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# Solitary waves for dispersive equations with inhomogeneous nonlinearities 

## Ola Isaac Høgåsen Mæhlen

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# Solitary waves for dispersive equations with inhomogeneous nonlinearities 

Ola Mæhlen
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## Abstract

We present an original proof for solitary-wave solutions of the PDE

$$
-\nu u+L u-n(u)=0
$$

by the means of variational calculus and functional analysis. Here $L$ is a Fourier multiplier with a symbol of positive order, and $n$ a nonlinear function. The proof is constructed by the author with inspiration from [2] and [7].

## Sammendrag

We gir et originalt bevis på soliton-bølge løsninger av følgende PDE

$$
-\nu u+L u-n(u)=0
$$

ved hjelp av variasjonskalkyl og funksjonell analyse. Her er $L$ en Fourier multiplikator av positiv orden, og $n$ en ikke-lineær function. Beviset er konstruert av forfatteren med inspirasjon hentet fra [2] og [7].

## Preface

This master thesis is written over the course of five months, and marks the end of my enrollment in the 5-year integrated master's programme, "Applied Physics and Mathematics" at the Norwegian University of Science and Technology. I thank my supervisor, Prof. Mats Ehrnström, for mathematical guidance and (just as important) his moral support. Moreover, I thank PhD. cand. Mathias Nikolai Arnesen for insightful discussions.

## Notation

The notation introduced here will be used extensively throughout the paper.

| $\mathbb{N}_{0}$ | We write $\mathbb{N}_{0}$ for the set of non-negative integers. |
| :---: | :---: |
| $L^{p}$ | For $1 \leq p \leq \infty$, we denote $L^{p}$ for the space of measurable functions $f: \mathbb{R} \rightarrow \mathbb{C}$, with finite $L^{p}$-norm: $\\|f\\|_{L^{p}}<\infty$. |
| $C^{k}, C^{\infty}$ | For $k=0,1, \ldots, C^{k}$ is the set of functions $f: \mathbb{R} \rightarrow \mathbb{C}$, that are $k$-times continuously differentiable, and $C^{\infty}=\cap_{k=0}^{\infty} C^{k}$. |
| $\mathscr{S}, \mathscr{S}^{\prime}$ | $\mathscr{S}$ is the Schwartz space, and $\mathscr{S}^{\prime}$ is the set of tempered distributions (see definition 2.6). |
| $V, V^{\infty}$ | $V$ is the set of tempered functions, and $V^{\infty}$ is the set of functions $g$ so that $g^{(k)} \in V$ for $k=0,1, \ldots$ (see definition 2.11). |
| $C^{k, \alpha}$ | The Hölder space (see definition 2.20). |
| $H^{s}, H^{s}(\mathbb{R})$ | $H^{s}$ is the Sobolev space of order $s$, and $H^{s}(\mathbb{R})$ is the set of real valued functions in $H^{s}$ (see definition 2.22). |
| $\lesssim, \gtrsim$ | For two functions $f, g: X \rightarrow \mathbb{R}$, we write $f \lesssim g$ if there exists a constant $C>0$ so that $f(x) \leq C g(x)$ for all $x \in X$. Similarly, we write $f \gtrsim g$ if $g \lesssim f$. |
| $\simeq$ | We write $f \simeq g$ if $f \lesssim g \lesssim f$. |
| $\hat{f}, \check{f}$ | For a function $f: \mathbb{R} \rightarrow \mathbb{R}$, we let $\hat{f}$ denote its Fourier transform and $\check{f}$ its inverse Fourier transform. |
| $\langle\cdot\rangle$ | For a real number $\xi \in \mathbb{R}$, we define $\langle\xi\rangle=\sqrt{1+\xi^{2}}$. |

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## 1 Introduction

Our goal is to find soliton-wave solutions $u(x, t)$, to the PDE

$$
\begin{equation*}
u_{t}+(L u-n(u))_{x}=0 \tag{1}
\end{equation*}
$$

A coarse description of the terms $L$ and $n$ will now be given; for a detailed description, see section 3 .

- $L$ is a Fourier multiplier with symbol $m(\xi)$; that is to say $\mathcal{F}\{L u\}(\xi)=m(\xi) \hat{u}(\xi)$ for a suitable function $u$. In addition, $m$ is continuous and of order $2 s$, that is, $m(\xi)$ grows like $|\xi|^{2 s}$.
- The term $n$ is the sum of the two continuous functions $n_{p}$ and $n_{r}$. Here $n_{p}$ takes either the form $n_{p}(x)=c_{p}|x|^{p}$ with $c_{p} \neq 0$ or $n_{p}(x)=c_{p} x|x|^{p-1}$ with $c_{p}>0$, while $n_{r}$ satisfies $n_{r}(x)=\mathcal{O}\left(|x|^{p+\delta}\right)$ for some $\delta>0$.

The constants $s$ and $p$ must satisfy $p>1$ and $2 s>\max \{1,(p-1) / 2\}$. We look for solutions of (1) of the form

$$
u(x, t)=\tilde{u}(x-\nu t)
$$

with $\tilde{u}(\lambda) \rightarrow 0$ as $|\lambda| \rightarrow \infty$; so called solitary-wave solutions. We will refer to $\nu$ in (1) as the velocity of $u$. Albeit a little abuse of notation, we will for simplicity not distinguish between $u$ and $\tilde{u}$; a notational convenience. If we insert for $u$ in (1) and perform an indefinite integral we obtain

$$
\begin{equation*}
-\nu u+L u-n(u)=0 \tag{2}
\end{equation*}
$$

where we have set the integrating factor to zero, in light of the assumption $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Note that (2) is a weaker form of (1); any continuously differentiable solution of (2) must also solve (1). We shall focus on proving existence of solitary wave solutions of the latter PDE; this is Theorem 1.1. We will find sufficient criteria on $L$ and $n$ for solutions to be continuously differentiable (Theorem 9.4); consequently also solving (1).

Similar problems have been studied previously. In work done by Arnesen [2] and Albert [1], existence of solitary-wave solutions of (2) has been proved, but the arguments requires nonlinearities of the form $n=n_{p}$. On the other hand, arguments presented in [7] proves existence of solitary wave solutions with nonlinearities of the form $n=n_{p}+n_{r}$, but requires negative order $\left(\mathrm{sj}_{\mathrm{j}} 0\right)$ of the symbol $m$.

In this paper we present an original proof by the means of variational calculus and functional analysis, inspired by [2] and [7]. Although many of the proofs are inspired by said sources, they are often simplified by the author or approached differently. Several results are also original; in particular section 2, 4.2, 7.1, 7.3, 8 and 9 , consist of mostly original work. We will devote most of this paper to prove the following theorem.

Theorem 1.1 (Existence). There exist $\mu_{*}>0, \rho>0$, and for every $\varepsilon>0 a$ constant $C_{\varepsilon}>0$, so that the following statement holds true:
For every $\mu \in\left(0, \mu_{*}\right)$ there is a continuous function $u \in H^{s}(\mathbb{R})$ and a constant $\nu \in \mathbb{R}$ satisfying
(i) $\|u\|_{L^{2}}^{2}=2 \mu$,
(ii) $u$ is a solution of (2), with velocity $\nu$,
(iii) $\nu$ satisfies

$$
\rho \mu^{\beta}<m(0)-\nu<C_{\varepsilon} \mu^{\beta-\varepsilon}
$$

where $\beta$ is given by

$$
\beta=\frac{2 s(p-1)}{4 s-(p-1)}
$$

## 2 Preliminaries

We here introduce relevant language and results. Every proof is original (constructed by the author), unless otherwise stated.

### 2.1 Some results on functional analysis

Some familiarity with general topology and Fourier theory is assumed. In the following definitions and results we let $X$ be a normed vector space (normed space for short) over $\mathbb{C}$ and $X^{\prime}$ its continuous dual.

Definition 2.1. We define the weak-* topology of $X^{\prime}$ as the one generated by the sub-basis

$$
U_{V, x}=\left\{x^{\prime} \in X^{\prime}: x^{\prime}(x) \in V\right\}
$$

for every $x \in X$ and open set $V \subset \mathbb{C}$.
By this topology, a net $\left(x_{\alpha}^{\prime}\right)$ will converge to $x^{\prime}$ exactly when $x_{\alpha}^{\prime}(x) \rightarrow x^{\prime}(x)$ for every $x \in X$. The definition of this topology is motivated by the BanachAlaoglu theorem. For convenience we state a special version of this theorem, whose proof follows by combining theorem 3.1. and 5.1. from [3]. We remind the reader that a separable topological space is one with a dense countable subset; in particular $L^{2}$ is separable as it can be given a countable orthogonal basis (see [8]).

Theorem 2.2 (Sequential Banach-Alaoglu theorem). If $X$ is a normed space, then the closed unit ball of $X^{\prime}$ is sequentially compact with respect to the weak-* topology, if and only if $X$ is separable.

It is easily seen that both translation and scaling on $X^{\prime}$ behaves like homeomorphisms in the weak-* topology, thus the preceding theorem holds for any closed ball in $X^{\prime}$. There is a similar concept of weak convergence in a Banach space $X$; a sequence $\left(x_{n}\right)$ converges weakly to $x$, denoted $x_{i} \rightharpoonup x$ if and only if $x^{\prime}\left(x_{n}\right) \rightarrow x^{\prime}(x)$ for every $x^{\prime} \in X^{\prime}$; in reflexive spaces, weak convergence coincides with weak-* convergence, and so we get the following corollary.

Corollary 2.3. Every bounded sequence $\left(x_{n}\right)$ in a reflexive separable Banach space $X$ has a subsequence, again denoted by $\left(x_{n}\right)$ which converges weakly in $X$.

We now show some properties of weak limits. The following proofs are constructed by the author; they are however very standard calculations.

Proposition 2.4. If a sequence $\left(x_{n}\right)$ converges weakly to $x$ in a Banach space $X$. Then

$$
\|x\|_{X} \leq \liminf _{n \rightarrow \infty}\left\|x_{n}\right\|_{X}
$$

Proof. By the Hahn-Banach theorem, there exist $x^{\prime} \in X^{\prime}$ with $\left\|x^{\prime}\right\|_{X^{\prime}}=1$ and $x^{\prime}(x)=\|x\|_{X}$. We then get

$$
\begin{aligned}
\|x\|_{X} & =x^{\prime}(x) \\
& =\liminf _{n \rightarrow \infty} x^{\prime}\left(x_{n}\right) \\
& \leq \liminf _{n \rightarrow \infty}\left\|x^{\prime}\right\|_{X^{\prime}}\left\|x_{n}\right\|_{X} \\
& =\liminf _{n \rightarrow \infty}\left\|x_{n}\right\|_{X} .
\end{aligned}
$$

Proposition 2.5. Let $X, Y$ be Banach spaces and $T: X \rightarrow Y$ a bounded linear map. Suppose also $\left(x_{n}\right)$ is a sequence in $X$. If $x_{n} \rightharpoonup x$ in $X$, then $T x_{n} \rightharpoonup T x$ in $Y$.

Proof. Pick $y^{\prime} \in Y^{\prime}$. As $T$ is a linear mapping, there exist $x^{\prime} \in X^{\prime}$ so that $x^{\prime}=y^{\prime} \circ T$. Consequently,

$$
y^{\prime}\left(T x_{n}\right)=x^{\prime}\left(x_{n}\right) \rightarrow x^{\prime}(x)=y^{\prime}\left(T_{x}\right)
$$

as $n \rightarrow \infty$.

### 2.2 The Schwartz space $\mathscr{S}$ and the tempered distributions $\mathscr{S}^{\prime}$

We start by introducing the Schwartz space; a natural space to work with when generalizing the Fourier transform.

Definition 2.6. The Schwartz space $\mathscr{S}$ is the topological vector space of infinitely differentiable functions $\varphi: \mathbb{R} \rightarrow \mathbb{C}$ satisfying

$$
\|\varphi\|_{n, k}:=\sup _{x \in \mathbb{R}}\left|x^{n} \varphi^{(k)}(x)\right|<\infty
$$

for all $n, k \in \mathbb{N}_{0}$. We equip $\mathscr{S}$ with the topology generated by the family of semi-norms $\|\cdot\|_{n, k}$. Thus a sequence $\left(\varphi_{m}\right) \subset \mathscr{S}$ converges to $\varphi \in \mathscr{S}$ (denoted $\varphi_{m} \rightarrow \varphi$ ) exactly when $\left\|\varphi_{m}-\varphi\right\|_{n, k} \rightarrow 0$ for all $n, k \in \mathbb{N}_{0}$.

In the previous definition we gave both a characterization of the open sets in $\mathscr{S}$ and a characterization of convergent sequences. Although the latter can be constructed from the first, it need not hold the other way around for general topological spaces. This is however true for first countable spaces; a property $\mathscr{S}$ possesses by the following remark.

Remark 2.7. It can be shown that $\mathscr{S}$ is a Fréchet space (see [10]) hence also first countable. This implies that a function from $\mathscr{S}$ is continuous if and only if it respects limits of sequences.

We give a sufficient criterion for a continuous operator on $\mathscr{S}$.
Proposition 2.8. Let $L$ be a linear operator $L: \mathscr{S} \mapsto \mathscr{S}$. Suppose for each pair of numbers $(n, k) \in \mathbb{N}_{0} \times \mathbb{N}_{0}$ there are a finite pair of numbers, $\left(n_{i}, k_{i}\right)$ for $i=1, \ldots, N$ (where the value of $N$ might depend on $(n, k)$ ), so that

$$
\begin{equation*}
\|L \varphi\|_{n, k} \lesssim \sum_{i=1}^{N}\|\varphi\|_{n_{i}, k_{i}} \tag{3}
\end{equation*}
$$

for every $\varphi \in \mathscr{S}$. Then $L$ is continuous.
Proof. Pick a pair $(n, k) \in \mathbb{N}_{0} \times \mathbb{N}_{0}$ and a corresponding finite pair of numbers satisfying (3). Pick a sequence $\left(\varphi_{n}\right) \subset \mathscr{S}$ converging to $\varphi \in \mathscr{S}$. A straight forward calculation shows

$$
\left\|L \varphi_{n}-L \varphi\right\|_{n, k}=\left\|L\left(\varphi_{n}-\varphi\right)\right\|_{n, k} \lesssim \sum_{i=1}^{N}\left\|\varphi_{n}-\varphi\right\|_{n_{i}, k_{i}} \rightarrow 0
$$

as $n \rightarrow \infty$. Thus $L$ is continuous.
As a first application of the above proposition we prove that differentiation and multiplying by monomials are continuous operations on $\mathscr{S}$.
Corollary 2.9. For $m \in N_{0}$, the operations $\varphi \mapsto(\cdot)^{m} \varphi$ and $\varphi \mapsto \varphi^{(m)}$ are continuous on $\mathscr{S}$.

Proof. A straight forward calculation shows

$$
\left\|(\cdot)^{m} \varphi\right\|_{n, k}=\|\varphi\|_{n, k+m} \quad \text { and } \quad\left\|\varphi^{(m)}\right\|_{n, k}=\|\varphi\|_{n+m, k}
$$

and so the result follows from proposition 2.8.

One of the main reasons the Schwartz space is so useful in Fourier theory, is the fact that $\mathcal{F}$ maps the Schwartz space continuously into itself, as established by the next proposition.

Proposition 2.10. The Fourier transform $\mathcal{F}$ is a linear homeomorphism from $\mathscr{S}$ to itself.
Proof. Pick $\varphi \in \mathscr{S}$, then by elementary properties of the Fourier transform we get

$$
\begin{aligned}
\|\hat{\varphi}\|_{n, k} & =\sup _{\xi \in \mathbb{R}}\left|\xi^{n} \hat{\varphi}^{(k)}(\xi)\right| \\
& =\sup _{\xi \in \mathbb{R}}\left|\widehat{x^{k} \varphi^{(n)}}(\xi)\right| \\
& =\sup _{\xi \in \mathbb{R}} \frac{1}{\sqrt{2 \pi}}\left|\int_{\mathbb{R}} x^{k} \varphi^{(n)} e^{-i \xi x} d x\right| \\
& \leq \frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}}\left|x^{k} \varphi^{(n)}\right| d x \\
& =\frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}} \frac{|x|^{k}}{1+|x|^{k+2}}\left[\left|\varphi^{(n)}\right|+\left|x^{k+2} \varphi^{(n)}\right|\right] d x \\
& \leq \frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}} \frac{|x|^{k}}{1+|x|^{k+2}}\left[\|\varphi\|_{0, n}+\|\varphi\|_{k+2, n}\right] d x \\
& \lesssim\|\varphi\|_{0, n}+\|\varphi\|_{k+2, n} .
\end{aligned}
$$

From this calculation we see that $\mathcal{F}$ maps $\mathscr{S}$ to itself; continuity follows from proposition 2.8. That $\mathcal{F}$ is invertible follows from the fact that $\mathscr{S} \subset L^{2}$ and the Fourier inversion theorem. The continuity of $\mathcal{F}^{-1}$ can be seen by a similar calculation as above, or the fact that $\mathcal{F}^{3}=\mathcal{F}^{-1}$.

We wish to study another class of continuous operators on $\mathscr{S}$; for this we introduce the tempered functions. We note that the definition here might differ from other sources.

Definition 2.11. We say a function $g: \mathbb{R} \rightarrow \mathbb{C}$ is tempered if there exist $N \in \mathbb{N}_{0}$ so that

$$
\frac{g(\cdot)}{1+|\cdot|^{N}} \in L^{1}
$$

We denote the set of tempered functions by $V$. We also define the subset $V^{\infty} \subset V$ of infinitely differentiable functions $g$ so that $g^{(k)}$ is tempered for each $k \in \mathbb{N}_{0}$.
Remark 2.12. Clearly any function in $L^{1}$ is tempered, and by multiplication with $\left(1+|\cdot|^{2}\right)^{-1}$ it is clear that any function in $L^{\infty}$ also is tempered. When $1<p<\infty$ and $g \in L^{p}$, we exploit Hölder's inequality to see that

$$
\int_{R} \frac{|g|}{1+|x|^{N}} d x \leq\left[\int_{\mathbb{R}} \frac{d x}{\left(1+|x|^{N}\right)^{\frac{p}{p-1}}}\right]^{1-\frac{1}{p}}\|g\|_{L^{p}}
$$

which clearly is less than infinity for $\frac{N p}{p-1}>1$. Thus for $1 \leq p \leq \infty, L^{p}$-functions are tempered.

We proved earlier that multiplication by monomials defined continuous operators on $\mathscr{S}$; we now prove a stronger result.

Proposition 2.13. If $g \in V^{\infty}$, then the operation $\varphi \mapsto g \varphi$ is continuous on $\mathscr{S}$.

Proof. As $g^{(k)}$ is tempered and continuous, there is a $N_{k} \in \mathbb{N}_{0}$ so that $g^{(k)} \lesssim$ $\left(1+|\cdot|^{N_{k}}\right)$. Consequently

$$
\begin{aligned}
\sup _{x \in \mathbb{R}}\left|x^{n} g^{(k)} \varphi^{(m)}\right| & \lesssim \sup _{x \in \mathbb{R}}\left|x^{n}\left(1+|x|^{N_{k}}\right) \varphi^{(m)}\right| \\
& \leq\|\varphi\|_{n, m}+\|\varphi\|_{n+N_{k}, m}
\end{aligned}
$$

We then calculate

$$
\begin{aligned}
\|g \varphi\|_{n, k} & =\sup _{x \in \mathbb{R}}\left|x^{n}\left(\frac{d}{d x}\right)^{k} g \varphi\right| \\
& =\sup _{x \in \mathbb{R}}\left|x^{n} \sum_{j=0}^{k}\binom{k}{j} g^{(j)} \varphi^{(k-j)}\right| \\
& \lesssim \sum_{j=0}^{k} \sup _{x \in \mathbb{R}}\left|x^{n} g^{(j)} \varphi^{(k-j)}\right| \\
& \lesssim \sum_{j=0}^{k}\|\varphi\|_{n, k-j}+\|\varphi\|_{n+N_{j}, k-j}
\end{aligned}
$$

where the last inequality follows from the previous calculation. The proof is now complete by proposition 2.8 .

In particular, every monomial $x \mapsto x^{n}$, for $n \in \mathbb{N}_{0}$, is an element of $V^{\infty}$. The following corollary

Definition 2.14. We denote the continuous dual of the Schwartz space by $\mathscr{S}^{\prime}$. This is the space of continuous linear functions $T: \mathscr{S} \rightarrow \mathbb{C}$; by remark 2.7 , a function $T$ is continuous if and only if

$$
\varphi_{m} \rightarrow \varphi \Longrightarrow T\left(\varphi_{m}\right) \rightarrow T(\varphi)
$$

for every convergent sequence $\left(\varphi_{m}\right) \subset \mathscr{S}$. We equip $\mathscr{S}^{\prime}$ with its corresponding weak-* topology: a net $\left(T_{i}\right) \subset \mathscr{S}^{\prime}$ converges to $T \in \mathscr{S}^{\prime}$ exactly when

$$
T_{i}(\varphi) \rightarrow T(\varphi)
$$

for all $\varphi \in \mathscr{S}^{\prime}$.

The set $\mathscr{S}^{\prime}$ will be referred to as the tempered distributions. We will refer to a tempered distribution $T$ as real valued, if and only if $T(\varphi) \in \mathbb{R}$, for every real valued $\varphi \in \mathbb{R}$. We now show that continuous operators on $\mathscr{S}$ naturally extends to $\mathscr{S}^{\prime}$.

