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Well-Posedness of a Fractional Mean Field Game System with Non-Local Coupling

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Abstract

We prove existence and uniqueness of classical solutions for a fractional Mean Field Game system with non-local coupling, where the fractional exponent is greater than 1/2. To our knowledge this is not proven before in the literature, and is therefore a new result. In addition, we show regularity in time and space for the fractional Hamilton-Jacobi equation, and use this result to show regularity for the fractional Fokker-Planck equation.

Sammendrag

Vi beviser eksistens og entydighet av klassiske løsninger for et fraksjonelt Mean Field Game system med ikke-lokal kobling, der den fraksjonelle eksponenten er større enn 1/2. Til vår kunnskap er dette ikke vist tidligere i litteraturen, og er dermed et nytt resultat. Vi viser også regularitet i tid og rom for den fraksjonelle Hamilton-Jacobi-ligningen, og bruker dette resultatet for å vise regularitet for den fraksjonelle Fokker-Planck-ligningen.

Preface

This thesis is the conclusion of my Master's project in Industrial Mathematics at the Applied Physics and Mathematics study programme at the Norwegian University of Science and Technology (NTNU). The project was carried out during the spring of 2017.

Techniques I have learned in functional analysis, and in courses on partial differential equations, have proved to be very useful during the work with this thesis. Much of the results have been obtained using standard techniques from functional analysis.

In the end, I would like to thank my supervisor Espen Robstad Jakobsen, professor at the Department of Mathematical Sciences at NTNU, for very good supervision and for pushing me in my work. Also a thanks to my fellow students for their company and coffe breaks, that made writing this thesis a very cheerful endeavour.

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Chapter 1

Introduction

1.1 Fractional Mean Field Games with non-local coupling

The object of this thesis is to prove existence and uniqueness of solutions for a fractional Mean Field Game (MFG) system with non-local coupling.

Mean Field Games is a relatively new field of mathematics, and was introduced almost simultaneously by Lasry and Lions [12] , and Caines, Huang and Malhamé [7]. The idea of Mean Field Games is to model differential games with instinguishable (symmetric) players, where the amount of players tend to infinity, and each player becomes accordingly small. The average player wants to optimize some cost function in a noisy environment, where the information available is the distribution of other players and the position of itself.

Until very recently, most of the litterature on MFG have modelled the noisy environment as a standard diffusion process, but a recent paper by Cesaroni et al. [4] discusses a stationary MFG system where the noisy environment is modelled by pure jump Lévy processes. They look at the stationary case, that is, where one assumes that a Nash equilibrium has occured: a state where no player would spontaneously change their position, knowing the distribution of the other players.

We look at the case where the players still want to change their positions, based on the information they receive on the density of other players: A time dependent case. This is something that, to the best of our knowledge, is not yet presented in the literature. The system of PDE's that describe this system is given by

$$
(1.1) \qquad \begin{cases} -\partial_t u + (-\Delta_{\mathbb{T}^d})^{\frac{\alpha}{2}} u + H(x, u, Du) = F(x, m(t)) & \text{in } (0,T) \times \mathbb{T}^d \\ \partial_t m + (-\Delta_{\mathbb{T}^d})^{\frac{\alpha}{2}} m - div\,(m D_p H(x, u, Du)) = 0 & \text{in } (0,T) \times \mathbb{T}^d \\ m(0) = m_0, u(x,T) = G(x, m(T)) \end{cases}
$$

where $\alpha \in (1, 2)$, and $(-\Delta_{\mathbb{T}^d})^{\alpha/2} u$ is the fractional Laplacian on the torus. The functions F and G are both non-local coupling functions. The function H is called the Hamiltonian, and is convex in the last variable.

The first equation in (1.1) is known as the fractional Hamilton-Jacobi equation, and is solved backwards in time, while the second one is the fractional Fokker-Planck equation, and is solved forwards in time. We seek classical solutions, that is, a pair $u, m \in C^{1,2}\left((0,T) \times \mathbb{T}^d\right)$ that solves the system (1.1) simultaneously.

The content of this thesis is as follows:

1.2 Outline of thesis

Chapter 2: Preliminaries

Here we present some theory on the fractional Laplacian, both on $\mathbb{R}^d$ and on the torus $\mathbb{T}^d$. We then present some theory on the probability space $P\left(\mathbb{T}^d\right)$ endowed with the Kantorovitch-Rubinstein metric d_1. The last part of the Preliminaries consists of presenting some fixed point theorems, Hölder spaces and compact embedding theorms.

Chapter 3: Fractional MFG systems with nonlocal coupling

In this chapter we prove the existence and uniqueness of classical solutions for the fractional MFG system (1.1), under suitable assumptions on the Hamiltonian H, the coupling functions F, G and the initial conditions m_0.

Chapter 4: Regularity for the fractional Hamilton-Jacobi equation

We present some regularity results for the fractional Hamilton-Jacobi equation, with a Hamiltonian H of a quite general form. We prove regularity in time and space by using Duhamel's formula, combining it with known regularity of the unique viscosity solution for the fractional Hamilton-Jacobi equation.

Chapter 5: The fractional Fokker-Planck equation

By rewriting the fractional Fokker-Planck equation into divergence free form, we can write it on the form of a fractional Hamilton-Jacobi equation. We then show under which conditions this system admits a unique solution with sufficient regularity for the MFG-existence proof.

Chapter 6: Estimates of $\partial_x^\beta H$

We show a way to represent the derivative $\partial_x^\beta H\left(s, x, u\left(s,x\right) w\left(s,x\right)\right)$, and use this representation to give some estimates that are used in Chapter 4.

Concluding remarks

The main results of this thesis is presented, along with suggestions for further work.

Appendix

We give the proof of some Lemma's stated in the report, that are a bit too long for being written in the main report.

Chapter 2

Preliminaries

2.1 The fractional Laplace operator

Assume that we have a function $u : \mathbb{R}^d \to \mathbb{R}$. There are several and equivalent ways of definining the fractional Laplace operator on this function, as shown in [11], but we will limit ourselves to only one of them.

One can define it is as a singular integral. Let $\alpha \in (0, 2)$. Then the fractional Laplacian $(-\Delta)^{\alpha/2} u$ can be written as (for an arbitrary $r > 0$)

$$(-\Delta)^{\alpha/2} u(x) = c(d, \alpha) \left(\int_{B_r} \frac{u(x+z) - u(x) - \nabla u(x) \cdot z}{|z|^{d+\alpha}} dz + \int_{\mathbb{R}^d \setminus B_r} \frac{u(x+z) - u(x)}{|z|^{d+\alpha}} dz \right) \tag{2.1}$$

where $c(d, \alpha)$ is a constant. For the case $\alpha \in (1, 2)$, the expression (2.1) one can simplifed to (Theorem 1. in [6])

$$(-\Delta)^{\alpha/2} u(x) = c(d, \alpha) \int_{\mathbb{R}^d} \frac{u(x+z) - u(x) - \nabla u(x) \cdot z}{|z|^{d+\alpha}} dz \tag{2.2}$$

Note that these integrals are singular near $z = 0$, so that they are understood in the sense of Cauchy principal value.

2.1.1 Fractional Laplacian on the torus

Having given a definition of the fractional Laplacian on the whole space $\mathbb{R}^d$, we want to look into how it is defined on the torus, $\mathbb{T}^d$. This is natural, since we will later look at a Mean Field Game system defined on the torus.

The d-dimensional torus can be defined as the quotient space

$$\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d \tag{2.3}$$

or equivalently, as the product space of d circles

$$\mathbb{T}^d := \underbrace{S^1 \times \cdots \times S^1}_{\text{d times}} \tag{2.4}$$

A thing worth knowing about the torus, is that

Lemma 2.1. *The torus $\mathbb{T}^d$ is compact.*

This follows easily from S^1 being compact due to Heine-Borel, and then from that product spaces of compact spaces are also compact.

If a function $f : \mathbb{T}^d \to \mathbb{R}$, then the function f has a periodic extension to $\mathbb{R}^d$, which we will just call f. For this function we have that for all $x \in \mathbb{R}^d$ and $z \in \mathbb{Z}^d$

$$f(x+z) = f(x)$$

Using the periodic extensions for functions defined on the torus, we can define the fractional Laplacian for functions on the torus. This is because the earlier definitions (2.1) and (2.2) still works for functions $u : \mathbb{T}^d \to \mathbb{R}$, when we look at their periodic extensions to $\mathbb{R}^d$. So, the definition is the same, with the only difference that now $x \in \mathbb{T}^d$

$$\begin{aligned}(-\Delta_{\mathbb{T}^d})^{\alpha/2} u(x) = c(d,\alpha) \Bigg(&\int_{B_r} \frac{u(x+z) - u(x) - \nabla u(x) \cdot z}{|z|^{d+\alpha}} dz \\ &+ \int_{\mathbb{R}^d \setminus B_r} \frac{u(x+z) - u(x)}{|z|^{d+\alpha}} dz \Bigg) \quad , \quad x \in \mathbb{T}^d\end{aligned}$$

For the torus, we also present another way to work with the fractional Laplacian, and that is through the use of Fourier series.

The Fourier series of the function $u : \mathbb{T}^d \to \mathbb{R}$ is given by (see [13])

$$u(x) = \sum_{n \in \mathbb{Z}^d} c_n(u) e^{in \cdot x} \quad , x \in \mathbb{T}^d$$

where $n \cdot x = n_1 x_1 + \cdots + n_d x_d$, and the Fourier coefficients are defined as

$$c_n(u) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} u(x) e^{-in \cdot x} dx$$

Then the Fourier series for the fractional Laplace on the torus is given by (for $\alpha \in (0,2)$)

$$(2.5)\qquad (-\Delta_{\mathbb{T}^d})^{\alpha/2} u(x) = \sum_{n\in\mathbb{Z}^d} c(\alpha,d)\,|n|^{\alpha} c_n(u)\, e^{in\cdot x} \quad , \quad x \in \mathbb{T}^d$$

We will now state some properties of the fractional Laplacian on the torus, and it begins with the following interpolation Lemma

Theorem 2.1. *(Hölder estimates, Theorem 2.6 in [14]). Assume that* $\alpha \in (0,2)$ *and* $\sigma \in (0,1]$.

Let $v \in C^{1,\sigma}(\mathbb{T}^d)$ *and* $\alpha \geq \sigma$, *with* $\sigma - \alpha + 1 > 0$. *Then* $(-\Delta_{\mathbb{T}^d})^{\frac{\alpha}{2}} v \in C^{0,\sigma-\alpha+1}(\mathbb{T}^d)$ *and*

$$(2.6)\qquad \|(-\Delta_{\mathbb{T}^d})^{\frac{\alpha}{2}} v\|_{C^{0,\sigma-\alpha+1}(\mathbb{T}^d)} \leq C\|v\|_{C^{1,\sigma}(\mathbb{T}^d)}$$

A consequence of Theorem 2.1 is that, if we have a function $v \in C^2(\mathbb{T}^d)$, we get the interpolation

$$(2.7)\qquad \|(-\Delta_{\mathbb{T}^d})^{\frac{\alpha}{2}} v\|_{L^\infty(\mathbb{T}^d)} \leq C\|v\|_{C^2(\mathbb{T}^d)}$$

which is an estimate that we use a lot.

The next thing we want to say about the fractional Laplacian on the torus, is something about the identity $\langle(-\Delta)^{\alpha/2} u, v\rangle_{L^2(\mathbb{T}^d)} = \langle u, (-\Delta)^{\alpha/2} v\rangle_{L^2(\mathbb{T}^d)}$, for $u, v \in C^2(\mathbb{T}^d)$. This is a result we need in the uniqueness proof for classical solutions of the MFG system.

Lemma 2.2. *Assume that* $f, g \in C^\infty(\mathbb{T}^d)$. *Then the following identity holds:*

$$\int_{\mathbb{T}^d} (-\Delta_{\mathbb{T}^d})^{\alpha/2} f(x)\, g(x) = \int_{\mathbb{T}^d} f(x)\, (-\Delta_{\mathbb{T}^d})^{\alpha/2} g(x)\, dx$$

Proof. Since $f, g \in C^\infty(\mathbb{T}^d)$, one can show that the corresponding Fourier series, and the Fourier series of $(-\Delta_{\mathbb{T}^d})^{\alpha/2} f$ and $(-\Delta_{\mathbb{T}^d})^{\alpha/2} g$ converges absolutely (see [13]).

Therefore, one can interchange integration and summation to obtain the result

$$\begin{aligned}
\int_{\mathbb{T}^d} (-\Delta_{\mathbb{T}^d})^{\frac{\alpha}{2}} f(x)\, g(x)\, dx &= \int_{\mathbb{T}^d} \left(\sum_{n,m\in\mathbb{Z}^d} c(\alpha,d)\, |n|^\alpha c_n(f)\, e^{inx} c_m(g)\, e^{imx} \right) dx \\
&= \sum_{n,m\in\mathbb{Z}^d} c(\alpha,d)\, |n|^\alpha c_n(f)\, c_m(g) \int_{\mathbb{T}^d} e^{i(n+m)\cdot x} dx \\
&\overset{(*)}{=} \sum_{n+m=0} c_n(f)\, c(\alpha,d)\, |m|^\alpha c_m(g) \int_{\mathbb{T}^d} e^{i(n+m)\cdot x} dx \\
&\overset{(*)}{=} \sum_{n,m\in\mathbb{Z}^d} c_n(f)\, c(\alpha,d)\, |m|^\alpha c_m(g) \int_{\mathbb{T}^d} e^{i(n+m)\cdot x} dx \\
&= \int_{\mathbb{T}^N} f(x)\, (-\Delta_{\mathbb{T}^d})^{\frac{\alpha}{2}} g(x)\, dx.
\end{aligned}$$

The equality marked with (*) comes from the fact that

$$\int_{\mathbb{T}^d} e^{i(n+m)\cdot x} dx = 0, \text{ for } n+m \neq 0$$

□

We can generalize the result from Lemma 2.2, to functions $u, v \in C^2(\mathbb{T}^d)$ by using a density argument.

Lemma 2.3. *Let $f, g \in C^2(\mathbb{T}^d)$. Then the following identity holds, for $\alpha \in (1,2)$.*

$$\int_{\mathbb{T}^d} (-\Delta_{\mathbb{T}^d})^{\alpha/2} f(x)\, g(x) = \int_{\mathbb{T}^d} f(x)\, (-\Delta_{\mathbb{T}^d})^{\alpha/2} g(x)\, dx$$

Proof. The proof is given in the Appendix. □

2.2 Measures and distance

In this section we want to say something about the space of Borel probability measures on the torus, and give a definition of a metric d_1 defined on this space. We will just list the results we need.

Definition 2.1. *Let X be a separable metric space. We denote $P(X)$ to be:*

$$P(X) := \textit{the family of all Borel probability measures on } X.$$

Theorem 2.2. *(Prokhorov, from Ambrosio thm 5.1.3) If a set $K \subset P(X)$ is tight, i.e.*

$$\forall \epsilon > 0 \;\; \exists K_\epsilon \textit{ compact in } X \textit{ such that } \; \mu(X \setminus K_\epsilon) \leq \epsilon \; \forall \mu \in K, \tag{2.8}$$

then K is relatively compact in $P(X)$. Conversely, if there exists an equivalent complete metric for X, i.e. X is a so called Polish space, then every relatively compact subset of $P(X)$ is tight.

Comment: What is meant, is that K is relatively compact with respect to the narrow topology on $P(X)$.

Definition 2.2. *(The set of Borel probability measures on the torus) We define* $\mathbf{P}\left(\mathbb{T}^d\right)$ *to be:*

$$\mathbf{P}\left(\mathbb{T}^d\right) := \textit{the set of Borel probability measures on } \mathbb{T}^d$$

On this set, we can define the following (Kantorovitch-Rubinstein) distance:

$$d_1(\mu,\nu) = \sup\left(\int_{\mathbb{T}^d} \phi(x)(\mu-\nu) \,|\, \phi : \mathbb{T}^d \to \mathbb{R} \ 1-\textit{Lipschitz continuous}\right).$$

which metricizes the weak topology on $\mathbf{P}\left(\mathbb{T}^d\right)$*.*

Lemma 2.4. $\left(P\left(\mathbb{T}^d\right), d_1\right)$ *is a compact metric space.*

Proof. We refer to Lemma 4.1.7 of [3], and recall that all r-moments of members of $P\left(\mathbb{T}^d\right)$ are finite. □

We also state the following property.

Lemma 2.5. *The metric* d_1 *can be defined equivalently as:*

$$d_1(\mu,\nu) = \sup\left(\int_{\mathbb{T}^d} \phi(x)(\mu-\nu) \,|\, \phi : \mathbb{T}^d \to \mathbb{R} \ 1-\textit{Lipschitz continuous}, \phi(0) = 0\right)$$

Proof. Recall the definition of d_1. Take any $\phi \in 1 - Lip$. Assume that $\phi(0) = k \in \mathbb{R}$. We can then define $\tilde{\phi}(x) = \phi(x) - k$. Then $\tilde{\phi} \in 1 - Lip$, since

$$|\tilde{\phi}(x) - \tilde{\phi}(y)| = |\phi(x) - \phi(y)| \leq 1 \cdot |x-y|$$

For any $\mu, \nu \in P\left(\mathbb{T}^d\right)$, we get

$$\begin{aligned}\int_{\mathbb{T}^d} \tilde{\phi}(x)\, d(\mu-\nu)(x) &= \int_{\mathbb{T}^d} \phi(x)\, d(\mu-\nu)(x) - k\int_{\mathbb{T}^d} d(\mu-\nu)(x)\\ &= \int_{\mathbb{T}^d} \phi(x)\, d(\mu-\nu)(x)\end{aligned}$$

This shows that the definitions are equivalent. □

2.3 Analysis

In this section we will present some results from analysis. We start with the fundamental theorem of Calculus.

Theorem 2.3. *(Fundamental theorem) Assume that* $f \in C^1\left(\mathbb{R}^N\right)$. *Then the following holds for* $x, y \in \mathbb{R}^N$:

$$f(x) - f(y) = \int_0^1 \frac{d}{dt} f(tx + (1-t)y)\, dt$$

By using the chain rule, one can also write this as:

$$f(x) - f(y) = \sum_{i=1}^{N} (x_i - y_i) \int_0^1 \frac{\partial}{\partial x_i} f(tx + (1-t)y)\, dt$$

We also need the following short result for some of the calculations later.

Lemma 2.6. *Suppose that* $f : [0,\infty) \times \mathbb{R}^N \to \mathbb{R}$ *is Lipschitz and uniformly bounded. In other words, there exists constants* $L, M > 0$ *such that:*

$$|f(s,x) - f(t,y)| \leq L(|s-t| + |x-y|)$$

$$\|f\|_\infty \leq M$$

Then there exists a constant $C > 0$ *such that* f *satisfies:*

$$|f(s,x) - f(t,y)| \leq C\left(|s-t|^{\frac{1}{2}} + |x-y|\right)$$

Proof. The proof consists of two cases, $|s-t| \leq 1$ and $|s-t| > 1$. The case $|s-t| \leq 1$ holds trivially as then $|s-t| \leq |s-t|^{\frac{1}{2}}$. For the case $|s-t| > 1$ one can compute

$$|f(s,x) - f(t,y)| \leq M + M \leq 2M\left(|s-t|^{\frac{1}{2}} + |x-y|\right)$$

One can then choose $C = \max(L, 2M)$. □

The next theorem we present is the Arzela-Ascoli theorem, which is a useful Theorem from functional analysis.

Theorem 2.4. *(Arzela-Ascoli) (p. 234 of [10]) Let* K *be a compact space, and* (E,d) *be a metric space. The space of continuous functions* $C(K,E)$ *from* K *to* E*, endowed with the uniform distance, is a metric space.*

A subset $A \subset C(K,E)$ *is relatively compact in* $C(K,E)$ *if and only if, for each point* $x \in K$:

- *(EQ) A is equicontinuous at x, that is for all $\epsilon > 0$, there exists a neighbourhood V of x such that:*

$$\forall f \in A \; \forall y \in V : \; d\left(f\left(x\right), f\left(y\right)\right) < \epsilon \tag{2.9}$$

- *(RC) The set $A\left(x\right) = \left\{f\left(x\right) | f \in A\right\}$ is relatively compact in (E, d).*

The following is a Lemma that is useful for proving convergence of sequences.

Lemma 2.7. *Let (X, d) a metric space and $K \subset\subset X$ a compact subset of X. Further, let $(x_n) \subset K$ be a sequence, such that all convergent subsequences have the same limit point $x^* \in K$. Then $x_n \to x^*$.*

Proof. By contradiction. Assume that there exists a subsequence (x_{n_k}) that doesn't converge towards x^*, i.e:

$$\exists \epsilon > 0 \; \forall N \in \mathbb{N} \; \exists n_\epsilon \geq N : \; d\left(x_{n_\epsilon}, x^*\right) > \epsilon \tag{2.10}$$

Starting from $N = 1, 2, 3, \ldots$ these x_{n_ϵ} defines a subsequence $(x_{n_\epsilon}) \subset (x_n)$. Since K is compact, it follows that (x_{n_ϵ}) has a convergent subsequence, with limit, say $\tilde{x} \in K$. However, since all convergent subsequences of (x_n) have the same limit point, it follows that $\tilde{x} = x^*$. But this is a contradiction to the construction (2.10). □

The following two fixed point theorems are really important for us, and play an important role in this thesis.

Theorem 2.5. *(Schauder's fixed point theorem)*
Let X be a Banach space, $K \subset X$ a convex, closed and compact subset. Further, let $T : K \to K$ be a continuous map. Then T has a fixed point in K.

Theorem 2.6. *(Banach's fixed point theorem)*
Let (X, d) be a complete metric space, and $T : X \to X$ a map. If there exists $q \in [0, 1)$ such that for all $x, y \in X$:

$$d\left(T\left(x\right), T\left(y\right)\right) \leq q d\left(x, y\right)$$

Then T has a unique fixed point $x \in X$.

2.4 Hölder continuity and Hölder spaces

Since concepts like Hölder continuity and Hölder spaces will be used later on, they will be presented here. We will define the Hölder-norm, Hölder spaces, and look at compact inclusion of Hölder spaces.

2.4.1 Definitions

Definition 2.3. *A function $f : \Omega \subset X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is Hölder-continuous with exponent $\alpha \in (0, 1]$ if there exists a constant $C > 0$ such that:*

$$\forall x, y \in \Omega : \quad d_Y \left(f(x), f(y)\right) \leq C \left(d_X (x, y)\right)^\alpha$$

Definition 2.4. *(reference: Def. 1.7 p. 46 of[1]) (Hölder space) Let $\Omega \subset \mathbb{R}^n$, $k \in \mathbb{N}$ and $\beta \in (0, 1]$. The Hölder space $C^{k,\beta}(\Omega)$ is the set of all functions $f : \Omega \to \mathbb{R}$ with $f \in C^k(\Omega)$, such that the following norm is finite:*

$$\|f\|_{C^{k,\beta}(\Omega)} := \sum_{|\alpha| \leq k} \|D^\alpha f\|_{C(\Omega)} + \sum_{|\alpha| = k} [D^\alpha f]_{C^{0,\beta}(\Omega)} .$$

Here, we denote by

$$\|D^\alpha f\|_{C(\Omega)} := \sup \{|D^\alpha f(x)| \,|x \in \Omega\}$$

the supremumsnorm, and

$$[D^\alpha f]_{C^{0,\beta}} := \sup \left\{ \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x - y|} \middle| x, y \in \Omega, x \neq y \right\}$$

a semi-norm.

