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Adaptive Stabilization and Set-Point Regulation of Linear 2 x 2 Hyperbolic Systems

With Application to Automatic Attenuation of
Kick and Loss in Managed Pressure Drilling

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HOVEDOPPGAVE

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Oppgavens tittel: Automatic Attenuation of Loss or Gain During Drilling

Background

Many interesting problems in the oil and gas industry face the challenge of responding to disturbances from afar. Typically, the disturbance occurs at the inlet of a pipeline or at the bottom of an oil well, while sensing and actuation equipment is installed at the outlet, only. A new method for removing the effect of the disturbance at the inlet boundary by co-located output feedback control at the outlet boundary was derived in [1], and has later been extended in various ways. The topic of this MSc thesis work is to develop the method further so that it can be applied for automatic attenuation of loss or gain during drilling. The following points should be addressed by the student:

Tasks:

1) Review relevant literature on the methods exploited in [1] and the developments that build on [1] (a number of recent papers, many of which have not yet been published, will be provided).
2) Rewrite the drilling model in Section IV of [1] to accommodate a simple production-index-based inflow at the bottom ($q_{res}=p_i(p_{res}-p_{bottom})$), where both the driving reservoir pressure (p_{res}) and the production index (p_i) are assumed unknown. Derive the transformation of your model into the (u,v) coordinates amenable for control design by backstepping, and write your model in these coordinates. Discuss to what extent it fits into available control design methods.
3) Design an adaptive control law (preferably output feedback with measurements restricted to be taken topside) that stabilizes the down-hole pressure at the (unknown) setpoint $p_{bottom}=p_{res}$.
4) Demonstrate the performance of the system in simulations. Matlab code that computes the controller and observer gains in [1] will be (partially) provided.
5) If time permits, the following optional tasks should be considered:
   a. Suppose p_{bottom} is available. Investigate whether this can be used to redesign the control strategy from 3) to improve performance.
   b. Consider writing a scientific paper on the results of the thesis work.
6) Write a report.

Faglærer/Veileder: Professor Ole Morten Aamo

[1] O.M. Aamo, "Disturbance rejection in 2x2 linear hyperbolic systems", IEEE Transactions on Automatic Control, 2013.

Preface

This Master Thesis has been carried out at the Department of Engineering Cybernetics at The Norwegian University of Science and Technology in the spring of 2017.

I would like to thank my supervisor Ole Morten Aamo for giving me the opportunity to have him as my supervisor and for all the encouraging guidance throughout the semester. I would also like to thank PhD Candidate Henrik Anfinsen for taking the time to answer all my questions and for valuable, candid feedback on both the theoretical part and the conference papers. A special thanks goes to my family for supporting me the last five years and to Ann Christin Reiersølmoen for always being there for me.

Haavard Holta
Trondheim, June 2017

Abstract

Linear 2×2 hyperbolic partial differential equations can be used to describe many real-world problems and have attracted considerable research interest in later years. This thesis considers adaptive set-point regulation of such systems by using the infinite-dimensional backstepping method. Two control methods are proposed. The first for a system where sensing is restricted to be collocated with actuation and anti-collocated with two unknown parameters in an affine boundary condition. The second for a system where sensing is non-collocated with actuation and where the uncertain boundary parameters, anti-collocated with actuation, appear in a bilinear form with two unknown parameters. Boundedness in the L_2-sense and point-wise in space are proved. Convergence to the selected set-point at the left boundary is shown for both methods. Because the boundary parameters are unknown, the steady state solution achieving this set-point is also unknown, transforming the problem to a tracking problem with a parameter-estimate-dependent tracking objective. The parameter estimates are generated by a parameter update law operating on linear parametric models relating the system state estimates, measurements and boundary parameters.

The theory is applied to the Kick and Loss Detection and Attenuation Problem in Managed Pressure Drilling, where the goal is to attenuate any sudden inflow into the well-bore or outflow into the reservoir by controlling the bottom-hole pressure from top-side actuation. In addition, the reservoir pressure and the relation between pressure difference and net inflow are unknown. Simulations show that compared to using constant top-side actuation, the derived methods provide a significant reduction in total inflow and convergence time before the kick/loss is completely attenuated. The method using non-collocated sensing performs marginally better than the method only using collocated sensing and control in terms of less overshoot and oscillations. The difference, however, is insignificant for the tested parameter values. Both methods are able to attenuate the kick/loss in a time close to the theoretical constraint imposed by the bottom-hole – top-side – bottom-hole propagation time.

Sammendrag

Lineære 2×2 hyperbolske partielle differensialligninger kan brukes til å beskrive mange fysiske systemer og har vært gjenstand for omfattende forskning i de senere år. Denne oppgaven omhandler adaptiv setpunkt-regulering av slike systemer ved å bruke metoden uendelig-dimensjonal-"backstepping". To kontrollmetoder er foreslått. Den første for systemer hvor måling er begrenset til å være samlokalisert med pådraget og anti-samlokalisert med to ukjente grenseparametere i en affin grensebetingelse. Den andre for systemer hvor måling er tillatt å være ikke-samlokalisert med pådraget og hvor de ukjente grenseparameterne, ikke-samlokalisert med pådraget, opptrer i en bilineær form med to ukjente parametere. Begrensethet i L_2 og punktvis i rom er bevist. Konvergens til det valgte settpunktet på venstre grense er vist for begge metodene. Fordi grenseparameterne er ukjente vil også stasjonærtilstanden som oppnår dette settpunktet være ukjent, og problemet blir essensielt transformert til et sporingsproblem med et parameterestimat-avhengig sporingsmål. Parameterestimatene er generert av en parameter-oppdateringslov som opererer på en lineær parametrisk modell som relaterer systemtilstandene, målingene og grenseparameterne til hverandre.

Teorien er anvendt på "Kick" og "Loss" Deteksjons- og Dempings-problemet i "Managed Pressure Drilling" hvor målet er å dempe enhver plutselig innstrømming inn i brønnhullet eller utstrømming ut i reservoaret ved å kontrollere trykket nedhulls ved hjelp av et toppside-pådrag. Reservoartrykket, og forholdet mellom trykkdifferanse og netto innstrømming er i tillegg ukjent. Sammenlignet med når et konstant topside-pådrag er brukt, viser simuleringer at begge metoder gir en betydelig reduksjon i total innstrømming og konvergenstid før "kick"-et eller "loss"-et er fullstendig dempet. Metoden som bruker ikke-samlokalisert måling og pådrag yter marginalt bedre enn metoden som kun bruker samlokalisert måling og pådrag i form av mindre oversvingninger og oscilleringer. Forskjellen er imidlertid ubetydelig med de valgte parameterverdiene. Begge metodene klarer å dempe "kick"-et/"loss"-et innen en tid som er nærme den teoretiske begrensingen ilagt av forplantningstiden nedhull – toppside – nedhull.

Contents

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List of Symbols

Hyperbolic system with collocated sensing and control

$\alpha(x,t), \beta(x,t)$	Target system state variables.
$\bar{v}(t), \psi(t)$	Parametric model, regressor.
$\eta(x,t)$	Target system state variable.
Γ	Adaptation gain.
λ, μ	System parameters: Transport speeds.
$\mathcal{G}, \mathcal{G}_0$	Backstepping operators.
$\mathcal{K}, \mathcal{K}_0$	Backstepping operators.
$\phi(x,t)$	Reference model state variable.
θ	Parameter vector.
θ_1, θ_2	Boundary parameters.
$\vartheta(t)$	Parametric model, regressand.
$\zeta(x,t)$	Target system state variable.
$c_1(x), c_2(x)$	System parameters: Source terms.
$D_1(x,\xi,t), D_2(x,\xi,t)$	Target system kernels.
d_α, d_β, t_F	Propagation times.
$g(x,t)$	Transformation kernel.
$G_0(x,\xi,t)$	Inverse transformation kernel.
$K^u(x,\xi), K^v(x,\xi)$	Controller transform kernels.
$P^u(x,\xi,t), P^v(x,\xi,t)$	Observer transform kernels.
$P_1(x,t), P_2(x,t)$	Injection gains.

r	Control objective constant.
t	Time variable.
$U(t)$	Actuation.
$u(x,t), v(x,t)$	State variables.
$V_i(t)$	Lyapunov function.
$w(x,t), z(x,t)$	Target system state variables.
x	Domain variable.
$y(t)$	Measurement collocated with actuation.

MPD system

β	Bulk modulus.
ρ	Mud density.
A_1	Annulus cross sectional area.
F_1	Friction factor.
g	Gravity constant.
J	Productivity index.
l	Well length.
$p(z,t)$	Pressure.
$p_r(t)$	Reservoir pressure.
$q(z,t)$	Volumetric flow.
q_{bit}	Drill bit flow.
t	Time variable.
z	Domain variable.

Hyperbolic system with non-collocated sensing and control

$\alpha(x,t), \beta(x,t)$	Target system state variables.
γ_1, γ_2	Adaptation gains.
λ, μ	Transport speeds.
$\mathcal{K}_1, \mathcal{K}_2$	Backstepping operators.
$\mathcal{P}_1, \mathcal{P}_2$	Backstepping operators.

$\nu(x,t), \eta(x,t)$	Deviation state variables.
$\omega(x,t), \zeta(x,t)$	Target system state variables.
$\varepsilon(t)$	Linear combination of the static error terms.
$\varphi(x,t), \phi(x,t)$	Reference model state variables.
$a(x,t), b(x,t)$	Swapping filter states.
$B_1(x,\xi,t), B_2(x,\xi,t)$	Target system kernels.
$c_1(x), c_2(x)$	System coefficients.
d_α, d_β, t_F	Propagation times.
$e(x,t), \epsilon(x,t)$	Static estimation error.
k, θ	Boundary parameters.
$K^{uu}, K^{uv}, K^{vu}, K^{vv}$	Transformation kernels.
$m(x,t), n(x,t)$	Swapping filter states.
$P^{uu}, P^{uv}, P^{vu}, P^{vv}$	Transformation kernels.
$P_1(x,t), P_2(x,t)$	Injection gains.
q, d	Alternative boundary parameters.
r	Controller objective constant.
t	Time variable.
$U(t)$	Actuation.
$u(x,t), v(x,t)$	State variables.
$V_i(t)$	Lyapunov function.
$w(x,t), z(x,t)$	Swapping filter states.
x	Domain variable.
$y_0(t), y_1(t)$	Measurements.

Sets

$\mathbb{R}$	Real numbers.
$\mathbb{R}^+$	Positive real numbers, including zero.
C	Continuous functions.
C^1	Functions with continuous first derivatives.

Abbreviations

ABP	Applied Back Pressure
BC	Boundary Condition
BVP	Boundary Value Problem
IVP	Initial Value Problem
LMI	Linear Matrix Inequality
MPD	Managed Pressure Drilling
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
PE	Persistently Exciting

Part I

Background

Chapter 1

Introduction

1.1 Motivation

This thesis concerns stabilization and adaptive set-point regulation of linear 2×2 partial differential equations (PDEs) of hyperbolic type. Many real world physical problems can be described by such systems. Examples include road traffic systems (Goatin, 2006; Fan et al., 2013), transmission lines (Curró et al., 2011), open fluid channels (de Halleux et al., 2003; Dos Santos and Prieur, 2008), gas pipeline networks (Gugat et al., 2011), and leak detection, estimation and localization in pipe flows (Aamo, 2016). Although the theoretical contributions in this thesis have many potential applications, the development is motivated by a specific problem encountered in oil and gas drilling operations; the *Kick & Loss Detection and Attenuation Problem in Managed Pressure Drilling*.

The drill system consists of a drill string with a drill bit at the bottom-hole end and a casting around the drill string called *annulus*. A drilling fluid called *mud* is circulated down the drill string, through the drill-bit and up the annulus to the surface where cuttings are removed and the mud recirculated down the drill string again. The purpose of the mud is not only to transport the cuttings out, but to provide pressure control throughout the well. If the pressure is too low, the well might collapse, and a too high bottom-hole pressure might lead to fracturing of the formation. Traditionally, pressure is controlled by varying the mud density, viscosity or circulation rate. In *managed pressure drilling* (MPD), with *applied back pressure* (ABP) in particular, the pressure in the annulus is controlled by using a *back pressure valve* top-side to limit the flow and a *back-pressure pump* in the case without circulation. The difficulty in MPD comes from the fact that actuation is located top-side, while the pressure of interest is bottom-hole usually several kilometers away. Sensing is only available at the boundaries and often only top-side. In other words, the task in MPD is to estimate and stabilize the pressure everywhere in the well within some bounds, when the only available measurement and control authority are at the surface.

Stabilizing the well pressure becomes even more challenging when considering the Kick & Loss Detection and Attenuation Problem. A kick is a sudden increase in flow through the annulus caused by a higher formation pressure than mud pressure. The result is formation fluids flowing up the annulus which, if not handled, might lead to uncontrolled blowouts on the surface. In the other case, a loss occur if the mud pressure is higher than the formation pressure, resulting in the loss of mud into the formation. Since the reservoir pressure is usually unknown, the problem is now to stabilize the bottom-hole pressure, using the already challenging MPD technique, so that both kicks and losses are prevented (or at least attenuated). This means that the bottom-hole pressure must be both estimated and controlled while at the same time estimating the reservoir pressure and then regulate the bottom-hole pressure based on this estimate.

The structure of this problem; with distributed states, and sensing and actuation only at boundaries, fits perfectly into the control framework of infinite-dimensional backstepping for PDEs. In addition, the unknown parameter part of the problem can be handled by combining the backstepping method with some kind of adaptive scheme. Previous work in the field of control of PDEs, backstepping control and adaptive control of PDEs are presented in the next section. Some previous work on kick/loss attenuation in MPD is also presented.

1.2 Previous Work

Early efforts in the field of PDE control date back to the late 1960s. The focus was mainly on optimal control and controllability. Research references include Curtain and Zwart (2012); Lasiecka and Triggiani (2000); Christofides (2012). Some early results in control of hyperbolic systems include methods using Riemann invariants along the characteristics (Greenberg and Tsien, 1984) and control Lyapunov functions (Coron et al., 2007).

The backstepping method was originally developed for ordinary differential equations, and is especially useful for nonlinear systems (Krstic et al., 1995; Khalil, 1996). The continuum version of the integrator backstepping method for ODEs uses a Volterra integral operator that "brings" the destabilizing in-domain terms to the boundary where they can be eliminated by an appropriate control law. An introduction to the backstepping method is given in the next chapter. Volterra equations were used as early as in the 1970s to solve PDEs and state controllability (Colton, 1977; Seidman, 1984). Using the same Volterra integral operator for control was not considered until around the year 2000, where an effort to develop the backstepping control technique for partial differential equations was initiated. The first attempt in Boskovic et al. (2001) involved a backstepping-like transformation and an explicit feedback law for a parabolic PDE. The design was, however, limited to systems with at maximum one open-loop unstable eigenvalue. A discretization based scheme was considered in Balogh and Krstic (2002); Bošković et al. (2003); Balogh and Krstic (2004), but this method turned out to be dependent on the discretization scheme and did not give convergent gain kernels. The infinite-dimensional backstepping method in its current form was first introduced for parabolic PDEs in Liu (2003) and further developed in Smyshlyaev and Krstic

(2004, 2005), where the gain kernel was expressed as a solution to a well-posed PDE.

The first result using backstepping applied on hyperbolic PDEs was for first order systems in Krstic and Smyshlyaev (2008). The method was later extended for second order hyperbolic systems in Smyshlyaev et al. (2010), and for two coupled first order hyperbolic systems in Vazquez et al. (2011). The results in the latter were used in Aamo (2013) for disturbance attenuation in managed pressure drilling which is similar to the system considered in this thesis.

While many results exist in the field of adaptive control for parabolic PDEs (Smyshlyaev and Krstic, 2010), adaptive control of hyperbolic PDEs is relatively new. Adaptive observers for $n + 1$ hyperbolic systems using non-collocated sensing can be found in Anfinsen et al. (2016) using swapping filers and in Bin and Di Meglio (2016) using a Lyapunov approach. The extension to general $m + n$ systems is given in Anfinsen et al. (2017). An adaptive observer for 2×2 systems using only collocated control is developed in Anfinsen and Aamo (2016). Adaptive stabilization of the same type of systems, but without the additive boundary condition is considered in Anfinsen and Aamo (2017b) and without the multiplicative boundary condition in Aamo (2013). Adaptive stabilization of the systems considered in Anfinsen et al. (2016) and Anfinsen et al. (2017) (with some modifications) are considered in Anfinsen and Aamo (2017c) and Anfinsen and Aamo (2017a) respectively. Stabilization of the system in Anfinsen and Aamo (2016) with both multiplicative and additive boundary parameters, i.e. an affine boundary condition, has to the best of our knowledge not previously been addressed.

Previous results on kick/loss detection and attenuation in MPD have mainly focused on using lumped drilling models. A lumped ODE model is applied on a gas kick detection and mitigation problem in Zhou et al. (2011) by using a method for switched control of the bottom-hole pressure. Another lumped model for estimation and control of in-/outflux is presented in Hauge et al. (2012) and Hauge et al. (2013a). An estimation scheme for reservoir influx and pore pressure, also based on a lumped model, is given in Ambrus et al. (2016). Kick handling methods for a first-order approximation to the PDE system are presented in Aarsnes et al. (2016a) using LMI (Linear Matrix Inequality) based controller design. In/outflux detection using an infinite-dimensional observer is presented in Hauge et al. (2013b). Detection and handling of kick & loss using a distributed PDE model incorporating a model of the reservoir inflow dynamics, has to the best of the authors knowledge not previously been addressed.

1.3 Contributions, Scope and Outline

The theoretical contribution in this thesis build on the results from Anfinsen and Aamo (2016, 2017b,a); Anfinsen et al. (2017). Two systems are considered; the first where sensing is restricted to be collocated with actuation at one boundary, and the second where sensing is allowed to be non-collocated with control, i.e. sensing at both boundaries. The first system is a 2×2 hyperbolic system with an affine boundary condition. The theoretical contribution is an adaptive control law that together with the observer from Anfinsen and Aamo (2016) stabilizes the system. The second system is also a 2×2 hyperbolic system, but with a bilinear boundary condition. The theoretical contribution is both a swapping-based estimation scheme for parameter and state estimation and a control law for stabilization.

The theory is applied on the kick and loss detection and attenuation problem in MPD by using a modification of the transformation given in Aamo (2013). Simulations of both the general 2×2 hyperbolic systems demonstrating the theory, and the MPD application with simulated kicks and losses, are provided.

The focus of this thesis is entirely on the theoretical development of mathematical estimation schemes, control laws and stability proofs as well as computer simulations. Implementation aspects for experimental testing are not considered; no model reduction or discretization schemes are discussed.

The thesis is separated into 5 parts as follows:

I) The first part consists of Chapters 1 and 2. Chapter 2 gives a short introduction to boundary control of PDEs with special emphasis on the backstepping method and methods in adaptive control of PDEs. A short description of different classes and properties of PDEs are also included.

II) The second part, consisting of Chapters 3 and 4, presents the main theoretical contribution of this thesis with the two control problems given in each chapter. Stability proofs are included in both chapter.

III) In the third part, the theory derived in Part II is applied to the kick and loss detection and attenuation problem in MPD. Transformations relating the drilling model to the systems in Part II as well as stability proofs are provided in Chapter 5. Simulation results are given in Chapter 6.

IV) The fourth part which consists of Chapter 7, provides some concluding remarks and possible areas for further work.

V) Appendix A includes some additional lemmas that are used to prove stability in Part II. Some additional material used throughout the report is included for reference in Appendix B. Finally, two conference papers based on the work in Parts II and III are included in Appendix C.

1.4 Notation

All vectors are column vectors. For a general set $\mathbb{F}$, the notation $\mathbb{F}^n$ means the set of n-dimensional vectors with elements in $\mathbb{F}$, and $\mathbb{F}^{m \times n}$ the set of matrices of size $m \times n$ with all elements in $\mathbb{F}$. The set $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}^+$ the set of real positive numbers including zero, C the set of continuous functions, and C^1 the set of functions with continuous first derivatives. Derivatives are usually stated using subscripts, i.e. $u_t(x,t)$, $u_x(x,t)$, etc, or with standard Leibniz notation, i.e. $\frac{\partial}{\partial x}u(x,t)$, $\frac{\partial}{\partial t}u(x,t)$, when using subscripts might be misleading. For monovariable functions of time, derivatives are stated using Newtons notation, i.e. $\dot{x}$ and $\ddot{x}$ for the first and second derivative respectively. For a signal $z(x,t)$ defined for $0 \leq x \leq 1$, $t \geq 0$, the norm $||z||$ denotes the L_2-norm, i.e

$$||z|| = \sqrt{\int_0^1 z^2(x,t)dx}, \tag{1.1}$$

if not otherwise specified. The argument of time is dropped when using the norm notation $||\cdot||$. For a time-varying, real signal $f(t)$, the following vector spaces are used:

$$f \in \mathcal{L}_p \leftrightarrow \left(\int_0^\infty |f(t)|^p dt\right)^{\frac{1}{p}} < \infty \tag{1.2}$$

for $p \geq 1$ with the particular case

$$f \in \mathcal{L}_\infty \leftrightarrow \sup_{t\geq 0}|f(t)| < \infty. \tag{1.3}$$

For a multivariable function $f(x,t)$, the notation $f(x,\cdot) \in \mathcal{L}_p$ is used to indicate the integration variable, in this case t.

Chapter 2

Boundary Control of PDEs

This chapter provides a brief introduction to PDEs and (boundary) control of PDEs. The design method of backstepping for PDEs is presented and a short overview of techniques for adaptive control of PDEs. The methods presented in this chapter will be applied in later chapters.

2.1 Introduction to PDEs

A partial differential equation is an equation involving one or more partial derivatives of an unknown function that depends on two or more variables (Kreyszig, 2011, Section 12.1). The *order* of a PDE is the order of the highest derivative.

A k^{th}-order PDE for some function $u : X \to \mathbb{R}$ in the independent variable $x \in X \subset \mathbb{R}^n$, can be written on the form (Evans, 2010, Section 1.1)

$$F\left(D^k u(x), D^{k-1}u(x), ..., Du(x), u(x), x\right) = 0 \tag{2.1}$$

where D^k denotes the set of all partial derivative of order k and

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}. \tag{2.2}$$

Similarly, a k^{th}-order *system* of PDEs, informally a collection of several PDEs with the *order* being the highest-order derivative occurring in any of its equations (Olver, 2014, Page 3), for some function $u : X \to \mathbb{R}^m$ in the independent variable $x \in X \subset \mathbb{R}^n$, can be written on the form (2.1) with

$$F : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \cdots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times U \to \mathbb{R}^m. \tag{2.3}$$

A PDE is said to be *linear* if F is linear in $u(x)$ and its derivatives. It is said to be *homogeneous* if each of its terms contains either $u(x)$ or one of its partial derivatives (Kreyszig, 2011, Section 12.1). In this thesis, only PDEs of two *independent* variables, systems of PDEs of at most 2 equations, and only linear PDEs are considered. The independent variables are time t and space x.

A system of PDEs for some function $u(x,t)$, $x \in X$, $t \in \mathbb{R}^+$, $u : X \times \mathbb{R}^+ \to \mathbb{R}^n$ with only first derivatives in time and space can be written on the form

$$A(x,t)u_t(x,t) + B(x,t)u_x(x,t) = C(x,t)u(x,t) \tag{2.4}$$

where A, B and C are quadratic matrices of dimension $n \times n$. System (2.4) is therefore refereed to as a *linear $n \times n$ system*. In this thesis, special emphasis is placed on the linear 2×2 system

$$u_t(x,t) + \lambda u_x(x,t) = c_1(x)v(x,t) \tag{2.5a}$$

$$v_t(x,t) - \mu v_x(x,t) = c_2(x)u(x,t) \tag{2.5b}$$

where $\lambda, \mu > 0$ are termed the *transport speed* and c_1, c_2 *source terms*. In that sense, u represents information convecting in the right direction (increase in the spatial variable x) and v information convecting in the left direction (decreasing x).

A PDE or system of PDEs are *well-posed* in the sense of Hadamard if (Renardy and Rogers, 2006, Section 1.1.5)

I) A solution exists.

II) The solution is unique.

III) The solution depends continuously on the data.

The problem is said to be *ill-posed* if these conditions do not hold.

2.1.1 Classes and Properties of PDEs

In contrast to (linear) ODEs, no general methodology for analysis nor control can be developed for PDEs (Krstic and Smyshlyaev, 2008, Section 1.5). The methods and tools used are therefore dependent on the type of PDE considered.

Consider the following general linear, second order PDE for a function $u(x,t)$ in two independent variables x, t:

$$a(x,t)u_{xx} + b(x,t)u_{xt} + c(x,t)u_{tt} = f(x,t,u_x,u_t). \tag{2.6}$$

This PDE can be classified by its *discriminant* given as

$$\Delta(x,t) = b^2(x,t) - 4a(x,t)c(x,t). \tag{2.7}$$

Similarly to how conic equations are classified, the linear second order PDE (2.6), at a point (x,t), is called (Olver, 2014, Definition 4.12)

I) *Hyperbolic* if $\Delta(x,t) > 0$,

II) *Parabolic* if $\Delta(x,t) = 0$,

III) *Elliptic* if $\Delta(x,t) < 0$,

IV) *Singular* if $a(x,t) = b(x,t) = c(x,t) = 0$.

If the sign of the discriminant varies through the domain, the PDE is said to be of *mixed* type. Most systems considered in this thesis will be of Type I; hyperbolic.

The complexity of the control problem varies not only by the class, but also by the following properties identified in Smyshlyaev and Krstic (2010):

I) *Stability of the open-loop system.* Open-loop unstable systems are more challenging than open-loop stable systems. In this thesis, both open loop stable and unstable systems are considered.

II) *Location of actuation and sensing.* Actuation and sensing can either be distributed in-domain, or localized, for instance at boundaries. Boundary control and sensing are considered to be physically more realistic, and also the harder problem because the input and output operators are unbounded (Krstic and Smyshlyaev, 2008, Section 1.1). This thesis exclusively considers boundary control and sensing.

III) *Uncertainty structure.* In the case of uncertain plant parameters, the structure of the uncertainty can either be *matched* or *mismatched.* A matched uncertainty can be directly canceled by the control input signal, while mismatched uncertainties require some form of transformation to relate the control gains to the uncertainties. This thesis focus exclusively on parameter uncertainties anti-collocated with the control signal, i.e. mismatched uncertainties.

IV) *Spatially constant or functional parameters?* The parametric uncertainties can be both spatially constant or varying. Only spatially constant uncertain parameters are considered in this thesis.

2.1.2 Boundary and Initial Conditions

A PDE problem together with a set of initial conditions (BC), usually at initial time $t = 0$, is called an *initial value problem* (IVP). A PDE problem with a set of *boundary conditions*, usually at boundaries of the spatial domain, is called a *boundary value problem* (BVP).

Some of the boundary conditions might be associated with an actuation signal. Within the field of boundary control, several types of actuation exists. Common types of boundary control are

I) *Dirichlet actuation* where the boundary value is associated with a state variable. For example in flow control where the actuation can be microjets or valves (Krstic and Smyshlyaev, 2008, Section 1.7).

II) *Neumann actuation* where the boundary value is associated with the gradient of a state variable. For example controlling the heat flux in thermal problems (Krstic and Smyshlyaev, 2008, Section 1.7).

Combinations of the two types mentioned above are often called *Robin* or *mixed* (Kreyszig, 2011, Page 564). The focus in this thesis is exclusively on Dirichlet actuation.

2.1.3 Control Objectives

Krstic and Smyshlyaev (2008, Section 1.5) identifies the following control objectives: performance improvement, stabilization and trajectory tracking. The selected control objective for a PDE problem is dependent on the specific application. A first and basic requirement is for the system to be stable (in the sense of some norm, see the next section). For an unstable open-loop system, *stabilization* is usually the first requirement and therefore a basic control objective. If the system is open-loop stable, the control objective can be *performance-improvements*; for instance in transient time or increased robustness margins. Another objective is stabilization to a pre-selected state trajectory, collectively called *trajectory tracking*. In this thesis, all three objectives presented above will be perused.

2.1.4 Lyapunov Stability

Finite dimensional systems, which includes systems of ODEs, have equivalence between vector norms, meaning that e.g. exponential stability in one norm will imply exponential stability in all norms. This is not the case for PDEs; when the state space is infinite-dimensional (in the spatial variable x), the state space is not Euclidean but a function space and the state norm is a function norm. Contrary to vector norms, function norms are not equivalent (Krstic et al., 1995, Chapter 2, page 14). No general concept of stability exists for PDEs and hence, stability must always be considered in the sense of some norm. Examples of common norms are the L_1, L_2 or L_∞-norm, which are function norms over the spatial domain (see Section 1.4), or the so-called *Sobolev* norms, which will not be considered in this thesis.

Lyapunov functions will usually be constructed for norms of a set of transformed state variables. Meaning that even if the Lyapunov function is a plain, diagonal, spatial norm in the transformed state variables, the Lyapunov function in the original state variables will often be complex and include nondiagonal and cross-term effects (Krstic et al., 1995, Section 2.4, page 14). The design methodology for transforming the state variables is presented in the next section.

2.2 Boundary Control by Backstepping

The conventional approach to PDE control is based on spatial discretization (Curtain and Zwart, 2012, Section 1.3), that is transforming the PDEs into finite dimensional ODEs for which standard control techniques can be applied. This step however is not trivial, and even if one is able to successfully transform the PDE into a set of ODEs, one can in general not guarantee that a control design for the approximated ODEs will be effective for the original PDE (Smyshlyaev and Krstic, 2010, Preface, page x). For this reason, control methods appreciating the distributed structure of PDEs must be considered.

Within the class of controllers in the continuum domain, early efforts includes optimal control and pole placement control. All of which requires a thorough understanding of PDEs and functional analysis (see e.g. Curtain and Zwart (2012) or Lasiecka and Triggiani (2000)). The control method used in this thesis is the *backstepping design method*, which is a continuum analog of integrator backstepping for ODEs. In contrast to optimal control that requires solving a Riccati equation in each iteration, or pole-placement control where the objective is to shift the location of the eigenvalues to some desirable location, backstepping for PDEs does not achieve optimally or precise assignment of any eigenvalues. Backstepping however, achieves Lyapunov stabilization by collectively shifting all eigenvalues in some desirable direction (Krstic and Smyshlyaev, 2008, Section 1.2), and is in a sense simpler to use and requires less background in PDE- and functional analysis. Smyshlyaev and Krstic (2010, Page 4) identifies in addition the following distinguishing features: First, the question of well-posedness is circumvented by transforming the plant into a well known, extensively studied PDE. Second, for many problems, closed form controller kernels and observers can be found avoiding the need for online calculation of the controller/observer gains. Third, the method extends naturally to adaptive control, discussed in Section 2.3.

2.2.1 ODE Backstepping

The method of backstepping was first developed for ODEs, in-particular non-linear ODEs (Kokotovic, 1992). Consider the following third order ODE system from Khalil (1996, Example 14.9):

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2 \tag{2.8a}$$
$$\dot{x}_2 = x_3 \tag{2.8b}$$
$$\dot{x}_3 = u \tag{2.8c}$$

where u is a control signal. This system has some characteristic features. First of all, the system is open-loop unstable. This can be seen from setting $u = 0$ and considering the $\dot{x}_1$ and $\dot{x}_2$ dynamics for non-zero initial conditions. Second, the system has a *triangular form*, meaning that a subsystem is only dependent on subsystems *below*. Third, with $\dot{x}_3$ entering at the *last* equation, it can be said to enter at the boundary of the system. Fourth, the control signal u can only stabilize the $\dot{x}_3$ dynamics directly; some form of transformation is needed to relate the control signal to the dynamics of the two other states. This is where the backstepping transformation is introduced. It can be shown that a specific change of variable together with an appropriate control law, transforms the system (2.8) into an equivalent stable target system. The transformation is obtained by recursively propagating the control law through each integrator, i.e. "stepping back", until the boundary, which is actuated, is reached.

2.2.2 The Main Idea of PDE Backstepping

The backstepping methodology can be extended to PDEs by considering an infinite-dimensional equivalent of (2.8). The backstepping procedure will now involve an infinite number of backstepping iterations, which can be represented as a Volterra equation (see Appendix B.3):

$$w(x,t) = u(x,t) - \int_0^x K(x,\xi)u(\xi)d\xi. \tag{2.9}$$

As an example of the backstepping procedure consider the following *reaction-diffusion* equation from Krstic and Smyshlyaev (2008, Section 4.1):

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t) \tag{2.10a}$$
$$u(0,t) = 0 \tag{2.10b}$$
$$u(1,t) = U(t) \tag{2.10c}$$

where λ is a constant and $U(t)$ the control signal. By using the transformation (2.9), with an appropriately selected K, along with the control law

$$u(1,t) = \int_0^1 K(1,t)u(\xi,t)d\xi, \tag{2.11}$$

the plant (2.10) can be transformed into the *target* system

$$w_t(x,t) = w_{xx}(x,t) \tag{2.12a}$$
$$w(0,t) = 0 \tag{2.12b}$$
$$w(1,t) = 0 \tag{2.12c}$$

which is the exponentially stable heat equation. It can be noted that the transformation (2.9) has the same lower triangular structure making it *spatially causal*, and with the boundary actuated, the framework of backstepping fits naturally into the field of boundary control of PDEs. However, the analogy between PDE and ODE backstepping is not a strict structural analogy; it is not true that the ODE backstepping method is a spatial discretization of the PDE backstepping method (Krstic and Smyshlyaev, 2008, Page 49).

2.2.3 Gain Kernel PDE and Solution Methods

The kernel K in (2.9) is called the *gain kernel*, and with $K(1,t)$ being the analog of the proportional gain in PID control for ODEs. For the control problem in Section 2.2.2, the gain kernel is found to satisfy the set of equations given by

$$K_{xx}(x,\xi) - K_{\xi\xi}(x,\xi) = \lambda K(x,\xi) \tag{2.13a}$$
$$K(x,0) = 0 \tag{2.13b}$$
$$K(x,x) = -\frac{\lambda}{2}x. \tag{2.13c}$$

This set is referred to as the *gain kernel PDE*. It can be shown that this PDE is well-posed – a necessary condition for the gain kernel PDE. System (2.13) is independent of time. The control law (2.11) can thus be calculated off-line once, and stored. This is often the case for control laws derived using backstepping. A closed form solution can be found for the gain kernel PDE (2.13)(see (Krstic et al., 1995, Section 4.3-4.4)). For other gain kernel PDEs, a closed form solution might not exist, and the solution must be found by numerical computations.

2.3 Adaptive Control of PDEs

Many physical, distributed systems have unknown or uncertain parameters varying with operating conditions. The uncertainty is often much larger than what can be handled by robust non-adaptive control designs and thus, a need for *parameter-adaptive* control techniques exist (Krstic and Smyshlyaev, 2008, Page 145).

Early result in adaptive control of distributed system focused on special classes of PDEs with relative degree one by high level gain tuning. A survey of the early efforts can be found in Logemann and Townley (1997). Parameter-adaptive control involves on-line estimation of the unknown parameters, and in turn re-computation of the controller gains. If the controller gains are found by solving a Riccati equation, which is the case in optimal control, a new Riccati equation must be solved at every iteration with every new parameter estimate. Solving a Riccati equation on-line is often not practical to do in real time . Backstepping on the other hand, often result in explicit controller gains or in a rapidly convergent numeric scheme which can be implemented on-line (Krstic and Smyshlyaev, 2008, Page 145).

Smyshlyaev and Krstic (2010) differentiates between two major classes of adaptive control schemes

I) *Lyapunov schemes*

II) *Certainty equivalence schemes*

and within the class of certainty equivalence schemes between

I) *Passivity-based identifiers*

II) *Swapping-based identifiers.*

The swapping based design methodology is explained by example in the next section. A swapping based design will be used in Chapter 4 to generate state and parameter estimates.

2.3.1 Swapping Identifiers

To introduce the concept of swapping based design, the following example from Anfinsen and Aamo (2017d) is presented: Consider the first order hyperbolic system

$$v_t(x,t) - v_x(x,t) = 0 \tag{2.14a}$$
$$v(1,t) = \theta v(0,t) + U(t) \tag{2.14b}$$

where v is the state variable in $x \in [0,1]$ and $t > 0$, θ is the unknown parameter and $U(t)$ a control signal. The signal $y(t) = v(0,t)$ is measured.

The idea behind the swapping method is to transform the dynamic parametrization given by (2.14) into a *static representation* by introducing a set of tilters; one parameter filter for each unknown parameter and one input filter. Consider the filters

$$\psi_t(x,t) - \psi_x(x,t) = 0, \qquad \psi(1,t) = U(t) \tag{2.15a}$$
$$\phi_t(x,t) - \phi_x(x,t) = 0, \qquad \phi(1,t) = v(0,t). \tag{2.15b}$$

The *non-adaptive estimate* $\bar{v}(x,t)$ of $v(x,t)$ can be generated from

$$\bar{v}(x,t) = \psi(x,t) + \theta\phi(x,t) \tag{2.16}$$

and the *non-adaptive estimation error* $e(x,t) = v(x,t) - \bar{v}(x,t)$ can be found to satisfy

$$e_t(x,t) - e_x(x,t) = 0 \tag{2.17a}$$
$$e(1,t) = 0 \tag{2.17b}$$

which is identically equal to zero $e(x,t) \equiv 0$ for all $x \in [0,1]$ and $t > 1$. This gives the *linear parametric model*

$$y(t) - \psi(0,t) = \theta\phi(0,t) \tag{2.18}$$

for $t > 1$. The linear parametric model can in turn be used together with any standard adaptive law to generate parameter estimates, for example by using the *gradient method* (see Appendix B.1). State estimates can then be obtained from the *adaptive state estimate*

$$\hat{v}(x,t) = \psi(x,t) + \hat{\theta}(t)\phi(x,t) \tag{2.19}$$

where $\hat{\theta}(t)$ is the parameter estimate. The *adaptive state estimation error* is $\hat{e}(x,t) = v(x,t) - \hat{v}(x,t)$.

Part II

Adaptive Set-Point Regulation

Chapter 3

Collocated Sensing and Control

This chapter considers adaptive stabilization and set-point regulation of a 2×2 linear hyperbolic system with sensing restricted to be collocated with the control and anti-collocated with an uncertain affine boundary condition.

3.1 Problem Statement

Consider the linear 2×2 first-order hyperbolic system

$$u_t(x,t) + \lambda u_x(x,t) = c_1(x)v(x,t) \tag{3.1a}$$
$$v_t(x,t) - \mu v_x(x,t) = c_2(x)u(x,t) \tag{3.1b}$$
$$u(0,t) = \theta_1 v(0,t) + \theta_2 \tag{3.1c}$$
$$v(1,t) = U(t) \tag{3.1d}$$

defined for $x \in [0,1]$, $t \geq 0$, where u, v are the system states and

$$\lambda, \mu > 0, \qquad c_1(x), c_2(x) \in C([0,1]) \tag{3.2}$$

are known, while

$$\theta_i \in \left[\underline{\theta}_i, \bar{\theta}_i\right] \subset \mathbb{R} \tag{3.3}$$

for $i \in \{1,2\}$, are unknown boundary parameters with known bounds

$$\underline{\theta}_1 \leq \theta_1 \leq \bar{\theta}_1 \tag{3.4a}$$
$$\underline{\theta}_2 \leq \theta_2 \leq \bar{\theta}_2. \tag{3.4b}$$

Sensing is restricted to the boundary collocated with actuation, that is

$$y(t) = u(1,t) \tag{3.5}$$

is the only available measurement. It is assumed that the initial conditions $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ satisfy

$$u_0,\, v_0 \in L_2([0,1]). \tag{3.6}$$

The objective is to design a control input $U(t)$ such that the system (3.1) is adaptively stabilized in the L_2-sense and such that the objective

$$\lim_{t\to\infty} \int_t^{t+T} |rv(0,\tau) - u(0,\tau)|\, d\tau = 0 \tag{3.7}$$

with

$$r \notin \left[\underline{\theta}_1, \bar{\theta}_1\right], \tag{3.8}$$

is achieved for some arbitrary $T > 0$.

The result in Aamo (2013) involves stabilization of (3.1) with the additive boundary parameter θ_2 unknown, but known multiplicative parameter θ_1. System (3.1) without the additive boundary parameter θ_2 is adaptively stabilized in Anfinsen and Aamo (2017b) by using the same observer as in Anfinsen and Aamo (2016). This is the first observer managing to estimate the unknown boundary parameters as well as generating on-line estimates of the system states. The design involves a backstepping technique with time-varying kernels. The same observer will be used in this chapter and is presented in Section 3.2 with the main result, involving an adaptive law and some additional properties, formally stated in Theorem 3.2. An adaptive control law will be derived in Section 3.3 with the main result stated in Theorem 3.4. Proof of Theorem 3.4, that is L_2-boundedness and point-wise boundedness of all signals in the closed loop system and convergence in the sense of (3.7), is given in Section 3.4.

3.2 Observer Design

In this section, the observer from Anfinsen and Aamo (2016) will be presented together with some additional properties needed for solving the adaptive control problem, that were not proven in Anfinsen and Aamo (2016).

3.2.1 Observer Equations

Consider the observer

$$\hat{u}_t(x,t) + \lambda \hat{u}_x(x,t) = c_1(x)\hat{v}(x,t) + P_1(x,t)\left(y(t) - \hat{u}(1,t)\right) \tag{3.9a}$$
$$\hat{v}_t(x,t) - \mu \hat{v}_x(x,t) = c_2(x)\hat{u}(x,t) + P_2(x,t)\left(y(t) - \hat{u}(1,t)\right) \tag{3.9b}$$
$$\hat{u}(0,t) = \hat{\theta}_1 \hat{v}(0,t) + \hat{\theta}_2 \tag{3.9c}$$
$$\hat{v}(1,t) = U(t) \tag{3.9d}$$

where $\hat{u}, \hat{v}$ are estimates of the system states with initial conditions $\hat{u}(x,0) = \hat{u}_0(x)$, $\hat{v}(x,0) = \hat{v}_0(x)$ satisfying

$$\hat{u}_0,\, \hat{v}_0 \in L_2([0,1]). \tag{3.10}$$

The parameters $\hat{\theta}_1$, $\hat{\theta}_2$ are estimates of the boundary parameters θ_1, θ_2 respectively, and P_1, P_2 are output injection gains to be specified.

Subtracting (3.9) from (3.1) gives the state estimation error dynamics

$$\tilde{u}_t(x,t) + \lambda \tilde{u}_x(x,t) = c_1(x)\tilde{v}(x,t) - P_1(x,t)\tilde{u}(1,t) \tag{3.11a}$$
$$\tilde{v}_t(x,t) - \mu \tilde{v}_x(x,t) = c_2(x)\tilde{u}(x,t) - P_2(x,t)\tilde{u}(1,t) \tag{3.11b}$$
$$\tilde{u}(0,t) = \hat{\theta}_1 \tilde{v}(0,t) + \tilde{\theta}_1 v(0,t) + \tilde{\theta}_2 \tag{3.11c}$$
$$\tilde{v}(1,t) = 0 \tag{3.11d}$$

where $\tilde{u} = u - \hat{u}$, $\tilde{v} = v - \hat{v}$, $\tilde{\theta}_1 = \theta_1 - \hat{\theta}_1$ and $\tilde{\theta}_2 = \theta_2 - \hat{\theta}_2$.

3.2.2 Decoupling the Observer Dynamics

The following lemma from Anfinsen and Aamo (2016) presents a backstepping transformation and corresponding target system that will facilitate the design of an adaptive law and the injection gains P_1, P_2 in (3.9).

Lemma 3.1 (Modified from Anfinsen and Aamo (2016, Lemma 1))**.** *The backstepping transformation*

$$\tilde{u}(x,t) = \alpha(x,t) + \int_x^1 P^u(x,\xi,t)\alpha(\xi,t)d\xi \tag{3.12a}$$
$$\tilde{v}(x,t) = \beta(x,t) + \int_x^1 P^v(x,\xi,t)\alpha(\xi,t)d\xi \tag{3.12b}$$

defined over

$$\mathcal{T}_1 = \{(x,\xi,t)\,|\,0 \leq x \leq \xi \leq 1 \cap t \geq 0\}, \tag{3.13}$$

with P^u, P^v satisfying the PDEs

$$P^u_t(x,\xi,t) + \lambda P^u_x(x,\xi) + \lambda P^u_\xi(x,\xi) = c_1(x)P^v(x,\xi) \tag{3.14a}$$
$$P^v_t(x,\xi,t) - \mu P^v_x(x,\xi) + \lambda P^v_\xi(x,\xi) = c_2(x)P^u(x,\xi) \tag{3.14b}$$
$$P^v(x,x) = \frac{c_2(x)}{\lambda+\mu} \tag{3.14c}$$
$$P^u(0,\xi) = \hat{\theta}_1 P^v(0,\xi), \tag{3.14d}$$

maps the state estimation error dynamics (3.11) *with*

$$P_1(x,t) = \lambda P^u(x,1,t) \tag{3.15a}$$
$$P_2(x,t) = \lambda P^v(x,1,t) \tag{3.15b}$$

into the target system

$$\alpha_t(x,t) + \lambda\alpha_x(x,t) = c_1(x)\beta(x,t) - \int_x^1 D_1(x,\xi)\beta(\xi,t)d\xi \tag{3.16a}$$

$$\beta_t(x,t) - \mu\beta_x(x,t) = -\int_x^1 D_2(x,\xi)\beta(\xi,t)d\xi \tag{3.16b}$$

$$\alpha(0,t) = \hat{\theta}_1\beta(0,t) + \tilde{\theta}_1 v(0,t) + \tilde{\theta}_2 \tag{3.16c}$$

$$\beta(1,t) = 0 \tag{3.16d}$$

where

$$D_1(x,\xi,t) = P^u(x,\xi,t)c_1(\xi) - \int_x^{x\xi} P^u(x,s,t)D_1(s,\xi,t)ds \tag{3.17a}$$

$$D_2(x,\xi,t) = P^v(x,\xi,t)c_1(\xi) - \int_x^{\xi} P^v(x,s,t)D_1(s,\xi,t)ds. \tag{3.17b}$$

Furthermore, the transformation (3.12) *is invertible, the kernel equation* (3.14) *has a unique, bounded solution for any bounded* $\hat{\theta}_1$*,* $\hat{\theta}_2$ *and initial states*

$$P^u(x,\xi,0),\ P^v(x,\xi,0) \in L_2(\mathcal{T}_1), \tag{3.18}$$

and the Volterra equation (3.17) *has a unique solution.*

Proof. For the derivation of (3.14)–(3.17), see the proof of Lemma 1 in Anfinsen and Aamo (2016). For uniqueness and boundedness of (3.14) see Anfinsen and Aamo (2016, Lemma 4). Following the well-behavedness of P^u, P^v, a solution to the Volterra equation (3.17) is ensured by Anfinsen and Aamo (2016, Lemma 5). □

Let

$$d_\alpha = \frac{1}{\lambda} \tag{3.19a}$$

$$d_\beta = \frac{1}{\mu} \tag{3.19b}$$

$$t_F = d_\alpha + d_\beta. \tag{3.20}$$

It should be noted that the β-subsystem in (3.16b) and (3.16d) is independent of α and will be $\beta \equiv 0$ for $t > d_\beta$. Thus, for $t > d_\beta$, the target system (3.16) is reduced to

$$\alpha_t(x,t) + \lambda\alpha_x(x,t) = 0 \tag{3.21a}$$

$$\alpha(0,t) = \tilde{\theta}_1 v(0,t) + \tilde{\theta}_2. \tag{3.21b}$$

3.2.3 Adaptive Law

In Anfinsen and Aamo (2016, Lemma 2), it was shown that for $t > t_F$, the signals

$$\vartheta(t) = y(t) - \hat{u}(1,t) + \hat{\theta}_1(t-d_\alpha)\bar{v}(t) + \hat{\theta}_2(t-d_\alpha) \tag{3.22a}$$

$$\begin{aligned}\bar{v}(t) =& \hat{v}(0, t-d_\alpha) \\ &+ \int_0^1 P^v(0,\xi,-t-d_\alpha)y(t-\frac{\xi}{\lambda})d\xi \\ &- \int_0^1 P^v(0,\xi,-t-d_\alpha)\hat{u}(1,t-\frac{\xi}{\lambda})d\xi\end{aligned} \tag{3.22b}$$

are related to the unknown parameters through the linear parametric model

$$\vartheta(t) = \psi^T(t)\theta \tag{3.23}$$

where

$$\psi(t) = \begin{bmatrix}\bar{v}(t) & 1\end{bmatrix}^T \tag{3.24a}$$

$$\theta = \begin{bmatrix}\theta_1 & \theta_2\end{bmatrix}^T. \tag{3.24b}$$

In addition, the relationship

$$\bar{v}(t) = v(0, t-d_\alpha) \tag{3.25}$$

holds for $t > t_F$.

The linear relationship (3.23) facilitates for the application of any standard identification law. We use the *gradient method with normalization and projection* (see Appendix B.1). State estimates can then be generated by combining the resulting parameter estimates with the observer (3.9). The adaptive law will be restated here together with some properties needed for adaptive control design. This is a modification of Anfinsen and Aamo (2017b, Theorem 3) with the additive boundary parameter θ_2 included.

Theorem 3.2 (Modified from Theorem 3 in Anfinsen and Aamo (2017b))**.** *Consider the adaptive law*

$$\dot{\hat{\theta}}(t) = \begin{cases} Proj_{\underline{\theta},\bar{\theta}}\left(\Gamma\frac{\vartheta(t)-\psi^T(t)\hat{\theta}(t)}{1+\psi^T(t)\psi(t)}\psi(t)\right) & \textit{for } t > t_F \\ 0 & \textit{otherwise}\end{cases} \tag{3.26}$$

for some adaptation gain $\Gamma = \Gamma^T > 0$*, where*

$$\hat{\theta}(t) = \begin{bmatrix}\hat{\theta}_1(t) & \hat{\theta}_2(t)\end{bmatrix}^T \tag{3.27a}$$

$$\underline{\theta}(t) = \begin{bmatrix}\underline{\theta}_1(t) & \underline{\theta}_2(t)\end{bmatrix}^T \tag{3.27b}$$

$$\bar{\theta}(t) = \begin{bmatrix}\bar{\theta}_1(t) & \bar{\theta}_2(t)\end{bmatrix}^T, \tag{3.27c}$$

the signals ϑ*,* ψ *generated from* (3.22a) *and* (3.22b) *respectively,* t_F *is defined in* (3.20)*, and* $\mathrm{Proj}(\cdot)$ *is the projection operator defined in Appendix B.1. Suppose system* (3.1) *and observer* (3.9) *have a unique solution* $u, v, \hat{u}, \hat{v} \in L_2([0,1])\ \forall t \geq 0$ *and the initial estimates* $\hat{\theta}_0 = \hat{\theta}(0)$ *are within the bounds* (3.3)*, then the adaptive law* (3.26) *has the following properties:*

I)

$$\hat{\theta}(t) \in [\underline{\theta}, \bar{\theta}] \tag{3.28}$$

for all $t > 0$.

