DIFFERENTIAL AND VARIATIONAL FORMALISM FOR AN ACOUSTICALLY–LEVITATING DROP

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Abstract

Starting with a most general problem on interface waves between two ideal compressible fluids, treated here as an ullage gas and a liquid, respectively, and separating fast and slow time scales, differential and variational formalism for an acoustically levitating drop and its time-averaged shape (the drop vibroequilibrium) is developed. The drop vibroequilibria can differ from spherical shape; stable vibroequilibria are associated with local minima of the quasipotential energy whose analytical form is also derived in the present paper.

1 Introduction

The acoustic levitation [1, 2, 3] has been developing from the 70-90's as a contactless technology in chemical and pharmaceutical industry [4, 5] of ultra-pure materials. The technology facilitates preventing the liquid contamination and intensifying the chemical reactions. The acoustic levitators are also used in physical measurements of the surface tension and the liquid viscosity [6, 7, 8]. A typical design of an acoustic levitator is schematically shown in Fig. 1. The levitator consists of an acoustic vibrator and a spheric reflector which create, altogether, an almost planar standing acoustic wave of the length λ. The acoustic wave yields the acoustic radiation pressure [9] which is a time-independent $\lambda/2$-periodic function along the vertical axis. Periodically changing positive (marked by '+') and negative ('$-$') radiation pressure zones enforce droplets to

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be located in a vicinity of a radiation pressure node with a possible downward shift d into the '+'-zone due to the vertical gravity force.

As long as the equivalent drop diameter $D_0 = 2R_0$ (the spherical drop diameter of the same volume) is much lower of the acoustic standing-wave length (see, Fig. 1 (a)), the acoustic radiation pressure does not deform the drop shape so that the drop oscillates relative to its spherical shape as if it levitates in the zero gravity. Those nonlinear drop oscillations have been extensively studied by many authors and we refer interested readers to [10, 11, 12, 13, 14] in which theoretical results are reported utilizing the Lagrange variational formalism.

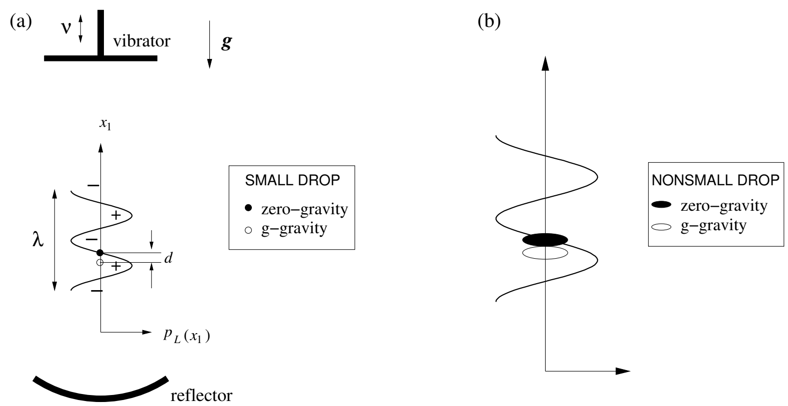

Figure 1: A schematic design of acoustic levitators in which an almost planar standing acoustic wave of the length λ is created along the gravity acceleration vector $\boldsymbol{g}$ and, therefore, there exists the time-independent acoustic radiation pressure, $p_L(x_1 + \frac{1}{2}\lambda) = p_L(x_1)$, which counteracts the gravity providing the drop levitation. In the zero-gravity, the drop locates at zeros of $p_L(x_1)$ but a downward drift d occurs when the gravitation vector is not zero moving the drop into the '+' zone of $p_L(x_1)$. The long standing acoustic wave (case a) does not deform the levitating drop, but when the one-fourth of the wave length is comparable with the drop size, the time-averaged drop shape becomes flattened (b).

In the contrast, when the vertical drop size and $\lambda/4$ are of the comparable order, the acoustic radiation pressure deforms the drop shape so that its averaged, visually observed geometry is far from a sphere as schematically illustrated in Fig. 1 (b). Those acoustically deformed drop shapes and their stability were investigated, *experimentally and theoretically*, for instance, in [9, 15, 16, 17]. The employed applied mathematical model in these references has been at the physical level of confidence. It empirically involves the free surface problem on the weightless drop dynamics in which the pressure (dynamic) boundary condition includes an extra quantity responsible for the acoustic radiation pressure generated by an external standing acoustic wave in gas. A feedback of the levitating drop shapes on the external acoustic field has been neglected – the acoustic field is assumed to be the same as for a solid levitating sphere. Appearance of the acoustic radiation pressure in this empirical model can be interpreted as the so-called vibrational force well-known from the vibrational mechanics [18]. The papers [19, 20, 21, 22, 23] considered the vibrational hydrodynamic problems of compressible liquids partly filling a container as an object of the *applied functional analysis*. They introduced the so-called *vibroequilibria*, the time-averaged liquid shapes occurring due to high-frequency vibrational loads. Furthermore, a series of theorems were proved on the spectral boundary problems describing the linear eigenoscillations relative to the vibroequilibria as well as the papers developed the Lagrangian formalism for the contained liquid vibromechanics.

The present paper follows the applied mathematical studies in [19, 20, 23] to construct a new, mathematically-justified model which describes slow-time motions of an acoustically levitating drop. The analysis starts with the "ulage gas–liquid drop" interface problem formulated within the framework of ideal compressible fluids with irrotational flows. Furthermore, fast and slow time scales are separated in both differential and variational statements. The fast-time averaged interface problem yields a free-surface problem in which the Langevin acoustic radiation pressure appears, in a natural way, in the dynamic boundary condition. The kinematic boundary condition of this problem implies that the free surface reflects the acoustic wave. Whereas there are no slow drop oscillations, the derived free-surface problem transforms to a static problem whose solution describes a visually-observed, acoustically deformed drop shape. The shape is called the *drop vibroequilibrium*. In contrast to the mathematical model from [15, 16, 17, 9], the drop vibroequilibria change the external vibrational field. This is the first main result of the present paper. Another main result consists of developing the averaged Lagrange variational formalism and deriving a functional which can be interpreted as a quasi-potential energy of the drop vibroeqilibria. The forthcoming studies should deal with generalizing the spectral theorems on the linear natural oscillations of the acoustically levitating drops relative to the drop vibroequilibria.