Later on, we often use the convention of writing $C^{k+\beta}(\Omega)$, instead of $C^{k,\beta}$, for $k \in \mathbb{N}$ and $\beta \in (0, 1]$.

2.4.2 Compact embedding theorems

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^n$ be a closed and bounded subset (compact by the Heine-Borel theorem), and let $0 \leq \alpha < \beta \leq 1$. Then the embedding:*

$$i : C^{0,\beta}(\Omega) \to C^{0,\alpha}(\Omega)$$
$$u \mapsto u$$

is continuous.
Further, if $D \subset C^{0,\beta}$ is a uniformly bounded subset, that is

$$\exists M > 0 \; s.t \; \forall f \in D : \quad \|f\|_{C^{0,\beta}(\Omega)} \leq M$$

then the set D is precompact in $C^{0,\alpha}$.

Proof. **Continuity:**
From the assumptions diam $(\Omega) < \infty$. Then, for any $u \in C^{0,\beta}(\Omega)$:

$$\begin{aligned}
[u]_{C^{0,\alpha}(\Omega)} &= \sup_{x,y\in\Omega, x\neq y} \frac{|u(x) - u(y)|}{|x-y|^{\alpha}} \\
&= \sup_{x,y\in\Omega, x\neq y} \frac{|x-y|^{\beta}}{|x-y|^{\alpha}} \frac{|u(x) - u(y)|}{|x-y|^{\beta}} \\
&\overset{\beta>\alpha}{\leq} (\text{diam } (\Omega))^{\beta-\alpha} [u]_{C^{0,\beta}(\Omega)}
\end{aligned}$$

it follows then that:

$$\|u\|_{0,\alpha;\Omega} = \|u\|_{0;\Omega} + [u]_{0,\alpha;\Omega} \leq \|u\|_{0;\Omega} + C(\Omega,\alpha,\beta)[u]_{0,\beta;\Omega} \leq c\|u\|_{0,\beta;\Omega}$$

with $c = \max\left(1, (\text{diam } (\Omega))^{\beta-\alpha}\right)$, and it follows that the inclusion is continuous.
Compactness:
We want to show that the set D is sequentially compact in $C^{0,\alpha}(\Omega)$. Take any sequence $(u_n) \subset D$. The sequence is uniformly bounded by the constant M:

$$\|u_n\|_{\infty} \leq \|u_n\|_{0,\beta;\Omega} \leq M$$

It is also equicontinuous, since $\forall \epsilon > 0$ choose $\delta = \left(\frac{\epsilon}{M}\right)^{1/\beta}$, so that

$$\|u_n(x) - u_n(y)\|_{\infty} \leq M|x-y|^{\beta} < \epsilon$$

whenever $|x-y| < \delta$.
Apply the Arzelà-Ascoli theorem: $\exists (u_{n_k}) \subset (u_n)$ a uniformly convergent subsequence. Denote the limit by u. Then it holds that $\|u_{n_k} - u\|_{\infty} \to 0$. Further, we have:

$$\begin{aligned}
&\frac{|(u_{n_k} - u)(x) - (u_{n_k} - u)(y)|}{|x-y|^{\alpha}} \\
&= \left(\frac{|(u_{n_k} - u)(x) - (u_{n_k} - u)(y)|}{|x-y|^{\beta}}\right)^{\alpha/\beta} |(u_{n_k} - u)(x) - (u_{n_k} - u)(y)|^{1-\beta/\alpha} \\
&\leq |u_{n_k} - u|_{C^{0,\beta}}^{\beta/\alpha} (2\|u_{n_k} - u\|_{\infty})^{1-\beta/\alpha} \leq (2M)^{\beta/\alpha} (2\|u_{n_k} - u\|_{\infty})^{1-\beta/\alpha} \to 0
\end{aligned}$$

By taking the supremum of $x, y \in \bar{\Omega}, x \neq y$ on the left side, we obtain:

$$[u_{n_k} - u]_{C^{0,\alpha}\bar{\Omega}} \to 0$$

In total we have

$$\|u_{n_k} - u\|_{0,\alpha;\Omega} \to 0$$

This shows that D is precompact in $C^{0,\alpha}$. □

Theorem 2.8. *(reference: see Thm 8.6, p. 338 of [1]) Let $\Omega \subset \mathbb{R}^n$ be a bounded and closed subset. Let $k \in 0, 1, 2, 3, \cdots$ and $0 \leq \alpha < \beta \leq 1$. If $D \subset C^{k,\beta}(\Omega)$ is a uniformly bounded subset, then D is precompact in $C^{k,\alpha}(\Omega)$.*

Proof. The result follows from repeated use of the Arzelà-Ascoli theorem. We have the following bound on D, for some $M > 0$:

$$\sup_{u \in D} \|u\|_{k,\beta;\Omega} \leq M$$

It follows directly for any multiindex $\alpha = (\alpha_1, \cdots, \alpha_n)$ that $\forall u \in D$:

$$\|D^\alpha u\|_\infty \leq M$$

We also have that for $|\alpha| < k$:

$$\|D^\alpha u(x) - D^\alpha u(y)\|_\infty \leq M|x - y|$$

and for $|\alpha| = k$:

$$\|D^\alpha u(x) - D^\alpha u(y)\|_\infty \leq M|x - y|^\beta$$

It follows that the family of functions $\{D^\alpha u : u \in D, |\alpha| \leq k\}$ is equicontinuos.
Let $(u_n) \subset D$ be a sequence. We want to show that there exists a convergent subsequence $(u_{n_k}) \subset (u_n)$ with limit u, such that $\|u_{n_k} - u\|_{k,\alpha;\Omega} \to 0$. The argument is an inductive one. Let $m = |\alpha| \leq k$. Assume that $\exists (u_{n_k}) \subset (u_n)$ a convergent subsequence, and a limit point u, such that $\forall |\alpha| \leq m$:

$$\|u_{n_k} - u\|_{m,0;\Omega} \to 0$$

Denote $\mathcal{A}_m = \left\{\alpha_{m_1}, \cdots, \alpha_{m_{N_m}}\right\}$ as the set of multi-indexes of size $|\alpha| = m$.
We start by looking at the multi-index $\alpha_{m_1} = \left(\alpha_{m_{1_1}}, \cdots, \alpha_{m_{1_n}}\right)$.
Denote $\alpha_{m_1,1} = \left(\alpha_{m_{1_1}} + 1, \cdots, \alpha_{m_{1_n}}\right)$.

For $|m| < k$, we observe that the sequence $D^{\alpha_{m_1,1}} u_{n_k}$ is uniformly bounded and equicontinuous. Thus, by Arzelà-Ascoli, there exists a convergent subsequence $\left(u_{n_{k_j}}\right) \subset (u_{n_k})$, and a limit (still denoted u) such that:

$$\|D^{\alpha_{m_1,1}} u_{n_k} - D^{\alpha_{m_1,1}} u\|_\infty \to 0$$

Also, if $|m| = k - 1$, we observe that

$$[D^{\alpha_{m_1,1}} u_{n_k} - D^{\alpha_{m_1,1}} u]_{0,\alpha;\Omega} \to 0$$

as shown in theorem 2.7).

Use this new subsequence $\left(u_{n_{k_j}}\right)$ and repeat the argument for $\alpha_{m_1,2},\cdots,\alpha_{m_1,n},\cdots,\alpha_{m_{N_m},1},\cdots,\alpha_{m_{N_m},n}$. Thus it holds for $m+1$ that $\exists\,(u_{n_k})\subset(u_n)$ a convergent subsequence, and a limit point u, such that $\forall|\alpha|\leq m+1$:

$$\|u_{n_k}-u\|_{m+1,0;\Omega}\to 0$$

and if $|m|=k-1$:

$$\|u_{n_k}-u\|_{k,\alpha;\Omega}\to 0$$

For $|m|=0$, we have that the sequence (u_n) is uniformly bounded and equicontinuos, thus there exists a convergent subsequence $(u_{n_k})\subset(u_n)$, with limit point u. So that

$$\|u_{n_k}-u\|_{\infty}\to 0$$

This concludes the proof. □

2.4.3 Parabolic Hölder spaces

We use the standard definition of parabolic Hölder spaces (see Krylov).

Definition

For points $z_1=(x_1,t_1),z_2=(x_2,t_2)$ in $\mathbb{R}^{d+1}$, define the parabolic distance between them as:

$$\rho(z_1,z_2)=|x_1-x_2|+|t_1-t_2|^{1/2}$$

Let $0<\alpha\leq 1$ and $Q\subset\mathbb{R}^{d+1}$. Then we denote

$$[u]_{\alpha,\alpha/2;Q}=\sup_{z_1,z_2\in Q,z_1\neq z_2}\frac{|u(z_1)-u(z_2)|}{\rho^{\alpha}(z_1,z_2)},\ \|u\|_{\alpha,\alpha/2;Q}=\|u\|_{0;Q}+[u]_{\alpha,\alpha/2;Q}$$

Definition 2.5. *Let $Q\subset\mathbb{R}^{d+1}$ and $\alpha\in(0,1]$. The parabolic Hölder space $C^{\alpha,\alpha/2}$ is the set of functions $u:Q\to\mathbb{R}$ such that*

$$\|u\|_{\alpha,\alpha/2;Q}<\infty$$

Definition 2.6. *Let $Q\subset\mathbb{R}^{d+1}$ and $\alpha\in(0,1]$. The parabolic Hölder space $C^{2+\alpha,1+\alpha/2}$ is the set of functions $u:Q\to\mathbb{R}$ such that*

$$[u]_{2+\alpha,1+\alpha/2;Q}:=[u_t]_{\alpha,\alpha/2;Q}+\sum_{i,j=1}^{d}\left[u_{x_ix_j}\right]_{\alpha,\alpha/2;Q}<\infty$$

and

$$\|u\|_{2+\alpha,1+\alpha/2;Q}:=|u|_{0;Q}+\sum_{i=1}^{d}|u_{x_i}|_{0;Q}+|u_t|_{0;Q}+\sum_{i,j=1}^{d}|u_{x_ix_j}|_{0;Q}+[u]_{2+\alpha,1+\alpha/2;Q}<\infty$$

Compactness

We just state the results here without proofs, since the proof method would be the same as in theorem 2.7 and theorem 2.8.

Theorem 2.9. *Let* $Q \subset \mathbb{R}^{d+1}$ *and* $0 \leq \alpha < \beta \leq 1$. *Let* $D \subset C^{2+\beta,1+\beta/2}(Q)$ *be a uniformly bounded subset. Then the set* D *is precompact in* $C^{2+\alpha,1+\alpha/2}(Q)$.

Proof. The same technique of proof as in Theorem 2.7 and Theorem 2.8. □

We also have the following Lemma that can be proved in a similar way:

Lemma 2.8. *Assume that* $\mathcal{U}$ *is a set of functions* $u : [0,T] \times \mathbb{T}^d \to \mathbb{R}$ *such that*

$$u, Du, D^2u, D^3u, \partial_t u, \partial_t Du, \partial_t D^2 u \in C_b\left((0,T) \times \mathbb{T}^d\right)$$

and

$$\partial_t u \in C_b^{\frac{1}{2},1}\left((0,T) \times \mathbb{T}^d\right)$$

Then the set $\mathcal{U}$ *is compact in* $C^{1,2}\left((0,T) \times \mathbb{T}^d\right)$.

Proof. The same technique of proof as in Theorem 2.7 and Theorem 2.8. □

Chapter 3

Fractional MFG systems with nonlocal coupling

In this chapter, we want to study a fractional Mean Field Game system with non-local couplings F and G. The system we study is of a quite general form, and to the best of our knowledge, no cases in the literature have proven the existence and uniqueness of solutions for this kind of Mean Field Game system. A newly submitted article by Cesaroni et al. [4], shows existence and uniqueness for the stationary case, where one assumes that the MFG system has reached an equilibrium state (Nash equilibrium) as $T \to \infty$. We look at the time-dependent case, where we don't assume the the players have settled to a steady equilibrium.

3.1 The fractional Mean Field Game system

Our aim is to study the following system of equations, which we call the fractional Mean Field Game system with non-local coupling. The system is on the following form:

$$\begin{cases} -\partial_t u + \left(-\Delta_{\mathbb{T}^d}\right)^{\frac{\alpha}{2}} u + H\left(x, u, Du\right) = F\left(x, m\left(t\right)\right) & \text{in } (0,T) \times \mathbb{T}^d \\ \partial_t m + \left(-\Delta_{\mathbb{T}^d}\right)^{\frac{\alpha}{2}} m - div\left(m D_p H\left(x, u, Du\right)\right) = 0 & \text{in } (0,T) \times \mathbb{T}^d \\ m\left(0\right) = m_0, u\left(x, T\right) = G\left(x, m\left(T\right)\right) \end{cases} \tag{3.1}$$

where $\alpha \in (1,2)$, and the operator $\left(-\Delta_{\mathbb{T}^d}\right)^{\frac{\alpha}{2}}$ is the fractional Laplace operator on the torus. The functions F and G are both non-local coupling.

We want to show that under certain assumptions on H, F, G and m_0, there exists at least one classical solution for the system (3.1). In other words, we look for a pair $(u,m) \in C^{1,2}\left((0,T) \times \mathbb{T}^d\right)$ that satisfies (3.1). Let us state our assumptions.

Assumptions

We make the following assumtions on the system:

1. (Bounds on F and G) F and G are continuous in $\mathbb{T}^d \times \mathbf{P}\left(\mathbb{T}^d\right)$.

2. (Lipschitz continuity of F and G) there exists a $C_0 > 0$ s.t.
$|F(x_1, m_1) - F(x_2, m_2)| \leq C_0 [|x_1 - x_2| + d_1(m_1, m_2)]$
$\forall (x_1, m_1), (x_2, m_2) \in \mathbb{T}^d \times P\left(\mathbb{T}^d\right)$, and

   $|G(x_1, m_1) - G(x_2, m_2)| \leq C_0 [|x_1 - x_2| + d_1(m_1, m_2)]$
$\forall (x_1, m_1), (x_2, m_2) \in \mathbb{T}^d \times P\left(\mathbb{T}^d\right)$.

3. (Uniform regularity of F and G) There exist constants $C_F, C_G > 0$, such that $\sup_{m \in P(\mathbb{T}^d)} \|F(\cdot, m)\|_{C_b^7(\mathbb{T}^d)} \leq C_F$ and $\sup_{m \in P(\mathbb{T}^d)} \|G(\cdot, m)\|_{W^{7,\infty}(\mathbb{T}^d)} \leq C_G$.

4. The Hamiltonian $H : \mathbb{T}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfies for $x \in \mathbb{T}^d, u \in [-R, R], p \in B_R$:
$$|D^\alpha H(x, u, p)| \leq C_R$$
with $|\alpha| \leq 7$ and $C_R > 0$ a positive constant dependent on R.

5. The Hamiltonian H satisfies for $x, y \in \mathbb{T}^d, u \in [-R, R], p \in \mathbb{R}^d$:
$$|H(x, u, p) - H(y, u, p)| \leq C_R (|p| + 1) |x - y|$$
Note that this assumption is automatically satisfied for Hamiltonians on the form $H(u, Du)$.

6. There exists $\gamma \in \mathbb{R}$ such that for all $x \in \mathbb{T}^d, u, v \in \mathbb{R}, u < v, p \in \mathbb{R}^d$,
$$H(x, v, p) - H(x, u, p) \geq \gamma (v - u)$$
This assumption is automatically satisfied for Hamiltonians on the form $H(x, Du)$, by choosing $\gamma = 0$.

7. The probability measure m_0 is absolutely continuous with respect to the Lebesgue measure (meaning $A \subset \mathbb{T}^d$ measurable: $m_0(A) = 0 \implies \lambda(A) = 0$), has a $W^{5,\infty}\left(\mathbb{T}^d\right)$-continuous density function (still denoted m_0).

Theorem 3.1. *(Existence of classical solution) Under the assumptions 1.-7., there exists at least one classical solution (u, m) to (3.1).*

By using the same type of approach like Cardaliaguet in the proof of Thm 3.1.1 in [3], we will show prove the existence of classical solutions. This result depends upon estimates we have made on the fractional Hamilton-Jacobi equation in later chapters.

First, we will start with a remark on weak solutions of the fractional Fokker-Planck equation.

3.2 On the fractional Fokker-Planck equation

The fractional Fokker-Planck equation can be written on the following form:

$$\begin{cases} \partial_t m + \nu \left(-\Delta_{\mathbb{T}^d}\right)^{\frac{\alpha}{2}} m - div\left(mb\right) = 0 \quad \text{in } \left(0,T\right) \times \mathbb{T}^d \\ m\left(0\right) = m_0 \end{cases} \tag{3.2}$$

where $\nu > 0$ is a constant, and $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a vector field that is continuous in time and Hölder continuous in space. In the following Lemma, we define what we say is a weak solution of the Fokker-Planck equation (3.2).

Lemma 3.1. *(Weak solutions of (3.2))*
A function $m \in L^1\left([0,T] \times \mathbb{T}^d\right)$ is said to be a weak solution to (3.2) if m satisfies the following for any test function $\phi \in C_c^\infty\left([0,T] \times \mathbb{R}^d\right)$

$$\begin{aligned} &\int_{\mathbb{T}^d} \phi\left(x,T\right) dm\left(T\right)\left(x\right) - \int_{\mathbb{T}^d} \phi\left(0,x\right) dm_0\left(x\right) \\ &\quad = \int_0^T \int_{\mathbb{T}^d} \left(\partial_t \phi\left(t,x\right) - \nu\left(-\Delta_{\mathbb{T}^d}\right)^{\frac{\alpha}{2}} \phi\left(t,x\right) + \left\langle D\phi\left(t,x\right), b\left(t,x\right)\right\rangle\right) dm\left(t\right)\left(x\right) \end{aligned}$$

Two important properties of weak solutions of the Fokker-Planck equation, is that

Lemma 3.2. *A classical solution m of (3.2) is also a weak solution.*

and

Lemma 3.3. *If m is a weak solution of (3.2), then it is unique.*

We didn't have time to prove these statements, but it should be possible, according to my supervisor. These Lemma's are essential to our analysis of the Fokker-Planck equation, as they allow us to say that a function m that solves Fokker-Planck classically, with m_0 being a probability density function, then m is also a probability density function. The method of proving Lemma 3.2 would probably be to insert a classical solution m into the definition of weak solution. To prove Lemma 3.3 one can probably use Holmgren's uniqueness theorem, or something similar.

Moving on, we will now introduce the stochastic differential equation (SDE) related to the fractional Fokker-Planck equation.

$$\begin{cases} dX_t = b\left(X_t, t\right) dt + \nu^{\frac{1}{\alpha}} dL_t, \quad t \in [0,T] \\ X_0 = Z_0 \end{cases} \tag{3.3}$$

where (L_t) is a d-dimensional α-stable pure jumps Lévy process, with Lévy measure

$$\nu\left(dz\right) = c_\alpha \frac{dz}{|z|^{d+\alpha}}$$

where

$$dL_t = \int_{|z|<1} z\tilde{N}(dt, dz) + \int_{|z|\geq 1} zN(dt, dz)$$

Here, N describes a poisson process, and $\tilde{N}$ describes a compensated poisson process.

One can prove the following Lemma to be true:

Lemma 3.4. *If $\mathcal{L}(Z_0) = m_0$, then $m(t) := \mathcal{L}(X_t)$ is a weak solution of (3.2)*

Proof. (Idea of proof)

The proof is a consequence of applying Itô's formula. If $\phi \in C_c^\infty([0,T] \times \mathbb{R}^d)$, then (see Applebaum [2], Thm. 4.4.7):

$$\begin{aligned}
&\phi(t, X_t) = \phi(0, Z_0) \\
&+ \int_0^t [\partial_t \phi(s, X_s) + \langle D\phi(s, X_s), b(X_s, s)\rangle] ds \\
&+ \int_0^t \int_{|z|\geq 1} [\phi(s-, X_{s-} + z) - \phi(s-, X_{s-})] N(ds, dx) \\
&+ \int_0^t \int_{|z|<1} [\phi(s-, X_{s-} + z) - \phi(s-, X_{s-})] \tilde{N}(ds, dx) \\
&+ \int_0^t \int_{|z|<1} [\phi(s-, X_{s-} + z) - \phi(s-, X_{s-}) - \langle \nabla\phi(s-, X_{s-}), z\rangle] \nu(dx) ds
\end{aligned}$$

where . When we take the expected value on both sides, the following term vanishes

$$\mathbb{E}\left[\int_0^t \int_{|z|<1} [\phi(s-, X_{s-} + z) - \phi(s-, X_{s-})] \tilde{N}(ds, dx)\right] = 0.$$

Then, from the definition of $m(t)$, as the Law of X_t, we should get the following, recalling the definition of the fractional Laplacian (2.1

$$\begin{aligned}
&\int_{\mathbb{T}^d} \phi(t, x) \, dm(t)(x) = \int_{\mathbb{T}^d} \phi(0, x) \, dm_0(x) \\
&+ \int_0^t \int_{\mathbb{T}^d} \partial_t \phi(s, x) + \langle D\phi(s, x), b(s, x)\rangle - \nu(-\Delta_{\mathbb{T}^d})^{\frac{\alpha}{2}} \phi(s, x) \, dm(s)(x)
\end{aligned}$$

This shows that m is a weak solution to (3.2).