Proposition 2.15. If $L: \mathscr{S} \rightarrow \mathscr{S}$ is a continuous linear operator, then the mapping $T \mapsto T \circ L$ must be a continuous linear mapping on $\mathscr{S}^{\prime}$.

Proof. Linearity is obvious. To prove that the mapping is continuous we pick a net $\left(T_{i}\right) \subset \mathscr{S}^{\prime}$ converging to $T$. For any $\varphi \in \mathscr{S}$ we then get

$$
T_{i} \circ L(\varphi)=T_{i}(L \varphi) \rightarrow T(L \varphi)=T \circ L(\varphi)
$$

and so by the definition of the topology on $\mathscr{S}^{\prime}$ we have $T_{i} \circ L \rightarrow T \circ L$; the mapping is continuous.

Definition 2.16. For a tempered distribution $T$, we define its Fourier transform $\mathcal{F}\{T\}=\hat{T}$, its derivatives $\left(\frac{d}{d x}\right)^{n} T=T^{(n)}$ and its product $T g=g T$ with a function $g \in V^{\infty}$, to be the tempered distributions:

$$
\begin{aligned}
\hat{T} & : \varphi \\
T^{(n)} & : \varphi \\
T g & \mapsto(-1)^{n} T\left(\varphi^{(n)}\right), \\
& \mapsto T(g \varphi)
\end{aligned}
$$

Note that all these three operations on $\mathscr{S}^{\prime}$ are continuous by proposition 2.15.
From the definition above, and the fact that the Fourier transform is invertible on $\mathscr{S}$, we get that it is invertible on $\mathscr{S}^{\prime}$ too. It is natural to ask how the Fourier transform on $\mathscr{S}^{\prime}$ relates to differentiation and multiplication by monomials. An educated guess would give the correct relationship, as established by the following proposition.

Proposition 2.17. For $T \in \mathscr{S}^{\prime}$ we have the two relationships

$$
\widehat{x^{n} T}=\left(i \frac{d}{d x}\right)^{n} \hat{T} \quad \text { and } \quad \widehat{T^{(n)}}=(i x)^{n} \hat{T}
$$

Proof. Two straight forward calculations show that

$$
\begin{aligned}
& \widehat{x^{n} T}(\varphi)=T\left(x^{n} \hat{\varphi}\right)=(-i)^{n} T\left(\widehat{\varphi^{(n)}}\right)=\left(i \frac{d}{d x}\right)^{n} \hat{T}(\varphi) \\
& \widehat{T^{(n)}}(\varphi)=(-1)^{n} T\left(\hat{\varphi}^{(n)}\right)=i^{n} T\left(\widehat{x^{n} \varphi}\right)=(i x)^{n} \hat{T}(\varphi)
\end{aligned}
$$

for any $\varphi \in \mathscr{S}$.
We next show that every tempered function $g$ has a natural corresponding tempered distribution $T_{g}$.

Proposition 2.18. If $g$ is a tempered function, then $T_{g}: \mathscr{S} \rightarrow \mathbb{C}$, defined by

$$
T_{g}(\varphi):=\int_{\mathbb{R}} g \varphi d x
$$

is a tempered distribution.
Proof. Linearity is obvious. As $g$ is tempered, there exist a constant $N \in \mathbb{N}_{0}$ so that $\left\|g /\left(1+|\cdot|^{N}\right)\right\|_{L^{1}}<\infty$. Thus

$$
\begin{aligned}
\left|T_{g}(\varphi)\right| & \leq \int_{\mathbb{R}}|g \varphi| d x \\
& \leq \int_{\mathbb{R}} \frac{g(x)}{1+|x|^{N}}\left[|\varphi|+\left|x^{N} \varphi\right|\right] d x \\
& \leq \int_{\mathbb{R}} \frac{g(x)}{1+|x|^{N}} d x\left[\|\varphi\|_{0, n}+\|\varphi\|_{k, n}\right] \\
& \lesssim\|\varphi\|_{0, n}+\|\varphi\|_{k, n},
\end{aligned}
$$

and so continuity of $T_{g}$ now follows by proposition 2.8 .
Now that $V$ naturally embeds in $\mathscr{S}^{\prime}$, one could how the three operations on $\mathscr{S}^{\prime}$ defined by 2.16 relates to corresponding operations on $V$, whenever the latter exist in some suitable sense.

For any $g \in V$ and $f \in V^{\infty}$, it follows straight from definitions and the observation $f g \in V$ that

$$
\begin{equation*}
f T_{g}=T_{f g} \tag{4}
\end{equation*}
$$

To obtain a corresponding result for the derivative, we define a function $g \in V$ is weakly differentiable in $V$, if there is a function $g^{\prime} \in V$ so that

$$
\begin{equation*}
\int_{\mathbb{R}} g(x) \varphi^{\prime}(x) d x=-\int_{\mathbb{R}} g^{\prime}(x) \varphi(x) d x \tag{5}
\end{equation*}
$$

for all $\varphi \in \mathscr{S}$. Notice that by integration by parts, this definition respects the classical derivative on functions in $\mathscr{S} \subset V$. It now follows immediately by definitions that if $g \in V$ has a weak derivative $g^{\prime} \in V$ then

$$
\begin{equation*}
T_{g}^{\prime}=T_{g^{\prime}} \tag{6}
\end{equation*}
$$

Turning to the Fourier transform, we recall that for $1 \leq p \leq 2$, the Fourier transform can canonically be defined on $L^{p}$ by interpolation and density arguments.

Proposition 2.19. For $g \in L^{p}$ and $1 \leq p \leq 2$, we have $\hat{T}_{g}=T_{\hat{g}}$, i.e.

$$
\begin{equation*}
\hat{T}_{g}(\varphi)=T_{\hat{g}}(\varphi) \tag{7}
\end{equation*}
$$

for all $\varphi \in \mathscr{S}$.

Proof. Pick $g \in L^{p}$ and $\varphi \in \mathscr{S}$. We have $\int_{N}^{N}|g| d x<\infty$ for every $N<\infty$ and $\varphi \in L^{1}$. Thus $(t, x) \mapsto g(t) \varphi(x)$ is absolutely integrable on $[-N, N] \times \mathbb{R}$ whenever $N<0$. We then use Fubini's theorem to calculate

$$
\begin{aligned}
\hat{T}_{g}(\varphi) & =\int_{\mathbb{R}} \hat{g} \varphi d x \\
& =\lim _{N \rightarrow \infty} \frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}} \int_{N}^{N} g(t) \hat{\varphi}(x) e^{-i x t} d t d x \\
& =\lim _{N \rightarrow \infty} \frac{1}{\sqrt{2 \pi}} \int_{N}^{N} g(t) \int_{\mathbb{R}} \hat{\varphi}(x) e^{-i x t} d x d t \\
& =\int_{\mathbb{R}} g(t) \hat{\varphi}(t) d t \\
& =T_{\hat{g}}(\varphi) .
\end{aligned}
$$

In light of the three preceding calculations, we now introduce a useful viewpoint. We let $T: V \rightarrow \mathscr{S}^{\prime}$ be the mapping $T: g \mapsto T_{g}$. We will view $V$ as a subset of $\mathscr{S}^{\prime}$, with $T$ the inclusion map; a viewpoint justified by the calculations (4),(6) and (7). In particular, we will simply write $g$ instead of $T_{g}$; this is slightly nonsensical, as $g$ and $T_{g}$ are very different objects, however this is convenient notation-wise. It is worth mentioning that although $L^{p} \subset \mathscr{S}^{\prime}$ as a set, we are not claiming that the topology of $L^{p}$ coincides with its subspace topology from $\mathscr{S}^{\prime}$.

### 2.3 The Hölder space $C^{n, \alpha}$

Definition 2.20. For $n \in \mathbb{N}_{0}$ and $0<\alpha<1$ we define the Hölder space $C^{n, \alpha}$ to be set of functions $f: \mathbb{R} \rightarrow \mathbb{C}$, with $f^{(k)}$ bounded and continuous for $k=0,1, \ldots, n$, and with finite Hölder coefficient

$$
\left|f^{(n)}\right|_{C^{n, \alpha}}:=\sup _{x \neq y} \frac{\left|f^{(n)}(x)-f^{(n)}(y)\right|}{|x-y|^{\alpha}}
$$

We equip $C^{n, \alpha}$ with the norm

$$
\|f\|_{C^{n, \alpha}}:=\|f\|_{C^{n}}+|f|_{C^{n, \alpha}},
$$

where

$$
\|f\|_{C^{n}}:=\sum_{k=0}^{n}\left\|f^{(k)}\right\|_{L^{\infty}} .
$$

Proposition 2.21. $C^{n, \alpha}$ is a Banach space.
Proof. See [5].

### 2.4 The Sobolev space $H^{s}$

Throughout this paper, we shall use the notation

$$
\langle\cdot\rangle:=\sqrt{1+(\cdot)^{2}} .
$$

We note that $\langle\cdot\rangle^{s} \in V^{\infty}$ for all $s \in \mathbb{R}$.
Definition 2.22. For $s \in \mathbb{R}$ we define the Sobolev space $H^{s}$, to be the vector space over $\mathbb{C}$, given by

$$
H^{s}:=\left\{f \in \mathscr{S}^{\prime}:\langle\cdot\rangle^{s} \hat{f} \in L^{2}\right\}
$$

We also define the real valued Sobolev space, $H^{s}(\mathbb{R}) \subset H^{s}$, to be vector space over $\mathbb{R}$ of the elements $f \in H^{s}$ that are real valued (in accordance with the discussion following definition 2.14). Both spaces are equipped with the norm

$$
\|f\|_{H^{s}}:=\left\|\langle\cdot\rangle^{s} \hat{f}\right\|_{L^{2}}=\left[\int_{\mathbb{R}}\langle\xi\rangle^{2 s}|\hat{f}|^{2} d \xi\right]^{\frac{1}{2}}
$$

Remark 2.23. Suppose $s \leq r$. From the definition it is immediate that $\|\cdot\|_{H^{s}} \leq$ $\|\cdot\|_{H^{r}}$ on $\mathscr{S}^{\prime}$ and so we have the continuous inclusion

$$
H^{r} \hookrightarrow H^{s}
$$

This implies in particular that $H^{s}$ is a function space for $s \geq 0$, since $H^{s} \subseteq$ $H^{0}=L^{2}$. Furthermore, one can easily see that $H^{s}(\mathbb{R})$ is the set of real valued functions in $H^{s}$ whenever $s \geq 0$.

We recall that for any function $f$, the Fourier transform satisfies $\check{\bar{f}}=\overline{\hat{f}}$ and $\hat{f}(x)=\check{f}(-x)$. Since a real valued function $f$ satisfies $f=\bar{f}$, we have for such a function

$$
\begin{equation*}
\hat{f}(x)=\overline{\hat{f}}(-x) . \tag{8}
\end{equation*}
$$

With our memory refreshed, we prove the next proposition.
Proposition 2.24. Let $s \geq 0$. For $f \in H^{s}$, let $f_{R}:=\operatorname{Re} f$ and $f_{I}:=\operatorname{Im} f$. Then

$$
\|f\|_{H^{s}}^{2}=\left\|f_{R}\right\|_{H^{s}}^{2}+\left\|f_{I}\right\|_{H^{s}}^{2}
$$

In particular, $f \in H^{s} \Leftrightarrow f_{R}, f_{I} \in H^{s}$.
Proof. By the preceding calculation we calculate

$$
\begin{aligned}
|\hat{f}|^{2} & =\left(\widehat{f_{R}}+i \widehat{f_{I}}\right) \overline{\left(\widehat{f_{R}}+i \widehat{f_{I}}\right)} \\
& =\left|\widehat{f_{R}}\right|^{2}+\left|\widehat{f_{I}}\right|^{2}+i[\underbrace{\widehat{f_{R}} \widehat{f_{I}}-\widehat{f_{R}} \widehat{\widehat{f_{I}}}}_{:=g(\xi)}] .
\end{aligned}
$$

By calculation (8) we have

$$
\widehat{\widehat{f_{R}}}(\xi) \widehat{f_{I}}(\xi)=\widehat{f_{R}}(-\xi) \widehat{\widehat{f_{I}}}(-\xi)
$$

and so $g$ is an odd function. As $\langle\cdot\rangle$ is an even function, we see that also $\langle\cdot\rangle^{2 s} g$ is an odd function. Then

$$
\begin{aligned}
\|f\|_{H^{s}}^{2} & =\lim _{N \rightarrow \infty} \int_{-N}^{N}\langle\xi\rangle^{2 s}|\hat{f}|^{2} d \xi \\
& =\lim _{N \rightarrow \infty} \int_{-N}^{N}\langle\xi\rangle^{2 s}\left[\left|\widehat{f_{R}}\right|^{2}+\left|\widehat{f_{I}}\right|^{2}\right] d \xi+i \lim _{N \rightarrow \infty} \underbrace{\int_{-N}^{N}\langle\xi\rangle^{2 s} g(\xi) d \xi}_{=0} \\
& =\left\|f_{R}\right\|_{H^{s}}^{2}+\left\|f_{I}\right\|_{H^{s}}^{2} .
\end{aligned}
$$

Note that the reason we restricted $s \geq 0$ in the previous proof, is simply because we have not generalized the concept of the real and imaginary part of a tempered distribution; Sobolev spaces of order $s<0$ will be of little importance in this paper.

Corollary 2.25. For $s \geq 0, H^{s}(\mathbb{R})$ is complete.
Proof. If a sequence of real valued functions $\left(\varphi_{n}\right) \subset H^{s}(\mathbb{R})$ converges to $\varphi \in H^{s}$, then the previous proposition shows that

$$
\left\|\varphi_{n}-\varphi\right\|_{H^{s}}^{2} \geq\|\operatorname{Im} \varphi\|_{H^{s}}^{2}
$$

thus $\operatorname{Im} \varphi=0$.
The intention of Sobolev spaces is to measure regularity of tempered distributions; to be an element of a Sobolev spaces of a high degree $(s \gg 1)$, requires a 'high' degree of regularity. This viewpoint reflects remark 2.23 , and is justified even further by the following proposition.

Proposition 2.26. Let $\frac{d}{d x}: \mathscr{S}^{\prime} \rightarrow \mathscr{S}^{\prime}$ be the derivative of tempered distributions (definition 2.16). Then $\frac{d}{d x}$ maps $H^{s}$ continuously into $H^{s-1}$.
Proof. We clearly have

$$
\langle x\rangle^{s-1} x \lesssim\langle x\rangle^{s}
$$

(but not $\gtrsim$ ) for all $s \in \mathbb{R}$. We now pick $f \in H^{s}$, and exploit proposition 2.17 to see that

$$
\begin{aligned}
\left\|f^{\prime}\right\|_{H^{s-1}} & =\left\|\langle\cdot\rangle^{s-1} \widehat{f}^{\prime}\right\|_{L^{2}} \\
& =\left\|\langle\cdot\rangle^{s-1}(\cdot) \hat{f}\right\|_{L^{2}} \\
& \lesssim\left\|\langle\cdot\rangle^{s} \hat{f}\right\|_{L^{2}} \\
& =\|f\|_{H^{s}} .
\end{aligned}
$$

Thus the claim is proved.

Proposition 2.27. $H^{s}$ is a Hilbert space.
Proof. For $s \in \mathbb{R}$ we define the linear continuous operator $\Lambda^{s}: \mathscr{S}^{\prime} \rightarrow \mathscr{S}^{\prime}$ by

$$
\Lambda^{s}(f)=\mathcal{F}^{-1}\left\{\langle\cdot\rangle^{s} \hat{f}\right\}
$$

Linearity of $\Lambda^{s}$ is obvious and continuity follows from the discussion after definition 2.16 and the fact that $\langle\cdot\rangle \in V^{\infty}$. We also note that the inverse of $\Lambda^{s}$ is $\Lambda^{-s}$. We calculate

$$
\begin{aligned}
\Lambda^{-s} L^{2} & =\left\{\Lambda^{-s} f: f \in L^{2}\right\} \\
& =\left\{f \in \mathscr{S}^{\prime}: \Lambda^{s} f \in L^{2}\right\} \\
& =\left\{f \in \mathscr{S}^{\prime}:\langle\cdot\rangle^{s} \hat{f} \in L^{2}\right\}=H^{s}
\end{aligned}
$$

where we in the third equality used that $\mathcal{F}$ is a unitary operator on $L^{2}$. Thus $\Lambda^{s} H^{s}=L^{2}$ and $\Lambda^{s}$ is an isomorphism between $H^{s}$ and $L^{2}$. The claim is proven if $\Lambda^{s}$ is isometric; a simple calculation shows for $f \in H^{s}$ that

$$
\|f\|_{H^{s}}=\left\|\langle\cdot\rangle^{s} \hat{f}\right\|_{L^{2}}=\left\|\mathcal{F}^{-1}\left\{\langle\cdot\rangle^{s} \hat{f}\right\}\right\|_{L^{2}}=\left\|\Lambda^{s} f\right\|_{L^{2}}
$$

It is a well known fact that $\mathscr{S}$ is dense in $L^{2}$ (see [9]), it is also easy to see that $\Lambda^{s}$ defined in the previous proposition is a continuous mapping from $\mathscr{S}$ to $\mathscr{S}$ (this follows from proposition 2.10 and 2.13). With this observation and the fact that $\Lambda^{s}$ is an isometric isomorphism between $H^{s}$ and $L^{2}$, we immediately get the following proposition.

Proposition 2.28. $\mathscr{S}$ is dense in $H^{s}$.
As $\mathscr{S}$ contains complex valued functions, we have $\mathscr{S} \not \subset H^{s}(\mathbb{R})$. However, by denoting $\mathscr{S}(\mathbb{R})$ for the real valued functions of $\mathscr{S}$, we easily see that $\mathscr{S}(\mathbb{R}) \subset$ $H^{s}(\mathbb{R})$.

Corollary 2.29. For $s \geq 0, \mathscr{S}(\mathbb{R})$ is dense in $H^{s}(\mathbb{R})$.
Proof. An elementary calculation shows that $\varphi \in \mathscr{S}$ implies that $\operatorname{Re} \varphi \in \mathscr{S}(\mathbb{R})$. As $H^{s}(\mathbb{R})$ is a subspace of $H^{s}$ it follows from the previous proposition that for any element $f \in H^{s}(\mathbb{R})$ there is a sequence $\left(\varphi_{n}\right) \subset \mathscr{S}$ so that $\varphi_{n} \rightarrow f$ in $H^{s}$. By proposition 2.24 we have $\left\|\operatorname{Re} \varphi_{n}-f\right\|_{H^{s}} \leq\left\|\varphi_{n}-f\right\|_{H^{s}}$, and so we conclude $\operatorname{Re} \varphi_{n} \rightarrow f$ in $H^{s}(\mathbb{R})$.

Again we note that the requirement $s \geq 0$, is because we have not generalized concept of real and imaginary part of tempered distributions; the proof given
above requires these definitions. Next, we pick arbitrary functions $\varphi, \psi \in \mathscr{S}$. By Hölder's inequality we observe that

$$
\begin{aligned}
\int_{\mathbb{R}} \varphi \psi d x & =\int_{R} \hat{\varphi} \check{\psi} d \xi \\
& =\int_{\mathbb{R}}\langle\xi\rangle^{s} \hat{\varphi}\langle\xi\rangle^{-s} \check{\psi} d \xi \\
& \leq\|\varphi\|_{H^{s}}\|\psi\|_{H^{-s}}
\end{aligned}
$$

By this observation, it follows from density that elements of $H^{s}$ (viewed as tempered distributions) can be extended to accept element of $H^{-s}$ as input, and vice versa. This next proposition should then not be too surprising.
Proposition 2.30. The dual space of $H^{s}$ is $H^{-s}$.
Proof. Denote $X=\left(H^{s}\right)^{\prime}$ for the continuous dual of $H^{s}$. By the discussion prior to this proposition it is clear that elements of $H^{-s}$ behave as bounded linear functionals on $H^{s}$. If we assume that $T \in H^{-s}$ vanish on $H^{s}$, that is $T(f)=0$ for all $f \in H^{s}$, then it must also vanish on $\mathscr{S} \subset H^{s}$, and so $T=0$. Thus $H^{-s} \subseteq X$. It remains to show that $X \subseteq H^{-s}$. Pick $T \in X$, and notice that $T \circ \Lambda^{s}$ is a continuous linear function on $L^{2}$. By Riesz representation theorem, there exist $f \in L^{2}$ so that $T \circ \Lambda^{s}=f$ (when $f$ is viewed as a tempered distribution). We rewrite $\Lambda^{s}$ in the fashion $\Lambda^{s}=\mathcal{F} M_{s} \mathcal{F}^{-1}$, where $M_{s}: \mathscr{S}^{\prime} \rightarrow \mathscr{S}^{\prime}$ is the continuous operation of multiplying by $\langle\cdot\rangle^{s}$. Note that $\Lambda^{s}$ is still the same operator; $\langle\cdot\rangle^{s}$ is symmetric about zero. We now see that the equation $T \circ \Lambda^{s}=T \circ \mathcal{F} M_{s} \mathcal{F}^{-1}=f$ is equivalent to

$$
\begin{equation*}
\hat{T}=T \circ \mathcal{F}=f \circ \mathcal{F} M_{-s}=\langle\cdot\rangle^{-s} \hat{f}, \tag{9}
\end{equation*}
$$

where we used that the distributional Fourier transform of $f$ coincides with its Fourier transform as an $L^{2}$ function: $f \circ \mathcal{F}=\hat{f}$, by proposition 2.19. Equation (9) shows that $T \in H^{s}$, and we are done.