2 Statement of the problem

Fig. 2 schematically shows the "ullage gas–liquid drop" mechanical system confined in a closed rigid box $Q = \{x \in \mathbb{R}^3 \mid W(x) < 0\}$ (acoustic levitator), where $W(x) = 0$ determines the piece-smooth box boundary and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the Cartesian coordinate system. The domain Q consists of the ullage gas $Q_1(t)$ and liquid $Q_2(t)$ time-dependent domains ($Q = Q_1(t) \cup Q_2(t)$) so that the interface $\Sigma(t) = \partial Q_2(t) = \{x \in Q_2 \mid \xi(x,t) = 0\}$ is defined by the unknown function $\xi(x_1, x_2, x_3, t) = 0$ so that $\nabla\xi/|\nabla\xi|$ is the exterior normal vector with respect to the drop domain $Q_2(t)$. Both gas and liquid are *compressible ideal and barotropic* fluids with *irrotational* flows. The box boundary $S = \partial Q$ falls into a reflecting surface $S_1 \subset S$ and acoustic vibrator $S_0 \subset S$, i.e. $S = S_0 \cup S_1$. The gravity acceleration vector is directed downward, against the Ox_3 axis.

We introduce the velocity potentials $\varphi_i = \varphi_i(x,t)$, the pressure $p_i = p_i(x,t)$

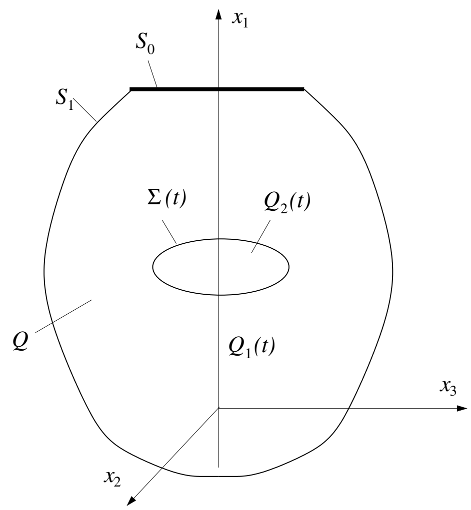

Figure 2: The "ullage gas–liquid drop" mechanical system located in a rigid box Q. The acoustic vibrator is marked by S_0, but S_1 is the box surface appearing as a reflector of the acoustic wave.

and the density $\rho_i = \rho_i(x,t)$ field defined in $Q_i(t),\ i = 1,2$. The *governing equations* for the ideal barotropic fluids [19] read as

$$\dot{\rho}_i + \mathrm{div}(\rho_i \nabla \varphi_i) = 0, \tag{1a}$$

$$\rho_i \nabla \left(\dot{\varphi}_i + \tfrac{1}{2}|\nabla \varphi_i|^2 + g x_1 \right) = -\nabla p_i, \tag{1b}$$

$$\rho_i = \rho_{0i} \left(\frac{p_i}{p_{0i}} \right)^{1/\gamma_i} \quad \text{in} \quad Q_i(t), \tag{1c}$$

where g is the gravity acceleration, ρ_{0i} are the mean densities, p_{0i} are the mean (static) pressures in the fluids ($i = 1,2$), and $\gamma_i,\quad i = 1,2$, are the adiabatic indexes for barotropic (by definition, the pressure is uniquely a function of the density) ullage gas and liquid, respectively. The time derivative is denoted by the dot. The two fluid domains should also satisfy the mass conservation condition

$$\int_{Q_i(t)} \rho_i \,\mathrm{d}Q = m_i, \quad i = 1,2, \tag{2}$$

where m_1 and m_2 are the constant masses of gas and liquid, respectively.

The *kinematic boundary conditions* are

$$\frac{\partial \varphi_i}{\partial n} = -\frac{\dot{\xi}}{|\nabla \xi|}, \quad i = 1,2 \quad \text{on} \quad \Sigma(t), \tag{3a}$$

$$\rho_1 \frac{\partial \varphi_1}{\partial n} = \rho_{01} V_0(x) \sin(\nu t) \quad \text{on} \quad S_0, \tag{3b}$$

$$\frac{\partial \varphi_1}{\partial n} = 0 \quad \text{on} \quad S_1. \tag{3c}$$

These conditions imply that fluid particles remain on the interface $\Sigma(t)$ (kinematic condition (3a)), define the normal velocity on the acoustic vibrator S_0 (condition (3b)) so that ν is the acoustic frequency and $V_0(x) \not\equiv 0$ determines the vibrator (S_0) shape, and (3c) implies that S_1 is a reflecting surface.

Finally, the compressible fluid interface problem requires the *dynamic boundary condition*

$$p_2 + T_s(k_1 + k_2) = p_1 \quad \text{on} \quad \Sigma(t) \tag{4}$$

expressing the pressure balance between the drop and the ullage gas, where the surface tension is associated with the $T_s(k_1 + k_2)$ quantity in which k_i, $i = 1, 2$ are the principal curvatures of $\Sigma(t)$ and T_s is the surface tension coefficient.

The problem (1)–(4) needs the initial conditions

$$\begin{aligned} &\xi(x,0) = \tilde{\xi}_0(x); \quad \dot{\xi}(x,0) = \tilde{\xi}_1(x), \\ &\varphi_i(x,0) = \upsilon_i(x); \quad \dot{\varphi}(x,0) = \upsilon_{1i}(x), \quad i = 1,2. \end{aligned} \tag{5}$$

3 The drop vibroequilibrium

3.1 Nondimensional statement

Henceforth, the free-interface problem (1)–(4) is considered in the nondimensional statement assuming the characteristic size $D_0 = 2R_0$ (the equivalent drop diameter) and the characteristic time ν^{-1} (ν is the circular acoustic frequency). The normalization suggests

$$x_{new} = D_0^{-1}x;\ \xi_{new} = D_0^{-1}\xi;\ \varphi_{i(new)} = D_0^{-2}\nu^{-1}\varphi_i;\ p_{i(new)} = \rho_{0i}D_0^{-2}\nu^{-2}p_i,$$
$$p_{0i(new)} = \rho_{0i}D_0^{-2}\nu^{-2}p_{0i}, \quad \rho_{i(new)} = \rho_{i0}/\rho_i, \quad m_{i(new)} = m_i D_0^{-3}/\rho_{0i}, \quad i = 1,2, \tag{6}$$

and introduces the following nondimensional parameters

$$b = \frac{gD_0^2\rho_{02}}{T_s}, \quad \delta = \frac{\rho_{01}}{\rho_{02}}, \quad \nu_*^2 = \frac{D_0^3\rho_{02}\nu^2}{T_s}, \quad k = \frac{\nu D_0}{c_g}, \text{ and } k_* = \frac{\nu D_0}{c_l}, \tag{7}$$

where δ is the "gas–liquid" mean densities ratio, ν_* is the nondimensional acoustic frequency, b is the Bond number, and k and k_* are the wave numbers of compressible wave motions in gas and liquid, respectively; c_g and c_l are speeds of sound in the corresponding media.