□

From this stochastic definition of $m(t)$, we can obtain the following estimate on the map $t \mapsto m(t) \in P(\mathbb{T}^d)$:

Lemma 3.5. *Let m be a weak solution of the fractional Fokker-Planck equation, with $\alpha \in (1,2)$. Then there exists a constant $c_0 > 0$ such that m satisfies:*

$$d_1\left(m\left(t\right), m\left(s\right)\right) \leq c_0\left(1 + \|b\|_\infty\right)|t-s|^{\frac{1}{2}} \qquad \forall s,t \in [0,T] \tag{3.4}$$

Proof. (Idea of proof)

We can use the SDE (3.3), to obtain estimates we want. One can write

$$X_t - X_s = \int_s^t b\left(\tau, X_\tau\right) d\tau + \nu^{1/\alpha}\int_s^t dL_t$$

Then, pick a $\phi \in 1 - Lip$, and compute

$$\begin{aligned} d_1\left(m\left(s\right), m\left(t\right)\right) &= \sup_{\phi\in 1-Lip}\left\{\int_{\mathbb{T}^d}\phi\left(x\right)\left(m\left(s\right)-m\left(t\right)\right)\left(dx\right)\right\} \\ &= \sup_{\phi\in 1-Lip}\left\{\mathbb{E}\left[\phi\left(X_t\right)-\phi\left(X_s\right)\right]\right\} \\ &\leq \mathbb{E}\left[\left|X_t - X_s\right|\right] \end{aligned}$$

We get that:

$$\mathbb{E}\left[\left|X_t - X_s\right|\right] \leq \mathbb{E}\left[\int_s^t \left|b\left(\tau, X_\tau\right)\right| d\tau + \nu^{1/\alpha}|L_t - L_s|\right]$$

Now, for the first term inside the expectation, it holds that:

$$\mathbb{E}\left[\int_s^t b\left(\tau, X_\tau\right) d\tau\right] \leq \|b\|_\infty |s-t|$$

For the second term, it should at least hold that (according to my supervisor)

$$\mathbb{E}\left[\nu^{1/\alpha}|L_t - L_s|\right] \leq c\nu^{1/\alpha}|s-t|^{\frac{1}{2}}$$

where $c > 0$ is some constant. So, finally we obtain

$$d_1\left(m\left(t\right), m\left(s\right)\right) \leq c_0\left(\|b\|_\infty + 1\right)|s-t|^{\frac{1}{2}}$$

for some constant $c_0 > 0$, which is what we wanted to show. □

3.3 Proof of existence

Having made clear our assumptions, and shown the estimates on the Fokker-Planck equation, we will now show that there exists a pair (u, m) that solves the Mean Field Game system (3.1) classically. We first begin with the idea of the proof.

3.3.1 Sketch of proof

The idea of the proof is the same as in the proof of Thm 3.1.1 in [3], where they proved existence of solutions for the same system of equations, but with a standard Laplacian instead of the fractional one.

We use Schauder's fixed point theorem (see Thm. 2.5) to show existence of solutions: We look at the Banach space $C^0\left([0,T],\mathbf{P}\left(\mathbb{T}^d\right)\right)$, and we define $\mathcal{C}\subset C^0\left([0,T],\mathbf{P}\left(\mathbb{T}^d\right)\right)$, which turns out to be a closed, convex and compact subset. Then we define a map $\psi:\mathcal{C}\to\mathcal{C}$ by using the fractional Mean Field Games equations, and we show that this is a well-defined and continuous map.

We can then apply Schauder's fixed point theorem to conclude that the map ψ has at least one fixed point, $m\in\mathcal{C}$, and then conclude that this fixed point is a classical solution of the system (3.1).

3.3.2 Proof

We begin the proof by defining the set $\mathcal{C}$.

The set $\mathcal{C}$

We consider the Banach space $\mathbf{C}^0\left([0,T],\mathbf{P}\left(\mathbb{T}^d\right)\right)$ endowed with the supremum metric $\tilde{d}\left(\mu,\nu\right)=\sup_{t\in[0,T]}d_1\left(\mu\left(t\right),\nu\left(t\right)\right)$, and we define the following subset

$$\mathcal{C}:=\left\{\mu\in\mathbf{C}^0\left([0,T],\mathbf{P}\left(\mathbb{T}^d\right)\right):\sup_{s\neq t}\frac{d_1\left(\mu\left(s\right),\mu\left(t\right)\right)}{|s-t|^{\frac{1}{2}}}\leq C_1\right\}\tag{3.5}$$

where the constant $C_1>0$ is later to be determined. For this subset $\mathcal{C}$ we will show the following properties.

Lemma 3.6. *$\mathcal{C}$ is a closed, convex and compact subset of $\mathbf{C}^0\left([0,T],\mathbf{P}\left(\mathbb{T}^d\right)\right)$.*

Proof. We prove each of the statements one by one.

Closed
We have to show that each limit point is contained in the set. Let $(\mu_n)\subset\mathcal{C}$ and $\mu_n\overset{\tilde{d}}{\to}\mu^*\in\mathbf{C}^0\left([0,T],\mathbf{P}\left(\mathbb{T}^d\right)\right)$. Then we find by the triangle inequality:

$$\begin{aligned}d_1\left(\mu^*\left(s\right),\mu^*\left(t\right)\right)&\leq d_1\left(\mu^*\left(s\right),\mu_n\left(s\right)\right)+d_1\left(\mu_n\left(s\right),\mu_n\left(t\right)\right)+d_1\left(\mu_n\left(t\right),\mu^*\left(t\right)\right)\\&\leq C_1|s-t|^{\frac{1}{2}}+d_1\left(\mu^*\left(s\right),\mu_n\left(s\right)\right)+d_1\left(\mu^*\left(t\right),\mu_n\left(t\right)\right)\to C_1|s-t|^{\frac{1}{2}}\end{aligned}$$

which shows that $\mu^*\in\mathcal{C}$.

Convex
Let $\mu, \nu \in \mathcal{C}$ and $\lambda \in (0,1)$. Then:

$$\begin{aligned}
&d_1\left(\lambda\mu(s)+(1-\lambda)\nu(s), \lambda\mu(t)+(1-\lambda)\nu(t)\right) = \\
&\sup_{\phi\in 1-Lip}\{\int_{\mathbb{T}^d}\phi(x)\left((\lambda\mu(s)+(1-\lambda)\nu(s))-(\lambda\mu(t)+(1-\lambda)\nu(t))\right)(dx)\} = \\
&\sup_{\phi\in 1-Lip}\left\{\lambda\int_{\mathbb{T}^d}\phi(x)(\mu(s)-\mu(t))(dx)+(1-\lambda)\int_{\mathbb{T}^d}\phi(x)(\nu(s)-\nu(t))(dx)\right\} \\
&\leq \lambda d_1(\mu(s),\mu(t))+(1-\lambda)d_1(\nu(s),\nu(t)) \\
&\leq \lambda C_1|s-t|^{\frac{1}{2}}+(1-\lambda)C_1|s-t|^{\frac{1}{2}} = C_1|s-t|^{\frac{1}{2}}.
\end{aligned}$$

This shows that $\lambda\mu+(1-\lambda)\nu \in \mathcal{C}$, so that convexity holds.

Compact
To show compactness we will use the Arzelá-Ascoli theorem (Thm. 2.4).

We need to show that $\mathcal{C} \in \mathbf{C}^0\left([0,T], \mathbf{P}\left(\mathbb{T}^d\right)\right)$ is equicontinuous and relatively compact.

EQ: Given an $\epsilon > 0$, we define $\delta = \left(\frac{\epsilon}{C_1}\right)^2$. Then we get $\forall s,t \in [0,T]$ and $\forall \mu \in \mathcal{C}$: $|s-t| < \delta \implies d_1(\mu(s),\mu(t)) \leq C_1|s-t|^{\frac{1}{2}} < C_1\sqrt{\left(\frac{\epsilon}{C_1}\right)^2} = \epsilon$, by use of the properties of $\mathcal{C}$.

RC: Let $s \in [0,T]$ and define $K_s := \{\mu(s) : \mu \in \mathcal{C}\}$. We have from definition, that $K_s \subset \mathbf{P}\left(\mathbb{T}^d\right)$, and thus it follows from set definitions that the closure $\overline{K_s}$ is closed in $\mathbf{P}\left(\mathbb{T}^d\right)$. From the compactness of $\mathbf{P}\left(\mathbb{T}^d\right)$ (**??**), it follows that $\overline{K_s}$ is compact, and thus from definition of relative compactness, that K_s is relatively compact.

Thus, by Arzela-Ascoli, we conclude that $\mathcal{C}$ is relatively compact in $\mathbf{C}^0\left([0,T], \mathbf{P}\left(\mathbb{T}^d\right)\right)$. Since $\mathcal{C}$ also is closed, it follows that it is compact.

□

The map ψ

Now, we define the map $\psi : \mathcal{C} \to \mathcal{C}$.

Let $\mu \in \mathcal{C}$ and define $m = \psi(\mu)$ as follows:
Let u be the solution to the fractional Hamilton-Jacobi equation given $\mu \in \mathcal{C}$

$$\begin{cases} -\partial_t u + (-\Delta_{\mathbb{T}^d})^{\frac{\alpha}{2}} u + H(x,u,Du) = F(x,\mu) & \text{in } (0,T)\times\mathbb{T}^d \\ u(x,T) = G(x,\mu(T)) \end{cases} \tag{3.6}$$

We define the following set:

$$\mathcal{U} := \{u : u = u(\mu), \mu \in \mathcal{C}\} \tag{3.7}$$

Further, we define $m = \psi(\mu)$ as the solution to the fractional Fokker-Planck equation:

$$\begin{cases} \partial_t m + (\Delta_{\mathbb{T}^d})^{\frac{\alpha}{2}} m - div\,(m D_p H(x, u, Du(t,x))) = 0 & \text{in } (0,T) \times \mathbb{T}^d \\ m(0,\cdot) = m_0(\cdot) & \text{in}\mathbb{T}^d \end{cases} \tag{3.8}$$

We also define the set:

$$\mathcal{M} := \{m : m = m(u), u \in \mathcal{U}\} \tag{3.9}$$

We need to show that the mapping ψ has two properties:

1. That the mapping $\mu \to \psi(\mu)$ is well defined.That is, show that $\mu \in \mathcal{C} \implies \psi(\mu) \in \mathcal{C}$
2. That the mapping is continuous.

Well defined

Lemma 3.7. *The map ψ is well-defined, that is, $\mathcal{M} \subset \mathcal{C}$. Furthermore the following holds for the sets $\mathcal{U}$ and $\mathcal{M}$:*

$$u, Du, \cdots, D^7u, \partial_t u, \partial_t Du, \partial_t D^2 u \in C_b\left((0,T) \times \mathbb{T}^d\right)$$
$$\partial_t u \in C_b^{\frac{1}{2},1}\left((0,T) \times \mathbb{T}^d\right)$$

All these quantities are uniformly bounded by a constant $U_1 > 0$, which depends on $\sup_{m \in P(\mathbb{T}^d)} \|G(\cdot, m)\|_{W^{7,\infty}(\mathbb{T}^d)}, \alpha, T, d$ and the local regularity of F and H.

For $\mathcal{M}$ the following estimates holds:

$$m, Dm, \cdots, D^5m, \partial_t m, \partial_t Dm, \partial_t D^2 m \in C_b\left((0,T) \times \mathbb{T}^d\right)$$
$$\partial_t m \in C_b^{\frac{1}{2},1}\left((0,T) \times \mathbb{T}^d\right)$$

where all these quantities are uniformly bounded by a constant $M_1 > 0$, only dependent on $\|m_0\|_{W^{5,\infty}(\mathbb{T}^d)}, \alpha, T, d, U_1$ and the local regularity of H.

Comment

We should give a short comment about why we need as much as $C^{1,7}$-regularity in u and $C^{1,5}$-regularity in m. The reason comes from our regularity results on the fractional Hamilton-Jacobi equation. We use these results both for the fractional Hamilton-Jacobi equation and the fractional Fokker-Planck equation. To obtain $\partial_t m \in C_b^{\frac{1}{2},1}\left(]0,T[\times \mathbb{T}^d\right)$, we need $m, Dm, \cdots, D^5m \in C_b\left(]0,T[\times \mathbb{T}^d\right)$, due to our computations. Since the Fokker-Planck equation is dependent on u, we need $u, Du, \cdots, D^7u \in C_b\left(]0,T[\times \mathbb{T}^d\right)$ in order to get enough regularity on m.

This is not ideal of course, but the best we can do for now.

Proof. **Hamilton-Jacobi**

Take a $\mu \in \mathcal{C}$, and look at the fractional Hamilton-Jacobi-equation (3.6). By setting $\tilde{H}(t,x,u,p) = H(x,u,p) - F(x,\mu(t))$ in (3.6) we get the expression:

$$\begin{cases} -\partial_t u + (-\Delta_{\mathbb{T}^d})^{\frac{\alpha}{2}} u + \tilde{H}(t,x,u,Du) = 0 & \text{in } (0,T)\times\mathbb{T}^d \\ u(x,T) = G(x,\mu(T)) \end{cases} \tag{3.10}$$

From the assumptions made on H, F and G, we get that (A0)-(A4) holds. Also the Hölder-assumption (4.2) on H holds, since for $s,t \in [0,T], x,y \in \mathbb{T}^d, u,v \in [-R,R], p,q \in B_R$

$$\begin{aligned} &|\tilde{H}(t,x,u,p) - \tilde{H}(s,y,v,q)| \\ &\quad \leq |H(x,u,p) - H(y,v,q)| + |F(x,\mu(t)) - F(y,\mu(s))| \\ &\quad \leq L_R(|x-y| + |u-v| + |p-q|) + C_0(|x-y| + d_1(\mu(t),\mu(s))) \\ &\quad \leq L_R\left(|s-t|^{1/2} + |x-y| + |u-v| + |p-q|\right) \end{aligned}$$

Then, by Theorem 4.2 from the chapter on the fractional HJ-equation, we get there exists a unique u that solves 3.10, and that it has the following regularity:

$$\begin{aligned} u, Du, \cdots, D^7u, \partial_t u, \partial_t Du, \partial_t D^2 u &\in C_b\left((0,T)\times\mathbb{T}^d\right) \\ \partial_t u &\in C_b^{\frac{1}{2},1}\left((0,T)\times\mathbb{T}^d\right) \end{aligned}$$

where all these quantities are uniformly bounded by a constant $U_1 > 0$, which depends on $\sup_{m\in P(\mathbb{T}^d)} \|G(\cdot,m)\|_{W^{7,\infty}(\mathbb{T}^d)}, \alpha, T, d$ and the local regularity of F and H.

We will now look at the Fokker-Planck equation, using the function u we obtained from the Hamilton-Jacobi equation.

Fokker-Planck

We look at the equation

$$\begin{cases} \partial_t m + (-\Delta_{\mathbb{T}^d})^{\frac{\alpha}{2}} m - div(mD_pH(x,u,Du(t,x))) = 0 & \text{in } (0,T)\times\mathbb{T}^d \\ m(0,\cdot) = m_0(\cdot) & \text{in} \mathbb{T}^d \end{cases} \tag{3.11}$$

We can directly apply Theorem 5.1 from the chapter on the fractional Fokker-Planck equation, to obtain a unique solution m of (3.11) that satisfies:

$$\begin{aligned} m, Dm, \cdots, D^5m, \partial_t m, \partial_t Dm, \partial_t D^2 m &\in C_b\left((0,T)\times\mathbb{T}^d\right) \\ \partial_t m &\in C_b^{\frac{1}{2},1}\left((0,T)\times\mathbb{T}^d\right) \end{aligned}$$

where all these quantities are uniformly bounded by a constant $M_1 > 0$, only dependent on $\|m_0\|_{W^{5,\infty}(\mathbb{T}^d)}, \alpha, T, d, U_1$ and the local regularity of H.

Recalling our discussion on the Fokker-Planck equation, we have that m is a weak solution of (3.11) (referring to Lemma 3.2 and Lemma 3.3). Thus, Lemma 3.5 gives us the following estimates on m:

$$d_1\left(m\left(t\right), m\left(s\right)\right) \leq c_0\left(1 + \|D_pH\left(\cdot, Du\right)\|_\infty\right)|t-s|^{\frac{1}{2}} \quad \forall s, t \in [0, T]$$

Since $\|Du\|_\infty \leq U_1$, it follows that $\|D_pH\left(\cdot, Du\right)\|_\infty \leq C_2$, where $C_2 > 0$ is a constant not dependent on μ. By setting $C_1 \geq c_0\left(1 + C_2\right)$ we get the sought after constant in the definition of $\mathcal{C}$. Further, we obtain that $m = \psi\left(\mu\right) \in \mathcal{C}$, which shows that the map ψ is well-defined. □

Continuity

We now want to check that the mapping is continuous, with respect to the metric $\tilde{d}$ defined on $\mathbf{C}^0\left([0,T], \mathbf{P}\left(\mathbb{T}^d\right)\right)$.

For this, let $\mu_n \in \mathcal{C}$ be a given sequence, that converges to a point $\mu \in \mathcal{C}$. Further, let (u_n, m_n) and (u, m) be the corresponding solutions of the system of equations. We want to show that $m_n \in \mathcal{C}$ converges to $m \in \mathcal{C}$, because this in turn implies continuity of ψ.

Hamilton-Jacobi

We first begin by looking at the pairs (μ_n, u_n) and (μ, u). We want to show that $u_n \xrightarrow{C^{1,2}} u$.

From the uniform bounds on functions $u \in \mathcal{U}$, it follows that $\mathcal{U} \subset\subset C^{1,2}\left((0,T) \times \mathbb{T}^d\right)$, by Lemma 2.8. A consequence of Lemma 2.7 is that, if every convergent subsequence of a sequence $(u_n) \subset \mathcal{U}$ converges, then the whole sequence converges in $C^{1,2}$ to the same limit point.

So, we only need to prove the following statement:

Lemma 3.8. *Every convergent subsequence* (u_{n_k}) *of* (u_n) *(convergent in* $C^{1,2}$*) converges to the same limit point,* $u = u\left(\mu\right)$.

Proof. Let (u_{n_k}) be a convergent subsequence of (u_n), and (μ_{n_k}) the corresponding μ-s. Assume that $u_{n_k} \xrightarrow{C^{2,1}} \tilde{u} \in \mathcal{U}$. From the assumptions μ_{n_k} converges to μ. The pair (μ_{n_k}, u_{n_k}) satisfies the fractional Hamilton-Jacobi equation:

$$\begin{cases} -\partial_t u_{n_k} + \left(-\Delta_{\mathbb{T}^d}\right)^{\frac{\alpha}{2}} u_{n_k} + H\left(x, u_{n_k}, Du_{n_k}\right) = F\left(x, \mu_{n_k}\left(t\right)\right) \\ u_{n_k}\left(x, T\right) = G\left(x, \mu_{n_k}\left(T\right)\right) \end{cases} \tag{3.12}$$

Also the limit point $(\mu, \tilde{u})$ satisfies the fractional Hamilton-Jacobi equation

$$\begin{cases} -\partial_t \tilde{u} - \left(-\Delta_{\mathbb{T}^d}\right)^{\frac{\alpha}{2}} \tilde{u} + H\left(x, \tilde{u} D\tilde{u}\right) = F\left(x, \mu\left(t\right)\right) \\ \tilde{u}\left(x, T\right) = G\left(x, \mu\left(T\right)\right) \end{cases} \tag{3.13}$$

To conclude the proof, we need to show that all the terms in equation (3.12) converges pointwise to the terms in equation (3.13).

From the assumption $u_{n_k} \xrightarrow{C^{1,2}} \tilde{u}$, we get directly that:

$$\|\partial_t u_{n_k} - \partial_t \tilde{u}\|_0 \to 0$$

By using Lemma 2.1 from the Preliminaries, we get

$$\begin{aligned}&\| \left(-\Delta_{\mathbb{T}^d}\right)^{\frac{\alpha}{2}} u_{n_k}(t,\cdot) - \left(-\Delta_{\mathbb{T}^d}\right)^{\frac{\alpha}{2}} \tilde{u}(t,\cdot) \|_{L^\infty\left(\mathbb{T}^d\right)} \overset{(2.6)}{\leq} \|u_{n_k}(t,\cdot) - \tilde{u}(t,\cdot)\|_{C^2\left(\mathbb{T}^d\right)} \\ &\leq \|u_{n_k} - \tilde{u}\|_{C^{1,2}\left((0,T)\mathbb{T}^d\right)}\end{aligned}$$

Further, we have for F that

$$\begin{aligned}\left|F\left(x,\mu_{n_k}(t)\right) - F\left(x,\mu(t)\right)\right| &\leq C_0\left[d_1\left(\mu_{n_k}(t),\mu(t)\right)\right] \\ &\leq C_0\left[\tilde{d}\left(\mu_{n_k},\mu\right)\right] \to 0\end{aligned}$$

and we get, by the same method, the same result for G:

$$\left|G\left(x,\mu_{n_k}(T)\right) - G\left(x,\mu(T)\right)\right| \to 0$$

The remaining term to look at is the $H\left(x, Du_{n_k}\right)$. We know that all $u \in \mathcal{U}$ satisfies $\|Du\|_\infty \leq U_1$ as shown in lemma 3.7, so we can use that H is locally Lipschitz continuous:

$$\left|H\left(x,u_{n_k},Du_{n_k}\right) - H\left(x,\tilde{u},D\tilde{u}\right)\right| \leq L_{H,U_1}\left(|u_{n_k}-\tilde{u}| + |Du_{n_k} - D\tilde{u}|\right) \to 0$$

This shows that every term in (3.12) converges pointwise to the corresponding terms in equations (3.13) . The equation (3.13) has a unique solution $\tilde{u} \in \mathcal{U}$, referring to theorem 4.2. Thus, all convergent subsequences of (u_n) have the same limit point. This concludes the proof. □

By Lemma 2.8, the set $\mathcal{U}$ is compact in $C^{1,2}$. Since $(u_n) \subset \mathcal{U}$, and every convergent subsequence of (u_n) has the same limit point $u \in \mathcal{U}$ (Lemma 3.8), we conclude by Lemma 2.7 that $u_n \xrightarrow{C^{1,2}} u$.

Fokker-Planck

Now, we want to show that $m_n \xrightarrow{C^{1,2}} m$, based on the result that $u_n \to u \in C^{1,2}$. The set $\mathcal{M} \subset\subset C^{1,2}\left((0,T)\times\mathbb{T}^d\right)$, as a consequence of Lemma 2.8. Thus, we will do the same as for the fractional Hamilton-Jacobi equation: We will show that every convergent subsequence $(m_{n_k}) \subset (m_n)$ converges to the same limit point, and from here conclude by Lemma 2.7 that $m_n \xrightarrow{C^{1,2}} m$. We can prove the following statement:

Lemma 3.9. *Every convergent subsequence* (m_{n_k}) *of* (m_n) *in* $C^{1,2}$ *converges to the same limit point,* $m = m(u)$.

Proof. Let (m_{n_k}) be a convergent subsequence, and (u_{n_k}) the corresponding sequence in u-s. Assume that $(m_{n_k}) \xrightarrow{C^{1,2}} \tilde{m} \in \mathcal{M}$. We know that $u_{n_k} \xrightarrow{C^{1,2}} u$ from lemma 3.8. Each pair (u_{n_k}, m_{n_k}) satisfy the fractional Fokker-Planck equation

(3.14)

$$\begin{cases} \partial_t m_{n_k} + (-\Delta_{\mathbb{T}^d})^{\alpha/2} m_{n_k} - \langle Dm_{n_k}, D_pH(x, u_{n_k}, Du_{n_k})\rangle - m_{n_k}\ \mathrm{div} D_pH(x, u_{n_k}, Du_{n_k}) = 0 \\ m_{n_k}(0) = m_0 \end{cases}$$

Also, the limit point satisfy the fractional Fokker-Planck equation

$$\begin{cases} \partial_t \tilde{m} + (-\Delta_{\mathbb{T}^d})^{\alpha/2} \tilde{m} - \langle D\tilde{m}, D_pH(x, u, Du)\rangle - \tilde{m}\ \mathrm{div} D_pH(x, u, Du) = 0 \\ \tilde{m}(0) = m_0 \end{cases} \tag{3.15}$$

It can be shown that every term in expression (3.14) converges pointwise to the corresponding limit in (3.15), where we again use that $\|Du\|_\infty \le U_1$ for all $u \in \mathcal{U}$, and that D_pH and $divD_pH$ are locally Lipschitz continuous. The solution for the limit equation (3.15) is unique when $u \in \mathcal{U}$ is given. Thus, we can conclude that all convergent subsequences of (m_n) have the same limit point. □

By theorem 2.9, the set $\mathcal{M}$ is compact in $C^{2,1}$. Since $(m_n) \subset \mathcal{M}$, and every convergent subsequence of (m_n) has the same limit point $m \in \mathcal{M}$ (lemma 3.9), we conclude by lemma 2.7 that $m_n \xrightarrow{C^{2,1}} m$.

