The final publication is available at Springer via http://dx.doi.org/10.1007/s11858-017-0882-4

A methodology for instructional design in mathematics—with the generic and epistemic student at the centre

Heidi Strømskag[1]

Norwegian University of Science and Technology; heidi.stromskag@ntnu.no

Abstract: This theoretical paper presents a methodology for instructional design in mathematics. It is a theoretical analysis of a proposed model for instructional design, where tasks are embedded in situations that preserve meaning with respect to particular pieces of mathematical knowledge. The model is applicable when there is an intention of teaching someone some particular mathematical knowledge. It is structured into four phases: epistemological analysis; development of an epistemological model; implementation; and, institutionalisation. The methodology is rooted in the theory of didactical situations in mathematics, and is built on two epistemological principles: the target knowledge should be an optimal solution to a task embedded in a situation; and, the milieu of the situation should provide feedback to students, whether their responses are adequate with respect to the target knowledge. These principles place the generic and epistemic student at the centre of instructional design in such a way that the focal point is on the student's opportunities to develop the knowledge aimed at—from an epistemological viewpoint. My research goal is to develop a model for instructional design where students will need some particular knowledge to solve a task and where the solution process is managed by features of the milieu with which the students interact. Through the concept of an epistemological model, the methodology enables a profound understanding of one of the phases of didactical engineering: conception and a priori analysis. Empirical data are provided to support the viability of the proposed model. The data are from an empirical investigation of student teachers' engagement with a situation where the target knowledge is a theorem in elementary number theory.

Keywords: mathematical task, methodology, epistemological model, adidactical situation, milieu, institutionalisation.

1 Introduction

The centrality of tasks as instruments in mathematics classroom instruction is reported in the TIMSS 1999 Video Study: In the eighth-grade classrooms that were investigated (in Australia, the Czech Republic, Hong Kong SAR, Japan, the Netherlands, Switzerland, and the United States), at least 80 percent of lesson time, on average, was spent on solving mathematical tasks (Hiebert et al. 2003). Shimizu et al. (2010) outline how the classroom performance of a task is a unique synthesis

[1] Department of Mathematical Sciences,
Norwegian University of Science and Technology,
Alfred Getz' vei 1, Sentralbygg 2, 7. Etg., Gløshaugen,
7491 Trondheim, Norway.

of task, teacher, students and situation. I aim to provide insight into such syntheses by the methodology proposed in this paper. The methodology deals with how tasks can be designed from a student perspective, focusing on students' perceptions of purpose and utility of the mathematical knowledge at stake.

Using the theory of didactical situations in mathematics, TDS (Brousseau 1997) as a framework, I present a theoretical analysis of a model for instructional design in mathematics (i.e., a methodology), where the generic and epistemic student is the focal point of design of situations. The proposed methodology is rooted in the methodological principle of TDS: the idea that a piece of mathematical knowledge is represented by a *situation* that involves a task (or tasks) that can be solved in an optimal manner using this knowledge. Brousseau (1997) refers to such a situation as a *fundamental situation*, and postulates that each item of knowledge can be characterised by some fundamental situation that preserves meaning.

TDS is a holistic theory that encompasses a research methodology, didactical engineering. According to Artigue (2015), didactical engineering as a research methodology is structured into four phases: preliminary analyses; conception and *a priori* analysis; realization, observation and data collection; and, *a posteriori* analysis and validation. Of these phases, the first two have to do with design of situations, and the last two have to do with implementation of, and carrying out research into, situations. The methodology for instructional design proposed in this paper is about design and implementation of mathematics instruction. It theorizes the phase of conception and *a priori* analysis (of didactical engineering) through the construct of an epistemological model. The research goal is to develop a model for instructional design in which students perceive tasks and the mathematical knowledge necessary to solve them, as meaningful.

Ainley and Margolinas (2015), in their advocacy for students' perspectives in task design, propose 'robustness' as a topic for future research on mathematical tasks: "The *robustness* of tasks might be intended as resistant to changes from the teacher but also understandable and useful for all the students" (p. 137). This feature is related to Ainley et al.'s (2006) constructs of *purpose* and *utility* in their framework for pedagogic task design. 'Purpose' refers to the student's perceptions, rather than to relevance of the mathematics at stake outside the classroom context. A purposeful task is defined as one that is meaningful for the student in terms of an actual or virtual outcome, or in terms of the solution of a stimulating problem (Ainley et al. 2006). 'Utility' of mathematical ideas refers to the feature that the learning of mathematics involves the construction of meaning for the ways in which those mathematical ideas are useful (Ainley et al., 2006). Purpose and utility are interconnected in the way that recognition of the utility of mathematical ideas can best be developed within purposeful tasks.

The methodology presented in this paper provides an analysis of how a student perspective on instructional design in mathematics is attended to by using TDS as a theoretical framework. The methodology is based on two principles that represent the student perspective: first, the target knowledge should be necessary or in some sense optimal to solve a task embedded in a *situation*; second, the material and intellectual reality on which the students act when solving the task (i.e., the *milieu*) should provide feedback to them, whether their responses are adequate with respect to the target knowledge. Purpose and utility are substantiated through the first principle, which means that the students will *need* the target knowledge to succeed—this provides a rationale for and meaning of the knowledge they develop. The second principle means that the students' engagement in the situation should be managed by features of the milieu rather than by teacher intervention. It can be noted that the purpose of engaging in a situation, as perceived by the student, likely is not the target knowledge, but to solve the task defined by the situation. Utility of the underlying mathematical idea—what it can be used *for*—is experienced by the students through solution of the task defined by the situation, and further reinforced by the teacher during decontextualisation of the situated knowledge (i.e., institutionalisation).

The methodology will be presented after TDS has been introduced in the next section.

2 The theory of didactical situations in mathematics

The theory of didactical situations in mathematics, TDS, provides a systemic framework for investigating teaching and learning processes and for supporting didactical design in mathematics, where the particularity of the knowledge taught plays a significant role. In TDS, knowledge is defined as solutions to problems, a principle influenced by the French philosopher Gaston Bachelard (1938/2001, p. 25): "For a scientific mind, all knowledge is an answer to a question. If there has been no question, there can be no scientific knowledge." TDS' methodology is therefore based on creating a *situation* with a task to be solved, where the knowledge aimed at is necessary or in some sense optimal to solve the given task. In order to explain this principle further, I need to explain some concepts of TDS first.[2]

An *adidactical situation* is a situation in which the student takes a mathematical task as his own and tries to solve it without the teacher's guidance and without didactical reasoning (i.e., not trying to interpret the teacher's intention with it). In addition to responsibility for handling the evolution of an adidactical situation, the teacher has two main roles in the broader didactical situation: One is *devolution* of an adidactical situation to the students. This means to introduce the task to be solved, inform about the rules for operating in the adidactical situation, and make the students accept the responsibility for solving the task. The other main role is *institutionalisation* of the knowledge developed by the students in the adidactical situation. This means to transform the responses produced by the students into scholarly knowledge in conventional notation, so that it can be reused in situations other than the one arranged by the teacher.