Corollary 2.31. For $s \geq 0$, the dual space of $H^{s}(\mathbb{R})$ is $H^{-s}(\mathbb{R})$.
Proof. Denote $X=\left(H^{s}(\mathbb{R})\right)^{\prime}$ for the continuous dual of $H^{s}(\mathbb{R})$. Together with the previous proposition, and the density of $\mathscr{S}(\mathbb{R})$ in $H^{s}(\mathbb{R})$ it is clear that any $T \in H^{-s}(\mathbb{R})$ defines a bounded linear map from $H^{s}(\mathbb{R})$ to $\mathbb{R}$. Pick $T \in$ $H^{-s}(\mathbb{R})$ and assume that it vanish on $H^{s}(\mathbb{R})$; it consequently vanish on $\mathscr{S}(\mathbb{R}) \subset$ $H^{s}(\mathbb{R})$. As earlier, an elementary calculation shows that $\varphi \in \mathscr{S}$ if, and only if, $\operatorname{Re} \varphi, \operatorname{Im} \varphi \in \mathscr{S}(\mathbb{R})$. Then for any $\varphi \in \mathscr{S}$, we have

$$
T(\varphi)=T(\operatorname{Re} \varphi)+i T(\operatorname{Im} \varphi)=0
$$

and so $T$ vanish on all of $\mathscr{S}$. Thus $H^{-s}(\mathbb{R}) \subseteq X$. Similarly, any $T \in X$ extends naturally to a bounded linear functional on $H^{s}$; pick any $f \in H^{s}$, then

$$
T(f)=T(\operatorname{Re} f)+i T(\operatorname{Im} f)
$$

where continuity follows from continuity of $T$ together with proposition 2.24 . Thus $X=H^{-s}(\mathbb{R})$ and we are done.

This next theorem establishes regularity properties of Sobolev spaces with rank $s>\frac{1}{2}$. The proof is inspired by [10], which assumed $\frac{1}{2}<s<\frac{3}{2}$; this proof however is carried out in greater detail, simplified and generalized to $s>\frac{3}{2}$ by the author.

Theorem 2.32 (Sobolev embedding). For $k \in \mathbb{N}_{0}$ and $0<\alpha<1$, let $s=k+\alpha+\frac{1}{2}$. Then the inclusion mapping

$$
H^{s} \hookrightarrow C^{k, \alpha}
$$

is continuous.
Proof. We prove this by induction.
Step 1: Proving the claim for $k=0$.
Let $k=0$, pick $0<\alpha<1$ and set $s=k+\alpha+\frac{1}{2}$. Pick $f \in H^{s}$. For the statement to be true we need

$$
\|f\|_{L^{\infty}} \lesssim\|f\|_{H^{s}} \quad \text { and } \quad[f]_{C^{0, \alpha}} \lesssim\|f\|_{H^{s}}
$$

The first part is straight forward; by the Fourier transform and Hölder's inequality we have (almost everywhere)

$$
\begin{align*}
|f(x)| & =\frac{1}{\sqrt{2 \pi}}\left|\int_{\mathbb{R}} \hat{f} e^{i x \xi} d \xi\right| \\
& \leq \frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}}|\hat{f}| d \xi  \tag{10}\\
& \leq \frac{1}{\sqrt{2 \pi}} \underbrace{\left[\int_{\mathbb{R}}\langle\xi\rangle^{-2 s} d \xi\right]^{\frac{1}{2}}}_{:=C} \underbrace{\left[\int_{\mathbb{R}}\langle\xi\rangle^{2 s} \hat{f}^{2} d \xi\right]^{\frac{1}{2}}}_{=\|f\|_{H^{s}}}
\end{align*}
$$

As $2 s>1,\langle\xi\rangle^{-2 s}$ is integrable. Consequently $C<\infty$ and the first part is proved. To prove the second part we start off similarly; it holds (almost everywhere) that

$$
\begin{aligned}
|f(x+y)-f(x)| & =\frac{1}{\sqrt{2 \pi}}\left|\int_{\mathbb{R}} \hat{f} e^{i x \xi}\left(e^{i y \xi}-1\right) d \xi\right| \\
& \leq \frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}}|\hat{f}|\left|e^{i y \xi}-1\right| d \xi \\
& \lesssim\|f\|_{H^{s}}\left[\int_{\mathbb{R}} \frac{\left|e^{i y \xi}-1\right|^{2}}{\langle\xi\rangle^{2 s}} d \xi\right]^{\frac{1}{2}} \quad \text { (by Hölder's inequality). }
\end{aligned}
$$

We wish to show that the latter integral is bounded by $C|y|^{2 s-1}$. To do this we start by noticing the two important bounds

$$
\left|e^{i t}-1\right|^{2} \leq 4, \quad \text { and } \quad\left|e^{i t}-1\right|^{2} \leq t^{2}
$$

where the latter follows from the mean value theorem. We now make the substitution $y \xi \mapsto t$ and exploit the first and second bound above for $|t| \geq 1$ and $|t| \leq 1$ respectively. We calculate

$$
\begin{align*}
\frac{1}{2|y|^{2 s-1}} \int_{\mathbb{R}} \frac{\left|e^{i y \xi}-1\right|^{2}}{\langle\xi\rangle^{2 s}} d \xi & =\int_{0}^{\infty} \frac{\left|e^{t}-1\right|^{2}}{\left[|y|^{2}+t^{2}\right]^{s}} d t \\
& \leq \int_{0}^{\infty} \frac{\left|e^{t}-1\right|^{2}}{t^{2 s}} d t  \tag{11}\\
& \leq \int_{0}^{1} t^{2-2 s} d t+\int_{1}^{\infty} 4 t^{-2 s} d t \\
& <\infty,
\end{align*}
$$

where the last inequality is valid as $1<2 s<3$. Consequently

$$
|f(x+y)-f(x)| \lesssim\left|\left|f \|_{H^{s}}\right| y\right|^{s-1 / 2}
$$

and the proof is complete for the case $k=0$ and $0<\alpha<1$.
Step 2: Proving the claim for the general case $k \in \mathbb{N}_{0}$.
Suppose the claim holds for some $k \in N_{0}$ and all $0<\alpha<1$. Then for $s-\frac{1}{2}=$ $k+1+\alpha$, we get

$$
\begin{aligned}
\|f\|_{C^{k+1, \alpha}} & =\|f\|_{L^{\infty}}+\left\|f^{\prime}\right\|_{C^{k, \alpha}} \\
& \lesssim\|f\|_{H^{s}}+\left\|f^{\prime}\right\|_{H^{s-1}} \\
& \lesssim\|f\|_{H^{s}}
\end{aligned}
$$

where we used the calculation (10) together with proposition 2.26. As the claim is true for $k=0$ and all $0<\alpha<1$ by the previous proposition, our inductive proof is complete.

As useful as this last theorem seems, we will make little use of its full power; we will be mostly interested in the continuity properties of functions in $H^{s}$ when $s>\frac{1}{2}$. As the inclusion mapping $H^{s^{\prime}} \hookrightarrow H^{s}$ is continuous for $s^{\prime} \geq s$ the preceding theorem implies the following result.
Corollary 2.33. For every $s>\frac{1}{2}$, there is a Hölder space $C^{0, \alpha}$ so that the inclusion map $H^{s} \hookrightarrow C^{0, \alpha}$ is continuous.

The next proposition will not be of use in this paper, but gives some insight in how the Sobolev norm measures regularity. The proof is inspired by [4], but is modified and carried out in greater detail.
Proposition 2.34. For $0<s<1$ we have $\|\cdot\|_{H^{s}}^{2} \simeq\|\cdot\| \|_{L^{2}}^{2}+[\cdot]_{H^{s}}$ where

$$
[f]_{H^{s}}:=\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(y)-f(x)|^{2}}{|y-x|^{1+2 s}} d x d y
$$

that is, the Sobolev norm is equivalent with the norm $\sqrt{\|\cdot\|_{L^{2}}^{2}+[\cdot]_{H^{s}}}$.

Proof. Pick $f \in H^{s}$. A straight forward calculation shows

$$
\begin{aligned}
{[f]_{H^{s}} } & =\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(y)-f(x)|^{2}}{|y-x|^{1+2 s}} d x d y \\
& =\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(z+x)-f(x)|^{2}}{|z|^{1+2 s}} d x d z \\
& =\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\left|e^{-i z \xi}-1\right|^{2}}{|z|^{1+2 s}}|\hat{f}(\xi)|^{2} d \xi d z \\
& =\underbrace{\int_{\mathbb{R}} \frac{\left|e^{-i t}-1\right|^{2}}{|t|^{1+2 s}} d t}_{:=C} \int_{\mathbb{R}}|\xi|^{2 s}|\hat{f}(\xi)|^{2} d \xi
\end{aligned}
$$

We notice that $C$ is the same integral that shows up in (11), whith $s$ shifted by a half; this coincides with the difference in the restrictions $0<s<1$ and $\frac{1}{2}<s<\frac{3}{2}$. Thus $C<\infty$, and consequently

$$
\|f\|_{L^{2}}^{2}+[f]_{H^{s}}=\int_{\mathbb{R}}\left(1+C|\xi|^{2 s}\right)|\hat{f}(\xi)|^{2} d \xi
$$

As $0<C<\infty$, we have $\langle\xi\rangle^{2 s} \simeq 1+C|\xi|^{2 s}$. Consequently $\|\cdot\|_{H^{s}}^{2} \simeq\|\cdot\|_{L^{2}}^{2}+[\cdot]_{H^{s}}$ and we are done.

The remanding theory in this subsection is of less general importance, but will play an important role in section 7 . Before we move on, we define the vector space $H^{\infty}$ by

$$
H^{\infty}=\bigcap_{s \in \mathbb{R}} H^{s}
$$

Combining some Fourier theory with the Sobolev embedding theorem, it is not hard to see that $H^{\infty}$ is the set of infinitely differentiable functions $f: \mathbb{R} \rightarrow \mathbb{C}$ so that $f^{(n)} \in L^{2}$, for all $n \in \mathbb{N}_{0}$. We also define for $r \in(-\infty, \infty]$ the vector space $\mathcal{F} H^{r} \subset \mathscr{S}^{\prime}$ to be the set

$$
f \in \mathcal{F} H^{r} \Leftrightarrow \hat{f} \in H^{r}
$$

If this definition seems asymmetrical to the reader, we recall that $\langle\cdot\rangle$ is symmetric about zero and consequently $\hat{f} \in H^{r} \Leftrightarrow \check{f} \in H^{r}$. We also stress that, the notation $H^{s}$ will never represent the space $H^{\infty}$; it is assumed $s \in \mathbb{R}$. Note that by definition, $\mathcal{F} H^{\infty}$ is the set of functions $\varphi: \mathbb{R} \rightarrow \mathbb{C}$ so that $\langle\cdot\rangle^{s} \varphi \in L^{2}$ for every $s \in \mathbb{R}$. Exploiting Hölders inequality, this implies that $\langle\cdot\rangle^{s} \varphi \in L^{1}$ for every $s \in \mathbb{R}$ and $\varphi \in \mathcal{F} H^{\infty}$. An interesting consequence of the characterization of $H^{\infty}$ and $\mathcal{F} H^{\infty}$ is that $H^{\infty} \cap \mathcal{F} H^{\infty}=\mathscr{S}$.

Proposition 2.35. Suppose $\varphi \in H^{\infty}$ and $f \in H^{s}$. Then $\varphi f \in H^{s}$, and

$$
\left.\|\varphi f\|_{H^{s}} \lesssim \|\langle\cdot\rangle\right\rangle^{|s|} \hat{\varphi}\left\|_{L^{1}}\right\| f \|_{H^{s}}
$$

Proof. A simple calculation shows that

$$
\begin{aligned}
\langle x+y\rangle^{2} & =1+(x+y)^{2} \\
& =1+x^{2}+y^{2}+2 x y \\
& \leq 2\left(1+x^{2}+y^{2}+(x y)^{2}\right) \\
& =2\left(1+x^{2}\right)\left(1+y^{2}\right) \\
& =2\langle x\rangle^{2}\langle y\rangle^{2}
\end{aligned}
$$

By first substituting $x=\xi$ and $y=t-\xi$, and then $x=t$ and $y=\xi-t$ the inequality above and the fact that $\langle\cdot\rangle$ is symmetric about zero, implies that the quantities $\langle\xi\rangle /\langle t\rangle$ and $\langle t\rangle /\langle\xi\rangle$ are both less than or equal to $\sqrt{2}\langle\xi-t\rangle$. And so since $s \in \mathbb{R}$ is fixed, we get

$$
\begin{equation*}
\frac{\langle\xi\rangle^{s}}{\langle t\rangle^{s}} \lesssim\langle\xi-t\rangle^{|s|} \tag{12}
\end{equation*}
$$

By factorizing, we can rewrite $|\hat{\varphi}(\xi-t) \hat{f}(t)|$ as

$$
\begin{equation*}
\frac{1}{\langle\xi\rangle^{s}}[\underbrace{\frac{\langle\xi\rangle^{s}}{\langle t\rangle^{s}\langle\xi-t\rangle^{|s|}}}_{\lesssim 1}][\underbrace{\langle\xi-t\rangle^{|s|}|\hat{\varphi}(\xi-t)|}_{:=\psi(\xi-t)}][\underbrace{\langle t\rangle^{s}|\hat{f}(t)|}_{:=g(t)}] \tag{13}
\end{equation*}
$$

It is not hard to see that $\|\psi\|_{L^{1}}=\left\|\langle\cdot\rangle^{|s|} \varphi\right\|_{L^{1}}$ and $\|g\|_{L^{2}}=\|f\|_{H^{s}}$. By the factorization in (13), we now calculate implies that

$$
\begin{align*}
|\hat{\varphi} * \hat{f}(\xi)| & \leq \int_{\mathbb{R}}|\hat{\varphi}(\xi-t) \hat{f}(t)| d t \\
& \lesssim \frac{1}{\langle\xi\rangle^{s}} \int_{\mathbb{R}} \psi(\xi-t) g(t) d t  \tag{14}\\
& =\frac{1}{\langle\xi\rangle^{s}}(\psi * g)(\xi) .
\end{align*}
$$

Before we move on, note that both $\psi$ and $g$ are elements of $L^{2}$. We finally obtain

$$
\begin{aligned}
\|\varphi f\|_{H^{s}} & =\left\|\langle\cdot\rangle^{s}(\hat{\varphi} * \hat{f})\right\|_{L^{2}} \\
& \lesssim\|\psi * g\|_{L^{2}} \\
& =\|\hat{\psi} \hat{g}\|_{L^{2}} \\
& \leq\|\hat{\psi}\|_{L^{\infty}}\|\hat{g}\|_{L^{2}} \\
& \leq\|\psi\|_{L^{1}}\|g\|_{L^{2}} \\
& =\left\|\langle\cdot\rangle^{|s|} \hat{\varphi}\right\|_{L^{1}}\|f\|_{H^{s}} .
\end{aligned}
$$

If we pick $\varphi, \psi \in H^{\infty}$, it follows by the previous proposition that $\varphi \psi \in H^{s}$ for all $s \in \mathbb{R}$, as $\psi \in H^{s}$ for all $s \in \mathbb{R}$. We have proved the next corollary. $\varphi \psi \in H^{\infty}$.

Corollary 2.36. If $\varphi, \psi \in H^{\infty}$, then also $\varphi \psi \in H^{\infty}$.
By the Fourier transform, the previous two results implies that $f * \varphi \in \mathcal{F} H^{r}$, whenever $f \in \mathcal{F} H^{r}$, for $r \in(-\infty, \infty]$, and $\varphi \in \mathcal{F} H^{\infty}$. It is in this form that the two results will most often be applied.

Before the next proposition, we refresh our memory. By definition 2.16 we can multiply an element $T \in \mathscr{S}^{\prime}$ with an element $g \in V^{\infty}$ to again get a tempered distribution $T g \in \mathscr{S}^{\prime}$. We restrict our attention to an element $T \in H^{s} \subset \mathscr{S}^{\prime}$ and $\varphi \in H^{\infty} \subset V^{\infty}$; again $T \varphi \in \mathscr{S}^{\prime}$. As $\hat{T}$ is a function, we could ask whether it holds that $\widehat{T \varphi}$ coincides with the tempered distribution represented by the function $\hat{T} * \hat{\varphi}$. This is indeed true as we now show.
Proposition 2.37. Viewing $T \in H^{s}$ as a tempered distribution, and picking $\varphi \in H^{\infty}$, then it holds that $\widehat{T \varphi}=\hat{T} * \hat{\varphi}$, i.e. for any $\psi \in \mathscr{S}$ we have

$$
\widehat{T \varphi}(\psi):=T(\varphi \hat{\psi})=\int_{\mathbb{R}}(\hat{T} * \hat{\varphi}) \psi d t=:(\hat{T} * \hat{\varphi})(\psi)
$$

Proof. We start with the calculation

$$
\begin{align*}
T(\varphi \hat{\psi}) & =\hat{T}(\check{\varphi} * \psi) \\
& =\int_{\mathbb{R}} \hat{T}(x)(\check{\varphi} * \psi)(x) d x  \tag{15}\\
& =\int_{\mathbb{R}} \int_{\mathbb{R}} \hat{T}(x) \check{\varphi}(x-t) \psi(t) d t d x
\end{align*}
$$

Taking the absolute value of the integrand we would get

$$
\begin{equation*}
\int_{\mathbb{R}}|\hat{T}(x)|(|\check{\varphi}| *|\psi|)(x) d x \tag{16}
\end{equation*}
$$

Since $|\check{\varphi}|,|\psi| \in \mathcal{F} H^{\infty}$, it follows from the previous corollary that $|\check{\varphi}| *|\psi| \in \mathcal{F} H^{\infty}$, which in particular implies that $\langle\cdot\rangle^{-s}|\check{\varphi}| *|\psi| \in L^{2}$. Since we also have $\langle\cdot\rangle^{s}|\hat{T}| \in L^{2}$ by definition, it is clear that the integral (16) is finite. Thus we may swap the integrals in (15) and rewrite $\check{\varphi}(x-t)$ as $\hat{\varphi}(t-x)$ to obtain

$$
T(\varphi \hat{\psi})=\int_{\mathbb{R}}(\hat{T} * \hat{\varphi}) \psi d t=:(\hat{T} * \hat{\varphi})(\psi)
$$

Our final proposition studies how translation of $\hat{u}$ affect the $H^{s}$-norm of $u$.
Proposition 2.38. For $u \in H^{s}$ we have

$$
\left\|\langle\cdot\rangle^{s} \hat{u}(\cdot-t)\right\|_{L^{2}} \lesssim\langle t\rangle^{|s|}\|u\|_{H^{s}}
$$

Proof. By a shift in variables, equation (12) implies that

$$
\langle\xi\rangle^{s}|\hat{u}(\xi-t)| \lesssim\langle t\rangle^{|s|}\langle\xi-t\rangle^{s}|\hat{u}(\xi-t)|
$$

and so we immediately get the conclusion

$$
\left\|\langle\cdot\rangle^{s} \hat{u}(\cdot-t)\right\|_{L^{2}} \lesssim\langle t\rangle^{|s|}\left\|\langle\cdot-t\rangle^{s} \hat{u}(\cdot-t)\right\|_{L^{2}}=\langle t\rangle^{|s|}\|u\|_{H^{s}}
$$

### 2.5 Calculus of variation

We give here the general definition of the Fréchet derivative of a function between Banach spaces.

Definition 2.39. Let $X$ and $Y$ be Banach spaces and $U \subseteq X$ and open subset. A function $f: U \rightarrow Y$ is said to be Fréchet differentiable at $x \in U$ if there exist a bounded linear operator $D f(x): X \rightarrow Y$ so that

$$
\lim _{h \rightarrow 0} \frac{\|f(x+h)-D f(x) h\|_{Y}}{\|h\|_{X}}=0
$$

The above limit is to be understood to hold for any sequence $\left(h_{n}\right) \subset X$ converging to 0 .

### 2.6 Strict subadditivity

This section consist of original proofs.
Definition 2.40. We say that $f: \mathbb{R} \rightarrow \mathbb{R}$ is strictly subhomogeneous on an interval $\left(0, x_{0}\right)$ if $f(t x)<t f(x)$ whenever $x \in\left(0, x_{0}\right)$ and $t \in\left(1, \frac{x_{0}}{x}\right)$.

Lemma 2.41. Suppose $f$ satisfies the condition for strict subhomogeneity on $\left(0, x_{0}\right)$, when $t \in\left(1, \min \left\{\epsilon, \frac{x_{0}}{x}\right\}\right)$ for some $\epsilon>0$. Then $f$ is truly strictly subhomogeneous on ( $0, x_{0}$ ).

Proof. For $x \in\left(0, x_{0}\right)$ and $t \in\left(1, \frac{x_{0}}{x}\right)$, we can find $k \in \mathbb{N}$ so that $t^{\frac{1}{k}}<\epsilon$. For such a $k$ we get

$$
f(t x)<t^{\frac{1}{k}} f\left(t^{\frac{k-1}{k}} x\right)<t^{\frac{2}{k}} f\left(t^{\frac{k-2}{k}} x\right)<\cdots<t f(x)
$$

Definition 2.42. We say that $f: \mathbb{R} \rightarrow \mathbb{R}$ is strictly subadditive on an interval $\left(0, x_{0}\right)$ if $f\left(x_{1}+x_{2}\right)<f\left(x_{1}\right)+f\left(x_{2}\right)$ whenever $x_{1} \in\left(0, x_{0}\right)$ and $x_{2} \in\left(0, x_{0}-x_{1}\right)$.