Conclusion of continuity

We have shown that $\mu_n \xrightarrow{\tilde{d}} \mu \implies m_n \xrightarrow{C^{2,1}} m$. We will show now that $m_n \xrightarrow{\tilde{d}} m$.

Since m_n, m are classical solutios of the fractional Fokker-Planck equation, they must also be a weak solution according to Lemma 3.2, and are unique weak solution by Lemma 3.3. Therefore, by Lemma 3.4, $m_n(t)$ and $m(t)$ are probability measures on $P(\mathbb{T}^d)$. This allows us to compute, recalling the equivalent definition of d_1 in Lemma 2.5

$$\begin{aligned} d_1(m_n(t), m(t)) &= \sup_{\phi \in 1-Lip, \phi(0)=0} \left\{ \int_{\mathbb{T}^d} \phi(x)\, d(m_n(t) - m(t))(x) \right\} \\ &= \sup_{\phi \in 1-Lip, \phi(0)=0} \left\{ \int_{\mathbb{T}^d} \phi(x)(m_n(t,x) - m(t,x))\, dx \right\} \\ &\le \int_{\mathbb{T}^d} 1 \cdot \|m_n - m\|_0 dx \to 0. \end{aligned}$$

This shows convergence in $C^0([0,T], P(\mathbb{T}^d))$, $m_n \xrightarrow{\tilde{d}} m$, and we can conclude that the mapping ψ is continuous.

Conclusion of existence proof

We have defined a Banach space $\mathbf{C}^0\left([0,T],\mathbf{P}\left(\mathbb{T}^d\right)\right)$, a compact, convex and closed subset $\mathcal{C}$, and a continuous map $\psi : \mathcal{C} \to \mathcal{C}$. Hence, by Schauder's fixed point theorem, the map $\psi : \mathcal{C} \to \mathcal{C}$ has a fixed point in $m \in \mathcal{C}$, and this fixed point is a solution of the MFG system (3.1).

Lemma 3.10. *A fixed point of ψ is a solution to the system (3.1)*

Proof. Let $m = \psi(m)$. Then it holds that $m \in C^{2,1}$. By inserting m into the Hamilton-Jacobi equation, we obtain a unique solution $u \in C^{2,1}$. This u in turn solves the Fokker-Planck equation uniquely, with $m \in C^{2,1}$. Thus, the pair (u, m) solves the fractional MFG system. □

3.4 Uniqueness

In this section, we assume that H is on the form $H(x, Du)$.

Assume that the following conditions hold on F and G, which we will refer to as the monotonicity condition:

$$\int_{\mathbb{T}^d} \left(F(x,m_1) - F(x,m_2)\right) d(m_1 - m_2)(x) \geq 0 \;\forall m_1, m_2 \in P\left(\mathbb{T}^d\right) \tag{3.16}$$

and

$$\int_{\mathbb{T}^d} \left(G(x,m_1) - G(x,m_2)\right) d(m_1 - m_2)(x) \geq 0 \;\forall m_1, m_2 \in P\left(\mathbb{T}^d\right) \tag{3.17}$$

Also, assume that H is uniformly convex with respect to it's last variable $p \in \mathbb{R}^d$:

$$\exists C > 0, \;\; \frac{1}{C} I_d \leq D^2_{pp} H(x,p) \leq C I_d \tag{3.18}$$

Theorem 3.2. *With the extra assumptions (3.16), (3.17) and (3.18) there is at most one classical solution of the mean field equation (3.1)*

The proof is exactly the same as in proof of Thm. 3.1.5 in Cardaliaguet [3], with the minor difference that we're dealing with the fractional Laplace operator, instead of the ususal one.

Proof. Let (u_1, m_1) and (u_2, m_2) be two classical solutions. We then set $\tilde{u} = u_1 - u_2$ and $\tilde{m} = m_1 - m_2$. Since both $\tilde{u}$ and $\tilde{m}$ are continuously differentiable and bounded on $\mathbb{T}^d$, which is compact, we can interchange the integration and differentiation operator:

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{T}^d} \tilde{u}\tilde{m} = \int_{\mathbb{T}^d} \frac{\partial}{\partial t}(\tilde{u}\tilde{m}) = \int_{\mathbb{T}^d} (\partial_t \tilde{u})\tilde{m} + \tilde{u}(\partial_t \tilde{m})\\
= \int_{\mathbb{T}^d} \Big(\Big((-\Delta_{\mathbb{T}^d})^{\alpha/2}\tilde{u} + H(x,Du_1) - H(x,Du_2) - F(x,m_1) + F(x,m_2)\Big)\tilde{m}\\
- \tilde{u}(-\Delta_{\mathbb{T}^d})^{\alpha/2}\tilde{m} - \langle D\tilde{u}, m_1 D_p H(x,Du_1) - m_2 D_p H(x,Du_2)\rangle\Big)
\end{aligned}$$

where the last term is obtained from partial integration.

We notice that

$$\int_{\mathbb{T}^d} \left((-\Delta_{\mathbb{T}^d})^{\frac{\alpha}{2}} \tilde{u} \right) \tilde{m} - \tilde{u} \left((-\Delta_{\mathbb{T}^d})^{\frac{\alpha}{2}} \tilde{m} \right) dx = 0. \tag{3.19}$$

which follows from Lemma 2.3 in the Preliminaries.

From the condition (3.16) on F, we also get the estimate:

$$\int_{\mathbb{T}^d} \left(-F(x, m_1) + F(x, m_2) \right) \tilde{m} = \int_{\mathbb{T}^d} \left(-F(x, m_1) + F(x, m_2) \right) (m_1 - m_2) \leq 0$$

We can rewrite the terms involving H in the following way:

$$\begin{aligned} &\int_{\mathbb{T}^d} \left(\left(H(x, Du_1) - H(x, Du_2) \right) \tilde{m} - \langle D\tilde{u}, m_1 D_p H(x, Du_1) - m_2 D_p H(x, Du_2) \rangle \right) \\ &\quad = -\int_{\mathbb{T}^d} m_1 \left(\left(H(x, Du_2) - H(x, Du_1) \right) - \langle D_p H(x, Du_1), Du_2 - Du_1 \rangle \right) \\ &\quad\quad - \int_{\mathbb{T}^d} m_2 \left(\left(H(x, Du_1) - H(x, Du_2) \right) - \langle D_p H(x, Du_2), Du_1 - Du_2 \rangle \right) \end{aligned} \tag{3.20}$$

Now we want to use Taylors theorem and the uniform convexity assumption to get estimates on (3.20). From Taylors theorem, and for any $v, w \in \mathbb{R}^d$ it holds that:

$$\begin{aligned} H(x, v) = H(x, w) &+ \langle D_p H(x, w), v - w \rangle \\ &+ \frac{1}{2} D^2_{pp} H(x, \xi) |v - w|^2 \end{aligned}$$

for some $\xi \in \mathbb{R}^d$ on the line between v and w. But from the uniform convexity assumption (3.18) it holds that $\forall \xi \in \mathbb{R}^d$:

$$\begin{aligned} &H(x, v) - H(x, w) - \langle D_p(x, w), v - w \rangle \\ &= \frac{1}{2} D^2_{pp}(x, \xi) |v - w|^2 \overset{(3.18)}{\geq} \frac{1}{2C} |v - w|^2 \end{aligned} \tag{3.21}$$

We can apply this to the equation (3.20): By applying this to (3.20), we get the estimate:

$$\begin{aligned} &\int_{\mathbb{T}^d} \left(\left(H(x, Du_1) - H(x, Du_2) \right) \tilde{m} - \langle D\tilde{u}, m_1 D_p H(x, Du_1) - m_2 D_p H(x, Du_2) \rangle \right) \\ &\quad = -\int_{\mathbb{T}^d} m_1 \left(\left(H(x, Du_2) - H(x, Du_1) \right) - \langle D_p H(x, Du_1), Du_2 - Du_1 \rangle \right) \\ &\quad\quad - \int_{\mathbb{T}^d} m_2 \left(\left(H(x, Du_1) - H(x, Du_2) \right) - \langle D_p H(x, Du_2), Du_1 - Du_2 \rangle \right) \\ &\overset{(3.21)}{\leq} -\int_{\mathbb{T}^d} \frac{m_1}{2C} |Du_2 - Du_1|^2 - \int_{\mathbb{T}^d} \frac{m_2}{2C} |Du_1 - Du_2|^2 = -\int_{\mathbb{T}^d} \frac{(m_1 + m_2)}{2C} |Du_1 - Du_2|^2 \end{aligned}$$

So, by combining all the estimates, we obtain the following inequality:

$$\frac{d}{dt}\int_{\mathbb{T}^d}\tilde{u}\tilde{m}\leq-\int_{\mathbb{T}^d}\frac{(m_1+m_2)}{2C}|Du_1-Du_2|^2.$$

We can integrate this inequality on the time interval $[0,T]$, to obtain:

$$\int_{\mathbb{T}^d}\tilde{u}\,(T)\,\tilde{m}\,(T)-\int_{\mathbb{T}^d}\tilde{u}\,(0)\,\tilde{m}\,(0)\leq-\int_0^T\int_{\mathbb{T}^d}\frac{m_1+m_2}{2C}|Du_1-Du_2|^2$$

From the initial conditions, we have that $\tilde{m}\,(0)=m_1\,(0)-m_2\,(0)=m_0-m_0=0$. We also have $\tilde{u}\,(T)=G\,(x,m_1\,(T))-G\,(x,m_2\,(T))$, so that:

$$\int_{\mathbb{T}^d}\tilde{u}\,(T)\,\tilde{m}\,(T)=\int_{\mathbb{T}^d}\left(G\,(x,m_1\,(T))-G\,(x,m_2\,(T))\right)(m_1\,(T)-m_2\,(T))\geq 0$$

from the assumptions on G (3.17). What we obtain then is the inequality:

$$0\leq-\int_0^T\int_{\mathbb{T}^d}\frac{m_1+m_2}{2C}|Du_1-Du_2|^2\leq 0$$

which means that the integrand must be zero, so that $Du_1=Du_2$ on the set $\{m_1>0\}\cup\{m_2>0\}$. In the fractional Fokker-Planck equation, m_1 then solves the same equation as m_2, since we have the same third term $D_pH\,(x,Du_1)=D_pH\,(x,Du_1)$. Therefore $m_1=m_2$ and it follows from uniqueness of the fractional Hamilton-Jacobi equation, that u_1 and u_2 must be the same.

□

Chapter 4

Regularity for the fractal hamilton-jacobi equation

In this chapter we want to prove regularity for solutions of the fractional Hamilton-Jacobi equation, under suitable assumptions on the Hamiltonian H. Many of our proofs get their inspiration from the article [8] by Imbert.

More precisely, the equation we study in this chapter is on the form:

$$\begin{cases} \partial_t u + (-\Delta)^{\lambda/2} u + H(t,x,u,Du) = 0 & \text{in } (0,T) \times \mathbb{R}^N \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{4.1}$$

where $1 < \lambda < 2$, and the non-local operator $(-\Delta)^{\lambda/2}$ is defined as:

$$(-\Delta)^{\lambda/2} \phi(x) = c_{N,\lambda} \int_{\mathbb{R}^N \setminus \{0\}} \frac{\phi(x+z) - \phi(x) - \nabla\phi(x) \cdot z}{|z|^{N+\lambda}} dz$$

where $c_{N,\lambda}$ is a universal constant. This non-local operator is also known as the fractional Laplace operator, and it is a linear operator. Now, we will list our assumptions, and then go on with main results and proofs of these results.

4.1 Assumptions

We have the following assumptions on the Hamiltonian H. For any $T > 0$,

- (A0) The function $H : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is continuous.
- (A1) There exists $\gamma \in \mathbb{R}$ such that for all $x \in \mathbb{R}^N$, $u, v \in \mathbb{R}$, $u < v$, $p \in \mathbb{R}^N$, $t \in [0,T)$,

$$H(t,x,v,p) - H(t,x,u,p) \geq \gamma (v - u)$$

- (A2) For any $R > 0$, there exists $C_R > 0$ such that for all $x, y \in \mathbb{R}^N, u \in [-R, R], p \in \mathbb{R}^N, t \in [0, T)$,

$$|H(t, x, u, p) - H(t, y, u, p)| \leq C_R(|p| + 1)|x - y|$$

- (A3) For any $R > 0$, there exists $C_R > 0$ such that for all $t \in [0, T), x \in \mathbb{R}^N, u, v \in [-R, R], p, q \in B_R$, the derivatives of H (up till the k-th derivative) are bounded by the constant C_R, that is,

$$|D^\alpha H(t, x, u, p)| \leq C_R$$

  with $\alpha = (\alpha_{x_1}, \ldots, \alpha_{x_N}, \alpha_u, \alpha_{p_1}, \ldots, \alpha_{p_N})$ a multi-index with $|\alpha| \leq k$.

- (A4) There exists $C_0 > 0$ such that

$$\sup_{t \in [0,T), x \in \mathbb{R}^N} |H(t, x, 0, 0)| \leq C_0$$

Comment:

Chapter 6 deals with how to estimate the spatial derivatives of $H(t, x, u(t, x), w(t, x))$, given an arbitrary multi-index $\beta = (\beta_1, \cdots, \beta_N)$ with $|\beta| \leq k$. Also we deal with how to estimate the difference $|\partial_x^\beta H(s, x, u(s, x), w_1(s, x)) - \partial_x^\beta H(s, x, u(s, x), w_2(s, x))|$. These estimates are necessary in order to show C_b^k-regularity in spaces for solutions of the fractional Hamilton-Jacobi equation, and plays an important part.

4.2 Main result

We state the main Theorems of this chapter:

Theorem 4.1. *Assume that $u_0 \in W^{k,\infty}(\mathbb{R}^N)$ with $k \geq 3$ and (A1)-(A4) holds. Then (4.1) admits a unique classical solution u that satisfies*

$$\partial_t u, u, Du, \cdots, D^k u \in C_b(]0, T[\times \mathbb{R}^N)$$

All these quantities are bounded by a constant c depending on $\|u_0\|_{W^{k,\infty}(\mathbb{R}^N)}$, λ, T, N and k.

We also achieve a bit more time regularity by adding to the assumptions on H.

Theorem 4.2. *Assume that $u_0 \in W^{k,\infty}(\mathbb{R}^N)$ with $k \geq 5$ and (A1)-(A4) holds. Furthermore, assume that for $t \in [0, T],\ x, y \in \mathbb{R}^N,\ u, v \in [-R, R],\ p, q \in B_R$ the Hamiltonian H satisfies*

$$|H(s, x, u, p) - H(t, y, v, q)| \leq L_{H,R}\left(|s - t|^{\frac{1}{2}} + |x - y| + |u - v| + |p - q|\right) \tag{4.2}$$

where $L_{H,R} > 0$ is a constant depending on R. Then (4.1) admits a unique classical solution u that satisfies

$$\partial_t u, u, Du, \cdots, D^k u \in C_b\left(]0,T[\times \mathbb{R}^N\right)$$

and

$$\partial_t u, D^2 u \in C_b^{\frac{1}{2},1}\left(]0,T[\times \mathbb{R}^N\right)$$

where $C_b^{\frac{1}{2},1}\left(]0,T[\times \mathbb{R}^N\right)$ is a parabolic Hölder space. All these quantities are uniformly bounded by a constant $c > 0$ depending on $\|u_0\|_{W^{k,\infty}(\mathbb{R}^N)}$, λ, T, N, k and $L_{H,R}$, where $R = \|u\|_0 + \|Du\|_0$.

Idea of proof:

The idea is to use the Duhamel's integral representation to show that the unique viscosity solution u of (4.1) has $C_b^{1,k}$-regularity on small intervals $(0, T_k^\epsilon)$ and $(t_0, t_0 + T_k)$, where $t_0 \in (0,T)$ and T_k^ϵ, T_k are strictly positive, independent of t_0. By patching intervals together, we can conclude that u belongs to $C_b^{1,k}\left((0,T)\times \mathbb{R}^N\right)$.

We can extend the time-regularity a bit further, by making the assumption (4.2) on the Hamiltonian H.

4.3 Unique viscosity solution

We begin by referring to the article [8], which states that (4.1) has a unique viscosity solution u.

Theorem 4.3. *(Theorem 3, Imbert [8]) Assume that (A0)-(A4) holds. For any $u_0 : \mathbb{R}^N \to \mathbb{R}$ bounded and uniformly continuous, there exists a (unique) viscosity solution u of (4.1) in $[0,+\infty)\times\mathbb{R}^N$ such that $u(0,x) = u_0(x)$.*

The next Lemma is from the same article, and gives some regularity result for the viscosity solution.

Lemma 4.1. *(Lemma 2 in [8]) Assume that (A0)-(A4) holds, and that $u_0 \in W^{1,\infty}\left(\mathbb{R}^N\right)$. The viscosity solution u satisfies: For any $t \in [0,T)$, $\|u(t,\cdot)\|_{W^{1,\infty}(\mathbb{R}^N)} \leq M_T$ with M_T that only depends on $\|u_0\|_{W^{1,\infty}(\mathbb{R}^N)}$, C_0 and T.*

Proof. The proof can be read in [8]. The constants that are presented in the Theorem are $M_T = \mathrm{e}^{KT/2}\left(K/8 + \|\nabla u_0\|_\infty^2\right)^{1/2}$. $K = 4C_R$, with $R = \|u\|_\infty$ from (A2), where $\|u\|_\infty \leq \|u_0\|_\infty + C_0 T$, which can be obtained directly from the comparison principle. □

Let us now forget for a little while about the viscosity solution u of (4.1). We want to look into Duhamel's formula, and to show that we can construct classical solutions v that satisfies (4.1) on small time intervals, given initial data v_0 that is smooth enough. Later we will combine these results with the viscosity solution u to show regularity on the whole time interval $(0,T)$.

4.4 Regularity by Duhamel's formula

Before embarking on the regularity theory for the fractional Hamilton-Jacobi equation, we need to say a few words on the heat kernel associated with the fractional Laplace operator.

4.4.1 The heat kernel of the fractional Laplacian

The semi-group generated by the non-local operator $(-\Delta)^{\lambda/2}$ is given by the convolution with the kernel (defined for $t > 0$ and $x \in \mathbb{R}^N$)

$$K(t,x) = \mathcal{F}\left(e^{-t|\cdot|^{\lambda}}\right)(x)$$

where $\mathcal{F}$ denotes the Fourier transform in $\mathbb{R}^N$. We list briefly some properties of K (see [8] and [5])

$$K \in C^{\infty}\left((0,+\infty)\times\mathbb{R}^N\right) \quad \text{and } K \geq 0 \tag{4.3}$$

$$\forall (t,x) \in (0,+\infty)\times\mathbb{R}^N,\ K(t,x) = t^{-N/\lambda}K\left(1, t^{-1/\lambda}x\right) \tag{4.4}$$

For all integers $m \geq 0$ and all multi-indexes α with $|\alpha| = m$, there exists a constant $B_m > 0$ such that

$$\forall (t,x) \in (0,+\infty)\times\mathbb{R}^N,\ |\partial_x^{\alpha} K(t,x)| \leq t^{(-N+m)/\lambda}\frac{B_m}{\left(1 + t^{-(N+1)/\lambda}|x|^{N+1}\right)} \tag{4.5}$$

Also, we have

$$\|K(t)\|_{L^1(\mathbb{R}^N)} = 1 \quad \text{and} \quad \|\nabla K(t)\|_{L^1(\mathbb{R}^N)} = \mathcal{K}_1 t^{-1/\lambda}. \tag{4.6}$$

where $\mathcal{K}_1 > 0$ is a constant. We also refer to the following useful result from [8].

Proposition 4.1. *(Proposition 1, Imbert [8]) Consider $u_0 \in C_b\left(\mathbb{R}^N\right)$. Then $K(t,\cdot) * u_0(\cdot)$ is a C^{∞} (in (t,x)) solution of*

$$\begin{aligned} &\partial_t u + (-\Delta)^{\lambda/2} u = 0 \quad \textit{in } (0,+\infty)\times\mathbb{R}^N \\ &u(0,\cdot) = u_0(\cdot) \quad \textit{in } \mathbb{R}^N \end{aligned}$$

We also should state Duhamel's formula for the equation (4.1). It is given by

$$v(t,x) = K(t,\cdot) * v_0(\cdot) - \int_0^t K(t-s,\cdot) * H(s,x,v(s,\cdot),\nabla v(s,\cdot))(x)\,ds$$

where K is the heat kernel, v_0 are initial conditions, and H is the Hamiltonian. We use Duhamel's formula to show that with initial conditions v_0 there exists a unique v that satisfies Duhamel's formula on a small time interval. We will also show that this v is in fact a classical solution of the equation (4.1) on this small time interval.

4.4.2 Starting point

We first begin, as in Imbert [8], to consider a small interval $]0,T_1[$, where $T_1>0$ is to be determined.

One can show that for given initial data $v_0 \in W^{1,\infty}\left(\mathbb{R}^N\right)$ there exists a unique v that satisfies Duhamel's formula on $]0,T_1[$, and that $v, \nabla v \in C_b\left(]0,T_1[\times\mathbb{R}^N\right)$, where Duhamel's formula is given by

$$\begin{aligned} v(t,x) &= K(t,\cdot) * v_0(\cdot) - \int_0^t K(t-s,\cdot) * H(s,x,v(s,\cdot),\nabla v(s,\cdot))(x)\,ds \\ v(0,\cdot) &= v_0(\cdot) \end{aligned} \tag{4.7}$$

4.4.3 C^1-regularity in x

The following Lemma is from Imbert [8].

Lemma 4.2. *Let $v_0 \in W^{1,\infty}\left(\mathbb{R}^N\right)$. Then there exists $T_1>0$, such that $v \in C_b\left(]0,T_1[\times\mathbb{R}^N\right)$ and $\nabla v \in C_b\left(]0,T_1[\times\mathbb{R}^N\right)$ and (4.7) holds.*

Proof. (Idea of proof) The proof is the same as in [8], so we will not go much into detail. The next proof we will do, we will use exactly the same method as Imbert did here, so we don't think it is necessary to do a complete proof in this case.