The *didactical contract* refers to the phenomenon that the interaction between the teacher and students in a didactical situation is regulated by rules related to the knowledge at stake. These rules form a set of reciprocal obligations. The didactical contract deals with relationships between the adidactical and didactical dimensions of a situation, and it is the teacher's role to organise them (Artigue et al. 2014). This takes place in devolution, where the teacher (implicitly) negotiates a contract that involves a temporary transfer of responsibility for the knowledge at stake, from the teacher to the students.

[2] The explication is based on the writings of Brousseau (1997), in cases where no other sources are listed.

The *milieu* represents the elements of the material and intellectual reality on which the students act when solving a task—these elements are conditions for the students' actions and reasoning (Laborde and Perrin-Glorian 2005). The milieu may comprise: the task to be solved; material or symbolic tools provided (artefacts, informative texts, data, etc.); students' prior knowledge; other students; and arrangement of the classroom and rules for operating in the situation (determinative of who is supposed to interact with whom). The milieu of an adidactical situation is called an *adidactical milieu*. An appropriate adidactical milieu provides feedback to the students, whether their responses are adequate with respect to the knowledge at stake. This involves that the milieu is designed with conditions that are incentives for the student to choose one "model" or strategy to solve the task rather than another, where the chosen one corresponds to the target knowledge. In this way, the adidactical milieu takes care of the student perspective in design of situations—the milieu is organized so as to make the student interact with it in a way that corresponds to *using* the target knowledge.

After devolution, four situations follow where the role of the teacher and the status of knowledge change: Situations of action, formulation, and validation are (intentionally) adidactical, whereas the situation of institutionalisation is didactical. The adidactical situations are designed with milieus that are supposed to give feedback to the students, as explained above. Particulars of how milieus are designed are explained in Section 3.2. The situation of *action* is where the students engage with the given task on the basis of its inner logic, without the teacher's intervention. The students construct a representation of the situation that serves as a "model" that guides them in their decisions. The knowledge represented by this implicit model has the status of a *protomathematical notion* (Chevallard 1990). The model is an example of relationships between certain objects, or rules that the students have perceived as relevant in the situation. The situation of *formulation* is where the students' formulations are useful in order to act indirectly on the material milieu—that is, to formulate a strategy (i.e., an explicit model) enabling somebody else to operate on the milieu. In the situation of formulation the teacher's role is to make different formulations "visible" in the classroom. The status of the knowledge is that of *paramathematical notions* (Brousseau 1997, p. 59), where an implicit model from the situation of action is made explicit. The situation of *validation* is where the students try to explain a phenomenon or verify a conjecture. In the situation of validation the teacher's role is to act as a chair of a scientific debate, and (ideally) intervene only to structure the debate and try to make the students express themselves in more precise mathematical language. Knowledge in the situation of validation appears as *mathematical notions*. The situation of *institutionalisation* is where the teacher connects the knowledge built by the students with forms of knowledge that are socially shared, culturally embedded and institutionally legitimised (Artigue et al. 2014).

The development of knowledge towards gradually more explicit and formal forms, as described above, is related to the way knowledge and learning is understood in TDS. Students' learning is seen as a combination of processes of *adaptation* and *acculturation* (Artigue et al. 2014). Adaptation is explained in the way that "[t]he student learns by adapting herself to a *milieu* which generates contradictions, difficulties and disequilibria, rather as human society does. This knowledge, the result of the students' adaptation, manifests itself by new responses which provide evidence for learning." (Brousseau 1997, p. 30). But this adaptation needs to be combined with acculturation, which links students' constructions to scholarly and decontextualised forms of knowledge. For such a change in the status of knowledge to take place, the teacher needs to carry out didactic interventions. The two processes are explained by Artigue et al. (2014) this way: independent adaptation is explained through the concepts of adidactical situation and milieu; acculturation is explained through the concepts of didactical situation and didactical contract; and the relationships between adaptation and acculturation are explained through the concepts of devolution and institutionalisation. In the next section, I use the presented framework in the analysis of a model for instructional design in mathematics.

3 A model for instructional design in mathematics

A model for instructional design is a theoretical model that gives a simplified representation of the reality of instructional design. It focuses on essential features of that reality, and leaves out other features. The theoretical model for instructional design presented here is centred on the construct of an *epistemological model* of some piece of knowledge. It is essential to explain the different meanings of 'model' used here; the distinction can be made between a 'model *for*' and a 'model *of*'. The theoretical *model for* instructional design is a hypothetical description of the system of elements to be used as guidance in instructional design, where these elements, and the relationships between them, are represented and explained by concepts and principles from TDS. An epistemological *model of* a piece of knowledge is one of the elements contained in the model for instructional design. It is a hypothetical description of three components that substantiate the knowledge at stake and an image of its learning: a model of the target knowledge (possibly an iconic representation); a situation that preserves meaning (with respect to the target knowledge); and, milieus of situations of action, formulation and validation (according to an image of a generic and epistemic student's learning of the target knowledge).

The model for instructional design in mathematics presented here contains four phases: epistemological analysis; development of an epistemological model; implementation; and, institutionalisation. The first two phases are about design of situations that preserve meaning for particular pieces of mathematical knowledge; the last two are about realization of these situations. Realization is part of the model for instructional design because this is where knowledge can progress towards increasingly explicit and mathematical forms. The four phases of the proposed model correspond to phases of didactical engineering (Artigue 2015): epistemological analysis corresponds to 'preliminary analyses';[3] development of an epistemological model corresponds to 'conception and *a priori* analysis'; and implementation followed by institutionalisation corresponds to 'realization'. The methodology for instructional design proposed here theorizes the phase of conception and *a priori* analysis, the aim of which is to enable a profound understanding of the construct and design of a *situation* that preserves meaning with respect to some particular knowledge. This is done through the analysis of an *epistemological model* of the target knowledge.

[3] Preliminary analyses (in didactical engineering as a *research* methodology) consist of a third dimension: an institutional analysis, the purpose of which is to identify the characteristics of the context in which the didactical engineering takes place, in terms of the conditions and constraints it faces.

The model for instructional design is illustrated by a flow chart in Figure 1. Its viability is supported by empirical data in Section 4. I now turn to a presentation of the four phases of the proposed model.

3.1 Epistemological analysis

The first phase involves an epistemological analysis of the knowledge at stake. It consists of two components: an analysis of the knowledge itself, and a didactical analysis. These components are informed by two of the three dimensions of preliminary analyses of didactical engineering (epistemological analysis, institutional analysis, and didactical analysis), as explained by Artigue (2015).

An analysis of the knowledge itself aims to identify possible epistemological obstacles, and it supports the search for fundamental situations that represents the knowledge. Drawing on the work of Bosch et al. (2006), central questions here are as follows: Where does this knowledge come from? What place does it have in school mathematics? Why should the students learn it? What is it *for*? What questions motivated its genesis? How is it related to other mathematical concepts or topics? How can its validity be justified? What *conditions* are conducive to students' *use* of the target knowledge? That is, what conditions must be fulfilled for a situation to implement the knowledge it defines? Answers to these questions provide information about what task(s) should be solved, and under which conditions.