II)

$$\dot{\hat{\theta}}_1,\ \dot{\hat{\theta}}_2, \in \mathcal{L}_\infty \cap \mathcal{L}_2. \tag{3.29}$$

III)

$$\frac{\tilde{\theta}_1(\cdot)v(0,\cdot) + \tilde{\theta}_2(\cdot)}{\sqrt{2 + v^2(0,\cdot)}} \in \mathcal{L}_2. \tag{3.30}$$

IV) If $\bar{v} \in \mathcal{L}_\infty$*, then*

$$\hat{u}(x,\cdot),\ \hat{v}(x,\cdot)\ \in \mathcal{L}_\infty \tag{3.31}$$

for all $x \in [0,1]$.

V) If $\psi(t)$ *is persistently exciting (PE), that is, there exist positive constants* T, T_1, k_1, k_2 *so that*

$$k_1 I \geq \frac{1}{T}\int_t^{t+T} \psi(\tau)\psi^T(\tau)d\tau \geq k_2 I \tag{3.32}$$

for $t > T_1$*, then the estimates* $\hat{\theta}_1, \hat{\theta}_2, \hat{u}(x,\cdot), \hat{v}(x,\cdot)$ *converge exponentially to their true values for all* $x \in [0,1]$.

Proof. From Ioannou and Sun (2012, Theorem 4.4.1), the gradient adaptive law (3.26) with the projection retain all properties that are established in the absence of projection. Therefore, in proving properties I through IV the unprojected adaptive law

$$\dot{\hat{\theta}}(t) = \Gamma\frac{\vartheta(t) - \psi^T(t)\hat{\theta}(t)}{1 + \psi^T(t)\psi(t)}\psi(t) \tag{3.33}$$

will be considered. Furthermore, the projection operator will guarantee that the estimates θ_1, θ_2 remain within the bounds (3.3) for all $t > 0$.

Inserting the parametric model (3.23) into the right hand side of (3.33) and using $\tilde{\theta} = \theta - \hat{\theta}$ for $t > t_F$ give

$$\dot{\hat{\theta}}(t) = \Gamma\frac{\psi^T(t)\tilde{\theta}(t)}{1 + \psi^T(t)\psi(t)}\psi(t). \tag{3.34}$$

Forming the Lyapunov function

$$V_0 = \frac{1}{2}\tilde{\theta}^T(t)\Gamma^{-1}\tilde{\theta}(t), \tag{3.35}$$

differentiating with respect to time and inserting (3.34) give

$$\begin{aligned} \dot{V}_0 =& -\tilde{\theta}^T(t)\Gamma^{-1}\dot{\hat{\theta}}(t) \\ =& -\tilde{\theta}^T(t)\frac{\psi^T(t)\tilde{\theta}(t)}{1+\psi^T(t)\psi(t)}\psi(t) \\ =& -\frac{\left(\psi^T(t)\tilde{\theta}(t)\right)^2}{1+\psi^T(t)\psi(t)} \\ \leq& 0. \end{aligned} \tag{3.36}$$

Hence,

$$V_0 \in \mathcal{L}_\infty \tag{3.37}$$

Noting that

$$\frac{\psi^T(t)}{\sqrt{1+\psi^T(t)\psi(t)}} \in \mathcal{L}_\infty \tag{3.38}$$

and using (3.37) imply

$$\frac{\psi^T(t)\tilde{\theta}(t)}{\sqrt{1+\psi^T(t)\psi(t)}} \in \mathcal{L}_\infty. \tag{3.39}$$

Integrating (3.36) from $t = 0$ to $t = \infty$, using that $V_0 \geq 0$ is a non-increasing function of time, gives

$$\int_0^\infty \frac{\left(\psi^T(\tau)\tilde{\theta}(\tau)\right)^2}{1+\psi^T(\tau)\psi(\tau)}d\tau = -\int_0^\infty \dot{V}_0(\tau)d\tau = V_0(0) - V_0(\infty) < \infty, \tag{3.40}$$

and therefore

$$\frac{\psi^T(\tau)\tilde{\theta}(\tau)}{\sqrt{1+\psi^T(\tau)\psi(\tau)}} \in \mathcal{L}_2. \tag{3.41}$$

From (3.34), one has

$$||\dot{\tilde{\theta}}|| = ||\dot{\hat{\theta}}|| \leq ||\Gamma|| \left|\left| \frac{\psi^T(\tau)\tilde{\theta}(\tau)}{\sqrt{1+\psi^T(\tau)\psi(\tau)}} \right|\right| \left|\left| \frac{\psi^T(t)}{\sqrt{1+\psi^T(t)\psi(t)}} \right|\right| \tag{3.42}$$

which together with (3.38), (3.39) and (3.41) give $\dot{\tilde{\theta}} = -\dot{\hat{\theta}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and property II.

Let $\gamma_{min}, \gamma_{max}$ be the smallest and largest eigenvalue of Γ, respectively. Starting from (3.36), a lower bound for $\dot{V}_0$ can be found as follows:

$$\dot{V}_0 = -\frac{\left(\psi^T(t)\tilde{\theta}(t)\right)^2}{1+\psi^T(t)\psi(t)}$$

$$
\begin{aligned}
&= -\tilde{\theta}^T(t)\frac{\psi(t)\psi^T(t)}{1+\psi^T(t)\psi(t)}\tilde{\theta}(t) \\
&\geq -\tilde{\theta}^T(t)\tilde{\theta}(t) \\
&\geq -2\gamma_{max}\frac{1}{2}\tilde{\theta}^T(t)\Gamma^{-1}\tilde{\theta}(t) \\
&\geq -2\gamma_{max}V_0.
\end{aligned} \tag{3.43}
$$

A lower bound for V_0 can now be found by using the method of *separation of variables* (see Appendix B.2) as

$$
\begin{aligned}
\dot{V}_0 &\geq -2\gamma_{max}V_0 \\
\frac{\dot{V}_0}{V_0} &\geq -2\gamma_{max} \\
\int_{V_0(t-d_\alpha)}^{V_0(t)} \frac{dV_0}{V_0} &\geq -\int_{t-d_\alpha}^{t} 2\gamma_{max}d\tau \\
\ln\left(\frac{V_0(t)}{V_0(t-d_\alpha)}\right) &\geq -2d_\alpha\gamma_{max}
\end{aligned} \tag{3.44}
$$

and solving for $V_0(t)$ to yield

$$
V_0(t) \geq e^{-2d_\alpha\gamma_{max}}V_0(t-d_\alpha), \tag{3.45}
$$

which shows that the decay rate of V_0 is at maximum exponential. Again, using the definition of V_0 in (3.35) one obtains

$$
\begin{aligned}
\tilde{\theta}^T(t)\tilde{\theta}(t) &\geq 2\gamma_{min}\frac{1}{2}\tilde{\theta}^T(t)\Gamma^{-1}\tilde{\theta}(t) \\
&= 2\gamma_{min}V_0(t) \\
&\geq 2\gamma_{min}e^{-2d_\alpha\gamma_{max}}V_0(t-d_\alpha) \\
&= 2\gamma_{min}e^{-2d_\alpha\gamma_{max}}\frac{1}{2}\tilde{\theta}^T(t-d_\alpha)\Gamma^{-1}\tilde{\theta}(t-d_\alpha) \\
&\geq \frac{\gamma_{min}}{\gamma_{max}}e^{-2d_\alpha\gamma_{max}}\tilde{\theta}^T(t-d_\alpha)\tilde{\theta}(t-d_\alpha)
\end{aligned} \tag{3.46}
$$

The relation (3.46) can now be substituted into (3.36) to yield

$$
\begin{aligned}
\dot{V}_0 &= -\frac{\left(\psi^T(t)\tilde{\theta}(t)\right)^2}{1+\psi^T(t)\psi(t)} \\
&\leq -\frac{\gamma_{min}}{\gamma_{max}}e^{-2d_\alpha\gamma_{max}}\frac{\left(\psi^T(t)\tilde{\theta}(t-d_\alpha)\right)^2}{1+\psi^T(t)\psi(t)}
\end{aligned} \tag{3.47}
$$

Integrating (3.47) from $t = 0$ to $t = \infty$ and using that $V_0 \geq 0$ is a non-increasing function of time give

$$\begin{aligned}\int_0^\infty \frac{\gamma_{min}}{\gamma_{max}} e^{-2d_\alpha \gamma_{max}} \frac{\left(\psi^T(t)\tilde{\theta}(t-d_\alpha)\right)^2}{1+\psi^T(t)\psi(t)} d\tau \leq & - \int_0^\infty \dot{V}_0(\tau) d\tau \\ =& V_0(0) - V_0(\infty) \\ <& \infty, \end{aligned} \tag{3.48}$$

which shows that

$$\int_0^\infty \frac{\left(\psi^T(t)\tilde{\theta}(t-d_\alpha)\right)^2}{1+\psi^T(t)\psi(t)} < \infty. \tag{3.49}$$

Inserting (3.24) and (3.25) into (3.49) give property III.

From (3.21), $\hat{\theta}_1, \hat{\theta}_2 \in \mathcal{L}_\infty$ and the assumption that $\bar{v}$ is bounded one gets $\alpha(x, \cdot) \in \mathcal{L}_\infty$ for all $x \in [0, 1]$. Boundedness of the kernels, stated in Lemma 3.1, give property IV.

For the PE property, if $\psi(t)$ is PE, then from Theorem 4.3.2 in Ioannou and Sun (2012), $\hat{\theta}$ converges exponentially to θ. From (3.21), $\alpha(x, \cdot)$ will converge exponentially to zero for all $x \in [0, 1]$, and from (3.12) and using that $\beta \equiv 0$ for $t > t_F$, $\hat{u}(x, \cdot), \hat{v}(x, \cdot)$ will converge exponentially to $u(x, \cdot), v(x, \cdot)$ for all $x \in [0, 1]$. □

3.3 Closed Loop Adaptive Control

The main result from this section will be a control law $U(t)$ that, together with Theorem 3.2, adaptively stabilizes (3.1) in the L_2-sense and achieves (3.7). Anfinsen and Aamo (2017b) consider adaptive control of (3.1) without the additive boundary term θ_2 and with control objective $u(x, \cdot) = v(x, \cdot) = 0$ for all $x \in [0, 1]$ asymptotically. This section will start off by restating some of the operators from Anfinsen and Aamo (2017b) and their properties, before the main theorem is presented. The stability proof is deferred to Section 3.4.

3.3.1 Backstepping Operators

Consider the operators from Anfinsen and Aamo (2017b)

$$\mathcal{K}, \mathcal{K}_0 \,:\, L_2([0, 1]) \times L_2([0, 1]) \to L_2([0, 1]) \tag{3.50}$$

given as

$$\mathcal{K}[a, b](x) = b(x) - \mathcal{K}_0[a, b](x) \tag{3.51a}$$

$$\mathcal{K}_0[a, b](x) = \int_0^x K^u(x, \xi) a(\xi) d\xi + \int_0^x K^v(x, \xi) b(\xi) d\xi \tag{3.51b}$$

where $a(x)$, $b(x)$ are two signals defined for $x \in [0,1]$ and (K^u, K^v) is the solution to the time-invariant system

$$\mu K^u_x(x,\xi) - \lambda K^u_\xi(x,\xi) = c_2(\xi)K^v(x,\xi) \tag{3.52a}$$
$$\mu K^v_x(x,\xi) + \mu K^v_\xi(x,\xi) = c_1(\xi)K^u(x,\xi) \tag{3.52b}$$
$$K^u(x,x) = -\frac{c_2(x)}{\lambda+\mu} \tag{3.52c}$$
$$K^v(x,0) = 0 \tag{3.52d}$$

defined over

$$\mathcal{T}_2 = \{(x,\xi) \,|\, 0 \le \xi \le x \le 1\}. \tag{3.53}$$

Consider also the operator from Anfinsen and Aamo (2017b)

$$\mathcal{G}[t], \mathcal{G}_0[t] \; : \; L_2([0,1]) \to L_2([0,1]), \tag{3.54}$$

given as

$$\mathcal{G}[a;t](x) = a(x) - \mathcal{G}_0[a;t](x) \tag{3.55a}$$
$$\mathcal{G}_0[a;t](x) = \frac{1}{\mu}\int_0^x g(x-\xi,t)a(\xi)d\xi \tag{3.55b}$$

where g is the on-line solution to the Volterra equation

$$g(x,t) = -\mathcal{G}[\hat{\theta}_1 H](x,t) \tag{3.56}$$

where

$$H(x) = -\lambda K^u(x,0). \tag{3.57}$$

The kernel (K^u, K^v) is time-invariant and can therefore be calculated off-line, while g is time-dependent and must be calculated on-line.

In the following lemma, which is taken almost verbatim from Anfinsen and Aamo (2016), some useful properties regarding the operators $\mathcal{K}_0, \mathcal{K}, \mathcal{G}_0, \mathcal{G}$ are stated.

Lemma 3.3 (Lemma 5 and 6 from Anfinsen and Aamo (2017b))**.** *The system* (3.52) *has a bounded, continuous and unique solution* (K^u, K^v)*. Moreover, the mapping* $(a,b) \to (\bar{a},\bar{b})$ *given by*

$$\begin{aligned}\bar{a}(x) &= a(x)\\ \bar{b}(x) &= \mathcal{K}[a,b](x)\end{aligned} \tag{3.58}$$

is invertible with unique and bounded inverse transformation kernels.[1]

For every bounded $\hat{\theta}_1$*, equation* (3.56) *has a unique solution that satisfies*

$$|g(x,t)| \le h_1|\hat{\theta}_1(t)| \tag{3.59}$$

[1]see Anfinsen and Aamo (2016) for details.

for some $h_1 > 0$. The time derivative of g satisfies

$$|g_t(x,t)| \leq h_2|\dot{\hat{\theta}}_1(t)| \tag{3.60}$$

for some $h_2 > 0$. The transformation $a \to \bar{a}$ given by

$$\bar{a}(x) = \mathcal{G}[a;t](x) \tag{3.61}$$

is invertible with inverse

$$a(x) = \bar{a}(x) + \int_0^x G_0(x,\xi,t)\bar{a}(\xi)d\xi \tag{3.62}$$

where G_0 is the solution to the Volterra equation

$$G_0(x,\xi,t) = \frac{1}{\mu}g(x-\xi) + \frac{1}{\mu}\int_\xi^x g(x-s)G_0(s,\xi,t)ds \tag{3.63}$$

which has a bounded, unique solution G_0 for every t.

Proof. See Anfinsen and Aamo (2017b). □

3.3.2 Main Result

Theorem 3.4. *Consider the system* (3.1)*, the observer* (3.9) *and the adaptive law* (3.26)*. The control law*

$$U(t) = \mathcal{K}_0[\hat{u},\hat{v}](1,t) + \mathcal{G}_0[\mathcal{K}[\hat{u},\hat{v}];t](1,t) + \frac{\hat{\theta}_2(t)}{r-\hat{\theta}_1(t)} - \frac{1}{\mu}\int_0^1 \mathcal{G}[H;t](\xi,t)d\xi\hat{\theta}_2(t) \tag{3.64}$$

where $\mathcal{K}, \mathcal{K}_0, \mathcal{G}, \mathcal{G}_0$ are the operators defined in (3.51) *and* (3.55)*, H is defined in* (3.57)*, r satisfies* (3.8)*, and $\hat{\theta}_1, \hat{\theta}_2$ are generated from the adaptive law* (3.26)*, guarantees* (3.7)*. Moreover, all signals in the closed loop system are bounded,*

$$\int_t^{t+T} |\tilde{u}(0,\tau)|\, d\tau \to 0 \tag{3.65}$$

and

$$\tilde{v}(0,t) \to 0. \tag{3.66}$$

It should be noted that for $\theta_2 = 0$ and $\hat{\theta}_2 \equiv 0$, the control law (3.64) reduces to the control law presented in Anfinsen and Aamo (2016, Theorem 4). In that case, the only solution satisfying (3.7) with $r \neq \theta_1$ is $u(0,t), v(0,t) \to 0$.

Proof of Theorem 3.4 is deferred to Section 3.4. The rest of this section will present the derivation of the control law (3.64). To improve readability, the control law $U(t)$ is decomposed into three parts

$$U(t) = U_1(t) + U_2(t) + U_3(t) \tag{3.67}$$

where U_1 facilitates decoupling of the observer dynamics, U_2 eliminates boundary terms and brings the system into an equivalent target system for which stability analysis is easier and U_3 implements reference tracking so that the objective (3.7) is achieved. Each term is presented in separate sections and lemmas.

3.3.3 Decoupling of the Observer Dynamics

Lemma 3.5. *Consider the observer* (3.9) *and the operators* $\mathcal{K}$ *and* $\mathcal{K}_0$ *from* (3.51)*. The transformation*

$$w(x,t) = \hat{u}(x,t) \tag{3.68a}$$
$$z(x,t) = \mathcal{K}[\hat{u},\hat{v}](x,t), \tag{3.68b}$$

and the control law (3.67) *with*

$$U_1(t) = \mathcal{K}_0[\hat{u},\hat{v}](1,t) \tag{3.69}$$

map (3.9) *into the target system*

$$w_t(x,t) + \lambda w_x(x,t) = c_1(x)z(x,t) + P_1(x,t)\alpha(1,t) + \int_0^x \kappa_1(x,\xi)w(\xi,t)d\xi + \int_0^x \kappa_2(x,\xi)z(\xi,t)d\xi \tag{3.70a}$$
$$z_t(x,t) - \mu z_x(x,t) = \Omega(x,t)\alpha(1,t) + \hat{\theta}_1 H(x)z(0,t) + \hat{\theta}_2 H(x) \tag{3.70b}$$
$$w(0,t) = \hat{\theta}_1 z(0,t) + \hat{\theta}_2 \tag{3.70c}$$
$$z(1,t) = U_2(t) + U_3(t) \tag{3.70d}$$

where

$$\Omega(x) = \mathcal{K}[P_1,P_2](x), \tag{3.71}$$

H *is defined in* (3.57)*,* κ_1 *and* κ_2 *are given by*

$$\kappa_1(x,\xi) = c_1(x)K^u(x,\xi) + \int_\xi^x \kappa_2(x,s)K^u(s,\xi)ds \tag{3.72a}$$
$$\kappa_2(x,\xi) = c_1(x)K^v(x,\xi) + \int_\xi^x \kappa_2(x,s)K^v(s,\xi)ds, \tag{3.72b}$$

α *is defined in* (3.12)*,* $\hat{\theta}_1$*,* $\hat{\theta}_2$ *are obtained from* (3.26) *in Theorem 3.2, and* U_2*,* U_3 *are control signals to be designed.*

Proof. From (3.68b) and the definition (3.51), partial differentiation with respect to time, inserting the dynamics (3.9) and integration by parts give

$$\begin{aligned}
&\hat{v}_t(x,t) - \int_0^x K^u(x,\xi)\hat{u}_t(\xi,t)d\xi - \int_0^x K^v(x,\xi)\hat{v}_t(\xi,t)d\xi \\
=&\hat{v}_t(x,t) \\
&- \int_0^x K^u(x,\xi)\left(-\lambda\hat{u}_\xi(\xi,t) + c_1(\xi)\hat{v}(\xi,t) + P_1(\xi,t)\tilde{u}(1,t)\right)d\xi \\
&- \int_0^x K^v(x,\xi)\left(\mu\hat{v}_\xi(\xi,t) + c_2(\xi)\hat{u}(\xi,t) + P_2(\xi,t)\tilde{u}(1,t)\right)d\xi \\
=&\hat{v}_t(x,t)
\end{aligned}$$

$$
\begin{aligned}
&+ K^u(x,x)\lambda\hat{u}(x,t) - K^u(x,0)\lambda\hat{u}(0,t) - \int_0^x K^u_\xi(x,\xi)\lambda\hat{u}(\xi,t)d\xi \\
&- \int_0^x K^u(x,\xi)c_1(\xi)\hat{v}(\xi,t)d\xi - \int_0^x K^u(x,\xi)P_1(\xi,t)\tilde{u}(1,t)d\xi \\
&- K^v(x,x)\mu\hat{v}(x,t) + K^v(x,0)\mu\hat{v}(0,t) + \int_0^x K^v_\xi(x,\xi)\mu\hat{v}(\xi,t)d\xi \\
&- \int_0^x K^v(x,\xi)c_2(\xi)\hat{u}(\xi,t)d\xi - \int_0^x K^v(x,\xi)P_2(\xi,t)\tilde{u}(1,t)d\xi.
\end{aligned} \tag{3.73}
$$

Similarly differentiating with respect to space and applying *Leibniz' differentiation rule* (see Appendix A.8) give

$$
\begin{aligned}
z_x(x,t) =& \hat{v}_x(x,t) \\
&- K^u(x,x)\hat{u}(x,t) - \int_0^x K^u_x(x,\xi)\hat{u}(\xi,t)d\xi \\
&- K^v(x,x)\hat{v}(x,t) - \int_0^x K^v_x(x,\xi)\hat{v}(\xi,t)d\xi.
\end{aligned} \tag{3.74}
$$

Substituting (3.73) and (3.74) into the dynamics (3.9b) and using (3.52), (3.57) and (3.71) one finds

$$
\begin{aligned}
&z_t(x,t) - \mu z_x(x,t) \\
=& K^u(x,x)\lambda\hat{u}(x,t) - K^u(x,0)\lambda\hat{u}(0,t) - \int_0^x K^u_\xi(x,\xi)\lambda\hat{u}(\xi,t)d\xi \\
&- \int_0^x K^u(x,\xi)c_1(\xi)\hat{v}(\xi,t)d\xi - \int_0^x K^u(x,\xi)P_1(\xi,t)\tilde{u}(1,t)d\xi \\
&- K^v(x,x)\mu\hat{v}(x,t) + K^v(x,0)\mu\hat{v}(0,t) + \int_0^x K^v_\xi(x,\xi)\mu\hat{v}(\xi,t)d\xi \\
&- \int_0^x K^v(x,\xi)c_2(\xi)\hat{u}(\xi,t)d\xi - \int_0^x K^v(x,\xi)P_2(\xi,t)\tilde{u}(1,t)d\xi \\
&+ \mu\Bigg(K^u(x,x)\hat{u}(x,t) + \int_0^x K^u_x(x,\xi)\hat{u}(\xi,t)d\xi \\
&\qquad + K^v(x,x)\hat{v}(x,t) + \int_0^x K^v_x(x,\xi)\hat{v}(\xi,t)d\xi\Bigg) \\
&+ c_2(x)\hat{u}(x,t) + P_2(x,t)\tilde{u}(1,t) \\
\\
=& -\int_0^x \underbrace{\left[K^u_\xi(x,\xi)\lambda\hat{u}(\xi,t) + K^v(x,\xi)c_2(\xi)\hat{u}(\xi,t) - \mu K^u_x(x,\xi)\right]}_{=0}\hat{u}(\xi,t)d\xi \\
&- \int_0^x \underbrace{\left[K^u(x,\xi)c_1(\xi)\hat{v}(\xi,t) - K^v_\xi(x,\xi)\mu\hat{v}(\xi,t) - \mu K^v_x(x,\xi)\right]}_{=0}\hat{v}(\xi,t)d\xi \\
&+ \underbrace{\left[K^u(x,x)\lambda + \mu K^u(x,x) + c_2(x)\right]}_{=0}\hat{u}(x,t)
\end{aligned}
$$

$$
\begin{aligned}
&+\underbrace{\left[P_2(x,t)-\int_0^x K^u(x,\xi)P_1(\xi,t)-\int_0^x K^v(x,\xi)P_2(\xi,t)\right]}_{=\Omega(x,t)}\alpha(1,t)\\
&-\left[\hat{\theta}_1\underbrace{K^u(x,0)\lambda}_{-H(x)}-\underbrace{K^v(x,0)\mu}_{=0}\right]z(0,t)\underbrace{-K^u(x,0)\lambda}_{H(x)}\hat{\theta}_2\\
=&\Omega(x,t)\alpha(1,t)+\hat{\theta}_1H(x)z(0,t)+\hat{\theta}_2H(x)
\end{aligned}
\tag{3.75}
$$

Inserting (3.68) into (3.70a) gives

$$
\begin{aligned}
w_t(x,t)+\lambda w_x(x,t)=&c_1(x)\hat{v}(x,t)+P_1(x,t)\left(y(t)-\hat{u}(1,t)\right)\\
=&c_1(x)z(x,t)+P_1(x,t)\alpha(1,t)\\
&+c_1(x)\int_0^x K^u(x,\xi)\hat{u}(\xi,t)d\xi+c_1(x)\int_0^x K^v(x,\xi)\hat{v}(\xi,t)d\xi\\
=&c_1(x)z(x,t)+P_1(x,t)\alpha(1,t)\\
&+\int_0^x \kappa_1(x,\xi)w(\xi,t)d\xi+\int_0^x \kappa_2(x,\xi)z(\xi,t)d\xi.
\end{aligned}
\tag{3.76}
$$

The relation (3.72) is obtained by subtracting (3.70a) from (3.76):

$$
\begin{aligned}
0=&-c_1(x)\int_0^x K^u(x,\xi)\hat{u}(\xi,t)d\xi-c_1(x)\int_0^x K^v(x,\xi)\hat{v}(\xi,t)d\xi\\
&+\int_0^x \kappa_1(x,\xi)w(\xi,t)d\xi+\int_0^x \kappa_2(x,\xi)z(\xi,t)d\xi\\
=&-\int_0^x c_1(x)K^u(x,\xi)\hat{u}(\xi,t)d\xi-\int_0^x c_1(x)K^v(x,\xi)\hat{v}(\xi,t)d\xi\\
&+\int_0^x \kappa_1(x,\xi)\hat{u}(\xi,t)d\xi\\
&+\int_0^x \kappa_2(x,\xi)\left(\hat{v}(\xi,t)-\int_0^\xi K^u(\xi,s)\hat{u}(s,t)ds-\int_0^\xi K^v(\xi,s)\hat{v}(s,t)ds\right)d\xi\\
=&\int_0^x\left[-c_1(x)K^u(x,\xi)+\kappa_1(x,\xi)\right]\hat{u}(\xi,t)d\xi\\
&+\int_0^x\left[-c_1(x)K^v(x,\xi)+\kappa_2(x,\xi)\right]\hat{v}(\xi,t)d\xi\\
&+\int_0^x \kappa_2(x,\xi)\left(-\int_0^\xi K^u(\xi,s)\hat{u}(s,t)ds-\int_0^\xi K^v(\xi,s)\hat{v}(s,t)ds\right)d\xi\\
=&\int_0^x\left[-c_1(x)K^u(x,\xi)+\kappa_1(x,\xi)\right]\hat{u}(\xi,t)d\xi\\
&+\int_0^x\left[-c_1(x)K^v(x,\xi)+\kappa_2(x,\xi)\right]\hat{v}(\xi,t)d\xi
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^x \int_\xi^x \kappa_2(x,s)K^u(s,\xi)\hat{u}(\xi,t)dsd\xi - \int_0^x \int_\xi^x \kappa_2(x,s)K^v(s,\xi)\hat{v}(\xi,t)dsd\xi \\
&= \int_0^x \underbrace{\left[-c_1(x)K^u(x,\xi) + \kappa_1(x,\xi) - \int_\xi^x \kappa_2(x,s)K^u(s,\xi)ds\right]}_{=0} \hat{u}(\xi,t)d\xi \\
&\quad + \int_0^x \underbrace{\left[-c_1(x)K^v(x,\xi) + \kappa_2(x,\xi) - \int_\xi^x \kappa_2(x,s)K^v(s,\xi)ds\right]}_{=0} \hat{v}(\xi,t)d\xi. \qquad (3.77)
\end{aligned}
$$

Evaluating (3.68b) at $x = 1$, and inserting (3.9b) and (3.69) give (3.70d):

$$
\begin{aligned}
z(1,t) =& \hat{v}(1,t) - \int_0^1 K^u(1,\xi)\hat{u}(\xi,t)d\xi - \int_0^1 K^v(1,\xi)\hat{v}(\xi,t)d\xi \\
=& U(t) - \int_0^1 K^u(1,\xi)\hat{u}(\xi,t)d\xi - \int_0^1 K^v(1,\xi)\hat{v}(\xi,t)d\xi \\
=& U_2(t) + U_3(t). \qquad (3.78)
\end{aligned}
$$

The last boundary condition (3.70c) follows from inserting (3.68) into (3.9c). □

The significance of Lemma 3.5 is that subsystem (3.70b) is independent of w. If $z, \alpha, \hat{\theta}_1, \hat{\theta}_2$ are bounded, then it can be noted from the transport equation (3.70a) and boundary condition (3.70c) that w will be bounded as well. Furthermore, $w(0,t)$ is uniquely determined by $\hat{\theta}_1, \hat{\theta}_2, z$ in (3.70c). The problem of stabilizing (3.1) is therefore reduced to stabilizing z and α.

3.3.4 Elimination of Boundary Terms

Lemma 3.6. *Consider the subsystem* (3.70b) *and* (3.70d) *and the operators* $\mathcal{G}, \mathcal{G}_0$ *from* (3.55). *The transformation*

$$\zeta(x,t) = \mathcal{G}[z;t](x,t), \qquad (3.79)$$

and control law

$$U_2 = \mathcal{G}_0[z;t](1,t), \qquad (3.80)$$

map the system (3.70b) *and* (3.70d) *into the target system*

$$
\begin{aligned}
\zeta_t(x,t) - \mu\zeta_x(x,t) =& \int_0^x B(x,\xi,t)\zeta(\xi,t)d\xi + \Omega_1(x,t)\alpha(1,t) + H_1(x,t)\hat{\theta}_2 \qquad (3.81a) \\
\zeta(1,t) =& U_3(t) \qquad (3.81b)
\end{aligned}
$$

where

$$\Omega_1(x,t) = \mathcal{G}[\Omega;t](x,t) \qquad (3.82)$$

$$H_1(x,t) = \mathcal{G}[H;t](x,t) \qquad (3.83)$$

and

$$B(x,\xi,t) = -\frac{1}{\mu}g_t(x-\xi,t) - \frac{1}{\mu}\int_\xi^x g_t(x-s,t)G_0(s,\xi,t)ds \tag{3.84}$$

has the property

$$||B||^2 = \int_0^1\int_0^x B^2(x,\xi,\cdot)d\xi dx \in \mathcal{L}_1 \cap \mathcal{L}_\infty \tag{3.85}$$

for all $(x,\xi) \in \mathcal{T}_2$ *with* $\mathcal{T}_2$ *defined in* (3.53).

Proof. Differentiating (3.79) with respect to time, inserting the dynamics (3.70b) and integrating by parts yield

$$\begin{aligned}
\zeta_t(x,t) =& z_t(x,t) \\
&- \int_0^x \frac{1}{\mu}g_t(x-\xi,t)z(\xi,t)d\xi - \int_0^x \frac{1}{\mu}g(x-\xi,t)z_t(\xi,t)d\xi \\
=& z_t(x,t) \\
&- \int_0^x \frac{1}{\mu}g_t(x-\xi,t)z(\xi,t)d\xi - \int_0^x \frac{1}{\mu}g(x-\xi,t) \\
&\quad \times \left(\mu z_\xi(\xi,t) + \Omega(\xi,t)\alpha(1,t) + \hat{\theta}_1 H(\xi)z(0,t) + \hat{\theta}_2 H(\xi)\right) d\xi \\
=& z_t(x,t) \\
&- \int_0^x \frac{1}{\mu}g_t(x-\xi,t)z(\xi,t)d\xi - g(0,t)z(x,t) + g(x,t)z(0,t) \\
&- \int_0^x g_\xi(x-\xi,t)z(\xi,t)d\xi - \int_0^x \frac{1}{\mu}g(x-\xi,t)\Omega(\xi,t)\alpha(1,t)d\xi \\
&- \int_0^x \frac{1}{\mu}g(x-\xi,t)\hat{\theta}_1 H(\xi)z(0,t)d\xi - \int_0^x \frac{1}{\mu}g(x-\xi,t)\hat{\theta}_2 H(\xi)d\xi.
\end{aligned} \tag{3.86}$$

Similarly, differentiating with respect to space yields

$$\zeta_x(x,t) = z_x(x,t) - \frac{1}{\mu}g(0,t)z(x,t) - \int_0^x \frac{1}{\mu}g_x(x-\xi,t)z(\xi,t)d\xi. \tag{3.87}$$

Inserting (3.82), (3.83), (3.86) and (3.87) into (3.70b) give (3.81a):

$$\begin{aligned}
\zeta_t(x,t) - \mu\zeta_x(x,t) =& -\int_0^x \frac{1}{\mu}g_t(x-\xi,t)z(\xi,t)d\xi - g(0,t)z(x,t) + g(x,t)z(0,t) \\
&- \int_0^x g_\xi(x-\xi,t)z(\xi,t)d\xi - \int_0^x \frac{1}{\mu}g(x-\xi,t)\Omega(\xi,t)\alpha(1,t)d\xi \\
&- \int_0^x \frac{1}{\mu}g(x-\xi,t)\hat{\theta}_1 H(\xi)z(0,t)d\xi \\
&- \int_0^x \frac{1}{\mu}g(x-\xi,t)\hat{\theta}_2 H(\xi)d\xi
\end{aligned}$$

$$
\begin{aligned}
& + \mu \left(\frac{1}{\mu} g(0,t) z(x,t) + \int_0^x \frac{1}{\mu} g_x(x-\xi,t) z(\xi,t) d\xi \right) \\
& + \Omega(x,t)\alpha(1,t) + \hat{\theta}_1 H(x) z(0,t) + \hat{\theta}_2 H(x) \\
= & - \int_0^x \frac{1}{\mu} g_t(x-\xi,t) z(\xi,t) d\xi \\
& + \left[\Omega(x,t) - \int_0^x \frac{1}{\mu} g(x-\xi,t)\Omega(\xi,t) d\xi \right] \alpha(1,t) \\
& + \left[\hat{\theta}_1 H(x) + g(x,t) - \hat{\theta}_1 \int_0^x \frac{1}{\mu} g(x-\xi,t) H(\xi) d\xi \right] z(0,t) \\
& + \left[H(x) - \int_0^x \frac{1}{\mu} g(x-\xi,t) H(\xi) d\xi \right] \hat{\theta}_2 \\
= & \int_0^x \left(-\frac{1}{\mu} g_t(x-\xi,t) - \int_\xi^x \frac{1}{\mu} g_t(x-s,t) G_0(s,\xi,t) ds \right) \\
& \qquad \times \zeta(\xi,t) d\xi + \Omega_1(x,t)\alpha(1,t) + H_1(x,t)\hat{\theta}_2 \\
= & \int_0^x B(x,\xi,t)\zeta(\xi,t) d\xi + \Omega_1(x,t)\alpha(1,t) + H_1(x,t)\hat{\theta}_2 \qquad (3.88)
\end{aligned}
$$

Evaluating (3.79) at $x = 1$, and inserting (3.70d) and (3.80) give (3.81b):

$$
\begin{aligned}
\zeta(1,t) = & z(1,t) - \int_0^1 \frac{g(1-\xi,t)}{\mu} z(\xi,t) d\xi \\
= & U_2(t) + U_3(t) - \int_0^1 \frac{g(1-\xi,t)}{\mu} z(\xi,t) d\xi \\
= & U_3(t) \qquad (3.89)
\end{aligned}
$$

Boundedness and square integrability of B in (3.84) follows from boundedness of G_0 and property (3.60) in Lemma 3.3:

$$
\begin{aligned}
||B||^2 \leq & \frac{1}{\mu} \int_0^1 \int_0^x |g_t|^2 d\xi dx + \frac{1}{\mu} \int_0^1 \int_0^x \int_\xi^x |g_t(x-s,t) G_0(s,\xi,t)|^2 ds d\xi dx \\
\leq & \frac{1}{\mu} \int_0^1 \int_0^x |g_t|^2 d\xi dx + \frac{1}{\mu} \bar{G}_0^2 \int_0^1 \int_0^x \int_\xi^x |g_t(x-s,t)|^2 ds d\xi dx \\
\leq & \frac{h_2}{\mu} \int_0^1 \int_0^x \left| \dot{\hat{\theta}}_1(t) \right|^2 d\xi dx + \frac{h_2}{\mu} \bar{G}_0^2 \int_0^1 \int_0^x \int_\xi^x \left| \dot{\hat{\theta}}_1(t) \right|^2 ds d\xi dx \\
= & \frac{h_2}{\mu} \left| \dot{\hat{\theta}}_1(t) \right|^2 \int_0^1 \int_0^x d\xi dx + \frac{h_2}{\mu} \bar{G}_0^2 \left| \dot{\hat{\theta}}_1(t) \right|^2 \int_0^1 \int_0^x \int_\xi^x ds d\xi dx \\
\leq & h_2' \left| \dot{\hat{\theta}}_1(t) \right|^2, \qquad (3.90)
\end{aligned}
$$

for some constant $h_2' > 0$. By the boundedness and square integrability property of $\dot{\hat{\theta}}_1(t)$ from Theorem 3.2, property (3.85) is obtained. □

3.3.5 Reference Signal and Tracking

The set-point regulation problem of achieving (3.7) for the system (3.1), can be transformed into a tracking problem for the ζ-system (3.81). Specifically, an equivalent objective is for the ζ-system to track a time-varying reference signal $\zeta^*(t)$ selected as

$$\zeta^*(t) = \frac{\hat{\theta}_2(t)}{r - \hat{\theta}_1(t)} \tag{3.91}$$

where $\hat{\theta}_1(t), \hat{\theta}_2(t)$ are generated using the adaptive law (3.26) in Theorem 3.2. The following lemma motivates the use of this reference signal.

Lemma 3.7. *Consider the reference signal* (3.91)*. If, for some $T > 0$,*

$$\int_t^{t+T} |\zeta(0,\tau) - \zeta^*(\tau)| \, d\tau \to 0 \tag{3.92}$$

and r satisfies (3.8)*, then*

$$\int_t^{t+T} |r\hat{v}(0,\tau) - \hat{u}(0,\tau)| \, d\tau \to 0. \tag{3.93}$$

If in addition

$$\int_t^{t+T} |\alpha(0,\tau)| \, d\tau \to 0 \tag{3.94}$$

and

$$||\alpha|| \to 0 \tag{3.95}$$

then the objective (3.7) *is satisfied,*

$$\int_t^{t+T} |\tilde{u}(0,\tau)| \, d\tau \to 0 \tag{3.96}$$

and

$$\tilde{v}(0,t) \to 0. \tag{3.97}$$

Proof. Starting with the integrand of (3.92), using transformation (3.79) and (3.68b) evaluated at $x = 0$, rearranging and inserting the boundary condition (3.9c) give

$$\begin{aligned} |\zeta(0,\tau) - \zeta^*(\tau)| &= |z(0,\tau) - \zeta^*(\tau)| \\ &= \left| \hat{v}(0,\tau) - \frac{\hat{\theta}_2(t)}{r - \hat{\theta}_1(t)} \right| \end{aligned}$$

$$
\begin{aligned}
&=\left|r-\hat{\theta}_1(t)\right| \frac{\left|\hat{v}(0,\tau)-\frac{\hat{\theta}_2(t)}{r-\hat{\theta}_1(t)}\right|}{\left|r-\hat{\theta}_1(t)\right|}\\
&=\frac{\left|\hat{v}(0,\tau)r-\hat{\theta}_1(t)\hat{v}(0,\tau)-\hat{\theta}_2(t)\right|}{\left|r-\hat{\theta}_1(t)\right|}\\
&=\frac{\left|\hat{v}(0,\tau)r-\hat{u}(0,t)\right|}{\left|r-\hat{\theta}_1(t)\right|}.
\end{aligned} \tag{3.98}
$$

Since $\hat{\theta}_1(t)$ is generated using projection, implying $\hat{\theta}_1(t) \in \left[\underline{\theta}_1, \bar{\theta}_1\right]$ and since, by assumption, $r \notin \left[\underline{\theta}_1, \bar{\theta}_1\right]$, there exists a $\delta > 0$ such that

$$
|\zeta(0,\tau)-\zeta^*(\tau)| = \frac{|\hat{v}(0,\tau)r-\hat{u}(0,t)|}{\left|r-\hat{\theta}_1(t)\right|} \geq \frac{1}{\delta}\left|\hat{v}(0,\tau)r-\hat{u}(0,t)\right| \tag{3.99}
$$

Integrating both sides from $\tau = t$ to $\tau = t+T$, it can be seen that (3.92) implies (3.94) and the first part of the proof is complete.

For the second part; from the backstepping transformation (3.12), the fact that $\beta \equiv 0$ for $t > t_\beta$, boundedness of the kernels P^u, P^v from Lemma 3.1, and using (3.94) and Cauchy-Schwarz' inequality (see Lemma A.3), one obtains

$$
\begin{aligned}
|\tilde{u}(0,t)| &\leq |\alpha(0,t)| + \int_0^1 |P^u(0,\xi,t)\alpha(\xi,t)|\, d\xi\\
&\leq |\alpha(0,t)| + \int_0^1 |P^u(0,\xi,t)||\alpha(\xi,t)|\, d\xi\\
&\leq |\alpha(0,t)| + \sqrt{\int_0^1 |P^u(0,\xi,t)|^2\, d\xi}\sqrt{\int_0^1 |\alpha(\xi,t)|^2\, d\xi}\\
&\leq |\alpha(0,t)| + h||\alpha||,
\end{aligned} \tag{3.100}
$$

for some $h > 0$, and similarly that

$$
\tilde{v}(0,t) \leq h||\alpha||. \tag{3.101}
$$

If (3.94) and (3.95) hold, then (3.96) and (3.97) follow trivially. Next, starting with (3.7) and substituting $u = \tilde{u} + \hat{u}$ and $v = \tilde{v} + \hat{v}$ give

$$
\int_t^{t+T} |rv(0,\tau)-u(0,\tau)|\, d\tau = \int_t^{t+T} |r\hat{v}(0,\tau)-\hat{u}(0,\tau)+r\tilde{v}(0,\tau)-\tilde{u}(0,\tau)|\, d\tau, \tag{3.102}
$$

Relation (3.100) and (3.101) can now be inserted to yield

$$\begin{aligned}\int_t^{t+T} |rv(0,\tau) - u(0,\tau)|\, d\tau \leq & \int_t^{t+T} |r\hat{v}(0,\tau) - \hat{u}(0,\tau)|\, d\tau \\ & + |r| \int_t^{t+T} |\tilde{v}(0,\tau)|\, d\tau \\ & + \int_t^{t+T} |\tilde{u}(0,\tau)|\, d\tau \\ \leq & \int_t^{t+T} |r\hat{v}(0,\tau) - \hat{u}(0,\tau)|\, d\tau \\ & + \int_t^{t+T} |\alpha(0,\tau)|\, d\tau \\ & + h' \int_t^{t+T} ||\alpha|| d\tau \end{aligned} \tag{3.103}$$

for some constant $h' > 0$. Finally, from (3.93)–(3.95), the right hand side will converge to zero asymptotically and the objective (3.7) follows. □

The problem of stabilizing (3.1) is now transformed to the problem of finding a controller that achieves (3.92), (3.94) and (3.95).

A time delayed version of the signal (3.91) can be modeled as the simple transport equation

$$\phi_t(x,t) - \mu\phi_x(x,t) = 0 \tag{3.104a}$$
$$\phi(1,t) = \zeta^*(t) \tag{3.104b}$$

with the explicit solution

$$\phi(0,t) = \zeta^*(t - d_\beta). \tag{3.105}$$

Lemma 3.8. *Consider system* (3.81) *and* (3.104)*. The linear transformation*

$$\eta(x,t) = \zeta(x,t) - \phi(x,t) + H_2(x,t)\hat{\theta}_2(t) \tag{3.106}$$

and control law

$$U_3(t) = \zeta^*(t) - H_2(1,t)\hat{\theta}_2(t), \tag{3.107}$$

map system (3.81) *and* (3.104) *into the target system*

$$\begin{aligned}\eta_t(x,t) - \mu\eta_x(x,t) =& H_2(x,t)\dot{\hat{\theta}}_2(t) + \left(\frac{\partial}{\partial t} H_2(x,t)\right)\hat{\theta}_2(t) + \Omega_1(x,t)\alpha(1,t) \\ & + \int_0^x B(x,\xi,t)\left(\eta(x,t) - H_2(\xi,t)\hat{\theta}_2(t) + \phi(\xi,t)\right) d\xi \end{aligned} \tag{3.108a}$$

$$\eta(1,t) = 0 \tag{3.108b}$$

where

$$H_2(x,t) = \frac{1}{\mu}\int_0^x H_1(\xi,t)d\xi \tag{3.109}$$

has the property

$$\left|\frac{\partial}{\partial t}H_2(x,t)\right| \leq h_3|\dot{\hat{\theta}}_1| \in \mathcal{L}_2 \cap \mathcal{L}_\infty \tag{3.110}$$

for some $h_3 > 0$*. The reference signal* ζ^* *is generated from* (3.91) *and* $\dot{\hat{\theta}}_2$ *from the adaptive law* (3.26)*.*

Proof. Differentiating (3.106) with respect to time and space and inserting the dynamics (3.81a) and (3.104a) yield

$$\begin{aligned}\eta_t(x,t) =&\zeta_t(x,t) - \phi_t(x,t)\\ &+ \frac{1}{\mu}\int_0^x H_1(\xi,t)d\xi\,\dot{\hat{\theta}}_2(t) + \frac{1}{\mu}\int_0^x \frac{\partial}{\partial t}H_1(\xi,t)d\xi\,\hat{\theta}_2(t)\end{aligned} \tag{3.111}$$

and

$$\eta_x(x,t) =\zeta_x(x,t) - \phi_x(x,t) + \frac{1}{\mu}H_1(x,t)\hat{\theta}_2(t). \tag{3.112}$$

Inserting (3.111) and (3.112) into (3.81a) and using (3.109) give (3.108a):

$$\begin{aligned}\eta_t(x,t) - \mu\eta_x(x,t) =&\zeta_t(x,t) - \phi_t(x,t) + \frac{1}{\mu}\int_0^x H_1(\xi,t)d\xi\,\dot{\hat{\theta}}_2(t)\\ &+ \frac{1}{\mu}\int_0^x \frac{\partial}{\partial t}H_1(\xi,t)d\xi\,\hat{\theta}_2(t)\\ &- \mu\left(\zeta_x(x,t) - \phi_x(x,t) + \frac{1}{\mu}H_1(x,t)\hat{\theta}_2(t)\right)\\ =&H_2(x,t)\dot{\hat{\theta}}_2(t) + \frac{\partial}{\partial t}H_2(x,t)\hat{\theta}_2(t) - H_1(x,t)\hat{\theta}_2(t) + \hat{\theta}_2(t)H_1(x,t)\\ &+ \int_0^x B(x,\xi,t)\zeta(\xi,t)d\xi + \Omega_1(x,t)\alpha(1,t)\\ =&H_2(x,t)\dot{\hat{\theta}}_2(t) + \frac{\partial}{\partial t}H_2(x,t)\hat{\theta}_2(t) + \Omega_1(x,t)\alpha(1,t)\\ &+ \int_0^x B(x,\xi,t)\left(\eta(x,t) - H_2(\xi,t)\hat{\theta}_2(t) + \phi(\xi,t)\right)d\xi\end{aligned} \tag{3.113}$$

Evaluating (3.106) at $x = 1$, and inserting (3.81b), (3.104b) and (3.107) give (3.108b):

$$\begin{aligned}\eta(1,t) =&\zeta(1,t) - \phi(1,t) + H_2(1,t)\,\hat{\theta}_2(t)\\ =&U_3(t) - \zeta^*(t) + H_2(1,t)\,\hat{\theta}_2(t)\\ =&0\end{aligned} \tag{3.114}$$

Property (3.110) can be seen from

$$\begin{aligned}\frac{\partial}{\partial t}H_2(x,t) =&\frac{1}{\mu}\int_0^x \frac{\partial}{\partial t}H_1(\xi,t)d\xi\\ =&\frac{1}{\mu}\int_0^x \frac{\partial}{\partial t}\left[H(\xi)-\int_0^\xi \frac{1}{\mu}g(\xi-s,t)H(s)ds\right]d\xi\\ =&-\frac{1}{\mu^2}\int_0^x\int_0^\xi g_t(\xi-s,t)H(s)dsd\xi\\ \leq&\frac{1}{\mu^2}\int_0^x\int_0^\xi |g_t(\xi-s,t)||H(s)|dsd\xi \end{aligned} \tag{3.115}$$

where the definitions of H_1 and H_2 from (3.83) and (3.109) have been substituted in. Using property (3.60) and boundedness of H from Lemma 3.3 give

$$\begin{aligned}\frac{\partial}{\partial t}H_2(x,t) \leq&\frac{1}{\mu^2}h_2\int_0^x\int_0^\xi |H(s)|dsd\xi|\hat{\theta}_1(t)|\\ \leq&\frac{1}{\mu^2}h_2\bar{H}_1\int_0^x\int_0^\xi dsd\xi|\hat{\theta}_1(t)|\\ =&\frac{1}{\mu^2}h_2\bar{H}_1\frac{1}{2}x^2|\hat{\theta}_1(t)|\\ \leq&\frac{1}{2\mu^2}h_2\bar{H}_1|\hat{\theta}_1(t)|\\ =&h_3|\hat{\theta}_1(t)|. \end{aligned} \tag{3.116}$$

Finally, from Theorem 3.2, using that $\hat{\theta}_1 \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ gives (3.110). □

From transformation (3.106) and definition (3.109), it can be seen that $\eta(0,t) = 0$ implies $\zeta(0,t) = \phi(0,t) = \zeta^*(t-d_\beta)$. A useful relationship between $\zeta^*(t)$ and $\zeta^*(t-d_\beta)$ will be stated next.