After omitting the subscript *new*, (1)-(4) transforms to the nondimensional form

$$\dot{\rho}_i + \mathrm{div}(\rho_i \nabla \varphi_i) = 0, \tag{8a}$$

$$\rho_i \nabla \left(\dot{\varphi}_i + \tfrac{1}{2}|\nabla \varphi_i|^2 + \nu_*^{-2} b x_1 \right) = -\nabla p_i, \tag{8b}$$

$$\rho_i = \left(\frac{p_i}{p_{0i}} \right)^{1/\gamma_i} \quad \text{in} \quad Q_i(t), \tag{8c}$$

$$\int_{Q_i(t)} \rho_i \,\mathrm{d}Q = m_i, \quad i = 1,2, \tag{8d}$$

$$\frac{\partial \varphi_1}{\partial n} = 0 \quad \text{on} \quad S_1, \tag{8e}$$

$$\rho_1 \frac{\partial \varphi_1}{\partial n} = \underbrace{\frac{\sup |V_0|}{c_g}}_{\epsilon} \underbrace{\frac{V_0(x)}{\sup |V_0|}}_{V(x)=O(1)} \frac{1}{k} \sin t \quad \text{on} \quad S_0, \tag{8f}$$

$$\frac{\partial \varphi_i}{\partial n} = -\frac{\dot{\xi}}{|\nabla \xi|}, \quad i = 1,2, \tag{8g}$$

$$p_2 + \underbrace{\nu_*^{-2}}_{\mu\mu_1\epsilon^3}(k_1 + k_2) = p_1 \underbrace{\delta}_{\mu_1\epsilon} \quad \text{on} \quad \Sigma(t). \tag{8h}$$

A set of *small* nondimensional parameters is introduced that are marked by the underbraces.

First, the primary, main small parameter is

$$\epsilon = \frac{\sup |V_0|}{c_g} \ll 1. \tag{9}$$

It implies the ratio between the maximum acoustic vibrator velocity and the sound speed in the ullage gas. *Secondly,* the density ratio

$$\frac{\rho_{01}}{\rho_{02}} = \delta = \mu_1 \epsilon, \quad \mu_1 \sim 1 \tag{10}$$

is assumed to be of the same order than ϵ ($\mu_1 = O(1)$ is the proportionality coefficient). *Thirdly,* the nondimensional acoustic frequency is chosen as high as to provide the asymptotic relation

$$\nu_*^{-2} = \mu\mu_1\epsilon^3, \quad \mu = O(1). \tag{11}$$

Fourthly, the wave numbers are

$$O(\epsilon) = k_*^2 \ll k^2 = O(1) \tag{12}$$

implying (from the physical point of view) that the acoustic frequency may be close to lower acoustic resonant frequencies in the ullage gas, $k = O(1)$, but, because speed of sound in the liquid is higher of that in the ullage gas, the compressible liquid motions are far from the resonant condition and, in the first approximation, the drop can be considered as an incompressible liquid.

3.2 Introducing slow and fast time variables

As it is usually accepted in the vibrational mechanics [18], the fast and slow time scales can be introduced so that the fast time is associated with the nondimensional time t appearing in the non-homogeneous condition (8f) expressing the input vibrational signal, but the slow time scale τ should be proportional to the square-root of the nondimensional potential type forces. The latter forces are contributed by the surface tension and the gravity. The related quantities appear in the dynamic interface condition (8h) accompanied by the $O(\epsilon^3)$-multiplier and, therefore, the slow time variable can be defined as $\tau = \epsilon^{3/2}t$. The nondimensional solution of (8) takes the form

$$\varphi_i = \varphi_i(x,t,\tau),\ p_i = p_i(x,t,\tau),\ \rho_i = \rho_i(x,t,\tau),\ \text{and}\ \xi = \xi(x,t,\tau). \tag{13}$$

The nondimensional problem (8) contains the small parameters ϵ, ϵ^3 and, because of the slow-time component in (13), $\epsilon^{3/2}$. The standard assumption of the asymptotic method employing the fast and slow time separation is that (13) can be posed in the asymptotic series

$$\begin{aligned} \varphi_i &= \sum_{k=0}^{\infty} \epsilon^{k/2}\varphi_i^{(k/2)}(x,t,\tau); \quad p_i = \sum_{k=0}^{\infty} \epsilon^{k/2} p_i^{(k/2)}(x,t,\tau), \\ \rho_i &= \sum_{k=0}^{\infty} \epsilon^{k/2}\rho^{(k/2)}(x,t,\tau); \quad \xi = \sum_{k=0}^{\infty} \epsilon^{k/2}\xi_{k/2}(x,t,\tau), \end{aligned} \tag{14}$$

where the coefficients are smooth functions of their variables. Specifically, the rational-number superscript indexes are introduced to link the functional coefficients with the small parameter powers. The sequence of the indexes are $0, 1/2, 1, 3/2, 2, 5/2, 3, \ldots$.

3.3 Separating slow and fast time variables

Substituting (14) into (8) leads to the k-family of boundary value problems with respect to $\varphi_i^{(k/2)}$, $p_i^{(k/2)}$, $\rho_i^{(k/2)}$, $\xi_{k/2}$, $i = 1,2$ starting with $k = 0$. The starting point implies the $O(1)$-*order approximation* which comes from the homogeneous problem

$$\begin{aligned} &\left(\rho_i^{(0)}\right)_t + \operatorname{div}\left(\rho_i^{(0)}\nabla\varphi_i^{(0)}\right) = 0; \quad \rho_i^{(0)}\nabla\left(\left(\varphi_i^{(0)}\right)_t + \tfrac{1}{2}\left(\nabla\varphi_i^{(0)}\right)^2\right) = -\nabla p_i^{(0)}, \\ &\rho_i^{(0)} = \left(\frac{p_i^{(0)}}{p_{0i}}\right)^{1/\gamma_i} \quad \text{in} \quad Q_i^{(0)}; \quad \frac{\partial \varphi_1^{(0)}}{\partial n} = 0 \quad \text{on} \quad S_1; \quad \frac{\partial \varphi_1^{(0)}}{\partial n} = 0 \quad \text{on} \quad S_0, \\ &\frac{\partial \varphi_i^{(0)}}{\partial n} = -\frac{(\xi_0)_t}{|\nabla \xi_0|}, \quad i = 1,2; \quad -p_2^{(0)} = 0 \quad \text{on} \quad \Sigma^{(0)}, \end{aligned}$$