Lemma 2.43. Suppose $f$ is strictly subhomogeneous on $\left(0, x_{0}\right)$, then $f$ is strictly subadditive on $\left(0, x_{0}\right)$.

Proof. We first assume $x_{1}=x_{2} \in\left(0, \frac{x_{0}}{2}\right)$. Then by strict subhomogeneity, $f\left(x_{1}+x_{2}\right)=f\left(2 x_{1}\right)<2 f\left(x_{1}\right)=f\left(x_{1}\right)+f\left(x_{2}\right)$. We assume next $0<x_{1}<x_{2}<\left(x_{0}-x_{1}\right)$. Then by strict subhomogeneity

$$
\begin{aligned}
f\left(x_{1}+x_{2}\right) & =f\left(\left[1+\frac{x_{1}}{x_{2}}\right] x_{2}\right) \\
& <\left[1+\frac{x_{1}}{x_{2}}\right] f\left(x_{2}\right) \\
& =f\left(x_{2}\right)+\frac{x_{1}}{x_{2}} f\left(x_{2}\right) \\
& <f\left(x_{2}\right)+f\left(x_{1}\right)
\end{aligned}
$$

where the last inequality also follows from subhomogeneity:

$$
\frac{x_{1}}{x_{2}} f\left(x_{2}\right)=\frac{x_{1}}{x_{2}} f\left(\frac{x_{2}}{x_{1}} x_{1}\right)<f\left(x_{1}\right)
$$

By symmetry, we have also proved for the case when $x_{1}>x_{2}$.

### 2.7 Concentration compactness principle

Theorem 2.44 ((Lions [6]) Concentration-compactness). Any sequence $\left(\rho_{n}\right) \subset$ $L^{1}(\mathbb{R})$ of non-negative functions with the property

$$
\int_{\mathbb{R}} \rho_{n} d x=\mu>0
$$

admits a subsequence, denoted again by $\left(\rho_{n}\right)$, for which one of the following phenomena occurs.

Vanishing: For each $r>0$ one has that

$$
\lim _{n \rightarrow \infty}\left(\sup _{x_{0} \in \mathbb{R}} \int_{B_{r}\left(x_{0}\right)} \rho_{n}\right)=0
$$

Concentration: There is a sequence $\left(x_{n}\right) \subset \mathbb{R}$ with the property that for each $\varepsilon>0$ there exists $r<\infty$ with

$$
\int_{B_{r}\left(x_{n}\right)} \rho_{n} d x \geq \mu-\varepsilon
$$

for all $n \in \mathbb{N}$.
Dichotomy: There exist $\bar{\mu} \in(0, \mu)$ so that for every $\varepsilon>0$ there exists a natural number $n_{0} \geq 1$ and two sequences of positive functions $\left(\rho_{n}^{(1)}\right),\left(\rho_{n}^{(2)}\right) \subset L^{1}$
satisfying for $n \geq n_{0}$

$$
\begin{aligned}
& \left\|\rho_{n}-\rho_{n}^{(1)}-\rho_{n}^{(2)}\right\|_{L^{1}} \leq \varepsilon, \\
& \left|\int_{\mathbb{R}} \rho_{n}^{(1)}-\bar{\mu}\right| \leq \varepsilon, \\
& \left|\int_{\mathbb{R}} \rho_{n}^{(2)}-(\mu-\bar{\mu})\right| \leq \varepsilon, \\
& \operatorname{dist}\left(\operatorname{supp}\left(\rho_{n}^{(1)}\right), \operatorname{supp}\left(\rho_{n}^{(2)}\right)\right) \rightarrow \infty .
\end{aligned}
$$

We also state without proof a corresponding lemma from [6]:
Lemma 2.45 ([6]). Suppose dichotomy occurs when applying the concentration compactness principle, then the two sequences $\left(\rho_{n}^{(1)}\right),\left(\rho_{n}^{(2)}\right)$ can be chosen to satisfy

$$
\begin{aligned}
& \operatorname{supp}\left(\rho_{n}^{(1)}\right) \subset\left(y_{n}-r_{n}, y_{n}+r_{n}\right) \\
& \operatorname{supp}\left(\rho_{n}^{(2)}\right) \subset \mathbb{R} \backslash\left(y_{n}-2 r_{n}, y_{n}+2 r_{n}\right)
\end{aligned}
$$

for two sequences $\left(y_{n}\right),\left(r_{n}\right) \in \mathbb{R}$.

## 3 Assumptions

The argument used to prove Theorem 1.1, requires the following assumption.

1. Assumptions on the nonlinear term $n: \mathbb{R} \rightarrow \mathbb{R}$.

We split $n$ up into $n=n_{p}+n_{r}$.
1.1 The function $n_{p}$ takes either of the two forms:
(a) $n_{p}(x)=c_{p}|x|^{p}$ and $c_{p} \neq 0$,
(b) $n_{p}(x)=c_{p} x|x|^{p-1}$ and $c_{p}>0$, for some real number $p>1$.
1.2 The function $n_{r}$ is continuous.
1.3 The function $n_{r}$ satisfies $n_{r}(x)=\mathcal{O}\left(|x|^{p+\delta}\right)$ for some $\delta>0$ as $x \rightarrow 0$.
2. Assumptions on the symbol $m: \mathbb{R} \rightarrow \mathbb{R}$, of the Fourier multiplier $L$.
2.1 Symmetry about zero: $m(\xi)=m(-\xi)$.
2.2 Non-negative at zero: $m(0) \geq 0$.
2.3 We have the the growth bound

$$
m(\xi)-m(0) \simeq|\xi|^{2 s}
$$

for some real number $s$ satisfying $2 s>\max \left\{1, \frac{p-1}{2}\right\}$.
2.4 There is a symmetric function $k \geq 0$, so that

$$
|m(\xi+t)-m(\xi)| \leq k(t)\langle\xi\rangle^{2 s}
$$

with $\lim _{t \rightarrow 0} k(t)=0$.
We make some observations. Assumption 1.1 implies that there are numbers $r \in \mathbb{R}$ satisfying $n_{p}(r) r>0$. Assumption 1.1 and 1.2 , implies that $n$ is continuous. We also note that assumption 2.2 implies that $m(\xi)>m(0)$ for $\xi \neq 0$, and together with assumption 2.3 it also implies $m(\xi)+1 \simeq\langle\xi\rangle^{2 s}$.

## 4 Strategy and preparation

We will search for solutions of (2) in $H^{s}(\mathbb{R})$; the set of real valued functions $u \in H^{s}$. We equip $H^{s}(\mathbb{R})$ with the $H^{s}$-norm; by corollary 2.25 , this space is complete. The search will be carried out by working on an appropriate constraint minimization problem of a functional $\mathcal{E}: H^{s}(\mathbb{R}) \rightarrow \mathbb{R}$ over a set $U \in H^{s}$. For practical reasons, it will be important that $\|\cdot\|_{L^{\infty}}$ is bounded on the set $U$. As we assume $s>\frac{1}{2}$, it follows by Sobolev embedding (Theorem 2.32) that $\|\cdot\|_{L^{\infty}} \lesssim\|\cdot\|_{H^{s}}$, and so we will work with the appropriate set $U \subset H^{s}(\mathbb{R})$, defined by

$$
U=\left\{u \in H^{s}(\mathbb{R}):\|u\|_{H^{s}}<R\right\}
$$

where $R>0$ is some fixed constant, whose exact size does not affect the main result. The constraint will be of the form $Q(u)=\mu$, where

$$
Q(u):=\frac{1}{2} \int_{\mathbb{R}} u^{2} d x
$$

and so we naturally define $U_{\mu}$ by

$$
U=\{u \in U: Q(u)=\mu\}
$$

Although the functional $\mathcal{E}$ is yet to be defined, we present the form of our constrained minimization problem:

$$
I_{\mu}:=\inf _{u \in U_{\mu}} \mathcal{E}(u)
$$

By the definition of $I_{\mu}$, there must be a minimizing sequence $\left(u_{n}\right) \in U_{\mu}$ so that

$$
\lim _{n \rightarrow \infty} \mathcal{E}\left(u_{n}\right)=I_{\mu}
$$

We will show that there is some $\mu_{*}>0$ so that when $\mu \in\left(0, \mu_{*}\right)$ we can 'build' a solution $u \in U_{\mu}$, i.e. $\mathcal{E}(u)=I_{\mu}$. The exact value of $\mu_{*}$ will not be specified; we will instead discover a finite set of positive upper bounds so that setting $\mu_{*}$ smaller than these will suffice in the desired property. An immediety upper
bound of $\mu_{*}$ is $2 R^{2}$; as $\|\cdot\|_{L^{2}} \leq\|\cdot\|_{H^{s}}<R$ on $U$, it is clear that $U_{\mu}$ is empty for $\mu \geq 2 R^{2}$. We have now introduced the relevant terminology, and so we provide a coarse overview of how theorem 1.1 will be proved from working with the corresponding constrained minimization problem.

1. We find lower and upper bounds of $I_{\mu}$. Functions $u \in U_{\mu}$ for which $\mathcal{E}(u)$ are within these bounds will be referred to as near minimizers.
2. We prove properties of near minimizers. These properties will suffice to show that any minimizing sequence $\left(u_{n}\right) \subset U_{\mu}$ must concentrate in accordance with the Concentration-Compactness principle (lemma 2.7).
3. From a 'concentrating' minimizing sequence, we build a function $u \in U_{\mu}$ so that $\mathcal{E}(u)=I_{\mu}$.
4. We prove that any minimizer of the constrained minimization problem must solve (2).

In addition, we find some sufficient conditions on $n(x)$ to establish regularity of a solution of (2). This is Theorem 9.4.

### 4.1 The functionals and the minimizing problem

For $x \in \mathbb{R}$ we define the functions

$$
N_{p}(x):=\int_{0}^{x} n_{p}(t) d t, \quad N_{r}(x):=\int_{0}^{x} n_{r}(t) d t
$$

Note that they are both continuous. As $n_{r}(x)=\mathcal{O}\left(|x|^{p+\delta}\right)$ as $|x| \rightarrow 0$ it follows that $N_{r}(x)=\mathcal{O}\left(|x|^{p+1+\delta}\right)$; a similar result obviously holds for $N_{p}$ too. We also define the corresponding functionals

$$
\mathcal{N}_{p}(u):=\int_{\mathbb{R}} N_{p}(u) d x, \quad \mathcal{N}_{r}(u):=\int_{\mathbb{R}} N_{r}(u) d x
$$

defined for $u \in H^{s}(\mathbb{R})$. We denote the sum of the two functionals by

$$
\mathcal{N}:=\mathcal{N}_{p}+\mathcal{N}_{r}
$$

In the next subsection we will prove that $L$ maps $H^{s}(\mathbb{R})$ continuously into $H^{-s}(\mathbb{R})$. This, together with proposition 2.31 , implies that $L u:=L(u)$ is a continuous linear map from $H^{s}(\mathbb{R})$ to $\mathbb{R}$ for any $u \in H^{s}(\mathbb{R})$. We will use the notation $L u[v]$ for the value of $L u$ with $v$ as argument. We can now define the next functional

$$
\mathcal{L}(u):=\frac{1}{2} L u[u],
$$

which we will show has the representation

$$
\mathcal{L}(u)=\frac{1}{2} \int_{\mathbb{R}} m(\xi)|\hat{u}|^{2} d \xi
$$

on $H^{s}(\mathbb{R})$; in particular, $\mathcal{L}$ is real valued. We finally arrive at the definition of $\mathcal{E}$,

$$
\mathcal{E}:=\mathcal{L}-\mathcal{N}
$$

With all the relevant notation, we restate the constrained minimization problem we wish to solve,

$$
\begin{equation*}
I_{\mu}=\inf _{u \in U_{\mu}} \mathcal{E}(u) \tag{17}
\end{equation*}
$$

As will be shown later, any minimizer $u \in U_{\mu}$ of $I_{\mu}$, must also solve (2) for some velocity $\nu$.

Remark 4.1. From here on, when a function $u$ and $\mu$ shows up in the same calculation, it is implied that $Q(u)=\mu$. We also stress that expressions of the form $f(u) \lesssim g(\mu)$ implies that $f(u) \leq C g(\mu)$, for some $C>0$ independent of $\mu$.

### 4.2 Some preliminary results for the functionals

Every proof in this subsection is original. We here show some general useful properties of the functionals $\mathcal{L}$ and $\mathcal{N}$. We will make repeated use of the boundness of $\|\cdot\|_{L^{\infty}}$ on $U$; a fact established at the beginning of this section. We will also exploit simple Fourier theory, similar to what was used to obtain (8).

Proposition 4.2. L is a bounded linear operator $L: H^{s}(\mathbb{R}) \rightarrow H^{-s}(\mathbb{R})$.
Proof. We start by proving linearity. Note that $m \in V$ (by the discussion following definition 2.11). As the Fourier transform, and multiplication by elements of $V$, are linear operations on $\mathscr{S}^{\prime}$, we get for $u, v \in H^{s}(\mathbb{R})$,

$$
L \widehat{(u+v)}=m(\xi)(\hat{u}+\hat{v})=m \hat{u}+m \hat{v}
$$

and by taking the Fourier inverse on each side and exploit its linearity, we obtain $L(u+v)=L(u)+L(v)$. To prove boundness we pick $u \in H^{s}(\mathbb{R})$. Then

$$
\begin{array}{rlr}
\|L u\|_{H^{-s}}^{2} & =\int_{\mathbb{R}}\langle\xi\rangle^{-2 s}|m(\xi) \hat{u}|^{2} d \xi \\
& =\int_{\mathbb{R}}\left[\frac{m(\xi)}{\langle\xi\rangle^{2 s}}\right]^{2}\langle\xi\rangle^{2 s}|\hat{u}|^{2} d \xi & \\
& \lesssim \int_{\mathbb{R}}\langle\xi\rangle^{2 s}|\hat{u}|^{2} d \xi & \quad \text { (by assumption on } m(\xi) \text { ) } \\
& =\|u\|_{H^{s}}^{2} . &
\end{array}
$$

It remains to show that $L(u)[v]$ is real valued for all $v \in H^{s}(\mathbb{R})$. We write

$$
\begin{equation*}
L u[v]=\widehat{L u}[\check{v}]=\int_{\mathbb{R}} m(\xi) \hat{u} \check{v} d \xi \tag{18}
\end{equation*}
$$

As the functions $u, v$ are real valued, $\hat{u} \check{v}$ is conjugate anti symmetric; that is

$$
\hat{u}(\xi) \check{v}(\xi)=\overline{\hat{u}(-\xi) \check{v}(-\xi)} .
$$

As $m$ is an even real valued function (by assumption), the integrand in (18) is conjugate anti symmetric. Consequently $L(u)[v]$ is real valued.

We also get a nice symmetrical property of $L$, as a consequence of the symmetry of $m$.
Proposition 4.3. For all $u, v \in H^{s}(\mathbb{R})$, we have $L u[v]=L v[u]$.
Proof. A straight forward calculations shows that

$$
\begin{aligned}
L u[v] & =\int_{\mathbb{R}} m(\xi) \hat{u} \check{v} d \xi \\
& \left.=\int_{\mathbb{R}} m(-\xi) \hat{u}(-\xi) \check{v}(-\xi) d \xi \quad \text { (by the substitution } \xi \mapsto-\xi\right) \\
& =\int_{\mathbb{R}} m(\xi) \check{u} \hat{v} d \xi \\
& =L v[u] .
\end{aligned}
$$

Proposition 4.4. $\mathcal{L}$ has the representation

$$
\mathcal{L}(u)=\frac{1}{2} \int_{\mathbb{R}} m|\hat{u}|^{2} d \xi
$$

on $H^{s}(\mathbb{R})$.
Proof. Exploiting $\check{u}=\overline{\hat{u}}$ for $u \in H^{s}(\mathbb{R})$, we obtain

$$
\begin{aligned}
2 \mathcal{L}(u) & =L u[u] \\
& =\widehat{L u}[\check{u}] \\
& =\int_{\mathbb{R}} m \hat{u} \check{u} d \xi \\
& =\int_{\mathbb{R}} m|\hat{u}|^{2} d \xi
\end{aligned}
$$

Proposition 4.5. For all $u \in H^{s}(\mathbb{R})$, we have $\mathcal{L}(u)>m(0) \mu$.
Proof. By the assumption, $m(\xi)>m(0)$, for $\xi \neq 0$. Consequently

$$
\begin{aligned}
\mathcal{L}(u) & =\frac{1}{2} \int_{\mathbb{R}} m(\xi)|\hat{u}|^{2} d \xi \\
& >\frac{1}{2} \int_{\mathbb{R}} m(0)|\hat{u}|^{2} d \xi \\
& =m(0) \mu
\end{aligned}
$$

Before the next proposition we note that as $\|\cdot\|_{L^{\infty}}$ is bounded on $U$, then so is $\|\cdot\|_{L^{p}}$ for $2 \leq p \leq \infty$; this can easily be seen from the calculation

$$
\begin{equation*}
\|f\|_{L^{p}}^{p}=\int_{\mathbb{R}}|f|^{p} d x \leq\|f\|_{L^{\infty}}^{p-2} \int_{\mathbb{R}}|f|^{2} d x=\|f\|_{L^{\infty}}^{p-2}\|f\|_{L^{2}}^{2}, \tag{19}
\end{equation*}
$$

together with the fact that $\|\cdot\|_{L^{2}}$ is bounded on $U$.
Proposition 4.6. For $u \in U$, we have the inequalities

$$
\begin{array}{ll}
\left|\mathcal{N}_{p}(u)\right| \lesssim\|u\|_{L^{p+1}}^{p+1} & \left|\mathcal{N}_{p}(u)\right| \lesssim \mu\|u\|_{L^{\infty}}^{p-1} \\
\left|\mathcal{N}_{r}(u)\right| \lesssim\|u\|_{L^{p+1+\delta}}^{p+1+\delta} & \left|\mathcal{N}_{r}(u)\right| \lesssim \mu\|u\|_{L^{\infty}}^{p-1+\delta}
\end{array}
$$

Proof. Note that by the discussion prior to this proposition, the two inequalities on the right follows from the two on the left. We prove upper left inequality; the remaining can be proved similarly. As $N_{p}$ is continuous and approaches zero like $|x|^{p+1}$, it follows that $\left|N_{p}(x)\right| \lesssim|x|^{p+1+\delta}$ on any compact interval $[-K, K]$, (for $N_{p}$ in particular, this is true on all of $\left.\mathbb{R}\right)$. Consequently, as $\|\cdot\|_{L^{\infty}}$ is bounded on $U$, there is some $C>0$ so that

$$
\left|\mathcal{N}_{p}(u)\right| \leq C\|u\|_{L^{p+1}}^{p+1}
$$

for every $u \in U$.
As both $\|\cdot\|_{L^{2}}$ and $\|\cdot\|_{L^{\infty}}$ is bounded on $U$, we immediately get the corollary.
Corollary 4.7. $\mathcal{N}$ is bounded on $U$.
Towards the end of this paper, regularity of $\mathcal{E}$ will be important; in particular we will need the following proposition.

Proposition 4.8. $\mathcal{E}$ is uniformly continuous on $U$.
Proof. We are claiming that for any $u, v \in U,|\mathcal{E}(u)-\mathcal{E}(v)|$ is bounded by a function $f$ of $\|u-v\|_{H^{s}}$, with $f(h) \rightarrow 0$ as $h \searrow 0$. We will not prove it directly for $\mathcal{E}$; the claim will follow by proving it for both $\mathcal{N}$ and $\mathcal{L}$. To structure the proof, we prove it in two steps.

Step 1: Proving that $\mathcal{N}$ is uniformly continuous on $U$. For notational convenience, we set $N:=N_{p}+N_{r}$. We define

$$
g(x, y):=\frac{N(x+y)-N(x)}{(x+y)^{2}+(x)^{2}},
$$

for $(x, y) \neq 0$, and $g(0,0):=0$. We have $N(x)=O\left(|x|^{p+1}\right)$ as $x \rightarrow 0$; consequently $N(x)=o\left(|x|^{2}\right)$ as $x \rightarrow 0$. By this observation and the fact that $N$ is continuous, we conclude that $g$ is continuous on $\mathbb{R}^{2}$. Let $K \subset \mathbb{R}^{2}$ be any
bounded neighbourhood of 0 . As $g$ is uniformly continuous on $K$, there is a modulus of continuity $\omega: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$with $\omega(h) \rightarrow 0$ as $h \searrow 0$, so that

$$
\begin{equation*}
\left|g\left(x_{1}, y_{1}\right)-g\left(x_{2}, y_{2}\right)\right| \leq \omega\left(\operatorname{dist}\left(\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right)\right)\right) \tag{20}
\end{equation*}
$$

whenever $(x, y) \in K$. As all norms on $\mathbb{R}^{2}$ are equivalent, we can assume the function $\operatorname{dist}(\cdot, \cdot)$ above is the metric induced by the 1 -norm: $\|(x, y)\|_{1}=|x|+|y|$. If we now insert the points $(x, y-x)$ and $(x, 0)$ in (20), and notice that $g(x, 0)=0$, we get the equation

$$
\begin{equation*}
|N(y)-N(x)| \leq\left(x^{2}+y^{2}\right) \omega(|y-x|) \tag{21}
\end{equation*}
$$

which is valid whenever $(x, y-x) \in K$. We set $K$ to be a disk centered at zero, whose radius exceeds $2\|u\|_{L^{\infty}}$ for any $u \in U$. We then calculate for $u, v \in U$

$$
\begin{align*}
|\mathcal{N}(u)-\mathcal{N}(v)| & \leq \int_{\mathbb{R}}|N(u)-N(v)| d x \\
& \leq \int_{\mathbb{R}}\left(u^{2}+v^{2}\right) \omega(|u-v|) d x  \tag{22}\\
& \leq\left[\|u\|_{L^{2}}^{2}+\|v\|_{L^{2}}^{2}\right] \omega\left(\|u-v\|_{L^{\infty}}\right) \\
& \lesssim \omega\left(\|u-v\|_{L^{\infty}}\right)
\end{align*}
$$

where we assumed (without loss of generality) that $\omega$ is non-decreasing, and the fact that $\|\cdot\|_{L^{2}} \leq\|\cdot\|_{H^{s}} \lesssim 1$ on $U$. As $\|\cdot\|_{L^{\infty}} \lesssim\|\cdot\|_{H^{s}}$ by Sobolev embedding (Theorem 2.32), the uniform continuity of $\mathcal{N}$ follows.