Lemma 3.9. *Consider the adaptive law* (3.26)*. If all the assumptions in Theorem 3.2 hold and* $r \notin \left[\underline{\theta}_1, \bar{\theta}_1\right]$*, then*

$$\zeta^*(t) \to \zeta^*(t-d_\beta) \tag{3.117}$$

where ζ^* *is defined in* (3.91)*.*

Proof. Consider the limit

$$\lim_{t\to\infty}\left[\hat{\theta}_i(t)-\hat{\theta}_i(t-d_\beta)\right] \tag{3.118}$$

where $i \in \{1,2\}$. Using Cauchy-Schwarz' inequality, the following relation can be found

$$0 \leq \lim_{t\to\infty}|\hat{\theta}_i(t)-\hat{\theta}_i(t-d_\beta)| \leq \lim_{t\to\infty}\int_{t-d_\beta}^t \left|\dot{\hat{\theta}}_i(\tau)\right|d\tau$$

$$\leq \lim_{t\to\infty} \left(\int_{t-d_\beta}^{t} d\tau \right)^{\frac{1}{2}} \left(\int_{t-d_\beta}^{t} \left| \dot{\hat{\theta}}_i(\tau) \right|^2 d\tau \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{\sqrt{\lambda}} \lim_{t\to\infty} \left(\int_{t-d_\beta}^{t} \left| \dot{\hat{\theta}}_i(\tau) \right|^2 d\tau \right)^{\frac{1}{2}}. \tag{3.119}$$

From Theorem 3.2, we have $\dot{\hat{\theta}}_i \in \mathcal{L}_2$ for $i \in \{1, 2\}$, meaning the last integral in (3.119) converges to zero and by the squeeze theorem (see Lemma A.4)

$$\lim_{t\to\infty} \left[\hat{\theta}_i(t) - \hat{\theta}_i(t - d_\beta) \right] = 0, \tag{3.120}$$

or equivalently

$$\hat{\theta}_i(t) \to \hat{\theta}_i(t - d_\beta). \tag{3.121}$$

Since $r \notin \left[\underline{\theta}_1, \bar{\theta}_1 \right]$, (3.117) follows from (3.91). □

3.4 Stability Proof

Stabilization of (3.1) and convergence in the sense of (3.7) will be proved by considering the coupled system consisting of

$$\alpha_t(x,t) + \lambda \alpha_x(x,t) = 0 \tag{3.122a}$$

$$\begin{aligned} \eta_t(x,t) - \mu \eta_x(x,t) =& H_2(x,t)\dot{\hat{\theta}}_2(t) + \frac{\partial}{\partial t} H_2(x,t)\hat{\theta}_2(t) \\ &+ \Omega_1(x,t)\alpha(1,t) \\ &+ \int_0^x B(x,\xi,t) \Big(\eta(x,t) - H_2(\xi,t)\hat{\theta}_2(t) + \phi(\xi,t) \Big) d\xi \end{aligned} \tag{3.122b}$$

$$\alpha(0,t) = \tilde{\theta}_1(t) v(0,t) + \tilde{\theta}_2(t) \tag{3.122c}$$

$$\eta(1,t) = 0 \tag{3.122d}$$

and the adaptive law (3.26), where $v(0,t)$ is related to ζ, ζ^* and α through

$$v(0,t) = \eta(0,t) + \zeta^*(t - d_\beta) + \int_0^1 P^v(0,\xi,t)\alpha(\xi,t) d\xi. \tag{3.123}$$

The relation (3.123) can be seen from using $\beta \equiv 0$ for $t > d_\beta$ in (3.12b), the transformations (3.68b), (3.79) and (3.106), and $\phi(0,t) = \zeta^*(t - d_\beta)$.

Before proving Theorem 3.4, boundedness of the system states in $L_2([0,1])$, boundedness pointwise in space and convergence of $||\alpha||, ||\eta||$ in $L_2([0,1])$ will be proved in separate lemmas in the following sections. The concluding proof of Theorem 3.4 is given in Section 3.4.4.

3.4.1 Boundedness in $L_2([0,1])$

Lemma 3.10. *Consider the Lyapunov function candidate*

$$V_3 = a_1 V_1 + V_2 \tag{3.124}$$

where $a_1 > 0$ is a constant to be decided, and

$$V_1 = \lambda^{-1} \int_0^1 e^{-\delta x} \alpha^2(x,t) dx \tag{3.125a}$$

$$V_2 = \mu^{-1} \int_0^1 e^{kx} \eta^2(x,t) dx \tag{3.125b}$$

where α, η are the system states in the coupled system (3.122)*, H_2 and Ω_1 are defined in* (3.109) *and* (3.83) *respectively, $\hat{\theta}_1, \hat{\theta}_2$ are obtained from* (3.26) *in Theorem 3.2 and v is related to the system states through* (3.123)*.*

With appropriately selected a_1, δ and k, then (3.124) *satisfies*

$$\dot{V}_3 \leq -h_4 V_3 + l_1(t) V_3(t) + l_2(t) - \left(1 - h_5 \frac{\left(\tilde{\theta}_1(t) v(0,t) + \tilde{\theta}_2(t)\right)^2}{2 + v^2(0,t)}\right) \eta^2(0,t) \tag{3.126}$$

for some constants $h_4, h_5 > 0$, and where $l_1(t), l_2(t) \geq 0$ are real valued functions given by

$$l_1(t) = 2a_1 \frac{\left(\tilde{\theta}_1(t) v(0,t) + \tilde{\theta}_2(t)\right)^2}{2 + v^2(0,t)} (\bar{P}^v)^2 e^{\delta} \lambda + e^k ||B||^2 \tag{3.127a}$$

$$\begin{aligned} l_2(t) = &\left(\left(\bar{H}_2 \bar{\theta}_2\right)^2 + \left(\bar{\zeta^*}\right)^2 \right) e^k \mu^{-1} ||B||^2 + \frac{\bar{H}_2}{\mu k} \left(e^k - 1\right) \dot{\hat{\theta}}_2^2(t) \\ &+ a_1 \frac{\left(\tilde{\theta}_1(t) v(0,t) + \tilde{\theta}_2(t)\right)^2}{2 + v^2(0,t)} \left(2 + 4(\bar{\zeta^*})^2\right) + \frac{\bar{\theta}_2 c}{\mu k} \left(e^k - 1\right) \dot{\hat{\theta}}_1^2(t) \end{aligned} \tag{3.127b}$$

satisfying

$$l_1, l_2 \in \mathcal{L}_1 \cap \mathcal{L}_\infty. \tag{3.128}$$

Furthermore,

$$V_3 \in \mathcal{L}_1 \cap \mathcal{L}_\infty \tag{3.129}$$

and

$$||\alpha||, ||\eta||, ||\hat{u}||, ||\hat{v}||, ||u||, ||v|| \in \mathcal{L}_\infty. \tag{3.130}$$

Proof. Differentiating (3.125a) with respect to time, inserting the dynamics (3.122a), (3.122c) and integration by parts give

$$\begin{aligned} \dot{V}_1 = &- 2 \int_0^1 e^{-\delta x} \alpha(x,t) \alpha_x(x,t) dx \\ = &- e^{-\delta} \alpha^2(1,t) + \alpha^2(0,t) - \delta \int_0^1 e^{-\delta x} \alpha^2(x,t) dx \end{aligned}$$

$$= -e^{-\delta}\alpha^2(1,t) + \left(\tilde{\theta}_1(t)v(0,t) + \tilde{\theta}_2(t)\right)^2 - \delta\lambda V_1 \tag{3.131}$$

Similarly, differentiating (3.125b) with respect to time and inserting the dynamics (3.122b) give

$$\begin{aligned}
\dot{V}_2 =& 2\int_0^1 e^{kx}\mu^{-1}\eta(x,t)\eta_t(x,t)\,dx \\
=& 2\int_0^1 e^{kx}\eta(x,t)\eta_x(x,t)\,dx \\
&+ 2\int_0^1 e^{kx}\mu^{-1}\eta(x,t)\int_0^x B(x,\xi,t)\eta(\xi,t)\,d\xi\,dx \\
&+ 2\int_0^1 e^{kx}\mu^{-1}\eta(x,t)\int_0^x B(x,\xi,t)(-H_2(\xi))d\xi\,dx\,\hat{\theta}_2(t) \\
&+ 2\int_0^1 e^{kx}\mu^{-1}\eta(x,t)\int_0^x B(x,\xi,t)\phi(\xi,t)\,d\xi\,dx \\
&+ 2\int_0^1 e^{kx}\mu^{-1}\eta(x,t)\Omega_1(x,t)\,dx\ \alpha(1,t) \\
&+ 2\int_0^1 e^{kx}\mu^{-1}\eta(x,t)H_2(x,t)dx\ \dot{\hat{\theta}}_2(t) \\
&+ 2\int_0^1 e^{kx}\mu^{-1}\eta(x,t)\frac{\partial}{\partial t}H_2(x,t)dx\ \hat{\theta}_2(t).
\end{aligned} \tag{3.132}$$

Each of the terms on the right hand side of (3.132) will be considered separately.

1st term: Integration by parts and using boundary condition (3.122d) give

$$\begin{aligned}
&2\int_0^1 e^{kx}\eta(x,t)\eta_x(x,t)dx \\
=& e^k\eta^2(1,t) - \eta^2(0,t) - \int_0^1 ke^{kx}\eta^2(x,t)dx \\
=& -\eta^2(0,t) - \mu k V_2.
\end{aligned} \tag{3.133}$$

2nd term: Using Young's inequality (see Lemma A.2) to separate the cross-term, and Cauchy-Schwarz' inequality on the last term give

$$\begin{aligned}
&2\int_0^1 e^{kx}\mu^{-1}\eta(x,t)\int_0^x B(x,\xi,t)\eta(\xi,t)d\xi dx \\
\leq& \int_0^1 e^{kx}\mu^{-1}\eta^2(x,t)dx + \int_0^1 e^{kx}\mu^{-1}\left(\int_0^x B(x,\xi,t)\eta(\xi,t)d\xi\right)^2 dx \\
\leq& \int_0^1 e^{kx}\mu^{-1}\eta^2(x,t)dx + e^k\mu^{-1}\int_0^1\left(\int_0^x B(x,\xi,t)\eta(\xi,t)d\xi\right)^2 dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 e^{kx}\mu^{-1}\eta^2(x,t)dx + e^k\mu^{-1}\int_0^1\left(\int_0^x B^2(x,\xi,t)d\xi\int_0^x \eta^2(\xi,t)d\xi\right)dx \\
&\leq \int_0^1 e^{kx}\mu^{-1}\eta^2(x,t)dx + e^k\mu^{-1}\int_0^1\left(\int_0^x B^2(x,\xi,t)d\xi\int_0^1 \eta^2(\xi,t)d\xi\right)dx \\
&\leq \int_0^1 e^{kx}\mu^{-1}\eta^2(x,t)dx + e^k\mu^{-1}\int_0^1\int_0^x B^2(x,\xi,t)d\xi dx\int_0^1 \eta^2(x,t)dx \\
&\leq \int_0^1 e^{kx}\mu^{-1}\eta^2(x,t)dx + e^k\int_0^1\int_0^x B^2(x,\xi,t)d\xi dx\int_0^1 e^{kx}\mu^{-1}\eta^2(x,t)dx \\
&\leq \left(1+e^k||B||\right)V_2.
\end{aligned}
\tag{3.134}
$$

3rd and 4th term: Again, separating the cross term using Young's inequality and using that H_2, ζ^* and $\hat{\theta}_2$ are bounded give similarly for the 3rd and 4th term

$$
\begin{aligned}
&2\int_0^1 e^{kx}\mu^{-1}\eta(x,t)\int_0^x B(x,\xi,t)(-H_2(\xi))d\xi\, dx\,\hat{\theta}_2(t) \\
=&2\int_0^1\int_0^x e^{kx}\mu^{-1}\eta(x,t)B(x,\xi,t)(-H_2(\xi))d\xi\, dx\,\hat{\theta}_2(t) \\
\leq&\int_0^1 e^{kx}\mu^{-1}\eta^2(x,t)\int_0^x d\xi\, dx + \int_0^1\int_0^x e^{kx}\mu^{-1}B^2(x,\xi,t)H_2^2(\xi))d\xi\, dx\,\hat{\theta}_2^2(t) \\
\leq&\int_0^1 e^{kx}\mu^{-1}\eta^2(x,t)\, x\, dx + \int_0^1\int_0^x e^{kx}\mu^{-1}B^2(x,\xi,t)H_2^2(\xi))d\xi\, dx\,\hat{\theta}_2^2(t) \\
\leq&\int_0^1 e^{kx}\mu^{-1}\eta^2(x,t)dx + \left(\bar{H}_2\bar{\theta}_2\right)^2 e^k\mu^{-1}\int_0^1\int_0^x B^2(x,\xi,t))d\xi\, dx \\
\leq&V_2 + \left(\bar{H}_2\bar{\theta}_2\right)^2 e^k\mu^{-1}||B||^2
\end{aligned}
\tag{3.135}
$$

and

$$
\begin{aligned}
&+2\int_0^1 e^{kx}\mu^{-1}\eta(x,t)\int_0^x B(x,\xi,t)\phi(\xi,t)\, d\xi\, dx \\
=&2\int_0^1\int_0^x e^{kx}\mu^{-1}\eta(x,t)B(x,\xi,t)\phi(\xi,t)\, d\xi\, dx \\
\leq&\int_0^1 e^{kx}\mu^{-1}\eta^2(x,t)\int_0^x d\xi\, dx + \int_0^1\int_0^x e^{kx}\mu^{-1}B^2(x,\xi,t)\phi^2(\xi,t)\, d\xi\, dx \\
\leq&\int_0^1 e^{kx}\mu^{-1}\eta^2(x,t)\, x\, dx + \int_0^1\int_0^x e^{kx}\mu^{-1}B^2(x,\xi,t)\phi^2(\xi,t)d\xi\, dx \\
\leq&\int_0^1 e^{kx}\mu^{-1}\eta^2(x,t)dx + \left(\bar{\zeta^*}\right)^2 e^k\mu^{-1}\int_0^1\int_0^x B^2(x,\xi,t))d\xi\, dx \\
\leq&V_2 + \left(\bar{\zeta^*}\right)^2 e^k\mu^{-1}||B||^2.
\end{aligned}
\tag{3.136}
$$

5th and 6th term: Using that Ω_1 and H_2 are bounded, and separating the cross-term using Young's inequality give similarly for the 5th and 6th term

$$\begin{aligned}
&+2\int_0^1 e^{kx}\mu^{-1}\eta(x,t)\Omega_1(x,t)\,dx\,\alpha(1,t) \\
\le & 2\bar{\Omega}_1\int_0^1 e^{kx}\mu^{-1}|\eta(x,t)|dx\,|\alpha(1,t)| \\
\le & \bar{\Omega}_1\int_0^1 e^{kx}\mu^{-1}\eta^2(x,t)dx + \bar{\Omega}_1\int_0^1 e^{kx}\mu^{-1}dx\,\alpha^2(1,t) \\
\le & \bar{\Omega}_1 V_2 + \frac{\bar{\Omega}_1}{\mu k}\left(e^k-1\right)\alpha^2(1,t)
\end{aligned} \tag{3.137}$$

and

$$\begin{aligned}
&+2\int_0^1 e^{kx}\mu^{-1}\eta(x,t)H_2(x)dx\,\dot{\hat{\theta}}_2(t) \\
\le & 2\bar{H}_2\int_0^1 e^{kx}\mu^{-1}|\eta(x,t)|dx\,|\dot{\hat{\theta}}_2(t)| \\
\le & \bar{H}_2\int_0^1 e^{kx}\mu^{-1}\eta^2(x,t)dx + \bar{H}_2\int_0^1 e^{kx}\mu^{-1}dx\,\dot{\hat{\theta}}_2^2(t) \\
\le & \bar{H}_2 V_2 + \frac{\bar{H}_2}{\mu k}\left(e^k-1\right)\dot{\hat{\theta}}_2^2(t).
\end{aligned} \tag{3.138}$$

7th term: From boundedness of $\hat{\theta}_2$, property (3.110) and Young's inequality on the cross term, one obtains

$$\begin{aligned}
&+2\int_0^1 e^{kx}\mu^{-1}\eta(x,t)\frac{\partial}{\partial t}H_2(x,t)dx\hat{\theta}_2(t) \\
\le\, & 2\bar{\theta}_2\int_0^1 e^{kx}\mu^{-1}|\eta(x,t)|\left|\frac{\partial}{\partial t}H_2(x,t)\right|dx \\
\le\, & 2\bar{\theta}_2 h_3\int_0^1 e^{kx}\mu^{-1}|\eta(x,t)||\dot{\hat{\theta}}_1(t)|dx \\
\le\, & \bar{\theta}_2 h_3\int_0^1 e^{kx}\mu^{-1}\eta^2(x,t)dx + \bar{\theta}_2 h_3\int_0^1 e^{kx}\mu^{-1}dx\dot{\hat{\theta}}_1^2(t) \\
\le\, & \bar{\theta}_2 h_3 V_2 + \frac{\bar{\theta}_2 h_3}{\mu k}\left(e^k-1\right)\dot{\hat{\theta}}_1^2(t).
\end{aligned} \tag{3.139}$$

Combining (3.131) and (3.132) to form $\dot{V}_3$ and inserting (3.133)–(3.139) into (3.132) yield

$$
\begin{aligned}
\dot{V}_3 \leq & a_1 \left(-e^{-\delta}\alpha^2(1,t) + \left(\tilde{\theta}_1(t)v(0,t) + \tilde{\theta}_2(t)\right)^2 - \delta\lambda V_1\right) \\
& - \eta^2(0,t) - \mu k V_2 \\
& + \left(1 + e^k ||B||^2\right) V_2 \\
& + V_2 + \left(\bar{H}_2\bar{\theta}_2\right)^2 e^k \mu^{-1} ||B||^2 \\
& + V_2 + \left(\bar{\zeta^*}\right)^2 e^k \mu^{-1} ||B||^2 \\
& + \bar{\Omega}_1 V_2 + \frac{\bar{\Omega}_1}{\mu k}\left(e^k - 1\right)\alpha^2(1,t) \\
& + \bar{H}_2 V_2 + \frac{\bar{H}_2}{\mu k}\left(e^k - 1\right)\dot{\hat{\theta}}_2^2(t) \\
& + \bar{\theta}_2 h_3 V_2 + \frac{\bar{\theta}_2 c}{\mu k}\left(e^k - 1\right)\dot{\hat{\theta}}_1^2(t).
\end{aligned}
\tag{3.140}
$$

Selecting $a_1 = e^{\delta}\frac{\bar{\Omega}_1}{k\mu}(e^k - 1)$ and $\delta = 1$, (3.140) can be simplified to

$$
\begin{aligned}
\dot{V}_3 \leq & a_1 \left(\tilde{\theta}_1(t)v(0,t) + \tilde{\theta}_2(t)\right)^2 - \eta^2(0,t) \\
& - a_1\delta\lambda V_1 - \left(\mu k - 3 - \bar{\Omega}_1 - \bar{H}_2 - \bar{\theta}_2 h_3\right) V_2 \\
& + e^k ||B||^2 V_2 + \left(\left(\bar{H}_2\bar{\theta}_2\right)^2 + \left(\bar{\zeta^*}\right)^2\right) e^k \mu^{-1} ||B||^2 \\
& + \frac{\bar{H}_2}{\mu k}\left(e^k - 1\right)\dot{\hat{\theta}}_2^2(t) + \frac{\bar{\theta}_2 h_3}{\mu k}\left(e^k - 1\right)\dot{\hat{\theta}}_1^2(t).
\end{aligned}
\tag{3.141}
$$

For $k > \frac{1}{\mu}\left(3 + \bar{\Omega}_1 + \bar{H}_2 + \bar{\theta}_2 c\right)$, the 4th term in parentheses will be positive, yielding

$$
\begin{aligned}
\dot{V}_3 \leq & a_1 \left(\tilde{\theta}_1(t)v(0,t) + \tilde{\theta}_2(t)\right)^2 - \eta^2(0,t) \\
& - h_4 V_3 \\
& + e^k ||B||^2 V_2 + \left(\left(\bar{H}_2\bar{\theta}_2\right)^2 + \left(\bar{\zeta^*}\right)^2\right) e^k \mu^{-1} ||B||^2 \\
& + \frac{\bar{H}_2}{\mu k}\left(e^k - 1\right)\dot{\hat{\theta}}_2^2(t) + \frac{\bar{\theta}_2 c}{\mu k}\left(e^k - 1\right)\dot{\hat{\theta}}_1^2(t).
\end{aligned}
\tag{3.142}
$$

The first term can be rewritten on the form considered in property III in Theorem 3.2 as follows

$$
\begin{aligned}
\dot{V}_3 \leq & a_1 \frac{\left(\tilde{\theta}_1(t)v(0,t) + \tilde{\theta}_2(t)\right)^2}{2 + v^2(0,t)}\left(2 + v^2(0,t)\right) - \eta^2(0,t) \\
& - h_4 V_3 \\
& + e^k ||B||^2 V_2 + \left(\left(\bar{H}_2\bar{\theta}_2\right)^2 + \left(\bar{\zeta^*}\right)^2\right) e^k \mu^{-1} ||B||^2
\end{aligned}
$$

$$+ \frac{\bar{H}_2}{\mu k}\left(e^k - 1\right)\dot{\hat{\theta}}_2^2(t) + \frac{\bar{\theta}_2 c}{\mu k}\left(e^k - 1\right)\dot{\hat{\theta}}_1^2(t). \tag{3.143}$$

Next, using relation (3.123), this can be written as

$$\begin{aligned}
\dot{V}_3 \leq & a_1 \frac{\left(\tilde{\theta}_1(t)v(0,t) + \tilde{\theta}_2(t)\right)^2}{2 + v^2(0,t)} \left(2 + 4\eta^2(0,t) + 4(\bar{\zeta^*})^2 + 2(\bar{P}^v)^2 \int_0^1 \alpha^2(\xi,t)d\xi\right) \\
& - \eta^2(0,t) - h_4 V_3 \\
& + e^k ||B||^2 V_2 + \left(\left(\bar{H}_2\bar{\theta}_2\right)^2 + \left(\bar{\zeta^*}\right)^2\right) e^k \mu^{-1} ||B||^2 \\
& + \frac{\bar{H}_2}{\mu k}\left(e^k - 1\right)\dot{\hat{\theta}}_2^2(t) + \frac{\bar{\theta}_2 c}{\mu k}\left(e^k - 1\right)\dot{\hat{\theta}}_1^2(t),
\end{aligned} \tag{3.144}$$

and after some reorganizing of the terms, we end up with

$$\begin{aligned}
\dot{V}_3 \leq & a_1 \frac{\left(\tilde{\theta}_1(t)v(0,t) + \tilde{\theta}_2(t)\right)^2}{2 + v^2(0,t)} \left(2 + 4(\bar{\zeta^*})^2\right) \\
& - \left(1 - 4a_1 \frac{\left(\tilde{\theta}_1(t)v(0,t) + \tilde{\theta}_2(t)\right)^2}{2 + v^2(0,t)}\right) \eta^2(0,t) \\
& + 2a_1 \frac{\left(\tilde{\theta}_1(t)v(0,t) + \tilde{\theta}_2(t)\right)^2}{2 + v^2(0,t)} (\bar{P}^v)^2 e^\delta \lambda V_1 \\
& - h_4 V_3 \\
& + e^k ||B||^2 V_2 \\
& + \left(\left(\bar{H}_2\bar{\theta}_2\right)^2 + \left(\bar{\zeta^*}\right)^2\right) e^k \mu^{-1} ||B||^2 \\
& + \frac{\bar{H}_2}{\mu k}\left(e^k - 1\right)\dot{\hat{\theta}}_2^2(t) + \frac{\bar{\theta}_2 c}{\mu k}\left(e^k - 1\right)\dot{\hat{\theta}}_1^2(t) \\
\leq & - h_4 V_3 - \left(1 - 4a_1 \frac{\left(\tilde{\theta}_1(t)v(0,t) + \tilde{\theta}_2(t)\right)^2}{2 + v^2(0,t)}\right) \eta^2(0,t) \\
& + \left(2a_1 \frac{\left(\tilde{\theta}_1(t)v(0,t) + \tilde{\theta}_2(t)\right)^2}{2 + v^2(0,t)} (\bar{P}^v)^2 e^\delta \lambda + e^k ||B||^2\right) V_3 \\
& + \left(\left(\bar{H}_2\bar{\theta}_2\right)^2 + \left(\bar{\zeta^*}\right)^2\right) e^k \mu^{-1} ||B||^2 \\
& + \frac{\bar{H}_2}{\mu k}\left(e^k - 1\right)\dot{\hat{\theta}}_2^2(t) + \frac{\bar{\theta}_2 c}{\mu k}\left(e^k - 1\right)\dot{\hat{\theta}}_1^2(t) \\
& + a_1 \frac{\left(\tilde{\theta}_1(t)v(0,t) + \tilde{\theta}_2(t)\right)^2}{2 + v^2(0,t)} \left(2 + 4(\bar{\zeta^*})^2\right).
\end{aligned} \tag{3.145}$$

From this last expression, substituting in $l_1(t)$ and $l_2(t)$ from (3.127) give the desired result (3.126) with some constants $h_4 > 0$ and $h_5 > 0$.

By using the definitions of $\hat{\theta}(t)$ and $\psi(t)$ from (3.27a) and (3.24a) respectively, the last term in (3.126) can be written in vector form as

$$\frac{\left(\tilde{\theta}_1(t)v(0,t)+\tilde{\theta}_2(t)\right)^2}{2+v^2(0,t)} = \hat{\theta}^T(t)\frac{\psi(t+d_\alpha)\psi^T(t+d_\alpha)}{1+\psi^T(t+d_\alpha)\psi(t+d_\alpha)}\hat{\theta}(t), \tag{3.146}$$

where the relation (3.25) has been used. Now, from (3.47), the term (3.146) can be recognized to satisfy

$$\dot{V}_0(t) \leq -h_5\hat{\theta}^T(t)\frac{\psi(t+d_\alpha)\psi^T(t+d_\alpha)}{1+\psi^T(t+d_\alpha)\psi(t+d_\alpha)}\hat{\theta}(t)$$

for $V_0 \geq 0$ defined in (3.33). Furthermore, from (3.124) we have that $V_3 \geq 0$, and since all terms are squared in (3.127) that $l_1(t), l_2(t) \geq 0$ for all $t > 0$. From Theorem 3.4 property I, we have that $\hat{\theta} \in \mathcal{L}_\infty$, which together with property II and III, and (3.85) in Lemma 3.6 give (3.128). Lastly, we have that

$$0 \leq \frac{\psi(t)\psi^T(t)}{1+\psi^T(t)\psi(t)} = \left(\frac{\psi(t)\psi^T(t)}{1+\psi^T(t)\psi(t)}\right)^T \leq I_{2\times 2}. \tag{3.147}$$

Lemma 8 from Anfinsen and Aamo (2017c) can now be applied, yielding (3.129). For reference, Anfinsen and Aamo (2017c, Lemma 8) is included in Appendix A.5 as Lemma A.5.

From (3.129) it follows that

$$||\alpha||, ||\eta|| \in \mathcal{L}_\infty \tag{3.148}$$

and from the invertibility of the transforms (3.12), (3.68), (3.79) and (3.106) that

$$||\hat{u}||, ||\hat{v}||, ||u||, ||v|| \in \mathcal{L}_\infty \tag{3.149}$$

which completes the proof. □

3.4.2 Boundedness Pointwise in Space

Lemma 3.11. *Consider the system* (3.1) *in closed loop with the observer* (3.9) *and adaptive law* (3.26)*. If the control signal $U(t)$ is selected according to* (3.64)*, then the states (u, v) will be bounded pointwise in space, that is*

$$u(x,\cdot), v(x,\cdot) \in \mathcal{L}_\infty, \quad \forall x \in [0,1] \tag{3.150}$$

Proof. Consider the backstepping transformation

$$\check{\alpha}(x,t) = u(x,t) - \int_0^x L^{uu}(x,\xi)u(\xi,t)d\xi - \int_0^x L^{uv}(x,\xi)v(\xi,t)d\xi \tag{3.151a}$$

$$\check{\beta}(x,t) = v(x,t) - \int_0^x L^{vu}(x,\xi)u(\xi,t)d\xi - \int_0^x L^{vv}(x,\xi)v(\xi,t)d\xi \tag{3.151b}$$

where $(L^{uu}, L^{uv}, L^{vu}, L^{vv})$ is the solution to the system

$$\lambda L_x^{uu} + \lambda L_\xi^{uu} = - c_2(\xi) L^{uv} \tag{3.152a}$$
$$\lambda L_x^{uv} - \mu L_\xi^{uv} = - c_1(\xi) L^{uu} \tag{3.152b}$$
$$\mu L_x^{vu} - \lambda L_\xi^{vu} = c_2(\xi) L^{vv} \tag{3.152c}$$
$$\mu L_x^{vv} + \mu L_\xi^{vv} = c_1(\xi) L^{vu} \tag{3.152d}$$
$$L^{uu}(x, 0) = h_1(x) \tag{3.152e}$$
$$L^{uv}(x, 0) = \frac{c_1(x)}{\lambda + \mu} \tag{3.152f}$$
$$L^{vu}(x, 0) = - \frac{c_2(x)}{\lambda + \mu} \tag{3.152g}$$
$$L^{vv}(x, 0) = \frac{\theta_1 \lambda}{\mu} L^{vu}(x, 0) \tag{3.152h}$$

and $h_1(x)$ is a design parameter. The backstepping transformation (3.151) and kernel PDE (3.152) were first used in Vazquez et al. (2011), where it is shown that (3.151) is invertible and there exist a unique bounded solution to (3.152) for all $(x, \xi) \in \mathcal{T}_2$ ($\mathcal{T}_2$ defined in (3.53)).

Differentiation of (3.151a) with respect to time and space, inserting the dynamics (3.1) and integration by parts yield

$$\begin{aligned}
\check{\alpha}_t(x,t) =& u_t(x,t) - \int_0^x L^{uu}(x,\xi) u_t(\xi,t) d\xi - \int_0^x L^{uv}(x,\xi) v_t(\xi,t) d\xi \\
=& u_t(x,t) - \int_0^x L^{uu}(x,\xi) \left(-\lambda u_\xi(\xi,t) + c_1(\xi) v(\xi,t)\right) d\xi \\
& - \int_0^x L^{uv}(x,\xi) \left(\mu v_\xi(\xi,t) + c_2(\xi) u(\xi,t)\right) d\xi \\
=& u_t(x,t) + L^{uu}(x,x) \lambda u(x,t) - L^{uu}(x,0) \lambda u(0,t) - \int_0^x L_\xi^{uu}(x,\xi) \lambda u(\xi,t) d\xi \\
& - L^{uv}(x,x) \mu v(x,t) + L^{uv}(x,0) \mu v(0,t) + \int_0^x L_\xi^{uv}(x,\xi) \mu v(\xi,t) d\xi \\
& - \int_0^x L^{uu}(x,\xi) c_1(\xi) v(\xi,t) d\xi - \int_0^x L^{uv}(x,\xi) c_2(\xi) u(\xi,t) d\xi,
\end{aligned} \tag{3.153}$$

and

$$\begin{aligned}
\check{\alpha}_x(x,t) =& u_x(x,t) - L^{uu}(x,x) u(x,t) - \int_0^x L_x^{uu}(x,\xi) u(\xi,t) d\xi \\
& - L^{uv}(x,x) v(x,t) - \int_0^x L_x^{uv}(x,\xi) v(\xi,t) d\xi.
\end{aligned} \tag{3.154}$$

Similarly for (3.151b), one obtains

$$\check{\beta}_t(x,t) = v_t(x,t) - \int_0^x L^{vu}(x,\xi) u_t(\xi,t) d\xi - \int_0^x L^{vv}(x,\xi) v_t(\xi,t) d\xi$$

$$
\begin{aligned}
&= v_t(x,t) - \int_0^x L^{vu}(x,\xi)\left(-\lambda u_\xi(\xi,t) + c_1(\xi)v(\xi,t)\right)d\xi \\
&\quad - \int_0^x L^{vv}(x,\xi)\left(\mu v_\xi(\xi,t) + c_2(\xi)u(\xi,t)\right)d\xi \\
&= v_t(x,t) + L^{vu}(x,x)\lambda u(x,t) - L^{vu}(x,0)\lambda u(0,t) - \int_0^x L^{vu}_\xi(x,\xi)\lambda u(\xi,t)d\xi \\
&\quad - L^{vv}(x,x)\mu v(x,t) + L^{vv}(x,0)\mu v(0,t) + \int_0^x L^{vv}_\xi(x,\xi)\mu v(\xi,t)d\xi \\
&\quad - \int_0^x L^{vu}(x,\xi)c_1(\xi)v(\xi,t)d\xi - \int_0^x L^{vv}(x,\xi)c_2(\xi)u(\xi,t)d\xi, \qquad (3.155)
\end{aligned}
$$

and

$$
\begin{aligned}
\check{\beta}_x(x,t) =& v_x(x,t) - L^{vu}(x,x)u(x,t) - \int_0^x L^{vu}_x(x,\xi)u(\xi,t)d\xi \\
& - L^{vv}(x,x)v(x,t) - \int_0^x L^{vv}_x(x,\xi)v(\xi,t)d\xi. \qquad (3.156)
\end{aligned}
$$

Combining (3.153) and (3.154) and inserting the kernel equations (3.152) give

$$
\begin{aligned}
\check{\alpha}_t(x,t) + \lambda\check{\alpha}_x(x,t) =& u_t(x,t) + L^{uu}(x,x)\lambda u(x,t) - L^{uu}(x,0)\lambda u(0,t) \\
& - \int_0^x L^{uu}_\xi(x,\xi)\lambda u(\xi,t)d\xi - L^{uv}(x,x)\mu v(x,t) \\
& + L^{uv}(x,0)\mu v(0,t) + \int_0^x L^{uv}_\xi(x,\xi)\mu v(\xi,t)d\xi \\
& - \int_0^x L^{uu}(x,\xi)c_1(\xi)v(\xi,t)d\xi - \int_0^x L^{uv}(x,\xi)c_2(\xi)u(\xi,t)d\xi \\
& + \lambda\Bigg(u_x(x,t) - L^{uu}(x,x)u(x,t) - \int_0^x L^{uu}_x(x,\xi)u(\xi,t)d\xi \\
& \quad - L^{uv}(x,x)v(x,t) - \int_0^x L^{uv}_x(x,\xi)v(\xi,t)d\xi\Bigg) \\
=& h_2(x)\beta(0,t) - h_1(x)\lambda\theta_2 \qquad (3.157)
\end{aligned}
$$

where

$$
h_2(x) = \frac{c_1(x)}{\lambda+\mu}\mu - h_1(x)\lambda\theta_1. \qquad (3.158)
$$

Similarly, combining (3.155) and (3.156) and inserting the kernel equations (3.152) give

$$
\begin{aligned}
\check{\beta}_t(x,t) - \mu\check{\beta}_x(x,t) =& v_t(x,t) + L^{vu}(x,x)\lambda u(x,t) - L^{vu}(x,0)\lambda u(0,t) \\
& - \int_0^x L^{vu}_\xi(x,\xi)\lambda u(\xi,t)d\xi - L^{vv}(x,x)\mu v(x,t) \\
& + L^{vv}(x,0)\mu v(0,t) + \int_0^x L^{vv}_\xi(x,\xi)\mu v(\xi,t)d\xi
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^x L^{vu}(x,\xi)c_1(\xi)v(\xi,t)d\xi - \int_0^x L^{vv}(x,\xi)c_2(\xi)u(\xi,t)d\xi \\
&- \mu\left(v_x(x,t) - L^{vu}(x,x)u(x,t) - \int_0^x L_x^{vu}(x,\xi)u(\xi,t)d\xi\right. \\
&\qquad \left. -L^{vv}(x,x)v(x,t) - \int_0^x L_x^{vv}(x,\xi)v(\xi,t)d\xi\right) \\
=&\frac{c_2(x)}{\lambda+\mu}\lambda\theta_2.
\end{aligned} \tag{3.159}
$$

Evaluating (3.151b) at $x = 1$ and inserting the dynamics (3.1c) and control law (3.64) expressed using the plant and observer states $u, v, \hat{u}, \hat{v}$, we obtain

$$
\begin{aligned}
\check{\beta}(1,t) =& \int_0^1 K^u(1,x)\hat{u}(x,t)dx + \int_0^1 K^v(1,x)\hat{v}(x,t)dx \\
&+ \int_0^1 \frac{g(1-x,t)}{\mu}\hat{v}(x,t)dx \\
&- \int_0^1 \frac{g(1-x,t)}{\mu}\int_0^x K^u(x,\xi)\hat{u}(\xi,t)d\xi dx \\
&- \int_0^1 \frac{g(1-x,t)}{\mu}\int_0^x K^v(x,\xi)\hat{v}(\xi,t)d\xi dx \\
&+ \zeta^*(t) - H_2(1)\hat{\theta}_2(t) \\
&- \int_0^1 L^{vu}(1,\xi)u(\xi,t)d\xi - \int_0^1 L^{vv}(1,\xi)v(\xi,t)d\xi.
\end{aligned} \tag{3.160}
$$

The other boundary condition at $x = 0$ follows trivially from inserting (3.151) evaluated at $x = 0$ into (3.1c). In summary, we have the target system

$$
\check{\alpha}_t(x,t) + \lambda\check{\alpha}_x(x,t) = h_2(x)\beta(0,t) - h_1(x)\lambda\theta_2 \tag{3.161a}
$$

$$
\check{\beta}_t(x,t) - \mu\check{\beta}_x(x,t) = \frac{c_2(x)}{\lambda+\mu}\lambda\theta_2 \tag{3.161b}
$$

$$
\check{\alpha}(0,t) = \theta_1\check{\beta}(0,t) + \theta_2 \tag{3.161c}
$$

$$
\begin{aligned}
\check{\beta}(1,t) =& \int_0^1 K^u(1,x)\hat{u}(x,t)dx + \int_0^1 K^v(1,x)\hat{v}(x,t)dx \\
&+ \int_0^1 \frac{g(1-x,t)}{\mu}\hat{v}(x,t)dx \\
&- \int_0^1 \frac{g(1-x,t)}{\mu}\int_0^x K^u(x,\xi)\hat{u}(\xi,t)d\xi dx \\
&- \int_0^1 \frac{g(1-x,t)}{\mu}\int_0^x K^v(x,\xi)\hat{v}(\xi,t)d\xi dx \\
&+ \zeta^*(t) - H_2(1)\hat{\theta}_2(t) \\
&- \int_0^1 L^{vu}(1,\xi)u(\xi,t)d\xi - \int_0^1 L^{vv}(1,\xi)v(\xi,t)d\xi
\end{aligned} \tag{3.161d}
$$

From (3.161d), we obtain

$$
\begin{aligned}
\check{\beta}(1,t) \leq & \int_0^1 |K^u(1,x)\hat{u}(x,t)|\, dx + \int_0^1 |K^v(1,x)\hat{v}(x,t)| dx \\
& + \int_0^1 \left| \frac{g(1-x,t)}{\mu} \hat{v}(x,t) \right| dx \\
& + \int_0^1 \left| \frac{g(1-x,t)}{\mu} \right| \int_0^x |K^u(x,\xi)\hat{u}(\xi,t)| d\xi dx \\
& + \int_0^1 \left| \frac{g(1-x,t)}{\mu} \right| \int_0^x |K^v(x,\xi)\hat{v}(\xi,t)| d\xi dx \\
& + |\zeta^*(t)| + |H_2(1)||\hat{\theta}_2(t)| \\
& + \int_0^1 |L^{vu}(1,\xi)u(\xi,t)| d\xi + \int_0^1 |L^{vv}(1,\xi)v(\xi,t)| d\xi.
\end{aligned} \tag{3.162}
$$

Using first Young's inequality and then Cauchy-Schwarz' inequality give

$$
\begin{aligned}
\check{\beta}(1,t) \leq & \sqrt{\int_0^1 |K^u(1,x)|^2\, dx} \sqrt{\int_0^1 |\hat{u}(x,t)|^2\, dx} \\
& + \sqrt{\int_0^1 |K^v(1,x)|^2 dx} \sqrt{\int_0^1 |\hat{v}(x,t)|^2 dx} \\
& + \sqrt{\int_0^1 \left| \frac{g(1-x,t)}{\mu} \right|^2 dx} \sqrt{\int_0^1 |\hat{v}(x,t)|^2\, dx} \\
& + \frac{1}{2} \int_0^1 \left| \frac{g(1-x,t)}{\mu} \right|^2 dx + \frac{1}{2} \int_0^1 \int_0^x |K^u(x,\xi)|^2 d\xi dx \int_0^1 |\hat{u}(x,t)|^2 dx \\
& + \frac{1}{2} \int_0^1 \left| \frac{g(1-x,t)}{\mu} \right|^2 dx + \frac{1}{2} \int_0^1 \int_0^x |K^v(x,\xi)|^2 d\xi dx \int_0^1 |\hat{v}(x,t)|^2 dx \\
& + |\zeta^*(t)| + |H_2(1)||\hat{\theta}_2(t)| \\
& + \sqrt{\int_0^1 |L^{vu}(1,x)|^2 dx} \sqrt{\int_0^1 |u(x,t)|^2 dx} \\
& + \sqrt{\int_0^1 |L^{vv}(1,x)|^2 dx} \sqrt{\int_0^1 |v(x,t)|^2 dx}
\end{aligned} \tag{3.163}
$$

Using that (L^{vu}, L^{vv}) is bounded and from Lemma 3.3 that (K^u, K^v) is bounded, this can be written simpler as

$$
\begin{aligned}
\check{\beta}(1,t) \leq & h_6||u|| + h_7||v|| + h_8||\hat{u}|| + h_9||\hat{v}|| \\
& + h_{10}||g|| + |\zeta^*(t)| + |H_2(1)||\hat{\theta}_2(t)|.
\end{aligned} \tag{3.164}
$$

for some positive constants $h_6, h_7, h_8, h_9, h_{10}$. Lastly, using that H_2 is bounded, from Lemma 3.3 and Theorem 3.2 that $|\zeta^*(t)|$, $||g||$, $|\hat{\theta}_2(t)|$ are bounded and from Lemma 3.10 that $||u||, ||v||, ||\hat{u}||, ||\hat{v}||$ are bounded, it follows that $\check{\beta}(1,t)$ is bounded.

Since system (3.161) consists of simple, cascaded transport equations, one must have $\check{\alpha}(x,\cdot), \check{\beta}(x,\cdot) \in \mathcal{L}_\infty$ for all $x \in [0,1]$. Furthermore, from the invertibility of transformation (3.151), property (3.150) follows. □

3.4.3 Convergence in $L_2([0,1])$

Lemma 3.12. *Consider the transformed system* (3.122) *where* H_2 *and* Ω_1 *are defined in* (3.109) *and* (3.83) *respectively, and* v *is related to the system states through* (3.123). *If* $\hat{\theta}_1, \hat{\theta}_2$ *are generated using* (3.26) *in Theorem 3.2, then* α, η *converge to zero in* $L_2([0,1])$*, that is*

$$||\alpha||, ||\eta|| \to 0. \tag{3.165}$$

Proof. By design, system (3.122) is obtained using the control law (3.64). Hence, all assumptions in Lemma 3.11 hold and the sates u, v are bounded pointwise in space. With v bounded, it follows from Theorem 3.2 property IV that $\hat{u}, \hat{v}$ are bounded pointwise in space, and from transformation (3.68), (3.79) and (3.106) that $\eta^2(0,t)$ is bounded. Now, since $\hat{\theta}_1, \hat{\theta}_2 \in \mathcal{L}_\infty$ from Theorem 3.2 and $V_3, l_1, l_2 \in \mathcal{L}_\infty$ from Lemma 3.10, the right hand side of (3.126) is bounded, implying $\dot{V}_3 \in \mathcal{L}_\infty$. This result, together with $V_3 \in \mathcal{L}_1 \cap \mathcal{L}_\infty$ from Lemma 3.10 gives, by Barbalat's Lemma (see Lemma A.6),

$$V_3 \to 0 \tag{3.166}$$

and (3.165) follows. □

3.4.4 Proof of Theorem 3.4

Proof of Theorem 3.4. Inserting (3.69), (3.80) and (3.107) from Lemma 3.5, 3.6 and 3.8 respectively, together with the operators (3.51) and (3.55), into (3.67) give (3.64). Boundedness of all signals in the closed loop system then follows from Lemma 3.10 and 3.11 and Theorem 3.2.

Consider the Lyapunov function candidate

$$V_4 = ||\eta||^2 = \int_0^1 \eta^2(x,t)dx. \tag{3.167}$$

Differentiating with respect to time

$$\begin{aligned}
\dot{V}_4 =& 2\int_0^1 \eta(x,t)\eta_t(x,t)dx \\
=& -\mu\eta^2(0,t) \\
&+ 2\int_0^1 \eta(x,t)\int_0^x B(x,\xi,t)\left(\eta(\xi,t) - H_2(\xi)\hat{\theta}_2(t) + \phi(\xi,t)\right) d\xi\, dx \\
&+ 2\int_0^1 \eta(x,t)\Omega_1(x,t)\, dx\ \alpha(1,t) + 2\int_0^1 \eta(x,t)H_2(x)dx\ \dot{\hat{\theta}}_2(t),
\end{aligned} \tag{3.168}$$

and then integrating from t to $t+T$ gives

$$\begin{aligned}
\int_t^{t+T} \dot{V}_4 d\tau =& V_4(t+T) - V_4(t) \\
=& -\mu \int_t^{t+T} \eta^2(0,\tau) d\tau \\
&+ 2\int_t^{t+T} \int_0^1 \eta(x,\tau) \int_0^x B(x,\xi,\tau) \\
&\quad \times \Big(\eta(\xi,\tau) - H_2(\xi)\hat{\theta}_2(\tau) + \phi(\xi,\tau)\Big)\, d\xi\, dx\, d\tau \\
&+ 2\int_t^{t+T} \int_0^1 \eta(x,\tau)\Omega_1(x,\tau)\, dx\ \alpha(1,\tau)\, d\tau \\
&+ 2\int_t^{t+T} \int_0^1 \eta(x,\tau) H_2(x) dx\ \dot{\hat{\theta}}_2(\tau)\, d\tau.
\end{aligned} \tag{3.169}$$

Rearranging the terms and applying Cauchy-Schwarz' inequality yield

$$\begin{aligned}
&V_4(t+T) - V_4(t) + \mu \int_t^{t+T} \eta^2(0,\tau) d\tau \\
\leq& 2\int_t^{t+T} \sqrt{\int_0^1 |\eta(x,\tau)|^2 dx} \\
&\quad \times \sqrt{\int_0^1 \left(\int_0^x B(x,\xi,\tau) \Big(\eta(\xi,\tau) - H_2(\xi)\hat{\theta}_2(\tau) + \phi(\xi,\tau)\Big)\, d\xi\right)^2 dx}\, d\tau \\
&+ 2\int_t^{t+T} \sqrt{\int_0^1 |\eta(x,\tau)|^2 dx} \sqrt{\int_0^1 |\Omega_1(x,\tau)|^2 dx}\, \alpha(1,\tau)\, d\tau \\
&+ 2\int_t^{t+T} \sqrt{\int_0^1 |\eta(x,\tau)|^2 dx} \sqrt{\int_0^1 |H_2(x)|^2 dx}\, \dot{\hat{\theta}}_2(\tau) d\tau
\end{aligned} \tag{3.170}$$

Since $||\eta||, V_4 \to 0$ and $\int_t^{t+T} \eta^2(0,t) d\tau, V_4 \geq 0$, all terms on the right hand side of (3.170) converge to zero, and the left hand side is bounded from below. Then, by the squeeze theorem, one obtains

$$\int_t^{t+T} \eta^2(0,\tau) d\tau \to 0, \tag{3.171}$$

and thereby

$$\int_t^{t+T} |\eta(0,\tau)| d\tau \to 0. \tag{3.172}$$

Regarding the α dynamics; since (3.122a) is a simple transport equation and from $||\alpha|| \to 0$, one obtains similarly

$$\int_t^{t+T} \alpha^2(0,\tau) d\tau \to 0, \tag{3.173}$$

from which it follows that

$$\int_t^{t+T} |\alpha(0,\tau)| d\tau \to 0 \tag{3.174}$$

and Lemma 3.7 gives (3.65) and (3.66). Inserting transformation (3.106) and the reference signal (3.91) into (3.172) yield

$$\begin{aligned}
&\int_t^{t+T} |\eta(0,\tau)| d\tau \to 0 \\
&\int_t^{t+T} |\zeta(0,\tau) - \phi(0,\tau)| d\tau \to 0 \\
&\int_t^{t+T} |\zeta(0,\tau) - \zeta^*(\tau - d_\beta)| d\tau \to 0
\end{aligned} \tag{3.175}$$

where the explicit solution (3.105) has been inserted. From Lemma 3.9 it then follows that

$$\int_t^{t+T} |\zeta(0,\tau) - \zeta^*(\tau)| d\tau \to 0, \tag{3.176}$$

and by Lemma 3.7, the objective (3.7) is satisfied.

□

Chapter 4

Non-Collocated Sensing and Control

This chapter considers adaptive stabilization and set-point regulation of a system similar to the system presented in Chapter 3, but with sensing also taken at the left boundary, anti-collocated with control, and with unknown boundary parameters appearing in a special bilinear form.