A didactical analysis aims at surveying what published research can provide regarding the teaching and learning of the mathematics at stake—knowledge likely to guide the design. The outcome of the epistemological analysis informs the next phase, the development of an epistemological model.

3.2 Development of an epistemological model

The second phase involves development of an epistemological model of the knowledge at stake, based on the outcome of the epistemological analysis. In this paper, an epistemological model is a construct that consists of three components: first, a model of the target knowledge—possibly an iconic representation; second, a *situation* that preserves meaning—involving a task that can be solved in an optimal manner using the target knowledge; and, third, milieus of situations of action, formulation and validation—designed so as to make students' knowledge progress towards gradually more explicit and mathematical forms, based on an image of students' adaptations to the milieus. The second and third component together can be considered a model of the generic and epistemic student's intended learning.

An epistemological model focuses on a generic and epistemic student's *opportunities* to learn the target knowledge, based on an epistemological analysis. It is designed so as to make students *use* the target knowledge through adaptation to a milieu.[4] The fact that different types of interaction with the milieu and different forms of knowledge are justified *a priori* (for epistemological reasons) allows the teacher to identify the properties of the milieu that are necessary in order to provoke the interactions and knowledge aimed at. Questions such as the following, "Why would the student do

[4] The singular 'milieu' is used as a generic form to refer to milieus of action, formulation and validation.

or say this rather than that?", "What must happen if the student does or does not do it?" are suitable for imposing important conditions on the milieu.

A model of students' intended learning is developed according to the conditions that must be fulfilled for a situation to implement the knowledge it defines, as identified through the epistemological analysis. The point is to *design milieus* of situations of action, formulation and validation with conditions so that the responses produced by the students in the successive situations will become gradually more explicit and mathematical (i.e., independent adaptation), and ultimately can be institutionalised to become the scholarly knowledge aimed at by the teacher (i.e., acculturation). The milieu of the situation of action is the material milieu, which is derived from the model of the target knowledge. The material milieu is something concrete (or iconic) for the students to act on. It can be manipulatives, diagrams, etc. An example of a material milieu would be the first few iconic elements (i.e., geometrical configurations) of a shape pattern, where the task given to the students might be to find the number of components (e.g., dots) of the general element of the pattern. The milieu of the situation of formulation is the outcome of the situation of action—that is, an implicit strategy to solve the task (cf. Section 2). The way of making the knowledge (the implicit strategy) more formal, is to make the students act indirectly on the material milieu. That is, to create a need for them to explain to someone else how to act on the material milieu. In the example with the shape pattern, it would involve explaining one's strategy in order to make someone else draw the next element of the pattern (e.g., by explaining how to add components to get the next element). The milieu of the situation of validation is the outcome of the situation of formulation—that is, an explicit strategy to solve the task. The way of making the knowledge (the explicit strategy) more formal is to use mathematical notions to explain how the strategy will solve the task. In the example with the shape pattern, it would involve verifying that the explicit strategy applies to all elements (e.g., providing a generic example that shows how the explicit strategy will keep the structure of the pattern).

The epistemological model is the basis for the teacher's devolution of a situation (including a task) aiming at students' adidactical interaction with its milieu. This is the next phase, implementation of an epistemological model.

3.3 Implementation

The third phase involves *implementation* in the classroom. This means devolution of a (fundamental) situation based on the epistemological model, followed by students' interaction with the milieus of situations of action, formulation and validation. The student perspective on the instructional design will facilitate the transfer of responsibility from the teacher to the students for solving the task (i.e., the devolution). This facilitation is explained by the following points: first, the *purpose* of the task involves that the students will *need* the target knowledge to succeed in the situation; second, *utility* of the underlying mathematical idea will be constructed by the students as their experience (in the fundamental situation) of what this knowledge is *for*.

Students' learning in this phase is understood as *independent adaptation* through the concepts of adidactical situation and milieu. The results are responses (forms of knowledge) that gradually develop towards more explicit and mathematical forms, as described in the previous section.

3.4. Institutionalisation

The fourth phase involves *institutionalisation* of the solution to the task into scholarly and decontextualised forms of knowledge. This is a didactical phase informed by the epistemological analysis, where the teacher compares the contextualised knowledge (students' solutions) with the scholarly knowledge aimed at by the institution. It involves informing students about formal mathematical terminology, definitions and results that are important in order for the contextualised knowledge to gain status as cultural knowledge that can be used in other situations. Students' learning in this phase is understood as *acculturation*, which enables them to know the place, importance, and future of the mathematical knowledge reached.

In didactical engineering, the phase of realization involves both implementation and institutionalisation. In the model for instructional design proposed here, I have chosen to structure them in two phases. The reason for this is that it makes the distinction between adaptation and acculturation clearer—that is, the distinction between adidactical and didactical phases. Figure 1 illustrates the model for instructional design. The dotted curve signifies that the epistemological analysis informs the institutionalisation of the target knowledge. The model gives an overview of the different elements involved in design and realization of particular pieces of mathematical knowledge. Further, it displays the teacher's roles in this enterprise: the process of *didactical transposition* (Chevallard 1989) transforms the epistemological analysis into an epistemological model; the process of *devolution* transforms an epistemological model into a task embedded in a situation; and, the process of *institutionalisation* transforms the solution to the task into scholarly and decontextualised forms of knowledge.

In the next section, I provide empirical data to support the viability of the proposed model. The data are from an investigation of student teachers' engagement with a situation, where the target knowledge is a theorem in elementary number theory.