Step 2: Proving that $\mathcal{L}$ is uniformly continuous on $U$.
Turning to $\mathcal{L}$, we start with the observation that for any two complex numbers $z, w \in \mathbb{C}$ we have by the triangle inequality $|z|-|w| \leq|z+w|$ and $|w|-|z| \leq|z+w| ;$ consequently

$$
\begin{equation*}
||z|-|w|| \leq|z-w| \tag{23}
\end{equation*}
$$

By this observation, the assumption $m \lesssim\langle\cdot\rangle^{2 s}$ and the representation of $\mathcal{L}$ on $U$, we calculate for $u, n \in U$

$$
\begin{aligned}
|\mathcal{L}(u)-\mathcal{L}(v)| & \leq\left.\int_{\mathbb{R}} m(\xi)| | \hat{u}\right|^{2}-|\hat{v}|^{2} \mid d \xi & & \\
& \lesssim \int_{\mathbb{R}}\langle\xi\rangle^{2 s}|\hat{u}+\hat{v} \| \hat{u}-\hat{v}| d \xi & & \text { (by }(23)) \\
& \leq\|u+v\|_{H^{s}}\|u-v\|_{H^{s}} & & \text { (by Hölder's inequality) } \\
& \lesssim\|u-v\|_{H^{s}} & &
\end{aligned}
$$

where we in the final step used $\|u+v\|_{H^{s}} \leq\|u\|_{H^{s}}+\|v\|_{H^{s}}$ and the boundness of $\|\cdot\|_{H^{s}}$ on $U$.

We end this section by calculating the Fréchet derivative of $Q, \mathcal{L}, \mathcal{N}$ and $\mathcal{E}$. As they are all functions from $H^{s}(\mathbb{R})$ to $\mathbb{R}$, their derivative at a 'point' $u \in H^{s}(\mathbb{R})$ will be elements of $H^{-s}(\mathbb{R})$ (in accordance with proposition 2.31).

Proposition 4.9. The Fréchet derivative of $Q$ at $u \in H^{s}(\mathbb{R})$ is given by

$$
Q^{\prime}(u)=u
$$

Proof. Viewing $u \in H^{-s}(\mathbb{R})$ as a dual element of $H^{s}(\mathbb{R})$, we have $u[v]=$ $\int_{\mathbb{R}} u v d x$, for $v \in H^{s}(\mathbb{R})$. Pick $v \in H^{s}(\mathbb{R})$; we observe that

$$
\begin{aligned}
2 Q(u+v) & =\int_{\mathbb{R}}(u+v) d x \\
& =\int_{\mathbb{R}} u^{2} d x+2 \int_{\mathbb{R}} u v d x+\int_{\mathbb{R}} v^{2} d x \\
& =2[Q(u)+u[v]+Q(v)] .
\end{aligned}
$$

An immediate calculation then gives

$$
\frac{|Q(u+v)-Q(u)-u[v]|}{\|v\|_{H^{s}}}=\frac{|Q(v)|}{\|v\|_{H^{s}}} \leq \frac{\|v\|_{L^{2}}^{2}}{2\|v\|_{H^{s}}} \leq \frac{\|v\|_{H^{s}}}{2} \rightarrow 0
$$

as $v \rightarrow 0$. Thus we have proved that indeed $Q^{\prime}(u)=u$.
Proposition 4.10. The Fréchet derivative of $\mathcal{L}$ at $u \in H^{s}(\mathbb{R})$ is given by

$$
\mathcal{L}^{\prime}(u)=L u
$$

Proof. We start by exploiting the bilinearity of $(x, y) \mapsto L x[y]$, to see that for $v \in H^{s}(\mathbb{R})$, we have

$$
\begin{aligned}
2 \mathcal{L}(u+v) & =L(u+v)[u+v] \\
& =L u[u]+L u[v]+L v[u]+L v[v] \\
& =2[\mathcal{L}(u)+L u[v]+\mathcal{L}(v)]
\end{aligned}
$$

where we used the symmetry of $L$, established by proposition 4.3. A straight forward calculation then yields

$$
\frac{|\mathcal{L}(u+v)-\mathcal{L}(u)-L u[v]|}{\|v\|_{H^{s}}}=\frac{\mathcal{L}(v)}{\|v\|_{H^{s}}} \lesssim\|v\|_{H^{s}} \rightarrow 0
$$

as $v \rightarrow 0$, where we used that $\mathcal{L} \lesssim\|\cdot\|_{H^{s}}^{2}$; this follows from proposition 4.2. We conclude that $\mathcal{L}^{\prime}(u)=L u$.

Proposition 4.11. The Fréchet derivative of $\mathcal{N}$ at $u \in H^{s}(\mathbb{R})$ is given by

$$
\mathcal{N}^{\prime}(u)=n(u)
$$

Proof. By assumption, $n$ is continuous and satisfies $n(x)=\left(|x|^{p}\right)$ as $x \rightarrow 0$, with $p>1$. By a similar argument as the one used to arrive at the bound (21), we can for any compact set $K \in \mathbb{R}^{2}$, find a modulus of continuity $\omega: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$, with $\omega(h) \rightarrow 0$ as $h \searrow 0$, so that

$$
\begin{equation*}
|n(x+y)-n(x)| \leq(|x|+|y|) \omega(|y|) \tag{24}
\end{equation*}
$$

whenever $(x, y) \in K$. Note that by the argument used to derive (21), we would expect the term $|x+y|+|y|$ in (24); this is however dominated by $2(|x|+|y|)$ and we can let $\omega$ absorb the constant 2 . We set $K$ to be a disc centered at zero, whose radius exceeds $\|u\|_{L^{\infty}}+1$. Pick $v \in H^{s}(\mathbb{R})$ with $\|v\|_{H^{s}}$ small enough so that $\|v\|_{L^{\infty}}<1$. A calculation then yields

$$
\begin{aligned}
\frac{|\mathcal{N}(u+v)-\mathcal{N}(u)-n(u)[v]|}{\|v\|_{H^{s}}} & =\frac{1}{\|v\|_{H^{s}}}\left|\int_{\mathbb{R}} N(u+v)-N(u)-n(u) v d x\right| \\
& =\frac{1}{\|v\|_{H^{s}}}\left|\int_{\mathbb{R}} v \int_{0}^{1}[n(u+t v)-n(u)] d t d x\right| \\
& \leq \frac{1}{\|v\|_{H^{s}}} \int_{\mathbb{R}}|v| \int_{0}^{1}|n(u+t v)-n(u)| d t d x \\
& \leq \frac{1}{\|v\|_{H^{s}}} \int_{\mathbb{R}}\left[|v u|+|v|^{2}\right] \omega\left(\|v\|_{L^{\infty}}\right) d x \\
& \leq\left[\|u\|_{L^{2}}+\|v\|_{L^{2}}\right] \omega\left(\|v\|_{L^{\infty}}\right) \\
& \rightarrow 0
\end{aligned}
$$

as $v \rightarrow 0$, in $H^{s}(\mathbb{R})$, where we used Hölder's inequality and that $\|\cdot\|_{L^{\infty}} \lesssim\|\cdot\|_{H^{s}}$, by Sobolev embedding. We conclude that $\mathcal{N}^{\prime}(u)=n(u)$.

Exploiting the triangle inequality, it becomes obvious that $\mathcal{E}^{\prime}=\mathcal{L}^{\prime}-\mathcal{N}^{\prime}$, and so we get the corollary.

Corollary 4.12. The Fréchet derivative of $\mathcal{E}$ at $u \in H^{s}(\mathbb{R})$ is given by

$$
\mathcal{E}^{\prime}(u)=L u-n(u) .
$$

## 5 Upper and lower bounds for $I_{\mu}$

In this section we prove that $\infty<I_{\mu}<m(0) \mu-\kappa \mu^{1+\beta}$, for some constants $\kappa$ and $\beta$ yet to be determined. The lower bound will turn out to be necessary when we show that a solution of (17) is a solution of (2). The upper bound will be used extensively to prove properties of minimizing sequences that which will guarantee a solution $u$ of (17). We also note that together, the upper and lower bound implies that $I_{m}$ can not satisfies equations of the form $I_{\mu}=I_{\mu}+c$ for $c \in \mathbb{R}$.

Proposition 5.1. $-\infty<I_{\mu}$.

Proof. Note that $\mathcal{L}(u) \geq 0$. Also by corollary $4.7,|\mathcal{N}|$ is bounded by some constant $C$ on $U$. Consequently, for $u \in U$ we have

$$
\mathcal{E}(u)=\mathcal{L}(u)-\mathcal{N}(u)>-C
$$

Thus $I_{\mu} \geq-C>-\infty$.
The following proof is inspired by [7], but is carried out in greater detail.
Proposition 5.2. There exist $\kappa>0$ so that $I_{\mu}<m(0) \mu-\kappa \mu^{1+\beta}$, where the exponent $\beta$ is given by

$$
\begin{equation*}
\beta=\frac{2 s(p-1)}{4 s-(p-1)} \tag{25}
\end{equation*}
$$

Proof. Pick a function $\varphi \in \mathscr{S}$ satisfying $\|\varphi\|_{H^{s}} \leq R, Q(\varphi)=1$ and $c_{p} \varphi(x) \geq 0$. This last inequality implies that

$$
\mathcal{N}_{p}(\varphi)=\frac{\left|c_{p}\right|}{p+1}\|\varphi\|_{L^{p+1}}^{p+1}
$$

We define the function $\varphi_{t}(x)=\sqrt{\frac{\mu}{t}} \varphi(x / t)$ and assume $t \geq 1$. We make two important calculations. First off,

$$
\left\|\varphi_{t}\right\|_{H^{s}}^{2}=\int_{\mathbb{R}}\langle\xi\rangle^{2 s} t \hat{\varphi}^{2}(t \xi) d \xi=\int_{\mathbb{R}}\langle\xi / t\rangle^{2 s} \hat{\varphi}^{2}(\xi) d \xi \leq\|\varphi\|_{H^{s}}^{2}
$$

which shows that $\varphi_{t} \in U$ for all $t \geq 1$. Secondly we have for any $k \geq 1$ that

$$
\begin{aligned}
\left\|\varphi_{t}\right\|_{L^{k}}^{k} & =\int_{\mathbb{R}}\left|\varphi_{t}\right|^{k} d x \\
& =\left[\frac{\mu}{t}\right]^{\frac{k}{2}} \int_{\mathbb{R}}|\varphi(x / t)|^{k} d x \\
& =t\left[\frac{\mu}{t}\right]^{\frac{k}{2}} \int_{\mathbb{R}}|\varphi(x)|^{k} d x \quad \text { (by the substitution } x \mapsto t x \text { ) } \\
& =t\left[\frac{\mu}{t}\right]^{\frac{k}{2}}\|\varphi\|_{L^{k}}^{k} .
\end{aligned}
$$

For $k=2$, this calculation shows that $Q\left(\varphi_{t}\right)=\mu$, and so together with the observation above we have $\varphi_{t} \in U_{\mu}$ for all $t \geq 1$. We wish to see how $\mathcal{E}=\mathcal{L}-\mathcal{N}$ acts on $\varphi_{t}$. We start by looking at $\mathcal{N}\left(\varphi_{t}\right)$;

$$
\begin{aligned}
& \mathcal{N}_{r}\left(\varphi_{t}\right)=\mathcal{O}\left(\left\|\varphi_{t}\right\|_{L^{p+1+\delta}}^{p+1+\delta}\right)=\mathcal{O}(t)\left[\frac{\mu}{t}\right]^{(p+1+\delta) / 2} \\
& \mathcal{N}_{p}\left(\varphi_{t}\right)=\frac{\left|c_{p}\right|}{p+1}\left\|\varphi_{t}\right\|_{L^{p+1}}^{p+1}=\underbrace{\frac{\left|c_{p}\|\mid \varphi\|_{L^{p+1}}^{p+1}\right.}{p+1}}_{:=C_{2}} t\left[\frac{\mu}{t}\right]^{(p+1) / 2}
\end{aligned}
$$

Turning to $\mathcal{L}\left(\varphi_{t}\right)$ we find that

$$
\begin{aligned}
\mathcal{L}\left(\varphi_{t}\right) & =\frac{\mu t}{2} \int_{R} m(\xi) \hat{\varphi}^{2}(\xi t) d \xi \\
& =\frac{\mu}{2} \int_{R} m(\xi / t) \hat{\varphi}^{2}(\xi) d \xi \\
& =m(0) \mu+\frac{\mu}{2} \int_{R}(m(\xi / t)-m(0)) \hat{\varphi}^{2}(\xi) d \xi \\
& \leq m(0) \mu+C \frac{\mu}{t^{2 s}} \int_{\mathbb{R}}|\xi|^{2 s} \hat{\varphi}^{2}(\xi) d \xi \\
& =m(0) \mu+\underbrace{C| | \xi^{s} \hat{\varphi} \|_{L^{2}}^{2}}_{:=C_{1}} \frac{\mu}{t^{2 s}} .
\end{aligned}
$$

We now get the bound

$$
\begin{aligned}
I_{\mu} \leq \mathcal{E}\left(\varphi_{t}\right) & \leq m(0) \mu+C_{1} \frac{\mu}{t^{2 s}}-C_{2} t\left[\frac{\mu}{t}\right]^{(p+1) / 2}+\mathcal{O}(t)\left[\frac{\mu}{t}\right]^{(p+1+\delta) / 2} \\
& =\mu\left[m(0)+C_{1} \frac{1}{t^{2 s}}-C_{2}\left[\frac{\mu}{t}\right]^{(p-1) / 2}\right]+\mathcal{O}(t)\left[\frac{\mu}{t}\right]^{(p+1+\delta) / 2}
\end{aligned}
$$

We set $t^{-1}=B \mu^{\beta / 2 s}$ where $\beta$ is given by (25). $B$ is yet to be characterized, but we require it to be positive and small enough so that $t \geq 1$. Our inequality becomes:

$$
I_{\mu} \leq m(0) \mu-[\underbrace{C_{2} B^{(p-1) / 2}-C_{1} B^{s}}_{:=2 \kappa}] \mu^{1+\beta}+B^{(p-1+\delta) / 2} \mathcal{O}\left[\mu^{1+\beta+(1+\alpha) \delta / 2}\right]
$$

We now pick $B$ small enough so that $\kappa$ is positive and $\kappa \mu^{1+\beta}$ is greater than the $\mathcal{O}$ term for all values of $\mu \in\left(0,2 R^{2}\right)$. This is possible as $2 s>(p-1) / 2$ and $\delta>0$. We then get the desired result:

$$
I_{\mu}<m(0) \mu-\kappa \mu^{1+\beta}
$$

Remark 5.3. From here on, we assume to have picked a constant $\kappa>0$ as described in the last proposition.

## 6 Near minimizers

As a consequence of the preceding proposition, there exist $u \in U$ so that

$$
\mathcal{E}(u)<m(0) \mu-\kappa \mu^{1+\beta}
$$

where (as always) it is assumed $Q(u)=\mu$. We will refer to these functions as near minimizers and denote the set of such functions by $\tilde{U} \subset U$.

The study of near minimizers is motivated by the following proposition.

Proposition 6.1. For any minimizing sequence $\left(u_{n}\right) \in U_{\mu}$ there is a number $N \geq 0$ so that $u_{k}$ is a near minimizer for $k \geq N$.

Proof. This follows immediately from the definition of a minimizing sequence and the definition of a near minimizer.

This section is devoted to proving important results concerning near minimizers. We list the three most important.

1. Proposition 6.2 , shows that the $\|u\|_{L^{\infty}}$ is uniformly bounded below for any near minimizer $u \in U_{\mu}$; this will be sufficient to exclude vanishing when applying the Concentration-Compactness principle to a minimizing sequence.
2. Proposition 6.5 shows that the $H^{s}$-norm is bounded above by the $L^{2}-$ norm on any near minimizer $u \in U_{\mu}$. This result will be used to show that if a minimizing sequence concentrates in accordance with the ConcentrationCompactness principle, then a subsequence converges weakly to a minimizer.
3. Corollary 6.10 , shows that for $\mu_{*}$ sufficiently small, $\mu \mapsto I_{\mu}$ is strictly sub-additive on $\left(0, \mu_{*}\right)$; this will be sufficient to exclude dichotomy when applying the Concentration-Compactness principle to a minimizing sequence.
Proposition 6.2. A near minimizer $u$ satisfies $\|u\|_{L^{\infty}} \gtrsim \mu^{\beta /(p-1)}$.
Proof. For $u \in U$, we have

$$
\mathcal{N}(u) \lesssim\|u\|_{L^{p+1}}^{p+1}
$$

which can be obtained by proposition 4.6, the calculation (19) and (yet again) the fact that $\|\cdot\|_{L^{\infty}}$ is bounded on $U$. We then calculate

$$
\begin{aligned}
2\|u\|_{L^{\infty}}^{p-1} \mu & \geq\|u\|_{L^{p+1}}^{p+1} \\
& \gtrsim \mathcal{N}(u) \\
& \geq \mathcal{N}(u)+m(0) \mu-\mathcal{L}(u) \\
& =m(0) \mu-\mathcal{E}(u) \\
& >\kappa \mu^{1+\beta} .
\end{aligned}
$$

Remark 6.3. In the preceding chain of inequalities we also notice the important inequality

$$
\mathcal{N}(u)>\kappa \mu^{1+\beta}
$$

valid for all near minimizers $u$.
This next lemma is not particularly important but will be used in several calculations.

Lemma 6.4. A near minimizer $u$ satisfies

$$
\int_{\mathbb{R}}[m(\xi)-m(0)] \hat{u}^{2} d \xi \lesssim \mu\|u\|_{L \infty}^{p-1}
$$

Proof. Rewriting the corresponding inequality of a near minimizer $u$ we get $\mathcal{L}(u)-m(0) \mu<\mathcal{N}(u)-\kappa \mu^{1+\beta}$. Writing this out in full we get

$$
\int_{\mathbb{R}}[m(\xi)-m(0)] \hat{u}^{2} d \xi<2 \mathcal{N}(u)-2 \kappa \mu^{1+\beta}<2 \mathcal{N}(u) \lesssim \mu\|u\|_{L^{\infty}}^{p-1}
$$

Proposition 6.5. A near minimizer $u$ satisfies $\|u\|_{H^{s}}^{2} \lesssim \mu$.
Proof. By assumption we have $\langle\xi\rangle^{2 s} \lesssim 1+m(\xi)-m(0)$, and so we calculate for a near minimizer $u$ :

$$
\begin{array}{rlr}
\|u\|_{H^{s}}^{2} & =\int_{\mathbb{R}}\langle\xi\rangle^{2 s} \hat{u}^{2} d \xi & \\
& \lesssim \mu+\int_{\mathbb{R}}[m(\xi)-m(0)] \hat{u}^{2} d \xi & \\
& \lesssim \mu+\mu\|u\|_{L^{\infty}}^{p-1} & \\
& & \text { (by lemma } 6.4) \\
& \lesssim \mu & \\
\text { (by boundness of } \left.\|\cdot\|_{L^{\infty}}\right) .
\end{array}
$$

This next remark will be very important towards the end; it will play a vital role in excluding the possibility of a minimizing sequence to converge to the boundary of $U$.

Remark 6.6. By the previous proposition, we can pick $\mu_{*}>0$ small enough so that for $\mu \in\left(0, \mu_{*}\right)$ and a near minimizer $u \in U_{\mu}$, we have $\|u\|_{H^{s}} \leq R / 2$. From here on we assume to have picked such a $\mu_{*}$.

This next lemma is greatly influenced by [7], but the proof is simplified and carried out in greater detail.