4.1 Problem Statement

Consider the linear 2×2 first-order hyperbolic system

$$u_t(x,t) + \lambda u_x(x,t) = c_1(x)v(x,t) \tag{4.1a}$$
$$v_t(x,t) - \mu v_x(x,t) = c_2(x)u(x,t) \tag{4.1b}$$
$$u(0,t) = rv(0,t) + k(\theta - y_0(t)) \tag{4.1c}$$
$$v(1,t) = U(t) \tag{4.1d}$$

defined for $x \in [0,1]$, $t \geq 0$, where u, v are the system states and

$$\lambda, \mu > 0, \qquad c_1(x), c_2(x) \in C([0,1]) \tag{4.2}$$

are known, while

$$k \in \mathbb{R}\ , \quad \theta \in \mathbb{R} \tag{4.3}$$

are unknown boundary parameters, but where $\text{sign}(k)$ is known. Sensing is allowed at both boundaries. The measurement collocated with actuation is on the form

$$y_1(t) = u(1,t) \tag{4.4}$$

while the measurement anti-collocated with actuation is generated as a linear combination of the system states. That is,

$$y_0(t) = a_0 u(0,t) + b_0 v(0,t). \tag{4.5}$$

with $a_0 \neq 0$. Furthermore, it is assumed that the initial conditions $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ satisfy

$$u_0,\, v_0 \in L_2([0,1]). \tag{4.6}$$

The objective is to design a control input $U(t)$ such that system (4.1) is adaptively stabilized in the L_2-sense and such that the objective

$$\lim_{t\to\infty} \int_t^{t+T} |rv(0,\tau) - u(0,\tau)|\, d\tau = 0 \tag{4.7}$$

with

$$r \neq -\frac{b_0}{a_0} \tag{4.8}$$

is achieved for some arbitrary $T > 0$.

Boundary condition (4.1c) can be written on the form

$$u(0,t) = qv(0,t) + d \tag{4.9}$$

where

$$q = \frac{r - b_0 k}{1 + a_0 k} \tag{4.10a}$$

$$d = \frac{k\theta}{1 + a_0 k} \tag{4.10b}$$

System (4.1), but with boundary condition on the form (4.10) is considered in Anfinsen et al. (2016) ($n+1$ case) and Anfinsen et al. (2017) ($m+n$ case). Here, the unknown boundary parameters q, d and system states u, v are estimated using a swapping-based design. The extension to stabilization, without the additive boundary parameter d and sensing at the left boundary restricted to the form $y_0(t) = v(0,t)$, is given in Anfinsen and Aamo (2017c) ($n+1$ case) and Anfinsen and Aamo (2017a) ($m+n$ case).

The contributions in this chapter are twofold. First, in Section 4.2, building on the results from Anfinsen et al. (2017), a swapping based observer exploiting the bilinear form in the boundary condition (4.1c) is derived. A bilinear parametric model together with a suitable adaptive law are used to estimate the unknown boundary parameters k, θ. Compared to the linear parametric model used in Anfinsen et al. (2017), the bilinear form has some desirable properties regarding parameter convergence. Properties of the adaptive law are formally stated in Theorem 4.4. Second, an adaptive control law stabilizing system (4.1) in the L_2-sense and achieving (4.7) are presented in Section 4.3 with the main result stated in Theorem 4.6. Proof of Theorem 4.6, that is L_2-boundedness and point-wise boundedness of all signals in the closed loop system and convergence in the sense of (4.7), is given in Section 4.4.

4.2 State and Parameter Estimation

In this section, swapping filters for state and parameter estimation are presented. Non-adaptive and adaptive relations between the system states and swapping filters are found. The adaptive estimation error will be used to generate on-line parameter updates and for control design in later sections.

4.2.1 Filter Design

Consider the input filters

$$
\begin{aligned}
a_t(x,t) + \lambda a_x(x,t) =& c_1(x)b(x,t) + P_1(x)(y_1(t) - a(1,t)) && (4.11a)\\
b_t(x,t) - \mu b_x(x,t) =& c_2(x)a(x,t) + P_2(x)(y_1(t) - a(1,t)) && (4.11b)\\
a(0,t) =& rb(0,t) && (4.11c)\\
b(1,t) =& U(t) && (4.11d)
\end{aligned}
$$

and parameter filters

$$
\begin{aligned}
m_t(x,t) + \lambda m_x(x,t) =& c_1(x,t)n(x,t) - P_1(x)m(1,t) && (4.12a)\\
n_t(x,t) - \mu n_x(x,t) =& c_2(x)m(x,t) - P_2(x)m(1,t) && (4.12b)\\
m(0) =& rn(0,t) + 1 && (4.12c)\\
n(1) =& 0 && (4.12d)
\end{aligned}
$$

and

$$
\begin{aligned}
w_t(x,t) + \lambda w_x(x,t) =& c_1(x)z(x,t) - P_1(x)w(1,t) && (4.13a)\\
z_t(x,t) - \mu z_x(x,t) =& c_2(x)w(x,t) - P_2(x)w(1,t) && (4.13b)\\
w(0,t) =& rz(0,t) - y_0(t) && (4.13c)\\
z(1,t) =& 0 && (4.13d)
\end{aligned}
$$

where P_1, P_2 are gains to be designed. The input filters model how the control signal $U(t)$ affect the system states u, v, while the parameter filters model the effect of the boundary parameters k and θ on the system states.

4.2.2 Relationship to the System States

The *non-adaptive* state estimates are defined as

$$
\begin{aligned}
\bar{u}(x,t) =& a(x,t) + k\left(\theta m(x,t) + w(x,t)\right) && (4.14a)\\
\bar{v}(x,t) =& b(x,t) + k\left(\theta n(x,t) + z(x,t)\right) && (4.14b)
\end{aligned}
$$

where the last term has the same bilinear form as boundary condition (4.1c). The non-adaptive state estimates are related to the system states through

$$
\begin{aligned}
u(x,t) =& \bar{u}(x,t) + e(x,t) && (4.15a)\\
v(x,t) =& \bar{v}(x,t) + \epsilon(x,t) && (4.15b)\\
& && (4.15c)
\end{aligned}
$$

where e, ϵ represent the *non-adaptive* estimation error.

Lemma 4.1. *The error terms e and ϵ in* (4.15) *have the dynamics*

$$
\begin{aligned}
e_t(x,t) + \lambda e_x(x,t) =& c_1(x)\epsilon(x,t) - P_1(x)e(1,t) && \text{(4.16a)}\\
\epsilon_t(x,t) - \mu\epsilon_x(x,t) =& c_2(x)e(x,t) - P_2(x)e(1,t) && \text{(4.16b)}\\
e(0,t) =& r\epsilon(0,t) && \text{(4.16c)}\\
\epsilon(1,t) =& 0. && \text{(4.16d)}
\end{aligned}
$$

Proof. Inserting the static estimates (4.14) into (4.15), rearranging, differentiating with respect to time and space and inserting the system dynamics (4.1a) and (4.1b) and filter dynamics (4.11a), (4.11b), (4.12a), (4.12b), (4.13a) and (4.13b), yield

$$
\begin{aligned}
e_t(x,t) + \lambda e_x(x,t) =& u_t(x,t) + \lambda u_x(x,t)\\
& - a_t(x,t) - \lambda a_x(x,t)\\
& - k\left(\theta\left(m_t(x,t) + \lambda m_x(x,t)\right) + w_t(x,t) + \lambda w_x(x,t)\right)\\
=& c_1(x)v(x,t)\\
& - c_1(x)b(x,t) - P_1(x)(y_1(t) - a(1,t))\\
& - k\left(\theta\left(c_1(x,t)n(x,t) - P_1(x)m(1,t)\right)\right.\\
& \qquad \left. + c_1(x)z(x,t) - P_1(x)w(1,t)\right)\\
=& c_1(x)\epsilon(x,t) - P_1 e(1,t)
\end{aligned} \tag{4.17}
$$

and

$$
\begin{aligned}
\epsilon_t(x,t) - \mu\epsilon_x(x,t) =& v_t(x,t) - \mu v_x(x,t)\\
& - b_t(x,t) + \mu b_x(x,t)\\
& - k\left(\theta\left(n_t(x,t) - \mu n_x(x,t)\right) + z_t(x,t) - \mu z_x(x,t)\right)\\
=& c_2(x)u(x,t)\\
& - c_2(x)a(x,t) - P_2(x)(y_1(t) - a(1,t))\\
& - k\left(\theta\left(c_2(x,t)m(x,t) - P_2(x)m(1,t)\right)\right.\\
& \qquad \left. + c_2(x)w(x,t) - P_2(x)w(1,t)\right)\\
=& c_2(x)e(x,t) - P_2 e(1,t).
\end{aligned} \tag{4.18}
$$

Boundary condition (4.16c) follows from evaluating (4.16c) at $x = 0$ and inserting (4.11c), (4.12c), (4.13c) and (4.16c):

$$
\begin{aligned}
e(0,t) =& u(0,t) - a(0,t) - k\left(\theta m(0,t) + w(0,t)\right)\\
=& rv(0,t) + k(\theta - y_0(t)) - rb(0,t) - k\left(\theta(rn(0,t) + 1) + rz(0,t) - y_0(t)\right)\\
=& r\left(v(0,t) - b(0,t) - k\left(\theta n(0,t) + z(0,t)\right)\right)\\
=& r\epsilon(0,t).
\end{aligned} \tag{4.19}
$$

Similarly, boundary condition (4.16d) follows from evaluating (4.16d) at $x = 1$ and inserting (4.11d), (4.12d), (4.13d) and (4.16d):

$$
\begin{aligned}
\epsilon(1,t) =& v(1,t) - b(1,t) - k\left(\theta n(1,t) + z(1,t)\right)\\
=& U(t) - U(t)\\
=& 0.
\end{aligned} \tag{4.20}
$$

□

If the error terms e, ϵ in (4.15) go to zero in finite time, then (4.14) is a static representation of the system states. Stability of the error system is addressed in the next section by first transforming (4.15) into an equivalent target system.

4.2.3 Error Dynamics Analysis

To facilitate the analysis, consider the operators

$$\mathcal{P}_1, \mathcal{P}_2 : L^2([0,1]) \times L^2([0,1]) \to L^2([0,1]) \tag{4.21}$$

given as

$$\mathcal{P}_1[a,b](x) = a(x) + \int_x^1 P^{uu}(x,\xi)a(\xi)d\xi + \int_x^1 P^{uv}(x,\xi)b(\xi)d\xi \tag{4.22a}$$

$$\mathcal{P}_2[a,b](x) = b(x) + \int_x^1 P^{vu}(x,\xi)a(\xi)d\xi + \int_x^1 P^{vv}(x,\xi)b(\xi)d\xi \tag{4.22b}$$

where $a(x), b(x)$ are two signals defined for $x \in [0,1]$ and $(P^{uu}, P^{uv}, P^{vu}, P^{vv})$ is the solution to

$$\begin{aligned}
\lambda P^{uu}_x(x,\xi) + \lambda P^{uu}_\xi(x,\xi) =& c_1(x)P^{vu}(x,\xi) && (4.23a)\\
\lambda P^{uv}_x(x,\xi) - \mu P^{uv}_\xi(x,\xi) =& c_1(x)P^{vv}(x,\xi) && (4.23b)\\
\mu P^{vu}_x(x,\xi) - \lambda P^{vu}_\xi(x,\xi) =& -c_2(x)P^{uu}(x,\xi) && (4.23c)\\
\mu P^{vv}_x(x,\xi) + \mu P^{vv}_\xi(x,\xi) =& -c_2(x)P^{uv}(x,\xi) && (4.23d)\\
P^{uv}(x,x)\lambda + P^{uv}(x,x)\mu =& -c_1(x) && (4.23e)\\
P^{vu}(x,x)\lambda + P^{vu}(x,x)\mu =& c_2(x) && (4.23f)\\
P^{uu}(0,\xi) =& rP^{vu}(0,\xi) && (4.23g)\\
P^{uv}(0,\xi) =& rP^{vv}(0,\xi). && (4.23h)
\end{aligned}$$

It is shown i Vazquez et al. (2011) that system (4.23) has a bounded, continuous and unique solution. Furthermore, it is shown that the mapping $(a,b) \to (\bar{a}, \bar{b})$ given by

$$\begin{aligned}
a(x) =& \mathcal{P}_1[\bar{a}, \bar{b}](x) && (4.24a)\\
b(x) =& \mathcal{P}_2[\bar{a}, \bar{b}](x) && (4.24b)
\end{aligned}$$

is invertible.

Using the operators (4.22), the non-adaptive error system can be transformed into an equivalent target system for which the stability analysis is easier. The backstepping transformation and corresponding target system used in the next lemma were first used in Vazquez et al. (2011).

Lemma 4.2. *Let*

$$d_\alpha = \frac{1}{\lambda} \tag{4.25a}$$

$$d_\beta = \frac{1}{\mu} \tag{4.25b}$$

and consider the non-adaptive error system (4.16)*. If the injection terms are selected as*

$$P_1(x) = \lambda P^{uu}(x,1) \tag{4.26a}$$

$$P_2(x) = \lambda P^{vu}(x,1), \tag{4.26b}$$

then the error terms e, ϵ will tend to zero in a finite time given by

$$t_F = d_\alpha + d_\beta \tag{4.27}$$

and (4.14) *is a static representation of the system states u, v.*

Proof. Consider the transformation

$$e(x,t) = \mathcal{P}_1[\alpha,\beta](x,t) \tag{4.28a}$$

$$\epsilon(x,t) = \mathcal{P}_2[\alpha,\beta](x,t) \tag{4.28b}$$

where $\mathcal{P}_1, \mathcal{P}_2$ are defined in (4.22). It is shown in Vazquez et al. (2011) that the transformation maps the non-adaptive error system (4.16) into the target system

$$\alpha_t(x,t) + \lambda\hat{\alpha}_x(x,t) = 0 \tag{4.29a}$$

$$\beta_t(x,t) - \mu\hat{\beta}_x(x,t) = 0 \tag{4.29b}$$

$$\alpha(0,t) = r\beta(0,t) \tag{4.29c}$$

$$\beta(1,t) = 0. \tag{4.29d}$$

The subsystem consisting of (4.29b) and (4.29d) is a simple transport equation and will be zero $\beta \equiv 0$ for all $t > d_\beta$, reducing the boundary condition (4.29c) to $\alpha(0,t) = 0$ and we have $\alpha \equiv 0$ for another $t \geq d_\alpha$. From the invertibility of transformation (4.28), $e, \epsilon \equiv 0$ for all $t \geq d_\alpha + d_\beta$ follows and the relation (4.15) is reduced to

$$u(x,t) = \bar{u}(x,t) \tag{4.30a}$$

$$v(x,t) = \bar{v}(x,t) \tag{4.30b}$$

$$\tag{4.30c}$$

for all $t \geq d_\alpha + d_\beta$. □

4.2.4 Adaptive Law

Before presenting the adaptive law and the main result of this section, an equivalent set of filter systems will be derived using a backstepping transformation. This equivalent set will be used to prove properties of the adaptive law.

Lemma 4.3. *If P_1, P_2 are selected according to* (4.26)*, the transformation*

$$m(x,t) = \mathcal{P}_1[\check{m}, \check{n}](x,t) \tag{4.31a}$$
$$n(x,t) = \mathcal{P}_2[\check{m}, \check{n}](x,t) \tag{4.31b}$$

map the filters (4.12) *into the target system*

$$\check{m}_t(x,t) + \lambda \check{m}_x(x,t) = 0 \tag{4.32a}$$
$$\check{n}_t(x,t) - \mu \check{n}_x(x,t) = 0 \tag{4.32b}$$
$$\check{m}(0,t) = r\check{n}(0,t) + 1 \tag{4.32c}$$
$$\check{n}(1,t) = 0, \tag{4.32d}$$

and the transformation

$$w(x,t) = \mathcal{P}_1[\check{w}, \check{z}](x,t) \tag{4.33a}$$
$$z(x,t) = \mathcal{P}_2[\check{w}, \check{z}](x,t) \tag{4.33b}$$

map the filters (4.13) *into the target system*

$$\check{w}_t(x,t) + \lambda \check{w}_x(x,t) = 0 \tag{4.34a}$$
$$\check{z}_t(x,t) - \mu \check{z}_x(x,t) = 0 \tag{4.34b}$$
$$\check{w}(0,t) = r\check{z}(0,t) - y_0(t) \tag{4.34c}$$
$$\check{z}(1,t) = 0 \tag{4.34d}$$

with $\mathcal{P}_1, \mathcal{P}_2$ defined in (4.22)*.*

Proof. Equations (4.32a) and (4.32b) follow from differentiating (4.31) and inserting (4.12a) and (4.12b). Similarly, (4.34a) and (4.34b) follow from differentiating (4.33) and inserting (4.13a) and (4.13b). Boundary condition (4.32c) and (4.34c) can be seen from

$$\begin{aligned}
\check{m}(0,t) =& m(0,t) - \int_0^1 P^{uu}(0,\xi)\check{m}(\xi,t)d\xi - \int_0^1 P^{uv}(0,\xi)\check{n}(\xi,t)d\xi \\
=& r\check{n}(0,t) + 1 - \int_0^1 \left(P^{uu}(0,\xi) - rP^{vu}(0,\xi)\right)\check{m}(\xi,t)d\xi \\
&- \int_0^1 \left(P^{uv}(0,\xi) - rP^{vv}(0,\xi)\right)\check{n}(\xi,t)d\xi \\
=& r\check{n}(0,t) + 1
\end{aligned} \tag{4.35}$$

and

$$\begin{aligned}
\check{w}(0,t) =& w(0,t) - \int_0^1 P^{uu}(0,\xi)\check{w}(\xi,t)d\xi - \int_0^1 P^{uv}(0,\xi)\check{z}(\xi,t)d\xi \\
=& r\check{z}(0,t) - y_0(t) - \int_0^1 \left(P^{uu}(0,\xi) - rP^{vu}(0,\xi)\right)\check{w}(\xi,t)d\xi \\
&- \int_0^1 \left(P^{uv}(0,\xi) - rP^{vv}(0,\xi)\right)\check{z}(\xi,t)d\xi
\end{aligned}$$

$$=r\check{z}(0,t) - y_0(t) \tag{4.36}$$

Boundary conditions (4.32d) and (4.34d) follow trivially from evaluating (4.31) and (4.33) at $x = 1$ and inserting (4.12d) and (4.33). □

Using that $e(1,t) = 0$ for all $t > t_F$ from Lemma 4.1 and inserting (4.4), the static relationship (4.15) evaluated at $x = 1$ can be written on the bilinear form

$$y_1(t) - a(1,t) = k(\theta m(1,t) + w(1,t)). \tag{4.37}$$

Motivated by this bilinear form of the static relationship, the following *adaptive* state estimates are generated:

$$\hat{u}(x,t) =a(x,t) + \hat{k}(t)\left(\hat{\theta}(t)m(x,t) + w(x,t)\right) \tag{4.38a}$$

$$\hat{v}(x,t) =b(x,t) + \hat{k}(t)\left(\hat{\theta}(t)n(x,t) + z(x,t)\right). \tag{4.38b}$$

The adaptive state estimates are related to the system states through

$$u(x,t) =\hat{u}(x,t) + \hat{e}(x,t) \tag{4.39a}$$

$$v(x,t) =\hat{v}(x,t) + \hat{\epsilon}(x,t) \tag{4.39b}$$

where $\hat{e}, \hat{\epsilon}$ represent the *adaptive* estimation error.

Evaluating (4.39a) at $x = 1$, inserting (4.4) and rearranging then give

$$\hat{e}(1,t) = y_1(t) - a(1,t) - \hat{k}(t)\left(\hat{\theta}(t)m(x,t) + w(x,t)\right) \tag{4.40}$$

Assuming the sign of k is known, the *gradient method for bilinear parametric models* in Ioannou and Sun (2012, Theorem 4.52) can be used to minimize a cost function based on the square error $\hat{e}^2(1,t)$ and thereby forming an adaptive law for the parameter estimates $\hat{\theta}, \hat{k}$ (see Appendix B.1.3). The following theorem presents the main results from Ioannou and Sun (2012, Theorem 4.52) together with some additional properties needed to prove stability of the closed loop system.

Theorem 4.4. *Consider the adaptive law*

$$\dot{\hat{\theta}}(t) = \begin{cases} \gamma_1 \mathrm{sign}(k) \dfrac{\hat{e}(1,t)}{1 + w^2(1,t)} m(1,t) & t \geq t_F \\ 0 & \textit{otherwise} \end{cases} \tag{4.41a}$$

$$\dot{\hat{k}}(t) = \begin{cases} \gamma_2 \left[\hat{\theta}(t)m(1,t) + w(1,t)\right] \dfrac{\hat{e}(1,t)}{1 + w^2(1,t)} & t \geq t_F \\ 0 & \textit{otherwise} \end{cases} \tag{4.41b}$$

for some adaptation gain $\gamma_1, \gamma_2 > 0$ where $m(1,t)$ and $w(1,t)$ are the filters given in (4.12) *and* (4.13), *$\hat{e}(1,t)$ is the adaptive estimation error in* (4.40) *and t_F is defined in* (4.27). *Suppose system* (4.1) *has a unique solution $u, v \in L_2([0,1])$ $\forall t \geq 0$ and* sign(k) *is known, then the adaptive law* (4.41) *has the following properties:*

I)

$$\hat{\theta},\ \hat{k},\ \in \mathcal{L}_\infty \tag{4.42}$$

II)

$$\dot{\hat{\theta}},\ \dot{\hat{k}},\ \in \mathcal{L}_\infty \cap \mathcal{L}_2 \tag{4.43}$$

III)

$$\hat{\theta}(t) \to \hat{\theta}(t - d_\beta) \tag{4.44a}$$

$$\hat{k}(t) \to \hat{k}(t - d_\beta) \tag{4.44b}$$

IV)

$$\frac{\tilde{k}\,(\theta - y_0) + \hat{k}\tilde{\theta}}{\sqrt{1 + w^2(1,\cdot)}} \in \mathcal{L}_2 \tag{4.45}$$

where $\tilde{\theta} = \theta - \hat{\theta}$ *and* $\tilde{k} = k - \hat{k}$.

V) If

$$\hat{\theta}m(1,\cdot) + w(1,\cdot) \in \mathcal{L}_2 \tag{4.46}$$

then $\hat{\theta}$ *converges to* θ *and* $\hat{k}$ *converges to some constant.*

Proof. Consider the Lyapunov function candidate

$$V_0 = |k|\frac{1}{2\gamma_1}\tilde{\theta}^2 + \frac{1}{2\gamma_2}\tilde{k}^2 \tag{4.47}$$

where $\tilde{\theta} = \theta - \hat{\theta}$ and $\tilde{k} = k - \hat{k}$. Differentiating and inserting the adaptive laws (4.41) for $t > t_F$ give

$$\begin{aligned}
\dot{V}_0 =& |k|\frac{1}{\gamma_1}\tilde{\theta}\dot{\hat{\theta}} + \frac{1}{\gamma_2}\tilde{k}\dot{\hat{k}} \\
=& |k|\tilde{\theta}\mathrm{sign}(k)\frac{\hat{e}(1,t)}{1 + w^2(1,t)}m(1,t) + \tilde{k}\left[\hat{\theta}(t)m(1,t) + w(1,t)\right]\frac{\hat{e}(1,t)}{1 + w^2(1,t)} \\
=& \frac{\hat{e}(1,t)}{1 + w^2(t)}\left(k\tilde{\theta}m(1,t) + \tilde{k}\left[\hat{\theta}(t)m(1,t) + w(1,t)\right]\right) \\
=& \frac{\hat{e}(1,t)}{1 + w^2(t)}\left(-k\left[\theta(t)m(1,t) + w(1,t)\right] + \hat{k}\left[\hat{\theta}(t)m(1,t) + w(1,t)\right]\right) \\
=& \frac{\hat{e}(1,t)}{1 + w^2(t)}\left(-u(1,t) + \hat{u}(1,t)\right) \\
=& -\frac{\hat{e}^2(1,t)}{1 + w^2(t)}
\end{aligned}$$

$$\leq 0 \tag{4.48}$$

which shows that

$$V_0, \tilde{\theta}, \tilde{k} \in \mathcal{L}_\infty \tag{4.49}$$

and Property I follows.

The transformed filter system $(\check{m}, \check{n})$ in (4.32) is a simple cascaded transport equation and we have $\check{m} \equiv 1$ and $\check{n} \equiv 0$ for all $x \in [0, 1]$ and $t > t_F$. From the invertibility of transformation (4.31), we have $m(x, \cdot), n(x, \cdot) \in \mathcal{L}_\infty$, which together with Property I give

$$\frac{\hat{e}(1, \cdot)}{\sqrt{1 + w^2(1, \cdot)}} \in \mathcal{L}_\infty \tag{4.50}$$

and

$$\frac{m(1, \cdot)}{\sqrt{1 + w^2(1, \cdot)}} \in \mathcal{L}_\infty \tag{4.51}$$

Integrating (4.48) from $t = 0$ to $t = \infty$ and using that $V_0 \geq 0$ is a non-increasing function of time give

$$\int_0^\infty \left(\frac{\hat{e}^2(1, \tau)}{1 + w^2(\tau)}\right)^2 d\tau = -\int_0^\infty \dot{V}_0(\tau) d\tau = V_0(0) - V_0(\infty) \leq \infty, \tag{4.52}$$

and therefore

$$\frac{\hat{e}(1, \cdot)}{\sqrt{1 + w^2(\cdot)}} \in \mathcal{L}_2. \tag{4.53}$$

From (4.41a), one has

$$\left|\dot{\hat{\theta}}(t)\right| \leq \gamma_1 \left|\frac{\hat{e}(1, t)}{\sqrt{1 + w^2(t)}}\right| \left|\frac{m(1, t)}{\sqrt{1 + w^2(1, t)}}\right| \tag{4.54}$$

which together with (4.50), (4.51) and (4.53) give $\dot{\hat{\theta}} \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and the first part of Property II. For the second part, one has similarly

$$\left|\dot{\hat{k}}(t)\right| \leq \gamma_2 \left|\frac{\hat{e}(1, t)}{\sqrt{1 + w^2(t)}}\right| \left|\frac{\hat{\theta} m(1, t) + w(1, t)}{\sqrt{1 + w^2(1, t)}}\right| \tag{4.55}$$

which together with (4.50), (4.51) and (4.53) give $\dot{\hat{k}} \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and the second part of Property II.

The proof of Property III is similar to the proof of Lemma 3.9 and therefore omitted.

Let

$$\Theta(t) = \left[\tilde{k}(t), \quad \sqrt{|k|}\tilde{\theta}(t)\right]^T \tag{4.56a}$$

$$\Psi(t) = \frac{1}{\sqrt{1 + w^2(1, t)}} \left[\hat{\theta} m(1, t) + w(1, t), \quad \operatorname{sign}(k)\sqrt{|k|} m(1, t)\right]^T \tag{4.56b}$$

$$\Gamma = \operatorname{diag}([\gamma_1, \gamma_2]) \tag{4.56c}$$

We then have

$$V_0 = \Theta^T(t)\Gamma^{-1}\Theta(t) \tag{4.57}$$

and

$$\dot{V}_0 = -\hat{e}^2(1,t) = \left(\Theta^T(t)\Psi(t)\right)^2 = \Theta^T(t)\Psi(t)\Psi^T(t)\Theta(t). \tag{4.58}$$

From Property I, a lower bound for $\dot{V}_0$ can be found as

$$\begin{aligned}\dot{V}_0 =& -\hat{e}^2(1,t) = \left(\Theta^T(t)\Psi(t)\right)^2 = \Theta^T(t)\Psi(t)\Psi^T(t)\Theta(t)\\ \geq& -h\Theta^T(t)\Theta(t)\\ \geq& -2h\gamma_{max}\frac{1}{2}\Theta^T(t)\Gamma^{-1}\Theta(t)\\ \geq& -2h\gamma_{max}V_0\end{aligned} \tag{4.59}$$

where $h > 0$ is a constant and γ_{max} the largest eigenvalue of Γ. A lower bound for V_0 can now be found by using the method of *separation of variables* (see Appendix B.2) as

$$\begin{aligned}\dot{V}_0 \geq& -2h\gamma_{max}V_0\\ \frac{\dot{V}_0}{V_0} \geq& -2h\gamma_{max}\\ \int_{V_0(t-d_\alpha)}^{V_0(t)}\frac{dV_0}{V_0} \geq& -\int_{t-d_\alpha}^{t}2h\gamma_{max}d\tau\\ \ln\left(\frac{V_0(t)}{V_0(t-d_\alpha)}\right) \geq& -2d_\alpha h\gamma_{max}\end{aligned} \tag{4.60}$$

and solving for $V_0(t)$ to yield

$$V_0(t) \geq e^{-2d_\alpha h\gamma_{max}}V_0(t-d_\alpha), \tag{4.61}$$

which shows that the decay rate of V_0 is at maximum exponential. Again, using the definition of V_0 in (4.57) one obtains

$$\begin{aligned}\tilde{\Theta}^T(t)\tilde{\Theta}(t) \geq& 2\gamma_{min}\frac{1}{2}\tilde{\theta}^T(t)\Gamma^{-1}\tilde{\theta}(t)\\ =& 2\gamma_{min}V_0(t)\\ \geq& 2\gamma_{min}e^{-2d_\alpha h\gamma_{max}}V_0(t-d_\alpha)\\ =& 2\gamma_{min}e^{-2d_\alpha h\gamma_{max}}\frac{1}{2}\tilde{\Theta}^T(t-d_\alpha)\Gamma^{-1}\tilde{\Theta}(t-d_\alpha)\\ \geq& \frac{\gamma_{min}}{\gamma_{max}}e^{-2d_\alpha h\gamma_{max}}\tilde{\Theta}^T(t-d_\alpha)\tilde{\Theta}(t-d_\alpha).\end{aligned} \tag{4.62}$$

The relation (4.62) can now be substituted into (4.58), yielding

$$\dot{V}_0 = -\left(\Theta^T(t)\Psi(t)\right)^2$$

$$\leq -\frac{\gamma_{min}}{\gamma_{max}} e^{-2d_\alpha h \gamma_{max}} \left(\tilde{\Theta}^T(t-d_\alpha)\Psi(t)\right)^2 \tag{4.63}$$

Integrating (4.63) from $t = 0$ to $t = \infty$ and using that $V_0 \geq 0$ is a non-increasing function of time give

$$\begin{aligned}\int_0^\infty \frac{\gamma_{min}}{\gamma_{max}} e^{-2d_\alpha h\gamma_{max}} \left(\tilde{\Theta}^T(\tau-d_\alpha)\Psi(\tau)\right)^2 d\tau \leq & -\int_0^\infty \dot{V}_0(\tau)d\tau \\ =& V_0(0) - V_0(\infty) \\ <& \infty, \end{aligned} \tag{4.64}$$

which shows that

$$\int_0^\infty \left(\tilde{\Theta}^T(\tau-d_\alpha)\Psi(\tau)\right)^2 d\tau < \infty. \tag{4.65}$$

Since the target system (4.32) is a set of simple cascaded transport equations and from transformation (4.31), it follows that $m(1,t) = \check{m}(1,t) = 1$ for all $t > t_F$ and similarly that $w(1,t) = \check{w}(1,t) = y_0(t-d_\alpha)$ for all $t > t_F$. Inserting (4.56a) and (4.56b) into (4.65) and rearranging the terms give Property IV.

Inserting (4.40) into (4.41a) yields

$$\dot{\tilde{\theta}}(t) = -\frac{\gamma_1 \mathrm{sign}(k)}{1+w^2(1,t)} \left(k\tilde{\theta}m(1,t) + \tilde{k}(t)\left(\hat{\theta}m(1,t) + w(1,t)\right)\right) m(1,t) \tag{4.66}$$

where the last term can be treated as an external input. Using that $m(1,t) \equiv 1$ for all $t > t_F$ and if the last term $\tilde{k}(t)\left(\hat{\theta}m(1,t) + w(1,t)\right)$ is square integrable, then (4.66) form an *exponentially stable* system and it follows that $\tilde{\theta} \to 0$ as $t \to \infty$ or equivalently the first part of Property V.

If $\hat{\theta}(\tau)m(1,\tau) + w(1,\tau) \in \mathcal{L}_2$, and by using Cauchy-Schwarz' inequality, we obtain the inequality

$$\begin{aligned}\int_0^\infty \left|\dot{\hat{k}}(\tau)\right| d\tau \leq& \gamma_2 \int_0^\infty \left| \left[\hat{\theta}(\tau)m(1,\tau) + w(1,\tau)\right] \frac{\hat{e}(1,\tau)}{1+w^2(1,\tau)}\right| d\tau \\ \leq& \gamma_2 \sqrt{\int_0^\infty \left|\hat{\theta}(\tau)m(1,\tau)+w(1,\tau)\right|^2 d\tau} \sqrt{\int_0^\infty \left|\frac{\hat{e}(1,\tau)}{1+w^2(1,\tau)}\right|^2 d\tau} \\ <& \infty \end{aligned} \tag{4.67}$$

which implies that $\dot{\hat{k}} \in \mathcal{L}_1$ and the second part of Property V follows. □

4.3 Closed Loop Adaptive Control

The main result from this section will be a control law $U(t)$ that, together with Theorem 4.4, adaptively stabilizes (4.1) in the L_2-sense and achieves (4.7). This section will start off by deriving the estimator dynamics and introduce a backstepping operator, before the main theorem is presented. The stability proof is deferred to Section 4.4.

4.3.1 Estimator Dynamics

Lemma 4.5. *The state estimates $\hat{u}, \hat{v}$ generated from* (4.38) *have the dynamics*

$$\begin{aligned}\hat{u}_t(x,t) + \lambda \hat{u}_x(x,t) =& c_1(x)\hat{v}(x,t) + P_1(x)\hat{e}(1,t) \\ &+ \dot{\hat{k}}(t)\left(\hat{\theta}(t)m(x,t) + w(x,t)\right) + \hat{k}(t)\dot{\hat{\theta}}(t)m(x,t) \end{aligned} \tag{4.68a}$$

$$\begin{aligned}\hat{v}_t(x,t) - \mu \hat{v}_x(x,t) =& c_2(x)\hat{u}(x,t) + P_2(x)\hat{e}(1,t) \\ &+ \dot{\hat{k}}(t)\left(\hat{\theta}(t)n(x,t) + z(x,t)\right) + \hat{k}(t)\dot{\hat{\theta}}(t)n(x,t) \end{aligned} \tag{4.68b}$$

$$\hat{u}(0,t) = r\hat{v}(0,t) + \hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right) \tag{4.68c}$$

$$\hat{v}(1,t) = U(t) \tag{4.68d}$$

Alternatively, boundary condition (4.68c) *can be written on the form*

$$\hat{u}(0,t) = q(t)\hat{v}(0,t) + d(t) + \kappa(t)\varepsilon(t) \tag{4.69}$$

where

$$q(t) = \frac{r - b_0\hat{k}(t)}{1 + a_0\hat{k}(t)} \tag{4.70a}$$

$$d(t) = \frac{\hat{k}(t)\hat{\theta}(t)}{1 + a_0\hat{k}(t)} \tag{4.70b}$$

$$\kappa(t) = -\frac{\hat{k}(t)}{1 + a_0\hat{k}(t)} \tag{4.70c}$$

$$\varepsilon(t) = a_0\hat{e}(0,t) + b_0\hat{\epsilon}(0,t). \tag{4.70d}$$

Proof. Differentiating (4.38a) with respect to time and space give

$$\begin{aligned}\hat{u}_t(x,t) =& a_t(x,t) + \dot{\hat{k}}(t)\left(\hat{\theta}(t)m(x,t) + w(x,t)\right) \\ &+ \hat{k}(t)\left(\dot{\hat{\theta}}(t)m(x,t) + \hat{\theta}(t)m_t(x,t) + w_t(x,t)\right), \end{aligned} \tag{4.71}$$

$$\hat{u}_x(x,t) = a_x(x,t) + \hat{k}(t)\left(\hat{\theta}(t)m_x(x,t) + w_x(x,t)\right). \tag{4.72}$$

Combining (4.71) and (4.72) and inserting the filter dynamics (4.11a), (4.12a) and (4.13a) yield

$$\begin{aligned}\hat{u}_t(x,t) + \lambda\hat{u}_x(x,t) =& a_t(x,t) + \dot{\hat{k}}(t)\left(\hat{\theta}(t)m(x,t) + w(x,t)\right) \\ &+ \hat{k}(t)\left(\dot{\hat{\theta}}(t)m(x,t) + \hat{\theta}(t)m_t(x,t) + w_t(x,t)\right) \\ &+ \lambda\left(a_x(x,t) + \hat{k}(t)\left(\hat{\theta}(t)m_x(x,t) + w_x(x,t)\right)\right) \\ =& a_t(x,t) + \lambda a_x(x,t)\end{aligned}$$

$$
\begin{aligned}
&+ \hat{k}(t)\left(\hat{\theta}(t)\left(m_t(x,t) + \lambda m_x(x,t)\right) + w_t(x,t) + \lambda w_x(x,t)\right) \\
&+ \dot{\hat{k}}(t)\left(\hat{\theta}(t)m(x,t) + w(x,t)\right) + \hat{k}(t)\dot{\hat{\theta}}(t)m(x,t) \\
=& c_1(x)b(x,t) + P_1(x)(y_1(t) - a(1,t)) \\
&+ \hat{k}(t)\Big(\hat{\theta}(t)\left(c_1(x,t)n(x,t) - P_1(x)m(1,t)\right) \\
&\qquad + c_1(x)z(x,t) - P_1(x)w(1,t)\Big) \\
&+ \dot{\hat{k}}(t)\left(\hat{\theta}(t)m(x,t) + w(x,t)\right) + \hat{k}(t)\dot{\hat{\theta}}(t)m(x,t) \\
=& c_1(x)\hat{v}(x,t) + P_1(x)\hat{e}(1,t) \\
&+ \dot{\hat{k}}(t)\left(\hat{\theta}(t)m(x,t) + w(x,t)\right) + \hat{k}(t)\dot{\hat{\theta}}(t)m(x,t).
\end{aligned} \tag{4.73}
$$

Similarly, differentiating (4.38b) with respect to time and space give

$$
\begin{aligned}
\hat{v}_t(x,t) =& b_t(x,t) + \dot{\hat{k}}(t)\left(\hat{\theta}(t)n(x,t) + z(x,t)\right) \\
&+ \hat{k}(t)\left(\dot{\hat{\theta}}(t)n(x,t) + \hat{\theta}(t)n_t(x,t) + z_t(x,t)\right),
\end{aligned} \tag{4.74}
$$

$$
\hat{v}_x(x,t) = b_x(x,t) + \hat{k}(t)\left(\hat{\theta}(t)n_x(x,t) + z_x(x,t)\right) \tag{4.75}
$$

and combining (4.74) and (4.75), and inserting the filter dynamics (4.11b), (4.12b) and (4.13b) yield

$$
\begin{aligned}
\hat{v}_t(x,t) - \mu\hat{v}_x(x,t) =& b_t(x,t) + \dot{\hat{k}}(t)\left(\hat{\theta}(t)n(x,t) + z(x,t)\right) \\
&+ \hat{k}(t)\left(\dot{\hat{\theta}}(t)n(x,t) + \hat{\theta}(t)n_t(x,t) + z_t(x,t)\right) \\
&- \mu\left(b_x(x,t) + \hat{k}(t)\left(\hat{\theta}(t)n_x(x,t) + z_x(x,t)\right)\right) \\
=& b_t(x,t) - \mu b_x(x,t) \\
&+ \hat{k}(t)\left(\hat{\theta}(t)\left(n_t(x,t) - \mu n_x(x,t)\right) + z_t(x,t) - \mu z_x(x,t)\right) \\
&+ \dot{\hat{k}}(t)\left(\hat{\theta}(t)n(x,t) + z(x,t)\right) + \hat{k}(t)\dot{\hat{\theta}}(t)n(x,t) \\
=& c_2(x)b(x,t) + P_2(x)(y_1(t) - a(1,t)) \\
&+ \hat{k}(t)\Big(\hat{\theta}(t)\left(c_2(x,t)n(x,t) - P_2(x)m(1,t)\right) \\
&\qquad + c_2(x)z(x,t) - P_2(x)w(1,t)\Big) \\
&+ \dot{\hat{k}}(t)\left(\hat{\theta}(t)n(x,t) + z(x,t)\right) + \hat{k}(t)\dot{\hat{\theta}}(t)n(x,t) \\
=& c_2(x)\hat{u}(x,t) + P_2(x)\hat{e}(1,t) \\
&+ \dot{\hat{k}}(t)\left(\hat{\theta}(t)n(x,t) + z(x,t)\right) + \hat{k}(t)\dot{\hat{\theta}}(t)n(x,t).
\end{aligned} \tag{4.76}
$$

Boundary condition (4.68c) can be seen from

$$
\begin{aligned}
\hat{u}(0,t) =& a(0,t) + \hat{k}(t)\left(\hat{\theta}(t)m(0,t) + w(0,t)\right) \\
=& rb(0,t) + \hat{k}(t)\left(\hat{\theta}(t)\left(rn(0,t)+1\right) + rz(0,t) - y_0(t)\right) \\
=& r\hat{v}(0,t) + \hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right),
\end{aligned} \tag{4.77}
$$

while the alternative boundary condition (4.69) is obtained by rearranging the terms and using relation (4.5) and (4.39). Boundary condition (4.68d) follows trivially from evaluating (4.38b) at $x = 1$:

$$
\begin{aligned}
\hat{v}(1,t) =& b(1,t) + \hat{k}(t)\left(\hat{\theta}(t)n(1,t) + z(1,t)\right) \\
=& U(t).
\end{aligned} \tag{4.78}
$$

□

4.3.2 Backstepping Operators and Main Result

Consider the operators

$$
\mathcal{K}_1, \mathcal{K}_2 \,:\, L_2([0,1]) \times L_2([0,1]) \to L_2([0,1]) \tag{4.79}
$$

given as

$$
\mathcal{K}_1[a,b](x) = a(x) - \int_0^x K^{uu}(x,\xi)a(\xi)d\xi + \int_0^x K^{uv}(x,\xi)b(\xi)d\xi \tag{4.80a}
$$

$$
\mathcal{K}_2[a,b](x) = b(x) - \int_0^x K^{vu}(x,\xi)a(\xi)d\xi + \int_0^x K^{vv}(x,\xi)b(\xi)d\xi \tag{4.80b}
$$

where $a(x)$, $b(x)$ are two signals defined for $x \in [0,1]$ and $(K^{uu}, K^{uv}, K^{vu}, K^{vv})$ is the solution to

$$
K_x^{uu}(x,\xi)\lambda + K_\xi^{uu}(x,\xi)\lambda = -K^{uv}(x,\xi)c_2(x) \tag{4.81a}
$$

$$
K_x^{uv}(x,\xi) - K_\xi^{uv}(x,\xi)\mu\lambda = -K^{uu}(x,\xi)c_1(x) \tag{4.81b}
$$

$$
K_x^{vu}(x,\xi)\mu - K_\xi^{vu}(x,\xi)\lambda = K^{vv}(x,\xi)c_2(x) \tag{4.81c}
$$

$$
K_x^{vv}(x,\xi)\mu + K_\xi^{vv}(x,\xi)\mu = K^{vu}(x,\xi)c_1(x) \tag{4.81d}
$$

$$
K^{uv}(x,x)\lambda + K^{uv}(x,x)\mu = c_1(x) \tag{4.81e}
$$

$$
K^{vu}(x,x)\lambda + K^{vu}(x,x)\mu = -c_2(x) \tag{4.81f}
$$

$$
K^{uu}(x,0)\lambda r = K^{uv}(x,0)\mu \tag{4.81g}
$$

$$
K^{vu}(x,0)\lambda r = K^{vv}(x,0)\mu \tag{4.81h}
$$

defined over

$$
\mathcal{T}_2 = \{(x,\xi)\,|\,0 \le \xi \le x \le 1\}. \tag{4.82}
$$

From Vazquez et al. (2011, Theorem 4), system (4.81) has a unique solution $K^{uu}, K^{uv}, K^{vu}, K^{vv}$. Moreover, the mapping $(a, b) \to (\bar{a}, \bar{b})$ given by

$$\begin{aligned}\bar{a}(x) =& \mathcal{K}_1[a, b](x) \\ \bar{b}(x) =& \mathcal{K}_2[a, b](x)\end{aligned} \tag{4.83}$$

is invertible with unique inverse transformation kernels (see Vazquez et al. (2011) for details).

Theorem 4.6. *Consider the system* (4.1)*, the state estimates* (4.38) *and the adaptive law* (4.41)*. The control law*

$$U(t) = \mathcal{K}_2[\hat{u}, \hat{v}](1) + \frac{1}{a_0 r + b_0}\hat{\theta}(t) \tag{4.84}$$

where $\mathcal{K}_2$ is defined in (4.80b)*, r satisfies* (4.8) *and $\hat{\theta}$ are generated from the adaptive law* (4.41)*, guarantees* (4.7)*. Moreover, all signals in the closed loop system are bounded and the parameter estimate $\hat{\theta}$ converges to its true value in the sense*

$$\int_t^{t+T} |\hat{\theta}(\tau) - \theta| d\tau \to 0 \tag{4.85}$$

for some $T > 0$.

Proof of Theorem 4.6 is deferred to Section 4.4. The rest of this section will present the derivation of the control law (4.84). To improve readability, the control law $U(t)$ is decomposed into two parts

$$U(t) = U_1(t) + U_2(t) \tag{4.86}$$

with each term presented in separate subsections and lemmas.

4.3.3 Decoupling of the Observer Dynamics

Lemma 4.7. *Consider the state estimate dynamics* (4.68) *generated from* (4.38) *and the operators $\mathcal{K}_1, \mathcal{K}_2$ from* (4.80)*. The transformation*

$$\omega(x, t) = \mathcal{K}_1[\hat{u}, \hat{v}](x, t) \tag{4.87a}$$

$$\zeta(x, t) = \mathcal{K}_2[\hat{u}, \hat{v}](x, t), \tag{4.87b}$$

and the control law (4.86) *with*

$$U_1(t) =\mathcal{K}_2[\hat{u}, \hat{v}](1, t) \tag{4.88}$$

map (4.68) *into the target system*

$$
\begin{aligned}
\omega_t(x,t) + \lambda\omega_x(x,t) =& \hat{\theta}(t)H_1(x,t)\dot{\hat{k}}(t) + G_1(x,t)\dot{\hat{k}}(t) + \hat{k}(t)H_1(x,t)\dot{\hat{\theta}}(t) \\
&+ \Omega_1(x)\hat{e}(1,t) + \Psi_1(x)\hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right) && (4.89a) \\
\zeta_t(x,t) - \mu\zeta_x(x,t) =& \hat{\theta}(t)H_2(x,t)\dot{\hat{k}}(t) + G_2(x,t)\dot{\hat{k}}(t) + \hat{k}(t)H_2(x,t)\dot{\hat{\theta}}(t) \\
&+ \Omega_2(x)\hat{e}(1,t) + \Psi_2(x,t)\hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right) && (4.89b) \\
\omega(0,t) =& \zeta(0,t)q(t) + d(t) + \kappa(t)\varepsilon(t) && (4.89c) \\
\zeta(1,t) =& U_2(t). && (4.89d)
\end{aligned}
$$

where

$$
\begin{aligned}
G_1(x,t) =& \mathcal{K}_1[w,z](x,t) && (4.90a) \\
G_2(x,t) =& \mathcal{K}_2[w,z](x,t) && (4.90b)
\end{aligned}
$$

$$
\begin{aligned}
H_1(x,t) =& \mathcal{K}_1[m,n](x,t) && (4.91a) \\
H_2(x,t) =& \mathcal{K}_2[m,n](x,t) && (4.91b)
\end{aligned}
$$

$$
\begin{aligned}
\Omega_1(x) =& \mathcal{K}_1[P_1,P_2](x) && (4.92a) \\
\Omega_2(x) =& \mathcal{K}_2[P_1,P_2](x) && (4.92b)
\end{aligned}
$$

$$
\begin{aligned}
\Psi_1(x) =& -K^{uu}(x,0)\lambda && (4.93a) \\
\Psi_2(x) =& -K^{vu}(x,0)\lambda && (4.93b)
\end{aligned}
$$

and $q, d, \kappa, \varepsilon$ *defined in* (4.70).