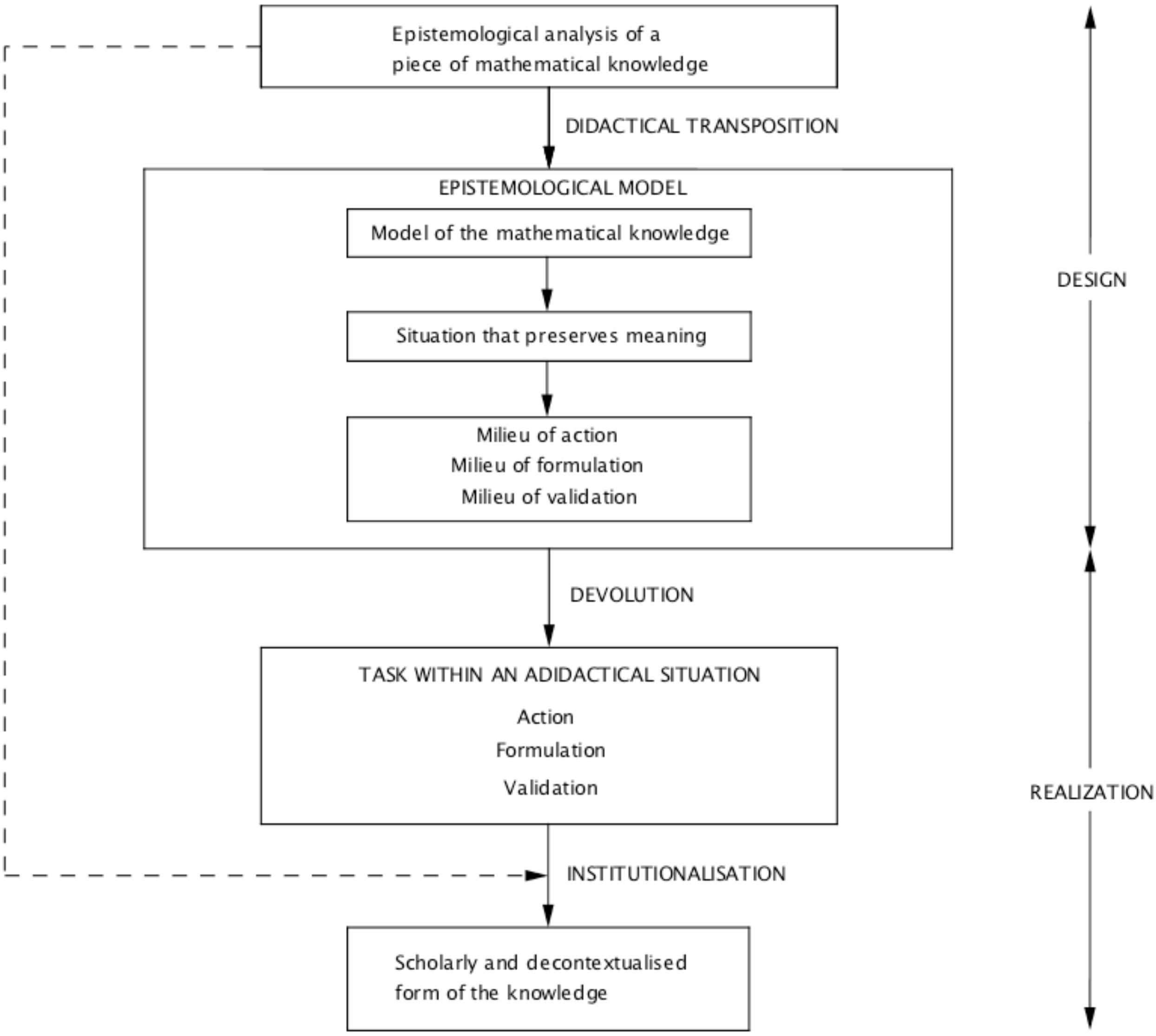

Figure 1. A model for instructional design in mathematics

4 Viability of the theoretical model supported by empirical data

In this section, I provide data from an empirical investigation to demonstrate how the model for instructional design is eligible to integrate a student perspective in the design and implementation of a task embedded in a situation. The target knowledge is a theorem expressing that the sum of the first n odd numbers is equal to the square of n. The investigation was carried out with 20 student teachers (henceforth 'students') within a mathematics education course that was part of two different Master's programmes in mathematics education at a Norwegian university college—one geared towards Grades 1-7, the other towards Grades 5-10. It can be noted that in Norway it is not required that students enrolled in such teacher education programmes have taken advanced mathematics courses at upper secondary school. Mathematics courses in such programmes therefore contain topics from both mathematics and didactics of mathematics. For the situation analysed here, it is relevant that the students were not familiar with the theorem aimed at. Hence, it was a didactical situation intended to teach them this theorem, so the data provided here should be comparable to data collected in school. However, because the observed students were *student teachers*, institutionalisation was extended by a meta-level, where the designed situation (and its realization) was taken as an object of discussion. This aspect is explained in Section 4.3.

I designed the instruction and taught the observed lesson (90 minutes) to test the viability of the theoretical model proposed in this paper. It is beyond the scope of this paper to present a complete report from the empirical investigation; only parts of the data and analysis of them are presented here as illustrations. The data of the investigation are as follows: an epistemological analysis of the target knowledge; an epistemological model of the target knowledge; classroom observations of implementation and institutionalisation; and, students' written solutions. The classroom observations were video recorded by three cameras on tripods. The first camera recorded the activity of John and Claire, the second camera recorded the activity of Tina and Anne, and the third camera was directed towards the blackboard to record activity in devolution, validation and institutionalisation. The names are pseudonyms. In the following sections, I discuss glimpses of the data.

4.1 Epistemological analysis of the target knowledge

The chosen piece of knowledge is the general numerical statement expressing that the sum of the first *n* odd numbers is equivalent to the square of *n*, possibly represented in algebraic notation as $\sum_{i=1}^{n}(2i-1)=n^2$, or $1+3+5+\cdots+(2n-1)=n^2$. It is related to *polygonal numbers*—a type of figurate numbers. A polygonal number is a generalisation of triangular numbers, square numbers, pentagonal numbers, etc., to an *n*-gon for *n* an arbitrary natural number (Weisstein 2009, Vol. 3).

According to Reed (1972), humans have a natural inclination to observe patterns, and to impose patterns on different experiences. Steen (1988) claims that mathematics is the science of patterns. Mathematicians seek patterns in different areas, including numbers (arithmetic and number theory), possibilities (probability theory), reasoning (logic), form (geometry), motion (calculus), and position (topology). The equivalence statement $\sum_{i=1}^{n}(2i-1)=n^2$ is a generalisation of a pattern in elementary number theory. Its basis is empirical and consists of a pattern of arithmetic equivalences: $1=1^2, 1+3=2^2, 1+3+5=3^2$, and so on. This pattern is illustrated by a sequence of geometrical configurations (a shape pattern), the first four elements of which are illustrated in Figure 2. The same figure also illustrates the recursive relationships: $2^2-1^2=2\cdot 2-1$; $3^2-2^2=2\cdot 3-1$; $4^2-3^2=2\cdot 4-1$; and so on.

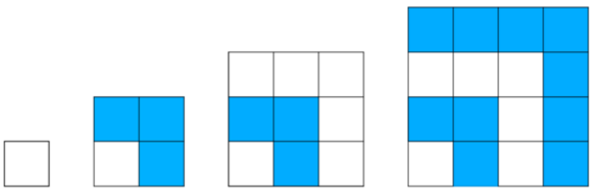

Figure 2. The first four members of a sequence of square numbers represented iconically as sums of odd numbers

The mathematical object at stake is a mathematical statement (theorem) of *equivalence* of the square of n and the sum of the first n odd numbers, $\sum_{i=1}^{n}(2i-1)=n^2$. The truth of the statement can be established geometrically or algebraically. Geometrically, it can be done by a representation-based proof (a generic example)—e.g., the fourth configuration in Figure 2. This diagram (the fourth configuration) satisfies Schifter's (2009) three criteria of validity of a proof by representation: (a) the meaning of the operation involved is represented in the diagram (the sum of the first four odd numbers is equivalent to the fourth square number); (b) this diagram is accessible for a class of instances (every square can be configured by adding odd numbers); and, (c) the conclusion of the claim follows from the structure of the diagram. Algebraically, the truth of the statement can be established in different ways (those described here do not constitute an exhaustive list): One way is by a direct proof, using the Gaussian method—that is, adding the first and the last term, then the second and the last but one term, and so on. Another way to prove the statement is to add the terms twice, one series with terms in reverse order, which yields twice the sum sought. Yet another proof involves using the property $\sum_{i=1}^{n}(2i-1)=2\sum_{i=1}^{n}i-\sum_{i=1}^{n}1$. Or, the statement can be proved by mathematical induction.