where $(\cdot)_t$ is the fast-time derivative. The last pressure condition on $\Sigma^{(0)}$ shows that the liquid motions are dynamically uncoupled with the compressible gas flows and, moreover, the drop is not affected by the surface tension. From physical point of view, this means that the $O(1)$-order drop motions can only slowly deform on the τ-scale and the zero-order solution takes the form

$$
\begin{aligned}
&\xi_0 = \xi_0(x,\tau); \quad \int_{Q_2^{(0)}(\tau)} \mathrm{d}Q = m_2; \quad \nabla\varphi_i^{(0)} = 0,\ i = 1,2, \\
&p_1^{(0)} = p_{01};\ \rho_1^{(0)} = 1;\ \rho_2^{(0)} = p_2^{(0)} = 0
\end{aligned}
$$

where $\xi_0(x,\tau) = 0$ defines the $O(1)$-order interface motions $\Sigma^{(0)} = \Sigma^{(0)}(\tau)$ which, in turn, defines the slowly-deforming domains $Q_i^{(0)}(\tau),\ i = 1,2$.

Henceforth, the $O(1)$-order drop motions are associated with the fast-time averaged drop shape, i.e., by definition:

$$
\Sigma_0(\tau) = \Sigma^0(\tau) = \langle\Sigma(t,\tau)\rangle_t; \quad Q_i^{(0)}(\tau) = \langle Q_i(t,\tau)\rangle_t,\ i = 1,2. \tag{15}
$$

Furthermore, the higher-order asymptotic problems with respect to $\varphi_i^{(k/2)}$, $p_i^{(k/2)}$, $\rho_i^{(k/2)}$, $\xi_{k/2}$, $k \geq 1$ would be formulated in the fast-time averaged domains $Q_1^{(0)}(\tau)$ and $Q_2^{(0)}(\tau)$ separated by $\Sigma_0(\tau)$.

The problem (8) contains three small input parameters of the order $O(\epsilon)$, $O(\epsilon^{3/2})$ and $O(\epsilon^3)$, but there are no the $O(\epsilon^{1/2})$-order input quantities. This means that the $O(\epsilon^{1/2})$-order approximation is zero. The $O(\epsilon)$-*order approximation* ($k = 2$) comes from the problem

$$
\begin{aligned}
&k^2\left(\varphi_1^{(1)}\right)_{tt} - \nabla^2\varphi_1^{(1)} = 0 \ \text{ in } \ Q_1^{(0)}(\tau); \quad \frac{\partial\varphi_1^{(1)}}{\partial n} = -\frac{(\xi_1)_t}{|\nabla\xi_0|} \ \text{ on } \ \Sigma_0(\tau), \\
&\frac{\partial\varphi_1^{(1)}}{\partial n} = 0 \ \text{ on } \ S_1; \quad \frac{\partial\varphi_1^{(1)}}{\partial n} = \frac{V(x)\sin t}{k} \ \text{ on } \ S_0, \\
&\frac{\partial\varphi_2^{(1)}}{\partial n} = -\frac{(\xi_1)_t}{|\nabla\xi_0|}; \ \ p_2^{(1)} = \mu_1 p_{01} \ \text{ on } \ \Sigma_0(\tau), \\
&\dot\rho_2^{(1)} + \nabla^2\varphi_2^{(1)} = 0; \quad \rho_2^{(1)} = 0 \ \text{ in } \ Q_2^{(0)}(\tau),
\end{aligned}
$$

where the last condition is due to $\rho_2^{(1)} = k_*^2 p_2^{(1)}$ and (12).

As it happened in the zero-order approximation, the dynamic interface condition (here, $p_2^{(1)} = \mu_1 p_{01} = const$) on the fast-time averaged interface $\Sigma_0(\tau)$ decouples the interface problem into two independent boundary value problems in $Q_2^{(0)}(\tau)$ and $Q_1^{(0)}(\tau)$, respectively. Analyzing the first boundary problem in $Q_2^{(0)}(\tau)$ shows that this approximation can only contribute a slow-time drop deformation which, due to definition (15), is already accounted for by the $O(1)$-order component. As a consequence,

$$
\xi_1 = 0; \quad \nabla\varphi_2^{(1)} = 0; \quad \rho_2^{(1)} = 0; \quad p_2^{(1)} = \mu_1 p_{01}.
$$

The second boundary value problem in $Q_1^{(0)}(\tau)$ has the solution

$$
\varphi_1^{(1)} = \Phi_1(x,\tau)\sin t; \quad p_1^{(1)} = \Phi_1(x,\tau)\cos t, \tag{16}
$$

where $\Phi_1(x)$ is the so-called *wave function* of the linear acoustic field in the ullage gas governed by the Neumann boundary value problem

$$\nabla^2\Phi_1 + k^2\Phi_1 = 0 \text{ in } Q_1^{(0)}(\tau); \quad \frac{\partial\Phi_1}{\partial n} = 0 \text{ on } S_1 \cup \Sigma_0(\tau);$$
$$\frac{\partial\Phi_1}{\partial n} = \frac{V(x)}{k} \text{ on } S_0 \quad (17)$$

and stated in the slowly-deforming gas domain; the fast-time averaged drop surface $\Sigma_0(\tau)$ plays the role of a reflector.

The interface problem remains decoupled in the $O(\epsilon^{3/2})$-order approximation. For the gas domain $Q_1^{(0)}(\tau)$, the homogeneous τ-dependent boundary problem takes the form

$$\nabla^2\varphi_1^{(3/2)} = 0; \;\; p_1^{(3/2)} = \left(\varphi_1^{(3/2)}\right)_t \text{ in } Q_1^{(0)}(\tau),$$
$$\frac{\partial\varphi_1^{(3/2)}}{\partial n} = 0 \text{ on } S_0 \cup S_1; \quad \frac{\partial\varphi_1^{(3/2)}}{\partial n} = -\frac{(\xi_0)_\tau + \left(\xi_{3/2}\right)_t}{|\nabla\xi_0|} \text{ on } \Sigma_0(\tau),$$

but

$$\begin{aligned}&\nabla^2\varphi_2^{(3/2)} = 0; \;\; p_2^{(3/2)} = \left(\varphi_2^{(3/2)}\right)_t; \;\; \rho_2^{(3/2)} = 0 \text{ in } Q_2^{(0)}(\tau),\\ &\frac{\partial\varphi_2^{(3/2)}}{\partial n} = -\frac{(\xi_0)_\tau + \left(\xi_{3/2}\right)_t}{|\nabla\xi_0|}; \;\; p_2^{(3/2)} = 0 \text{ on } \Sigma_0(\tau)\end{aligned} \quad (18)$$

describes the $O(\epsilon^{3/2})$-contribution to the drop motions which also is τ-dependent. This means that $\varphi_i^{(3/2)} = \varphi_i^{(3/2)}(x,\tau)$, $i = 1, 2$.