First, we define the map ψ_1 as

$$\psi_1(v)(t,x) = K(t,\cdot) * v_0(\cdot)(x) - \int_0^t K(t-s,\cdot) * H(s,x,v(s,\cdot),\nabla v(s,\cdot))(x)\,ds$$

and we consider the space

$$E_1 = \left\{v \in C_b\left(]0,T_1[\times\mathbb{R}^N\right), \nabla v \in C_b\left(]0,T_1[\times\mathbb{R}^N\right)\right\}$$

which has the natural norm $\|v\|_{E_1} = \|v\|_{C_b(]0,T_1[\times\mathbb{R}^N)} + \|\nabla v\|_{C_b(]0,T_1[\times\mathbb{R}^N)}$. From here, it can be shown that

1. ψ_1 is well-defined: $\psi : E_1 \to E_1$, and
2. There exists a unique fixed point v, so that $v = \psi(v)$.

To show well-definedness, we assume that the initial data is bounded by some $R_0>0$, that is:

$$\|v_0\|_{W^{1,\infty}(\mathbb{R}^N)} = \|v_0\|_\infty + \|Dv_0\|_\infty \leq R_0$$

Also, we assume that we look at $v \in E_1$ such that $\|v\|_{E_1} \leq R_1$, where $R_1 \geq R_0 > 0$ is to be determined.

Following the computations of [8], he estimates the value of $\|\psi_1(v)\|_{E_1}$, which turns out to be

$$\|\psi_1(v)\|_{E_1} \leq R_0 + (C_0 + C_{R_1})\left(T_1 + \mathcal{K}_1 \frac{\lambda}{\lambda-1} T_1^{(\lambda-1)/\lambda}\right)$$

Then he chooses that $R_1 = 2R_0$, and further selects $T_1 > 0$ such that the following condition holds:

$$(C_0 + C_{R_1} R_1)\left(T_1 + \mathcal{K}_1 \frac{\lambda}{\lambda-1} T_1^{(\lambda-1)/\lambda}\right) \leq R_0 \tag{4.8}$$

It holds then that $\psi_1 : B_{R_1} \to B_{R_1}$, and he shows that the map is in fact a contraction in the E_1-norm. Thus, by Banach's fixed point theorem, one can conclude that the map ψ_1 has a unique fixed point $v \in B_{R_1}$. □

Remark

If we rewrite the expression (4.8) as

$$T_1 + \mathcal{K}_1 \frac{\lambda}{\lambda-1} T_1^{(\lambda-1)/\lambda} \leq \frac{R_0}{(C_0 + 2C_{R_1} R_0)}$$

using that $R_1 = 2R_0$, we notice that, if R_0 is bounded from below, say $R_0 \geq 1$, and above, say by another constant $K > 0$, we can find a $T_1 > 0$ that satisfies (4.8), only dependent on K, C_0, C_K, $\mathcal{K}_1$ and λ.

4.5 C_b^k-regularity in x

In this section we will show that we can gain extra space-regularity by the use of Duhamel's Formula, and by use of a fixed point argument. The assumption we make, is that we already have a function v that satisfies Duhamel's formula, with initial condition v_0, on some small time interval and that this function has C_b^k-regularity in space. From here we go to show (depending on the regularity of v_0), that the function v have C_b^{k+1}-regularity in space on a even smaller time interval than the original one.

Imbert proves in his article [8] that one have C_b^2-regularity in space for a Hamiltonian on the form $H(t, x, u, Du)$, and then shows C_b^k-regularity for a simpler Hamiltonian $H(Du)$.

We will go directly to show C_b^k-regularity for the more general Hamiltonian $H(t, x, u, Du)$, as this is neccessary in the order to prove well-posedness for the MFG system (3.1). Also, this allows us to use the same estimates later for the Fokker-Planck equation.

4.5.1 Starting assumptions and framework

We will start with our assumptions, and then afterwards state the results.

Assume that there exists constants $R_0 > 0$ and $T_0 > 0$ such that the function $\bar{w} = \nabla \bar{v}$ satisfies Duhamel's formula on $]0, T_0[\times \mathbb{R}^N$, and that it belongs to the space $F_k(0, T_0) := \left\{w, Dw, \cdots D^{k-1}w \in C_b\left(]0, T_0[\times \mathbb{R}^N\right)\right\}$. In other words, $\bar{w}$ satisfies

$$\begin{aligned} &\bar{w}(t,x) = K(t,\cdot) * w_0(\cdot) - \int_0^t \nabla K(t-s,\cdot) * H(s,\cdot,\bar{v}(s,\cdot),\bar{w}(s,\cdot))(x)\,ds \\ &\|\bar{w}\|_{F_k} \le R_0. \end{aligned} \tag{4.9}$$

We also assume that the initial condition v_0 satisfies

$$\sum_{|\beta|\le k} \|\partial^\beta v_0\|_\infty \le R_0.$$

We will work with two different spaces, namely

$$F_{k+1}(0, T_{k+1}) = \left\{w, Dw, \cdots, D^k w \in C_b\left(]0, T_{k+1}[\times \mathbb{R}^N\right)\right\}$$

and

$$E_{k+1}(0, T_{k+1}) = \left\{w, Dw, \cdots, D^{k-1}w, t^{1/\lambda} D^k w \in C_b\left(]0, T_{k+1}[\times \mathbb{R}^N\right)\right\}.$$

The norms of these spaces we define as, for $w = (w_1, \cdots, w_N)$

$$\|w\|_{F_{k+1}} := \sum_{i=1}^N \left(\|w_i\|_0 + \sum_{1\le|\beta|\le k} \|\partial_x^\beta w_i\|_0 \right)$$

and

$$\|w\|_{E_{k+1}} := \sum_{i=1}^N \left(\|w_i\|_0 + \sum_{1\le|\beta|\le k-1} \|\partial_x^\beta w_i\|_0 + \sum_{|\beta|=k} \|t^{1/\lambda}\|\partial_x^\beta w_i\|_0 \right)$$

where we have defined $\|\cdot\|_0 = \|\cdot\|_{C_b(]0,T_k[\times\mathbb{R}^N)}$. We refer to these spaces as F_{k+1} and E_{k+1}, unless otherwise is stated.

Note that for $k \ge 1$ $\|w\|_{F_k} \le \|w\|_{E_{k+1}}$. This follows directly from the definitions.
We also define the following map ψ_2 as:

$$\psi_2(w)(t,x) = K(t,\cdot) * w_0(\cdot)(x) - \int_0^t \nabla K(t-s,\cdot) * H(s,\cdot,v(s,\cdot),w(s,\cdot))(x)\,ds \tag{4.10}$$

Having stated our assumptions, and defined E_{k+1}, F_{k+1} and ψ_2, we give the results of this section.

Lemma 4.3. *Assume that (4.9) holds for $R_0, T_0 > 0$, and that $w_0 \in W^{k-1,\infty}$. Then there exists $R_{k+1} > 0$ and $T_{k+1} > 0$, with $R_{k+1} \geq R_0$ and $T_{k+1} \leq T_0$, such that the map ψ_2 has a unique fixed point w in $E_{k+1}(0, T_{k+1})$ with $\|\psi_2(w)\|_{E_{k+1}} \leq R_{k+1}$. Also, T_{k+1} depends only on the quantities R_0, N, λ and C_{R_0} (and T_0).*

and

Lemma 4.4. *Assume that (4.9) holds, and that $w_0 \in W^{k,\infty}$. Then there exists $R^{\epsilon}_{k+1} > 0$ and $T^{\epsilon}_{k+1} > 0$, with $R^{\epsilon}_{k+1} \geq R_0$ and $T^{\epsilon}_{k+1} \leq T_0$, such that the map ψ_2 has a unique fixed point w^{ϵ} in $F_{k+1}\left(0, T^{\epsilon}_{k+1}\right)$ with $\|\psi_2(w)\|_{F_{k+1}} \leq R^{\epsilon}_{k+1}$.*

The main difference between the Lemmas 4.3 and 4.4 is the assumptions we make on the initial conditions w_0. We need Lemma 4.4 to prove regularity of the viscosity solution close to $t = 0$, and Lemma 4.3 to prove regularity of the viscosity solution on time intervals $(t_0, t_0 + T_{k+1})$, for $t_0 \in [0, T)$.

Now, we go to the proofs of these Lemmas.

4.5.2 Proof of Lemma 4.3

We start by looking at the space $E_{k+1}(0, T_{k+1})$.

We assume that $\nabla v \in F_k(0, T_0)$, with $\|\nabla v\|_{F_k} \leq R_0$, and we consider the map:

$$\psi_{2,i}(w)(t,x) = K(t,\cdot) * w_{0,i}(\cdot)(x) - \int_0^t \partial_i K(t-s,\cdot) * H(s,\cdot,v(s,\cdot),w(s,\cdot))(x)\, ds$$

where we analyse the i-th component of the vector, since this is easier to think about. We look at the space $E_{k+1}(0, T_{k+1})$, where $T_{k+1} \leq T_0$ is to be determined, and pick w such that $\|w\|_{E_{k+1}} \leq R_{k+1}$, where $R_{k+1} \geq R_0$ is to be determined. The fact that $R_{k+1} \geq R_0$ implies that $\|\nabla v\|_{F_k} \leq R_{k+1}$.

We first begin with some calculations, which will come in handy.

Estimates on H

In the chapter “Estimates of $\partial^{\beta} H$”, we have laid the foundation for the following calculations. In the following β is a multi-index on the form $\beta = (\beta_1, \cdots, \beta_N)$.

Bounds

From the assumptions $\|\nabla v\|_{F_k}, \|w\|_{E_{k+1}} \leq R_{k+1}$, and the assumptions (A0)-(A4), we calculate:

$$|H(s,x,v(s,x),w(s,x))| \leq C_0 + 2C_{R_{k+1}} R_{k+1} =: c_0(R_{k+1}) \tag{4.11}$$

From Lemma 6.2 we have for $1 \le |\beta| \le k-1$ that

$$|\partial_x^\beta H(s,x,v(s,x),w(s,x))| \le c(N,|\beta|,R_{k+1}) \tag{4.12}$$

and by the same Lemma, for $|\beta| = k$:

$$|\partial_x^\beta H(s,x,v(s,x),w(s,x))| \le c_1(k,N,R_{k+1}) + s^{-1/\lambda} c_2(k,N,R_{k+1}) \tag{4.13}$$

Difference

Assume that we have $w_p, w_q \in E_{k+1}$, that $\|\nabla v\|_{F_k}, \|w_p\|_{E_{k+1}}, \|w_q\|_{E_{k+1}} \le R_{k+1}$, and that the assumptions (A0)-(A4) holds. Also, assume that $R_{k+1} \ge k^2$. We can then calculate:

$$\begin{aligned} &|H(s,x,v,w_p) - H(s,x,v,w_q)| \\ &\quad \le C_{R_{k+1}} \|w_p - w_q\|_0 \le (N+1)\left(C_0 + 2C_{R_{k+1}} R_{k+1}\right) \|w_p - w_q\|_{E_{k+1}} \\ &\quad \le (N+1)\, c_0 \|w_p - w_q\|_{E_{k+1}} \end{aligned} \tag{4.14}$$

where c_0 is the same constant as in (4.11).

Referring to the results of Lemma 6.14, we compute for $1 \le |\beta| \le k-1$ that

(4.15)

$$\begin{aligned} |\partial_x^\beta H(s,x,u,w_p) - \partial_x^\beta H(s,x,u,w_q)| &\le \left(\frac{k^2}{R_{k+1}} + N\right) c(N,|\beta|,R_{k+1}) \|w_p - w_q\|_{E_{k+1}} \\ &\le (N+1)\, c(N,|\beta|,R_{k+1}) \|w_p - w_q\|_{E_{k+1}} \end{aligned}$$

where c is the same constant as in (4.12)

For $|\beta| = k$ we obtain by use of Lemma 6.15:

(4.16)

$$\begin{aligned} |\partial_x^\beta H(s,x,u,w_p) - \partial_x^\beta H(s,x,u,w_q)| &\le \left(\frac{k^2}{R} + N\right) \left(c_1 + s^{-1/\lambda} c_2\right) \|w_p - w_q\|_{E_{k+1}} \\ &\le (N+1) \left(c_1 + s^{-1/\lambda} c_2\right) \|w_p - w_q\|_{E_{k+1}} \end{aligned}$$

where c_1 and c_2 are the same constants as in (4.13)

Estimates on ψ_2

We proceed by calculating the spatial derivatives of ψ_2, using our estimates on H. We have

$$\begin{aligned} |\psi_{2,i}(w)(x,t)| &\le |K(t,\cdot) * w_{0,i}(\cdot)(x)| + \int_0^t \int_{\mathbb{R}^N} |\partial_i K(t-s,x-y)| |H(s,y,v(s,y),w(s,y))| dyds \\ &\overset{(4.6)}{\le} \|K(t,\cdot)\|_{L^1(\mathbb{R}^N)} \|w_{0,i}\|_\infty + c_0(R_{k+1}) \int_0^t \mathcal{K}_1 (t-s)^{-1/\lambda} ds \\ &\overset{(4.6)}{\le} \|w_{0,i}\|_\infty + c_0(R_{k+1}) \mathcal{K}_1 \frac{\lambda}{\lambda-1} T_{k+1}^{(\lambda-1)/\lambda} \end{aligned}$$

Similarily for $1 \leq |\beta| \leq k-1$:

$$\begin{aligned}|\partial_x^\beta \psi_{2,i}(w)(x,t)| &\leq |K(t-s,\cdot) * \partial_x^\beta w_{0,i}(\cdot)(x)| \\ &+ \int_0^t \int_{\mathbb{R}^N} |\partial_i K(t-s,x-y)||\partial_x^\beta H(s,y,v(s,y),w(s,y))|dyds \\ &\leq \|\partial_x^\beta w_{0,i}\|_\infty + c(N,|\beta|,R_{k+1})\mathcal{K}_1 \frac{\lambda}{\lambda-1} T_{k+1}^{(\lambda-1)/\lambda}\end{aligned}$$

$|\beta| = k$: Here we can only differentiate the initial conditions $k-1$ times. We will denote this as $\partial_x^{\beta-1} w_0$.

$$\begin{aligned}&|t^{1/\lambda}\partial_x^\beta \psi_{2,i}(w)(x,t)| \leq |t^{1/\lambda}\partial_x^\beta (K(t,\cdot) * w_{0,i}(\cdot))| \\ &\quad + t^{1/\lambda}\int_0^t \int_{\mathbb{R}^N} |\partial_i K(t-s,x-y)||\partial_x^\beta H(s,y,v(s,y),w(s,y))|dyds \\ &\leq t^{1/\lambda}\mathcal{K}_1 t^{-1/\lambda}\|\partial_x^{\beta-1} w_0\|_\infty + t^{1/\lambda}\int_0^t \left(c_1 + c_2 s^{-1/\lambda}\right)\int_{\mathbb{R}^N} |\partial_i K(t-s,x-y)|dyds \\ &= \mathcal{K}_1\|\partial_x^{\beta-1} w_{0,i}\|_\infty + c_1\mathcal{K}_1 t^{1/\lambda}\int_0^t (t-s)^{-1/\lambda} ds + c_2\mathcal{K}_1 t^{1/\lambda}\int_0^t s^{-1/\lambda}(t-s)^{-1/\lambda} ds\end{aligned}$$

For the second integral term, we use the substitution $\tau = s/t, d\tau = ds/t$, which yields:

$$\begin{aligned}&c_2\mathcal{K}_1 t^{1/\lambda}\int_0^t s^{-1/\lambda}(t-s)^{-1/\lambda} ds \\ &= c_2\mathcal{K}_1 t^{1/\lambda}\int_0^1 (t\tau)^{-1/\lambda}(t-t\tau)^{-1/\lambda}(td\tau) \\ &= c_2\mathcal{K}_1 t^{1-1/\lambda}\int_0^1 \tau^{-1/\lambda}(1-\tau)^{-1/\lambda} d\tau\end{aligned}$$

So, we get in total

$$|t^{1/\lambda}\partial^\beta \psi_2(w)(t,x)| \leq \mathcal{K}_1\|\partial_x^{\beta-1} w_{0,i}\|_\infty + c_1\mathcal{K}_1\frac{\lambda}{\lambda-1}T_{k+1} + c_2\gamma_\lambda T_{k+1}^{(\lambda-1)/\lambda}$$

where $\gamma_\lambda = \int_0^1 s^{-1/\lambda}(1-s)^{-1/\lambda} ds$.

By summing these expression, we get in total:

$$\begin{aligned}\|\psi_2(w)\|_{E_{k+1}} &\le (1+\mathcal{K}_1)R_0\\ &+\sum_{i=1}^{N}\left[\mathcal{K}_1\frac{\lambda}{\lambda-1}T_{k+1}^{(\lambda-1)/\lambda}\left(c_0(R_{k+1})+\sum_{1\le|\beta|\le k-1}c(N,|\beta|,R_{k+1})\right)\right.\\ &\quad\left.+\sum_{|\beta|=k}c_1\mathcal{K}_1\frac{\lambda}{\lambda-1}T_{k+1}+c_2\gamma_\lambda T_{k+1}^{(\lambda-1)/\lambda}\right]\end{aligned}$$

We now choose R_{k+1} to be $R_{k+1}:=\max\left(2(1+\mathcal{K}_1)R_0,2k^2\right)$, and choose $T_{k+1}>0$ such that:

$$\begin{aligned}&\sum_{i=1}^{N}\left[\mathcal{K}_1\frac{\lambda}{\lambda-1}T_{k+1}^{(\lambda-1)/\lambda}\left(c_0(R_{k+1})+\sum_{1\le|\beta|\le k-1}c(N,|\beta|,R_{k+1})\right)\right.\\ &\quad\left.+\sum_{|\beta|=k}c_1\mathcal{K}_1\frac{\lambda}{\lambda-1}T_{k+1}+c_2\gamma_\lambda T_{k+1}^{(\lambda-1)/\lambda}\right]\le\frac{1}{2}\frac{1}{N+1}\end{aligned}\tag{4.17}$$

This choice of T_{k+1} is possible, since the LHS of (4.17) goes towards zero for fixed R_{k+1} as $T_{k+1}\to 0$. Applying this condition, we obtain:

$$\|\psi_2(w)\|_{E_{k+1}}\le(1+\mathcal{K}_1)R_0+\frac{1}{2}\frac{1}{N+1}\le\frac{1}{2}R_{k+1}+\frac{1}{2}R_{k+1}=R_{k+1}.$$

This shows that the map ψ_2 is well-defined, and that $\psi_2:B_{R_{k+1}}\to B_{R_{k+1}}$.

Contraction

We will compute the difference $\|\psi_2(w_p)-\psi_2(w_q)\|_{E_{k+1}}$ for $w_p,w_q\in B_{R_{k+1}}$, to show that the map ψ_2 is a contraction. We use the estimates (4.14), (4.15) and (4.16) to compute the following:

For $|\beta|=0$:

$$\begin{aligned}&|\psi_{2,i}(w_p)(x,t)-\psi_{2,i}(w_q)(x,t)|\\ &\le\int_0^t\int_{\mathbb{R}^N}|\partial_iK(t-s,x-y)||H(s,y,v,w_p)-H(s,y,v,w_q)|dyds\\ &\overset{(4.14)}{\le}(N+1)c_0\|w_p-w_q\|_{E_{k+1}}\int_0^t\int_{\mathbb{R}^N}|\partial_iK(t-s,x-y)|dyds\\ &=(N+1)c_0(R_{k+1})\mathcal{K}_1\frac{\lambda}{\lambda-1}T_{k+1}^{(\lambda-1)/\lambda}\|w_p-w_q\|_{E_{k+1}}\end{aligned}$$

For $1\le|\beta|\le k-1$:

$$\begin{aligned}
&|\partial_x^\beta \psi_{2,i}(w_p)(x,t) - \partial_x^\beta \psi_{2,i}(w_q)(x,t)| \\
&\leq \int_0^t \int_{\mathbb{R}^N} |\partial_i K(t-s,x-y)||\partial_x^\beta H(s,y,v,w_p) - \partial_x^\beta H(s,y,v,w_q)|dyds \\
&\overset{(4.15)}{\leq} (N+1)\, c(N,|\beta|,R_{k+1}) \|w_p - w_q\|_{E_{k+1}} \int_0^t \int_{\mathbb{R}^N} |\partial_i K(t-s,x-y)|dyds \\
&= (N+1)\, c(N,|\beta|,R_{k+1}) \mathcal{K}_1 \frac{\lambda}{\lambda-1} T_{k+1}^{(\lambda-1)/\lambda} \|w_p - w_q\|_{E_{k+1}}
\end{aligned}$$

For $|\beta| = k$ we get

$$\begin{aligned}
&|t^{1/\lambda}\partial_x^\beta \psi_{2,i}(w_p)(x,t) - t^{1/\lambda}\partial_x^\beta \psi_{2,i}(w_q)(x,t)| \\
&\leq t^{1/\lambda} \int_0^t \int_{\mathbb{R}^N} |\partial_i K(t-s,x-y)||\partial_x^\beta H(s,y,v,w_p) - \partial_x^\beta H(s,y,v,w_q)|dyds \\
&\overset{(4.16)}{\leq} t^{1/\lambda} \int_0^t \int_{\mathbb{R}^N} |K(t-s,x-y)|(N+1)\left(c_1 C_R + s^{-1/\lambda} c_2 C_R\right) \|w_p - w_q\|_{E_{k+1}} dyds \\
&\leq (N+1)\left(c_1 \mathcal{K}_1 \frac{\lambda}{\lambda-1} T_{k+1} + c_2 \gamma_\lambda T_{k+1}^{(\lambda-1)/\lambda}\right) \|w_p - w_q\|_{E_{k+1}}
\end{aligned}$$

where the last inequality comes from the same technique as in the proof of boundedness of $|t^{1/\lambda}\psi_2(w)(t,x)|$.

Summing everything together, we obtain

$$\begin{aligned}
&\|\psi_2(w_p) - \psi_2(w_q)\|_{E_{k+1}} \\
&\leq (N+1)\sum_{i=1}^N \left[\mathcal{K}_1 \frac{\lambda}{\lambda-1} T_{k+1}^{(\lambda-1)/\lambda}\left(c_0(R_{k+1}) + \sum_{1\leq|\beta|\leq k-1} c(N,|\beta|,R_{k+1})\right)\right. \\
&\quad \left. + \sum_{|\beta|=k} c_1 \mathcal{K}_1 \frac{\lambda}{\lambda-1} T_{k+1} + c_2 \gamma_\lambda T_{k+1}^{(\lambda-1)/\lambda}\right] \|w_p - w_q\|_{E_{k+1}} \\
&\overset{(4.17)}{\leq} (N+1)\frac{1}{2}\frac{1}{N+1}\|w_p - w_q\|_{E_{k+1}} \\
&= \frac{1}{2}\|w_p - w_q\|_{E_{k+1}}
\end{aligned}$$

This inequality shows that ψ_2 is a contraction in the E_{k+1}-norm, with the chosen values of R_{k+1} and T_{k+1}. By Banach's fixed point theorem there exists a unique fixed point $w \in B_{R_{k+1}}$ that solves $\|w - \psi_2(w)\|_{E_{k+1}} = 0$.