Proof. From (4.87a) and definition (4.80a), differentiating with respect to time, inserting the dynamics (4.68) and integration by parts give

$$
\begin{aligned}
\omega_t(x,t) =& \hat{u}_t(x,t) - \int_0^x K^{uu}(x,\xi)\hat{u}_t(\xi,t)d\xi - \int_0^x K^{uv}(x,\xi)\hat{v}_t(\xi,t)d\xi \\
=& \hat{u}_t(x,t) \\
&- \int_0^x K^{uu}(x,\xi)\Big(-\lambda\hat{u}_\xi(\xi,t) + c_1(\xi)\hat{v}(\xi,t) + P_1(\xi)\hat{e}(1,t) \\
&\qquad + \dot{\hat{k}}(t)\left(\hat{\theta}(t)m(\xi,t) + w(\xi,t)\right) + \hat{k}(t)\dot{\hat{\theta}}(t)m(\xi,t)\Big)\, d\xi \\
&- \int_0^x K^{uv}(x,\xi)\Big(+ \mu\hat{v}_x(x,t) + c_2(x)\hat{u}(x,t) + P_2(x)\hat{e}(1,t)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + \dot{\hat{k}}(t)\left(\hat{\theta}(t)n(x,t) + z(x,t)\right) + \hat{k}(t)\dot{\hat{\theta}}(t)n(x,t)\Big) d\xi \\
=&\hat{u}_t(x,t) \\
&+ K^{uu}(x,x)\lambda\hat{u}(x,t) - K^{uu}(x,0)\lambda\hat{u}(0,t) - \int_0^x K^{uu}_\xi(x,\xi)\lambda\hat{u}(\xi,t)d\xi \\
&- \int_0^x K^{uu}(x,\xi)\Big(c_1(\xi)\hat{v}(\xi,t) + P_1(\xi)\hat{e}(1,t) \\
&\qquad\qquad + \dot{\hat{k}}(t)\left(\hat{\theta}(t)m(\xi,t) + w(\xi,t)\right) + \hat{k}(t)\dot{\hat{\theta}}(t)m(\xi,t)\Big) d\xi \\
&- K^{uv}(x,x)\mu\hat{v}(x,t) + K^{uv}(x,0)\mu\hat{v}(0,t) + \int_0^x K^{uv}_\xi(x,\xi)\mu\hat{v}(\xi,t)d\xi \\
&- \int_0^x K^{uv}(x,\xi)\Big(c_2(\xi)\hat{u}(\xi,t) + P_2(\xi)\hat{e}(1,t) \\
&\qquad\qquad + \dot{\hat{k}}(t)\left(\hat{\theta}(t)n(\xi,t) + z(\xi,t)\right) + \hat{k}(t)\dot{\hat{\theta}}(t)n(\xi,t)\Big) d\xi.
\end{aligned}
\tag{4.94}
$$

Differentiating with respect to space and applying Leibniz' differentiation rule (see Appendix A.8) give

$$
\begin{aligned}
\omega_x(x,t) =&\hat{u}_x(x,t) \\
&- K^{uu}(x,x)\hat{u}(x,t) - \int_0^x K^{uu}_x(x,\xi)\hat{u}(\xi,t)d\xi \\
&- K^{uv}(x,x)\hat{v}(x,t) - \int_0^x K^{uv}_x(x,\xi)\hat{v}(\xi,t)d\xi.
\end{aligned}
\tag{4.95}
$$

Substituting (4.94) and (4.95) into (4.68a) and using (4.81) and (4.90a)–(4.93a) one finds

$$
\begin{aligned}
&\omega_t(x,t) + \lambda\omega_x(x,t) \\
=&K^{uu}(x,x)\lambda\hat{u}(x,t) - K^{uu}(x,0)\lambda\hat{u}(0,t) - \int_0^x K^{uu}_\xi(x,\xi)\lambda\hat{u}(\xi,t)d\xi \\
&- \int_0^x K^{uu}(x,\xi)\Big(c_1(\xi)\hat{v}(\xi,t) + P_1(\xi)\hat{e}(1,t) \\
&\qquad\qquad + \dot{\hat{k}}(t)\left(\hat{\theta}(t)m(\xi,t) + w(\xi,t)\right) + \hat{k}(t)\dot{\hat{\theta}}(t)m(\xi,t)\Big) d\xi \\
&- K^{uv}(x,x)\mu\hat{v}(x,t) + K^{uv}(x,0)\mu\hat{v}(0,t) + \int_0^x K^{uv}_\xi(x,\xi)\mu\hat{v}(\xi,t)d\xi \\
&- \int_0^x K^{uv}(x,\xi)\Big(c_2(\xi)\hat{u}(\xi,t) + P_2(\xi)\hat{e}(1,t) \\
&\qquad\qquad + \dot{\hat{k}}(t)\left(\hat{\theta}(t)n(\xi,t) + z(\xi,t)\right) + \hat{k}(t)\dot{\hat{\theta}}(t)n(\xi,t)\Big) d\xi \\
&+ \lambda\Big(-K^{uu}(x,x)\hat{u}(x,t) - \int_0^x K^{uu}_x(x,\xi)\hat{u}(\xi,t)d\xi
\end{aligned}
$$

$$
\begin{aligned}
&\qquad -K^{uv}(x,x)\hat{v}(x,t) - \int_0^x K_x^{uv}(x,\xi)\hat{v}(\xi,t)d\xi\Big) \\
&\quad + c_1(x)\hat{v}(x,t) + P_1(x)\hat{e}(1,t) \\
&\quad + \dot{\hat{k}}(t)\left(\hat{\theta}(t)m(x,t) + w(x,t)\right) + \hat{k}(t)\dot{\hat{\theta}}(t)m(x,t) \\
=& -\int_0^x \underbrace{\left[K_\xi^{uu}(x,\xi)\lambda + K_x^{uu}(x,\xi)\lambda + K^{uv}(x,\xi)c_2(\xi)\right]}_{=0}\hat{u}(\xi,t)d\xi \\
&\quad -\int_0^x \underbrace{\left[-K_\xi^{uv}(x,\xi)\mu + K_x^{uv}(x,\xi)\lambda + K^{uu}(x,\xi)c_1(\xi)\right]}_{=0}\hat{v}(\xi,t)d\xi \\
&\quad - \underbrace{\left[K^{uv}(x,x)\mu + K^{uv}(x,x)\lambda - c_1(x)\right]}_{=0}\hat{v}(x,t) \\
&\quad - \underbrace{\left[K^{uu}(x,0)\lambda r - K^{uv}(x,0)\mu\right]}_{=0}\hat{v}(0,t) \\
&\quad + \underbrace{\left[-K^{uu}(x,0)\lambda\right]}_{\Psi_1(x)}\hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right) \\
&\quad + \underbrace{\left[P_1(x) - \int_0^x K^{uu}(x,\xi)P_1(\xi)d\xi - \int_0^x K^{uv}(x,\xi)P_2(\xi)d\xi\right]}_{=\Omega_1(x)}\hat{e}(1,t) \\
&\quad + \hat{\theta}(t)\underbrace{\left[m(x,t) - \int_0^x K^{uu}(x,\xi)m(\xi,t)d\xi - \int_0^x K^{uv}(x,\xi)n(\xi,t)d\xi\right]}_{H_1(x,t)}\dot{\hat{k}}(t) \\
&\quad + \underbrace{\left[w(x,t) - \int_0^x K^{uu}(x,\xi)w(\xi,t)d\xi - \int_0^x K^{uv}(x,\xi)z(\xi,t)d\xi\right]}_{G_1(x,t)}\dot{\hat{k}}(t) \\
&\quad + \hat{k}(t)\underbrace{\left[m(x,t) - \int_0^x K^{uu}(x,\xi)m(\xi,t)d\xi - \int_0^x K^{uv}(x,\xi)(t)n(\xi,t)d\xi\right]}_{H_1(x,t)}\dot{\hat{\theta}}(t) \\
=&\hat{\theta}(t)H_1(x,t)\dot{\hat{k}}(t) + G_1(x,t)\dot{\hat{k}}(t) + \hat{k}(t)H_1(x,t)\dot{\hat{\theta}}(t) \\
&\quad + \Omega_1(x)\hat{e}(1,t) + \Psi_1(x)\hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right). \qquad (4.96)
\end{aligned}
$$

Similarly, from (4.87b) and definition (4.80b), differentiating with respect to time, inserting the dynamics (4.68) and integration by parts give

$$
\begin{aligned}
\zeta_t(x,t) =&\hat{v}_t(x,t) - \int_0^x K^{vu}(x,\xi)\hat{u}_t(\xi,t)d\xi - \int_0^x K^{vv}(x,\xi)\hat{v}_t(\xi,t)d\xi \\
=&\hat{v}_t(x,t)
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^x K^{vu}(x,\xi) \Big(- \lambda \hat{u}_\xi(\xi,t) + c_1(\xi)\hat{v}(\xi,t) + P_1(\xi)\hat{e}(1,t) \\
&\qquad\qquad + \dot{\hat{k}}(t) \left(\hat{\theta}(t) m(\xi,t) + w(\xi,t) \right) + \hat{k}(t)\dot{\hat{\theta}}(t) m(\xi,t) \Big) d\xi \\
&- \int_0^x K^{vv}(x,\xi) \Big(+ \mu \hat{v}_x(x,t) + c_2(x)\hat{u}(x,t) + P_2(x)\hat{e}(1,t) \\
&\qquad\qquad + \dot{\hat{k}}(t) \left(\hat{\theta}(t) n(x,t) + z(x,t) \right) + \hat{k}(t)\dot{\hat{\theta}}(t) n(x,t) \Big) d\xi \\
=& \hat{v}_t(x,t) \\
&+ K^{vu}(x,x)\lambda\hat{u}(x,t) - K^{vu}(x,0)\lambda\hat{u}(0,t) - \int_0^x K^{vu}_\xi(x,\xi)\lambda\hat{u}(\xi,t) d\xi \\
&- \int_0^x K^{vu}(x,\xi) \Big(c_1(\xi)\hat{v}(\xi,t) + P_1(\xi)\hat{e}(1,t) \\
&\qquad\qquad + \dot{\hat{k}}(t) \left(\hat{\theta}(t) m(\xi,t) + w(\xi,t) \right) + \hat{k}(t)\dot{\hat{\theta}}(t) m(\xi,t) \Big) d\xi \\
&- K^{vv}(x,x)\mu\hat{v}(x,t) + K^{vv}(x,0)\mu\hat{v}(0,t) + \int_0^x K^{vv}_\xi(x,\xi)\mu\hat{v}(\xi,t) d\xi \\
&- \int_0^x K^{vv}(x,\xi) \Big(c_2(\xi)\hat{u}(\xi,t) + P_2(\xi)\hat{e}(1,t) \\
&\qquad\qquad + \dot{\hat{k}}(t) \left(\hat{\theta}(t) n(\xi,t) + z(\xi,t) \right) + \hat{k}(t)\dot{\hat{\theta}}(t) n(\xi,t) \Big) d\xi.
\end{aligned} \tag{4.97}
$$

Differentiating with respect to space and applying Leibniz' differentiation rule give

$$
\begin{aligned}
\zeta_x(x,t) =& \hat{v}_x(x,t) \\
&- K^{vu}(x,x)\hat{u}(x,t) - \int_0^x K^{vu}_x(x,\xi)\hat{u}(\xi,t) d\xi \\
&- K^{vv}(x,x)\hat{v}(x,t) - \int_0^x K^{vv}_x(x,\xi)\hat{v}(\xi,t) d\xi.
\end{aligned} \tag{4.98}
$$

Substituting (4.97) and (4.98) into (4.68b) and using (4.81) and (4.90b)–(4.93b) one finds

$$
\begin{aligned}
&\zeta_t(x,t) - \mu\zeta_x(x,t) \\
=& K^{vu}(x,x)\lambda\hat{u}(x,t) - K^{vu}(x,0)\lambda\hat{u}(0,t) - \int_0^x K^{vu}_\xi(x,\xi)\lambda\hat{u}(\xi,t) d\xi \\
&- \int_0^x K^{vu}(x,\xi) \Big(c_1(\xi)\hat{v}(\xi,t) + P_1(\xi)\hat{e}(1,t) \\
&\qquad\qquad + \dot{\hat{k}}(t) \left(\hat{\theta}(t) m(\xi,t) + w(\xi,t) \right) + \hat{k}(t)\dot{\hat{\theta}}(t) m(\xi,t) \Big) d\xi \\
&- K^{vv}(x,x)\mu\hat{v}(x,t) + K^{vv}(x,0)\mu\hat{v}(0,t) + \int_0^x K^{vv}_\xi(x,\xi)\mu\hat{v}(\xi,t) d\xi \\
&- \int_0^x K^{vv}(x,\xi) \Big(c_2(\xi)\hat{u}(\xi,t) + P_2(\xi)\hat{e}(1,t)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + \dot{\hat{k}}(t)\left(\hat{\theta}(t)n(\xi,t) + z(\xi,t)\right) + \hat{k}(t)\dot{\hat{\theta}}(t)n(\xi,t)\Big) d\xi \\
&\quad - \mu\Bigg(-K^{vu}(x,x)\hat{u}(x,t) - \int_0^x K_x^{vu}(x,\xi)\hat{u}(\xi,t)d\xi \\
&\qquad -K^{vv}(x,x)\hat{v}(x,t) - \int_0^x K_x^{vv}(x,\xi)\hat{v}(\xi,t)d\xi\Bigg) \\
&\quad + c_2(x)\hat{u}(x,t) + P_2(x)\hat{e}(1,t) \\
&\quad + \dot{\hat{k}}(t)\left(\hat{\theta}(t)n(x,t) + z(x,t)\right) + \hat{k}(t)\dot{\hat{\theta}}(t)n(x,t) \\
=& - \int_0^x \underbrace{\left[K_\xi^{vu}(x,\xi)\lambda - K_x^{vu}(x,\xi)\mu + K^{vv}(x,\xi)c_2(\xi)\right]}_{=0}\hat{u}(\xi,t)d\xi \\
&\quad - \int_0^x \underbrace{\left[-K_\xi^{vv}(x,\xi)\mu - K_x^{vv}(x,\xi)\mu + K^{vu}(x,\xi)c_1(\xi)\right]}_{=0}\hat{v}(\xi,t)d\xi \\
&\quad + \underbrace{\left[K^{vu}(x,x)\mu + K^{vu}(x,x)\lambda + c_2(x)\right]}_{=0}\hat{u}(x,t) \\
&\quad - \underbrace{\left[K^{vu}(x,0)\lambda r - K^{vv}(x,0)\mu\right]}_{=0}\hat{v}(0,t) \\
&\quad + \underbrace{\left[-K^{vu}(x,0)\lambda\right]}_{=\Psi_2(x)}\hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right) \\
&\quad + \underbrace{\left[P_2(x) - \int_0^x K^{vu}(x,\xi)P_1(\xi)d\xi - \int_0^x K^{vv}(x,\xi)P_2(\xi)d\xi\right]}_{=\Omega_2(x)}\hat{e}(1,t) \\
&\quad + \hat{\theta}(t)\underbrace{\left[n(x,t) - \int_0^x K^{vu}(x,\xi)m(\xi,t)d\xi - \int_0^x K^{vv}(x,\xi)n(\xi,t)d\xi\right]}_{H_2(x,t)}\dot{\hat{k}}(t) \\
&\quad + \underbrace{\left[z(x,t) - \int_0^x K^{vu}(x,\xi)w(\xi,t)d\xi - \int_0^x K^{vv}(x,\xi)z(\xi,t)d\xi\right]}_{G_2(x,t)}\dot{\hat{k}}(t) \\
&\quad + \hat{k}(t)\underbrace{\left[n(x,t) - \int_0^x K^{vu}(x,\xi)m(\xi,t)d\xi - \int_0^x K^{vv}(x,\xi)(t)n(\xi,t)d\xi\right]}_{H_2(x,t)}\dot{\hat{\theta}}(t) \\
=&\hat{\theta}(t)H_2(x,t)\dot{\hat{k}}(t) + G_2(x,t)\dot{\hat{k}}(t) + \hat{k}(t)H_2(x,t)\dot{\hat{\theta}}(t) \\
&+\Omega_2(x)\hat{e}(1,t) + \Psi_2(x,t)\hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right).
\end{aligned}
\tag{4.99}
$$

The boundary condition (4.89c) can be seen from evaluating (4.87) at $x = 0$ and inserting (4.69):

$$\omega(0,t) = \hat{u}(0,t)$$

$$
\begin{aligned}
&= \hat{v}(0,t)q(t) + d(t) + \kappa(t)\varepsilon(t) \\
&= \zeta(0,t)q(t) + d(t) + \kappa(t)\varepsilon(t). \qquad (4.100)
\end{aligned}
$$

Selecting the control law according to (4.86) and (4.88) give the boundary condition (4.89d):

$$
\begin{aligned}
\zeta(1,t) =& \hat{v}(1,t) - \int_0^1 K^{vu}(1,\xi,t)\hat{u}(\xi,t)d\xi - \int_0^1 K^{vv}(1,\xi,t)\hat{v}(\xi,t)d\xi \\
\zeta(1,t) =& U_2(t). \qquad (4.101)
\end{aligned}
$$

□

4.3.4 Reference Model and Tracking

Motivated by the structure of system (4.89), consider the reference model

$$
\begin{aligned}
\varphi_t(x,t) + \lambda\varphi_x(x,t) =& 0 && (4.102a)\\
\phi_t(x,t) - \mu\phi_x(x,t) =& 0 && (4.102b)\\
\varphi(0,t) =& q(t)\phi(0,t) + d(t) && (4.102c)\\
\phi(1,t) =& \zeta^*(t) && (4.102d)
\end{aligned}
$$

where $\zeta^*(t)$ is a reference signal to be designed. The initial conditions $\varphi(x,0) = \varphi_0(x)$, $\phi(x,0) = \phi_0(x)$ satisfy

$$
\varphi_0,\ \phi_0 \in L_2([0,1]). \qquad (4.103)
$$

Lemma 4.8. *Consider system* (4.89) *and the reference model* (4.102)*. The deviation signals*

$$
\begin{aligned}
\nu(x,t) =& \omega(x,t) - \varphi(x,t) && (4.104a)\\
\eta(x,t) =& \zeta(x,t) - \phi(x,t) && (4.104b)
\end{aligned}
$$

with the control law selected as

$$
U_2(t) = \zeta^*(t) \qquad (4.105)
$$

satisfy the dynamics

$$
\begin{aligned}
\nu_t(x,t) + \lambda\nu_x(x,t) =& \hat{\theta}(t)H_1(x,t)\dot{\hat{k}}(t) + G_1(x,t)\dot{\hat{k}}(t) + \hat{k}(t)H_1(x,t)\dot{\hat{\theta}}(t) \\
&+ \Omega_1(x)\hat{e}(1,t) + \Psi_1(x)\hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right) && (4.106a)\\
\eta_t(x,t) - \mu\eta_x(x,t) =& \hat{\theta}(t)H_2(x,t)\dot{\hat{k}}(t) + G_2(x,t)\dot{\hat{k}}(t) + \hat{k}(t)H_2(x,t)\dot{\hat{\theta}}(t) \\
&+ \Omega_2(x)\hat{e}(1,t) + \Psi_2(x)\hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right) && (4.106b)\\
\nu(0,t) =& \eta(0,t)q(t) + \kappa(t)\varepsilon(t) && (4.106c)\\
\eta(1,t) =& 0 && (4.106d)
\end{aligned}
$$

where $G_i, H_i, \Omega_i, \Psi_i$ $i \in [1,2]$ are defined in (4.90)–(4.93) *and $d, q, \kappa, \varepsilon$ in* (4.70).

Proof. The dynamics (4.106a) and (4.106b) follow straight forward by differentiating (4.104) and inserting (4.89) and (4.102). Direct substitution of the boundary conditions and control law (4.105) give (4.106c) and (4.106d). $\square$

4.4 Stability Proof

4.4.1 Estimation Error Dynamics

The *adaptive* estimation error dynamics can be found by substituting in the state dynamics (4.1) and estimator dynamics (4.68) into (4.39). The Boundary condition (for $t > t_F$) can be seen from

$$\begin{aligned}\hat{e}(0,t) =& u(0,t) - \hat{u}(0,t)\\ =& rv(0,t) + k(\theta - y_0(t)) - r\hat{v}(0,t) - \hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right)\\ =& r\hat{\epsilon}(0,t) + \tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t)\tilde{\theta}(t).\end{aligned} \tag{4.107}$$

This gives the dynamics:

$$\begin{aligned}\hat{e}_t(x,t) + \lambda\hat{e}_x(x,t) =& c_1(x)\hat{\epsilon}(x,t) - P_1\hat{e}(1,t)\\ & - \dot{\hat{k}}\left(\hat{\theta}(t)m(x,t) - w(x,t)\right) - \hat{k}\dot{\hat{\theta}}m(x,t)\end{aligned} \tag{4.108a}$$

$$\begin{aligned}\hat{\epsilon}_t(x,t) + \lambda\hat{\epsilon}_x(x,t) =& c_2(x)\hat{e}(x,t) - P_2\hat{e}(1,t)\\ & - \dot{\hat{k}}\left(\hat{\theta}(t)n(x,t) - z(x,t)\right) - \hat{k}\dot{\hat{\theta}}n(x,t)\end{aligned} \tag{4.108b}$$

$$\hat{e}(0,t) = r\hat{\epsilon}(0,t) + \tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t)\tilde{\theta}(t) \tag{4.108c}$$

$$\hat{\epsilon}(1,t) = 0 \tag{4.108d}$$

where P_1, P_2 are output injection gains originating from the *static* estimation error system (4.16) found by solving (4.23). To facilitate the Lyapunov analysis, the estimation error system is transformed into an equivalent target system in the next lemma.

Lemma 4.9. *Consider the operators $\mathcal{P}_1, \mathcal{P}_2$ in* (4.22)*. The backstepping transformation*

$$\hat{e}(x,t) = \mathcal{P}_1[\hat{\alpha},\hat{\beta}](x,t) \tag{4.109a}$$

$$\hat{\epsilon}(x,t) = \mathcal{P}_2[\hat{\alpha},\hat{\beta}](x,t) \tag{4.109b}$$

maps the error dynamics (4.108) *into the target system*

$$\hat{\alpha}_t(x,t) + \lambda\hat{\alpha}_x(x,t) = B_1(x,t) \tag{4.110a}$$

$$\hat{\beta}_t(x,t) - \mu\hat{\beta}_x(x,t) = B_2(x,t) \tag{4.110b}$$

$$\hat{\alpha}(0,t) = r\hat{\beta}(0,t) + \tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t)\tilde{\theta}(t) \tag{4.110c}$$

$$\hat{\beta}(1,t) = 0 \tag{4.110d}$$

where (B_1, B_2) *is given by the* 2×2 *Volterra equation*

$$\begin{aligned}B_1(x,t) =&\dot{\hat{k}}\left(\hat{\theta}(t)m(x,t) - w(x,t)\right) + \hat{k}\dot{\hat{\theta}}m(x,t)\\&+\int_x^1 P^{uu}(x,\xi)B_1(\xi,t)d\xi + \int_x^1 P^{uv}(x,\xi)B_2(\xi,t)d\xi\end{aligned} \tag{4.111a}$$

$$\begin{aligned}B_2(x,t) =&\dot{\hat{k}}\left(\hat{\theta}(t)n(x,t) - z(x,t)\right) + \hat{k}\dot{\hat{\theta}}n(x,t)\\&+\int_x^1 P^{vu}(x,\xi)B_1(\xi,t)d\xi + \int_x^1 P^{vv}(x,\xi)B_2(\xi,t)d\xi.\end{aligned} \tag{4.111b}$$

Proof. Differentiating (4.109a) with respect to time, inserting the dynamics (4.110a) and (4.110b) and integration by parts give

$$\begin{aligned}\hat{e}_t(x,t) =&\hat{\alpha}_t(x,t) + \int_x^1 P^{uu}(x,\xi)\hat{\alpha}_t(\xi,t)d\xi + \int_x^1 P^{uv}(x,\xi)\hat{\beta}_t(\xi,t)d\xi\\
=&\hat{\alpha}_t(x,t)\\
&-\lambda\int_x^1 P^{uu}(x,\xi)\hat{\alpha}_\xi(\xi,t)d\xi\\
&+\int_x^1 P^{uu}(x,\xi)B_1(\xi,t)d\xi\\
&+\mu\int_x^1 P^{uv}(x,\xi)\hat{\beta}_\xi(\xi,t)d\xi\\
&+\int_x^1 P^{uv}(x,\xi)B_2(\xi,t)d\xi\\
=&\hat{\alpha}_t(x,t)\\
&-P^{uu}(x,1)\lambda\hat{\alpha}(1,t) + P^{uu}(x,x)\lambda\hat{\alpha}(x,t) + \lambda\int_x^1 P^{uu}_\xi(x,\xi)\hat{\alpha}(\xi,t)d\xi\\
&+\int_x^1 P^{uu}(x,\xi)B_1(\xi,t)d\xi\\
&+P^{uv}(x,1)\mu\hat{\beta}(1,t) - P^{uv}(x,x)\mu\hat{\beta}(x,t) - \mu\int_x^1 P^{uv}_\xi(x,\xi)\hat{\beta}(\xi,t)d\xi\\
&+\int_x^1 P^{uv}(x,\xi)B_2(\xi,t)d\xi.\end{aligned} \tag{4.112}$$

Differentiating with respect to space and applying Leibniz' differentiation rule give

$$\begin{aligned}\hat{e}_x(x,t) =&\hat{\alpha}_x(x,t) - P^{uu}(x,x)\hat{\alpha}(x,t) + \int_x^1 P^{uu}_x(x,\xi)\hat{\alpha}(\xi,t)d\xi\\&- P^{uv}(x,x)\hat{\beta}(x,t) + \int_x^1 P^{uv}_x(x,\xi)\hat{\beta}(\xi,t)d\xi.\end{aligned} \tag{4.113}$$

Substituting (4.112) and (4.113) into (4.108a) and using (4.23) and the definitions (4.111) yield

$$
\begin{aligned}
&\hat{\alpha}_t(x,t) + \lambda\hat{\alpha}_x(x,t) \\
=&P^{uu}(x,1)\lambda\hat{\alpha}(1,t) - P^{uu}(x,x)\lambda\hat{\alpha}(x,t) - \lambda\int_x^1 P^{uu}_\xi(x,\xi)\hat{\alpha}(\xi,t)d\xi \\
&-\int_x^1 P^{uu}(x,\xi)B_1(\xi,t)d\xi \\
&-P^{uv}(x,1)\mu\hat{\beta}(1,t) + P^{uv}(x,x)\mu\hat{\beta}(x,t) + \mu\int_x^1 P^{uv}_\xi(x,\xi)\hat{\beta}(\xi,t)d\xi \\
&-\int_x^1 P^{uv}(x,\xi)B_2(\xi,t)d\xi \\
&+\lambda P^{uu}(x,x)\hat{\alpha}(x,t) - \lambda\int_x^1 P^{uu}_x(x,\xi)\hat{\alpha}(\xi,t)d\xi \\
&+\lambda P^{uv}(x,x)\hat{\beta}(x,t) - \lambda\int_x^1 P^{uv}_x(x,\xi)\hat{\beta}(\xi,t)d\xi \\
&+c_1(x)\hat{\beta}(x,t) + c_1(x)\int_x^1 P^{vu}(x,\xi)\hat{\alpha}(\xi,t)d\xi + c_1(x)\int_x^1 P^{vv}(x,\xi)\hat{\beta}(\xi,t)d\xi \\
&-P_1\hat{\alpha}(1,t) - \dot{\hat{k}}\left(\hat{\theta}(t)m(x,t) - w(x,t)\right) - \hat{k}\dot{\hat{\theta}}m(x,t) \\
\\
=&-\int_x^1 \underbrace{\left[P^{uu}_x(x,\xi)\lambda + P^{uu}_\xi(x,\xi)\lambda - c_1(x)P^{vu}(x,\xi)\right]}_{=0}\hat{\alpha}(\xi,t)d\xi \\
&-\int_x^1 \underbrace{\left[P^{uv}_x(x,\xi)\lambda - P^{uv}_\xi(x,\xi)\mu - c_1(x)P^{vv}(x,\xi)\right]}_{=0}\hat{\beta}(\xi,t)d\xi \\
&+\underbrace{\left[P^{uv}(x,x)\mu + \lambda P^{uv}(x,x) + c_1(x)\right]}_{=0}\hat{\beta}(x,t) \\
&+\underbrace{\left[P^{uu}(x,1)\lambda - P_1\right]}_{=0}\hat{\alpha}(1,t) \\
&-\left[\dot{\hat{k}}\left(\hat{\theta}(t)m(x,t) - w(x,t)\right) + \hat{k}\dot{\hat{\theta}}m(x,t)\right. \\
&\qquad\left. + \int_x^1 P^{uu}(x,\xi)B_1(\xi,t)d\xi + \int_x^1 P^{uv}(x,\xi)B_2(\xi,t)d\xi\right] \\
=&B_1(x,t).
\end{aligned}
\tag{4.114}
$$

Equation (4.110b) can be found in a similar way. Boundary condition (4.110c) is obtained by evaluating (4.109) at $x = 0$, inserting (4.108c) and using (4.23). Boundary condition (4.110d) follows trivially from evaluating (4.109a) at $x = 1$.

□

4.4.2 Boundedness in $L_2([0,1])$

To ease the Lyapunov proof in this section, some additional signals are introduced. Let $\bar{w}$ be a signal defined by the auxiliary filter

$$\bar{w}_t(x,t) + \lambda \bar{w}_x(x,t) = 0 \tag{4.115a}$$

$$\bar{w}(0,t) = -\hat{\theta}(t). \tag{4.115b}$$

Next, let $\tilde{w}$ describe the deviation between the transformed filter system (4.34) and the auxiliary system (4.115). That is

$$\tilde{w}(x,t) = \check{w}(x,t) - \bar{w}(x,t). \tag{4.116}$$

Differentiating (4.116) and substituting in (4.34) and (4.115) give the dynamics

$$\tilde{w}_t(x,t) + \lambda \tilde{w}_x(x,t) = 0 \tag{4.117a}$$

$$\tilde{w}(0,t) = \hat{\theta}(t) - y_0(t). \tag{4.117b}$$

System (4.117) will be included in the Lyapunov function candidate and used to prove boundedness of all signals in the closed loop system.

In the following lemma, some properties needed in the Lyapunov analysis are presented.

Lemma 4.10. *Consider G_1, G_2, H_1, H_2, Ω_1 and Ω_2 given in* (4.90)–(4.92)*, ε in* (4.70d)*, $\tilde{w}$ by* (4.117) *and B_i in* (4.111)*. The following properties hold for $t > t_F$ with t_F given by* (4.27)*:*

I)

$$H_1(x,\cdot),\ H_2(x,\cdot) \in \mathcal{L}_\infty. \tag{4.118}$$

II)

$$\Omega_1(x,\cdot),\ \Omega_2(x,\cdot) \in \mathcal{L}_\infty. \tag{4.119}$$

III)

$$\begin{aligned} ||G_1(t)|| &\leq h_1 ||\check{w}(t)|| \\ ||G_2(t)|| &\leq h_2 ||\check{w}(t)|| \end{aligned} \tag{4.120}$$

IV)

$$\begin{aligned} \varepsilon^2(t) \leq{}& h_3 \beta^2(0,t) + h_4 ||\check{\alpha}(t)||^2 + h_5 ||\check{\beta}||^2 \\ &+ h_6 \left(\tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t)\tilde{\theta}(t)\right)^2 \end{aligned} \tag{4.121}$$

V)

$$|\tilde{w}0,t)| \leq h_\theta + h_7 |\eta(0,t)| + |\varepsilon(t)| \tag{4.122}$$

VI)

$$||B_i(x,t)||^2 \leq h_8 \left|\dot{\hat{k}}(t)\right|^2 + h_9 \left|\dot{\hat{\theta}}(t)\right|^2 + h_{10} \left|\dot{\hat{k}}(t)\right|^2 ||\tilde{w}||^2. \tag{4.123}$$

for some constants h_θ and $h_i > 0$ $i \in [1,10]$.

Proof. The transformed filter system $(\check{m}, \check{n})$ in (4.32) is a simple cascaded transport equation and we have $\check{m} \equiv 1$ and $\check{n} \equiv 0$ for all $x \in [0,1]$ and $t > t_F$. From the invertibility of transformation (4.31), we have $m(x,\cdot), n(x,\cdot) \in \mathcal{L}_\infty$, leaving all signals in H_1, H_2 bounded and Property I follows. Boundedness of the kernels in (4.22) gives Property II. The subsystem (4.34b) is a simple transport equation and we have $\check{z} \equiv 0$ for all $x \in [0,1]$ and $t > t_F$. From the backstepping transformation (4.33) it then follows that

$$w(x,t) = \check{w}(x,t) + \int_x^1 P^{uu}(x,\xi)\check{w}(\xi,t)d\xi \tag{4.124a}$$

$$z(x,t) = \int_x^1 P^{vu}(x,\xi)\check{w}(\xi,t)d\xi \tag{4.124b}$$

for $t > t_F$, and by applying Cauchy-Schwarz' inequality that

$$||w| \leq h_1'||\check{w}|| \tag{4.125a}$$

$$||z|| \leq h_2'||\check{w}|| \tag{4.125b}$$

for some constants $h_1', h_2' > 0$. Boundedness of the kernels in (4.80), using Cauchy-Schwarz' inequality on G_1, G_2 and the relation (4.125) then give Property III.

From transformation (4.109), inserting boundary condition (4.110c), and using Cauchy-Schwarz' inequality on the integral terms, we get

$$\hat{e}(0,t) \leq r\hat{\beta}(0,t) + c_1||\hat{\alpha}|| + c_2||\hat{\beta}|| + \tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t)\tilde{\theta}(t) \tag{4.126a}$$

$$\hat{\epsilon}(0,t) \leq \hat{\beta}(0,t) + c_3||\hat{\alpha}|| + c_4||\hat{\beta}|| \tag{4.126b}$$

Squaring both terms and substituting the result into (4.70d) give Property IV.

The following relation between $\tilde{\omega}(0,t)$ and $\eta(0,t), \varepsilon(t)$ can be found:

$$\begin{aligned}
\tilde{w}(0,t) =& \hat{\theta}(t) - y_0(t) \\
=& \hat{\theta}(t) - a_0 u(0,t) - b_0 v(0,t) \\
=& \hat{\theta}(t) - a_0 \hat{u}(0,t) - b_0 \hat{v}(0,t) - a_0 \hat{e}(0,t) - b_0 \hat{\epsilon}(0,t) \\
=& \hat{\theta}(t) - a_0 \omega(0,t) - b_0 \zeta(0,t) - \varepsilon(t) \\
=& \hat{\theta}(t) - a_0 \nu(0,t) - a_0 \varphi(0,t) - b_0 \eta(0,t) - b_0 \phi(0,t) - \varepsilon(t) \\
=& \hat{\theta}(t) - a_0 d(t) - (a_0 q(t) + b_0)\, \eta(0,t) - (a_0 q(t) + b_0)\, \phi(0,t) - \varepsilon(t) \\
=& \hat{\theta}(t) - a_0 d(t) - (a_0 q(t) + b_0)\, \eta(0,t) - (a_0 q(t) + b_0)\, \zeta^*(t - d_\beta) - \varepsilon(t) \\
=& \hat{\theta}(t) - a_0 \frac{\hat{k}(t)\hat{\theta}(t)}{1 + a_0\hat{k}(t)} - \left(a_0 \frac{r - b_0\hat{k}(t)}{1 + a_0\hat{k}(t)} + b_0 \right) \frac{\hat{\theta}(t - d_\beta)}{a_0 r + b_0}
\end{aligned}$$

$$
\begin{aligned}
&- (a_0 q(t) + b_0)\, \eta(0,t) - \varepsilon(t) \\
=& \frac{\hat{\theta}(t) + a_0 \hat{k}(t)\hat{\theta}(t) - a_0 \hat{k}(t)\hat{\theta}(t)}{1 + a_0 \hat{k}(t)} - \left(\frac{a_0 r + b_0}{1 + a_0 \hat{k}(t)} \right) \frac{\hat{\theta}(t - d_\beta)}{a_0 r + b_0} \\
&- (a_0 q(t) + b_0)\, \eta(0,t) - \varepsilon(t) \\
=& \frac{\hat{\theta}(t) - \hat{\theta}(t - d_\beta)}{1 + a_0 \hat{k}(t)} - (a_0 q(t) + b_0)\, \eta(0,t) - \varepsilon(t)
\end{aligned} \tag{4.127}
$$

Using Property I of Theorem 4.4 gives Property V.

From Lemma A.1 we have

$$
B_i(x,t) \leq f(x,t) + \bar{g}||f||e^{2\bar{g}(1-x)} \tag{4.128}
$$

for $i \in [1,2]$ with

$$
\bar{g} = \max_{x,\xi \in [0,1]} (|P^{uu}(x,\xi)|, |P^{uv}(x,\xi)|, |P^{vu}(x,\xi)|, |P^{vv}(x,\xi)|) \tag{4.129}
$$

and

$$
\begin{aligned}
f(x,t) =& \left| \dot{\hat{k}} \left(\hat{\theta}(t) m(x,t) - w(x,t) \right) + \hat{k}\dot{\hat{\theta}} m(x,t) \right| \\
&+ \left| \dot{\hat{k}} \left(\hat{\theta}(t) n(x,t) - z(x,t) \right) + \hat{k}\dot{\hat{\theta}} n(x,t) \right|.
\end{aligned} \tag{4.130}
$$

Let

$$
f_a(x,t) = \left| \dot{\hat{k}}\hat{\theta}(t) m(x,t) + \hat{k}\dot{\hat{\theta}} m(x,t) \right| + \left| \dot{\hat{k}}\hat{\theta}(t) n(x,t) + \hat{k}\dot{\hat{\theta}} n(x,t) \right| \tag{4.131a}
$$

$$
f_b(x,t) = \left| \dot{\hat{k}} w(x,t) \right| + \left| \dot{\hat{k}} z(x,t) \right|, \tag{4.131b}
$$

then

$$
B_i(x,t) \leq |f_a(x,t)| + |f_b(x,t)| + (||f_a|| + ||f_b||)\bar{g}e^{2\bar{g}(1-x)}. \tag{4.132}
$$

Boundedness of $m, n, \hat{k}, \hat{\theta}$ and square integrability of $\dot{\hat{k}}, \dot{\hat{\theta}}$ from Theorem 4.4 give

$$
f_a(x,\cdot) \leq h_8' \left| \dot{\hat{k}}(t) \right| + h_9' \left| \dot{\hat{\theta}}(t) \right| \in \mathcal{L}_1 \cap \mathcal{L}_\infty, \tag{4.133}
$$

and using (4.125)

$$
||f_b||^2 \leq h_{10}' \left| \dot{\hat{k}}(t) \right|^2 ||\check{w}||^2 \tag{4.134}
$$

for some constants $h_8', h_9', h_{10}' > 0$. It then follows that

$$
||B_i(x,t)||^2 \leq h_8'' \left| \dot{\hat{k}}(t) \right|^2 + h_9 \left| \dot{\hat{\theta}}(t) \right|^2 + h_{10} \left| \dot{\hat{k}}(t) \right|^2 ||\check{w}||^2 \tag{4.135}
$$

for some other constant $h_8'' > 0$. Inserting relation (4.116) and boundedness of (4.115) give Property VI.

□

Lemma 4.11. *Consider the Lyapunov function candidate*

$$V_6(t) = \sum_{i=1}^{5} a_i V_i \tag{4.136}$$

where $a_i > 0\ \forall i \in [1,5]$ *are constants to be decided,*

$$V_1(t) = \lambda^{-1} \int_0^1 e^{-\delta x} \nu^2(x,t) dx \tag{4.137a}$$

$$V_2(t) = \mu^{-1} \int_0^1 e^{\sigma x} \eta^2(x,t) dx \tag{4.137b}$$

$$V_3(t) = \lambda^{-1} \int_0^1 e^{-\pi x} \tilde{w}^2(x,t) dx \tag{4.137c}$$

$$V_4(t) = \lambda^{-1} \int_0^1 e^{-\delta x} \hat{\alpha}^2(x,t) dx \tag{4.137d}$$

$$V_5(t) = \mu^{-1} \int_0^1 e^{\sigma x} \hat{\beta}^2(x,t) dx \tag{4.137e}$$

and ν, η *are given by* (4.106), $\tilde{w}$ *defined in* (4.117) *and* $\hat{\alpha}, \hat{\beta}$ *given by* (4.110).
With appropriately selected $a_i\ \forall i \in [1,5]$, δ, σ *and* π, *then* (4.136) *satisfies*

$$\begin{aligned} \dot{V}_6 \leq & - h_{66} V_6 + l_1(t) V_6 + l_2(t) \\ & - e^{-\pi} \left(1 - 2e^{\pi} h_{67} \frac{\left(\tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t)\tilde{\theta}(t) \right)^2}{1 + w^2(1,t)} \right) \tilde{w}^2(1,t) \end{aligned} \tag{4.138}$$

for some constants $h_{66}, h_{67} > 0$, *and where* $l_1(t), l_2(t) \geq 0$ *are real valued functions given by*

$$l_1(t) = a_1 2h_1 \left|\dot{\hat{k}}(t)\right|^2 + a_2 h_2 \left|\dot{\hat{k}}(t)\right|^2 + a_4 e^{\pi} \left|\dot{\hat{k}}(t)\right|^2 + a_5 e^{\sigma+\pi} \left|\dot{\hat{k}}(t)\right|^2 \tag{4.139a}$$

$$\begin{aligned} l_2(t) = & + a_4 \lambda^{-1} h_8 \left|\dot{\hat{k}}(t)\right|^2 + a_4 \lambda^{-1} h_9 \left|\dot{\hat{\theta}}(t)\right|^2 \\ & + a_1 \left(\frac{h_{14}}{\lambda\delta} \left(1 - e^{-\delta}\right) + \frac{1}{\delta} \left(1 - e^{-\delta}\right) h_{13} \right) \left|\dot{\hat{k}}(t)\right|^2 \\ & + a_1 \frac{h_{15}}{\lambda\delta} \left(1 - e^{-\delta}\right) \left|\dot{\hat{\theta}}(t)\right|^2 \\ & + a_1 \frac{\bar{\Omega}_1}{\lambda\delta} \left(1 - e^{-\delta}\right) |\hat{e}(1,t)|^2 \\ & + a_2 \left(\frac{h_{24}}{\mu\sigma} \left(e^{\sigma} - 1\right) + \frac{1}{\sigma} \left(e^{\sigma} - 1\right) h_{23} \right) \left|\dot{\hat{k}}(t)\right|^2 \\ & + a_2 \frac{h_{25}}{\mu\sigma} \left(e^{\sigma} - 1\right) \left|\dot{\hat{\theta}}(t)\right|^2 \end{aligned}$$

$$+ a_2 \frac{\bar{\Omega}_2}{\mu\sigma} \left(e^{\sigma} - 1\right) |\hat{e}(1,t)|^2$$
$$+ a_5 \mu^{-1} e^{\sigma} h_8 \left|\dot{\hat{k}}(t)\right|^2 + a_5 \mu^{-1} e^{\sigma} h_9 \left|\dot{\hat{\theta}}(t)\right|^2$$
$$+ h_{67} \frac{\left(\tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t)\tilde{\theta}(t)\right)^2}{1 + w^2(1,t)} \left(1 + \bar{w}^2(1,t)\right). \tag{4.139b}$$

satisfying

$$l_1, l_2 \in \mathcal{L}_1 \cap \mathcal{L}_\infty. \tag{4.140}$$

Furthermore,

$$V_3 \in \mathcal{L}_1 \cap \mathcal{L}_\infty \tag{4.141}$$

and

$$||\nu||, ||\eta||, ||\tilde{w}||, ||\hat{\alpha}||, ||\hat{\beta}|| \in \mathcal{L}_\infty. \tag{4.142}$$

Before proving Lemma 4.11, derivatives of all the terms in (4.137) will be calculated separately in the next sections.

Derivations regarding V_1

From (4.137a) and inserting the dynamics (4.106a), we get

$$\begin{aligned}
\dot{V}_1(t) =& 2\lambda^{-1} \int_0^1 e^{-\delta x} \nu(x,t)\nu_t(x,t)dx \\
=& - 2\lambda^{-1} \int_0^1 e^{-\delta x} \nu(x,t)\lambda\nu_x(x,t)dx \\
&+ 2\lambda^{-1} \int_0^1 e^{-\delta x} \nu(x,t)G_1(x,t)\dot{\hat{k}}(t)dx \\
&+ 2\lambda^{-1} \int_0^1 e^{-\delta x} \nu(x,t)\hat{\theta}(t)H_1(x,t)\dot{\hat{k}}(t)dx \\
&+ 2\lambda^{-1} \int_0^1 e^{-\delta x} \nu(x,t)\hat{k}(t)H_1(x,t)\dot{\hat{\theta}}(t)dx \\
&+ 2\lambda^{-1} \int_0^1 e^{-\delta x} \nu(x,t)\Omega_1(x)\hat{e}(1,t)dx \\
&+ 2\lambda^{-1} \int_0^1 e^{-\delta x} \nu(x,t)\Psi_1(x)\hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right)dx
\end{aligned} \tag{4.143}$$

1st term: Integration by parts and using boundary condition (4.106c) give

$$\begin{aligned}
&- 2\lambda^{-1} \int_0^1 e^{-\delta x} \nu(x,t)\lambda\nu_x(x,t)dx \\
=& - 2 \int_0^1 e^{-\delta x} \nu(x,t)\nu_x(x,t)dx
\end{aligned}$$

$$
\begin{aligned}
&= - e^{-\delta}\nu^2(1,t) + \nu^2(0,t) - \delta \int_0^1 e^{-\delta x}\nu^2(x,t)dx \\
&= - e^{-\delta}\nu^2(1,t) + (\eta(0,t)q(t) + \kappa(t)\varepsilon(t))^2 - \delta \int_0^1 e^{-\delta x}\nu^2(x,t)dx \\
&\leq 2\eta^2(0,t)q^2(t) + 2\kappa^2(t)\varepsilon^2(t) - \delta\lambda V_1. \qquad (4.144)
\end{aligned}
$$

2nd term: Substituting in the relation (4.116), separating the cross terms using Young's inequality (see Lemma A.2) and from Theorem 4.4 using that $\hat{\theta}$ and $\dot{\hat{k}}$ are bounded give

$$
\begin{aligned}
&2\lambda^{-1} \int_0^1 e^{-\delta x}\nu(x,t)G_1(x,t)\dot{\hat{k}}(t)dx \\
\leq& 2\lambda^{-1} \int_0^1 e^{-\delta x}|\nu(x,t)||G_1(x,t)|dx \left|\dot{\hat{k}}(t)\right| \\
\leq& 2\lambda^{-1}h_1 \left|\dot{\hat{k}}(t)\right| \int_0^1 e^{-\delta x}|\nu(x,t)|dx||\check{w}(t)|| \\
\leq& 2\lambda^{-1}h_1 \left|\dot{\hat{k}}(t)\right| \int_0^1 e^{-\delta x}|\nu(x,t)|dx||(\tilde{w}+\bar{w})(t)|| \\
\leq& \lambda^{-1}h_1 \left|\dot{\hat{k}}(t)\right|^2 \int_0^1 e^{-\delta x}\nu^2(x,t)dx + \lambda^{-1}h_1 \int_0^1 e^{-\delta x}dx||\check{w}(t)||^2 \\
&+ \lambda^{-1}h_1||\bar{w}|| \int_0^1 e^{-\delta x}\nu^2(x,t)dx + \lambda^{-1}h_1||\bar{w}|| \int_0^1 e^{-\delta x}dx \left|\dot{\hat{k}}(t)\right|^2 \\
\leq& 2h_1 \left|\dot{\hat{k}}(t)\right|^2 V_1 + 2h_1\frac{1}{\delta}\left(1-e^{-\delta}\right)V_3 + 2h_1||\bar{w}||V_1 + 2h_1\frac{||\bar{w}||}{\lambda\delta}\left(1-e^{-\delta}\right)\left|\dot{\hat{k}}(t)\right|^2 \\
\leq& 2h_1 \left|\dot{\hat{k}}(t)\right|^2 V_1 + h_{11}V_1 + \frac{1}{\delta}\left(1-e^{-\delta}\right)h_{12}V_3 + \frac{1}{\delta}\left(1-e^{-\delta}\right)h_{13}\left|\dot{\hat{k}}(t)\right|^2 \qquad (4.145)
\end{aligned}
$$

for some constants $h_{11}, h_{12}, h_{13} > 0$.

3rd, 4th and 5th term: Separating the cross terms using Young's inequality and using that $H_1, \Omega_1\hat{\theta}, \hat{k}, \dot{\hat{\theta}}, \dot{\hat{k}}$ are bounded (Lemma 4.10 and Theorem 4.4) give similarly for the 3rd, 4th and 5th term

$$
\begin{aligned}
&2\lambda^{-1} \int_0^1 e^{-\delta x}\nu(x,t)\hat{\theta}(t)H_1(x,t)\dot{\hat{k}}(t)dx \\
\leq& h_{14}2\lambda^{-1} \int_0^1 e^{-\delta x}|\nu(x,t)|dx \left|\dot{\hat{k}}(t)\right| \\
\leq& h_{14}\lambda^{-1} \int_0^1 e^{-\delta x}|\nu(x,t)|^2dx + h_{14}\lambda^{-1} \int_0^1 e^{-\delta x}dx \left|\dot{\hat{k}}(t)\right|^2 \\
\leq& h_{14}V_1 + \frac{h_{14}}{\lambda\delta}\left(1-e^{-\delta}\right)\left|\dot{\hat{k}}(t)\right|^2, \qquad (4.146)
\end{aligned}
$$

$$
\begin{aligned}
&2\lambda^{-1}\int_0^1 e^{-\delta x}\nu(x,t)\hat{k}(t)H_1(x,t)\dot{\hat{\theta}}(t)dx \\
\le & h_{15}2\lambda^{-1}\int_0^1 e^{-\delta x}|\nu(x,t)|dx\left|\dot{\hat{\theta}}(t)\right| \\
\le & h_{15}\lambda^{-1}\int_0^1 e^{-\delta x}|\nu(x,t)|^2dx + h_{15}\lambda^{-1}\int_0^1 e^{-\delta x}dx\left|\dot{\hat{\theta}}(t)\right|^2 \\
\le & h_{15}V_1 + \frac{h_{15}}{\lambda\delta}\left(1-e^{-\delta}\right)\left|\dot{\hat{\theta}}(t)\right|^2
\end{aligned}
\tag{4.147}
$$

and

$$
\begin{aligned}
&2\lambda^{-1}\int_0^1 e^{-\delta x}\nu(x,t)\Omega_1(x)\hat{e}(1,t)dx \\
\le & 2\lambda^{-1}\bar{\Omega}_1\int_0^1 e^{-\delta x}\left|\nu(x,t)\right|\left|\hat{e}(1,t)\right|dx \\
\le & \lambda^{-1}\bar{\Omega}_1\int_0^1 e^{-\delta x}\nu^2(x,t)dx + \lambda^{-1}\bar{\Omega}_1\int_0^1 e^{-\delta x}dx\left|\hat{e}(1,t)\right|^2 \\
\le & \bar{\Omega}_1V_1 + \frac{\bar{\Omega}_1}{\lambda\delta}\left(1-e^{-\delta}\right)\left|\hat{e}(1,t)\right|^2
\end{aligned}
\tag{4.148}
$$

for some constants $h_{14}, h_{15} > 0$.

6th term: Separating the cross terms using Young's inequality, inserting the boundary condition (4.117b), Property V and using that Ψ_1 and $\hat{k}$ are bounded give

$$
\begin{aligned}
&+2\lambda^{-1}\int_0^1 e^{-\delta x}\nu(x,t)\Psi_1(x)\hat{k}(t)\left(\hat{\theta}(t)-y_0(t)\right)dx \\
\le & 2\lambda^{-1}\bar{\Psi}_1\int_0^1 e^{-\delta x}\left|\nu(x,t)\right|\left|\hat{k}(t)\right|\left|\hat{\theta}(t)-y_0(t)\right|dx \\
\le & 2\lambda^{-1}\bar{\Psi}_1\int_0^1 e^{-\delta x}\left|\nu(x,t)\right|\left|\hat{k}(t)\right|\left|h_\theta + h_7|\eta(0,t)| + |\varepsilon(t)|\right|dx \\
\le & \left(1+h_\theta^2\right)\lambda^{-1}\bar{\Psi}_1\left|\hat{k}(t)\right|\int_0^1 e^{-\delta x}\nu^2(x,t)dx \\
&+\lambda^{-1}\bar{\Psi}_1\left|\hat{k}(t)\right|\int_0^1 e^{-\delta x}dx\left|h_7|\eta(0,t)| + |\varepsilon(t)|\right|^2 \\
\le & \left(1+h_\theta^2\right)\bar{\Psi}_1\left|\hat{k}(t)\right|V_1 + \frac{\bar{\Psi}_1}{\lambda\delta}\left|\hat{k}(t)\right|\left(1-e^{-\delta}\right)\left|h_7|\eta(0,t)| + |\varepsilon(t)|\right|^2 \\
\le & h_{16}V_1 + \frac{1}{\delta}\left(1-e^{-\delta}\right)h_{17}|\eta(0,t)|^2 + \frac{1}{\delta}\left(1-e^{-\delta}\right)h_{18}|\varepsilon(t)|^2
\end{aligned}
\tag{4.149}
$$

for some constants $h_{16}, h_{17}, h_{18} > 0$.