Summarizing all asymptotic quantities obtained from the constructed approximations gives

$$\varphi_2(x,t,\tau) = \epsilon^{3/2} \underbrace{\varphi_2^{(3/2)}(x,\tau)}_{\varphi(x,\tau)} + o(\epsilon^{3/2}), \quad (19a)$$

$$\xi(x,t,\tau) = \underbrace{\xi_0(x,\tau)}_{\zeta(x,\tau)} + o(\epsilon^{3/2}), \quad (19b)$$

$$\varphi_1(x,t,\tau) = \epsilon \underbrace{\Phi_1(x,\tau)}_{\Phi(x,\tau)} \sin t + \epsilon^{3/2}\varphi_1^{(3/2)}(x,\tau) + o(\epsilon^{3/2}). \quad (19c)$$

This shows that the lowest-order component of the velocity field in the drop domain is of the order $O(\epsilon^{3/2})$; the velocity field does not depend on the fast time t. In the contrast, the lowest-order component of the velocity field in the gas domain describes the linear acoustic standing wave for which the slowly-varying drop surface $\Sigma_0(\tau) : \zeta(x,\tau) = 0$ is a reflector.

Because the right-hand side of the dynamic interface condition (8h) has the $O(\epsilon)$-multiplier, the drop oscillates on the fast-time scale caused by the linear acoustic field (16) so that $\varphi_2^{(2)} = \sin t\, F_1(x,\tau)$. The velocity potential in $Q_1^{(0)}(\tau)$ takes the form $\varphi_1^{(2)} = \sin(2t)\, F_2(x,\tau) + \cos(2t)\, F_3(x,\tau)$. However, due to quadratic terms, the second-order pressure component in $Q_1^{(0)}(\tau)$ contains the fast-time averaged quantity

$$\langle p_1^{(2)} \rangle_t(x,\tau) = \tfrac{1}{4}\left(k^2(\Phi_1)^2 - (\nabla\Phi_1)^2\right) + const \tag{20}$$

expressing the so-called *Langevin acoustic radiation pressure.*

The $O(\epsilon^{5/2})$-order component is of more complicated structure, but it does not affect the $O(\epsilon^3)$-order approximation which yields the *fast-time averaged* dynamic boundary condition

$$\begin{aligned}\left(\varphi_2^{(3/2)}\right)_\tau + \tfrac{1}{2}\left(\nabla\varphi_2^{(3/2)}\right)^2 - \mu\mu_1(k_1+k_2) + \mu_1\mu b x_1 \\ + \tfrac{1}{4}\mu_1\left(k^2(\Phi_1)^2 - (\nabla\Phi_1)^2\right) = const \ \text{ on } \ \Sigma_0(\tau).\end{aligned} \tag{21}$$

3.4 Slow-time oscillations with respect to the drop vibroequilibrium

Accounting for the asymptotic solution (19), the fast-time averaged dynamic condition (21) as well as the governing boundary value problems for the lowest-order quantities in (19), we arrive, finally, at the following free-interface problem with respect to $\zeta(x,\tau) = \xi_0(x,\tau),\ \varphi(x,\tau) = \varphi_2^{(3/2)}(x,\tau)$ and $\Phi(x,\tau) = \Phi_1(x,\tau)$

$$\begin{aligned}&\nabla^2\varphi = 0 \ \text{ in } \ \Omega_2(\tau); \ \ \frac{\partial\varphi}{\partial n} = -\frac{\zeta_\tau}{|\nabla\zeta|} \ \text{ on } \ \Gamma(\tau); \ \ \int_{\Omega_2(\tau)} \mathrm{d}\Omega = m_2,\\ &\varphi_\tau + \tfrac{1}{2}(\nabla\varphi)^2 - \mu\mu_1(k_1+k_2) + \mu\mu_1 b x_1 \\ &\qquad + \tfrac{1}{4}\mu_1\left(k^2(\Phi)^2 - (\nabla\Phi)^2\right) = const \ \text{ on } \ \Gamma(\tau),\end{aligned} \tag{22a}$$

$$\begin{aligned}&\nabla^2\Phi + k^2\Phi = 0 \ \text{ in } \ \Omega_1(\tau); \ \ \frac{\partial\Phi}{\partial n} = 0 \ \text{ on } \ S_1 \cup \Gamma(\tau),\\ &\qquad\qquad \frac{\partial\Phi}{\partial n} = \frac{V(x)}{k} \ \text{ on } \ S_0\end{aligned} \tag{22b}$$

where $\Omega_1(\tau) = Q_1^{(0)}(\tau), \Omega_2(\tau) = Q_2^{(0)}(\tau)$, and $\Gamma(\tau) = \Sigma_0(\tau)$.

In fact, we have proved the following proposition:

Proposition 1. *If the original interface problem (8) has the asymptotic solution (14), the lowest order terms in (19) depend only on the slow time $\tau = \epsilon^{3/2}t$ and these terms are governed by the free-surface problem (22).*

The free-interface problem (22) is the announced *mathematical model* for the acoustically levitating drops. It describes *slow-time oscillations* of an acoustically levitating drop. The problem (22) is similar to the earlier empirical mathematical model in [15, 16, 17, 9] and should, perhaps, theoretically clarify the vertical vibrations and shape oscillations of droplets [25]. A *difference* consists of an extra term in the dynamic interface condition on $\Gamma(\tau)$ expressing the Langevin radiation pressure which becomes now parametrically depending on the τ-instant drop shape (due to the zero-Neumann boundary condition (22b) on $\Gamma(\tau)$). The latter boundary condition means that the slowly-oscillating drop surface is, in the lowest-order approximation, a reflector for the linear acoustic field in the ullage gas.