In the end, we also need to show that w and $\bar{w}$ coincides in $F_k\left(0,T_{k+1}\right)$. The following interpolation statement holds by the definition of the spaces $F_k\left(0,T_{k+1}\right)$ and $E_{k+1}\left(0,T_{k+1}\right)$

$$\|w\|_{F_k(0,T_{k+1})} \leq \|w\|_{E_{k+1}(0,T_{k+1})}. \tag{4.18}$$

Our starting assumption was that $\bar{w}$ satisfied $\|\bar{w}-\psi_2\left(\bar{w}\right)\|_{F_k}=0$, and we obtain from the inequality (4.18) that also w satisfies $\|w-\psi_2\left(w\right)\|_{F_k}=0$. One can show that ψ_2 is a contraction in $F_k\left(0,T_{k+1}\right)$, by doing the same calculations as before, only skipping the case $|\beta|=k$ and using the F_k-norm instead of the E_{k+1}-norm. By choosing R_{k+1} and T_{k+1} to be the same as before, one can show that

$$\|\psi_2\left(w_p\right)-\psi_2\left(w_q\right)\|_{F_k(0,T_{k+1})} \leq \frac{1}{2}\|w_p-w_q\|_{F_k(0,T_{k+1})}$$

and hence that ψ_2 is a contraction in $F_k\left(0,T_{k+1}\right)$. Due to uniqueness from Banach's fixed point theorem, we get that w and $\bar{w}$ must coincide in $F_k\left(0,T_{k+1}\right)$

$$\|w-\bar{w}\|_{F_k(0,T_{k+1})}=0.$$

This concludes the proof. □

4.5.3 Proof of Lemma 4.4

This proof of is simpler than the proof of Lemma 4.3, and it follows the same idea. Therefore, we will just state the main results, to save space.

Here, we consider the space:

$$F_{k+1}:=\left\{w,Dw,\cdots,D^k w\in C_b\left(\left]0,T_{k+1}^{\epsilon}\right[\times\mathbb{R}^N\right)\right\}$$

equipped with the norm

$$\|w\|_{F_{k+1}}:=\sum_{i=1}^{N}\left(\|w_i\|_0+\sum_{1\leq|\beta|\leq k}\|\partial_x^{\beta}w_i\|_0\right)$$

As before, we look at the map:

$$\psi_{2,i}\left(w\right)\left(t,x\right)=K\left(t,\cdot\right)*w_{0,i}\left(\cdot\right)\left(x\right)-\int_0^t\partial_i K\left(t-s,\cdot\right)*H\left(s,x,v\left(s,x\right),w\left(s,x\right)\right)\left(x\right)ds$$

where the initial condition $w_0=\nabla v_0$ now satisfies

$$\sum_{|\beta|\leq k+1}\|\partial_x^{\beta}v_0\|_0\leq R_0$$

and ∇v satisfies $\|\nabla v\|_{F_k(0,T_0)}\leq R_0$.
As before, one can calculate $\|\psi_2\left(w\right)\|_{F_{k+1}}$ and $\|\psi_2\left(w_1\right)-\psi_2\left(w_2\right)\|_{F_{k+1}}$. Then one can find constants R_{k+1}^{ϵ} and $T_{k+1}^{\epsilon}>0$ such that $\psi_2:B_{R_{k+1}}\to B_{R_{k+1}}$ and ψ_2 is a contraction in the F_{k+1}-norm. Furthermore, one can also that this fixed point coincides with $\bar{w}$ in $F_k\left(0,T_{k+1}^{\epsilon}\right)$, which concludes the proof. □

4.6 Time regularity

Now, we want to discuss time regularity. We will show that when a function v satisfies Duhamel's formula on some small time interval, and has enough space regularity, it is C^1 in t. In addition, v is also then a classical solution to the fractional Hamilton-Jacobi equation on this small time interval.

We begin by citing a Lemma from Imbert [8] (with a bit modified notation), which will help us to establish time regularity.

Lemma 4.5. *(Lemma 5, [8]). Suppose that* $f \in C_b\left(]0,T_0[\times\mathbb{R}^N\right)$ *is* C^2 *in* x *such that* $\nabla f, D^2 f \in C_b\left(]0,T_0[\times\mathbb{R}^N\right)$. *Then* $\Phi(f)(t,x) = \int_0^t K(t-s,\cdot) * f(s,\cdot)(x)\,ds$ *is* C^1 *w.r.t.* $t \in]0,T_0[$ *and* $\partial_t \Phi(f)(t,x) = f(t,x) - (-\Delta)^{\lambda/2}[\Phi(f)](t,x)$.

Proof. The proof can be read in [8]. □

We use this Lemma to prove time-regularity.

Lemma 4.6. *Assume that* v *satisfies Duhamels formula on the time interval* $]0,T_0[$ *(for initial data* v_0*) and that* $v, Dv, D^2v, D^3v \in C_b\left(]0,T_0[\times\mathbb{R}^N\right)$. *Then* v *is* C_b^1 *in* t*, and is a classical solution of (4.1) on* $]0,T_0[$.

Proof. Assume that $v, \nabla v, D^2 v, D^3 v \in C_b\left(]0,T_0[\times\mathbb{R}^N\right)$ satisfies Duhamel's formula on $]0,T_0[\times\mathbb{R}^N$:

$$v(t,x) = K(t,\cdot) * v_0(\cdot)(x) - \int_0^t K(t-s,\cdot) * H(s,\cdot,v(s,\cdot),\nabla v(s,\cdot))(x)\,ds$$

Taking the derivative with respect to t and applying Lemma 4.5 to the function $f(t,x) = H(t,x,v(t,x),\nabla v(t,x))$ yields:

$$\begin{aligned}
\partial_t v(t,x) &= \partial_t\left(K(t,\cdot) * v_0(\cdot)(x)\right) - \partial_t\Phi(w)(t,x) \\
&= -(-\Delta)^{\lambda/2}\left[K(t,\cdot) * v_0(\cdot)\right](x) - \partial_t\Phi(w)(t,x) \\
&\overset{\text{Lemma 4.5}}{=} -(-\Delta)^{\lambda/2}\left[K(t,\cdot) * v_0(\cdot)\right](x) - H(t,x,v(t,x),\nabla v(t,x)) \\
&+ (-\Delta)^{\lambda/2}\left[\int_0^t K(t-s,\cdot) * H(s,x,v(s,\cdot),\nabla v(s,\cdot))(x)\,ds\right] \\
&\overset{(-\Delta)^{\lambda/2}\text{ lin.}}{=} -H(t,x,v(t,x)\nabla v(t,x)) \\
&- (-\Delta)^{\lambda/2}\left[K(t,\cdot) * v_0(\cdot) - \int_0^t K(t-s,\cdot) * H(s,x,v(s,\cdot),\nabla v(s,\cdot))(x)\,ds\right] \\
&= -H(t,x,v(t,x),\nabla v(t,x)) - (-\Delta)^{\lambda/2}\left[v(t,\cdot)\right](x)
\end{aligned}$$

or

$$\partial_t v(t,x) + (-\Delta)^{\lambda/2} v(t,x) + H(t,x,v(t,x),\nabla v(t,x)) = 0$$

This shows that v is a classical solution of the Hamilton-Jacobi equation in $]0,T_0[\times\mathbb{R}^N$, and that the time derivative is bounded. $\square$

Next, we turn to conditions for when $D^k v$ is C^1 in t.

4.6.1 The k-th spatial derivative is C^1 in time

Lemma 4.7. *Assume that v satisfies Duhamel's formula on $]0,T_0[\times\mathbb{R}^N$ (for initial data v_0) and that $v,Dv,\cdots,D^{k+3}v\in C_b\left(]0,T_0[\times\mathbb{R}^N\right)$. Then $D^k v$ is C_b^1 in time.*

Proof. Again, we will apply Lemma 4.5 to show that the k-th time derivative is C^1 with respect to t. Assume that $v,Dv,\cdots,D^{k+3}v\in C_b\left(]0,T_0[\times\mathbb{R}^N\right)$. We differentiate both sides of Duhamel's formula to obtain:

$$D^k v\left(t,x\right)=K\left(t,\cdot\right)*D^k v_0\left(\cdot\right)\left(x\right)-\int_0^t K\left(t-s,\cdot\right)*D^k H\left(s,\cdot,v\left(s,\cdot\right),\nabla v\left(s,\cdot\right)\right)\left(x\right)ds$$

Since $v,Dv,\cdots,D^{k+3}v$ are bounded, we can apply Lemma 4.5. By taking the time derivative on both sides we obtain

$$\begin{aligned}\partial_t D^k v\left(t,x\right)&=-\left(-\Delta\right)^{\lambda/2}\left[K\left(t,\cdot\right)*D^k v_0\left(\cdot\right)\right]\left(x\right)-D^k H\left(t,x,v\left(t,x\right),\nabla v\left(t,x\right)\right)\\&\quad+\left(-\Delta\right)^{\lambda/2}\left[\int_0^t K\left(t-s,\cdot\right)*D^k H\left(s,\cdot,v\left(s,\cdot\right),\nabla v\left(s,\cdot\right)\right)\left(x\right)ds\right]\\&=-D^k H\left(t,x,v\left(t,x\right),\nabla v\left(t,x\right)\right)-\left(-\Delta\right)^{\lambda/2}\left[D^k v\right]\left(t,x\right)\end{aligned}$$

This concludes the proof. $\square$

4.6.2 Hölder continuity

We will give a bit finer time estimates for $\partial_t v$.

Lemma 4.8. *Assume that v satisfies Duhamel's formula on $]0,T_0[\times\mathbb{R}^N$ (for initial data v_0) and that $v,Dv,\cdots,D^5 v\in C_b\left(]0,T_0[\times\mathbb{R}^N\right)$. Assume also that for $s,t\in[0,T]$, $x,y\in\mathbb{R}^N$, $u_1,u_2\in[-R,R]$, $p,q\in B_R$ the Hamiltonian H satisfies*

$$\left|H\left(t,x,u_1,p\right)-H\left(s,y,u_2,q\right)\right|\le L_{H,R}\left(|t-s|^{\frac{1}{2}}+|x-y|+|u_1-u_2|+|p-q|\right)$$

for some constant $L_{H,R}>0$. Then

$$\partial_t v\in C_b^{\frac{1}{2},1}\left(]0,T_0[\times\mathbb{R}^N\right)$$

and

$$v,Dv,D^2v\in C^{1,1}\left(]0,T_0[\times\mathbb{R}^N\right)$$

Proof. By inserting $k = 2$ into Lemma 4.7, we get that $D^2 v \in C_b^{1,1}\left(]0, T_0[\times \mathbb{R}^N\right)$. Further, let $t, s \in [0, T]$ and $x, y \in \mathbb{R}^N$. By using the equation (4.1), we get:

$$\begin{aligned}
&|\partial_t v\,(t,x) - \partial_t v\,(s,y)\,| \\
&= |H\,(t,x,v\,(t,x)\,, \nabla v\,(t,x)) - H\,(s,y,v\,(s,y)\,, \nabla v\,(s,y))\,| \\
&\qquad + |\,(-\Delta)^{\lambda/2}\, v\,(t,x) - (-\Delta)^{\lambda/2}\, v\,(s,y)\,| \\
&\le L_H \left(|s-t|^{1/2} + |x-y| + |v\,(t,x) - v\,(s,y)\,| + |\nabla v\,(t,x) - \nabla v\,(s,y)\,|\right) \\
&+ R\left(|s-t|^{\frac{1}{2}} + |x-y|\right) \le C\left(|s-t|^{1/2} + |x-y|\right)
\end{aligned}$$

where $C > 0$ and $R > 0$ are some constant. This concludes the proof. □

To conclude, we will prove sufficient regularity for the unique viscosity solution of the fractional Hamilton-Jacobi equation (4.1), by using these estimates that we found using the Duhamel's formula.

4.7 Regularity on the unique viscosity solution

The aim of this section is to show regularity for the unique viscosity solution of the fractional Hamilton-Jacobi equation (4.1). The idea is to use Lemma 4.3 and 4.4 to show C_b^k space regularity for $k \ge 3$. Then we can start applying Lemma 4.6, 4.7 and 4.8 to establish time-regularity.

4.7.1 Close to zero

Suppose that the initial conditions u_0 from (4.1) belongs to $W^{k,\infty}\left(\mathbb{R}^N\right)$. We can start applying Lemma 4.4 with initial condition u_0 iteratively to obtain:

$$0 < T_k^\epsilon \le \cdots \le T_2^\epsilon \le T_1$$

where $T_1 > 0$ was obtained from Lemma 4.2. Our result is that there exists a unique v that satisfies Duhamel's formula with

$$v, Dv, \cdots, D^k v \in C_b\left(]0, T_k^\epsilon[\times \mathbb{R}^N\right)$$

and when $k \ge 3$, we have from Lemma 4.6 that

$$\partial_t v \in C_b\left(]0, T_k^\epsilon[\times \mathbb{R}^N\right)$$

This mean that v is a classical solution on $]0, T_k^\epsilon[$. Therefore $v = u$ on $]0, T_k^\epsilon[$, since any classical solution is a viscosity solution.

4.7.2 Away from zero

Consider some arbitrary $t_0 \in [0, T)$. We will use the viscosity solution u of (4.1) as initial conditions, by setting $v_0(\cdot) = u(t_0, \cdot)$. Recalling Lemma 4.1 we know that

$$\|u(t_0, \cdot)\|_{W^{1,\infty}(\mathbb{R}^N)} \leq M_T. \tag{4.19}$$

We start by applying Lemma 4.2 to obtain $T_1 > 0$, independent of t_0 due to (4.19), such that for the interval

$$]t_0, t_0 + T_1[$$

we have unique v that satisfies Duhamel's formula with

$$v, \nabla v \in C_b\left(]t_0, t_0 + T_1[\times \mathbb{R}^N\right)$$

Continuing this process, Lemma 4.3 gives us $T_2 \leq T_1$ such that on the interval

$$]t_0, t_0 + T_2[$$

there exists a unique v that satisfies Duhamel's formula with

$$v, \nabla v, t^{1/\lambda} D^2 v \in C_b\left(]t_0, t_0 + T_2[\times \mathbb{R}^N\right)$$

By considering this v, we can use it as initial conditions, if we move a small distance $\delta_1 > 0$ to the right. Namely

$$v, \nabla v, D^2 v \in C_b\left(]t_0 + \delta_1, t_0 + T_2[\times \mathbb{R}^N\right)$$

Using $v(t_0 + \delta_1, \cdot)$ as initial conditions we get $\delta_1 < T_3 \leq T_2 - \delta_1$ such that on the interval

$$]t_0 + \delta_1, t_0 + T_3[$$

there exists unique v that satisfies Duhamel's formula with

$$v, \nabla v, D^2 v, t^{1/\lambda} D^3 v \in C_b\left(]t_0 + \delta_1, t_0 + T_3[\times \mathbb{R}^N\right)$$

We can iterate this process, until we reach k. That is, we get that there exists $\delta_1 + \cdots + \delta_{k-1} < T_k \leq T_{k-1} - \delta_{k-1}$ such that on the interval

$$]t_0 + \delta_1 + \cdots + \delta_{k-1}, t_0 + T_k[$$

there exists a unique v that satisfies Duhamel's formula with

$$v, Dv, \cdots, D^{k-1} v, t^{1/\lambda} D^k v \in C_b\left(]t_0 + \delta_1 + \cdots + \delta_{k-1}, t_0 + T_k[\times \mathbb{R}^N\right)$$

In the end, we can pick a $\delta_k > 0$ such that on the interval

$$]t_0 + \delta_1 + \cdots + \delta_k, T_k[$$

we have a unique v that satisfies Duhamel's formula with

$$v, Dv, \cdots, D^k v \in C_b\left(]t_0 + \delta_1 + \cdots + \delta_k, T_k[\times \mathbb{R}^N\right)$$

When $k \geq 3$, we have by Lemma 4.6 that v is a classical solution on $]t_0 + \delta_1 + \cdots + \delta_k, T_k[$, and for $k \geq 5$ we have that

$$\partial_t v \in C_b^{\frac{1}{2},1}\left(]t_0 + \delta_1 + \cdots + \delta_k, T_k[\times \mathbb{R}^N\right)$$
$$v, Dv, D^2 v \in C_b^{1,1}\left(]t_0 + \delta_1 + \cdots + \delta_k, T_k[\times \mathbb{R}^N\right)$$

Now, for the delta's, we enforce the following condition

$$\delta_1 + \cdots + \delta_k \leq T_k^{\epsilon}$$

which we are free to do, since we can choose them as small as we like. By denoting

$$t_0^{'} = t_0 + \delta_1 + \cdots + \delta_k$$

we get that for $k \geq 3$ (since $v = u$ here)

$$\forall t_0^{'} \in [T_k^{\epsilon}/2, T],\ \exists T_k > 0 \ \text{ s.t. } u, Du, \cdots, D^k u, \partial_t u \in C_b\left(\Big]t_0^{'}, T_k\Big[\times \mathbb{R}^N\right)$$

4.7.3 Patching

Now we can patch everyting together. With initial data $u_0 \in W^{k,\infty}\left(\mathbb{R}^N\right)$, for $k \geq 3$, we get that the unique viscosity solution u is a classical solution on the intervals

$$]0, T_k^{\epsilon}[\,,]T_k^{\epsilon}/2, T_k^{\epsilon}/2 + T_k[\,,]T_k^{\epsilon}/2 + T_k/2, T_k^{\epsilon}/2 + 3T_k/2[\,, \cdots$$

until we reach $T > 0$, satisfying Theorem 4.1. For $k \geq 5$ we get the regularity results wanted in Theorem 4.2. Thus, we have proven Theorem 4.1 and Theorem 4.2. □

Chapter 5

The fractional Fokker-Planck equation

In this chapter, we will study the fractional Fokker-Planck equation. When the fractional Fokker-Planck equation is written on divergence free form, it pretty much looks like the Hamilton-Jacobi equation. This is a fact we exploit to show existence and uniqueness of a solution for this equation.

The fractional Fokker-Planck equation is on the form:

$$\begin{cases} \partial_t m + (-\Delta)^{\frac{\alpha}{2}} m - div\left(m D_p H\left(x, u, Du\right)\right) = 0 & \text{in } (0,T) \times \mathbb{R}^N \\ m\left(0, x\right) = m_0\left(x\right) & \text{in } \mathbb{R}^N \end{cases} \tag{5.1}$$

where $\alpha \in (1,2)$. We write it on divergence free form as follows

$$\partial_t m + (-\Delta)^{\frac{\alpha}{2}} m - \sum_{i=1}^{N} \left[\frac{\partial m}{\partial x_i} f_i\left(t,x\right) + m \frac{\partial}{\partial x_i} f_i\left(t,x\right) \right] = 0$$

where $f_i\left(t,x\right) = \frac{\partial H}{\partial p_i}\left(x, u\left(t,x\right), Du\left(t,x\right)\right)$. We notice then that this equation is essentially of the form:

$$\partial_t m + (-\Delta)^{\frac{\alpha}{2}} m + B\left(t, x, m, Dm\right) = 0$$

where $B\left(t,x,m,Dm\right) = -\sum_{i=1}^{N} \left[\frac{\partial m}{\partial x_i} f_i\left(t,x\right) + m \frac{\partial}{\partial x_i} f_i\left(t,x\right) \right]$.

One can now notice that this equation has the same form as the Hamilton-Jacobi equation, and given the right properties of B, which we will now show, we get a unique classical solution m of (5.1) with sufficient regularity in t and x.

We need to verify that the assumptions (A0)-(A4) from the chapter on the Fractional Hamilton-Jacobi equation holds. Then, we wish to use Theorem 4.2. The Hamiltonian B needs to satisfy:

- (A0) The function $B : [0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is continuous.
- (A1) There exists $\gamma \in \mathbb{R}$ such that for all $x \in \mathbb{R}^N$, $m, n \in \mathbb{R}$, $m < n$, $p \in \mathbb{R}^N$, $t \in [0, T)$,
$$B(t, x, n, p) - B(t, x, m, p) \geq \gamma (n - m)$$
- (A2) For any $R > 0$, there exists $C_R > 0$ such that for all $x, y \in \mathbb{R}^N, m \in [-R, R]\, p \in \mathbb{R}^N, t \in [0, T)$,
$$|B(t, x, m, p) - B(t, y, m, p)| \leq C_R (|p| + 1) |x - y|$$
- (A3) For any $R > 0$, there exists $C_R > 0$ such that for all $x \in \mathbb{R}^N, m \in [-R, R], p \in B_R, t \in [0, T)$, the derivatives of B (up till the k-th derivative) are bounded by the constant C_R, that is,
$$|D^\alpha B(t, x, m, p)| \leq C_R$$
with $\alpha = (\alpha_{x_1}, \ldots, \alpha_{x_N}, \alpha_m, \alpha_{p_1}, \ldots, \alpha_{p_N})$ a multi-index with $|\alpha| \leq k$.
- (A4) There exists $C_0 > 0$ such that
$$\sup_{t \in [0,T), x \in \mathbb{R}^N} |B(t, x, 0, 0)| \leq C_0$$

Theorem 5.1. *Assume the following (for $k \geq 5$*

1. *There exists $C_R > 0$ such that for $t \in (0, T), x \in \mathbb{R}^N, u \in [-R, R], p \in B_R$ and for all multi-indexes α_H with $|\alpha_H| \leq k + 2$*
$$|D^{\alpha_H} H(x, u, p)| \leq C_R$$
2. *We have that $u, Du, \cdots, D^{k+2}u, \partial_t u, \partial_t Du, \partial_t D^2 u \in C_b\left(]0, T[\times \mathbb{R}^N\right)$, and all these quantities are uniformly bounded by some constant, say $R > 0$.*
3. $m_0 \in W^{k,\infty}\left(\mathbb{R}^N\right)$

Then the following holds true:

1. *There exists a unique classical solution m of (5.1).*
2. *The following quantities are uniformly bounded:*
$$m, Dm, \cdots, D^5 m, \partial_t m, \partial_t Dm, \partial_t D^2 m \in C_b\left(]0, T[\times \mathbb{R}^N\right)$$
and
$$\partial_t m \in C_b^{\frac{1}{2},1}\left(]0, T[\times \mathbb{R}^N\right)$$
where $C_b^{\frac{1}{2},1}$ is a parabolic Hölder-space.

Proof. We need to show that the assumptions (A0)-(A4) and the Hölder condition 4.2 holds. We begin by showing (A3), and then proceed to show the others in the normal order.