A purpose of engagement with shape patterns in school mathematics is to provide physical or iconic reference contexts for generalisation and algebraic thinking. Generalisation of shape patterns and numerical sequences is part of the elementary and secondary curriculum in many countries, for example England (Department for Education 2014), Canada (Ontario Ministry of Education and Training 2005), Norway (Directorate for Education and Training 2013), and it is included in curriculum guidelines in the United States (National Council of Teachers of Mathematics 2000). Several studies have documented students' difficulties in establishing algebraic formulae from patterns and tables (e.g., Barbosa and Vale 2015; MacGregor and Stacey 1993; Orton and Orton 1996; Warren et al. 2006). Further, research indicates that it is not generalisation tasks in themselves that are difficult; the problems that students encounter are rather due to the way tasks are designed and limitations of the teaching approaches employed (Moss and Beatty 2006; Noss et al. 1997). These findings motivated me to conduct a study of conditions that constitute obstacles to students' establishment of algebraic generalisations of patterns. Results from this study are reported in my dissertation (Strømskag Måsøval 2011): the tasks with which the students engaged were not sharply focused on the target knowledge; and, the milieus did not provide appropriate feedback whether students' responses were adequate with regard to the target knowledge (see also Strømskag Måsøval 2013). These phenomena encouraged me to explore different ways of designing tasks and milieus, the result of which is reported here.

Based on the above epistemological analysis, I developed an epistemological model of the knowledge at stake, the result of which is presented in the next section.

4.2 An epistemological model of the target knowledge

As explained in Section 3, an epistemological model of a piece of knowledge consists of a model of the knowledge and a model of the students' intended learning, where the latter is made of two parts—a situation that preserves meaning, and milieus of action, formulation and validation.

4.2.1 A model of the theorem $\sum_{i=1}^{n}(2i\text{-}1) = n^2$

A model of the target knowledge is created using a dissection of a square into L-forms consisting of consecutive odd numbers of unit squares (where 1 is represented by one unit square, hence a degenerated L). A generic example is given in Figure 3, illustrating that $\sum_{i=1}^{4}(2i - 1) = 4^2$.

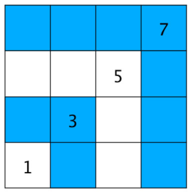

Figure 3. A model of the target knowledge

It is made of a dissection of the fourth square into the first four odd numbers. This model is not to be shown to the students. It is a tool for developing a model of the students' intended learning, of which I give an account below.

4.2.2 A situation that preserves meaning of the target knowledge

I invented a situation based on an imaginary company called TILEL, which sells a special kind of tile formations that can be used to cover squares. The tile formations have shapes as L-forms, and consist of an odd number of unit squares. The idea of this situation is derived from the model of the target knowledge, shown in Figure 3.

The task has two main parts:

- Find a method for building a square of side length a natural number, using L-forms from TILEL.
- Explain why the method will work for *any* natural number.

It can be noted that the theorem at stake was not known to the students involved in the investigation. The intention was that the students' methods should be such that when they argued that their method applies to any natural number, they would have to *make use* of the fact that the L-forms represent odd numbers. This means that the target knowledge—the statement that *n* squared is equivalent to the sum of the first *n* odd numbers—should be an optimal solution to the task. For this to happen I designed milieus to which the generic and epistemic student would adapt and thereby develop increasingly formal responses consistent with the target knowledge. The designed milieus provide a student perspective to the TILEL situation in the way the milieus afford purpose for the task, and utility of the intended theorem, to the students.

4.2.3 The milieus

Here I explain how the milieus are formed to make the knowledge needed in the adidactical situations progress towards gradually more explicit and formal forms.

The milieu of action

The milieu of action is the *material milieu* on which the students are supposed to operate. I created a material milieu in terms of ten paper cut-outs, consisting of $1, 3, 5, \cdots, 19$ unit squares that represent the first ten odd numbers, as illustrated in Figure 4.

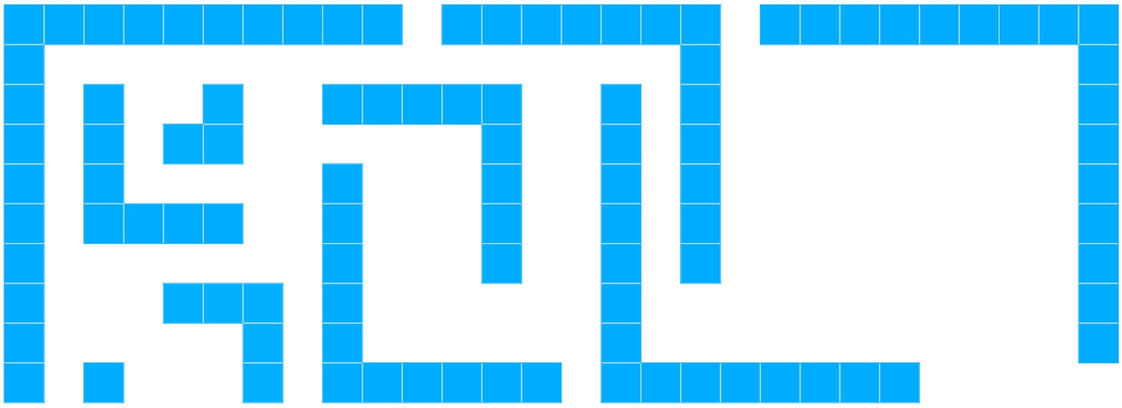

Figure 4. Material milieu for the intended theorem (the first ten L-forms)

The task in the situation of action is to *find a method of building a square of side length a natural number up to ten, using L-forms of different sizes*. The L-forms should be available for the students to access as needed. The adidactical features of the milieu (common for action, formulation, and validation) are the following:

- The material milieu made of the L-forms does provide feedback: it is visible for the students whether or not they succeed in building a square, and whether or not the square is of the intended size.
- The obligation to use L-forms of different sizes is to ensure that the students use consecutive odd number—that is, that they engage with the knowledge at stake. This obligation is part of the didactical contract.
- There is a principle to be followed: it is only the size of the resulting square that matters, which means that the students need not distinguish between different configurations of L-forms. This is to ensure that the students focus on the intended knowledge (and not on, say, combinatorics). This principle is part of the didactical contract. Two (of many possible) configurations of a square of size 4 are illustrated in Figures 5 and 6—these are *not* to be presented to the students.