When assuming that the fast-time averaged drop shape does *not* oscillate, we arrive at the static free-interface problem

$$-\mu(k_1+k_2)+b\mu x_1+\tfrac{1}{4}\left(k^2(\Phi)^2-(\nabla\Phi)^2\right)=const \quad \text{on} \quad \Gamma_0,$$
$$\int_{\Omega_{20}} \mathrm{d}\Omega = m_2, \tag{23a}$$

$$\nabla^2\Phi+k^2\Phi=0 \quad \text{in} \quad \Omega_{10}; \quad \frac{\partial\Phi}{\partial n}=0 \quad \text{on} \quad S_1\cup\Gamma_0; \quad \frac{\partial\Phi}{\partial n}=\frac{V(x)}{k} \quad \text{on} \quad S_0. \tag{23b}$$

The drop shape Γ_0 is called the *drop vibroequilibria.*

The drop vibroequilibria shape is what one can see in acoustic levitators but the evolution problem (22) describes, in fact, nonlinear motions with respect to the vibroequilibria. The drop vibroequilibria can be stable or not depending on input parameters. The stability analysis should normally involve the spectral problem on linear natural (eigen) oscillations with respect to Γ_0, or, alternatively, the extremal problem on the quasi-potential energy as in section 4.

The aforementioned spectral problem has the classical exact Rayleigh solution [26] for the weightless drop when the acoustic field is absent. The acoustically-deformed levitating drops are not the case and dedicated studies are required on the natural (eigen) modes and frequencies which differ from those in [26].

4 Lagrangian formalism for (1)–(4)

We will follow [19] to prove two theorems providing equivalence of (1)–(4) to the classical Lagrange and the Bateman–Luke variational formulations. The first case is the classical Lagrange principle.

Theorem 1. *When functions ξ, φ_i and ρ_i, $i=1,2$ are smooth enough, the free-interface problem (1)-(4) is equivalent to the necessary condition of the extremal points of the action*

$$G(\xi,\varphi_i,\rho_i)=\int_{t_1}^{t_2}[T-U-\Pi]\,\mathrm{d}t$$
$$=\int_{t_1}^{t_2}\left\{\sum_{i=1}^{2}\int_{Q_i(t)}\rho_i\left[\tfrac{1}{2}(\nabla\varphi_i)^2-U_i(\rho_i)-gx_1\right]\,\mathrm{d}Q-T_s|\Sigma|\right\}\mathrm{d}t \tag{24}$$

subject to the kinematic constraint (1) and assuming the smooth isochronous variations

$$\delta\xi|_{t_1,t_2} = 0; \quad \delta\rho_i|_{t_1,t_2} = 0. \tag{25}$$

Here, T is the kinetic energy, Π is the potential energy, and $U_i(\rho_i)$ is the inner energy of gas and liquid, respectively. The area is denoted as $|\cdot|$. The inner energy of barotropic fluids defines the pressure as

$$p_i \overset{def}{=} \rho_i^2 \frac{dU_i}{d\rho_i}. \tag{26}$$

Remark 1. *Because of constraint (1), the action is a function of ξ and ρ_i.*

Proof. We employ the formula

$$\int_{\Omega(t)} [\dot\rho + (\nabla\varphi\cdot\nabla\psi)]\,\mathrm{d}Q + \frac{d}{dt}\int_{\Omega(t)} \rho\varphi\,\mathrm{d}Q - \int_{S_0} \rho_{01}V_0\varphi\sin(\nu t)\,\mathrm{d}S$$
$$= -\int_{\Omega(t)} [\dot\rho + \mathrm{div}(\rho\nabla\psi)]\varphi\,\mathrm{d}Q + \int_{S_1} \rho\frac{\partial\psi}{\partial n}\varphi\,\mathrm{d}S + \int_{\Sigma(t)} \rho\frac{\partial\psi}{\partial n}\varphi\,\mathrm{d}S$$
$$+ \int_{\Sigma(t)} \rho\frac{\dot\xi}{|\nabla\xi|}\varphi\,\mathrm{d}S + \int_{S_0}\left[\rho\frac{\partial\psi}{\partial n} - \rho_{01}V_0\sin(\nu t)\right]\varphi\,\mathrm{d}S \tag{27}$$

following from the Reynolds transport theorem and the Green formulas when $\Omega(t)$, $\partial\Omega(t) = \Sigma(t)\cup S_1\cup S_0$ is an arbitrary domain, $\Sigma(t)$ ($\xi(x,t)=0$) is a piece of the time-dependent boundary, but $\varphi(x,t)$ and $\psi(x,t)$ are smooth functions.

Using the kinematic constraint (1) with $\varphi=\varphi_1,\ \psi=\psi_1$ for $\Omega(t) = Q_1(t)$, the right-hand side of (27) equals to zero. Analogously, when $\varphi=\varphi_2$ and $\psi=\psi_2$ in $\Omega(t)=Q_1(t)$, $\partial\Omega(t)=\Sigma(t)$, the right-hand-side is also zero. After integration by t from t_1 to t_2 of the remaining left-hand sides and subtracting the results from the action, we come to

$$G(\xi,\varphi_i,\rho_i) = \int_{t_1}^{t_2}\left\{\sum_{i=1}^{2}\int_{Q_i(t)}\rho_i\left[-\dot\varphi_i - \tfrac12(\nabla\varphi_i)^2 - U_i(\rho_i) - gx_1\right]\mathrm{d}Q\right.$$
$$\left.-T_s|\Sigma(t)| + \int_{S_0}\rho_{01}V_0\varphi_1\sin(\nu t)\,\mathrm{d}S\right\}\mathrm{d}t - \sum_{i=1}^{2}(\rho_i\varphi_i)|_{t_1}^{t_2}. \tag{28}$$

Now, assuming the kinematic constraint (1) is satisfied, one can compute variations of G by ρ_i and ξ employing (28). Variations by ρ_i give

$$\delta_{\rho_i}G = \int_{t_1}^{t_2}\left[\int_{Q_j(t)}\delta\rho_j\left[-\dot\varphi_j - \tfrac12(\nabla\varphi_j)^2 - gx_1 - U_j(\rho_j) - \rho_j\frac{dU_j}{d\rho_j}\right]\mathrm{d}Q\right.$$
$$\left.-\int_{Q_j(t)}\rho_j[\delta\dot\varphi_j + (\nabla\varphi_j\cdot\nabla\delta\varphi_j)]\,\mathrm{d}Q + \int_{S_0}\rho_{01}V_0\delta\varphi_1\sin(\nu t)\,\mathrm{d}S\right]\mathrm{d}t$$
$$-\left[\delta\rho_j\varphi_j + \rho_j\delta\varphi_j\right]|_{t_1}^{t_2} = 0,\ j=1,2. \tag{29}$$

Accounting for (27), (25) and (1) leads to

$$-\dot{\varphi}_j - \tfrac{1}{2}(\nabla\varphi_j)^2 - gx_1 - U_j(\rho_j) - \rho_j \frac{dU_j}{d\rho_j} = 0. \tag{30}$$

Taking the gradient action and using (26) give (1b).