(A3)

Assume that $t \in [0,T), x \in \mathbb{R}^N, m \in [-R,R], p \in B_R$. Then we compute the following for a multi-index $\alpha = (\alpha_{x_1}, \dots, \alpha_{x_N}, \alpha_m, \alpha_{p_1}, \dots, \alpha_{p_N})$ with $|\alpha| \le k$:

$$\begin{aligned}
|D^\alpha B(t,x,m,p)| &= \left| \sum_{i=1}^N D^\alpha \left[p_i f_i(t,x) + m \frac{\partial}{\partial x_i} f_i(t,x) \right] \right| \\
&\le \sum_{i=1}^N \left[R \sum_{0 \le |\beta| \le k} \left(\|\partial^\beta f_i\|_0 \right) + R \sum_{0 \le |\beta| \le k} \left(\|\partial^\beta \frac{\partial}{\partial x_i} f_i\|_0 \right) \right] \\
&\le 2R \sum_{i=1}^N \sum_{0 \le |\beta| \le k+1} \|\partial^\beta f_i\|_0
\end{aligned}$$

recalling that $\|\cdot\|_0 = \|\cdot\|_{C_b(]0,T[\times\mathbb{R}^N)}$. Remember that f_i was given by:

$$f_i(t,x) = \frac{\partial H}{\partial p_i}(x, u(t,x), Du(t,x))$$

So, if $u, Du, \cdots, D^{k+2}u$ are uniformly bounded, and assumption 1. from Theorem 5.1 holds, the following sum is uniformly bounded

$$\sum_{0 \le |\beta| \le k+1} \|\partial^\beta f_i\|_0 \le K \tag{5.2}$$

where $K > 0$ is some constant. This shows that (A3) holds.

(A0)

This assumption holds, since f_i, $\frac{\partial}{\partial x_i} f_i$ are continuous, from the assumptions made on H and u.

(A1)

Since $\frac{\partial}{\partial x_i} f_i$ is uniformly bounded by some constant $K > 0$, as shown in (5.2), we can compute:

$$\begin{aligned}
&B(t,x,n,p) - B(t,x,m,p) \\
&= -\sum_{i=1}^N \left[p_i f_i(t,x) + n \frac{\partial}{\partial x_i} f_i(t,x) \right] + \sum_{i=1}^N \left[p_i f_i(t,x) + m \frac{\partial}{\partial x_i} f_i(t,x) \right] \\
&= \sum_{i=1}^N (m-n) \frac{\partial}{\partial x_i} f_i(t,x) \ge N(m-n)(-K) = (n-m) NK
\end{aligned}$$

which shows that the assumption holds.

(A2)

We calculate:

$$\begin{aligned}
&|B(t,x,u,p) - B(t,y,u,p)| \\
&= \left| \sum_{i=1}^{N} \left[p_i f_i(t,x) + u\partial_i f_i(t,x)\right] - \sum_{i=1}^{N} \left[p_i f_i(t,y) + u\partial_i f_i(t,y)\right] \right| \\
&= \left| \sum_{i=1}^{N} p_i \left[f_i(t,x) - f_i(t,y)\right] + u \sum_{i=1}^{N} \left[\partial_i f_i(t,x) - \partial_i f_i(t,y)\right] \right| \\
&\leq |p| \sum_{i=1}^{N} |f_i(t,x) - f_i(t,y)| + R \sum_{i=1}^{N} |\partial_i f_i(t,x) - \partial_i f_i(t,y)| \\
&\leq C_R(|p|+1)|x-y|
\end{aligned}$$

where the last inequality follows from f_i and $\partial_i f_i$ being continuously differentiable in x and bounded (referring to (5.2)). This implies that both functions are Lipschitz in x, with some Lipschitz constant, which yields the last inequality.

(A4)

Inserting $u = p = 0$, we end up with

$$|B(t,x,0,0)| = 0$$

so this assumption holds trivially.

Hölder-condition

The last thing to do, is to show that B satisfies the following condition:

For all $s,t \in \left]0,T\right[, x,y \in \mathbb{R}^N, u,v \in [-R,R], p,q \in B_R$ there exists a constant $L_R > 0$ such that:

$$|B(s,x,u,p) - B(t,y,v,q)| \leq L_R\left(|s-t|^{\frac{1}{2}} + |x-y| + |u-v| + |p-q|\right) \tag{5.3}$$

We compute

$$\begin{aligned}
&|B(s,x,m,p) - B(t,y,n,q)| \\
&= \left| \sum_{i=1}^{N} \left[p_i f_i(s,x) + m\partial_i f_i(s,x)\right] - \sum_{i=1}^{N} \left[q_i f_i(t,y) + n\partial_i f_i(t,y)\right] \right| \\
&\leq \sum_{i=1}^{N} \Big(|p_i f_i(s,x) - q_i f_i(t,y)| + |m\partial_i f_i(s,x) - n\partial_i f_i(t,y)| \Big)
\end{aligned}$$

Now, we can use the triangle inequality on the first term to obtain

$$\begin{aligned}
&|p_i f_i(s,x) - q_i f_i(t,y)| \\
&\quad \leq |p_i f_i(s,x) - q_i f_i(s,x)| + |q_i f_i(s,x) - q_i f_i(t,y)| \\
&\quad \leq |p_i - q_i||f_i(s,x)| + |q_i||f_i(s,x) - f_i(t,y)|
\end{aligned}$$

and in the same manner, we obtain for the second term

$$|m\partial_i f_i(s,x) - n\partial_i f_i(t,y)| \leq |m-n||\partial_i f_i(s,x)| + |n||\partial_i f_i(s,x) - \partial_i f_i(t,y)|.$$

Recall that f_i and $\partial_i f_i$ are uniformly bounded by some constant $K > 0$, defined in (5.2). Also, recall our assumptions that $u, Du, D^2u \in C_b^{1,1}\left(]0,T[\times\mathbb{R}^N\right)$. From this, one can show that

$$f_i, \partial_i f_i \in C_b^{1,1}\left(]0,T[\times\mathbb{R}^N\right),$$

since the expression $\partial_i f_i$ involves the functions u, Du, D^2u (but not $D^3u, D^4u, \cdots$).

Recalling Lemma 2.6 (regarding bounded C^1 functions also being $C^{1/2}$ functions), we continue the calculations to obtain:

$$\begin{aligned}
&|B(s,x,m,p) - B(t,y,n,q)| \\
&\leq \sum_{i=1}^{N}\Big(|p_i - q_i||f_i(s,x)| + |q_i||f_i(s,x) - f_i(t,y)| \\
&\qquad + |m-n||\partial_i f_i(s,x)| + |n||\partial_i f_i(s,x) - \partial_i f_i(t,y)|\Big) \\
&\leq \sum_{i=1}^{N} K|p_i - q_i| + RK\left(|s-t|^{\frac{1}{2}} + |x-y|\right) + K|m-n| + RK\left(|s-t|^{\frac{1}{2}} + |x-y|\right) \\
&\leq L_R\left(|s-t|^{\frac{1}{2}} + |x-y| + |m-n| + |p-q|\right)
\end{aligned}$$

where $L_R > 0$ is some constant, which is what we wanted to show.

Conclusion

Having established that (A0)-(A4) holds, and that B satisfies (5.3), we conclude by Theorem 4.2 that there exists a unique m that solves (5.1), with the stated regularity. □

Chapter 6

Estimates of $\partial_x^\beta H$

The Hamiltonian H is on the following form:

$$H : [0, \infty) \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$$
$$(s, x, u, p) \mapsto H(s, x, u, p)$$

However, in the analysis of the fractional Hamilton-Jacobi equation, we usually look at the composite function

$$(s, x) \mapsto H(s, x, u(s, x), w(s, x))$$

where u and w are functions mapping to $\mathbb{R}$ and $\mathbb{R}^N$, respectively. Now, taking spatial derivatives of this function involves using the chain rule, but due to the complexity of the function, this turns out to be a quite complicated process. As a start example, we compute $\partial_{x_j} h$, to show how the complexity grows.

(6.1)

$$\begin{aligned}
&\partial_{x_j} H(s, x, u(s, x), w(s, x)) = \frac{\partial H}{\partial x_j}(s, x, u(s, x), w(s, x)) \\
&+ \frac{\partial H}{\partial u}(s, x, u(s, x), w(s, x)) \frac{\partial u}{\partial x_j}(s, x) \\
&+ \frac{\partial H}{\partial p_1}(s, x, u(s, x), w(s, x)) \frac{\partial w_1}{\partial x_j}(s, x) + \cdots + \frac{\partial H}{\partial p_N}(s, x, u(s, x), w(s, x)) \frac{\partial w_N}{\partial x_j}(s, x)
\end{aligned}$$

As one can see, this expression is quite big. If we want to continue, calculating the second derivative, $\partial_{x_i} \partial_{x_j} h$, we would have to differentiate each of the terms in (6.1), and so on for higher order derivatives.

Historically, there have been invented methods for problems of this kind. For example

Faà di Bruno's Formula, which says that if $f, g : \mathbb{R} \to \mathbb{R}$ are both regular enough, then

$$\frac{d^m}{dt^m} g(f(x)) = \sum \frac{m!}{b_1! b_2! \cdots b_m!} g^{(k)}(f(t)) \left(\frac{f'(t)}{1!}\right)^{b_1} \left(\frac{f''(t)}{2!}\right)^{b_2} \cdots \left(\frac{f^{(m)}(t)}{m!}\right)^{b_m} \tag{6.2}$$

where the sum is over all different solutions in nonnegative integers $b_1, \cdots, b_m$ of $b_1 + 2b_2 + \cdots + mb_m = m$, and k is defined as $k := b_1 + \cdots + b_m$ (Johnson, [9]).

The problem is that the expression (6.2) is only valid for quite simple functions f and g (mapping from $\mathbb{R}$ to $\mathbb{R}$), while the expression we deal is more complicated. The bad news is that such complexity makes it difficult to find an exact expression for the derivative. The good news is that we don't need to know the exact expression. Finding out which properties the derivative has is, as we will show, sufficient for our case. We have the following aims for this chapter:

Aim

For a multi-index $\beta = (\beta_1, \cdots, \beta_N)$, we want to

1. find a useful representation of $\partial^\beta H(s, x, u(s,x), w(s,x))$,
2. find an upper bound for $|\partial^\beta H|$
3. find an estimate for $|\partial^\beta H(s, x, u(s,x), w_p(s,x)) - \partial^\beta H(s, x, u(s,x), w_q(s,x))|$

Resolution

By investigating expressions like (6.1), it is clear that each term can be divided in two parts: One part that contains some derivative with respect to H, and one part that is a polynomial consisting of derivatives of u and w_i. Like for example the term:

$$\frac{\partial H}{\partial p_2} \frac{\partial w_2}{\partial x_j}$$

If we take the second derivative, e.g. computing $\partial_{x_i} \partial_{x_j} h$, the expression contains the term:

$$\left(\frac{\partial w_1}{\partial x_j} \frac{\partial w_2}{\partial x_i} + \frac{\partial w_1}{\partial x_i} \frac{\partial w_2}{\partial x_j}\right) \frac{\partial^2 H}{\partial p_1 \partial p_2}(s, x, u, w)$$

When we take the k-th order derivative, we are no longer sure what these polynomials look like, but we know something about their structure. The first thing we can show is that each polynomial has order less or equal to k. The second thing we can show is that u and w_i are differentiated maximally k times. The easy solution is to say that "there is some polynomial in front of each $D^\alpha H$, and this polynomial has order less or equal to k. This is what we try to formalize in the next lemma.

Lemma 6.1. *(Representation of $\partial^{\beta}H$). Let $\beta = (\beta_1, \cdots, \beta_N)$ be a multi-index with $|\beta| \leq k$. Then the derivative of H can be written on the form:*

$$\partial^{\beta} H(s,x,u,w) = \sum_{|\alpha|\leq k} \mathcal{P}^1_{\alpha,\beta}(u,w) \frac{\partial^{|\alpha|} H}{\partial_{x_1}^{\alpha_{x_1}} \cdots \partial_{x_N}^{\alpha_{x_N}} \partial_u^{\alpha_u} \partial_{p_1}^{\alpha_{p_1}} \cdots \partial_{p_N}^{\alpha_{p_N}}} \tag{6.3}$$

with

(6.4)

$$\mathcal{P}^1_{\alpha,\beta}(u,w) = \sum_{|\gamma|+|\eta_1|+\ldots|\eta_N|\leq k} K(\gamma,\eta_1,+\ldots,\eta_N;\alpha,\beta) \left(\prod_{|\beta_0|\leq k} \left(\partial^{\beta_0} u\right)^{\gamma(\beta_0)}\right) \left(\prod_{|\beta_1|\leq k} \left(\partial^{\beta_1} w_1\right)^{\eta_1(\beta_1)}\right)$$

$$\cdots \left(\prod_{|\beta_N|\leq k} \left(\partial^{\beta_N} w_N\right)^{\eta_N(\beta_N)}\right) \tag{6.5}$$

where K is a function taking multi-indexes (or tuples), and returning a number in $\{0,1,2,3,\ldots\}$.

The derivative can also be represented as:

$$\partial^{\beta} H(s,x,u,w) = \sum_{|\alpha|\leq k} \mathcal{P}^2_{\alpha,\beta}(u,w) \frac{\partial^{|\alpha|} H}{\partial_{x_1}^{\alpha_{x_1}} \cdots \partial_{x_N}^{\alpha_{x_N}} \partial_u^{\alpha_u} \partial_{p_1}^{\alpha_{p_1}} \cdots \partial_{p_N}^{\alpha_{p_N}}} + \sum_{i=1}^{N} \partial^{\beta} w_i \frac{\partial H}{\partial p_i} \tag{6.6}$$

with

(6.7)

$$\mathcal{P}^2_{\alpha,\beta}(u,w) = \sum_{|\gamma|+|\eta_1|+\ldots|\eta_N|\leq k} K(\gamma,\eta_1,+\ldots,\eta_N;\alpha,\beta) \left(\prod_{|\beta_0|\leq k} \left(\partial^{\beta_0} u\right)^{\gamma(\beta_0)}\right) \left(\prod_{|\beta_1|\leq k-1} \left(\partial^{\beta_1} w_1\right)^{\eta_1(\beta_1)}\right)$$

$$\cdots \left(\prod_{|\beta_N|\leq k-1} \left(\partial^{\beta_N} w_N\right)^{\eta_N(\beta_N)}\right) \tag{6.8}$$

We have that $\gamma, \eta_1, \cdots \eta_N$ are functions taking multi-indexes and returning a natural number, $\{0,1,2,3,\cdots\}$. We also define $|\gamma| = \sum_{\beta_0 \leq k} \gamma(\beta_0)$, and likewise for $\eta_1, \cdots, \eta_N$.

Proof. We want to prove that the derivative of H has this form, and that the polynomials $\mathcal{P}^1_{\alpha,\beta}(u,w)$ and $\mathcal{P}^2_{\alpha,\beta}(u,w)$ has order less or equal to k. One can prove this by induction. The case $k=1$ is clear from the expression (6.1), so what remains is to show that if it holds for k, then it holds for $k+1$.

If the expression holds for $|\beta| = k$, then $\partial^\beta H$ is on the form:

$$\partial^\beta H(s,x,u,w) = \sum_{|\alpha|\le k} \mathcal{P}^1_{\alpha,\beta}(u,w) \frac{\partial^{|\alpha|} H}{\partial_{x_1}^{\alpha_{x_1}} \cdots \partial_{x_N}^{\alpha_{x_N}} \partial_u^{\alpha_u} \partial_{p_1}^{\alpha_{p_1}} \cdots \partial_{p_N}^{\alpha_{p_N}}}$$

If we now differentiate this expression, e.g. in the direction x_i, we get (since differentiation is a linear operation):

$$\begin{aligned}
\partial_{x_i}\partial^\beta H(s,x,u,w) &\overset{\text{lin.}}{=} \sum_{|\alpha|\le k} \partial_{x_i}\left(\mathcal{P}^1_{\alpha,\beta}(u,w) \frac{\partial^{|\alpha|} H}{\partial_{x_1}^{\alpha_{x_1}} \cdots \partial_{x_N}^{\alpha_{x_N}} \partial_u^{\alpha_u} \partial_{p_1}^{\alpha_{p_1}} \cdots \partial_{p_N}^{\alpha_{p_N}}}\right) \\
&= \sum_{|\alpha|\le k} \Bigg\{ \left(\partial_{x_i}\mathcal{P}^1_{\alpha,\beta}(u,w)\right) \frac{\partial^{|\alpha|} H}{\partial_{x_1}^{\alpha_{x_1}} \cdots \partial_{x_N}^{\alpha_{x_N}} \partial_u^{\alpha_u} \partial_{p_1}^{\alpha_{p_1}} \cdots \partial_{p_N}^{\alpha_{p_N}}} \\
&\qquad + \mathcal{P}^1_{\alpha,\beta}(u,w) \left(\partial_{x_i} \frac{\partial^{|\alpha|} H}{\partial_{x_1}^{\alpha_{x_1}} \cdots \partial_{x_N}^{\alpha_{x_N}} \partial_u^{\alpha_u} \partial_{p_1}^{\alpha_{p_1}} \cdots \partial_{p_N}^{\alpha_{p_N}}}\right) \Bigg\}
\end{aligned}$$

Differenting the expression $\mathcal{P}^1_{\alpha,\beta}(u,w)$, does not increase the order of the polynomial, so that it still has order less or equal to k.

Differenting the expression $D^\alpha H$, leads maximally (by the chain rule) to an increase of 1 power in the polynomial in front of it, so that the new polynomials have order less or equal to $k+1$. Also, $\partial^\beta H$ is still on the form we proposed, as can be easily seen.

The case for the second representation (6.6) follows from looking at the expression (6.1) and realizing that the incidents of w_i being differentiated k times only happens in N different cases (the last N last terms in (6.1). This concludes the proof.

□

Using this representation of $\partial^\beta H$ leads to several useful results, starting with the next lemma.

Lemma 6.2. *(Upper bound of $|\partial^\beta H|$) Suppose that u and w and their derivatives up till k-th order are bounded by a constant R. Assume further that there exists a constant C_R such that for $|\alpha| \le k+1$, $|\partial^\alpha H(s,x,u,w)| \le C_R$. Then*

$$\exists c = c(N,k,R) \text{ s.t. } |\partial^\beta H(s,x,u,w)| \le c \tag{6.9}$$

furthermore, if $\|\partial^{\beta_0} u\|_\infty \le R$ for all $|\beta_0| \le k$, $\|\partial^{\beta_1} w_j\|_\infty \le R$ for all $|\beta_1| \le k-1$ and $\|t^{1/\lambda}\partial^{\beta_2} w_j\|_\infty \le R$ for all $|\beta_2| = k$, then

$$\exists c_1 = c(N,k,R), c_2 = c_2(N,R) \text{ s.t. } |\partial^\beta H(s,x,u,w)| \le c_1 + s^{-1/\lambda} c_2 \tag{6.10}$$

Proof. **Part I:**
This follows from using the representation in lemma 6.1. Pick u and w as described, then

$$|\partial^{\beta} H(s,x,u,w)| \leq \sum_{|\alpha|\leq k} \mathcal{P}^{1,\max}_{\alpha,\beta} C_R = c(N,k,R)$$

where

$$\mathcal{P}^{1,\max}_{\alpha,\beta} = \sum_{|\gamma|+|\eta_1|+\ldots|\eta_N|\leq k} K(\gamma,\eta_1,\ldots,\eta_N;\alpha,\beta) R^{|\gamma|+|\eta_1|+\ldots|\eta_N|}$$

which is what we wanted to show.

Part II:
By using the representation (6.6), we have

$$\partial^{\beta} H(s,x,u,w) = \sum_{|\alpha|\leq k} \mathcal{P}^{2}_{\alpha,\beta}(u,w) \frac{\partial^{|\alpha|} H}{\partial^{\alpha_{x_1}}_{x_1}\cdots\partial^{\alpha_{x_N}}_{x_N}\partial^{\alpha_u}_{u}\partial^{\alpha_{p_1}}_{p_1}\cdots\partial^{\alpha_{p_N}}_{p_N}} + \sum_{i=1}^{N} \partial^{\beta} w_i \frac{\partial H}{\partial p_i} \tag{6.11}$$

Picking u and w as described gives us

$$\begin{aligned}|\partial^{\beta} H(s,x,u,w)| &\leq \sum_{|\alpha|\leq k} \mathcal{P}^{2,\max}_{\alpha,\beta} C_R + \sum_{i=1}^{N} s^{-1/\lambda} R C_R \\ &= c_1(N,k,R) + s^{-1/\lambda} c_2(N,R)\end{aligned}$$

where

$$\mathcal{P}^{2,\max}_{\alpha,\beta} = \sum_{|\gamma|+|\eta_1|+\ldots|\eta_N|\leq k} K(\gamma,\eta_1,\ldots,\eta_N;\alpha,\beta) R^{|\gamma|+|\eta_1|+\ldots|\eta_N|}$$

which yields the necessary estimate. □

6.1 Estimates on the difference

Here is a small lemma that will help us trough the calculations.

Lemma 6.3. *Consider the polynomials $\mathcal{P}^1_{\alpha,\beta}$ and $\mathcal{P}^2_{\alpha,\beta}$. We have that for $w_1, w_2 \in F_{k+1}$ (or $p_1, p_2 \in E_{k+1}$) such that $\|w_1\|_{F_{k+1}}, \|w_2\|_{F_{k+1}} \leq R$ ($\|w_1\|_{E_{k+1}}, \|w_2\|_{E_{k+1}} \leq R$) that*

$$|\mathcal{P}^1_{\alpha,\beta}(u,w_1) - \mathcal{P}^1_{\alpha,\beta}(u,w_2)| \leq \frac{k^2}{R}\mathcal{P}^{1,\max}_{\alpha,\beta}\|w_1-w_2\|_{F_{k+1}} \tag{6.12}$$

and

$$|\mathcal{P}^2_{\alpha,\beta}(u,p_1) - \mathcal{P}^2_{\alpha,\beta}(u,p_2)| \leq \frac{k^2}{R}\mathcal{P}^{2,\max}_{\alpha,\beta}\|w_1-w_2\|_{E_{k+1}} \tag{6.13}$$

Proof. We can think of $\mathcal{P}^1_{\alpha,\beta}$ and $\mathcal{P}^2_{\alpha,\beta}$ as polynomials of degree k or less. For a polynomial $f : \mathbb{R}^d \to \mathbb{R}$ of degree k or less, the following holds by the fundamental theorem of calculus

$$|f(x) - f(y)| \leq \|Df\|_{L^\infty} \|x-y\|_1 \leq k\|Df\|_{L^\infty}\|x-y\|_\infty$$

where $\|\cdot\|_1$ and $\|\cdot\|_\infty$ is the 1-norm and infinity-norm of $\mathbb{R}^d$, respectively. Also, since f is a polynomial of degree k or less:

$$\|Df\|_{L^\infty(B_R)} \leq \frac{k}{R}\|f\|_{L^\infty(B_R)}$$

since $f \in P_k \implies Df \in P_{k-1}$. We conclude:

$$\|f(x) - f(y)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{k^2}{R}\|f\|_{L^\infty(\mathbb{R}^d)}\|x-y\|_\infty$$

Using this result, we easily obtain the estimates (6.12) and (6.13). □

6.1.1 Computations

Using our knowledge of $\partial^\beta H$, we can compute $|\partial^\beta H(s,x,u,w_1) - \partial^\beta H(s,x,u,w_2)|$. We will do two different kinds of computations, depending on which space w_1 and w_2 belong to. We will consider the spaces F_{k+1} and E_{k+1}, which is defined as:

$$F_{k+1}(T_s, T_e) := \left\{w, \cdots, D^k w \in C_b\left(]T_s, T_e[\times \mathbb{R}^N\right)\right\}$$

and

$$E_{k+1}(T_s, T_e) := \left\{w, \cdots, D^{k-1}w, t^{1/\lambda}D^k w \in C_b\left(]T_s, T_e[\times \mathbb{R}^N\right)\right\}$$

where $T_e > T_s \geq 0$. We start with the first case, when $w_1, w_2 \in F_{k+1}$.