Combining all the terms yield an expression for the derivative $\dot{V}_1$:

$$
\dot{V}_1 \le 2\eta^2(0,t)q^2(t) + 2\kappa^2(t)\varepsilon^2(t) - \delta\lambda V_1
$$

$$
\begin{aligned}
&+ 2h_1 \left|\dot{\hat{k}}(t)\right| V_1 + h_{11} V_1 + \frac{1}{\delta}\left(1-e^{-\delta}\right) h_{12} V_3 + \frac{1}{\delta}\left(1-e^{-\delta}\right) h_{13} \left|\dot{\hat{k}}(t)\right|^2 \\
&+ h_{14} V_1 + \frac{h_{14}}{\lambda\delta}\left(1-e^{-\delta}\right)\left|\dot{\hat{k}}(t)\right|^2 \\
&+ h_{15} V_1 + \frac{h_{15}}{\lambda\delta}\left(1-e^{-\delta}\right)\left|\dot{\hat{\theta}}(t)\right|^2 \\
&+ \bar{\Omega}_1 V_1 + \frac{\bar{\Omega}_1}{\lambda\delta}\left(1-e^{-\delta}\right)\left|\hat{e}(1,t)\right|^2 \\
&+ h_{16} V_1 + \frac{1}{\delta}\left(1-e^{-\delta}\right) h_{17}\left|\tilde{w}(0,t)\right|^2 \\
\leq & h_{16} V_1 + \frac{1}{\delta}\left(1-e^{-\delta}\right) h_{17}|\eta(0,t)|^2 + \frac{1}{\delta}\left(1-e^{-\delta}\right) h_{18}|\varepsilon(t)|^2 \\
&- \left(\delta\lambda - h_{11} - h_{14} - h_{15} - \bar{\Omega}_1 - h_{16}\right) V_1 \\
&+ 2h_1 \left|\dot{\hat{k}}(t)\right| V_1 \\
&+ \frac{1}{\delta}\left(1-e^{-\delta}\right) h_{12} V_3 \\
&+ \left(\frac{h_{14}}{\lambda\delta}\left(1-e^{-\delta}\right) + \frac{1}{\delta}\left(1-e^{-\delta}\right) h_{13}\right)\left|\dot{\hat{k}}(t)\right|^2 \\
&+ \frac{h_{15}}{\lambda\delta}\left(1-e^{-\delta}\right)\left|\dot{\hat{\theta}}(t)\right|^2 \\
&+ \frac{\bar{\Omega}_1}{\lambda\delta}\left(1-e^{-\delta}\right)\left|\hat{e}(1,t)\right|^2 \\
&+ \frac{1}{\delta}\left(1-e^{-\delta}\right) h_{17}|\eta(0,t)|^2 + \frac{1}{\delta}\left(1-e^{-\delta}\right) h_{18}|\varepsilon(t)|^2
\end{aligned} \tag{4.150}
$$

Derivations regarding V_2

From (4.137b) and inserting the dynamics (4.106b), we get

$$
\begin{aligned}
\dot{V}_2 =& 2\mu^{-1}\int_0^1 e^{\sigma x}\eta(x,t)\eta_t(x,t)dx \\
=& + 2\mu^{-1}\int_0^1 e^{\sigma x}\eta(x,t)\mu\eta_x(x,t)dx \\
&+ 2\mu^{-1}\int_0^1 e^{\sigma x}\eta(x,t)\hat{\theta}(t)H_2(x,t)\dot{\hat{k}}(t)dx \\
&+ 2\mu^{-1}\int_0^1 e^{\sigma x}\eta(x,t)G_2(x,t)\dot{\hat{k}}(t)dx \\
&+ 2\mu^{-1}\int_0^1 e^{\sigma x}\eta(x,t)\hat{k}(t)H_2(x,t)\dot{\hat{\theta}}(t)dx \\
&+ 2\mu^{-1}\int_0^1 e^{\sigma x}\eta(x,t)\Omega_2(x)\hat{e}(1,t)dx
\end{aligned}
$$

$$+ 2\mu^{-1} \int_0^1 e^{\sigma x}\eta(x,t)\Psi_2(x)\hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right) dx \tag{4.151}$$

1st term: Integration by parts and using boundary condition (4.106d) give

$$\begin{aligned}
&2\mu^{-1} \int_0^1 e^{\sigma x}\eta(x,t)\mu\eta_x(x,t)dx \\
=&2\int_0^1 e^{\sigma x}\eta(x,t)\eta_x(x,t)dx \\
=&e^{\sigma}\eta^2(1,t) - \eta^2(0,t) - \sigma\int_0^1 e^{\sigma x}\eta^2(x,t)dx \\
\leq& -\eta^2(0,t) - \sigma\mu V_2.
\end{aligned} \tag{4.152}$$

2nd term: Substituting in the relation (4.116), separating the cross terms using Young's inequality and from Theorem 4.4 using that $\hat{\theta}$ and $\dot{\hat{k}}$ are bounded give

$$\begin{aligned}
&2\mu^{-1}\int_0^1 e^{\sigma x}\eta(x,t)G_2(x,t)\dot{\hat{k}}(t)dx \\
\leq&2\mu^{-1}\int_0^1 e^{\sigma x}|\eta(x,t)||G_2(x,t)|dx\left|\dot{\hat{k}}(t)\right| \\
\leq&2\mu^{-1}h_2\left|\dot{\hat{k}}(t)\right|\int_0^1 e^{\sigma x}|\eta(x,t)|dx||\check{w}(t)|| \\
\leq&2\mu^{-1}h_2\left|\dot{\hat{k}}(t)\right|\int_0^1 e^{\sigma x}|\eta(x,t)|dx||(\tilde{w}+\bar{w})(t)|| \\
\leq&\mu^{-1}h_2\left|\dot{\hat{k}}(t)\right|^2\int_0^1 e^{\sigma x}\eta^2(x,t)dx + \mu^{-1}h_2\int_0^1 e^{\sigma x}dx||\check{w}(t)||^2 \\
&+\mu^{-1}h_2||\bar{w}||\int_0^1 e^{\sigma x}\eta^2(x,t)dx + \mu^{-1}h_2||\bar{w}||\int_0^1 e^{\sigma x}dx\left|\dot{\hat{k}}(t)\right|^2 \\
\leq&h_2\left|\dot{\hat{k}}(t)\right|^2 V_2 + h_2\frac{1}{\sigma}\left(e^{\sigma}-1\right)V_3 + h_2||\bar{w}||V_2 + h_2\frac{||\bar{w}||}{\mu\sigma}\left(e^{\sigma}-1\right)\left|\dot{\hat{k}}(t)\right|^2 \\
\leq&h_2\left|\dot{\hat{k}}(t)\right|^2 V_2 + h_{21}V_2 + \frac{1}{\sigma}\left(e^{\sigma}-1\right)h_{22}V_3 + \frac{1}{\sigma}\left(e^{\sigma}-1\right)h_{23}\left|\dot{\hat{k}}(t)\right|^2
\end{aligned} \tag{4.153}$$

for some constant $h_{21}, h_{22}, h_{23} > 0$.

3rd, 4th and 5th term: Separating the cross terms using Young's inequality and using that $H_2, \Omega_2\hat{\theta}, \hat{k}, \dot{\hat{\theta}}, \dot{\hat{k}}$ are bounded (Lemma 4.10 and Theorem 4.4) give similarly for the 3rd, 4th and 5th term

$$\begin{aligned}
&2\mu^{-1}\int_0^1 e^{\sigma x}\eta(x,t)\hat{\theta}(t)H_2(x,t)\dot{\hat{k}}(t)dx \\
\leq&h_{24}2\mu^{-1}\int_0^1 e^{\sigma x}|\eta(x,t)|dx\left|\dot{\hat{k}}(t)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq h_{24}\mu^{-1}\int_0^1 e^{\sigma x}|\eta(x,t)|^2 dx + h_{24}\mu^{-1}\int_0^1 e^{\sigma x}dx \left|\dot{\hat{k}}(t)\right|^2 \\
&\leq h_{24}V_2 + \frac{h_{24}}{\mu\sigma}\left(e^{\sigma}-1\right)\left|\dot{\hat{k}}(t)\right|^2,
\end{aligned} \tag{4.154}
$$

$$
\begin{aligned}
&2\mu^{-1}\int_0^1 e^{\sigma x}\eta(x,t)\hat{k}(t)H_2(x,t)\dot{\hat{\theta}}(t)dx \\
&\leq h_{25}2\mu^{-1}\int_0^1 e^{\sigma x}|\eta(x,t)|dx\left|\dot{\hat{\theta}}(t)\right| \\
&\leq h_{25}\mu^{-1}\int_0^1 e^{\sigma x}|\eta(x,t)|^2 dx + h_{25}\mu^{-1}\int_0^1 e^{\sigma x}dx\left|\dot{\hat{\theta}}(t)\right|^2 \\
&\leq h_{25}V_2 + \frac{h_{25}}{\mu\sigma}\left(e^{\sigma}-1\right)\left|\dot{\hat{\theta}}(t)\right|^2
\end{aligned} \tag{4.155}
$$

and

$$
\begin{aligned}
&2\mu^{-1}\int_0^1 e^{\sigma x}\eta(x,t)\Omega_2(x)\hat{e}(1,t)dx \\
&\leq 2\mu^{-1}\bar{\Omega}_2\int_0^1 e^{\sigma x}\left|\eta(x,t)\right|\left|\hat{e}(1,t)\right|dx \\
&\leq \mu^{-1}\bar{\Omega}_2\int_0^1 e^{\sigma x}\eta^2(x,t)dx + \mu^{-1}\bar{\Omega}_2\int_0^1 e^{\sigma x}dx\left|\hat{e}(1,t)\right|^2 \\
&\leq \bar{\Omega}_2 V_2 + \frac{\bar{\Omega}_2}{\mu\sigma}\left(e^{\sigma}-1\right)\left|\hat{e}(1,t)\right|^2.
\end{aligned} \tag{4.156}
$$

for some constants $h_{24}, h_{25} > 0$.

6th term: Separating the cross terms using Young's inequality, inserting the boundary condition (4.117b) and using that Ψ_2 and $\hat{k}$ are bounded give

$$
\begin{aligned}
&+2\mu^{-1}\int_0^1 e^{\sigma x}\eta(x,t)\Psi_2(x)\hat{k}(t)\left(\hat{\theta}(t)-y_0(t)\right)dx \\
&\leq 2\mu^{-1}\bar{\Psi}_2\int_0^1 e^{\sigma x}\left|\eta(x,t)\right|\left|\hat{k}(t)\right|\left|\hat{\theta}(t)-y_0(t)\right|dx \\
&\leq 2\mu^{-1}\bar{\Psi}_2\int_0^1 e^{\sigma x}\left|\eta(x,t)\right|\left|\hat{k}(t)\right|\left|h_\theta + h_7|\eta(0,t)| + |\varepsilon(t)|\right|dx \\
&\leq \left(1+h_\theta^2\right)\mu^{-1}\bar{\Psi}_2\left|\hat{k}(t)\right|\int_0^1 e^{\sigma x}\eta^2(x,t)dx \\
&\quad + \mu^{-1}\bar{\Psi}_2\left|\hat{k}(t)\right|\int_0^1 e^{\sigma x}dx\left|h_7|\eta(0,t)| + |\varepsilon(t)|\right|^2 \\
&\leq \left(1+h_\theta^2\right)\bar{\Psi}_2\left|\hat{k}(t)\right|V_2 + \frac{\bar{\Psi}_2}{\mu\sigma}\left|\hat{k}(t)\right|\left(e^{\sigma}-1\right)\left|h_7|\eta(0,t)| + |\varepsilon(t)|\right|^2
\end{aligned}
$$

$$\leq h_{26} V_2 + \frac{1}{\sigma}\left(e^{\sigma}-1\right) h_{27} \left|\eta(0,t)|\right|^2 + \frac{1}{\sigma}\left(e^{\sigma}-1\right) h_{28} \left|\varepsilon(t)\right|^2. \tag{4.157}$$

for some constants $h_{26}, h_{27}, h_{28} > 0$.

Combining all the terms yields an expression for the derivative $\dot{V}_2$:

$$\begin{aligned}
\dot{V}_2 \leq & -\eta^2(0,t) - \sigma\mu V_2 \\
& + h_2 \left|\dot{\hat{k}}(t)\right| V_2 + h_{21} V_2 + \frac{1}{\sigma}\left(e^{\sigma}-1\right) h_{22} V_3 + \frac{1}{\sigma}\left(e^{\sigma}-1\right) h_{23} \left|\dot{\hat{k}}(t)\right|^2 \\
& + h_{24} V_2 + \frac{h_{24}}{\mu\sigma}\left(e^{\sigma}-1\right) \left|\dot{\hat{k}}(t)\right|^2 \\
& + h_{25} V_2 + \frac{h_{25}}{\mu\sigma}\left(e^{\sigma}-1\right) \left|\dot{\hat{\theta}}(t)\right|^2 \\
& + \bar{\Omega}_2 V_2 + \frac{\bar{\Omega}_2}{\mu\sigma}\left(e^{\sigma}-1\right) \left|\hat{e}(1,t)\right|^2 \\
& + h_{26} V_2 + \frac{1}{\sigma}\left(e^{\sigma}-1\right) h_{27} \left|\eta(0,t)|\right|^2 + \frac{1}{\sigma}\left(e^{\sigma}-1\right) h_{28} \left|\varepsilon(t)\right|^2 \\
\leq & -\eta^2(0,t) \\
& - \left(\sigma\mu - h_{24} - h_6 - h_{25} - \bar{\Omega}_2 - h_{26}\right) V_2 \\
& + h_2 \left|\dot{\hat{k}}(t)\right| V_2 \\
& + \frac{1}{\sigma}\left(e^{\sigma}-1\right) h_{22} V_3 \\
& + \left(\frac{h_{24}}{\mu\sigma}\left(e^{\sigma}-1\right) + \frac{1}{\sigma}\left(e^{\sigma}-1\right) h_{23}\right) \left|\dot{\hat{k}}(t)\right|^2 \\
& + \frac{h_{25}}{\mu\sigma}\left(e^{\sigma}-1\right) \left|\dot{\hat{\theta}}(t)\right|^2 \\
& + \frac{\bar{\Omega}_2}{\mu\sigma}\left(e^{\sigma}-1\right) \left|\hat{e}(1,t)\right|^2 \\
& + \frac{1}{\sigma}\left(e^{\sigma}-1\right) h_{27} \left|\eta(0,t)|\right|^2 + \frac{1}{\sigma}\left(e^{\sigma}-1\right) h_{28} \left|\varepsilon(t)\right|^2.
\end{aligned} \tag{4.158}$$

Derivations regarding V_3

Differentiating (4.137c), inserting the dynamics (4.117), integration by parts and using Property V in Lemma 4.10 yield

$$\begin{aligned}
\dot{V}_3 = & 2\lambda^{-1} \int_0^1 e^{-\pi x} \tilde{w}(x,t) \tilde{w}_t(x,t) dx \\
= & -2 \int_0^1 e^{-\pi x} \tilde{w}(x,t) \tilde{w}_x(x,t) dx \\
= & -e^{-\pi} \tilde{w}^2(1,t) + \tilde{w}^2(0,t) - \pi \int_0^1 \tilde{w}^2(x,t) dx \\
\leq & -e^{-\pi} \tilde{w}^2(1,t) + 2h_7 \eta^2(0,t) + 2\varepsilon^2(t) - \pi\lambda V_3
\end{aligned} \tag{4.159}$$

Derivations regarding V_4

From (4.137d) and inserting the dynamics (4.110a), we get

$$\begin{aligned}\dot{V}_4 =& 2\lambda^{-1}\int_0^1 e^{-\delta x}\hat{\alpha}(x,t)\hat{\alpha}_t(x,t)dx \\ =& -2\int_0^1 e^{-\delta x}\hat{\alpha}(x,t)\hat{\alpha}_x(x,t)dx \\ & + 2\lambda^{-1}\int_0^1 e^{-\delta x}\hat{\alpha}(x,t)B_1(x,t)dx \end{aligned} \tag{4.160}$$

1st term: Integration by parts and inserting boundary condition (4.110c) give

$$\begin{aligned} & -2\int_0^1 e^{-\delta x}\hat{\alpha}(x,t)\hat{\alpha}_x(x,t)dx \\ =& -e^{-\delta}\hat{\alpha}^2(1,t)+\hat{\alpha}^2(0,t)-\delta\int_0^1 e^{-\delta x}\hat{\alpha}^2(x,t)dx \\ \leq & 2r^2\hat{\beta}^2(0,t)+2\left(\tilde{k}(t)(\theta-y_0(t))+\hat{k}(t)\tilde{\theta}(t)\right)^2-\delta V_4 \end{aligned} \tag{4.161}$$

2nd term: Separating the cross terms using Young's inequality and using Property VI of Lemma 4.10 give

$$\begin{aligned} & 2\lambda^{-1}\int_0^1 e^{-\delta x}\hat{\alpha}(x,t)B_1(x,t)dx \\ \leq & 2\lambda^{-1}\int_0^1 e^{-\delta x}|\hat{\alpha}(x,t)||B_1(x,t)|dx \\ \leq & \lambda^{-1}\int_0^1 e^{-\delta x}\hat{\alpha}^2(x,t)dx+\lambda^{-1}\int_0^1 |B_1(x,t)|^2dx \\ \leq & V_4+\lambda^{-1}||B_1(t)||^2 \\ \leq & V_4+\lambda^{-1}h_8\left|\dot{\hat{k}}(t)\right|^2+\lambda^{-1}h_9\left|\dot{\hat{\theta}}(t)\right|^2+\lambda^{-1}h_{10}\left|\dot{\hat{k}}(t)\right|||\tilde{w}||^2 \\ \leq & V_4+\lambda^{-1}h_8\left|\dot{\hat{k}}(t)\right|^2+\lambda^{-1}h_9\left|\dot{\hat{\theta}}(t)\right|^2+e^{\pi}\left|\dot{\hat{k}}(t)\right|^2V_3 \end{aligned} \tag{4.162}$$

Combining the two terms yields and expression for $\dot{V}_4$

$$\begin{aligned}\dot{V}_4 \leq & 2r^2\hat{\beta}^2(0,t)+2\left(\tilde{k}(t)(\theta-y_0(t))+\hat{k}(t)\tilde{\theta}(t)\right)^2-\delta V_4 \\ & + V_4+\lambda^{-1}h_8\left|\dot{\hat{k}}(t)\right|^2+\lambda^{-1}h_9\left|\dot{\hat{\theta}}(t)\right|^2+e^{\pi}\left|\dot{\hat{k}}(t)\right|^2V_3 \\ \leq & -(\delta-1)V_4+e^{\pi}\left|\dot{\hat{k}}(t)\right|^2V_3+2r^2\hat{\beta}^2(0,t) \\ & + 2\left(\tilde{k}(t)(\theta-y_0(t))+\hat{k}(t)\tilde{\theta}(t)\right)^2+\lambda^{-1}h_8\left|\dot{\hat{k}}(t)\right|^2+\lambda^{-1}h_9\left|\dot{\hat{\theta}}(t)\right|^2. \end{aligned} \tag{4.163}$$

Derivations regarding V_5

From (4.137e) and inserting the dynamics (4.110b), we get

$$\begin{aligned}\dot{V}_5 =& 2\mu^{-1}\int_0^1 e^{\sigma x}\hat{\beta}(x,t)\hat{\beta}_t(x,t)dx\\ =& 2\int_0^1 e^{\sigma x}\hat{\beta}(x,t)\hat{\beta}_x(x,t)dx\\ &+ 2\mu^{-1}\int_0^1 e^{\sigma x}\hat{\beta}(x,t)B_2(x,t)dx\end{aligned} \tag{4.164}$$

1st term: Integration by parts and inserting boundary condition (4.110d) give

$$\begin{aligned}&2\int_0^1 e^{\sigma x}\hat{\beta}(x,t)\hat{\beta}_x(x,t)dx\\ =& e^{\sigma}\hat{\beta}^2(1,t) - \hat{\beta}^2(0,t) - \sigma\int_0^1 e^{\sigma x}\hat{\beta}^2(x,t)dx\\ =& -\hat{\beta}^2(0,t) - \sigma V_5\end{aligned} \tag{4.165}$$

2nd term: Separating the cross terms using Young's inequality and using Property VI of Lemma 4.10 give

$$\begin{aligned}&2\mu^{-1}\int_0^1 e^{\sigma x}\hat{\beta}(x,t)B_2(x,t)dx\\ \le& 2\mu^{-1}\int_0^1 e^{\sigma x}|\hat{\beta}(x,t)||B_2(x,t)|dx\\ \le& \mu^{-1}\int_0^1 e^{\sigma x}\hat{\beta}^2(x,t)dx + \mu^{-1}e^{\sigma}\int_0^1 |B_2(x,t)|^2dx\\ \le& V_5 + \mu^{-1}e^{\sigma}||B_2(t)||^2\\ \le& V_5 + \mu^{-1}e^{\sigma}h_8\left|\dot{\hat{k}}(t)\right|^2 + \mu^{-1}e^{\sigma}h_9\left|\dot{\hat{\theta}}(t)\right|^2 + \mu^{-1}e^{\sigma}h_{10}\left|\dot{\hat{k}}(t)\right|||\tilde{w}||^2\\ \le& V_5 + \mu^{-1}e^{\sigma}h_8\left|\dot{\hat{k}}(t)\right|^2 + \mu^{-1}e^{\sigma}h_9\left|\dot{\hat{\theta}}(t)\right|^2 + e^{\sigma+\pi}\left|\dot{\hat{k}}(t)\right|^2 V_3\end{aligned} \tag{4.166}$$

Combining the two terms yield and expression for $\dot{V}_4$

$$\begin{aligned}\dot{V}_5 \le& -\hat{\beta}^2(0,t) - \sigma V_5 + V_5 + \mu^{-1}e^{\sigma}h_8\left|\dot{\hat{k}}(t)\right|^2 + \mu^{-1}e^{\sigma}h_9\left|\dot{\hat{\theta}}(t)\right|^2 + e^{\sigma+\pi}\left|\dot{\hat{k}}(t)\right|^2 V_3\\ \le& -\hat{\beta}^2(0,t) - (\sigma-1)V_5 + e^{\sigma+\pi}\left|\dot{\hat{k}}(t)\right|^2 V_3\\ &+ \mu^{-1}e^{\sigma}h_8\left|\dot{\hat{k}}(t)\right|^2 + \mu^{-1}e^{\sigma}h_9\left|\dot{\hat{\theta}}(t)\right|^2.\end{aligned} \tag{4.167}$$

Derivations regarding V_6

Having calculated the derivative of all the terms in (4.137), we are ready to prove Lemma 4.11.

Proof of Lemma 4.11. Combining (4.150), (4.158), (4.159), (4.163) and (4.167) yield

$$
\begin{aligned}
\dot{V}_6 \leq & a_1 2\eta^2(0,t)q^2(t) + a_1 2\kappa^2(t)\varepsilon^2(t) \\
& - a_1\left(\delta\lambda - h_{11} - h_{14} - h_{15} - \bar{\Omega}_1 - h_{16}\right)V_1 \\
& + a_1 2h_1\left|\dot{\hat{k}}(t)\right|^2 V_1 \\
& + a_1\frac{1}{\delta}\left(1-e^{-\delta}\right)h_{12}V_3 \\
& + a_1\left(\frac{h_{14}}{\lambda\delta}\left(1-e^{-\delta}\right) + \frac{1}{\delta}\left(1-e^{-\delta}\right)h_{13}\right)\left|\dot{\hat{k}}(t)\right|^2 \\
& + a_1\frac{h_{15}}{\lambda\delta}\left(1-e^{-\delta}\right)\left|\dot{\hat{\theta}}(t)\right|^2 \\
& + a_1\frac{\bar{\Omega}_1}{\lambda\delta}\left(1-e^{-\delta}\right)|\hat{e}(1,t)|^2 \\
& + \frac{1}{\delta}\left(1-e^{-\delta}\right)h_{17}|\eta(0,t)|^2 + \frac{1}{\delta}\left(1-e^{-\delta}\right)h_{18}|\varepsilon(t)|^2 \\
& - a_2\eta^2(0,t) \\
& - a_2\left(\sigma\mu - h_{24} - h_6 - h_{25} - \bar{\Omega}_2 - h_{26}\right)V_2 \\
& + a_2 h_2\left|\dot{\hat{k}}(t)\right|^2 V_2 \\
& + a_2\frac{1}{\sigma}\left(e^{\sigma}-1\right)h_{22}V_3 \\
& + a_2\left(\frac{h_{24}}{\mu\sigma}\left(e^{\sigma}-1\right) + \frac{1}{\sigma}\left(e^{\sigma}-1\right)h_{23}\right)\left|\dot{\hat{k}}(t)\right|^2 \\
& + a_2\frac{h_{25}}{\mu\sigma}\left(e^{\sigma}-1\right)\left|\dot{\hat{\theta}}(t)\right|^2 \\
& + a_2\frac{\bar{\Omega}_2}{\mu\sigma}\left(e^{\sigma}-1\right)|\hat{e}(1,t)|^2 \\
& + \frac{1}{\sigma}\left(e^{\sigma}-1\right)h_{27}\left|\eta(0,t)|\right|^2 + \frac{1}{\sigma}\left(e^{\sigma}-1\right)h_{28}\left|\varepsilon(t)\right|^2 \\
& - e^{-\pi}\tilde{w}^2(1,t) + a_3 h_7\eta^2(0,t) + a_3\varepsilon^2(t) - a_3\pi\lambda V_3 \\
& - a_4(\delta-1)V_4 + a_4 e^{\pi}\left|\dot{\hat{k}}(t)\right|^2 V_3 + a_4 2r^2\hat{\beta}^2(0,t) \\
& + a_4 2\left(\tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t)\tilde{\theta}(t)\right)^2 + a_4\lambda^{-1}h_8\left|\dot{\hat{k}}(t)\right|^2 \\
& + a_4\lambda^{-1}h_9\left|\dot{\hat{\theta}}(t)\right|^2
\end{aligned}
$$

$$
- \hat{\beta}^2(0,t) - a_5 (\sigma - 1) V_5 + a_5 e^{\sigma+\pi} \left|\dot{\hat{k}}(t)\right|^2 V_3 + a_5 \mu^{-1} e^{\sigma} h_8 \left|\dot{\hat{k}}(t)\right|^2
+ a_5 \mu^{-1} e^{\sigma} h_9 \left|\dot{\hat{\theta}}(t)\right|^2 . \tag{4.168}
$$

Reorganizing the terms and using Property IV from Lemma 4.10 give

$$
\begin{aligned}
\dot{V}_6 \leq & - \left(a_2 - a_1 2 q^2(t) - a_3 h_7 - h_7 a_1 \frac{1}{\delta} \left(1 - e^{-\delta}\right) h_{17} \right) \eta^2(0,t) \\
& - \left(a_5 - a_4 2 r^2 - h_3 \left(a_1 2 \kappa^2(t) + a_3 \right) - a_1 \frac{1}{\delta} \left(1 - e^{-\delta}\right) h_{17} \right) \hat{\beta}^2(0,t) \\
& - a_1 \left(\delta\lambda - h_{11} - h_{14} - h_{15} - \bar{\Omega}_1 - h_{16} \right) V_1 \\
& - a_2 \left(\sigma\mu - h_{24} - h_6 - h_{25} - \bar{\Omega}_2 - h_{26} \right) V_2 \\
& - \left(a_3 \pi\lambda - a_1 \frac{1}{\delta} \left(1 - e^{-\delta}\right) h_{12} - a_2 \frac{1}{\sigma} \left(e^{\sigma} - 1\right) h_{22} \right) V_3 \\
& - \left(a_4 (\delta - 1) - h_4 e^{\delta} \left(a_1 2 \kappa^2(t) + a_3 + a_1 \frac{1}{\delta} \left(1 - e^{-\delta}\right) h_{17} \right) \right) V_4 \\
& - \left(a_5 (\sigma - 1) - h_5 \left(a_1 2 \kappa^2(t) + a_3 + a_1 \frac{1}{\delta} \left(1 - e^{-\delta}\right) h_{17} \right) \right) V_5 \\
& + a_1 2 h_1 \left|\dot{\hat{k}}(t)\right|^2 V_1 + a_2 h_2 \left|\dot{\hat{k}}(t)\right|^2 V_2 + a_4 e^{\pi} \left|\dot{\hat{k}}(t)\right|^2 V_3 + a_5 e^{\sigma+\pi} \left|\dot{\hat{k}}(t)\right|^2 V_3 \\
& - e^{-\pi} \tilde{w}^2(1,t) \\
& + \left(2 a_4 + h_6 \left(a_1 2 \kappa^2(t) + a_3 + a_1 \frac{1}{\delta} \left(1 - e^{-\delta}\right) h_{17} \right) \right) \\
& \qquad \times \left(\tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t) \tilde{\theta}(t) \right)^2 \\
& + a_4 \lambda^{-1} h_8 \left|\dot{\hat{k}}(t)\right|^2 + a_4 \lambda^{-1} h_9 \left|\dot{\hat{\theta}}(t)\right|^2 \\
& + a_1 \left(\frac{h_{14}}{\lambda\delta} \left(1 - e^{-\delta}\right) + \frac{1}{\delta} \left(1 - e^{-\delta}\right) h_{13} \right) \left|\dot{\hat{k}}(t)\right|^2 \\
& + a_1 \frac{h_{15}}{\lambda\delta} \left(1 - e^{-\delta}\right) \left|\dot{\hat{\theta}}(t)\right|^2 \\
& + a_1 \frac{\bar{\Omega}_1}{\lambda\delta} \left(1 - e^{-\delta}\right) \left|\hat{e}(1,t)\right|^2 \\
& + a_2 \left(\frac{h_{24}}{\mu\sigma} \left(e^{\sigma} - 1\right) + \frac{1}{\sigma} \left(e^{\sigma} - 1\right) h_{23} \right) \left|\dot{\hat{k}}(t)\right|^2 \\
& + a_2 \frac{h_{25}}{\mu\sigma} \left(e^{\sigma} - 1\right) \left|\dot{\hat{\theta}}(t)\right|^2 \\
& + a_2 \frac{\bar{\Omega}_2}{\mu\sigma} \left(e^{\sigma} - 1\right) \left|\hat{e}(1,t)\right|^2 \\
& + a_5 \mu^{-1} e^{\sigma} h_8 \left|\dot{\hat{k}}(t)\right|^2 + a_5 \mu^{-1} e^{\sigma} h_9 \left|\dot{\hat{\theta}}(t)\right|^2 .
\end{aligned} \tag{4.169}
$$

Let

$$a_{51} =a_4 2r^2 + h_3 \left(a_1 2\kappa^2(t) + a_3\right) + a_1 \frac{1}{\delta}\left(1 - e^{-\delta}\right) h_{17} \tag{4.170a}$$

$$a_{52} =\frac{1}{\sigma_1}\left(h_5 \left(a_1 2\kappa^2(t) + a_3 + a_1 \frac{1}{\delta}\left(1 - e^{-\delta}\right) h_{17}\right)\right) \tag{4.170b}$$

$$h_{67} =\left(2a_4 + h_6 \left(a_1 2\kappa^2(t) + a_3 + a_1 \frac{1}{\delta}\left(1 - e^{-\delta}\right) h_{17}\right)\right). \tag{4.170c}$$

Selecting

$$\delta \geq \max\left(\frac{1}{\lambda}\left(h_{11} + h_{14} + h_{15} + \bar{\Omega}_1 + h_{16}\right), 1\right) \tag{4.171a}$$

$$\sigma \geq \max\left(\frac{1}{\mu}\left(h_{24} + h_6 + h_{25} + \bar{\Omega}_2 + h_{26}\right), 1\right) \tag{4.171b}$$

$$a_1 =1 \tag{4.171c}$$

$$a_3 =1 \tag{4.171d}$$

$$a_2 \geq a_1 2q^2(t) + a_3 h_7 + h_7 a_1 \frac{1}{\delta}\left(1 - e^{-\delta}\right) h_{17} \tag{4.171e}$$

$$a_4 \geq \frac{1}{\delta - 1}\left(h_4 e^{\delta}\left(a_1 2\kappa^2(t) + a_3 + a_1 \frac{1}{\delta}\left(1 - e^{-\delta}\right) h_{17}\right)\right) \tag{4.171f}$$

$$a_5 \geq \max(a_{51}, a_{52}) \tag{4.171g}$$

$$\pi \geq \frac{1}{a_3 \lambda}\left(a_1 \frac{1}{\delta}\left(1 - e^{-\delta}\right) h_{12} + a_2 \frac{1}{\sigma}\left(e^{\sigma} - 1\right) h_{22}\right) \tag{4.171h}$$

yield

$$\begin{aligned}
\dot{V}_6 \leq & - h_{61} V_1 - h_{62} V_2 - h_{63} V_3 - h_{64} V_4 - h_{65}) V_5 \\
& + a_1 2h_1 \left|\dot{\hat{k}}(t)\right|^2 V_1 + a_2 h_2 \left|\dot{\hat{k}}(t)\right|^2 V_2 + a_4 e^{\pi} \left|\dot{\hat{k}}(t)\right|^2 V_3 + a_5 e^{\sigma+\pi} \left|\dot{\hat{k}}(t)\right|^2 V_3 \\
& - e^{-\pi} \tilde{w}^2(1,t) \\
& + h_{67}\left(\tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t)\tilde{\theta}(t)\right)^2 \\
& + a_4 \lambda^{-1} h_8 \left|\dot{\hat{k}}(t)\right|^2 + a_4 \lambda^{-1} h_9 \left|\dot{\hat{\theta}}(t)\right|^2 \\
& + a_1 \left(\frac{h_{14}}{\lambda\delta}\left(1 - e^{-\delta}\right) + \frac{1}{\delta}\left(1 - e^{-\delta}\right) h_{13}\right) \left|\dot{\hat{k}}(t)\right|^2 \\
& + a_1 \frac{h_{15}}{\lambda\delta}\left(1 - e^{-\delta}\right) \left|\dot{\hat{\theta}}(t)\right|^2 \\
& + a_1 \frac{\bar{\Omega}_1}{\lambda\delta}\left(1 - e^{-\delta}\right) \left|\hat{e}(1,t)\right|^2 \\
& + a_2 \left(\frac{h_{24}}{\mu\sigma}\left(e^{\sigma} - 1\right) + \frac{1}{\sigma}\left(e^{\sigma} - 1\right) h_{23}\right) \left|\dot{\hat{k}}(t)\right|^2 \\
& + a_2 \frac{h_{25}}{\mu\sigma}\left(e^{\sigma} - 1\right) \left|\dot{\hat{\theta}}(t)\right|^2
\end{aligned}$$

$$
\begin{aligned}
&+ a_2 \frac{\bar{\Omega}_2}{\mu\sigma} \left(e^{\sigma} - 1\right) |\hat{e}(1,t)|^2 \\
&+ a_5 \mu^{-1} e^{\sigma} h_8 \left|\dot{\hat{k}}(t)\right|^2 + a_5 \mu^{-1} e^{\sigma} h_9 \left|\dot{\hat{\theta}}(t)\right|^2 \\
\leq &- h_{66} V_6 \\
&+ \left(a_1 2 h_1 \left|\dot{\hat{k}}(t)\right|^2 + a_2 h_2 \left|\dot{\hat{k}}(t)\right|^2 + a_4 e^{\pi} \left|\dot{\hat{k}}(t)\right|^2 + a_5 e^{\sigma+\pi} \left|\dot{\hat{k}}(t)\right|^2 \right) V_6 \\
&- e^{-\pi} \tilde{w}^2(1,t) \\
&+ h_{67} \left(\tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t)\tilde{\theta}(t) \right)^2 \\
&+ a_4 \lambda^{-1} h_8 \left|\dot{\hat{k}}(t)\right|^2 + a_4 \lambda^{-1} h_9 \left|\dot{\hat{\theta}}(t)\right|^2 \\
&+ a_1 \left(\frac{h_{14}}{\lambda\delta} \left(1 - e^{-\delta}\right) + \frac{1}{\delta} \left(1 - e^{-\delta}\right) h_{13} \right) \left|\dot{\hat{k}}(t)\right|^2 \\
&+ a_1 \frac{h_{15}}{\lambda\delta} \left(1 - e^{-\delta}\right) \left|\dot{\hat{\theta}}(t)\right|^2 \\
&+ a_1 \frac{\bar{\Omega}_1}{\lambda\delta} \left(1 - e^{-\delta}\right) |\hat{e}(1,t)|^2 \\
&+ a_2 \left(\frac{h_{24}}{\mu\sigma} \left(e^{\sigma} - 1\right) + \frac{1}{\sigma} \left(e^{\sigma} - 1\right) h_{23} \right) \left|\dot{\hat{k}}(t)\right|^2 \\
&+ a_2 \frac{h_{25}}{\mu\sigma} \left(e^{\sigma} - 1\right) \left|\dot{\hat{\theta}}(t)\right|^2 \\
&+ a_2 \frac{\bar{\Omega}_2}{\mu\sigma} \left(e^{\sigma} - 1\right) |\hat{e}(1,t)|^2 \\
&+ a_5 \mu^{-1} e^{\sigma} h_8 \left|\dot{\hat{k}}(t)\right|^2 + a_5 \mu^{-1} e^{\sigma} h_9 \left|\dot{\hat{\theta}}(t)\right|^2 .
\end{aligned} \tag{4.172}
$$

Using (4.139), this expression can be simplified to

$$
\begin{aligned}
\dot{V}_6 \leq &- h_{66} V_6 + l_1(t) V_6 + l_2(t) \\
&- h_{67} \frac{\left(\tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t)\tilde{\theta}(t) \right)^2}{1 + w^2(1,t)} \left(1 + 2\bar{w}^2(1,t)\right) \\
&- e^{-\pi} \tilde{w}^2(1,t) + h_{67} \left(\tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t)\tilde{\theta}(t) \right)^2 .
\end{aligned} \tag{4.173}
$$

The last term can be written on the form considered in Property IV in Theorem 4.4 by dividing and multiplying by $\left(1 + w^2(1,t)\right)$ to yield

$$
\begin{aligned}
\dot{V}_6 \leq &- h_{66} V_6 + l_1(t) V_6 + l_2(t) - e^{-\pi} \tilde{w}^2(1,t) \\
&+ h_{67} \frac{\left(\tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t)\tilde{\theta}(t) \right)^2}{1 + w^2(1,t)} \left(1 + w^2(1,t)\right)
\end{aligned}
$$

$$
\begin{aligned}
&- h_{67}\frac{\left(\tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t)\tilde{\theta}(t)\right)^2}{1 + w^2(1,t)}\left(1 + 2\bar{w}^2(1,t)\right) \\
\leq &- h_{66}V_6 + l_1(t)V_6 + l_2(t) - e^{-\pi}\tilde{w}^2(1,t) \\
&+ h_{67}\frac{\left(\tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t)\tilde{\theta}(t)\right)^2}{1 + w^2(1,t)}\left(1 + \check{w}^2(1,t)\right) \\
\leq &- h_{66}V_6 + l_1(t)V_6 + l_2(t) - e^{-\pi}\tilde{w}^2(1,t) \\
&+ h_{67}\frac{\left(\tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t)\tilde{\theta}(t)\right)^2}{1 + w^2(1,t)}\left(1 + 2\bar{w}^2(1,t) - 1 - 2\bar{w}^2(1,t) + 2\tilde{w}^2(1,t)\right) \\
\leq &- h_{66}V_6 + l_1(t)V_6 + l_2(t) \\
&- e^{-\pi}\left(1 - 2e^{\pi}h_{67}\frac{\left(\tilde{k}(t)(\theta - y_0(t)) + \hat{k}(t)\tilde{\theta}(t)\right)^2}{1 + w^2(1,t)}\right)\tilde{w}^2(1,t)
\end{aligned} \tag{4.174}
$$

which is equal to (4.138).

Boundedness and integrability of l_1 and l_2 follow from Property I, II, IV and boundedness of (4.115). From (4.63) we have that

$$\dot{V}_0 \leq -h_{68}\left(\tilde{\Theta}^T(t)\Psi(t + d_\alpha)\right)^2 \tag{4.175}$$

where Ψ, Θ are defined in (4.56), $h_{68} > 0$ a constant, and with $h_{68}\Psi(t)\Psi^T(t) \leq I_{2\times 2}$. Lemma 8 from Anfinsen and Aamo (2017c) can then be applied, yielding (4.141). For reference, Anfinsen and Aamo (2017c, Lemma 8) is included in Appendix A.5 as Lemma A.5.

From (4.141) it follows that

$$||\nu||, ||\eta||, ||\tilde{w}||, ||\hat{\alpha}||, ||\hat{\beta}|| \in \mathcal{L}_\infty. \tag{4.176}$$

and from the invertibility of the transforms (4.87), (4.104) and (4.109) that

$$||\hat{u}||, ||\hat{v}||, ||u||, ||v|| \in \mathcal{L}_\infty \tag{4.177}$$

which completes the proof. □

4.4.3 Boundedness Point-wise in Space

Lemma 4.12. *Consider the system* (4.1) *with state estimates generated by* (4.38) *and the adaptive law* (4.41). *If the control signal $U(t)$ is selected according to* (4.84), *then the states u, v will be bounded point wise in space, that is*

$$u(x,\cdot), v(x,\cdot) \in \mathcal{L}_\infty, \quad \forall x \in [0,1]. \tag{4.178}$$

Proof. Using the same backstepping transformation as in the proof of Lemma 3.11, the proof is similar and therefore omitted. □

4.4.4 Convergence in $L_2([0,1])$

To prove convergence in $L_2([0,1])$, Lemma 3.1 from Liu and Krstić (2001) will be utilized. For reference, the lemma is restated in Appendix A.7 as Lemma A.7.

Lemma 4.13. *Consider the transformed system* (4.106) *in Lemma 4.8, the filer system* (4.117) *and the transformed error system* (4.110) *in Lemma 4.9. If* $\hat{k}, \hat{\theta}$ *are generated using* (4.41) *in Theorem 4.4, then* $\nu, \eta, \tilde{w}, \hat{\alpha}, \hat{\beta}$ *converge to zero in* $L_2([0,1])$*, that is*

$$||\nu||, ||\eta||, ||\tilde{w}||, ||\hat{\alpha}||, ||\hat{\beta}|| \to 0. \tag{4.179}$$

Proof. By design, system (4.106) is obtained using the control law (4.84). Hence, all assumptions in Lemma 4.12 hold and the sates u, v are bounded point-wise in space. From the definition (4.5) it follows that y_0 is bounded and from (4.117) and Property I in Theorem 4.4 that $\tilde{w}(1,t)$ is bounded. Now, since $V_3, l_1, l_2 \in \mathcal{L}_\infty$ from Lemma 4.11, the right hand side of (4.138) is bounded from above and there exists a constant M such that $\dot{V}_3 \leq M$. This result, together with $V_3 \in \mathcal{L}_1 \cap \mathcal{L}_\infty$ from Lemma 4.11 gives, by Lemma A.7,

$$V_3 \to 0 \tag{4.180}$$

and (4.179) follows. □

4.4.5 Proof of Theorem 4.6

Proof of Theorem 4.6. Inserting (4.88) and (4.105) from Lemma 4.7 and 4.8 respectively into (4.86), together with the operator (4.80b) and ζ^* selected as

$$\zeta^*(t) = \frac{1}{a_0 r + b_0}\hat{\theta}(t) \tag{4.181}$$

give (4.84). Boundedness of all signals in the closed loop system then follows from Lemma 4.11 and 4.12 and Theorem 4.4.

Consider the Lyapunov function candidate

$$V_7 = ||\eta||^2 = \int_0^1 \eta^2(x,t)dx. \tag{4.182}$$

Differentiating with respect to time

$$\begin{aligned}
\dot{V}_7 =& 2\int_0^1 \eta(x,t)\eta_t(x,t)dx \\
=& -\mu\eta^2(0,t) \\
&+ 2\int_0^1 \eta(x,t)\hat{\theta}(t)H_2(x,t)\dot{\hat{k}}(t)dx \\
&+ 2\int_0^1 \eta(x,t)G_2(x,t)\dot{\hat{k}}(t)dx
\end{aligned}$$

$$
\begin{aligned}
&+ 2\int_0^1 \eta(x,t)\hat{k}(t)H_2(x,t)\dot{\hat{\theta}}(t)dx \\
&+ 2\int_0^1 \eta(x,t)\Omega_2(x)\hat{e}(1,t)dx \\
&+ 2\int_0^1 \eta(x,t)\Psi_2(x)\hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right)dx
\end{aligned} \tag{4.183}
$$

and then integrating from t to $t+T$ gives

$$
\begin{aligned}
\int_t^{t+T} \dot{V}_7 d\tau =& V_7(t+T) - V_7(t) \\
=& -\mu \int_t^{t+T} \eta^2(0,\tau)d\tau \\
&+ 2\int_t^{t+T}\int_0^1 \eta(x,t)\hat{\theta}(t)H_2(x,t)\dot{\hat{k}}(t)dx\,d\tau \\
&+ 2\int_t^{t+T}\int_0^1 \eta(x,t)G_2(x,t)\dot{\hat{k}}(t)dx\,d\tau \\
&+ 2\int_t^{t+T}\int_0^1 \eta(x,t)\hat{k}(t)H_2(x,t)\dot{\hat{\theta}}(t)dx\,d\tau \\
&+ 2\int_t^{t+T}\int_0^1 \eta(x,t)\Omega_2(x)\hat{e}(1,t)dx\,d\tau \\
&+ 2\int_t^{t+T}\int_0^1 \eta(x,t)\Psi_2(x)\hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right)dx\,d\tau.
\end{aligned} \tag{4.184}
$$

Rearranging the terms and applying Cauchy-Schwarz' inequality yield

$$
\begin{aligned}
&V_7(t+T) - V_7(t) + \mu\int_t^{t+T} \eta^2(0,\tau)d\tau \\
&\le + 2\int_t^{t+T} \sqrt{\int_0^1 |\eta(x,t)|^2 dx}\sqrt{\int_t^{t+T} \left|\hat{\theta}(t)H_2(x,t)\dot{\hat{k}}(t)\right|^2 dx}\,d\tau \\
&\quad + 2\int_t^{t+T} \sqrt{\int_0^1 |\eta(x,t)|^2 dx}\sqrt{\int_t^{t+T} \left|G_2(x,t)\dot{\hat{k}}(t)\right|^2 dx}\,d\tau \\
&\quad + 2\int_t^{t+T} \sqrt{\int_0^1 |\eta(x,t)|^2 dx}\sqrt{\int_t^{t+T} \left|\hat{k}(t)H_2(x,t)\dot{\hat{\theta}}(t)\right|^2 dx}\,d\tau \\
&\quad + 2\int_t^{t+T} \sqrt{\int_0^1 |\eta(x,t)|^2 dx}\sqrt{\int_t^{t+T} \left|\Omega_2(x)\hat{e}(1,t)\right|^2 dx}\,d\tau \\
&\quad + 2\int_t^{t+T} \sqrt{\int_0^1 |\eta(x,t)|^2 dx}\sqrt{\int_t^{t+T} \left|\Psi_2(x)\hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right)\right|^2 dx}\,d\tau.
\end{aligned} \tag{4.185}
$$

Since $||\eta||, V_7 \to 0$ and $\int_t^{t+T} \eta^2(0,t)d\tau, V_7 \geq 0$, all terms on the right hand side of (4.185) converge to zero, and the left hand side is bounded from below. Then, by the squeeze theorem, one obtains

$$\int_t^{t+T} \eta^2(0,\tau)d\tau \to 0, \tag{4.186}$$

and thereby

$$\int_t^{t+T} |\eta(0,\tau)|d\tau \to 0. \tag{4.187}$$

Consider the Lyapunov function candidate

$$V_8 = ||\eta||^2 = \int_0^1 \hat{\beta}^2(x,t)dx. \tag{4.188}$$

Differentiating with respect to time

$$\begin{aligned} \dot{V}_8 =& 2\int_0^1 \hat{\beta}(x,t)\hat{\beta}_t(x,t)dx \\ =& -\mu\hat{\beta}^2(0,t) + 2\int_0^1 \hat{\beta}(x,t)B_2(x,t)dx \end{aligned} \tag{4.189}$$

and then integrating from t to $t+T$ give

$$\begin{aligned} \int_t^{t+T} \dot{V}_8 d\tau =& V_8(t+T) - V_8(t) \\ =& -\mu\int_t^{t+T} \hat{\beta}^2(0,\tau)d\tau + 2\int_t^{t+T}\int_0^1 \hat{\beta}(x,t)\Psi_2(x)B_2(x,t)dx\, d\tau. \end{aligned} \tag{4.190}$$

Rearranging the terms and applying Cauchy-Schwarz' inequality yield

$$\begin{aligned} & V_8(t+T) - V_8(t) + \mu\int_t^{t+T} \hat{\beta}^2(0,\tau)d\tau \\ \leq & 2\int_t^{t+T} \sqrt{\int_0^1 |\hat{\beta}(x,t)|^2 dx}\sqrt{\int_t^{t+T} |B_2(x,t)|^2\, dx}\, d\tau. \end{aligned} \tag{4.191}$$

Since $||\hat{\beta}||, V_8 \to 0$ and $\int_t^{t+T} \hat{\beta}^2(0,t)d\tau, V_8 \geq 0$, all terms on the right hand side of (4.191) converge to zero, and the left hand side is bounded from below. Then, by the squeeze theorem, one obtains

$$\int_t^{t+T} \hat{\beta}^2(0,\tau)d\tau \to 0, \tag{4.192}$$

and thereby

$$\int_t^{t+T} |\hat{\beta}(0,\tau)|d\tau \to 0. \tag{4.193}$$

Using transformation (4.87) and (4.104) and boundary conditions (4.106c) and (4.89c), the following relation can be found:

$$\begin{aligned}
&|r\hat{v}(0,t)-\hat{u}(0,t)|\\
=&|r\zeta(0,t)-\omega(0,t)|\\
=&|r\eta(0,t)-\nu(0,t)+r\phi(0,t)-\varphi(0,t)|\\
=&|(r-q(t))\eta(0,t)-\kappa(t)\varepsilon(t)+(r-q(t))\phi(0,t)-d(t)|\\
=&|(r-q(t))\eta(0,t)-\kappa(t)\varepsilon(t)+(r-q(t))\zeta^*(t-d_\beta)-d(t)|\\
=&|(r-q(t))\eta(0,t)-\kappa(t)\varepsilon(t)+(r-q(t))\frac{d(t-d_\beta)}{r-q(t-d_\beta)}-d(t)|\\
\leq&|r-q(t)||\eta(0,t)|+|\kappa(t)||\varepsilon(t)|+|(r-q(t))\frac{d(t-d_\beta)}{r-q(t-d_\beta)}-d(t)|
\end{aligned} \tag{4.194}$$

Integrating both sides from $\tau=t$ to $\tau=t+T$ yields

$$\begin{aligned}
\int_t^{t+T}|r\hat{v}(0,\tau)-\hat{u}(0,\tau)|d\tau\leq&\int_t^{t+T}|r-q(t)||\eta(0,\tau)|d\tau+\int_t^{t+T}|\kappa(t)\varepsilon(t)|\\
&+d\tau\int_t^{t+T}|(r-q(\tau))\frac{d(\tau-d_\beta)}{r-q(\tau-d_\beta)}-d(\tau)|d\tau
\end{aligned} \tag{4.195}$$

From (4.187), (4.193), invertibility of the transform (4.109) and Property I and III of Theorem 4.4, the right hand side will converge to zero and by the squeeze theorem

$$\int_t^{t+T}|r\hat{v}(0,\tau)-\hat{u}(0,\tau)|d\tau\to 0. \tag{4.196}$$

Consider

$$\begin{aligned}
\int_t^{t+T}|rv(0,\tau)-u(0,\tau)|d\tau\leq&\int_t^{t+T}|r\hat{v}(0,\tau)-\hat{u}(0,\tau)|d\tau\\
&+\int_t^{t+T}|r\hat{\epsilon}(0,\tau)-\hat{e}(0,\tau)|d\tau\\
\leq&\int_t^{t+T}|r\hat{v}(0,\tau)-\hat{u}(0,\tau)|d\tau\\
&+\int_t^{t+T}|\tilde{k}(\tau)(\theta-y_0(\tau))+\hat{k}(\tau)\tilde{\theta}(\tau)|d\tau.
\end{aligned} \tag{4.197}$$

From Property IV of Theorem 4.4, the last term is square integrable, implying that the last term will converge to zero, and again by the squeeze theorem, we have

$$\int_t^{t+T}|rv(0,\tau)-u(0,\tau)|d\tau\to 0. \tag{4.198}$$

Lastly, from the results (4.196) and (4.198), and boundary conditions (4.1c) and (4.68c), we obtain (4.85). □

Part III

Application and Simulation

Chapter 5

Application to Kick and Loss Attenuation in MPD

The theory derived in Part II will be applied to the Kick and Loss Detection and Attenuation problem in Managed Pressure Drilling. The MPD technique and kick and loss application were briefly presented in Chapter 1. A more detailed introduction to well control is given in Section 5.1. A model of the drilling system is presented in Section 5.2. Transformations relating the drilling system to the systems in Chapters 3 and 4 and accompanying stability proofs are presented in Sections 5.3 and 5.4 respectively. Section 5.5 presents a simple control method that will be used for benchmarking of the two other methods.