Computing the ξ-variation of (28), accounting for (25) and using formulas[24] for variations of the $\Sigma(t)$ area by ξ give, altogether,

$$\begin{aligned}\delta_\xi G = \int_{t_1}^{t_2} \Bigg[\sum_{i=1}^{2} \int_{Q_i(t)} \rho_i[-\delta\dot{\varphi}_i - (\nabla\varphi_i \cdot \nabla\delta\varphi_i)]\,\mathrm{d}Q \\ + \int_{\Sigma(t)} \sum_{i=1}^{2} (-1)^i \frac{\delta\xi}{|\nabla\xi|} \rho_i \left[-\dot{\varphi}_i - \tfrac{1}{2}(\nabla\varphi_i)^2 - gx_1 - U_i(\rho_i)\right]\,\mathrm{d}S \\ -T_s \int_{\Sigma(t)} [-k_1 - k_2] \frac{\delta\xi}{|\nabla\xi|}\,\mathrm{d}S + \int_{S_0} \rho\delta\varphi_1 V_0 \sin(\nu t)\,\mathrm{d}S \Bigg]\,\mathrm{d}t - \sum_{i=1}^{2} (\rho_i\delta\varphi_i)|_{t_1}^{t_2} = 0.\end{aligned} \tag{31}$$

Employing the formula (27) within φ and $\delta\varphi$ transforms (31) to the form

$$\begin{aligned}\delta_\xi G = \int_{t_1}^{t_2} \Bigg[\Bigg\{ \int_{\Sigma(t)} \sum_{i=1}^{2} (-1)^i \rho_i \left[-\dot{\varphi}_i - \tfrac{1}{2}(\nabla\varphi_i)^2 - gx_1 - U_i(\rho_i)\right] \\ + T_s[k_1 + k_2] \Bigg\} \frac{\delta\xi}{|\nabla\xi|}\,\mathrm{d}S \Bigg]\,\mathrm{d}t = 0,\end{aligned} \tag{32}$$

which leads to the dynamic condition (4) provided by (30) (following from the condition $\delta_{\rho_j} G = 0,\ j = 1, 2$). □

Another variational formulation is associated with the so-called *Bateman–Luke variational principle [24]* for a compressible fluid. Specifically, this variational principle is not restricted to the kinematic constraint. The Bateman–Luke action takes the form

$$\begin{aligned}B(\xi, \varphi_i, \rho_i) = \int_{t_1}^{t_2} \Bigg\{ \sum_{i=1}^{2} \int_{Q_i(t)} \rho_i \left[-\dot{\varphi}_i - \tfrac{1}{2}(\nabla\varphi_i)^2 - gx_1 - U_i(\rho_i)\right]\,\mathrm{d}Q \\ -T_s|\Sigma(t)| + \int_{S_0} \rho_{01} V_0 \varphi_1 \sin(\nu t)\,\mathrm{d}S \Bigg\}\,\mathrm{d}t\end{aligned} \tag{33}$$

which is the same as expression (28) but without the last summand.

Theorem 2. *When functions ξ, φ_i and ρ_i, $i = 1, 2$ are smooth enough, the free-interface problem (1)-(4) follows from the necessary condition of the extremal points of the action (33) subject to the isochronous smooth variations*

$$\delta\xi|_{t_1,t_2} = 0;\ \delta\varphi_i|_{t_1,t_2} = 0;\ \delta\rho_i|_{t_1,t_2} = 0. \tag{34}$$

Proof. The theorem immediately follows from the already computed variations of (28) by ρ_j, ξ as well as the formula for variations by φ_j:

$$\begin{aligned}\delta_{\varphi_j} B = \int_{t_1}^{t_2} \left[-\int_{Q_j(t)} \rho_j \left[\delta\dot{\varphi}_j + (\nabla\varphi_j \cdot \nabla\delta\varphi_j)\right] \mathrm{d}Q + \int_{S_0} \rho_{01}\delta\varphi_j V_0 \sin(\nu t)\, \mathrm{d}S \right] \mathrm{d}t \\ = \int_{t_1}^{t_2} \left[\int_{Q_j(t)} [\dot{\rho}_j + \mathrm{div}(\rho_j \nabla\varphi_j)]\delta\varphi_j \, \mathrm{d}Q \right. \\ - \int_{S_1} \rho_j \frac{\partial\varphi_j}{\partial n}\delta\varphi_j \, \mathrm{d}S - \int_{\Sigma(t)} \rho_j \left[\frac{\partial\varphi_j}{\partial n} + \frac{\dot{\xi}}{|\nabla\xi|} \right] \delta\varphi_j \, \mathrm{d}S \\ \left. - \int_{S_0} \left(\rho_1 \frac{\partial\varphi_1}{\partial n} - \rho_{01} V_0 \sin(\nu t) \right) \delta\varphi_1 \, \mathrm{d}S \right] \mathrm{d}t + \rho_j \delta\varphi_j |_{t_1}^{t_2} = 0. \quad (35)\end{aligned}$$

We should account for (34) and the fact that $S_0 = S_1 = \emptyset$ for $j = 2$ in (35). □

5 Quasipotential energy of the drop vibroequilibrium

In section 3.4, we showed that the nondimensional problem (8) has the asymptotic solution (19) whose lowest-order terms describe slow-time motions with respect to the drop vibroequilibrium. The slow time variable is $\tau = \epsilon^{3/2} t$ and the lowest-order terms are governed by (22). In this section, we separate slow and fast time variables in the variational formulations from section (4) to derive the quasi-potential energy of the drop vibroequilibrium governed by (23).