Lemma 6.7. Restricting $\|\cdot\|_{L^{\infty}}$ to the set of near minimizers $\tilde{U}$, we obtain

$$
\|\cdot\|_{L^{\infty}} \lesssim \mu^{(1+\tau) / 2}
$$

for every $\tau$ satisfying $2 s \tau<\beta$.
Proof. Combining the assumption on $m(\xi)$ and lemma 6.4 we notice

$$
\int_{\mathbb{R}}|\xi|^{2 s} \hat{u}^{2} d \xi \lesssim \int_{\mathbb{R}}[m(\xi)-m(0)] \hat{u}^{2} d \xi \lesssim \mu\|u\|_{L^{\infty}}^{p-1}
$$

Using this observation together with Hölder's inequality, we obtain for any $\tau \in \mathbb{R}$

$$
\begin{align*}
\|u\|_{L^{\infty}}^{2} & \leq\left(\int_{\mathbb{R}}|\hat{u}| d \xi\right)^{2} \\
& \leq \int_{\mathbb{R}}\left[1+\left(\frac{|\xi|}{\mu^{\tau}}\right)^{2 s}\right]^{-1} d \xi \int_{\mathbb{R}}\left[1+\left(\frac{|\xi|}{\mu^{\tau}}\right)^{2 s}\right]|\hat{u}|^{2} d \xi  \tag{26}\\
& \lesssim \mu^{1+\tau}[1+\underbrace{\frac{\|u\|_{L^{\infty}}^{p-1}}{\mu^{2 s \tau}}}_{:=f_{\tau}(u)}]
\end{align*}
$$

Suppose for fixed $\tau$ that $f_{\tau}$ is bounded above on near minimizers, then (26) shows that $\|u\|_{L^{\infty}}^{2} \lesssim \mu^{1+\tau}$. We define $Q \subset \mathbb{R}$ to be the set such that

$$
\tau \in Q \Leftrightarrow f_{\tau} \text { is bounded on } \tilde{U} \text {. }
$$

This set is certainly not empty; as $\|\cdot\|_{L^{\infty}}$ is bounded on $U$ and thus on $\tilde{U}$, it is clear that $0 \in Q$. Looking at the relation $f_{\tau_{1}}(u)=\mu^{2 s\left(\tau_{2}-\tau_{1}\right)} f_{\tau_{2}}(u)$, we can conclude that if $\tau_{1}<\tau_{2}$, then $\tau_{2} \in Q \Longrightarrow \tau_{1} \in Q$. Let $\tau_{*}=\sup Q$. Suppose $2 s \tau_{*}<\beta$; we then pick $\varepsilon>0$ so that $2 s\left(\tau_{*}+\varepsilon\right)<\beta(1-2 \varepsilon)$. As $\left(\tau_{*}-\varepsilon\right) \in Q$, a near minimizer $u$ satisfies $\|u\|_{L^{\infty}}^{2} \lesssim \mu^{1+\tau_{*}-\varepsilon}$. We then get the calculation

$$
\begin{equation*}
f_{\tau_{*}+\varepsilon}(u) \lesssim \mu^{\left(1+\tau_{*}-\varepsilon\right)(p-1) / 2-2 s\left(\tau_{*}+\varepsilon\right)} . \tag{27}
\end{equation*}
$$

Recalling that $\beta=\frac{2 s(p-1)}{4 s-(p-1)}$, we rewrite the exponent of $\mu$ in (27) to obtain

$$
\left(1+\tau_{*}-\varepsilon\right) \frac{(p-1)}{2}-2 s\left(\tau_{*}+\varepsilon\right)=\left[(1-2 \varepsilon) \beta-2 s\left(\tau_{*}+\varepsilon\right)\right] \frac{(p-1)}{2 \beta}>0
$$

Since $\mu$ is bounded, we conclude $f_{\tau_{*}+\varepsilon}$ is bounded above on $\tilde{U}$ and so $\left(\tau_{*}+\varepsilon\right) \in Q$; a contradiction. Thus $2 s \tau_{*} \geq \beta$ and we are done.
Corollary 6.8. $\|u\|_{L^{\infty}}^{p-1+\delta}=o\left(\mu^{\beta}\right)$ for $u \in \tilde{U}$.
Proof. For $\varepsilon>0$, set $\tau=\frac{\beta}{2 s}-\varepsilon=\frac{2 \beta}{(p-1)}-1-\varepsilon$. Then

$$
\|u\|_{L^{\infty}}^{p-1+\delta} \lesssim \mu^{(1+\tau)(p-1+\delta) / 2}=\mu^{\beta+[\beta \delta /(p-1)-\varepsilon(p-1+\delta) / 2]} .
$$

Picking $\varepsilon$ small enough so that

$$
\frac{\beta \delta}{(p-1)}-\varepsilon \frac{(p-1+\delta)}{2}>0
$$

the result immediately follows.
With the preceding bound on $\|\cdot\|_{L^{\infty}}$, we now build an argument showing that $\mu \mapsto I_{\mu}$ is strictly subadditive which will be the main tool when excluding dichotomy.

We need the following lemma whose proof is original.

Lemma 6.9. There is exist $\mu_{*}>0$ and $\gamma>0$ so that for any near minimizer $u$ with $Q(u)=\mu \in\left(0, \mu_{*}\right)$ and $t \in(1,2)$ we have the inequality

$$
I_{t \mu}<t \mathcal{E}(u)-\gamma(t-1) \mu^{1+\beta}
$$

Proof. We start by noticing $\mathcal{L}(\sqrt{t} u)=t \mathcal{L}(u)$ and $\mathcal{N}_{p}(\sqrt{t} u)=t^{(p+1) / 2} \mathcal{N}_{p}(u)$. As $Q(\sqrt{t} u)=t \mu$ We calculate

$$
\begin{align*}
I_{t \mu} & \leq \mathcal{L}(\sqrt{t} u)-\mathcal{N}(\sqrt{t} u) \\
& =t \mathcal{L}(u)-t^{(p+1) / 2} \mathcal{N}(u)+t^{(p+1) / 2} \mathcal{N}_{r}(\sqrt{t} u)-\mathcal{N}_{r}(\sqrt{t} u) \\
& =t \mathcal{E}(u)-\underbrace{\left[t^{(p+1) / 2}-t\right] \mathcal{N}(u)}_{:=\varphi(t, u)}+\underbrace{t^{(p+1) / 2} \mathcal{N}_{r}(u)-\mathcal{N}_{r}(\sqrt{t} u)}_{:=\phi(t, u)} \tag{28}
\end{align*}
$$

Using remark 6.3 we perform the calculation

$$
\begin{aligned}
\varphi(t, u) & >\left[t^{(p+1) / 2}-t\right] \kappa \mu^{1+\beta} \\
& =\kappa t \frac{\left[t^{(p-1) / 2}-1\right]}{[t-1]}(t-1) \mu^{1+\beta} \\
& >\kappa \frac{(p-1)}{2}(t-1) \mu^{1+\beta} \\
& \gtrsim(t-1) \mu^{1+\beta}
\end{aligned}
$$

where we exploited that $t \mapsto t \frac{\left[t^{(p-1) / 2}-1\right]}{[t-1]}$ is strictly increasing on $t \in(1,2)$ with the limit $(p-1) / 2$ as $t \rightarrow 1$.

As for $\phi$, we see that $\phi(1, u)=0$ and so we use the mean value theorem for some $t_{*} \in(1, t) \subset(1,2)$ to calculate

$$
\begin{array}{rlr}
\phi(t, u) & =(t-1)\left[\frac{d \phi}{d t}\left(t_{*}, u\right)\right] \\
& =(t-1)\left[\int_{\mathbb{R}} \frac{(p-1)}{2} t_{*}^{(p-1) / 2} N_{r}(u)-\frac{u}{2 \sqrt{t_{*}}} n_{r}\left(\sqrt{t_{*}} u\right) d x\right] \\
& \lesssim(t-1)\left[\mu\|u\|_{L^{\infty}}^{p-1+\delta}\right] & \text { (by assumption on } \left.n_{r}\right) \\
& =(t-1) o\left(\mu^{1+\beta}\right) & \text { (by corollary 6.8) }
\end{array}
$$

where the interchange of derivative and integral performed when calculating $\frac{d \phi}{d t}$ is justified by Leibniz integral rule together with the observation

$$
\left|u n_{r}(\sqrt{t} u) / 2 \sqrt{t}\right|<C|u(x)|^{p+1+\delta} \in L^{1}(\mathbb{R})
$$

for some $C$ whenever $t \in(1,2)$.
Looking back at (28) we can rewrite the last line in the following manner

$$
\begin{equation*}
I_{t \mu} \leq t \mathcal{E}(u)-\varphi\left[1-\frac{\phi}{\varphi}\right] \tag{29}
\end{equation*}
$$

By the previous calculations we find

$$
\left[1-\frac{\phi}{\varphi}\right] \gtrsim\left[1-\frac{o\left(\mu^{1+\beta}\right)}{\mu^{1+\beta}}\right]
$$

and so it should be clear that there is some $\mu_{*}>0$ so that for $\mu \in\left(0, \mu_{*}\right)$, we have $\left[1-\frac{\varphi}{\phi}\right]>r$ for some $r>0$ and consequently that $\varphi\left[1-\frac{\varphi}{\phi}\right] \gtrsim \varphi \gtrsim(t-1) \mu^{1+\beta}$. Adding this last inequality to (29), our lemma is proved.

Corollary 6.10. For a $\mu_{*}$ as described in the preceding lemma, $\mu \mapsto I_{\mu}$ is strictly subadditive on $\left(0, \mu_{*}\right)$.

Proof. Fix $\mu \in\left(0, \mu_{*}\right)$. Pick a minimizing sequence $\left(u_{n}\right)$ from $U_{\mu}$ consisting solely of near minimizers. Now for $t \in\left(1, \min \left\{2, \frac{\mu_{*}}{\mu}\right\}\right)$ the sequence $\left(\sqrt{t} u_{n}\right)$ lies in $U_{t \mu}$. By the preceding lemma there is some $\gamma>0$ so that

$$
I_{t \mu}<t \mathcal{E}\left(u_{n}\right)-\gamma(t-1) \mu^{1+\beta}
$$

and taking the limit we get

$$
\begin{equation*}
I_{t \mu} \leq t I_{\mu}-\gamma(t-1) \mu^{1+\beta} \tag{30}
\end{equation*}
$$

By the bounds obtained in section 5, it is clear that $I_{\mu}$ is a real valued number, and consequently (30) implies that $I_{t \mu}<t I_{\mu}$. As we did not specify $\mu \in\left(0, \mu_{*}\right)$ or $t \in\left(1, \min \left\{2, \frac{\mu}{\mu_{*}}\right\}\right)$, it follows from combining lemma 2.41 and 2.43 that $\mu \mapsto I_{\mu}$ is strictly subadditive on $\left(0, \mu_{*}\right)$.

## 7 Applying Concentration-Compactness

In this section, we study what happens when we apply concentration-compactness to minimizing sequences. For a minimizing sequence $\left(u_{n}\right) \subset U_{\mu}$, it follows by the Concentration-Compactness principle 2.7, that $\left(\left|u_{n}\right|^{2}\right)$ must admit a subsequence with the property vanishing, concentration or dichotomy.

We stress that when studying a minimizing sequence $\left(u_{n}\right) \subset U_{\mu}$, there is no harm in assuming $\left(u_{n}\right)$ to consist solely of near minimizers; by proposition 6.1 we can always remove the finite number of elements $u_{n}$ that fail to be near minimizers.

### 7.1 Excluding vanishing

This section is completely original. We introduce some notation. For $f \in U$ we define the seminorm

$$
|f|_{2}:=\sqrt{\int_{-1}^{1}|f|^{2} d x}
$$

We wish to show that $|f|_{2}$ is bounded below by the value of $f(0)$. We define the sets $U^{h}=\{f \in U:|f(0)|=h\}$, and the function

$$
g(h):=\inf _{f \in U^{h}}|f|_{2}^{2},
$$

where we define $\inf \varnothing=\infty$. We now prove the necessary lower bound on $|f|_{2}$.
Lemma 7.1. $g(h)$ is a non-decreasing function and positive for $h>0$.
Proof. For $0 \leq c \leq 1$ it is clear that $c U^{h}=\left\{c f: f \in U^{h}\right\} \subseteq U^{c h}$. Thus for such $c$ we get

$$
g(c h)=\inf _{f \in U^{c h}}|f|_{2}^{2} \leq \inf _{f \in c U^{h}}|f|_{2}^{2}=\inf _{f \in U^{h}}|c f|_{2}^{2}=c^{2} g(h),
$$

implying $g$ is non-decreasing. It remains to show that $g(h)>0$ for $h>0$. Since $f \in U \Leftrightarrow\|f\|_{H^{s}}<R$, we conclude from corollary 2.33 that there is some $C>0$ and $\alpha>0$ so that $|f(x)-f(y)| \leq C|x-y|^{\alpha}$ for all $x, y \in \mathbb{R}$ and $f \in U$. We pick $h \in[0,2 C]$, and define

$$
\epsilon:=\left(\frac{h}{2 C}\right)^{1 / \alpha}
$$

which by construction satisfies $\epsilon \leq 1$. We pick $f \in U^{h}$ and calculate

$$
\begin{aligned}
\int_{-1}^{1}|f(x)|^{2} d x & \geq \int_{-\epsilon}^{\epsilon}|f(x)|^{2} d x \\
& \geq \int_{-\epsilon}^{\epsilon}|f(x)-f(0)|^{2}-2|f(x)-f(0)| h+h^{2} d x \\
& \geq \int_{-\epsilon}^{\epsilon} h^{2}-2|f(x)-f(0)| h d x \\
& \geq \int_{-\epsilon}^{\epsilon} h^{2}-2 C|x|^{\alpha} h d x \\
& =\left[2 \epsilon h^{2}-\frac{4 C \epsilon^{1+\alpha}}{1+\alpha} h\right] \\
& =\left[\frac{2 \alpha}{(1+\alpha)(2 C)^{1 / \alpha}}\right] h^{2+1 / \alpha}
\end{aligned}
$$

and consequently $g(h) \gtrsim h^{2+1 / \alpha}$ for $h \in[0,2 C]$. This, together with the fact that $g$ is non-decreasing proves the lemma.

Proposition 7.2. Vanishing does not occur.
Proof. By corollary 6.2, for fixed $\mu$, there is some $c>0$ so that for any near minimizer $u \in U_{\mu}$ we have $\|u\|_{L^{\infty}}>c$. Pick a near minimizer $u \in U_{\mu}$, and a point $x_{0} \in \mathbb{R}$ so that $\left|u\left(x_{0}\right)\right|=\|u\|_{L^{\infty}}$ (this is possible as $u$ is continuous and $u(x) \rightarrow 0$ as $|x| \rightarrow \infty)$. Then by the preceding lemma

$$
\sup _{y \in \mathbb{R}} \int_{y-1}^{y+1}|u|^{2} d x \geq \int_{x_{0}-1}^{x_{0}+1}|u|^{2} d x \geq g\left(\|u\|_{L^{\infty}}\right) \geq g(c)
$$

and since $g(c)>0$, it is clear that vanishing can not occur.

### 7.2 Excluding dichotomy

This section is inspired by [2], but the proofs are approached differently and in much greater detail.

Lemma 7.3. Fix $\varphi \in H^{\infty}$ and define $\varphi_{r}(x)=\varphi(x / r)$ for $r \geq 1$. We define the bilinear function $B_{r}: U \times U \rightarrow \mathbb{R}$ by

$$
B_{r}(u, v):=L \varphi_{r} u[v]-L u\left[\varphi_{r} v\right],
$$

where $u, v \in U$ and $L$ is the Fourier multiplier of (2). $B_{r}$ satisfies

$$
\left|B_{r}\right| \rightarrow 0,
$$

as $r \rightarrow \infty$.
The key importance of this lemma is that $B_{r}$ tends to zero uniformly on $U \times U$. An essential part of the proof is that $\|\cdot\|_{H^{s}}$ is bounded on $U$, or equivalently that $\|\cdot\|_{H^{s}} \lesssim 1$. We also note that $\widehat{\varphi_{r}}=r \hat{\varphi}(r \xi)$; this together with the fact that $\varphi$ is fixed, gives us

$$
\begin{align*}
\left\|\langle\cdot\rangle^{s} \widehat{\varphi_{r}}\right\|_{L^{1}} & =\left\|\langle\cdot / r\rangle^{s} \hat{\varphi}\right\|_{L^{1}} \\
& \leq\left\|\langle\cdot\rangle^{s} \hat{\varphi}\right\|_{L^{1}}  \tag{31}\\
& \lesssim 1,
\end{align*}
$$

where $\langle\cdot / r\rangle^{s} \leq\langle\cdot\rangle^{s}$ follows as $s>0$ and $r \geq 1$. This inequality will be used throughout the proof.

Proof of lemma 7.3. For clear structure, we divide this proof up into steps.
Step 1: Defining the integral $E$.
We start by rewriting the the quantity $L u\left[\varphi_{r}^{2} u\right]$ as $\varphi_{r} L u\left[\varphi_{r} u\right]$ where $\varphi_{r} L u$ is the product (in accordance with definition 2.16) of the tempered distributions $\varphi_{r} \in H^{\infty}$ and $L u \in H^{\infty}$. Exploiting proposition 2.37 we get for $u \in U$

$$
\begin{align*}
B_{r}(u, v) & =L \varphi_{r} u[v]-\varphi_{r} L u[v] \\
& =\int_{\mathbb{R}} \check{v}\left[\left(\widehat{\varphi_{r}} * \hat{u}\right) m-\widehat{\varphi_{r}} *(m \hat{u})\right] d \xi  \tag{32}\\
& =\int_{\mathbb{R}} \check{v} \int_{\mathbb{R}} \widehat{\varphi_{r}}(t) \hat{u}(\xi-t)[m(\xi)-m(\xi-t)] d t d \xi
\end{align*}
$$

We want to interchange the two integrals; by Fubini's theorem this can be done if the integral is absolutely integrable. We denote $I_{r}$ for the quantity obtained when integrating the absolute value of the integrand above. It is not hard to see that

$$
I_{r} \leq \int_{\mathbb{R}}|\check{v}|\left[\left(\left|\widehat{\varphi_{r}}\right| *|\hat{u}|\right) m+\left|\widehat{\varphi_{r}}\right| *|m \hat{u}|\right] d \xi
$$

We also note that for $t \in(-\infty, \infty]$, the space $\mathcal{F} H^{t}$, is closed under the operations $f \mapsto|f|$; this together with $m \lesssim\langle\cdot\rangle^{2 s}$ and proposition 2.35 , implies

$$
|\check{v}| \in \mathcal{F} H^{s}, \quad \text { and } \quad\left[\left(\left|\widehat{\varphi_{r}}\right| *|\hat{u}|\right) m+\left|\widehat{\varphi_{r}}\right| *|m \hat{u}|\right] \in \mathcal{F} H^{-s}
$$

Consequently $I_{r}$ is finite, and we may swap the integrals in (32). We get

$$
\begin{equation*}
B_{r}(u, v)=\int_{\mathbb{R}} \widehat{\varphi}_{r} \underbrace{\int_{\mathbb{R}} \check{v}(\xi) \hat{u}(\xi-t)[m(\xi)-m(\xi-t)] d \xi}_{:=E} d t \tag{33}
\end{equation*}
$$

where the integral $E$ was named for notational convenience.
Step 2: Obtaining bounds for $E$.
There are two ways to obtain a bound for $E$. Firstly, we have the "coarse" bound given by

$$
\begin{align*}
|E| & \leq \int_{\mathbb{R}}|\check{v}(\xi) \hat{u}(\xi-t)||m(\xi)-m(\xi-t)| d \xi \\
& \lesssim \int_{\mathbb{R}}|\check{v}(\xi) \hat{u}(\xi-t)|\left[\langle\xi\rangle^{2 s}+\langle\xi-t\rangle^{2 s}\right] d \xi  \tag{34}\\
& \lesssim\langle t\rangle^{s}| | u\left\|_{H^{s}}| | v\right\|_{H^{s}} \\
& \lesssim\langle t\rangle^{s}
\end{align*}
$$

where the third inequality exploits Hölder's inequality, proposition 2.38 and the symmetry of $\langle\cdot\rangle$ about zero. Secondly, we have by assumption that

$$
|m(\xi-t)-m(\xi)| \leq k(t)\langle\xi\rangle^{2 s}
$$

and so substituting this into $E$ we get

$$
\begin{align*}
|E| & \leq \int_{\mathbb{R}}|\check{v}(\xi) \hat{u}(\xi-t)||m(\xi)-m(\xi-t)| d \xi \\
& \leq k(t) \int_{\mathbb{R}}|\check{v}(\xi) \hat{u}(\xi-t)|\langle\xi\rangle^{2 s} d \xi  \tag{35}\\
& \lesssim k(t)\langle t\rangle^{s}\|u\|_{H^{s}}\|v\|_{H^{s}} \\
& \lesssim k(t)\langle t\rangle^{s}
\end{align*}
$$

where we the third inequality uses Hölder's inequality and proposition 2.38.
Step 3: Obtaining a bound for $B_{r}$.
We pick $0<\alpha<1$. Using the bound (34) when $|t| \leq r^{-\alpha}$ and the bound (35)
when $|t|>r^{-\alpha}$ we get

$$
\begin{aligned}
\left|B_{r}(u, v)\right| & \leq \int_{\mathbb{R}}\left|\widehat{\varphi_{r}} E\right| d t \\
& \lesssim \int_{|t| \leq r^{-\alpha}} k(t)\left|\widehat{\varphi_{r}}\right|\langle t\rangle^{2 s} d t+\int_{|t|>r^{-\alpha}}\left|\widehat{\varphi_{r}}\right|\langle t\rangle^{2 s} d t \\
& =\int_{|t| \leq r^{1-\alpha}} k(t / r)|\hat{\varphi}(t)|\langle t / r\rangle^{2 s} d t+\int_{|t|>r^{-\alpha}}|\hat{\varphi}(t)|\langle t / r\rangle^{2 s} d t \\
& \leq\left[\sup _{|t| \leq r^{-\alpha}} k(t)\right]\left\|\hat{\varphi}\langle\cdot\rangle^{s}\right\|_{L^{1}}+\int_{|t|>r^{1-\alpha}}|\hat{\varphi}(t)|\langle t\rangle^{s} d t .
\end{aligned}
$$

By assumption, $k(t) \rightarrow 0$ as $t \rightarrow 0$, and since $|\hat{\varphi}| \in \mathcal{F} H^{-\infty}$, it follows that $|\hat{\varphi}|\langle\cdot\rangle^{s} \in L^{1}$. As $0<\alpha<1$, it should then be clear by the previous calculation that $\left|B_{r}\right|$ tends to zero as $r \rightarrow \infty$, independently of $u, v \in U$.