5.1 Well Control

When drilling, a fluid called *mud* is circulated down the drill-string, through the *drill-bit* and up the casing around the drill string called *annulus*. This is illustrated in Figure 5.1. The purpose of the drill mud is not only to carry cuttings out to the surface, but also to prevent fracturing of the formation and collapse of the well (Aamo, 2013). All formations penetrated during drilling are porous and permeable to some degree (Lyons and Plisga, 2011, Section 4.14.1). If the reservoir pressure, often called the *pore pressure* or *formation pressure*, is higher than the bottom-hole pressure of the drilling fluid, the formation fluid will enter the well and, if not controlled, will traverse all the way up the annulus and cause an uncontrolled release of formation fluid into free air known as a *blowout*. The sudden inflow of formation fluid into the well is called a *kick*. Kicks may be the result of many causes, among them; an abnormally high formation pressure when drilling into a new formation, loss of circulation, too low mud weight and swabbing while tripping[1] (Lyons and Plisga, 2011, Section 4.14.1). In this thesis, only kicks caused by drilling into reservoirs with unknown fraction pressure is considered. If the bottom-hole pressure

[1] A temporarily pressure reduction created when pulling out the drill string.

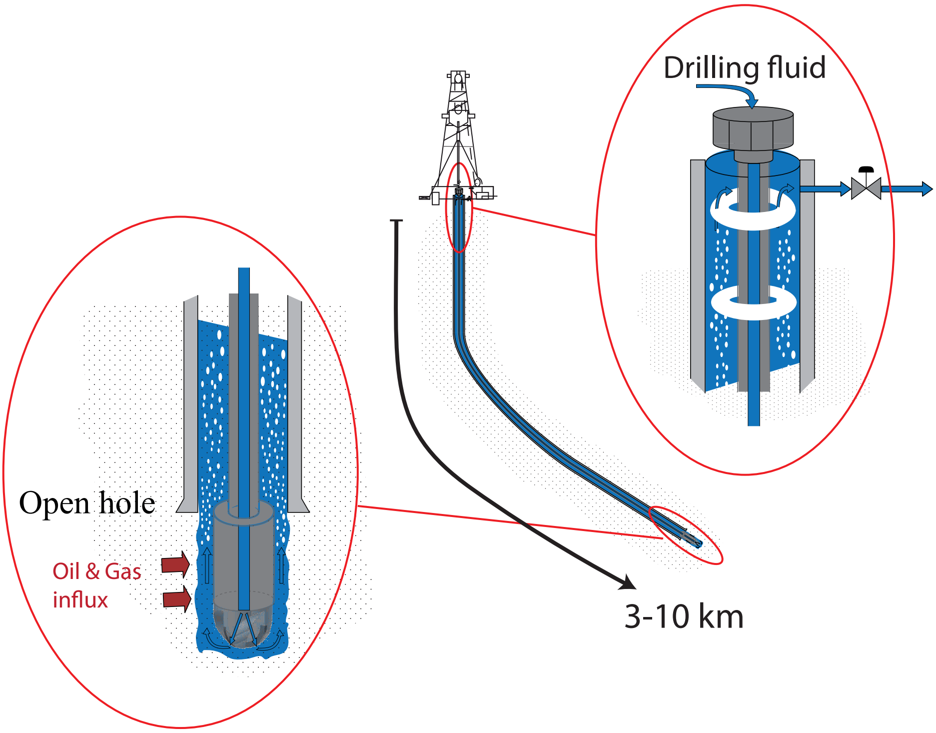


Figure 5.1: Schematic of a well being drilled. Courtesy of Ulf Jacob Aarsnes (Aarsnes et al. (2016b)).

is higher than the formation pressure, a situation known as a *loss* might occur where the drill fluid starts flowing into the formation, i.e. a loss of drilling fluid. This situation is also unwanted as it frequently leads to formation damage, decrease in penetration rate and in the extreme case fracturing and loss of circulation (Ostroot et al., 2007). In view of this, preventing or at least attenuating both kicks and losses should be the main concern in well pressure management.

In *conventional drilling operations* the drilling mud is circulated from an open pit, down the drill string, though the drill bit, up the annulus through a *bell nipple*, through a flow-line for separation of the mud and fraction fluids, and back to the open pit again (Malloy et al., 2009). Since the mud pit is open, drilling needs to be done with a higher bottom-hole pressure than pore pressure everywhere in the formation to avoid blowouts. The bottom-hole pressure can be controlled by varying the mud density, but because of the low bandwidth of this control method, the drilling pressure needs to be overly conservative. The result is a consistently *overbalanced* situation with all the negative consequences that entails. To achieve higher reservoir productivity, drilling with a pressure closer to the pore pressure is necessary. In *underbalanced drilling* the bottom-hole pressure is intentionally lower than the pore pressure in all parts of the formations. This way, all the

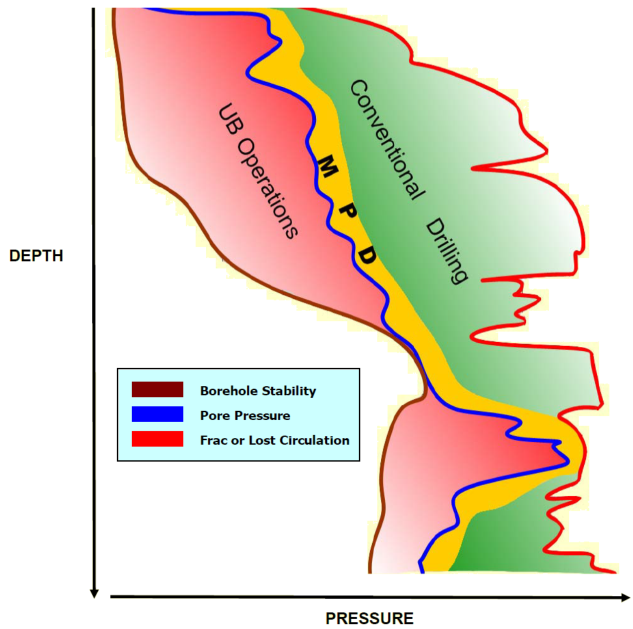


Figure 5.2: Illustration showing region of operation for conventional, underbalanced and managed pressure drilling together with borehole stability limits, pore pressure and fractioning limits. Adapted from Malloy et al. (2009).

negative effects associated with formation invasion are avoided. Drilling with a low bottom-hole pressure leads to formation fluids flowing up the annulus and to the surface. In underbalanced drilling, this is handled by sealing the top-side and diverting the produced fluids into a separator. Some problem-wells might be impossible or uneconomical to drill with conventional or underbalanced drilling. This might for instance be the case if the well stability pressure exceeds the pore pressure or if the fractioning pressure is close to the pore pressure. By controlling the back pressure through a valve (and pump if the circulation is stopped), the *applied back pressure* (ABP) method is able to control the pressure throughout the well. ABT is a method within managed pressure drilling where the goal is to control the bottom-hole pressure closer to the pore pressure. This is illustrated in Figure 5.2.

Well control can be divided into two classes defined in NORSOK (2004) as either *primary* or *secondary* barriers. Primary barriers are operational control methods and prevent formations from flowing into the wellbore by using the mud pressure weight to control the bottom-hole pressure. Secondary barriers are only used if the primary barrier fails. It uses a *blow out preventer* to stop the inflow and prevent blowouts. MPD can be used both for reactive MPD, i.e. as a secondary barrier to handle incidents after they occur, or as a proactive control method acting as a primary barrier. Only the latter use is considered in this thesis.

5.2 Problem Statement

To model the annular pressure and flow in a well using managed pressure drilling, a modification of the model presented in Landet et al. (2013) is used. This model was used to describe the *heave problem* in offshore drilling without circulation. To model the reservoir relation, the bottom-hole boundary condition is replaced by a simple *productivity index based inflow* model where the flow of produced formation fluid is directly proportional to the difference in pressure between the bottom-hole and reservoir. The proportional constant is called the *productivity index* (Ahmed, 2006, Chapter 7). This gives the following model:

$$p_t(z,t) = -\frac{\beta}{A_1} q_z(z,t) \tag{5.1a}$$

$$q_t(z,t) = -\frac{A_1}{\rho} p_z(z,t) - \frac{F_1}{\rho} q(z,t) - A_1 g \tag{5.1b}$$

$$q(0,t) = J\left(p_r - p(0,t)\right) + q_{bit} \tag{5.1c}$$

$$p(l,t) = p_l(t) \tag{5.1d}$$

where $z \in [0,l]$ and $t \geq 0$ are independent variables of space and time respectively, l is the well depth, $p(z,t)$ is pressure, $q(z,t)$ is volumetric flow, β is the bulk modulus of the mud, ρ is the density of the mud, A_1 is the cross sectional area of the annulus, F_1 is the friction factor, g is the acceleration of gravity, $J > 0$ is the productivity index, p_r the reservoir pressure and q_{bit} the flow through the drill bit. It is assumed that p_r satisfy

$$0 < p_r \leq \bar{p}_r \tag{5.2}$$

where $\bar{p}_r$ is some known upper bound for the reservoir pressure. Moreover, it is assumed that the choke controller have significantly faster dynamics than the rest of the system so that the actuation dynamics can be ignored and the top-side pressure p_l regarded as a control input. The design goal is to keep the down-hole pressure equal to the unknown reservoir pressure, that is

$$p(0,t) = p_r, \tag{5.3}$$

such that flow from the reservoir into the drill string is zero. This implies that the flow through the annulus is equal to the drill bit flow. Based on (5.3), the following control objective is selected:

$$\lim_{t\to\infty} \int_t^{t+T} |p(0,t) - p_r| \, d\tau = 0. \tag{5.4}$$

5.3 Feasibility of Design: Collocated Sensing and Control

If sensing is restricted to be taken top-side, that is collocated with the actuation, the theory derived in Chapter 3 can be applied to achieve the control objective (5.4). Let the top-side measurement be given as

$$q_l(t) = q(l,t). \tag{5.5}$$

Lemma 5.1. *The coordinate transformation*

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(q(xl,t) - q_{bit} + \frac{A_1}{\sqrt{\beta\rho}}(p(xl,t) + \rho g l x + \frac{F_1}{A_1} q_{bit} l x)\right) \\ &\quad \times \exp(\frac{lF_1}{2\sqrt{\beta\rho}}x) && (5.6a) \\ v(x,t) &= \frac{1}{2}\left(q(xl,t) - q_{bit} - \frac{A_1}{\sqrt{\beta\rho}}(p(xl,t) + \rho g l x + \frac{F_1}{A_1} q_{bit} l x)\right) \\ &\quad \times \exp(-\frac{lF_1}{2\sqrt{\beta\rho}}x) && (5.6b) \end{aligned}$$

where

$$x = \frac{z}{l} \tag{5.7}$$

maps the system (5.1) *into the form* (3.1) *with*

$$\lambda = \sqrt{\frac{\beta}{\rho}}\frac{1}{l} \tag{5.8a}$$

$$\mu = \sqrt{\frac{\beta}{\rho}}\frac{1}{l} \tag{5.8b}$$

$$c_1(x) = -\frac{F_1}{2\rho}\exp(\frac{lF_1}{\sqrt{\beta\rho}}x) \tag{5.8c}$$

$$c_2(x) = -\frac{F_1}{2\rho}\exp(-\frac{lF_1}{\sqrt{\beta\rho}}x) \tag{5.8d}$$

$$\theta_1 = \frac{\left(J\frac{\sqrt{\beta\rho}}{A_1} - 1\right)}{\left(J\frac{\sqrt{\beta\rho}}{A_1} + 1\right)} \tag{5.8e}$$

$$\theta_2 = \frac{J}{\left(J\frac{\sqrt{\beta\rho}}{A_1} + 1\right)}p_r \tag{5.8f}$$

and

$$U(t) = \frac{1}{2}\left(q(l,t) - q_{bit} - \frac{A_1}{\sqrt{\beta\rho}}(p(l,t) + \rho g l + \frac{F_1}{A_1}q_{bit}l)\right)\exp(-\frac{lF_1}{2\sqrt{\beta\rho}}) \tag{5.9a}$$

$$y(t) = \frac{1}{2}\left(q(l,t) - q_{bit} + \frac{A_1}{\sqrt{\beta\rho}}(p(l,t) + \rho g l + \frac{F_1}{A_1}q_{bit}l)\right)\exp(\frac{lF_1}{2\sqrt{\beta\rho}}). \tag{5.9b}$$

Moreover, the control objective (5.4) *is transformed to* (3.7) *with* $r = -1$.

Proof. The constant terms are removed and the origin shifted by defining

$$\bar{p}(z,t) = p(z,t) + \rho g z + \frac{F_1}{A_1}q_{bit}z \tag{5.10a}$$

$$\bar{q}(z,t) = q(z,t) - q_{bit}. \tag{5.10b}$$

Differentiating with respect to time gives

$$\bar{p}_t(z,t) = p_t(z,t) \tag{5.11}$$

and

$$\begin{aligned}
\bar{q}_t(z,t) &= -\frac{A_1}{\rho}p_z(z,t) - \frac{F_1}{\rho}q(z,t) - A_1 g \\
&= -\frac{A_1}{\rho}\left(p_z(z,t) + \rho g\right) - \frac{F_1}{\rho}(\bar{q}(z,t) + q_{bit}) \\
&= -\frac{A_1}{\rho}\left(p_z(z,t) + \rho g + \frac{F_1}{A_1}q_{bit}\right) - \frac{F_1}{\rho}\bar{q}(z,t) \\
&= -\frac{A_1}{\rho}\bar{p}_z(z,t) - \frac{F_1}{\rho}\bar{q}(z,t).
\end{aligned} \tag{5.12}$$

Similarly, differentiating with respect to space (z) gives

$$\bar{p}_z(z,t) = p_z(z,t) + \rho g + \frac{F_1}{A_1}q_{bit} \tag{5.13}$$

and

$$\bar{q}_z(z,t) = q_z(z,t). \tag{5.14}$$

Next, introducing the diagonalizing change of variables

$$\bar{u}(z,t) = \frac{1}{2}\left(\bar{q}(z,t) + \frac{A_1}{\sqrt{\beta\rho}}\bar{p}(z,t)\right) \tag{5.15a}$$

$$\bar{v}(z,t) = \frac{1}{2}\left(\bar{q}(z,t) - \frac{A_1}{\sqrt{\beta\rho}}\bar{p}(z,t)\right) \tag{5.15b}$$

and differentiating (5.15) with respect to space (z) give

$$\begin{aligned}
\bar{u}_z(z,t) &= \frac{1}{2}\bar{q}_z(z,t) + \frac{A_1}{2\sqrt{\beta\rho}}\bar{p}_z(z,t) \\
-\sqrt{\frac{\beta}{\rho}}\bar{u}_z(z,t) &= \frac{1}{2}\left(-\sqrt{\frac{\beta}{\rho}}\bar{q}_z(z,t) - \frac{A_1}{\rho}\bar{p}_z(z,t)\right)
\end{aligned} \tag{5.16}$$

and

$$
\begin{aligned}
\bar{v}_z(z,t) &= \frac{1}{2}\bar{q}_z(z,t) - \frac{A_1}{2\sqrt{\beta\rho}}\bar{p}_z(z,t) \\
\sqrt{\frac{\beta}{\rho}}\bar{u}_z(z,t) &= \frac{1}{2}\left(\sqrt{\frac{\beta}{\rho}}\bar{q}_z(z,t) - \frac{A_1}{\rho}\bar{p}_z(z,t)\right).
\end{aligned}
\tag{5.17}
$$

Differentiating with respect to time and inserting (5.16) and (5.17) into (5.15) give the dynamics in the new $(\bar{u}, \bar{v})$-coordinates:

$$
\begin{aligned}
\bar{u}_t(z,t) &= \frac{1}{2}\bar{q}_t(z,t) + \frac{A_1}{2\sqrt{\beta\rho}}\bar{p}_t(z,t) \\
&= \frac{1}{2}\left(-\frac{A_1}{\rho}\bar{p}_z(z,t) - \frac{F_1}{\rho}\bar{q}(z,t)\right) - \frac{A_1}{2\sqrt{\beta\rho}}\left(\frac{\beta}{A_1}\bar{q}_z(z,t)\right) \\
&= \frac{1}{2}\left(-\frac{A_1}{\rho}\bar{p}_z(z,t) - \sqrt{\frac{\beta}{\rho}}\bar{q}_z(z,t)\right) - \frac{F_1}{2\rho}\bar{q}(z,t) \\
&= -\sqrt{\frac{\beta}{\rho}}\bar{u}_z(z,t) - \frac{F_1}{2\rho}\left(\bar{u}(z,t) + \bar{v}(z,t)\right)
\end{aligned}
\tag{5.18}
$$

and

$$
\begin{aligned}
\bar{v}_t(z,t) &= \frac{1}{2}\bar{q}_t(z,t) - \frac{A_1}{2\sqrt{\beta\rho}}\bar{p}_t(z,t) \\
&= \frac{1}{2}\left(-\frac{A_1}{\rho}\bar{p}_z(z,t) - \frac{F_1}{\rho}\bar{q}(z,t)\right) + \frac{A_1}{2\sqrt{\beta\rho}}\left(\frac{\beta}{A_1}\bar{q}_z(z,t)\right) \\
&= \frac{1}{2}\left(-\frac{A_1}{\rho}\bar{p}_z(z,t) + \sqrt{\frac{\beta}{\rho}}\bar{q}_z(z,t)\right) - \frac{F_1}{2\rho}\bar{q}(z,t) \\
&= \sqrt{\frac{\beta}{\rho}}\bar{v}_z(z,t) - \frac{F_1}{2\rho}\left(\bar{u}(z,t) + \bar{v}(z,t)\right).
\end{aligned}
\tag{5.19}
$$

To remove the dependence of $\bar{u}$ from (5.18), $\bar{v}$ from (5.19) and scale the domain to $[0,1]$, the following transformation is defined:

$$
u(x,t) = \bar{u}(xl,t)\exp\left(\frac{lF_1}{2\sqrt{\beta\rho}}x\right) \tag{5.20a}
$$

$$
v(x,t) = \bar{v}(xl,t)\exp\left(-\frac{lF_1}{2\sqrt{\beta\rho}}x\right) \tag{5.20b}
$$

where $x = z/l$. Differentiating (5.20) with respect to space (x) gives

$$\begin{aligned}
\frac{1}{l}u_x(x,t) &= \frac{1}{l}\frac{\partial}{\partial x}\left(\bar{u}(z,t)\exp(\frac{lF_1}{2\sqrt{\beta\rho}}x)\right) \\
&= \frac{1}{l}\frac{\partial}{\partial z}\left(\bar{u}(z,t)\right)\frac{\partial z}{\partial x}\exp(\frac{lF_1}{2\sqrt{\beta\rho}}x) + \frac{1}{l}\bar{u}(z,t)\frac{\partial}{\partial x}\left(\exp(\frac{lF_1}{2\sqrt{\beta\rho}}x)\right) \\
&= \bar{u}_z(z,t)\exp(\frac{lF_1}{2\sqrt{\beta\rho}}x) + \bar{u}(z,t)\exp(\frac{lF_1}{2\sqrt{\beta\rho}}x)\frac{F_1}{2\sqrt{\beta\rho}} \\
&= \exp(\frac{lF_1}{2\sqrt{\beta\rho}}x)\left(\bar{u}_z(z,t) + \bar{u}(z,t)\frac{F_1}{2\sqrt{\beta\rho}}\right) \qquad (5.21)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{l}v_x(x,t) &= \frac{1}{l}\frac{\partial}{\partial x}\left(\bar{v}(z,t)\exp(-\frac{lF_1}{2\sqrt{\beta\rho}}x)\right) \\
&= \frac{1}{l}\frac{\partial}{\partial z}\left(\bar{v}(z,t)\right)\frac{\partial z}{\partial x}\exp(-\frac{lF_1}{2\sqrt{\beta\rho}}x) + \frac{1}{l}\bar{v}(z,t)\frac{\partial}{\partial x}\left(\exp(-\frac{lF_1}{2\sqrt{\beta\rho}}x)\right) \\
&= \bar{v}_z(z,t)\exp(-\frac{lF_1}{2\sqrt{\beta\rho}}x) - \bar{v}(z,t)\exp(-\frac{lF_1}{2\sqrt{\beta\rho}}x)\frac{F_1}{2\sqrt{\beta\rho}} \\
&= \exp(-\frac{lF_1}{2\sqrt{\beta\rho}}x)\left(\bar{v}_z(z,t) - \bar{v}(z,t)\frac{F_1}{2\sqrt{\beta\rho}}\right). \qquad (5.22)
\end{aligned}$$

Differentiating (5.20) with respect to time and inserting (5.21) and (5.22) yield

$$\begin{aligned}
u_t(x,t) &= \bar{u}_t(xl,t)\exp(\frac{lF_1}{2\sqrt{\beta\rho}}x) \\
&= \left(-\sqrt{\frac{\beta}{\rho}}\bar{u}_z(z,t) - \frac{F_1}{2\rho}\left(\bar{u}(z,t)+\bar{v}(z,t)\right)\right)\exp(\frac{lF_1}{2\sqrt{\beta\rho}}x) \\
&= -\sqrt{\frac{\beta}{\rho}}\left(\bar{u}_z(z,t) + \frac{F_1}{2\sqrt{\beta\rho}}\bar{u}(z,t)\right)\exp(\frac{lF_1}{2\sqrt{\beta\rho}}x) - \frac{F_1}{2\rho}\bar{v}(z,t)\exp(\frac{lF_1}{2\sqrt{\beta\rho}}x) \\
&= -\sqrt{\frac{\beta}{\rho}}\frac{1}{l}u_x(x,t) - \frac{F_1}{2\rho}\exp(\frac{lF_1}{\sqrt{\beta\rho}}x)v(x,t) \qquad (5.23)
\end{aligned}$$

and

$$\begin{aligned}
v_t(x,t) &= \bar{v}_t(xl,t)\exp(-\frac{lF_1}{2\sqrt{\beta\rho}}x) \\
&= \left(\sqrt{\frac{\beta}{\rho}}\bar{v}_z(z,t) - \frac{F_1}{2\rho}\left(\bar{v}(z,t)+\bar{u}(z,t)\right)\right)\exp(-\frac{lF_1}{2\sqrt{\beta\rho}}x) \\
&= \sqrt{\frac{\beta}{\rho}}\left(\bar{v}_z(z,t) - \frac{F_1}{2\sqrt{\beta\rho}}\bar{v}(z,t)\right)\exp(-\frac{lF_1}{2\sqrt{\beta\rho}}x) - \frac{F_1}{2\rho}\bar{u}(z,t)\exp(-\frac{lF_1}{2\sqrt{\beta\rho}}x) \\
&= \sqrt{\frac{\beta}{\rho}}\frac{1}{l}v_x(x,t) - \frac{F_1}{2\rho}\exp(-\frac{lF_1}{\sqrt{\beta\rho}}x)u(x,t). \qquad (5.24)
\end{aligned}$$

Inserting the definitions (5.8a)–(5.8d) give the dynamics (3.1a) and (3.1b).

From (5.15), the following relations can be found:

$$\begin{aligned}\bar{u}(z,t)+\bar{v}(z,t) =& \frac{1}{2}\left(\bar{q}(z,t)+\frac{A_1}{\sqrt{\beta\rho}}\bar{p}(z,t)\right)\\ &+\frac{1}{2}\left(\bar{q}(z,t)-\frac{A_1}{\sqrt{\beta\rho}}\bar{p}(z,t)\right)\\ =&\bar{q}(z,t)\end{aligned} \tag{5.25}$$

and

$$\begin{aligned}\frac{\sqrt{\beta\rho}}{A_1}\left(\bar{u}(z,t)-\bar{v}(z,t)\right) =& \frac{\sqrt{\beta\rho}}{2A_1}\left(\bar{q}(z,t)+\frac{A_1}{\sqrt{\beta\rho}}\bar{p}(z,t)\right)\\ &-\frac{\sqrt{\beta\rho}}{2A_1}\left(\bar{q}(z,t)-\frac{A_1}{\sqrt{\beta\rho}}\bar{p}(z,t)\right)\\ =&\bar{p}(z,t).\end{aligned} \tag{5.26}$$

Evaluating (5.10b) at $z=0$ gives

$$\begin{aligned}\bar{q}(0,t) &= q(0,t)-q_{bit}\\ &= J\left(p_r-p(0,t)\right)+q_{bit}-q_{bit}\\ &= -J\bar{p}(0,t)+Jp_r,\end{aligned} \tag{5.27}$$

inserting the relations (5.25) and (5.26) yield

$$\begin{aligned}\bar{u}(0,t)+\bar{v}(0,t) &= \bar{q}(0,t)\\ &= -J\bar{p}(0,t)+Jp_r\\ &= -J\frac{\sqrt{\beta\rho}}{A_1}\left(\bar{u}(0,t)-\bar{v}(0,t)\right)+Jp_r\end{aligned} \tag{5.28}$$

and by reorganizing the terms and using definitions (5.8e) and (5.8f), one obtains (3.1c).

Evaluating (5.6a) and (5.6b) at $x=l$ give trivially (5.9a) and (5.9b). From (5.28), it can be seen that $p(0,t)=p_r$ corresponds to $u(0,t)+v(0,t)=0$ and the objective (5.4) is transformed to (3.7) with $r=-1$. The complete transformation (5.6) can be seen from inserting (5.10) and (5.15) into (5.20) which completes the proof. □

From (5.8e) and the fact that $J>0$, it can be seen that θ_1 satisfy

$$-1<\theta_1<1 \tag{5.29}$$

which together with $r=-1$ means that the constraint (3.8) is satisfied. Inequality (5.29) can also be used as lower and upper bounds for θ_1. Lower and upper bounds for θ_2 can be found from (5.2) as $\underline{\theta}_2=0$ and $\bar{\theta}_2=\bar{p}_r$ respectively.

Since the system (5.1) takes the form of (3.1), the results from Theorem 3.2 and 3.4 can be applied.

Theorem 5.2. *Consider the system* (5.1)*. Let* $(\hat{p}, \hat{q})$ *be estimates of the states* (p, q) *generated from the observer* (3.9) *and transformation* (5.6)*, and let* $\hat{J}$ *and* $\hat{p}_r$ *be estimates of the unknown system parameters* J *and* p_r *generated using the adaptive law* (3.26) *and definitions* (5.8e)–(5.8f)*. If the system parameters and* r *are selected according to Lemma 5.1, the control law*

$$p_l(t) = \frac{\sqrt{\beta\rho}}{A_1}\left(q_l(t) - q_{bit} - 2U(t)\exp(\frac{lF_1}{2\sqrt{\beta\rho}})\right) - \rho g l - \frac{F_1}{A_1} q_{bit} l \tag{5.30}$$

with $U(t)$ *given by* (3.64)*, guarantees* (5.4) *and all signals in the closed loop system are bounded. Moreover, the estimate* $\hat{J}$ *converges to some steady state value and the estimate* $\hat{p}_r$ *converges to its true value* p_r *in the sense*

$$\int_t^{t+T} |\hat{p}_r(\tau) - p_r| \, d\tau \to 0. \tag{5.31}$$

Proof. For the first part of the theorem; by Lemma 5.1, having established that the system (5.1) takes the form (3.1), it suffices to show that the actuation $p_l(t)$ is related to $U(t)$ through (5.30). Solving (5.6b) for $p(xl, t)$ and evaluating the resulting equation at $x = 1$ give trivially the control law (5.30). By Theorem 3.4, the control objective (5.4) is achieved for some $T > 0$ and all signals in the closed loop are bounded. For the second part; from (3.65) and (3.66) it follows that

$$\int_t^{t+T} |\hat{p}(0, \tau) - p(0, \tau)| \, d\tau \to 0. \tag{5.32}$$

and since the control objective (5.4) is satisfied, we obtain (5.31). Convergence of $\hat{J}$ to some steady state value follows from Theorem 3.2. □

5.4 Feasibility of Design: Non-Collocated Sensing and Control

If sensing is allowed to be taken both top-side and bottom-hole, the theory derived in Chapter 4 can be applied to achieve the control objective (5.4). Consider the case where measurement of both top-side flow and bottom-hole pressure are available. Let the top-side measurement be given as

$$q_l(t) = q(l, t) \tag{5.33}$$

and the bottom-hole pressure as

$$p_0(t) = p(0, t). \tag{5.34}$$

Lemma 5.3. *The coordinate transformation*

$$u(x,t) = \frac{1}{2}\left(q(xl,t) - q_{bit} + \frac{A_1}{\sqrt{\beta\rho}}(p(xl,t) + \rho g l x + \frac{F_1}{A_1} q_{bit} l x)\right) \\ \times \exp(\frac{lF_1}{2\sqrt{\beta\rho}}x) \tag{5.35a}$$

$$v(x,t) = \frac{1}{2}\left(q(xl,t) - q_{bit} - \frac{A_1}{\sqrt{\beta\rho}}(p(xl,t) + \rho g l x + \frac{F_1}{A_1} q_{bit} l x)\right) \\ \times \exp(-\frac{lF_1}{2\sqrt{\beta\rho}}x) \tag{5.35b}$$

where

$$x = \frac{z}{l}, \tag{5.36}$$

maps the system (5.1) *into the form* (3.1) *with*

$$\lambda = \sqrt{\frac{\beta}{\rho}}\frac{1}{l} \tag{5.37a}$$

$$\mu = \sqrt{\frac{\beta}{\rho}}\frac{1}{l} \tag{5.37b}$$

$$c_1(x) = -\frac{F_1}{2\rho}\exp(\frac{lF_1}{\sqrt{\beta\rho}}x) \tag{5.37c}$$

$$c_2(x) = -\frac{F_1}{2\rho}\exp(-\frac{lF_1}{\sqrt{\beta\rho}}x) \tag{5.37d}$$

$$k = J\frac{\sqrt{\beta\rho}}{A_1} \tag{5.37e}$$

$$\theta = \frac{A_1}{\sqrt{\beta\rho}}p_r \tag{5.37f}$$

and

$$U(t) = \frac{1}{2}\left(q_l(t) - q_{bit} - \frac{A_1}{\sqrt{\beta\rho}}(p_l(t) + \rho g l + \frac{F_1}{A_1}q_{bit} l)\right)\exp(-\frac{lF_1}{2\sqrt{\beta\rho}}) \tag{5.38a}$$

$$y_1(t) = \frac{1}{2}\left(q_l(t) - q_{bit} + \frac{A_1}{\sqrt{\beta\rho}}(p_l(t) + \rho g l + \frac{F_1}{A_1}q_{bit} l)\right)\exp(\frac{lF_1}{2\sqrt{\beta\rho}}) \tag{5.38b}$$

$$y_0(t) = \frac{A_1}{\sqrt{\beta\rho}}p_0(t). \tag{5.38c}$$

The measurement y_0 is related to (u, v) by

$$y_0(t) = u(0,t) - v(0,t) \tag{5.39}$$

implying $a_0 = 1$ and $b_0 = -1$. Moreover, the control objective (5.4) *is transformed to* (4.7) *with $r = -1$.*

Proof. The proof of the mapping (5.1a)–(5.1b) to (4.1a)–(4.1b) is identical to the proof of Lemma 5.1 and is therefore omitted.

Evaluating (5.35a) and (5.35b) at $x = 0$ and adding them together yield

$$\begin{aligned} u(0,t) + v(0,t) = q(0,t) - q_{bit} =& J\left(p_r - p(0,t)\right) \\ =& J\frac{\sqrt{\beta\rho}}{A_1}\left(\frac{A_1}{\sqrt{\beta\rho}}p_r - \frac{A_1}{\sqrt{\beta\rho}}p(0,t)\right) \end{aligned} \tag{5.40}$$

and the boundary condition (4.1c) is obtained with θ and k given in (5.37e) and (5.37f). Subtracting (5.35a) evaluated at $x = 0$ from (5.35b) evaluated at $x = 0$ gives

$$u(0,t) - v(0,t) = \frac{A_1}{\sqrt{\beta\rho}}p(0,t) \tag{5.41}$$

and the measurement (4.5) is obtained with y_0 given by (5.38c), $a_0 = 1$ and $b_0 = -1$. From (5.40), it can be seen that $p(0,t) = p_r$ corresponds to $u(0,t)+v(0,t) = 0$ and the objective (5.4) is transformed to (4.7) with $r = -1$. □

From (5.37e) and $J > 0$, we have that $\mathrm{sign}(k)$ is known and positive. Furthermore, it can be seen that the selected a_1, b_1 and r satisfy the constraint (4.8).

Since the system (5.1) takes the form of (4.1), the results from Theorem 4.4 and 4.6 can be applied.

Theorem 5.4. *Consider the system* (5.1)*. Let* $(\hat{p},\hat{q})$ *be estimates of the states* (p,q) *generated from* (4.38)*, the update law in Theorem 4.4 and transformation* (5.35)*. Let* $\hat{J}$ *and* $\hat{p}_r$ *be estimates of the unknown system parameters* J *and* p_r *generated using the adaptive law* (4.41) *and definitions* (5.37f)*–*(5.37e)*. If the system parameters and* r *are selected according to Lemma 5.3, the control law*

$$p_l(t) = \frac{\sqrt{\beta\rho}}{A_1}\left(q_l(t) - q_{bit} - 2U(t)\exp\left(\frac{lF_1}{2\sqrt{\beta\rho}}\right)\right) - \rho g l - \frac{F_1}{A_1}q_{bit}l \tag{5.42}$$

with $U(t)$ *given by* (4.84)*, guarantees* (5.4) *and all signals in the closed loop system are bounded. Moreover, the estimate* $\hat{J}$ *converges to some steady state value and the estimate* $\hat{p}_r$ *converges to its true value* p_r*, that is*

$$\hat{p}_r \to p_r. \tag{5.43}$$

Proof. For the first part of the theorem; by Lemma 5.3, having established that the system (5.1) takes the form (4.1), it suffices to show that the actuation $p_l(t)$ is related to $U(t)$ through (5.42). Solving (5.35b) for $p(xl,t)$ and evaluating the resulting equation at $x = 1$ give trivially the control law (5.42). By Theorem 4.6, the control objective (5.4) is achieved for some $T > 0$ and all signals in the closed loop are bounded. For the second part; from Theorem 4.6 it follows that

$$\hat{\theta} \to \theta \tag{5.44a}$$

$$\hat{k} \to \bar{k} \tag{5.44b}$$

for some constant $\bar{k}$. Convergence in $\hat{p}_r$ and $\hat{J}$ then follow from definitions (5.37e) and (5.37f). □

5.5 Alternative Control Scheme: Constant Top-Side Actuation

In addition to the two controllers of Sections 5.3 and 5.4, a simple controller with constant top-side actuation is presented in this section. This is essentially the same control method used in Zhou et al. (2011) for kick attenuation in MPD. The main purpose of introducing this controller is for benchmarking of the two other methods. No stability or convergence proofs are offered[1]. Due to its simplicity, the methods of Sections 5.3 and 5.4 should, if successful, offer significant performance improvements over this controller. This however, can not be proven theoretically, and must be demonstrated by simulation or experiment. The following lemma presents the top-side control law and how it is implemented.

Lemma 5.5. *Consider the model* (5.1)*, transformation* (5.6)*, control signal* (5.9a) *and measurement* (5.9b)*. Let*

$$q_l(t) = q_{bit}, \qquad \forall t > 0 \tag{5.45}$$

where q_l *is the top-side flow and* q_{bit} *is the flow through the drill bit. The control law* (5.45) *can be implemented as*

$$U(t) = -y(t)\exp\left(-\frac{lF_1}{\sqrt{\beta\rho}}\right). \tag{5.46}$$

Proof. Direct substitution of (5.9a) and (5.9b) into the left and right hand side of (5.46) give (5.45) trivially. □

[1]Although the methods derived in Part II are able to stabilize open-loop unstable systems, benchmarking of the methods against this simple controller will only be performed on open-loop stable system. This is because the open loop system (5.1) with $U(t) \equiv 0$ is inherently stable for all physically realistic system parameters.

Chapter 6

Simulations

In this chapter, the MPD system (5.1) is implemented in `MATLAB` with the controller schemes of Theorem 5.2 and 5.4 and Lemma 5.5. The theory is applied to the Kick and Loss Detection and Attenuation application described in Section 5.1.

In addition, since the open loop system (5.1) with $U(t) \equiv 0$ is stable for all realistic system parameters, a mock-example with non-physical parameters is included to demonstrate the stabilizing capability of the controller schemes derived in Part II.

Four test cases are designed: The first one on a mock-example, the other three on the MPD model with the Kick and Loss Attenuation application. A description of the test cases are given in Section 6.1, performance metrics for comparing and benchmarking the methods are presented in Section 6.2, some details regarding how the systems are implemented are offered in Section 6.3, and simulation results and discussions can be found in Section 6.4.

6.1 Design of Test Cases

Four test cases are simulated using three different control methods. They will throughout this chapter be referred to as

I) The *collocated* method. Using the theory of Theorem 3.4 and the application in Theorem 5.2 where sensing is restricted to be collocated with actuation.

II) The *non-collocated* method. Using the theory of Theorem 4.6 and the application in Theorem 5.4 where sensing is allowed to be non-collocated with actuation.

III) The *constant* method. Using constant top-side actuation equal to the drill bit flow as described in Lemma 5.5.

The adaptation gains are selected as $\gamma_1 = \gamma_2 = 5$ in all test cases. For the **first test case**, systems (3.1) and (4.1) are simulated with the collocated and non-collocated

Table 6.1: Mock system parameters used in Case 1.

Parameter	Description	Value
λ	Transport speed	3
μ	Transport speed	3
$c_1(x)$	Source term	$3e^{-2x}$
$c_2(x)$	Source term	$3e^{2x}$
k	Boundary Parameter	4
θ	Boundary Parameter	1
r	Control objective coefficient	-1

Table 6.2: Well and drill system parameters used in Case 2, 3 and 4.

Parameter	Description	Value	Unit
β	Bulk modulus	7317×10^5	Pa
A_1	Annulus cross sectional area	0.024	m^2
ρ	Mud density	1250	$\mathrm{kg\,m^{-3}}$
F_1	Friction factor	10	$\mathrm{kg\,m^{-3}}$
g	Gravity constant	9.81	$\mathrm{m\,s^{-2}}$
l	Well length	2500	m
q_{bit}	Drill bit flow	1/60	$\mathrm{m^3\,s^{-1}}$
J	Productivity index	1.068×10^{-8}	$\mathrm{m^3\,s^{-1}\,Pa^{-1}}$

method respectively using mock-parameters, that is, parameters without any physical meaning. Numerical values for the system parameters, defined in Section 3.1 and/or Section 4.1 are given in Table 6.1[1] The purpose is to demonstrate the theory of Theorem 3.2, 3.4, 4.4 and 4.6. In accordance with that purpose, no comparison of performance between the two methods are made. No a priori information is assumed known about system sates or parameters; all initial estimates are zero. That is, $\hat{u}(x,0) = 0$, $\hat{v}(x,0)$ for all $x =\in [0,1]$, $\hat{\theta}_1(0) = \hat{\theta}_2(0) = 0$ (for the collocated method) and $\hat{\theta}(0) = \hat{k}(0) = 0$ (for the non-collocated method). Initial conditions for the system states are selected as

$$u(x,0) = 1, \quad \forall x \in [0,1] \tag{6.1a}$$

$$v(x,0) = \sin(x). \tag{6.1b}$$

In the other three test cases, system (5.1) is simulated with the physical well and drill parameters given in Table 6.2. The well length is 2500 meters, the productivity index is 40 stock tank barrels per day per psi[2] and the drill bit flow 1000 liters per minute. All additional parameters are the same as used in Aamo (2013).

[1] Numerical values for θ_1, θ_2 can be found by using (4.10).

[2] The productivity index is often referred to in terms of *stock tank barrels per day per psi* (STB/Day/Psi). In SI units: 40 STB/Day/Psi = $1.068 \times 10^{-8}\ m^3/s/Pa$.

The purpose of the **second test case** is to demonstrate stability and convergence of the collocated and non-collocated method applied on the MPD system (5.1) by using Theorem 5.2 and 5.4 respectively. In accordance with that purpose, no comparison of performance between the two methods are made. No a priori information is assumed known about the well and drill states. All initial state estimates are zero, and initial parameter estimates are randomly selected within some reasonable range as $\hat{J}(0) = 1.5 \times 10^{-4}\,\mathrm{m}^3$ and $\hat{p}_r(0) = 100 \times 10^5\,\mathrm{Pa}$. The reservoir pressure is kept constant and equal to $p_r(t) = 450 \times 10^5\,\mathrm{Pa}, \quad \forall\, t > 0$. Initial conditions for the system states are selected as[1]

$$
\begin{aligned}
p(z,0) =& 0, \quad \forall\, z \in [0,l] \\
q(z,0) =& 0.1\,\mathrm{m}^3\,\mathrm{s}^{-1}.
\end{aligned}
$$

The purpose of the **third and forth test case** is to compare the performance of the collocated method and non-collocated method up against each other and against the simple method. The performance is evaluated by how well the methods attenuate a simulated kick and loss. In the **third test case** a step drop in reservoir pressure is simulated, i.e. a loss. In the **forth test case** a step increase in reservoir pressure is simulated, i.e. a kick. In order to better isolate the effects of the simulated kick/loss, the simulation is run for an initial 10 seconds in steady state, before the step in reservoir pressure is simulated. The system is initialized with perfect knowledge about the system states and parameters, that is $\hat{p}(z,0) = p(z,0)$, $\hat{q}(z,0) = q(z,0)$ for all $z \in [0,l]$, $\hat{J}(0) = J$ and $\hat{p}_r(0) = p_r(0)$. The reservoir pressure for the third test case is selected as

$$
p_r(t) = \begin{cases} 450 \times 10^5\,\mathrm{Pa}, & \text{for } t < 10 \\ 400 \times 10^5\,\mathrm{Pa}, & \text{for } t > 10, \end{cases} \tag{6.3}
$$

and for the fourth case as

$$
p_r(t) = \begin{cases} 400 \times 10^5\,\mathrm{Pa}, & \text{for } t < 10 \\ 450 \times 10^5\,\mathrm{Pa}, & \text{for } t > 10. \end{cases} \tag{6.4}
$$

Initial steady state conditions are found by setting $p_t(z,t) = q_t(z,t) = 0$ in (5.1) and solving the resulting IVP, yielding

$$
p(z,0) = p_r - \frac{F_1}{A_1} q_{bit} z - \rho g z := p_{ss}(z) \tag{6.5a}
$$

$$
q(x,0) = q_{bit} := q_{ss}(z). \tag{6.5b}
$$

A summary of all initial conditions used in test Case 2, 3 and 4 can be found in Table 6.3.

[1]The initial pressure distribution is highly non-physical, but is selected to better illustrate specific features of the control methods.

Table 6.3: Initial conditions used in Case 2, 3 and 4

	Case 2	Case 3	Case 4
$p(z,0)$	0	p_{ss}	p_{ss}
$q(z,0)$	$0.1\,\mathrm{m^3\,s^{-1}}$	q_{ss}	q_{ss}
$p_r(0)$	$450\times10^5\,\mathrm{Pa}$	$450\times10^5\,\mathrm{Pa}$	$400\times10^5\,\mathrm{Pa}$
$\hat{p}(z,t)$	0	$p(z,0)$	$p(z,0)$
$\hat{q}(z,t)$	0	$q(z,0)$	$q(z,0)$
$\hat{p}_r(0)$	$100\times10^5\,\mathrm{Pa}$	$p_r(0)$	$p_r(0)$
$\hat{J}(0)$	$1.5\times10^{-8}\,\mathrm{m^3\,s^{-1}\,Pa^{-1}}$	J	J

6.2 Performance Metrics

In the kick and loss application, the goal is to counteract changes in reservoir pressure by controlling the bottom-hole pressure and thereby preventing fluids from flowing into or out of the well. A suitable measure of performance is thus the total amount of fluid flowing into/out of the well before the kick/loss is attenuated.

Let the *excess* flow in or out of the well be denoted

$$q_{exs}(t) = \max\left(0, |q(0,\tau) - q_{bit}|\right) \tag{6.6}$$

The performance of the control methods will be evaluated in terms of the L_1 and L_∞ norm of q_{exs} over the time interval t_0 to t_1, where t_0 is the step time and t_1 is the total simulation horizon. That is

$$||q_{exs}||_1 = \int_{t_0}^{t_1} |q_{exs}(\tau)|d\tau \tag{6.7a}$$

$$||q_{exs}||_\infty = \sup_{t\in[t_0,t_1]} |q_{exs}(\tau)|. \tag{6.7b}$$

6.3 Implementation

Instead of implementing system (5.1) directly, systems (3.1) and (4.1) are actually the ones implemented. The system sates (p,q) are calculated from post-processing of the (u,v) states by inverting the transform (5.6). This is also the procedure used in Anfinsen (2013); Aamo (2013). Since system (5.1) is not on Riemann form, implementing (5.1) would require the use of more advanced *finite element methods* (Sonnendrücker, 2015). Instead, system (3.1) and (4.1) can be implemented using the *method of lines* (Schiesser, 1991; Hamdi et al., 2007) where spatial derivatives in x are approximated using a 2nd order *upwind scheme* (Kreyszig, 2011, Section 21.7) and the explicit Runge-Kutta `MATLAB` solver `ode23()` used for the resulting IVP in time t. The spatial domain is discretized using $N = 100$ discretization points.

The kernel PDEs (3.14), (3.52) and (4.81) are solved using the same solvers developed for Anfinsen and Aamo (2016, 2017b). The kernel PDE (4.23) and injection gains (4.26) are found explicitly using the solution given in Vazquez and Krstic (2014) and applied on the MPD model in Aamo (2016).

6.4 Simulation of Test Cases

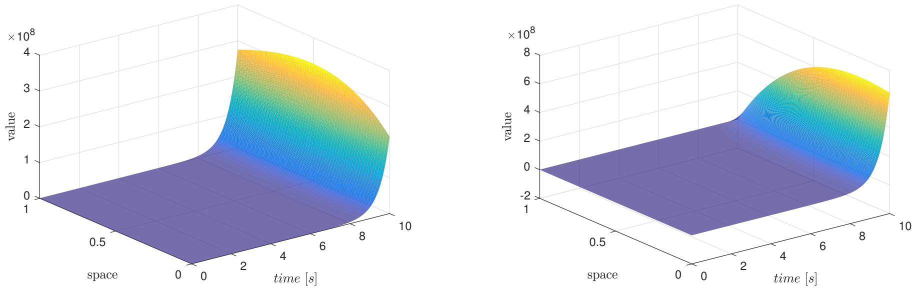
Each of the sections below contain a description of the figures included, discussion of the results, and for test case 3 and 4; the performance metrics.