Theorem 3. *Finding the fast-time averaged solution from the classical Lagrange variational formulation (Theorem 1) is equivalent to description of the extremal points of the nondimensional functional*

$$\langle G^*(\xi, \varphi_i, \rho_i) \rangle_t = const + \epsilon^{3/2} \mathcal{G}(\zeta, \varphi) + O(\epsilon^2),$$

within

$$\begin{aligned}\mathcal{G}(\zeta, \varphi, \Phi) = \int_{\tau_1}^{\tau_2} \left\{ \int_{\Omega_2(\tau)} \left[\tfrac{1}{2} (\nabla\varphi)^2 - \mu\mu_1 b x_1 \right] \mathrm{d}Q - \mu\mu_1 |\Gamma(\tau)| \right. \\ \left. + \frac{\mu_1}{4} \int_{\Omega_1(\tau)} \left[k^2 \Phi^2 - (\nabla\Phi)^2 \right] \mathrm{d}Q - \frac{\mu_1}{2k} \int_{S_0} \Phi V(x)\, \mathrm{d}S \right\} \mathrm{d}\tau \quad (36)\end{aligned}$$

subject to the kinematic constraint

$$\nabla^2 \varphi = 0 \ \text{in } \Omega_2(\tau); \quad \frac{\partial\varphi}{\partial n} = -\frac{\zeta_\tau}{|\nabla\zeta|} \ \text{on } \Gamma(\tau) \quad (37a)$$

$$\nabla^2 \Phi + k^2 \Phi = 0 \ \text{in } \Omega_1(\tau); \ \frac{\partial\Phi}{\partial n} = 0 \ \text{on } S_1 \cup \Gamma(\tau); \ \frac{\partial\Phi}{\partial n} = \frac{V(x)}{k} \ \text{on } S_0 \quad (37b)$$

for isochronous smooth variations $\delta\zeta|_{\tau_1,\tau_2} = 0$ *where* $\zeta(x,\tau) = 0$ *governs the slow-time oscillations of the drop surface* $\Gamma(\tau)$ *(*$\Omega_2(\tau)$ *and* $\Omega_1(\tau)$ *are liquid and gas domains, respectively, separated by* $\Gamma(\tau)$*) on the slow-time scale.*

Proof. According to Theorem 1, finding the solution of (1)-(4) (nondimensional statement (8)) is equivalent to description of the extremal points of the action (24). Adopting the nondimensional variational statement, substituting (19) into variational and differential formulations of Theorem 1 and choosing $|t_2 - t_1| > \epsilon^{-3/2}$, we get $\langle G(\xi, \varphi_i, \rho_i)\rangle_t = const + \epsilon^{3/2}\mathcal{G}(\zeta, \varphi) + O(\epsilon^2)$ and the kinematic constraint (37).

Let ζ, φ be a local extrema point of the action (36) subject to (37). Obviously, ζ and φ satisfy (22). Taking (19) in the nondimensional formulation of Theorem 1 gives, within to higher-order terms, an extremal point of G_*. $\square$

Theorem 4. *Finding the fast-time averaged solution from the Bateman–Luke variational formulation (Theorem 2) is equivalent to finding the extremal points of the time-averaged nondimensional action*

$$\langle B^*(\xi, \varphi_i, \rho_i)\rangle_t = const + \epsilon^{3/2}\mathcal{B}(\zeta, \varphi, \Phi) + O(\epsilon^2),$$

where

$$\mathcal{B}(\zeta, \varphi, \Phi) = \int_{\tau_1}^{\tau_2} \left\{ \int_{\Omega_2(\tau)} \left[-\varphi_\tau - \tfrac{1}{2}(\nabla\varphi)^2 - \mu\mu_1 b x_1 \right] \mathrm{d}Q - \mu\mu_1 |\Gamma(\tau)| \right. \\ \left. + \frac{\mu_1}{4} \int_{\Omega_1(\tau)} \left[k^2\Phi^2 - (\nabla\Phi)^2 \right] \mathrm{d}Q - \frac{\mu_1}{2k} \int_{S_0} \Phi\, V(x)\, \mathrm{d}S \right\} \mathrm{d}\tau, \quad (38)$$

subject to isochronous smooth variations

$$\delta\zeta|_{\tau_1,\tau_2} = 0; \;\; \delta\varphi|_{\tau_1,\tau_2} = 0; \;\;\; \delta\Phi|_{\tau_1,\tau_2} = 0.$$

Proof. The proof is similar to that in the previous theorem. $\square$

Remark 2. *The fast-time averaged variational formulation of the Bateman–Luke type leads to Theorem 4 which can be treated as the Bateman–Luke variational formulation for the weightless drop dynamics levitating in the zero-gravity and affected, altogether, by the surface tension and the Langevin radiation pressure.*

Assuming the τ-independent solutions in Theorems 3 and 4 leads to the quasi-potential energy of the mechanical system. This means that:

Theorem 5. *Finding the stable drop vibroequilibria from (23) is equivalent to finding the local minima of the quasi-potential energy functional*

$$U = \mu|\Gamma_0| + \mu b \int_{\Omega_{10}} x_1 \,\mathrm{d}Q - \frac{1}{4} \int_{\Omega_{10}} \left(k^2\Phi^2 - (\nabla\Phi)^2 \right) \mathrm{d}Q + \frac{1}{2k} \int_{S_0} V(x)\, \Phi \,\mathrm{d}S \quad (39)$$

subject to

$$\int_{\Omega_{20}} \mathrm{d}Q = m_2 = const \tag{40}$$

and

$$\nabla^2\Phi + k^2\Phi = 0 \ in \ \Omega_{10}; \ \ \frac{\partial \Phi_1}{\partial n} = 0 \ on \ S_1 \cup \Gamma_0; \ \ \frac{\partial \Phi_1}{\partial n} = \frac{V(x)}{k} \ on \ S_0. \tag{41}$$

6 Conclusions

Employing the differential and variational formulations of an interface problem for two compressible fluids, we studied the fast-time averaged motions of an acoustically levitated drop. A new mathematical model is derived describing slow-time motions of the drop with respect to the visually-observed quasi-static drop shapes which are called the drop vibroequilibria. The derived mathematical model is qualitatively similar to the physically-postulated models in [15, 16, 17, 9]. They all introduce the Langevin radiation pressure quantity appearing in the dynamic boundary condition on the drop surface. However, there is a novelty in our new mathematical model – it expresses the important fact that the acoustical field geometry parametrically depends on the drop shape.

Along with the differential formulation of the mathematical model, we present a series of theorems on the Lagrange variational formalism and derive a functional responsible for the quasipotential energy of the mechanical system.

The forthcoming analysis should, probably, focus on the small-magnitude drop oscillations with respect to the vibroequlibria, i.e. on the corresponding spectral theorems which can be considered as a generalization of the famous Rayleigh [26] results. Another open problem consists of appropriate numerical methods for solving the problem on the drop vibroequilibria. Theorem 5 should facilitate constructing the numerical methods.

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