Proposition 7.4. Suppose dichotomy occurs on a minimizing sequence $\left(u_{n}\right) \subset U_{\mu}$. For $\mu_{*}$ sufficiently small, there exist $0<\lambda<\mu$ and two sequences $\left(u_{n}^{1}\right) \subset U_{\lambda}$, and $\left(u_{n}^{2}\right) \subset U_{\mu-\lambda}$ so that

$$
\mathcal{E}(u)-\mathcal{E}\left(u_{n}^{1}\right)-\mathcal{E}\left(u_{n}^{2}\right) \rightarrow 0
$$

Proof. For clarity, we divide this proof up into steps.
Step 1: Defining the functions $\varphi, \psi$ and $\phi$, and the necessary bound on $\mu_{*}$. We pick two symmetrical functions $\varphi, \psi \in C^{\infty}$ so that $0 \leq \varphi, \psi \leq 1, \varphi^{2}+\psi^{2}=1$ and

$$
\varphi(x)= \begin{cases}1 & |x| \leq 1 \\ 0 & |x| \geq 2\end{cases}
$$

We also define $\phi:=1-\psi$; we note that $\varphi, \phi \in H^{\infty}$. For $r \geq 1$ we define $\varphi_{r}(x):=\varphi(x / r)$, and similarly for $\psi_{r}$ and $\phi_{r}$. Pick any $u \in U$. By the triangle inequality and proposition 2.35, we have the bounds

$$
\begin{aligned}
\left\|\varphi_{r} u\right\|_{H^{s}} & \lesssim\left\|\langle\cdot\rangle^{s} \widehat{\varphi_{r}}\right\|_{L^{1}}\|u\|_{H^{s}} \\
& \leq\left\|\langle\cdot\rangle^{s} \hat{\varphi}\right\|_{L^{1}}\|u\|_{H^{s}} \\
& \lesssim\|u\|_{H^{s}} \\
\left\|\psi_{r} u\right\|_{H^{s}} & \leq\|u\|_{H^{s}}+\left\|\phi_{r} u\right\|_{H^{s}} \\
& \lesssim\left[1+\left\|\langle\cdot\rangle^{s} \widehat{\phi_{r}}\right\|_{L^{1}}\right]\|u\|_{H^{s}} \\
& \leq\left[1+\left\|\langle\cdot\rangle^{s} \hat{\phi}\right\|_{L^{1}}\right]\|u\|_{H^{s}} \\
& \lesssim\|u\|_{H^{s}}
\end{aligned}
$$

where we exploited that $r \geq 1$ and a calculation similar to (31). By proposition 6.5, $\|u\|_{H^{s}}^{2} \lesssim \mu$ for any near minimizer $u \in U_{\mu}$; thus we can pick $\mu_{*}>0$ small enough so that whenever $\mu \in\left(0, \mu_{*}\right)$, a near minimizer $u \in U_{\mu}$ satisfies $\left\|\varphi_{r} u\right\|_{H^{s}},\left\|\psi_{r} u\right\|_{H^{s}} \leq R / 2$ for all $r \geq 1$. From here on we assume such a $\mu_{*}$ has been picked.

Step 2: Constructing the auxiliary sequences $\left(v_{n}^{1}\right),\left(v_{n}^{2}\right) \subset U$.
We now turn to a minimizing sequence $\left(u_{n}\right) \subset U_{\mu}$, for which dichotomy occurs. By the Concentration-Compactness principle, there exist sequences $\left(x_{n}\right),\left(r_{n}\right) \subset \mathbb{R}$ so that when $n \rightarrow \infty$, we have $r_{n} \rightarrow \infty$ and

$$
\begin{align*}
& \frac{1}{2} \int_{|x| \leq r_{n}}\left|u_{n}\left(x-x_{n}\right)\right|^{2} d x \rightarrow \lambda>0, \\
& \frac{1}{2} \int_{|x| \geq 2 r_{n}}\left|u_{n}\left(x-x_{n}\right)\right|^{2} d x \rightarrow \mu-\lambda>0 . \tag{36}
\end{align*}
$$

Without loss of generality we assume $x_{n}=0$ for all $n$; notice that the value of $\mathcal{E}(u)$ and $\|u\|_{H^{s}}$ are unaffected by translation of $u$. We can also assume without loss of generality that $r_{n} \geq 1$, and that $\left(u_{n}\right)$ consist solely of near minimizers.

We define $\varphi_{n}(x)=\varphi\left(x / r_{n}\right)$ and similarly for $\psi_{n}$ and $\phi_{n}$. We now define two new sequences $\left(v_{n}^{1}\right),\left(v_{n}^{2}\right)$ by

$$
v_{n}^{1}:=\varphi_{n} u_{n}, \quad v_{n}^{2}:=\psi_{n} u_{n}
$$

Without loss of generality, we assume $\left(u_{n}\right)$ to consist solely of near minimizers. By step 1 , we consequently have $\left\|v_{n}^{1}\right\|_{H^{s}},\left\|v_{n}^{2}\right\|_{H^{s}} \leq R / 2$, for all $n$.

Step 3: Constructing the sequences $\left(u_{n}^{1}\right) \subset U_{\lambda},\left(u_{n}^{2}\right) \subset U_{\mu-\lambda}$. In addition to the two limits (36), we also have

$$
\frac{1}{2} \int_{r_{n}<|x|<2 r_{n}}\left|u_{n}\right|^{2} d x \rightarrow 0
$$

As $\varphi_{n}(x)=1$ for $|x| \leq r_{n}, \varphi_{n}(x)=0$ for $|x| \geq 2 r_{n}$ and $0 \leq \varphi_{n}(x) \leq 1$ for $r_{n}<|x|<2 r_{n}$, we calculate

$$
\begin{aligned}
Q\left(v_{n}^{1}\right) & =\frac{1}{2} \int_{\mathbb{R}}\left|\varphi_{n} u_{n}\right|^{2} d x \\
& =\underbrace{\frac{1}{2} \int_{|x| \leq r_{n}}\left|u_{n}\right|^{2} d x}_{\rightarrow \lambda}+\underbrace{\frac{1}{2} \int_{r_{n}<|x|<2 r_{n}}\left|\varphi_{n} u_{n}\right|^{2} d x}_{\rightarrow 0}
\end{aligned}
$$

as $n \rightarrow \infty$. By a very similar calculation, we also have

$$
Q\left(v_{n}^{2}\right) \rightarrow \mu-\lambda
$$

as $n \rightarrow \infty$. By the two limits above, and the fact that $0<\lambda<\mu$, we can assume without loss of generality that $Q\left(v_{n}^{1}\right)>\frac{\lambda}{2}$ and $Q\left(v_{n}^{2}\right)>\frac{\mu-\lambda}{2}$ for all $n$. We now define the sequences $\left(a_{n}\right),\left(b_{n}\right) \subset \mathbb{R}^{+}$by

$$
a_{n}:=\frac{\lambda}{Q\left(v_{n}^{1}\right)}, \quad b_{n}:=\frac{\mu-\lambda}{Q\left(v_{n}^{2}\right)},
$$

which by assumption satisfies $\left|a_{n}\right|,\left|b_{n}\right|<2$ for all $n$. We can now define the sub-sequences $\left(u_{n}^{1}\right),\left(u_{n}^{2}\right)$, by

$$
u_{n}^{1}:=a_{n} v_{n}^{1}, \quad u_{n}^{2}:=b_{n} v_{n}^{2}
$$

where we by construction have $\left\|u_{n}^{1}\right\|_{H^{s}},\left\|u_{n}^{2}\right\|_{H^{s}}<R$ and $Q\left(u_{n}^{1}\right)=\lambda, Q\left(u_{n}^{2}\right)=$ $\mu-\lambda$ for all $n$. Thus $\left(u_{n}^{1}\right) \subset U_{\lambda}$ and $\left(u_{n}^{2}\right) \subset U_{\mu-\lambda}$. We also note that since $a_{n}, b_{n} \rightarrow 1$, we have $\left\|v_{n}^{1}-u_{n}^{1}\right\|_{H^{s}} \rightarrow 0$ and $\left\|v_{n}^{2}-u_{n}^{2}\right\|_{H^{s}} \rightarrow 0$ as $n \rightarrow \infty$.

Step 4: Proving that $\mathcal{E}\left(u_{n}^{1}\right)+\mathcal{E}\left(u_{n}^{2}\right) \rightarrow I_{\mu}$ as $n \rightarrow \infty$.
In this section we will only show directly that $\mathcal{E}\left(v_{n}^{1}\right)+\mathcal{E}\left(v_{n}^{2}\right) \rightarrow I_{\mu}$ as $n \rightarrow \infty$. By the uniform continuity of $\mathcal{E}$ on $U$ (proposition 4.8) and the fact that both $\left\|v_{n}^{1}-u_{n}^{1}\right\|_{H^{s}} \rightarrow 0$ and $\left\|v_{n}^{2}-u_{n}^{2}\right\|_{H^{s}} \rightarrow 0$, we must also have $\mathcal{E}\left(u_{n}^{1}\right)+\mathcal{E}\left(u_{n}^{2}\right) \rightarrow I_{\mu}$ as $n \rightarrow \infty$.

We start with the calculation

$$
\mathcal{L}\left(v_{n}^{1}\right)=L u_{n}\left[\varphi_{n}^{2} u_{n}\right]+[\underbrace{L \varphi_{n} u_{n}\left[\varphi_{n} u_{n}\right]-L u_{n}\left[\varphi_{n}^{2} u_{n}\right]}_{:=A_{n}}]
$$

We recognize that $A_{n}$ is exactly the expression studied in lemma 7.3; the quantities $(\varphi, u, v)$ in lemma 7.3 can be replaced by the quantities $\left(\varphi, u_{n}, \varphi_{n} u_{n}\right)$. Thus $A_{n} \rightarrow 0$ as $n \rightarrow \infty$. Similarly

$$
\mathcal{L}\left(v_{n}^{2}\right)=L u_{n}\left[\psi_{n}^{2} u_{n}\right]+[\underbrace{L \psi_{n} u_{n}\left[\varphi_{r} u_{n}\right]-L u_{n}\left[\psi_{n}^{2} u_{n}\right]}_{:=B_{n}}]
$$

By exploiting the bilinearity of $(u, v) \mapsto L u[v]$, we can rewrite $B_{n}$ as follows

$$
\begin{aligned}
B_{n} & =L \psi_{n} u_{n}\left[\psi_{r} u_{n}\right]-L u_{n}\left[\psi_{n}^{2} u_{n}\right] \\
& =L\left(1-\phi_{n}\right) u_{n}\left[\left(1-\phi_{n}\right) u_{n}\right]-L u_{n}\left[\left(1-\phi_{n}\right)^{2} u_{n}\right] \\
& =L u_{n}\left[\phi_{n} u_{n}\right]-L \phi_{n} u_{n}\left[u_{n}\right] .
\end{aligned}
$$

Again we recognize that $B_{n}$ is exactly the expression studied in lemma 7.3 ; the quantities $(\varphi, u, v)$ in lemma 7.3 can now be replaced by the quantities $\left(\phi, u_{n}, u_{n}\right)$. Thus also $B_{n} \rightarrow 0$ as $n \rightarrow \infty$.

We then have the calculation

$$
\mathcal{L}\left(v_{n}^{1}\right)+\mathcal{L}\left(v_{n}^{2}\right)=\mathcal{L}\left(u_{n}\right)+A_{n}+B_{n}
$$

We turn next to $\mathcal{N}$. For notational convenience we define the two sets $V_{n}$ and $W_{n}$ by $x \in V_{n} \Leftrightarrow r_{n}<|x|<2 r_{n}$ and $W_{n}:=\mathbb{R} \backslash V_{n}$. We now calculate

$$
\begin{aligned}
\mathcal{N}\left(v_{n}^{1}\right)+\mathcal{N}\left(v_{n}^{2}\right) & =\int_{\mathbb{R}} N\left(v_{n}^{1}\right)+N\left(v_{n}^{2}\right) d x \\
& =\int_{W_{n}} N\left(u_{n}\right) d x+\int_{V_{n}} N\left(\varphi_{n} u_{n}\right)+N\left(\psi_{n} u_{n}\right) d x \\
& =\mathcal{N}\left(u_{n}\right)+\underbrace{\int_{V_{n}} N\left(\varphi_{n} u_{n}\right)+N\left(\psi_{n} u_{n}\right)-N\left(u_{n}\right) d x}_{:=C_{n}} .
\end{aligned}
$$

By the fact that $N$ is continuous, $N(x)=O\left(|x|^{p+1}\right)$ as $x \rightarrow 0$, and the fact that $\|\cdot\|_{L^{\infty}}$ is bounded on $U$, we get

$$
\begin{aligned}
\left|C_{n}\right| & \leq \int_{V_{n}}\left|N\left(\varphi_{n} u_{n}\right)\right|+\left|N\left(\psi_{n} u_{n}\right)\right|+\left|N\left(u_{n}\right)\right| d x \\
& \lesssim \int_{V_{n}}\left|\varphi_{n} u_{n}\right|^{2}+\left|\psi_{n} u_{n}\right|^{2}+\left|u_{n}\right|^{2} d x \\
& \lesssim \int_{V_{n}}\left|u_{n}\right|^{2} d x \\
& \rightarrow 0
\end{aligned}
$$

as $n \rightarrow \infty$. Putting everything together, and recalling that $\left(u_{n}\right)$ is a minimizing sequence for $I_{\mu}$, we get

$$
\mathcal{E}\left(v_{n}^{1}\right)+\mathcal{E}\left(v_{n}^{2}\right)=\underbrace{\mathcal{E}\left(u_{n}\right)}_{\rightarrow I_{\mu}}+\underbrace{A_{n}+B_{n}-C_{n}}_{\rightarrow 0},
$$

as $n \rightarrow \infty$. By the discussion at the start of this step, the proof is complete.
Proposition 7.5. Dichotomy does not occur.
Assuming dichotomy occurs on a minimizing sequence $\left(u_{n}\right) \subset U_{\mu}$, it follows from the previous proposition that there is a constant $0<\lambda<\mu$, and sequences $\left(u_{n}^{1}\right) \subset U_{\lambda},\left(u_{n}^{2}\right) \subset U_{\mu-\lambda}$ so that

$$
\begin{equation*}
\mathcal{E}\left(u_{n}^{1}\right)+\mathcal{E}\left(u_{n}^{2}\right) \rightarrow I_{\mu}, \tag{37}
\end{equation*}
$$

as $n \rightarrow \infty$. From $u_{n}^{1} \in U_{\lambda}$ and $u_{n}^{2} \in U_{\mu-\lambda}$, it follows that $I_{\lambda} \leq \mathcal{E}\left(u_{n}^{1}\right)$ and $I_{\mu-\lambda} \leq \mathcal{E}\left(u_{n}^{2}\right)$ for all $n$. By corollary 6.10 and the fact that $0<\lambda<\mu$, we get the calculation

$$
I_{\mu}<I_{\lambda}+I_{\mu-\lambda} \leq \mathcal{E}\left(u_{n}^{1}\right)+\mathcal{E}\left(u_{n}^{2}\right)
$$

for all $n$. By now taking the limit as $n \rightarrow \infty$ on each side we get contradiction

$$
I_{\mu}<I_{\mu} .
$$

Consequently, dichotomy can not occur.

### 7.3 Convergence from concentration

This whole subsection is original. By the two preceding subsections we get the proposition:

Proposition 7.6. For $\mu_{*}>0$ sufficiently small, every minimizing sequence $\left(u_{n}\right) \subset U_{\mu}$ has a subsequence that concentrates.

We are now prepared to prove existence of a minimizer.
Proposition 7.7. If a minimizing sequence $\left(u_{n}\right) \subset U_{\mu}$ concentrates, then it admits a subsequence, denoted again by $\left(u_{n}\right)$, which can be translated by a sequence $\left(x_{n}\right) \subset \mathbb{R}$, to satisfy $u_{n}\left(\cdot-x_{n}\right) \rightharpoonup u \in U_{\mu}$ with $\mathcal{E}(u)=I_{\mu}$.

Proof. As the sequence $\left(u_{n}\right)$ concentrates, it follows from the ConcentrationCompactness principle that there is a sequence of real numbers $\left(x_{n}\right) \subset \mathbb{R}$ so that

$$
\int_{-r}^{r}\left|u_{n}\left(x-x_{n}\right)\right|^{2} d x \geq 2 \mu-\varepsilon_{r}
$$

where $\varepsilon_{r} \rightarrow 0$ as $r \rightarrow \infty$. As in proposition 7.4 , we assume without loss of generality that $x_{n}=0$ for all $n$. As the sequence is bounded with respect to $\|\cdot\|_{H^{s}}$, we can apply proposition 2.3 , to obtain yet another subsequence, again denoted by $\left(u_{n}\right)$, which converges weakly in $H^{s}(\mathbb{R})$; i.e. $u_{n} \rightharpoonup u$ for some element $u \in H^{s}(\mathbb{R})$. By proposition 2.4 and remark 6.6, it follows that

$$
\|u\|_{H^{s}} \leq \liminf _{n \rightarrow \infty}\|u\|_{H^{s}} \leq R / 2
$$

and so $u \in U$. As the inclusion map $H^{s}(\mathbb{R}) \hookrightarrow L^{2}$ is linear and continuous, it follows by proposition 2.5 that $u_{n} \rightharpoonup u$ in $L^{2}$ also. For clarity, we divide the remainder of the proof into steps.

Step 1: We prove that $u_{n} \rightarrow u$ in $L^{2}$ and that $\left(u_{n}\right)$ converges uniformly to $u$ on compact intervals $[-r, r]$.
We now show that $u_{n}$ converges strongly to $u$ in $L^{2}$. For notational convenience we define

$$
v_{n}:=u_{n}-u
$$

and so $u_{n} \rightarrow u$ if and only if $v_{n} \rightarrow 0$ in $L^{2}$. We begin by proving uniform convergence on a compact interval $[-r, r]$. Seeking a contradiction, assume there exist a sequence $\left(x_{n}\right) \subset[-r, r]$ so that $\lim \sup _{n \rightarrow \infty}\left|v_{n}\left(x_{n}\right)\right|=: \epsilon>0$. By compactness, we pick an accumulation point $x \in[-r, r]$ of $\left(x_{n}\right)$, and without loss of generality, we assume $x=0$. As the $H^{s}(\mathbb{R})$-norm of the elements $v_{n}$ are uniformly bounded, we yet again take use of Sobolev embedding to obtain constants $C, \alpha>0$ so that

$$
\begin{equation*}
\left|v_{n}(x)-v_{n}(y)\right|>C|y-x|^{\alpha} \tag{38}
\end{equation*}
$$

With $\epsilon$ as above, we pick $\rho>0$ so that

$$
\epsilon-C \rho^{\alpha}>0
$$

We wish to show that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left|\int_{\frac{-\rho}{2}}^{\frac{\rho}{2}} v_{n} d x\right| \geq c>0 \tag{39}
\end{equation*}
$$

for some $c>0$; this will be enough to reach the desired contradiction. As 0 is an accumulation point of $\left(x_{n}\right)$, there is no loss of generality in assuming that $\left(x_{n}\right) \subset[-\rho / 2, \rho / 2]$; alternatively, we could drop to a subsequence of $\left(x_{n}\right)$ and $\left(v_{n}\right)$. Exploiting the triangle inequality, we now calculate

$$
\begin{aligned}
\left|\int_{\frac{-\rho}{2}}^{\frac{\rho}{2}} v_{n} d x\right| & \geq\left|\int_{\frac{-\rho}{2}}^{\frac{\rho}{2}} v_{n}\left(x_{n}\right) d x\right|-\left|\int_{\frac{-\rho}{2}}^{\frac{\rho}{2}} v_{n}\left(x_{n}\right)-v_{n}(x) d x\right| \\
& \geq \rho\left|v_{n}\left(x_{n}\right)\right|-\int_{\frac{-\rho}{2}}^{\frac{\rho}{2}}\left|v_{n}\left(x_{n}\right)-v_{n}(x)\right| d x \\
& \geq \rho\left|v_{n}\left(x_{n}\right)\right|-C \int_{\frac{-\rho}{2}}^{\frac{\rho}{2}}\left|x_{n}-x\right|^{\alpha} d x \\
& \geq \rho\left|v_{n}\left(x_{n}\right)\right|-C \int_{\frac{-\rho}{2}}^{\frac{\rho}{2}} \rho^{\alpha} d x \\
& =\rho\left|v_{n}\left(x_{n}\right)\right|-C \rho^{\alpha+1} .
\end{aligned}
$$

Now (39) follows immediately, from the calculation

$$
\limsup _{n \rightarrow \infty}\left|\int_{\frac{-\rho}{2}}^{\frac{\rho}{2}} v_{n} d x\right| \geq \limsup _{n \rightarrow \infty} \rho\left|v_{n}\left(x_{n}\right)\right|-C \rho^{\alpha+1}=\rho\left[\epsilon-C \rho^{\alpha}\right]>0
$$

where the last inequality follows from the choice of $\rho$. Next, we define $f$ to be the function

$$
f(x)= \begin{cases}1 & |x| \leq \frac{\rho}{2} \\ 0 & x>\frac{\rho}{2}\end{cases}
$$

Clearly $f \in L^{2}$, and since $v_{n} \rightharpoonup 0$, it follows by the definition of weak convergence (and the Riesz representation theorem) that

$$
\int_{\frac{-\rho}{2}}^{\frac{\rho}{2}} v_{n} d x=\int_{\mathbb{R}} v_{n} \bar{f} d x=\left\langle v_{n}, f\right\rangle \rightarrow 0
$$

as $n \rightarrow \infty$. This conflicts with (39), and so we have reach our desired contradiction; consequently $u_{n}$ must converge uniformly to $u$ on $[-r, r]$. Combining this with the concentration of $u_{n}$, we conclude that

$$
\int_{-r}^{r}|u|^{2} d x \geq 2 \mu-\varepsilon_{r}
$$

This observation, together with the fact $\|u\|_{L^{2}}^{2} \leq \liminf _{n \rightarrow \infty}\left\|u_{n}\right\|_{L^{2}}^{2}=2 \mu$ (by proposition 2.4) implies that

$$
\int_{|x|>r}|u|^{2} d x<\varepsilon_{r}
$$

which obviously also hold too for all $u_{n}$. Picking $r>0$ we calculate
$\limsup _{n \rightarrow \infty}\left\|u_{n}-u\right\|_{L^{2}}^{2} \leq \underbrace{\limsup _{n \rightarrow \infty} \int_{|x| \leq r}\left|u_{n}-u\right|^{2} d x}_{=0}+\underbrace{\limsup _{n \rightarrow \infty} \int_{|x|>r} 2\left|u_{n}\right|^{2}+2|u|^{2} d x}_{\leq 4 \varepsilon_{r}}$.
As $\varepsilon_{r} \rightarrow 0$ as $r \rightarrow \infty$, we are done. We now know $u \in U_{\mu}$, and so $u$ is truly a candidate for a minimizer of $I_{\mu}$.