Lemma 6.4. *Let $w_1, w_2 \in F_{k+1}(T_s, T_e)$ so that $\|w_1\|_{F_{k+1}}, \|w_2\|_{F_{k+1}} \leq R$, and let $\|\nabla u\|_{F_k} \leq R$. Then it holds that:*

$$|\partial^\beta H(s,x,u,w_1) - \partial^\beta H(s,x,u,w_2)| \leq \left(\frac{k^2}{R} + N\right) c\|w_1 - w_2\|_{F_{k+1}} \tag{6.14}$$

where c is the constant from (6.9)

Proof. By using Lemma 6.1, we get that

$$\begin{aligned}
&|\partial^\beta H(s,x,u,w_1) - \partial^\beta H(s,x,u,w_2)| \\
&\overset{(6.3)}{=} \left| \sum_{|\alpha|\leq k} \mathcal{P}^1_{\alpha,\beta}(u,w_1)\,\partial^\alpha H(s,x,u,w_1) - \sum_{|\alpha|\leq k} \mathcal{P}^1_{\alpha,\beta}(u,w_2)\,\partial^\alpha H(s,x,u,w_2)\right| \\
&\leq \sum_{|\alpha|\leq k} \left|\mathcal{P}^1_{\alpha,\beta}(u,w_1)\,\partial^\alpha H(s,x,u,w_1) - \mathcal{P}^1_{\alpha,\beta}(u,w_2)\,\partial^\alpha H(s,x,u,w_2)\right|
\end{aligned}$$

By use of the mean value theorem, the following holds:

$$\partial^{\alpha} H\left(s, x, u, w_1\right) = \partial^{\alpha} H\left(s, x, u, w_2\right) + \partial^{\alpha+1} H\left(s, x, u, w_c\right)\left(w_1 - w_2\right)$$

where w_c lies on the line between w_1 and w_2. By using this, and (6.12), we get

$$\begin{aligned}
&\leq \sum_{|\alpha|\leq k} \Big| \mathcal{P}^1_{\alpha,\beta}\left(u, w_1\right) \partial^{\alpha} H\left(s, x, u, w_2\right) - \mathcal{P}^1_{\alpha,\beta}\left(u, w_2\right) \partial^{\alpha} H\left(s, x, u, w_2\right) \\
&\quad + \mathcal{P}^1_{\alpha,\beta}\left(u, w_1\right) \partial^{\alpha+1} H\left(s, x, u, w_c\right)\left(w_1 - w_2\right) \Big| \\
&\leq \sum_{|\alpha|\leq k} C_R \big| \mathcal{P}^1_{\alpha,\beta}\left(u, w_1\right) - \mathcal{P}^1_{\alpha,\beta}\left(u, w_2\right) \big| + \mathcal{P}^{1,\max}_{\alpha,\beta} C_R N \|w_1 - w_2\|_{\infty} \\
&\overset{(6.12)}{\leq} \sum_{|\alpha|\leq k} C_R \frac{k^2}{R} \mathcal{P}^{1,\max}_{\alpha,\beta} \|w_1 - w_2\|_{F_{k+1}} + N \mathcal{P}^{1,\max}_{\alpha,\beta} C_R \|w_1 - w_2\|_{F_{k+1}} \\
&= \left(\frac{k^2}{R} + N\right) \|w_1 - w_2\|_{F_{k+1}} \sum_{|\alpha|\leq k} C_R \mathcal{P}^{1,\max}_{\alpha,\beta} \\
&\overset{\text{Lemma 6.2}}{=} \left(\frac{k^2}{R} + N\right) c\left(N, k, R\right) C_R \|w_1 - w_2\|_{F_{k+1}}
\end{aligned}$$

This concludes the proof. □

We continue by looking at the space E_{k+1}.

Lemma 6.5. *Let $w_1, w_2 \in E_{k+1}\left(T_s, T_e\right)$ so that $\|w_1\|_{E_{k+1}}, \|w_2\|_{E_{k+1}} \leq R$, and let $\|\nabla u\|_{F_k} \leq R$. Then it holds for $|\beta| = k$ that:*

(6.15)
$$\left|\partial^{\beta} H\left(s, x, u, w_1\right) - \partial^{\beta} H\left(s, x, u, w_2\right)\right| \leq \left(\frac{k^2}{R} + N\right)\left(c_1 + s^{-1/\lambda} c_2\right) \|w_1 - w_2\|_{E_{k+1}}$$

where c_1 and c_2 are the constant from (6.10)

Proof. Let $w_1, w_2 \in E_k$ such that $\|w_1\|_{E_k}, \|w_2\|_{E_k} \leq R$. For $|\beta| = k$, we can use the representation (6.6) from Lemma 6.1 to obtain:

$$
\begin{aligned}
&\left|\partial^\beta H(s,x,u,w_1) - \partial^\beta H(s,x,u,w_2)\right| \\
&\overset{(6.6)}{\leq} \sum_{|\alpha|\leq k} \left| \mathcal{P}^2_{\alpha,\beta}(u,w_1)\,\partial^\alpha H(s,x,u,w_1) - \mathcal{P}^2_{\alpha,\beta}(u,w_2)\,\partial^\alpha H(s,x,u,w_2) \right| \\
&\quad + \sum_{i=1}^N \left| \partial^\beta (w_1)_i \frac{\partial H}{\partial p_i}(s,x,u,w_1) - \partial^\beta (w_2)_i \frac{\partial H}{\partial p_i}(s,x,u,w_2) \right| \\
&\leq \left(\frac{k^2}{R} + N\right) c_1 \|w_1 - w_2\|_{E_{k+1}} \\
&\quad + s^{-1/\lambda} \sum_{i=1}^N \left| s^{1/\lambda}\partial^\beta (w_1)_i \frac{\partial H}{\partial p_i}(s,x,u,w_1) - s^{1/\lambda}\partial^\beta (w_2)_i \frac{\partial H}{\partial p_i}(s,x,u,w_2) \right| \\
&\leq \left(\frac{k^2}{R} + N\right) c_1 \|w_1 - w_2\|_{E_{k+1}} + \left(\frac{k^2}{R} + N\right) s^{-1/\lambda} c_2 \|w_1 - w_2\|_{E_{k+1}} \\
&= \left(\frac{k^2}{R} + N\right) \left(c_1 + s^{-1/\lambda} c_2\right) \|w_1 - w_2\|_{E_{k+1}}
\end{aligned}
$$

where the two last inequalities follows from the exact same techniques as in the proof of Lemma 6.4. The constants c_1, c_2 are the same as in (6.10). □

Chapter 7

Concluding remarks

We have shown that a fractional Mean Field Games system with non-local coupling (system (3.1) admits a unique classical solution under certain assumptions on the initial conditions, the Hamiltonian H, and on F and G. In the process, we needed to prove statements for the fractional Laplacian on the torus, define weak solutions for the fractional Fokker-Planck equation, and to use regularity results for the fractional Hamilton-Jacobi equation and the fractional Fokker-Planck equation.

A large part of the thesis deals with regularity theory for the fractional Hamilton-Jacobi equation, with a Hamiltonian of a quite general form. We show that this equation admits bounded classical solutions, under suitable assumptions. We also show higher-order regularity in time and space for the fractional Hamilton-Jacobi equation. These estimates are necessary for proving existence of solutions for the fractional MFG system.

The only problem with our results, is that we need to make pretty strong assumptions on the differentiability of H,F,G and m_0 in the Mean Field Game system. This is due to our regularity estimates for the fractional Hamilton-Jacobi equation. Hopefully, we can find a way to lessen the assumptions.

There are some things that can be further improved upon, and we list them as follows:

- Check whether it is possible to lessen the assumptions on the Hamiltonian H and the initial conditions u_0, when dealing with the fractional Hamilton-Jacobi equation.
- Use another way of showing regularity for the fractional Fokker-Planck equation, that demands less of H and the solution u from the fractional Hamilton-Jacobi equation.
- Show existence and uniqueness for the same MFG-system, but with local coupling. For doing this, we need some estimates for weak solutions on the fractional Fokker-Planck equation on divergence form, which we don't have by now.

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Appendix

Proof of Lemma 2.3

In this appendix, we will prove Lemma 2.3 stated in the Preliminaries. It is stated like follows.

Lemma .1. *Let $f, g \in C^2\left(\mathbb{T}^d\right)$. Then the following identity holds, for $\alpha \in (1,2)$.*

$$\int_{\mathbb{T}^d} \left(-\Delta_{\mathbb{T}^d}\right)^{\alpha/2} f\left(x\right) g\left(x\right) = \int_{\mathbb{T}^d} f\left(x\right)\left(-\Delta_{\mathbb{T}^d}\right)^{\alpha/2} g\left(x\right) dx$$

Proof. The proof follows by a density argument.

We associate f and g with their periodic extensions, that is, $f, g \in C^2\left(\mathbb{R}^d\right)$ with

$$f\left(x+z\right) = f\left(x\right),\ g\left(x+z\right) = g\left(x\right) \quad \forall x \in \mathbb{R}^d,\ z \in \mathbb{Z}^d$$

From the compactness of the torus, we get by interpolation, that

$$\|f\|_{L^2\left(\mathbb{T}^d\right)} \leq C\|f\|_{L^\infty\left(\mathbb{T}^d\right)}$$

Let $\phi \in C_c^\infty\left(\mathbb{R}^d\right)$ be a positive mollifier, let $\phi_\epsilon = \epsilon^{-d}\phi\left(x/\epsilon\right)$, and define

$$f_\epsilon := \left(f * \phi_\epsilon\right)\left(x\right)$$
$$g_\epsilon := \left(g * \phi_\epsilon\right)\left(x\right)$$

An important property of the positive mollifier, is that, for any function $u \in L^2\left(\mathbb{R}^d\right)$, we have

$$\|u - u * \phi_\epsilon\|_{L^2\left(\mathbb{R}^d\right)} \xrightarrow{\epsilon \to 0} 0$$

A proof for this can be found in Rudin.

A consequence of this is that, for any function $u \in L^2\left(\mathbb{T}^d\right)$, we get that

$$\|u - u * \phi_\epsilon\|_{L^2\left(\mathbb{T}^d\right)} \xrightarrow{\epsilon \to 0} 0 \tag{1}$$

This statement is not obvious as the spaces $L^2\left(\mathbb{R}^d\right)$ and $L^2\left(\mathbb{T}^d\right)$ does not share the same set of functions at all (I think $f = 0$ would be the only common function). However, since the function ϕ has compact support, say *supp* $\phi \subset B_R$ for some $R > 0$, and x is confined to $\mathbb{T}^d$, the convolution

$$(u * \phi)(x) = \int_{\mathbb{R}^d} u(x-y)\,\phi(y)\,dy$$
$$= \int_{B_R\left(\mathbb{T}^d\right)} u(x-y)\,\phi(y)\,dy$$

is not over the whole of $\mathbb{R}^d$, so that we probably won't have any trouble with claiming (1) to be true.

Now, the functions f_ϵ and g_ϵ are periodic, due to the periodicity of f and g. To show this, pick $x \in \mathbb{R}^d, z \in \mathbb{Z}^d$:

$$f_\epsilon(x+z) = \int_{\mathbb{R}^d} f(x+z-y)\,\phi_\epsilon(y)\,dy$$
$$= \int_{\mathbb{R}^d} f(x-y)\,\phi_\epsilon(y)\,dy = f_\epsilon(x)$$

The same holds for g. This shows that $f_\epsilon, g_\epsilon \in C^\infty\left(\mathbb{T}^d\right)$, since $\phi \in C_c^\infty\left(\mathbb{R}^d\right)$. By Youngs inequality we obtain the following bounds, and using the rules for differentiation of a convolution, we get

$$\|f_\epsilon\|_{L^\infty\left(\mathbb{T}^d\right)} = \|f * \phi_\epsilon\|_{L^\infty\left(\mathbb{T}^d\right)} \leq \|f\|_{L^\infty\left(\mathbb{T}^d\right)} \|\phi_\epsilon\|_{L^1\mathbb{R}^d} = \|f\|_{L^\infty\left(\mathbb{T}^d\right)}$$

$$\|\left(-\Delta\right)^{\sigma/2} f_\epsilon\|_{L^\infty(\mathbb{T}^n)} \overset{(2.6)}{\leq} C\|f_\epsilon\|_{C^2\left(\mathbb{T}^d\right)} \leq C\|f\|_{C^2\left(\mathbb{R}^d\right)}$$

Since the torus is compact, we get the following interpolation for any $u \in L^2\left(\mathbb{T}^d\right)$:

$$\|u\|^2_{L^2\left(\mathbb{T}^d\right)} = \int_{\mathbb{T}^d} |u|^2 dx \leq |\mathbb{T}^d| \|u\|^2_{L^\infty\left(\mathbb{T}^d\right)} = C\|u\|^2_{L^\infty\left(\mathbb{T}^d\right)}$$
$$\implies \|u\|_{L^2\left(\mathbb{T}^d\right)} \leq C\|u\|_{L^\infty\left(\mathbb{T}^d\right)}$$

This implies that

$$\|f_\epsilon\|_{L^2\left(\mathbb{T}^d\right)} \leq C\|f\|_{L^\infty\left(\mathbb{R}^d\right)}$$

$$\|\left(-\Delta\right)^{\sigma/2} f_\epsilon\|_{L^2(\mathbb{T}^n)} \leq C\|f\|_{C^2\left(\mathbb{R}^d\right)}$$

and the same result for g_ϵ.

Since $f_\epsilon, g_\epsilon \in C^\infty\left(\mathbb{T}^d\right)$, we obtain from Lemma 2.2 that

$$\int_{\mathbb{T}^d} (-\Delta)^{\alpha/2} f_\epsilon(x)\, g_\epsilon(x) = \int_{\mathbb{T}^d} f_\epsilon(x)\, (-\Delta)^{\alpha/2} g_\epsilon(x)\, dx \tag{2}$$

To conclude the proof, we need to show that

$$\int_{\mathbb{T}^d} (-\Delta)^{\alpha/2} f_\epsilon(x)\, g_\epsilon(x)\, dx \xrightarrow{\epsilon\to 0} \int_{\mathbb{T}^d} (-\Delta)^{\alpha/2} f(x)\, g(x)\, dx$$

and

$$\int_{\mathbb{T}^d} f_\epsilon(x)\, (-\Delta)^{\alpha/2} g_\epsilon(x)\, dx \xrightarrow{\epsilon\to 0} \int_{\mathbb{T}^d} f(x)\, (-\Delta)^{\alpha/2} g(x)\, dx$$

It suffices to show the first one of them, since the proof will be the same.

Note that the interpolation results earlier gives

$$\begin{aligned} |\, (-\Delta)^{\alpha/2} f_\epsilon(x)\, g_\epsilon(x)\, | &\le \|\, (-\Delta)^{\alpha/2} f_\epsilon\|_{L^\infty\left(\mathbb{T}^d\right)} \|g_\epsilon\|_{L^\infty\left(\mathbb{T}^d\right)} \\ &\le \|f\|_{C^2\left(\mathbb{T}^d\right)} \|g\|_{L^\infty\left(\mathbb{T}^d\right)}. \end{aligned}$$

This allows us to use the dominated convergence theorem, exchanging the integral sign and the limit. We then obtain

$$\begin{aligned}
&\left| \int_{\mathbb{T}^d} (-\Delta)^{\alpha/2} fg\,dx - \lim_{\epsilon\to 0} \int_{\mathbb{T}^d} (-\Delta)^{\alpha/2} f_\epsilon g_\epsilon\,dx \right| \\
&= \left| \int_{\mathbb{T}^d} (-\Delta)^{\alpha/2} fg\,dx - \int_{\mathbb{T}^d} \lim_{\epsilon\to 0} \left[(-\Delta)^{\alpha/2} f_\epsilon g_\epsilon \right] dx \right| \\
&\le \lim_{\epsilon\to 0} \int_{\mathbb{T}^d} \left[\left|\, (-\Delta)^{\alpha/2} fg - (-\Delta)^{\alpha/2} f_\epsilon g_\epsilon \right| \right] dx \\
&\le \int_{\mathbb{T}^d} \lim_{\epsilon\to 0} \left[\left|\, (-\Delta)^{\sigma/2} fg - (-\Delta)^{\sigma/2} f_\epsilon g \right| \right. \\
&\quad \left. + \left|\, (-\Delta)^{\sigma/2} f_\epsilon g - (-\Delta)^{\sigma/2} f_\epsilon g_\epsilon \right| \right] dx \\
&\quad \le \lim_{\epsilon\to 0} \int_{\mathbb{T}^d} \left[\left|\, (g) \left((-\Delta)^{\sigma/2} f - (-\Delta)^{\sigma/2} f_\epsilon \right) \right| \right. \\
&\quad \left. + \left| \left((-\Delta)^{\sigma/2} f_\epsilon \right) (g - g_\epsilon) \right| \right] dx \\
&\quad \overset{\text{Hölder}}{\le} \lim_{\epsilon\to 0} \left(\|g\|_{L^2\left(\mathbb{T}^d\right)} \|\, (-\Delta)^{\sigma/2} f - (-\Delta)^{\sigma/2} (f * \phi_\epsilon)\, \|_{L^2\left(\mathbb{T}^d\right)} \right. \\
&\quad \left. + \|\, (-\Delta)^{\sigma/2} f_\epsilon\|_{L^2\left(\mathbb{T}^d\right)} \|g - g * \phi_\epsilon\|_{L^2\left(\mathbb{T}^d\right)} \right) \\
&\quad \overset{(*)}{=} \lim_{\epsilon\to 0} \left(\|g\|_{L^2\left(\mathbb{T}^d\right)} \|\, (-\Delta)^{\sigma/2} f - \left((-\Delta)^{\sigma/2} f \right) * \phi_\epsilon\|_{L^2\left(\mathbb{T}^d\right)} \right. \\
&\quad \left. + \|\, (-\Delta)^{\sigma/2} f_\epsilon\|_{L^2\left(\mathbb{T}^d\right)} \|g - g * \phi_\epsilon\|_{L^2\left(\mathbb{T}^d\right)} \right) \overset{(1)}{=} 0
\end{aligned}$$

This holds from that v and $(-\Delta)^{\sigma/2} u_\epsilon$ are uniformly bounded, independent of ϵ, and the convergence properties of ϕ_ϵ stated in (1). Thus, we get

$$\int_{\mathbb{T}^d} (-\Delta)^{\sigma/2} f(x)\, g(x)\, dx = \lim_{\epsilon\to 0} \int_{\mathbb{T}^d} (-\Delta)^{\sigma/2} f_\epsilon(x)\, g_\epsilon(x)\, dx$$
$$= \lim_{\epsilon\to 0} \int_{\mathbb{T}^d} f_\epsilon(x) (-\Delta)^{\sigma/2} g_\epsilon(x)\, dx = \int_{\mathbb{T}^d} f(x) (-\Delta)^{\sigma/2} g(x)\, dx$$

which is what we wanted to prove.

The last thing we need to do is to prove (∗), that

$$(-\Delta)^{\alpha/2} (u * \phi)(x) = \left((-\Delta)^{\alpha/2} u\right) * \phi_\epsilon(x)$$

We compute:

$$(-\Delta)^{\alpha/2} (u * \phi_\epsilon)(x) = \int_{\mathbb{R}^d} \frac{u * \phi_\epsilon(x+z) - u * \phi_\epsilon(x) - \nabla(u * \phi_\epsilon(x)) \cdot z}{|z|^{d+\alpha}} dz$$
$$= \int_{\mathbb{R}} \frac{1}{|z|^{d+\alpha}} \left(\int_{\mathbb{R}^d} u(x-y+z)\,\phi_\epsilon(y) - u(x-y)\,\phi_\epsilon(y) - \nabla u(x-y)\,\phi_\epsilon(y) \cdot z dy \right) dz$$
$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|z|^{d+\alpha}} \phi_\epsilon(y) \left[u(x-y+z) - u(x-y) - \nabla u(x-y) \cdot z \right] dy dz$$

If we are allowed to change the order of integration, we obtain

$$= \int_{\mathbb{R}^d} \phi_\epsilon(y) \int_{\mathbb{R}^d} \frac{1}{|z|^{d+\alpha}} \left[u(x-y+z) - u(x-y) - \nabla u(x-y) \cdot z \right] dz dy$$
$$= \int_{\mathbb{R}^d} \phi_\epsilon(y) (-\Delta)^{\alpha/2} u(x-y)\, dy = \left((-\Delta)^{\alpha/2} u\right) * \phi_\epsilon(x)$$

which is the statement we wanted to show.

To prove that we can change the order of integration, we use the Fubini-Tonelli theorem. First, by Taylor expansion, we have that

$$u(x-y+z) - u(x-y) - \nabla(u(x)) \cdot z \leq |z|^2 \|D^2 u\|_{L^\infty(\mathbb{R}^d)}$$

We also have the estimate:

$$u(x-y+z) - u(x-y) - \nabla(u(x)) \cdot z \leq 2\|u\|_{L^\infty(\mathbb{R}^d)} + |z| \|Du\|_{L^\infty(\mathbb{R}^d)}$$

By splitting the integral over $\mathbb{R}^d$ into a integral over B_1 and $\mathbb{R}^d \setminus B_1$, we use these estimates to obtain

$$
\begin{aligned}
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\frac{1}{|z|^{d+\alpha}}\phi(y)\left[u(x-y+z)-u(x-y)-\nabla u(x-y)\cdot z\right]\right|dydz \\
&\leq \int_{\mathbb{R}^d}\frac{1}{|z|^{d+\alpha}}\int_{\mathbb{R}^d}|\phi(y)|\left|u(x-y+z)-u(x-y)-\nabla u(x-y)\cdot z\right|dydz \\
&\leq \|D^2u\|_{L^\infty(\mathbb{R}^d)}\int_{B_1}\frac{|z|^2}{|z|^{d+\alpha}}\|\phi_\epsilon\|_{L^1(\mathbb{R}^d)}dz + 2\|u\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d\setminus B_1}\frac{1}{|z|^{d+\alpha}}\|\phi_\epsilon\|_{L^1(\mathbb{R}^d)}dz \\
&\quad + \|Du\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d\setminus B_1}\frac{|z|}{|z|^{d+\alpha}}\|\phi_\epsilon\|_{L^1(\mathbb{R}^d)}dz \\
&\leq C\left(\int_0^1\frac{1}{|\rho|^{\alpha-1}}d\rho+\int_1^\infty\frac{1}{|\rho|^{\alpha+1}}d\rho+\int_1^\infty\frac{1}{|\rho|^{\alpha}}d\rho\right)<\infty
\end{aligned}
$$

where we have used polar coordinates, $|z|=|\rho|$, $dz=\rho^{d-1}d\rho$. Thus, by the Fubini-Tonelli theorem, the order of integration can be changed. □