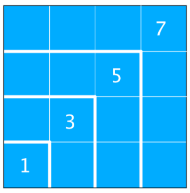

Figure 5. Example 1 of a configuration of L-forms to make a 4 x 4 square

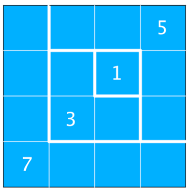

Figure 6. Example 2 of a configuration of L-forms to make a 4 x 4 square

The purpose of the situation of action is to create an *implicit model* of the knowledge at stake, through direct operation on the material milieu. After having chosen L-forms of different sizes, the students will try to arrange them into a square. It is expected that they will find out that they will have to use consecutive L-forms from 1 and upwards (not skipping any). Further, they will need to solve the rank problem—that is, observing that for a square of a particular size, they will need the L-forms up to and including the one with the same rank as the chosen square.

The milieu of formulation

The purpose of the situation of formulation is to create a need for an *explicit model* of the solution to the task (see Section 3.2). Such a need is created through students' *indirect* operation on the material milieu. This is done through the task of requiring a student (from one pair) to direct another student (from a second pair) to build a square of a certain size using Ls. The milieu of formulation is the outcome of the situation of action—this is, an implicit model of the solution to the task. The point here is that the one who is following the directions needs to operate directly on the L-forms, whereas the one who is directing does not. Further, the task for both parts is to write down the method in the general case—how to build an $n \, x \, n$ square using L-forms (for *n* a natural number). This may create a need for labelling the L-forms, possibly by $L_1, L_2, L_3, \cdots, L_{10}$, and in the general case, formulating that to build a square of a particular size, say *n*, one will need all the L-forms from L_1 up to and including L_n. The solution at this stage may be written as $L_1 + L_2 + L_3 + \cdots + L_n = n^2$.

The milieu of validation

The purpose of the situation of validation is to create a need for a further change in the status of the knowledge—that is, to turn the solution to the task into mathematical form. Such a need is created through the task of making one pair of students explain to another pair *why* their method (described in the situation of formulation) will work for *any* natural number. So, why do we get a square when adding L-forms from 1 and upwards? Here it is necessary to look at, and argue on the basis of, the nature of the Ls. This was crucial in all the proofs of the theorem, $\sum_{i=1}^{n}(2i-1) = n^2$, which were presented in Section 4.1.

In the next section, I present a brief account of how the epistemological model described above was realized in the classroom.

4.3 Realization in the classroom

4.3.1 Implementation

Here I present glimpses from realization in the classroom of the epistemological model described in Section 4.2. I will refer to the L-forms as $L_1, \cdots, L_{10}$, where L_k consists of $2k - 1$ unit squares, even if the students may have used different denotations. The material milieu (piles of different L-forms) was placed on a table at the back of the classroom. In devolution, I presented the problem situation, the material milieu, and the rules for operating on the milieu.

Under each of the adidactical situations, I give a brief, general description of the students' activity, and present excerpts of the transcript of John, Claire, Tina and Anne's utterances. These excerpts illustrate how the knowledge appears (its status), and how it evolves towards gradually more explicit and formal forms. To show the variety of reasoning in the situation of validation, I use the collected material (students' written solutions) to present Selma and Lucas' justification of their method (theorem).

Action realized

The students in each pair decided on the size of a square and went to the table at the back of the classroom to get a selection of L-forms. Further, they experimented (some needed supplements of L-forms) and found a method for precisely covering the square they set out to build.

John and Claire have collected one piece of each L-form (L_1 up to L_{10}).

John: If we start with the L that has 5 squares on the side, we can build a square of size 5. We just place them inside each other. [Claire arranges the first five L-forms similarly to Figure 5].

Claire: Is this the only way to arrange them? If we start with the first, the one… We can place the second and third like this [arranges the first three L-forms similarly to Figure 6].

John: But this is easier [refers to his own method].

Anne and Tina have also collected one piece of each L-form (L_1 up to L_{10}).

Tina: We just start with 1 and take the next L, place it to the right, take the next and do the same. Each time we get a bigger square [arranges the L-forms similarly to Figure 5].

Both groups' utterances in the situation of action show that the knowledge appears as implicit models. That is, it appears as manipulations of and pointing at L-forms and squares. The status of the knowledge is that of protomathematical notions, characterized by students' use of informal language to explain their methods.

Formulation realized

I introduced the situation of formulation by saying that the task here was to explain to somebody else how to solve the task. Each pair should write a *recipe* that would enable another student to build a square of an *arbitrary* integer size, imagining that they had L-forms of all ranks.

John: You start with the L that has the same number of [unit] squares on the side as [the size of] the square you have chosen, and continue with all L-forms down to 1. That's the recipe.

Claire: But we must tell how to arrange them. The recipe will be [writes]:

"Start with the L that has the same size as the square you want. Place it like an L with the opening to the right. Then take all the smaller L-forms down to 1 and place them inside each other in descending order".

Claire then symbolized the relationship for a $10\ x\ 10$ square: $L1 + L2 + \cdots + L10 = 10^2$.

Anne and Tina have in front of them a $10\ x\ 10$ square, arranged similarly to the one in Figure 5.

Tina: We can call them O_1 up to O_{10}. Because they are odd numbers.

Anne: OK. The recipe is that O_1 plus O_2 plus O_3 up to O_{10} is equal to ten squared. [Writes $O_1 + O_2 + O_3 + \cdots + O_{10} = 10^2$].

Tina: But this is just for a ten by ten square. It should be for any square. A recipe… the square has side *x*.

Anne: Then you take all L-forms up to x.

Tina: Yes. O_1 up to O_x. This is equal to *x* squared. [Writes $O_1 + O_2 + \cdots + O_x = x^2$].

Anne: Yes, the recipe is that you choose the side of a square and then you'll have to get all the L-forms from 1 up to the L with the same side.

When Claire used the notion of the "size" of an L, she probably had in mind its rank. Her recipe is a generic example ($n = 10$). Anne and Tina presented an explicit model for the general case, and explained in natural language what it means. In the situation of formulation—through the requirement of indirect operation on the material milieu—the need for creating references to the L-forms arose. Here, knowledge appears as students' explicit models. Its status is that of paramathematical notions, evolved from protomathematical notions as an adaptation to the need for more formal language.

Validation realized

The task in validation was to justify that their method (recipe) would work for *any* square of an integer side length. The students worked in pairs on this, followed by a whole-class conversation led by me.

Claire and John's reasoning was as follows:

Claire: Take the square you have and then pick the next L and place it on the other side. Then you'll get the next square. [She illustrates by extending a 5 x 5 square with L_6].

John: This will work because if you take this [points at L_6], it is five and five and one for the corner.

Anne and Tina's outcome of formulation was $O_1 + O_2 + \cdots + O_x = x^2$, which they have verified for $x = 10$. Anne's justification of the general case was as follows:

Anne: We'll show that it's true for 100, a 100 square. You just take all L-forms from 1 and upwards to the 100th. It is true for $x = 10$ and it is just about getting all the next L-forms and do the same up to O_{100}.