Step 2: We show that $\mathcal{E}(u)=I_{\mu}$.
Since $u \in U_{\mu}$ it follows that $I_{\mu} \leq \mathcal{E}(u)$. Thus the proposition is proved if we can establish that $\mathcal{E}(u) \leq I_{\mu}$. Combining the second inequality in (22) with the inequality $x^{2} \leq 2(x-y)^{2}+2 y^{2}$, we have that $\mathcal{N}$ satisfies

$$
\begin{equation*}
\left|\mathcal{N}\left(u_{n}\right)-\mathcal{N}(u)\right| \leq \int_{\mathbb{R}}\left(2\left(u_{n}-u\right)^{2}+3 u^{2}\right) \omega\left(\left|u_{n}-u\right|\right) d x \tag{40}
\end{equation*}
$$

for some modulus of continuity $\omega$. Note that the quantities $\left\|u_{n}-u\right\|_{L^{2}}$ and $\omega\left(\left\|u_{n}-u\right\|_{L^{\infty}}\right)$ are uniformly bounded in $n$, as $u_{n} \in U$ for all $n$. For fixed $r>0$, we split the integral (40) up into $|x| \leq r$ and $|x|>r$ to get

$$
\begin{equation*}
\left|\mathcal{N}\left(u_{n}\right)-\mathcal{N}(u)\right| \lesssim \sup _{|x| \leq r} \omega\left(\left|u_{n}-u\right|\right)+\left\|u_{n}-u\right\|_{L^{2}}+\int_{|x|>r} u^{2} d x \tag{41}
\end{equation*}
$$

where we stress that the implicit constant of $\lesssim$ is not dependent on $r$. As $\left(u_{n}\right)$ converges uniformly to $u$ on compact intervals, and strongly in $L^{2}$, it is clear that (for fixed $r$ ) the first two terms on the right in (41) tends to zero as $n \rightarrow \infty$. The last term can be made arbitrarily small as $r \rightarrow \infty$, and so we conclude that

$$
\mathcal{N}\left(u_{n}\right) \rightarrow \mathcal{N}(u)
$$

as $n \rightarrow \infty$. It remains to show

$$
\begin{equation*}
\mathcal{L}(u) \leq \liminf _{n \rightarrow \infty} \mathcal{L}\left(u_{n}\right) \tag{42}
\end{equation*}
$$

We first define the norm $\|\cdot\|_{m}$ on $H^{s}(\mathbb{R})$ by

$$
\|u\|_{m}:=\int_{\mathbb{R}}(m(\xi)+1)|\hat{u}|^{2} d \xi
$$

Since $m+1 \underset{\tilde{H}^{\prime}}{\sim}\langle\cdot\rangle^{2 s}$ it follows that the norms $\|\cdot\|_{m}$ and $\|\cdot\|_{H^{s}}$ are equivalent. If we define $\tilde{H}^{s}$ to be the vector space $H^{s}(\mathbb{R})$ equipped with $\|\cdot\|_{m}$ as norm, it
follows from the previous observation that the inclusion map $H^{s}(\mathbb{R}) \hookrightarrow \tilde{H}^{s}$ is a continuous linear mapping. Consequently by proposition 2.5 , we have $u_{n} \rightharpoonup u$ in $\tilde{H}^{s}$ and so

$$
\begin{aligned}
\int_{\mathbb{R}} m|\hat{u}|^{2} d \xi+2 \mu & =\|u\|_{m}^{2} \\
& \leq \liminf _{n \rightarrow \infty}\left\|u_{n}\right\|_{m}^{2} \quad \quad \text { (by proposition 2.4) } \\
& =\liminf _{n \rightarrow \infty} \int_{\mathbb{R}} m\left|\hat{u}_{n}\right|^{2} d \xi+2 \mu
\end{aligned}
$$

Subtracting $2 \mu$ on each side in the inequality above, we are left with the representation of $\mathcal{L}$ on real valued functions (proposition 4.4), and thus we have established (42). A final calculation gives the desired result

$$
\begin{aligned}
\mathcal{E}(u) & =\mathcal{L}(u)-\mathcal{N}(u) \\
& \leq \liminf _{n \rightarrow \infty} \mathcal{L}\left(u_{n}\right)-\mathcal{N}(u) \\
& =\liminf _{n \rightarrow \infty}\left[\mathcal{L}\left(u_{n}\right)-\mathcal{N}\left(u_{n}\right)\right] \\
& =I_{\mu}
\end{aligned}
$$

Thus $I_{\mu} \leq \mathcal{E}(u) \leq I_{\mu}$, and so $u$ is a minimizer of $I_{\mu}$.

## 8 From minimizers to solutions

This section is completely original, that is, the author has not taken inspiration from other sources. We now establish the connection between the constrained minimization problem and our partial differential equation (2).

Proposition 8.1. Any minimizer $u \in U_{\mu}$ of $I_{\mu}$ solves (2), with velocity $\nu$ given by

$$
\begin{equation*}
\nu=\frac{\mathcal{E}^{\prime}(u)[u]}{2 \mu} \tag{43}
\end{equation*}
$$

Proof. By proposition 4.9 we have $Q^{\prime}(u)=u$ in $H^{-s}(\mathbb{R})$. Pick any real valued function $v \in H^{s}(\mathbb{R})$, so that $Q^{\prime}(u)[v]=\int_{\mathbb{R}} u v d x=0$. Then

$$
a(t):=\frac{Q(u+t v)}{\mu}=1+c t^{2},
$$

where we have defined $c=Q(v) / \mu$. We next define $w_{t} \in H^{s}(\mathbb{R})$

$$
w_{t}:=\frac{u+t v}{a(t)}-u
$$

and notice that $u+w_{t} \in U_{\mu}$ for $t$ sufficiently close to 0 . We can also rewrite $w_{t}$ to obtain

$$
w_{t}=t v+\left[\frac{1-a(t)}{a(t)}\right](t v+u)=t v+O\left(t^{2}\right)(t v+u)
$$

A quick calculation shows that

$$
\begin{aligned}
\mathcal{E}\left(u+w_{t}\right) & =\mathcal{E}(u)+\mathcal{E}^{\prime}(u)\left[w_{t}\right]+o\left(\left\|w_{t}\right\|_{H^{s}}\right) \\
& =I_{\mu}+t \mathcal{E}^{\prime}(u)[v]+\mathcal{O}\left(t^{2}\right) \mathcal{E}^{\prime}(u)[t v+u]+o\left(t\|v\|_{H^{s}}+t^{2}\|u\|_{H^{s}}\right) \\
& =I_{\mu}+t \mathcal{E}^{\prime}(u)[v]+o(t)
\end{aligned}
$$

Pick $\varepsilon>0$ so that $u+w_{t} \in U_{\mu}$, whenever $t \in(-\varepsilon, \varepsilon)$. For such $t$ we have by definition $\mathcal{E}\left(u+w_{t}\right) \geq I_{\mu}$. If $\mathcal{E}^{\prime}(u)[v] \neq 0$, then the above calculation shows that we can find $t \in(-\varepsilon, \varepsilon)$ so that $\mathcal{E}\left(u+w_{t}\right)<I_{\mu}$; a contradiction. Thus for a real valued function $v \in H^{s}(\mathbb{R})$ we have

$$
\begin{equation*}
Q^{\prime}(u)[v] \Longrightarrow \mathcal{E}^{\prime}(u)[v] . \tag{44}
\end{equation*}
$$

By proposition 2.24, any function $v \in H^{s}(\mathbb{R})$ can be split up into its real and imaginary part, $v:=v_{R}+i v_{I}$, where $v_{R}, v_{I}$ are real valued functions in $H^{s}(\mathbb{R})$. As $u \in U_{\mu}$, we have that $u$ is real valued; by linearity of $Q^{\prime}(u)$ we conclude

$$
\operatorname{Re} Q^{\prime}(u)[v]=Q^{\prime}(u)\left[v_{R}\right], \quad \text { and } \quad \operatorname{Im} Q^{\prime}(u)[v]=Q^{\prime}(u)\left[v_{I}\right]
$$

These equations imply that $Q^{\prime}(u)[v]=0$ exactly when $Q^{\prime}(u)\left[v_{R}\right]=0$ and $Q^{\prime}(u)\left[v_{I}\right]=0$. By the previous observation, the implication (44), and the linearity of $\mathcal{E}^{\prime}(u)$ we conclude that

$$
\begin{equation*}
\operatorname{ker} Q^{\prime}(u) \subseteq \operatorname{ker} \mathcal{E}^{\prime}(u) \tag{45}
\end{equation*}
$$

We note that $2 \mu=Q^{\prime}(u)[u]$, and calculate for $v \in H^{s}(\mathbb{R})$,

$$
\mathcal{E}^{\prime}(u)[v]=\mathcal{E}^{\prime}(u)[\underbrace{v-\frac{Q^{\prime}(u)[v]}{2 \mu} u}_{\in \operatorname{ker} Q^{\prime}(u)}]+\underbrace{\frac{\mathcal{E}^{\prime}(u)[u]}{2 \mu}}_{:=\nu} Q^{\prime}(u)[v]
$$

As $v \in H^{s}(\mathbb{R})$ was arbitrary, we conclude from (45) that $\mathcal{E}^{\prime}(u)=\nu Q^{\prime}(u)$; by corollary 4.12 , we insert for $\mathcal{E}^{\prime}(u)$ and $Q^{\prime}(u)$, and rearrange to obtain

$$
-\nu u+L u-n(u)=0
$$

As a minimizer $u \in U_{\mu}$ is also a near minimizer, we can exploit properties of near minimizers to establish the following corollary.

Corollary 8.2. For $\mu_{*}>0$ small enough, there exist a constant $\rho>0$ satisfying the following property:

Pick $\mu \in\left(0, \mu_{*}\right)$ and let $u \in U_{\mu}$ be both solution of (2) and a minimizer of $I_{\mu}$ with corresponding velocity $\nu$; then $\nu$ satisfies

$$
\rho \mu^{\beta}<m(0)-\nu<C_{\varepsilon} \mu^{\beta-\varepsilon}
$$

where $\beta$ is as in proposition 5.2, $\varepsilon>0$ is arbitrary and $C_{\varepsilon}$ is a constant only dependent of $\varepsilon$.
Proof. For $\mu_{*}>0$, let $u \in U_{\mu}$ be a solution of (2) and a minimizer of $I_{\mu}$ with corresponding velocity $\nu$. We naturally divide the proof in two.

Step 1: Proving $m(0)-\nu<C_{\varepsilon} \mu^{\beta-\varepsilon}$.
For this bound, no requirements need to be put on $\mu_{*}$. For $\varepsilon>0$ we set

$$
\tau:=\frac{\beta}{2 s}-2 \varepsilon=\frac{2 \beta}{(p-1)}-1-2 \varepsilon
$$

then it follows from lemma 6.7, that

$$
\begin{equation*}
\|\cdot\|_{L^{\infty}}^{p-1} \lesssim \mu^{(p+1)(1+\tau) / 2}=\mu^{\beta-\varepsilon} \tag{46}
\end{equation*}
$$

on $\tilde{U}$. By similar methods used to prove proposition 4.6 , we see that

$$
\begin{equation*}
\left|\int_{\mathbb{R}} n(u) u d x\right| \lesssim \mu\|u\|_{L^{\infty}}^{p-1} \lesssim \mu^{1+\beta-\varepsilon} \tag{47}
\end{equation*}
$$

where the use of (46) is justified as $u$ is a near minimizer. Thus, for some $C_{\varepsilon}>0$ we get

$$
\begin{aligned}
\nu & =\frac{\mathcal{E}^{\prime}(u)[u]}{2 \mu} \\
& =\frac{1}{\mu} \mathcal{L}(u)-\frac{1}{2 \mu} \int_{\mathbb{R}} n(u) u d x \\
& >m(0)-C_{\varepsilon} \mu^{\beta-\varepsilon}
\end{aligned}
$$

where we used (47) and proposition 4.5.
Step 2: Proving $\rho \mu^{\beta}<m(0)-\nu$.
By the definition of $n_{p}$ and its primitive $N_{p}$, we see that

$$
n_{p}(x) x=(p+1) N_{p}(x)
$$

and so

$$
\int_{\mathbb{R}} n(u) u d x=(p+1) \mathcal{N}(u)+[\underbrace{\int_{\mathbb{R}} n_{r}(u) u-(p+1) N_{r}(u) d x}_{:=g(u)}]
$$

By similar methods used to prove proposition 4.6, see that

$$
|g(u)| \lesssim \mu\|u\|_{L^{\infty}}^{p-1+\delta}=o\left(\mu^{1+\beta}\right)
$$

where the latter equality follows from corollary 6.8 and the fact that $u$ is a near minimizer. By (43), we then calculate

$$
\begin{aligned}
\nu & =\frac{1}{\mu} \mathcal{L}(u)-\frac{1}{2 \mu} \int_{\mathbb{R}} n(u) u d x \\
& =\frac{1}{\mu} \mathcal{L}(u)-\frac{p+1}{2 \mu} \mathcal{N}(u)+o\left(\mu^{\beta}\right) \\
& <m(0)-\frac{p+1}{2} \kappa \mu^{\beta}+o\left(\mu^{\beta}\right)
\end{aligned}
$$

where the last inequality follows by proposition 4.5 and remark 6.3 . By defining $2 \rho:=\kappa(p+1) / 2$, we arrive at

$$
2 \rho \mu^{\beta}+o\left(\mu^{\beta}\right)<m(0)-\nu
$$

As $\rho>0$ is independent of $\mu$, it is clear from the above inequality, that we can set $\mu_{*}>0$ small enough so that that the absolute value of the $o$-term is dominated by $\rho \mu^{\beta}$ whenever $\mu \in\left(0, \mu_{*}\right)$. For such a $\mu_{*}$ we get

$$
\rho \mu^{\beta}<m(0)-\nu .
$$

### 8.1 Proof of Theorem 1.1

We have proved everything that is needed for this theorem to come together. For $\mu_{*}>0$ small enough, and $\mu \in\left(0, \mu_{*}\right)$, we can sum the proof up as follows: Proposition 7.6 implies every minimizing sequence in $U_{\mu}$ admits a subsequence that concentrates. Corollary 7.7 further implies that every concentrating minimizing sequence admits a subsequence that converges weakly to a minimizer of $I_{\mu}$. Proposition 8.1 shows that a minimizer of $I_{\mu}$ solves (2), while corollary 8.2 proves the relevant bounds for the corresponding velocity.

## 9 Regularity of solutions

This section aims to describe regularity of functions described by Theorem 1.1. For a function $g: \mathbb{R} \rightarrow \mathbb{R}$, let $P_{g}$ be the operator

$$
\begin{equation*}
P_{g}: f \mapsto g(f), \tag{48}
\end{equation*}
$$

for any function $f: \mathbb{R} \rightarrow \mathbb{R}$. We introduce a sufficient criterion for $P_{g}$ to map $H^{m}(\mathbb{R})$ continuously into $H^{m}(\mathbb{R})$, when $m=1,2, \ldots$. This result is due to [11]; it is obtained by combining Theorem. (ii) (p.267) and Theorem 1. (p.268), together with some simplifications (i.e. stronger assumptions).

Proposition 9.1 ([11]). Let $P_{g}$ be an operator as described by (48) and suppose $g(0)=0$. If $g \in C^{k+1}$, for $k \in \mathbb{N}_{0}$, then $P_{g}$ maps $H^{k+1}(\mathbb{R})$ continuously into $H^{k+1}(\mathbb{R})$.

Next we prove an original result.
Proposition 9.2. If $m(0)-\nu>0$, then the operator $L-\nu$ has a bounded inverse $(L-\nu)^{-1}$, that maps $H^{r}(\mathbb{R})$ continuously into $H^{r+2 s}(\mathbb{R})$.

Proof. Define $\tilde{m}(\xi):=m(\xi)-\nu$, and notice that $\tilde{m}(\xi) \geq m(0)-\nu$ for all $\xi$. When $m(0)-\nu>0$, it follows by the assumption on $m$, that $\tilde{m} \gtrsim\langle\cdot\rangle^{2 s}$, and consequently $\tilde{m}^{-1} \lesssim\langle\cdot\rangle^{-2 s}$. We define the two continuous linear operators $\tilde{L}, \tilde{L}^{-1}: \mathscr{S}^{\prime} \rightarrow \mathscr{S}^{\prime}$ by

$$
\mathcal{F}\{\tilde{L} f\}=\tilde{m} \hat{f}, \quad \text { and } \quad \mathcal{F}\left\{\tilde{L}^{-1} f\right\}=\tilde{m}^{-1} \hat{f}
$$

where linearity and continuity follows from similar arguments as those used to prove proposition 4.2 . From this definition, it is clear that $\tilde{L}^{-1}$ is the inverse of $\tilde{L}$. Together with the bound $\tilde{m}^{-1} \lesssim\langle\cdot\rangle^{-2 s}$, and the fact that $\tilde{m}$ is a real valued even function, we can again use similar arguments as those in the proof of proposition 4.2 , we get that $\tilde{L}^{-1} \operatorname{maps} H^{r}(\mathbb{R})$ continuously into $H^{r+2 s}(\mathbb{R})$ for every $r \in \mathbb{R}$. We finally notice that $\tilde{L}=L-\nu$, and the proof is complete.

Before we prove the main theorem of this section, we prove a proposition which in many ways is just as interesting.

Proposition 9.3. Suppose $n \in C^{k}$, for some $k \in\{1,2, \ldots\}$. Let $s \geq 1$, and suppose $u \in H^{s}(\mathbb{R})$ is a solution of (2), with velocity $\nu$ satisfying $m(0)-\nu>0$. Then $u \in H^{2 s+k}(\mathbb{R})$.

Proof. We rewrite (2), to obtain $(L-\nu) u=n(u)$. By proposition 9.2 and the fact that $m(0)-\nu>0$, we can further rewrite the preceding equation to obtain

$$
\begin{equation*}
u=\underbrace{(L-\nu)^{-1} \circ P_{n}}_{:=\Psi}(u), \tag{49}
\end{equation*}
$$

where $P_{n}$ is the operator $P_{n}(f)=n(f)$ for any $f \in H^{r}(\mathbb{R})$ with $r>\frac{1}{2}$. By assumption, $n$ is continuous and satisfies $n(x)=\left(|x|^{p-1}\right)$ as $x \rightarrow 0$;
thus $n(0)=0$. Set $j=1$. Since $n \in C^{k}$ implies that $n \in C^{j}$, it follows by proposition 9.1 and 9.2 , that the operator $\Psi$ maps $H^{j}(\mathbb{R})$ into $H^{j+2 s}$. As $s \geq 1$, we have the inclusion $H^{s} \hookrightarrow H^{1}$ (remark 2.23), and consequently $\Psi(u) \in$ $H^{1+2 s}$. By (49), this implies that $u \in H^{1+2 s}$. If $k \geq 2$, we can repeat the previous argument for $j=2$ and exploit the embedding $H^{1+2 s} \hookrightarrow H^{2}$, to obtain $\Psi(u) \in H^{2+2 s}$ which again implies $u \in H^{2+2 s}$. Repeating this process until $j=k$, we reach the desired conclusion.

We are now ready to prove the main theorem of this section.

Theorem 9.4 (Regularity of solutions). Assume $s \geq 1$ and that $n \in C^{k}$, for $k \in\{1,2, \ldots\}$. Write $2 s+k=m+\alpha+\frac{1}{2}$, where $m \in \mathbb{N}_{0}$ and $0<\alpha \leq 1$. The solutions $u \in H^{s}$ of (2) described by Theorem 1.1, satisfies the following:
(i) If $\alpha \neq 1$, then $u \in C^{m, \alpha}$.
(ii) If $\alpha=1$, then $u \in C^{m, \gamma}$, for every $\gamma \in(0,1)$.

Proof. By proposition 9.3, $u \in H^{k+2 s}$. The first part now follows directly by Sobolev embedding (Theorem 2.32). For the second part, we can for every $\varepsilon>0$ continuously embed $H^{k+2 s} \hookrightarrow H^{k+2 s-\varepsilon}$; we consequently also get the second part by Sobolev embedding.

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