well. Comparing the experimental and theoretical response curves, [213] showed a satisfactory agreement for the planar wave regime but, even though speculative damping ratios were included to fit the experimental data, a discrepancy was still evident for swirling. An attempt to derive a Narimanov–Moiseev WNMS for a non–central pile position was conducted in [212]. Sloshing in an upright ***compartment*** tank of circular and annular cross–sections was studied in [149, 150] by using the same research scheme as in [140, 144].

For tanks with ***non-vertical walls***, there are no exact analytical natural sloshing modes and the normal presentation of the free surface fails. The later problem could be resolved by utilising the *nonconformal mapping technique* proposed by Lukovsky [135, 136, 138, 176] in 1975. The technique was combined with the Narimanov scheme [136,176], used in the Perko like simulations [119–121], and incorporated into the Miles–Lukovsky variational method [63, 140, 144]. However, the analytically–given multidimensional WNMSs were derived only for *conical* and *spherical* tanks. The main difficulty for a further expansion onto other tank shapes remains the absence of the required analytically approximate natural sloshing modes which exactly satisfy the Laplace equation and the slip conditions on the non-vertical walls as well as allow for an *analytical continuation over the mean free surface* (see, limitations of the multimodal method discussed in [134, 144] and Ch. 9 of [58]).

According to [10,11,36], the flat mean free surface in a ***circular conical*** tank was replaced by a spherical cap and, thereby, an approximate analytical solution of NSP in terms of spherical functions was constructed. The corresponding multidimensional WNMS was derived in [146, 147]. The latter result was step-by-step improved in [73, 74, 76, 81, 132, 140, 142] (see, also references therein). Analytically approximate natural sloshing modes without the aforementioned replacement were constructed in [73, 81, 132]. Based on these modes, the Narimanov–

Moiseev type five–dimensional WNMSs for *non–truncated and truncated circular V–conical tanks* were derived and studied in [75, 81]. The systems contain extra (relative to the upright circular tank) nonlinear terms expressing the so-called *geometric nonlinearity* but the steady–state analysis leads to the same qualitative results extracting the planar and swirling wave regimes as well as the chaos in a certain frequency range. [78] focused on the nodal lines motions generalising the results from [77]. An emphasis was on the secondary resonance expectations. The theoretical secondary resonance analysis [75, 81] and a comparison of the theoretical results with experiments [26, 74, 75, 153] showed that the higher harmonics due to the secondary resonance indeed matter. The ANMs are required for almost all semi–apex angles. This is a challenge.

The required analytically approximate natural sloshing modes for ***circular*** and ***spherical*** tanks were constructed in [7, 8, 61, 62]. Based on those solutions from [8, 61], a complete infinite–dimensional Narimanov–Moiseev type WNMS (a generalisation of [130]) was explicitly derived in [63, 64] whose steady–state solutions were classified as well. The classification results were supported by experiments [203] for the depth–to–radius ratios ≤ 0.5. However, multiple secondary resonances and an experimentally–observed free–surface fragmentation (accompanied by overturning waves) made this WNMS inapplicable for higher tank fillings and with increasing the forcing amplitude. The Narimanov–Moiseev type infinite–dimensional WNMS for a circular tank was constructed but not analysed in [30]. The circular tank shape causes multiple secondary resonances for almost all tank fillings [16, 59]. This makes [30] weakly applicable for strongly nonlinear sloshing.

4. The future. There are *extensive* and *intensive* challenges of the nonlinear multimodal method. Whereas ***extensive*** challenges are mainly associated with generalisation and expansion of the multimodal method onto new practically–important sloshing problems (see, e.g., in Ch. 1 of [58] and [31, 144]), ***intensive*** ones imply improvement of the method and WNMSs (from physical and mathematical points of view) as well as dedicated rigorous mathematical studies (uniquely presented by [68, 86, 87] which deal with the modal systems from [50, 56]). Obvious generalisations are related to tanks of complex shape. One should construct analytically approximate natural sloshing modes of special kind and develop the tensor algebra related to the curvilinear coordinates [144] employed for these tank shapes. Finally, because derivations of WNMSs for complex tank shapes are especially tedious, a challenge is to code a computer algebra which enables a computer–based derivation as it was done for an upright circular cylindrical tank in [72].

The two important physical challenges are an adequate account for damping and surface tension. Ch. 6 of [58] reviews analytical methods which could be used for incorporating the damping–related terms in WNMSs. However, these were only realised for a linear damping due to the laminar boundary layer and the bulk viscosity, and, recently, for perforated screens [47, 48, 122]. Damping due to baffles and piles should be the next goal. The big challenge is to account for damping due to free–surface fragmentation, overturning and breaking waves. For nonlinear sloshing, the surface tension requires including the dynamic contact angle effect [13, 14]. It is not clear yet, how to include this effect into the multimodal method.

Finally, we believe that the multimodal method may help to describe an attractive swirling–induced V-constant rotation of the liquid [43] which was observed in famous experiments [12, 89, 90, 188, 194] but is not explained yet.

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