A third example of students' justification is reproduced from the written material collected from the class. Selma and Lucas' solution is as follows: To build a square of size *n* you'll need the first *n* odd numbers, which corresponds to L_1 up to L_n. They justified it by writing down $1 + 3 + \cdots + (2n - 3) + (2n - 1)$ and explaining that they added the first and last term, the second and last but one term, and so on. Further, they explained that this gives $\frac{n}{2}$ sums of $2n$. That is, $1 + 3 + \cdots + (2n - 3) + (2n - 1) = \frac{n}{2} \cdot 2n = n^2$, which completes the proof.

It is relevant to comment on the proofs given by the students. Claire and John's proof has elements of a generic example. However, to become a valid proof, it would be necessary for them to explain that it is the property of *odd numbers* that makes this work in general: Starting with a square of size *n*, a square of size $(n + 1)$ is built by adding two times *n* plus one for the corner. That is, adding the next odd number, $2n + 1 = 2(n + 1) - 1 = L_{n+1}$. This is also the step in an induction proof (it would be necessary though to show that it is true for, say, $n = 1$). Anne's argument can be interpreted as a *crucial experiment* (Balacheff 1988). It is a kind of naïve empiricism—it does not explain *why* their method works for all natural numbers. Anne does not utilise that the odd numbers are involved. Selma and Lucas' explanation is a valid proof—a direct algebraic proof, using the Gaussian method.

Regarding the status of knowledge, I consider that Anne does not use mathematical notions, whereas the other pairs do use mathematical notions (even if Claire and John's knowledge lacks some specification). During the whole-class discussion, I made the different arguments visible in the class. I got the students to explain their reasoning, to defend some justifications and act as critics of others. This served as a basis for the next phase, institutionalisation.

4.3.2 Institutionalisation

Institutionalisation is where students' constructions resulting from adaptation are combined with acculturation. It is the teacher's task to decontextualise the situated knowledge and link it to the scholarly knowledge aimed at by the institution. During institutionalisation I displayed students' various methods for solving the task—that is, the recipes and justifications that had appeared in situations of formulation and validation. The representations used by the students ($L1 + L2 + \cdots + L10 = 10^2$; $O_1 + O_2 + \cdots + O_x = x^2$; etc.) I compared with the target theorem, which I symbolized by $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ and commented that this might be written as $\sum_{i=1}^{n}(2i - 1) = n^2$. The students acknowledged that it stated that the sum of the first *n* odd numbers is equivalent to the *n*-th square number. Further, I categorised the different types of justifications used: naïve empiricism; generic example (using a diagram); and, direct algebraic proof. I discussed the distinction between naïve empiricism and a generic example—that explaining the *reason* why something works in a particular example makes it generic and valid as a proof (unlike naïve empiricism). Here, I drew on Schifter's (2009) three criteria for the validity of a representation-based proof (as explained in Section 4.1).

An objective of institutionalisation is enabling the students to know the *place, importance, and future* of the knowledge they have developed. This part was informed by elements from the epistemological analysis (see Section 4.1). Further, I compared the knowledge they had developed through the TILEL situation (i.e., a *theorem* in elementary number theory) to knowledge they had previously developed through another type of shape patterns (i.e., a *functional relationship* between position and numerical value of elements). (For an analysis of the two types of shape patterns, see Strømskag 2015).

The last part of institutionalisation was related to the fact that the students were *student teachers*, enrolled in teacher education programmes for Grades 1-7 or 5-10. It was a meta-level expansion where the TILEL situation (and the students' experiences with it) was taken as an object of discussion. I explained the two principles behind the design of the situation and its adidactical milieu. Further, it was discussed what adaptations would be necessary in order to implement the TILEL situation in various grades in school. For these students, the TILEL situation served as an introduction to later work on TDS.

5 Discussion

The data analysed in Section 4 showed that it was possible to use the proposed model for instructional design to create a situation that preserved meaning for the theorem at stake. The student perspective was taken care of through the two principles of task design: The first principle was applied in the way that the solution to the task of building a square in the TILEL situation was adding consecutive L-forms from 1 and upwards; this was situated knowledge, subsequently institutionalised to become the theorem that was the aim of the task. Purpose and utility (Ainley et al. 2006) were substantiated through the first principle in the way the students *needed* the target knowledge to succeed—this provided a rationale for, and meaning of, the knowledge they developed. The second principle of task design was applied in the way that the material milieu and the clauses of the didactical contract (described in Section 4.2) made it possible for the students to develop the intended knowledge through adidactical engagement in the TILEL situation. The didactical contract (shaped by the knowledge at stake) governed what the students were allowed to do to solve the task, and the milieu gave feedback whether they succeeded or not.

The proposed model for instructional design is applicable when there is an intention of teaching someone some particular mathematical knowledge, whether it is for students in school or college. To strengthen the claim about its relevance at tertiary level, it can be mentioned that TDS design principles have been used at the university. For instance, González-Martín et al. (2014) discuss three recent research cases—two on calculus and one on proof. These are stimulating with respect to the feasibility of designing epistemological models to teach university level mathematics (see also Artigue 2014). When the model is used for student teachers (who are also learning the mathematics at stake), it is relevant to extend the institutionalisation by a meta-level (didactical) discussion, similar to the one related to the TILEL situation described above.

The methodology presented here is for instructional design, where the focal point is the generic and epistemic students' opportunities to develop particular pieces of mathematical knowledge—from an epistemological viewpoint. The methodology might be extended to a research methodology by including phases of 'observation and data collection', and '*a posteriori* analysis and validation'—as in didactical engineering (Artigue 2015). An extension to a research methodology is possible, whether the instructional design is for teaching school mathematics or university level mathematics (be it teacher education or other programmes). Validation would be based on comparison between *a priori* and *a posteriori* analyses of the situations involved. The situation presented in Section 4 resulted from such an extension of the methodology. Analysis of the data showed that the TILEL situation was valid for the theorem $\sum_{i=1}^{n}(2i-1)=n^2$, because the students developed the intended theorem as a consequence of solving the task embedded in the situation.

I plan to use the model for instructional design in a teacher education programme, in which I give student teachers the assignment of design and implementation (with students in secondary school) of epistemological models that preserve meaning for particular pieces of mathematical knowledge (of their own choice, using the model proposed here). Further, they will be given the task of doing research into the situations, and judging their validity based on comparison of *a priori* and *a posteriori* analyses. One goal is to give student teachers experiences with instructional design that aims at meaningful mathematics (i.e., perceived utility), learned in meaningful ways (i.e., perceived purpose). Another goal—through iterated cycles of design, implementation and development—is to establish a stock of robust epistemological models of various pieces of mathematical knowledge that can be used in school. I conjecture that the theoretical analysis of the model presented in this paper will enable other researchers to use the model as a tool to do something similar to that outlined in this paragraph, beyond the local level described here.

Acknowledgement

I would like to thank two anonymous reviewers whose insightful comments resulted in significant improvements to this paper.

References

Ainley, J., & Margolinas, C. (2015). Accounting for students' perspective in task design. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education: An ICMI study 22* (pp. 130-156). Cham, Switzerland: Springer. doi:10.1007/978-3-319-09629-2_4.

Ainley, J., Pratt, D., & Hansen, A. (2006). Connecting engagement and focus in pedagogic task design. *British Educational Research Journal*, *32*(1), 23-38. doi:10.1080/01411920500401971.

Artigue, M. (2014). Potentialities and limitations of the Theory of Didactic Situations for addressing the teaching and learning at university level. *Research in Mathematics Education*, *16*, 135-138. doi: 10/1080/14794802.2014.918348.

Artigue, M. (2015). Perspectives on design research: The case of didactical engineering. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education* (pp. 467-496). Dordrecht, The Netherlands: Springer. doi: 10.1007/978-94-017-9181-6_17.

Artigue, M., Haspekian, M., & Corblin-Lenfant, A. (2014). Introduction to the theory of didactical situations (TDS). In A. Bikner-Ahsbahs & S. Prediger (Eds.), *Networking of theories as a research practice in mathematics education* (pp. 47-65). Cham, Switzerland: Springer. doi:10.1007/978-3-319-05389-9_4.

Bachelard, G. (2001). *The formation of the scientific mind: A contribution to a psychoanalysis of objective knowledge* (M. M. Jones, Trans.). Manchester, UK: Clinamen Press. (Original work published 1938)

Balacheff, N. (1988). Aspects of proof in pupil's practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216-235). London: Hodder & Stoughton.

Barbosa, A., & Vale, I. (2015). Visualization in pattern generalization: Potential and challenges. *Journal of the European Teacher Education Network*, *10*, 57-70. http://jeten-online.org /index.php/jeten/article/view/67. Accessed 10 March 2017.

Bosch, M., Chevallard, Y., & Gascón, J. (2006). Science or magic? The use of models and theories in didactics of mathematics. In M. Bosch (Ed.), *Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education* (pp. 1254-1263). Barcelona, Spain: Universitat Ramon Llull Editions.

Brousseau, G. (1997). *The theory of didactical situations in mathematics*: *Didactique des mathématiques, 1970-1990* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Eds. & Trans.). Dordrecht, The Netherlands: Kluwer.

Chevallard, Y. (1989). On didactic transposition theory: Some introductory notes. In H. G. Steiner & M. Hejny (Eds.), *Proceedings of the International Symposium on Selected Domains of Research and Development in Mathematics Education* (pp. 51-62). University of Bielefeld, Germany, and University of Bratislava, Slovakia.

Chevallard, Y. (1990). On mathematics education and culture: Critical afterthoughts. *Educational Studies in Mathematics*, *21*, 3-27.

Department for Education. (2014). *National curriculum in England: Mathematics programmes of study*. https://www.gov.uk/government/publications/national-curriculum-in-england-mathematics-programmes-of-study. Accessed 12 March 2017.

Directorate for Education and Training. (2013). *Læreplan i matematikk fellesfag* [Curriculum for the common core subject of mathematics]. http://www.udir.no/kl06/MAT1-04/. Accessed 12 March 2017.

González-Martín, A. S., Bloch, I., Durand-Guerrier, V., & Maschietto, M. (2014). Didactic Situations and Didactical Engineering in university mathematics: Cases from the study of Calculus and proof. *Research in Mathematics Education*, *16*, 117-134. doi: 10.1080/14794802.2014.918347.

Hiebert, J., Gallimore, R., Garnier, H., Givvin, K. B., Hollingsworth, H., Jacobs, J., … Stigler, J. W. (2003). *Teaching mathematics in seven countries: Results from the TIMSS 1999 video study*. Washington, DC: National Center for Education Statistics.

Laborde, C., & Perrin-Glorian, M.-J. (2005). Introduction. Teaching situations as object of research: Empirical studies within theoretical perspectives. *Educational Studies in Mathematics*, *59*, 1–12. doi:10.1007/s10649-005-5761-1.

MacGregor, M., & Stacey, K. (1993). Seeing a pattern and writing a rule. In J. Hirayabashi, N. Nohda, K. Shigematsu, & F.-L. Lin (Eds.), *Proceedings of the 17th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 181-188). Tsukuba, Japan: PME.

Moss, J., & Beatty, R. (2006). Knowledge building in mathematics: Supporting collaborative learning in pattern problems. *Computer-Supported Collaborative Learning*, *1*, 441-465. doi:10.1007/s11412-006-9003-z.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Noss, R., Healy, L., & Hoyles, C. (1997). The construction of mathematical meanings: Connecting the visual with the symbolic. *Educational Studies in Mathematics*, *33*, 203-233. doi:10.1023/A:1002943821419.

Ontario Ministry of Education and Training. (2005). *The Ontario curriculum, Grades 1–8: Mathematics* (Rev. ed.). http://www.edu.gov.on.ca/eng/curriculum/elementary/math18curr.pdf. Accessed 12 March 2017.

Orton, A., & Orton, J. (1996). Making sense of children's patterning. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 83-90). València, Spain: PME.

Reed, S. K. (1972). Pattern recognition and categorization. *Cognitive Psychology*, *3*, 383-407.

Schifter, D. (2009). Representation-based proof in the elementary grades. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proofs across the grades* (pp. 71-86). New York: Routledge.

Shimizu, Y., Kaur, B., Huang, R., & Clarke, D. (Eds.). (2010). *Mathematical tasks in classrooms around the world*. Rotterdam, The Netherlands: Sense Publishers.

Steen, L. A. (1988). The science of patterns. *Science*, *29*, 611-616.

Strømskag Måsøval, H. (2011). *Factors constraining students' establishment of algebraic generality in shape patterns: A case study of didactical situations in mathematics at a university college.* Doctoral dissertation at University of Agder, Kristiansand, Norway. https://brage.bibsys.no/xmlui/handle/11250/2394000. Accessed 18 May 2017.

Strømskag Måsøval, H. (2013). Shortcomings in the milieu for algebraic generalisation arising from task design and vagueness in mathematical discourse. In C. Margolinas (Ed.), *Task design in mathematics education. Proceedings of ICMI Study 22* (pp. 233-241). Oxford. https://hal.archives-ouvertes.fr/hal-00834054v2/document. Accessed 14 March 2017.

Strømskag, H. (2015). A pattern-based approach to elementary algebra. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education (CERME9)* (pp. 474-480). Prague, Czech Republic: European Society for Research in Mathematics Education. https://hal.archives-ouvertes.fr/hal-01286944/document. Accessed 18 May 2017.

Warren, E., Cooper, T. J., & Lamb, J. T. (2006). Investigating functional thinking in the elementary classroom: Foundations for early algebraic thinking. *Journal of Mathematical Behavior*, *25*, 208-223. doi:10.1016/j.jmathb.2006.09.006.

Weisstein, E. W. (Ed.). (2009). *CRC Encyclopedia of mathematics* (3rd ed., Vol. 3). Boca Raton, FL: Chapman & Hall/CRC.