6.4.1 Case 1: Stabilization of 2×2 Hyperbolic Systems

This test case shows simulation results using both the collocated and the non-collocated method. The systems are simulated for 15 seconds. The following figures are included:

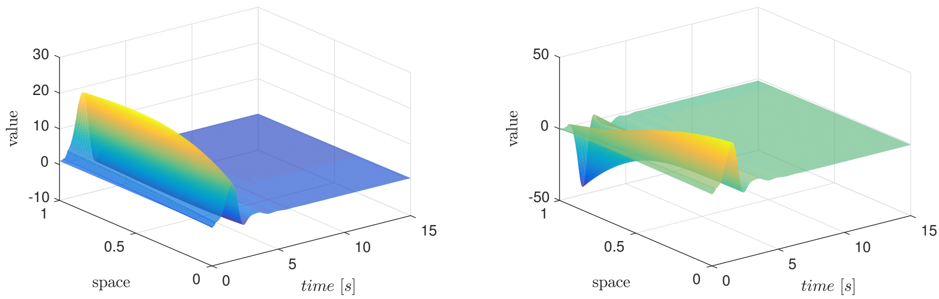
1. A 3D representation of the open loop system states $u(x,t), v(x,t)$ for all $x \in [0,1]$, that is with no control input ($U(t) \equiv 0$).
2. A 3D representation of the closed loop system states $u(x,t), v(x,t)$ for all $x \in [0,1]$ using the collocated method.
3. A 3D representation of the closed loop system states $u(x,t), v(x,t)$ for all $x \in [0,1]$ using the non-collocated method.
4. A 3D representation of the state estimation error $\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t)$, $\tilde{v}(x,t) = v(x,t) - \hat{v}(x,t)$ when using the collocated method.
5. A 3D representation of the state estimation error $\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t)$, $\tilde{v}(x,t) = v(x,t) - \hat{v}(x,t)$ when using the non-collocated method.
6. Parameter estimates $\hat{\theta}_1(t), \hat{\theta}_2(t)$ and actual boundary parameters θ_1, θ_2 for the collocated method.
7. Parameter estimates $\hat{k}(t), \hat{\theta}(t)$ and actual boundary parameters k, θ for the non-collocated method.
8. The applied actuation signal $U(t)$ for both the collocated method and the non-collocated method.
9. A linear combination of the system states at the boundary, namely $u(0,t) - rv(0,t)$. This is a stricter version of (3.7) and will be referred to as the *pointwise objective*. The pointwise objective is shown for both the collocated method and the non-collocated method.

Figure 6.1 shows that both sates are unbounded and the system is open loop unstable. Introducing the controllers (3.64) and (4.84) in Figures 6.2 and 6.3, show that all states now are bounded and that both the collocated method and non-collocated method are able to stabilize the system. All state estimates are shown to converge to zero in Figures 6.4 and 6.5, and from Figure 6.9 it can be seen that the objectives (3.7) and (4.7) are satisfied. Hence, the results from Theorem 3.4 and 4.6 are demonstrated by simulation. Furthermore, from Figures 6.6 and 6.7 it can be seen that all parameter estimates converge to a steady state value and that $\hat{\theta}$ in Figure 6.7b converge to its true value. This is in line with what was shown theoretically in Theorem 3.2 and 4.4. In addition, it can be seen from Figure 6.8 that the control signal converges to a constant non-zero value.



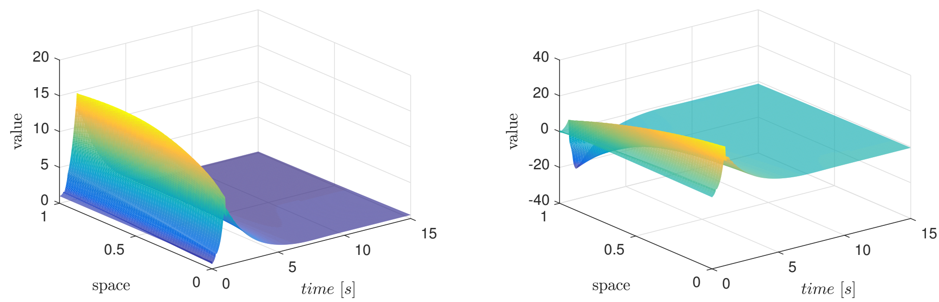
(a) State u **(b)** State v

Figure 6.1: States, open loop system ($U(t) \equiv 0$).



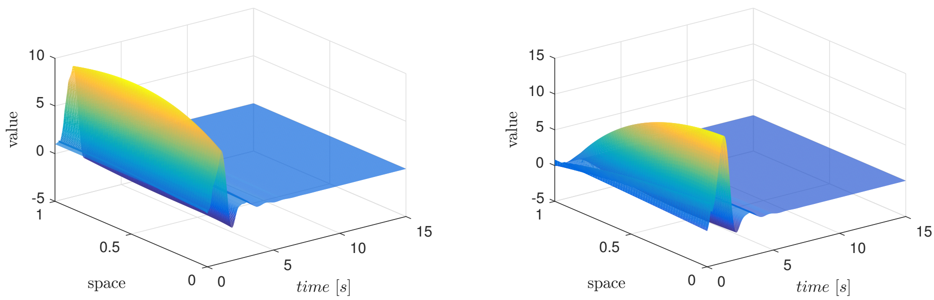
(a) State u **(b)** State v

Figure 6.2: States, closed loop system using collocated method.



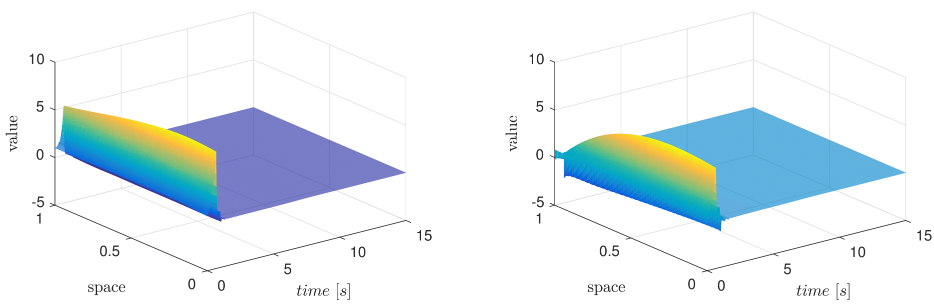
(a) State u **(b)** State v

Figure 6.3: States, closed loop system using non-collocated method.



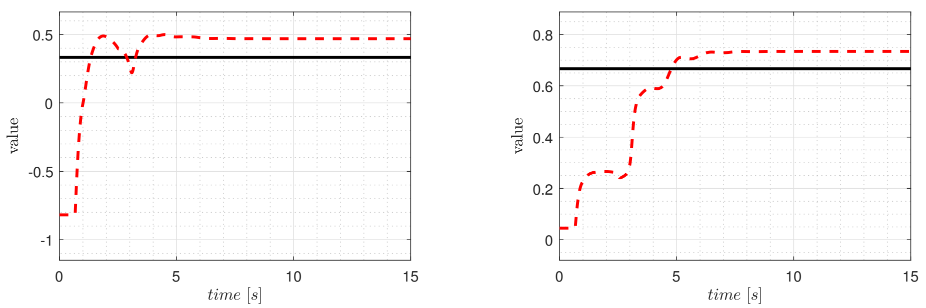
(a) Estimation error $\tilde{u}$ **(b)** Estimation error $\tilde{v}$

Figure 6.4: Estimation error using collocated method.



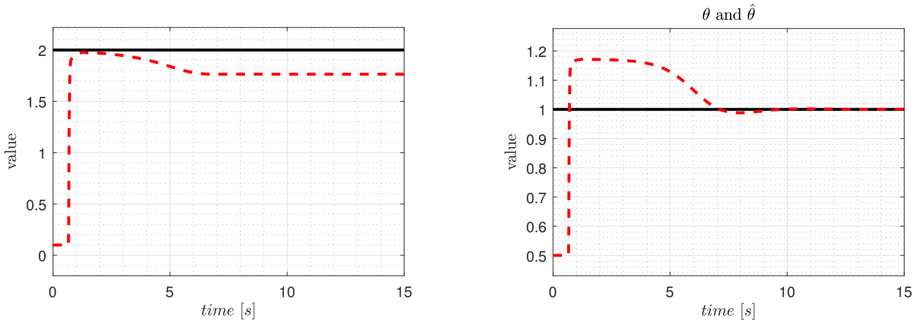
(a) Estimation error $\tilde{u}$ **(b)** Estimation error $\tilde{v}$

Figure 6.5: Estimation error using non-collocated method.



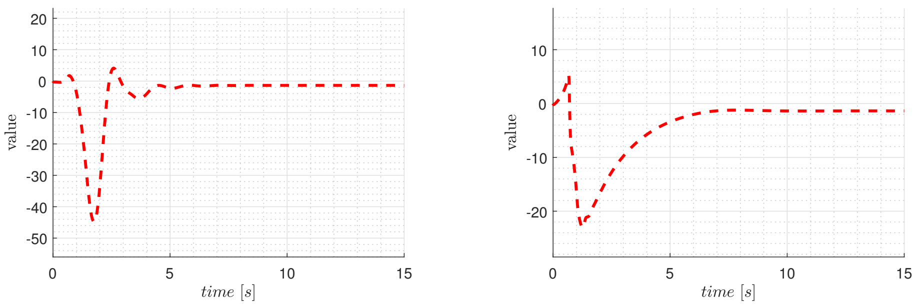
(a) $\hat{\theta}_1$ **(b)** $\hat{\theta}_2$

Figure 6.6: Parameter estimates (dashed red) and actual parameters (solid black) for the collocated method.



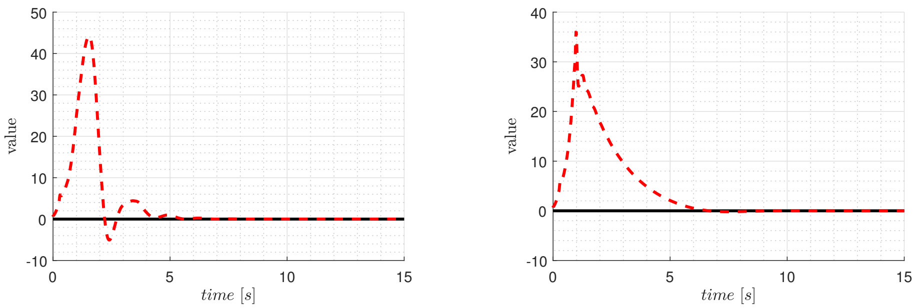
(a) $\hat{k}$ **(b)** $\hat{\theta}$

Figure 6.7: Parameter estimates (dashed red) and actual parameters (dashed black) for the non-collocated method.



(a) $U(t)$, collocated method **(b)** $U(t)$, non-collocated method

Figure 6.8: Control signal $U(t)$ for the collocated method (left) and non-collocated method (right).



(a) Point-wise control objective, collocated method. **(b)** Point-wise control objective, non-collocated method.

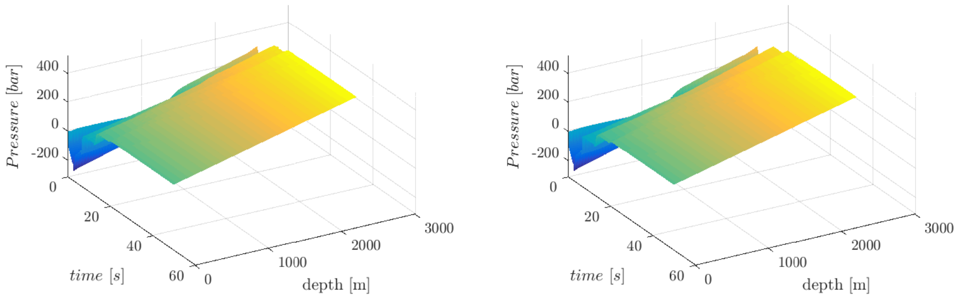
Figure 6.9: Point-wise control objective $u(0,t) - rv(0,t)$ for the collocated method (left) and non-collocated method (right).

6.4.2 Case 2: Stabilization of MPD System

Both the collocated and the non-collocated method are tested on the MPD model. The systems are simulated for 50 seconds. All figures show both methods side by side. The following figures are included:

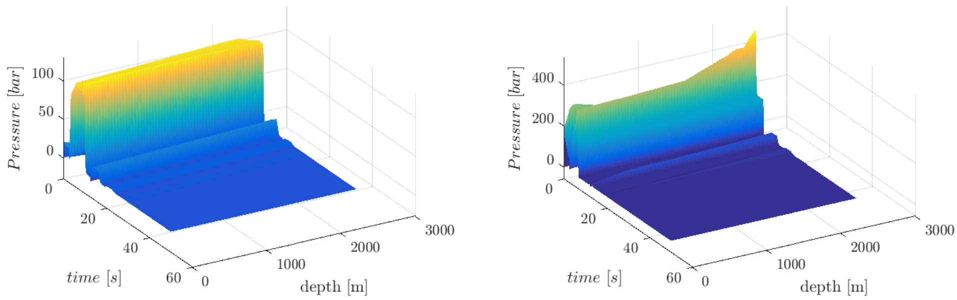
1. A 3D representation of the well pressure distribution $p(z,t)$ for all $z \in [0,l]$.
2. A 3D representation of the well pressure estimation error $\tilde{p}(z,t) = p(z,t) - \hat{p}(z,t)$.
3. Estimated reservoir pressure $\hat{p}_r(t)$ and actual reservoir pressure $p_r(t)$.
4. Estimated productivity index $\hat{J}(t)$ and actual productivity index J.
5. The applied actuation signal $p_l(t)$.
6. The bottom-hole pressure $p(0,t)$ and reservoir pressure $p_r(t)$
7. The bottom-hole volumetric flow $q(0,t)$ and drill bit flow q_{bit}.

Figure 6.10 shows that the well pressure is stabilized for all $z \in [0,l]$ for both methods. The pressure distributions converge to a linear steady state profile with the highest pressure bottom-hole. Both methods are also able to estimate the pressure everywhere in the well; where Figure 6.11 shows that the estimation errors converge to zero for all $z \in [0,l]$. In accordance with Theorem 5.2 and 5.4, the reservoir pressure estimates converge to their true value and the productivity index estimates to some constant steady sate values, as can be seen from Figures 6.12 and 6.13. Figures 6.14 and 6.15 show that the bottom-hole pressures are stabilized at the reservoir pressure and the net flows from the reservoir into the well converge to zero, meaning that both methods are able to attenuate the gain, and Theorem 5.2 and 5.4 are demonstrated by simulation.



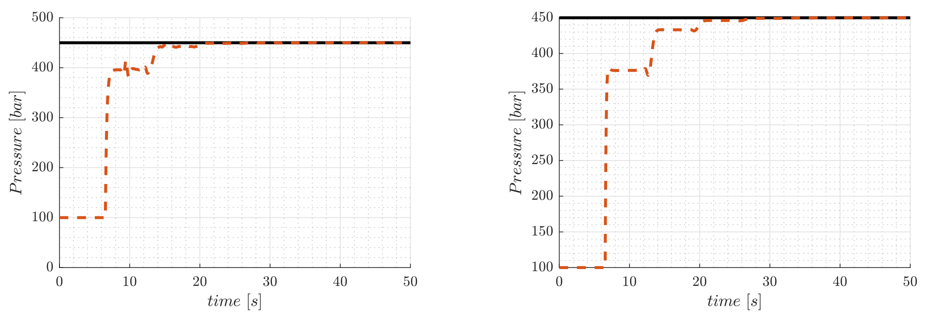
(a) Collocated method. **(b)** Non-collocated method.

Figure 6.10: Pressure distribution in well $p(z,t)$ for all $z \in [0,l]$.



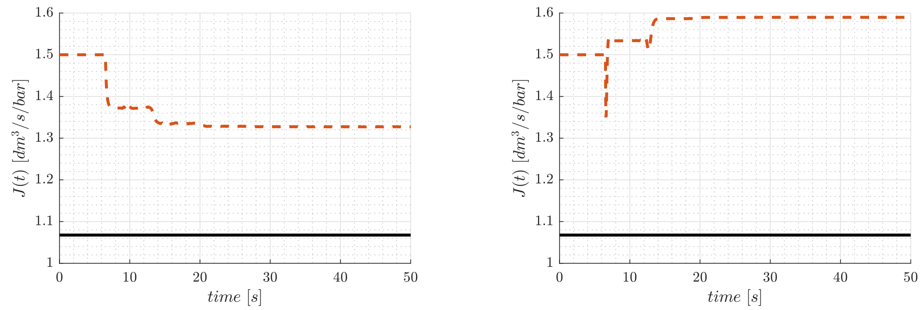
(a) Collocated method. **(b)** Non-collocated method.

Figure 6.11: Pressure estimation error in well $\tilde{p}(z,t)$ for all $z \in [0,l]$.



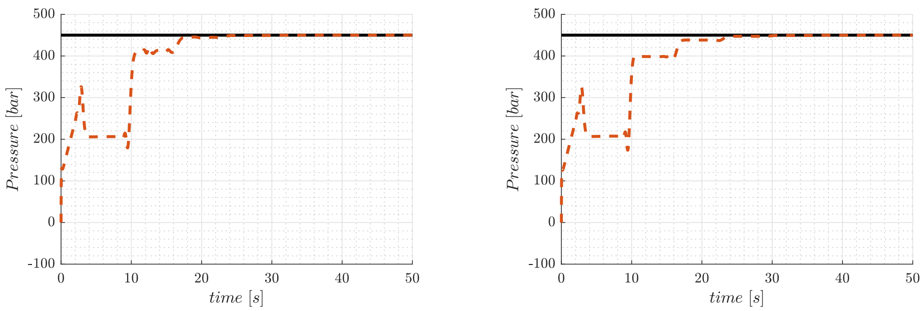
(a) Collocated method. **(b)** Non-collocated method.

Figure 6.12: Estimated reservoir pressure $\hat{p}_r(t)$ and actual reservoir pressure $p_r(t)$.



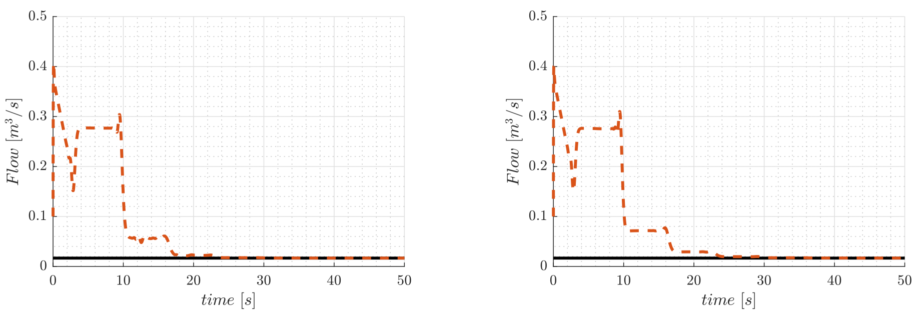
(a) Collocated method. **(b)** Non-collocated method.

Figure 6.13: Estimated productivity index $\hat{J}(t)$ and actual productivity index J.



(a) Collocated method. **(b)** Non-collocated method.

Figure 6.14: Bottom-hole pressure $p(0,t)$ and reservoir pressure $p_r(t)$.



(a) Collocated method. **(b)** Non-collocated method.

Figure 6.15: Bottom-hole volumetric flow $q(0,t)$ and drill bit flow q_{bit}.

6.4.3 Case 3: Loss Attenuation

A loss is simulated as a step drop in reservoir pressure at $t_0 = 10\,\mathrm{s}$ for all three methods; the collocated, non-collocated and simple method. The systems are simulated for a total of $t_1 = 60\,\mathrm{s}$. All methods are compared in the same figures. The following two figures are included:

1. Bottom-hole pressure $p(0,t)$ for all methods and reservoir pressure $p_r(t)$. Two plots are included; an overview for $t \in [0, 60\,\mathrm{s}]$ and a close up view for $t \in [16\,\mathrm{s}, 33\,\mathrm{s}]$. The latter is included to better show the difference between the collocated and non-collocated method.

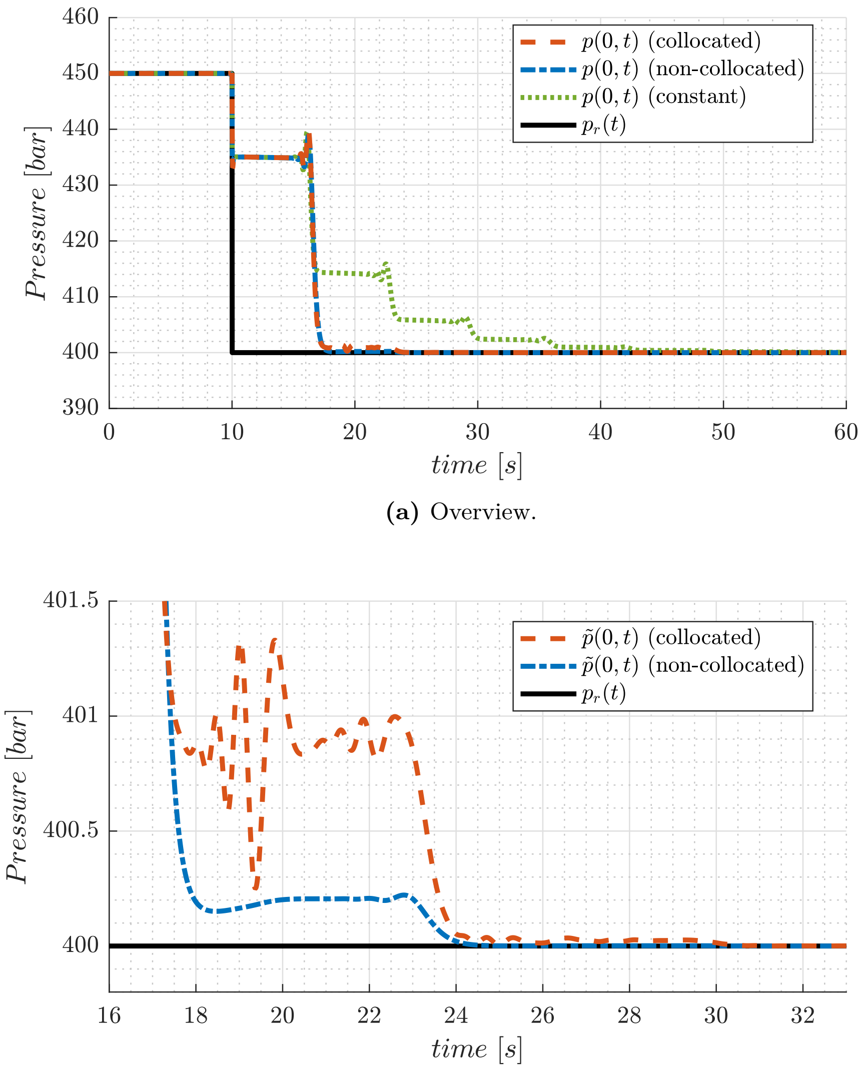
2. Bottom-hole volumetric flow $q(0,t)$ for all methods and drill bit flow q_{bit}. Both an overview and close up view are also here included.

Figures 6.16 and 6.17 show that all three methods are able to attenuate the loss. The bottom-hole pressure is stabilized at the reservoir pressure and the net loss out of the well converges to zero. It is seen that both the collocated and non-collocated method are significantly faster than the simple method. The performance results in Table 6.4 show that using the collocated method or non-collocated method offer a $\sim 40\%$ reduction in total out-flow, compared with the constant method. The performance of the collocated method and non-collocated method however are nearly identical, both in terms of total out-flow and in transient response. It would be natural to assume that the non-collocated method should offer some performance improvement over the collocated method because of the additional bottom-hole measurement and thereby better state estimates. The simulation results, however, show that the non-collocated method is only marginally better. Examining Figures 6.16 and 6.17 further show that the bottom-hole pressure and flow converge in almost discrete steps every $\sim 6\,\mathrm{s}$. This corresponds to the propagation time[1] from the bottom of the well to the top-side measurement and actuation and down again to the bottom of the well. This propagation time is a theoretical limit for how fast a bottom-hole loss can be attenuated by top-side actuation and sensing only. Using bottom-hole sensing however, bottom-hole pressure estimates are instantly available and the theoretical limit can be reduced to half of that. Inspecting the close up view in Figures 6.16b and 6.17b show that the non-located method offer some performance improvements over the collocated method in terms of faster convergence and less oscillations. In view of the theoretical limit of using non-collocated sensing, the additional bottom-hole measurement is nonetheless not satisfactory utilized. This is however to be expected from the structure of the control scheme: The bottom-hole measurement enters the swapping filter at one boundary and must propagate through the swapping filter up to the other boundary before the new information is available to the adaptive law and the estimate $\hat{p}_r$ can be updated. In conclusion, taking the structural design of the method into account, the non-collocated method performed satisfactory.

[1]The propagation time can be calculated from the transport speeds (5.8a) and (5.8b) with the parameters in Table 6.2.

Table 6.4: Performance metrics for Case 3.

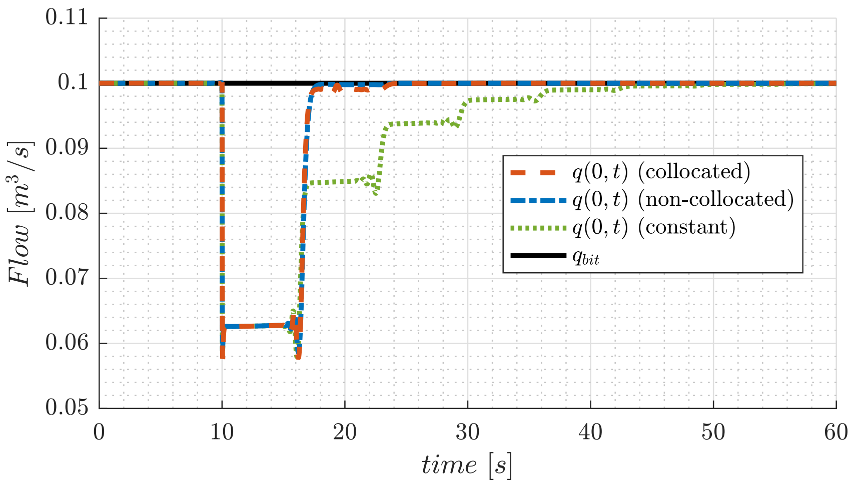
	$\|\|q_{exs}\|\|_1$	$\|\|q_{exs}\|\|_\infty$
Collocated	0.257 m^3	0.042 $\mathrm{m}^3\,\mathrm{s}^{-1}$
Non-collcoated	0.253 m^3	0.041 $\mathrm{m}^3\,\mathrm{s}^{-1}$
Simple	0.410 m^3	0.042 $\mathrm{m}^3\,\mathrm{s}^{-1}$



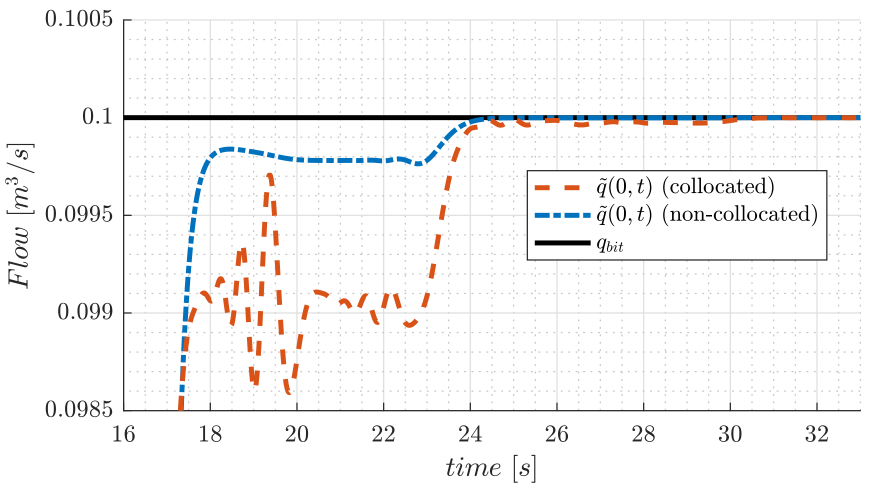
(a) Overview.

(b) Close up view.

Figure 6.16: Bottom-hole pressure $p(0, t)$ and reservoir pressure $p_r(t)$.



(a) Overview.



(b) Close up view.

Figure 6.17: Bottom-hole volumetric flow $q(0,t)$ and drill bit flow q_{bit}.

Table 6.5: Performance metrics for Case 2.

	$\|\|q_{exs}\|\|_1$	$\|\|q_{exs}\|\|_\infty$
Collocated	$0.250\ \mathrm{m^3}$	$0.042\ \mathrm{m^3\,s^{-1}}$
Non-collcoated	$0.254\ \mathrm{m^3}$	$0.041\ \mathrm{m^3\,s^{-1}}$
Simple	$0.410\ \mathrm{m^3}$	$0.042\ \mathrm{m^3\,s^{-1}}$

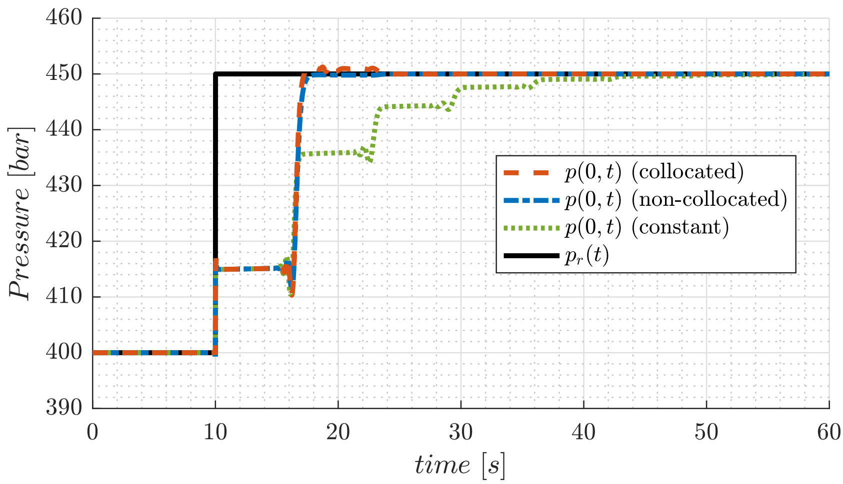
6.4.4 Case 4: Kick Attenuation

A kick is simulated as a step increase in reservoir pressure at $t_0 = 10\,\mathrm{s}$ for all three methods; the collocated, non-collocated and simple method. The systems are simulated for a total of $t_1 = 60\,\mathrm{s}$. All methods are compared in the same figures. The following two figures are included:

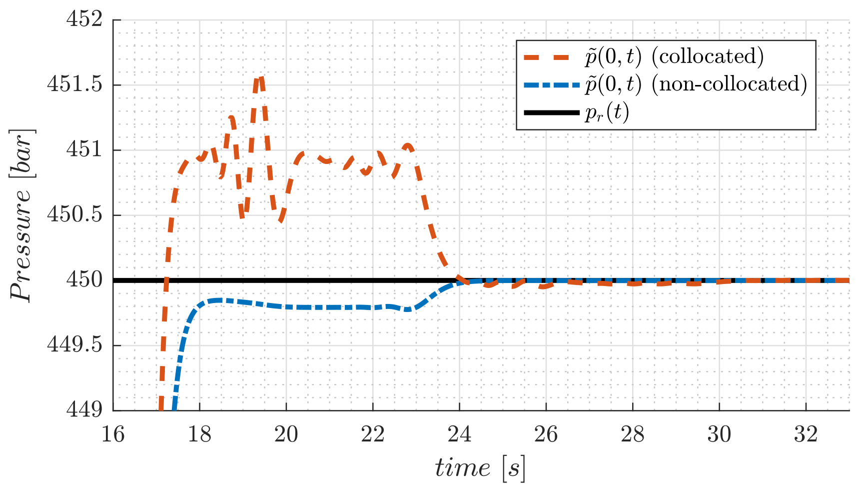
1. Bottom-hole pressure $p(0,t)$ for all methods and reservoir pressure $p_r(t)$. Two plots are included; an overview for $t \in [0, 60\,\mathrm{s}]$ and a close up view for $t \in [16\,\mathrm{s}, 33\,\mathrm{s}]$. The latter is included to better show the difference between the collocated and non-collocated method.

2. Bottom-hole volumetric flow $q(0,t)$ for all methods and drill bit flow q_{bit}. Both an overview and close up view are also here included.

Figures 6.18 and 6.19 show that all three methods are able to attenuate the kick. The bottom-hole pressure is stabilized at the reservoir pressure and the net gain into the well converge to zero. It is seen that both the collocated and non-collocated method, are significantly faster than the simple method. The performance results in Table 6.5 show that using the collocated method or non-collocated method also in this case offer a $\sim 40\%$ reduction in total in-flow, compared with the constant method. Similarly to the loss attenuation case, the performance of the collocated method and non-collocated method are nearly identical. The same discrete steps in convergence every $\sim 6\,\mathrm{s}$ can be found in Figures 6.18 and 6.19, and thus the same arguments about utilization of the bottom-hole measurement for the non-collocated method applies here. From Figures 6.18b and 6.19b and contrary to the loss attenuation case, an overshoot in the bottom-hole pressure and net out-flow can be observed when using the collocated method. Compared to the non-collocated method, the response is also highly oscillatory around $17\,\mathrm{s} - 24\,\mathrm{s}$ just before the bottom-hole pressure is stabilized at the set-point. The same conclusion as in Case 3 can be made; namely that the non-collocated method offer some performance improvement compared to the collocated method, and that both methods operate close to their structural theoretical limit.

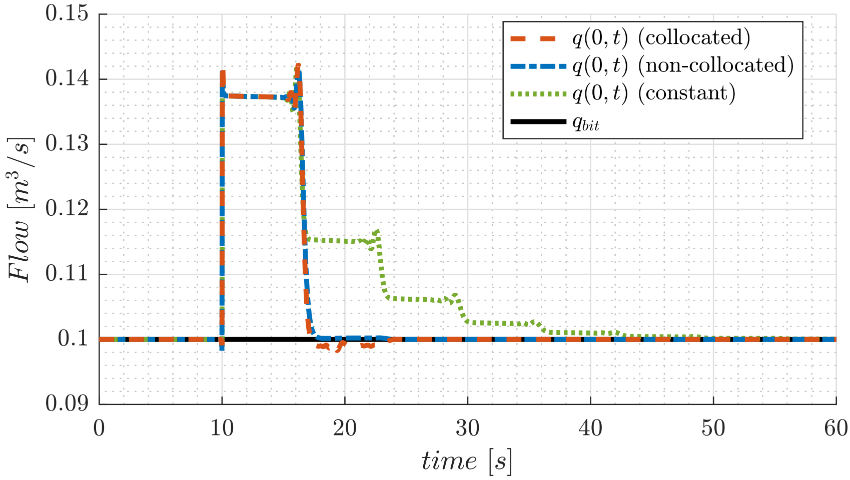


(a) Overview.

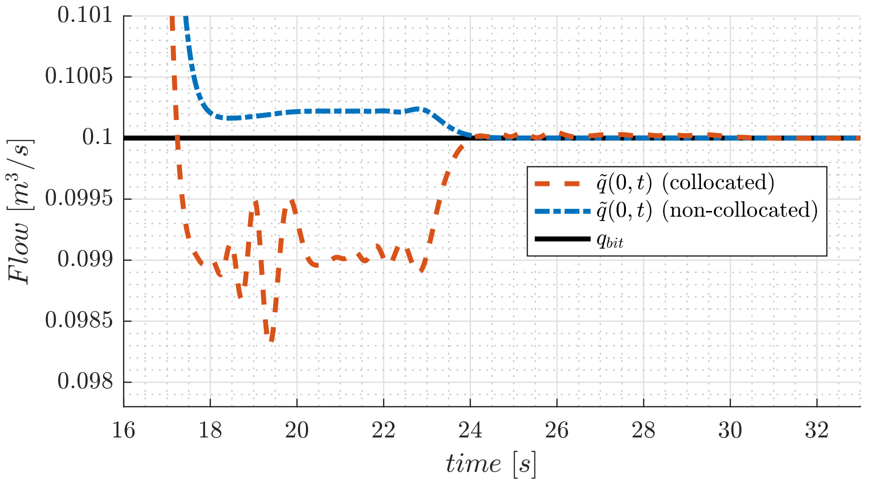


(b) Close up view.

Figure 6.18: Bottom-hole pressure $p(0,t)$ and reservoir pressure $p_r(t)$.



(a) Overview.



(b) Close up view.

Figure 6.19: Bottom-hole volumetric flow $q(0,t)$ and drill bit flow q_{bit}.

Part IV
Conclusions

Chapter 7

Conclusion and Further Work

In this thesis, two controller schemes for adaptive set-point regulation of linear 2×2 hyperbolic systems are derived. In the first scheme, the only available measurement is collocated with actuation. Furthermore, the boundary condition anti-collocated with the control is affine and includes two uncertain parameters. In the second scheme, sensing is allowed to be taken at both boundaries, i.e. non-collated with actuation. Moreover, the boundary condition anti-collocated with control has a special bilinear form which also includes two uncertain parameters.

Proof of boundedness in the L_2-sense and point-wise in space are proved for all signals in the closed loop system. State estimates are shown to converge to zero and parameter estimates to some constant steady state value. For the non-collocated method, one of the parameter estimates was also shown to converge to its true value. Convergence to some specified set-point, where the set-point is a function dependent on the estimated parameters, was also shown for both methods.

The theory was applied on the Kick and Loss Detection and Attenuation Problem in Managed Pressure Drilling. A transformation relating the general 2×2 hyperbolic systems to a model of the drilling system was found. Stabilization of the pressure and flow throughout the well were proved. Furthermore, it was shown that the bottom-hole pressure can be stabilized at the unknown reservoir pressure and precise estimates of the reservoir pressure obtained for both methods.

Compared with using simple constant top-side flow, the simulation results showed a significant reduction in total in/out-flow and attenuation time when using the two derived methods. The two methods performed comparatively equal, with the non-collocated method performing only slightly better; with less oscillations and less overshoot. The performance gain was however negligible when compared to using constant top-side actuation. Both methods have essentially the same controller structure and differ mainly in how state and parameter estimates are generated. This suggest that the estimation part is not the limiting factor. Indeed, the methods performed close to the theoretical limit imposed by the propagation time (bottom-hole – top-side – bottom-hole). Using a higher adaptation gain might increase the slope of convergence of the estimation error, but the gain in convergence time will therefore only be marginal.

The non-collocated method has some other advantages over the collocated method: Due to the special bilinear form, a simpler control law with fewer backstepping transformations and integrators can be found for the non-collocated method. This comes at the expense of a slightly more involved stability proof. Furthermore, the controller kernels used in the non-collocated method can be implemented off-line, significantly improving computation time compared with the collocated method.

Suggested areas for further work are:

I) Prove the stronger result of point-wise convergence in space, that is $u(0,t) - rv(0,t) \to 0$.

II) Design a non-adaptive method better utilizing the additional anti-collocated measurement by incorporating information from the left boundary condition of the swapping filters into the adaptive law.

III) Investigate the robustness of the proposed design to parameter uncertainties. For example in terms of the generic 2×2 hyperbolic parameters; transport speed and source terms, or in terms of the drilling parameters; friction factor or drill bit flow.

IV) Perform experimental lab testing of the derived controller methods.

V) Consider a system where the anti-collated measurement is available only with a delay.

VI) Consider a system where the drill bit flow is controllable.

VII) Generalize the proposed designs to include general $n+1$ and $m+n$ systems. This can be used to model gas kicks.

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Appendix A

Additional Lemmas

This appendix presents some useful additional lemmas used throughout the thesis.

A.1 Upper Bound on 2×2 Volterra Equations

Lemma A.1. *Let $f_1, f_2 : [0,1] \to \mathbb{R}$ be any known functions, $g_1, g_2, g_3, g_4 : [0,1] \times [0,1] \to \mathbb{R}$ bounded known functions and $B_1, B_2 : [0,1] \to \mathbb{R}$ satisfying the, possibly time-varying, set of Volterra equations*

$$B_1(x) = f_1(x) + \int_x^1 g_1(x,\xi)B_1(\xi) + g_2(x,\xi)B_2(\xi)d\xi \tag{A.1a}$$

$$B_2(x) = f_2(x) + \int_x^1 g_3(x,\xi)B_1(\xi) + g_4(x,\xi)B_2(\xi)d\xi. \tag{A.1b}$$

The unknown functions B_1 and B_2 are bounded by

$$B_1(x)| \leq |f_1(x)| + \bar{g}||f||e^{2\bar{g}(1-x)} \tag{A.2a}$$

$$B_2(x)| \leq |f_2(x)| + \bar{g}||f||e^{2\bar{g}(1-x)} \tag{A.2b}$$

where

$$\bar{g} = \max_{x,\xi \in [0,1]} (|g_1(x,\xi)|, |g_2(x,\xi)|, |g_3(x,\xi)|, |g_4(x,\xi)|) \tag{A.3}$$

and

$$f(x) = |f_1(x)| + |f_2(x)|. \tag{A.4}$$

Proof. Consider the sequence

$$B_1^0(x) = f_1(x) \tag{A.5a}$$

$$B_1^{n+1}(x) = f_1(x) + \int_x^1 g_1(x,\xi)B_1^n(\xi) + g_2(x,\xi)B_2^n(\xi)d\xi \tag{A.5b}$$

and

$$B_2^0(x) = f_2(x) \tag{A.6a}$$

$$B_2^{n+1}(x) = f_2(x) + \int_x^1 g_3(x,\xi)B_1^n(\xi) + g_4(x,\xi)B_2^n(\xi)d\xi. \tag{A.6b}$$

Next, define the differences

$$S_1^0(x) = B_1^0(x) = f_1(x) \tag{A.7a}$$

$$\begin{aligned} S_1^n(x) =& B_1^n(x) - B_1^{n-1}(x) \\ =& f_1(x) + \int_x^1 \left(g_1(x,\xi)B_1^{n-1}(\xi) + g_2(x,\xi)B_2^{n-1}(\xi)\right) d\xi \\ & - f_1(x) - \int_x^1 \left(g_1(x,\xi)B_1^{n-2}(\xi) + g_2(x,\xi)B_2^{n-1}(\xi)\right) d\xi \\ =& \int_x^1 g_1(x,\xi)\left(B_1^{n-1}(\xi) - B_1^{n-2}(\xi)\right) \\ & + g_2(x,\xi)\left(B_2^{n-1}(\xi) - B_2^{n-2}(\xi)\right) d\xi \\ =& \int_x^1 g_1(x,\xi)S_1^{n-1}(\xi) + g_2(x,\xi)S_2^{n-1}(\xi)d\xi \end{aligned} \tag{A.7b}$$

and similarly,

$$S_2^0(x) = B_1^0(x) = f_2(x) \tag{A.8a}$$

$$S_2^n(x) = \int_x^1 g_3(x,\xi)S_1^{n-1}(\xi) + g_4(x,\xi)S_2^{n-1}(\xi)d\xi. \tag{A.8b}$$

If the sum exists, we have

$$B_1(x) = \sum_{n=0}^{\infty} S_1^n(x) \tag{A.9}$$

and

$$B_2(x) = \sum_{n=0}^{\infty} S_2^n(x). \tag{A.10}$$

Assume S_1^n and S_2^n satisfy the bounds

$$|S_1^n(x)| \leq \bar{g}||f||\frac{(2\bar{g})^{n-1}}{(n-1)!}(1-x)^{n-1} \tag{A.11}$$

$$|S_2^n(x)| \leq \bar{g}||f||\frac{(2\bar{g})^{n-1}}{(n-1)!}(1-x)^{n-1} \tag{A.12}$$

where $\bar{g}$ and f are defined in (A.3) and (A.4), respectively. For $n = 1$, we have

$$\begin{aligned}
S_1^1(x) &= \int_x^1 g_1(x,\xi)S_1^0(\xi) + g_2(x,\xi)S_2^0(\xi)d\xi \\
&= \int_x^1 g_1(x,\xi)f_1(\xi) + g_2(x,\xi)f_2(\xi)d\xi \\
&\leq \int_x^1 |g_1(x,\xi)||f_1(\xi)| + |g_2(x,\xi)||f_2(\xi)|d\xi \\
&\leq \int_x^1 \bar{g}|f_1(\xi)| + \bar{g}|f_2(\xi)|d\xi \\
&= \int_x^1 \bar{g}f(\xi)d\xi \\
&\leq \sqrt{\int_x^1 |\bar{g}|^2 d\xi}\sqrt{\int_0^1 |f(\xi)|^2 d\xi} \\
&= \bar{g}\sqrt{1-x}||f|| \\
&= \bar{g}||f|| \qquad \text{(A.13)}
\end{aligned}$$

and similarly

$$S_2^1(x) \leq \bar{g}||f|| \qquad \text{(A.14)}$$

which shows that (A.11) and (A.12) hold for $n = 1$. Furthermore,

$$\begin{aligned}
S_1^{n+1}(x) &= \int_x^1 g_1(x,\xi)S_1^n(\xi) + g_2(x,\xi)S_2^n(\xi)d\xi \\
&\leq \bar{g}\int_x^1 |S_1^n(\xi)| + |S_2^n(\xi)|d\xi \\
&\leq \bar{g}||f||\frac{(2\bar{g})^n}{(n-1)!}\int_x^1 (1-\xi)^{n-1}d\xi \\
&\leq \bar{g}||f||\frac{(2\bar{g})^n}{(n)!}(1-x)^n \qquad \text{(A.15)}
\end{aligned}$$

and similarly

$$S_2^{n+1}(x) = \bar{g}||f||\frac{(2\bar{g})^n}{(n)!}(1-x)^n \qquad \text{(A.16)}$$

which shows that (A.11) and (A.12) hold for for all n. An upper bound for (A.1) can now be found as

$$\begin{aligned}
B_1(x)| &= \sum_{n=0}^{\infty} S_1^n(x) \\
&\leq |f_1(x)| + \bar{g}||f||\sum_{n=0}^{\infty}\frac{(2\bar{g})^{n-1}}{(n-1)!}(1-x)^{n-1} \\
&\leq |f_1(x)| + \bar{g}||f||e^{2\bar{g}(1-x)} \qquad \text{(A.17)}
\end{aligned}$$

and

$$B_2(x)| \leq +|f_2(x)| + \bar{g}||f||e^{2\bar{g}(1-x)} \tag{A.18}$$

□

A.2 Young's Inequality

Lemma A.2 (Young's Inequality)**.** *For two numbers* $a, b \in \mathbb{R}$*,* $a, b \geq 0$*. If* $p, q \in \mathbb{R}$*,* $p, q > 0$ *such that*

$$\frac{1}{p} = \frac{1}{q} = 1, \tag{A.19}$$

then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{A.20}$$

Proof. See e.g. Patty (2015). □

A.3 Cauchy–Schwarz' inequality

The following form of the Cauchy–Schwarz' inequality is used extensively in this thesis:

Lemma A.3 (Cauchy–Schwarz' inequality)**.** *Let* $f, g \in \mathbb{R}$ *be any two square-integrable functions in* $[a, b]$*, then*

$$\int_a^b |f(\tau)g(\tau)|d\tau \leq \left(\int_a^b |f(\tau)|^2 d\tau\right)^{\frac{1}{2}} \left(\int_a^b |g(\tau)|^2 d\tau\right)^{\frac{1}{2}} \tag{A.21}$$

Proof. See e.g. Patty (2015). □

A.4 Squeeze Lemma

Lemma A.4 (Squeeze Lemma)**.** *Let* f, g, h *be real-valued functions satisfying*

$$g(t) \leq f(t) \leq h(t) \tag{A.22}$$

for all t near a*, except possibly at* a*. If*

$$\lim_{t \to a} g(t) = \lim_{t \to a} h(t) = L, \tag{A.23}$$

then

$$\lim_{t \to a} f(t) = L \tag{A.24}$$

Proof. See e.g. Sohrab (2003). □

A.5 Stability Lemma

The following Lemma presents Lemma 8 from Anfinsen and Aamo (2017c), which is a modification of Lemma D.3 from Smyshlyaev and Krstic (2010).

Lemma A.5 (Lemma 8 from Anfinsen and Aamo (2017c))**.** *Let $V_1(t)$, $V_2(t)$, $l_1(t)$, $l_2(t)$ and $f(t)$ be real-valued functions and $G(t)$ a real-valued matrix of dimension $n \times n$ defined for $t \geq 0$, with*

$$V_1(t) = \frac{1}{2}\nu^T(t)\nu(t) \tag{A.25}$$

for a signal vector ν of length n. Suppose

$$\begin{aligned}
0 \leq& V_1(t), V_2(t), l_1(t), l_2(t), f(t) \ \ \forall t \geq 0 && \text{(A.26a)}\\
l_1, l_2 \in& \mathcal{L}_1 && \text{(A.26b)}\\
|\nu| \in& \mathcal{L}_\infty && \text{(A.26c)}\\
0 \leq& G(t) = G^T(t) \leq I_{n\times n} && \text{(A.26d)}\\
\int_0^t f(s)ds \leq& Ae^{Bt} && \text{(A.26e)}\\
\dot{V}_1 \leq& -\nu^T(t)G(t)\nu(t) && \text{(A.26f)}\\
\dot{V}_2 \leq& -cV_2(t) + l_1(t)V_2(t) + l_2(t) - a\left(1 - b\nu^T(t)G(t)\nu(t)\right)f(t) && \text{(A.26g)}
\end{aligned}$$

for some positive constants A, B, a, b and c. Then $V_2 \in \mathcal{L}_1 \cap \mathcal{L}_\infty$.

Proof. See Anfinsen and Aamo (2017c, Lemma 8). □

A.6 Barbalat's Lemma

Lemma A.6 (Corollary of Barbalat's Lemma)**.** *Consider the function $f : \mathbb{R}_+ \to \mathbb{R}$. If $f, \dot{f} \in \mathcal{L}_\infty$ and $f \in \mathcal{L}_p$ for some $p \in [1, \infty)$, then*

$$\lim_{t\to\infty} f(t) = 0. \tag{A.27}$$

Proof. See e.g. (Krstic et al., 1995, Corollary A.7). □

A.7 Alternative Convergence Lemma

The following lemma from Liu and Krstić (2001) presents an alternative to Barbalat's lemma which only requires the derivative of the function to be upper bounded.

Lemma A.7 (Lemma 3.1 from Liu and Krstić (2001))**.** *Suppose that the function $f(t)$ defined on $[0, \infty)$ satisfies the following conditions:*

1. *$f(t) \geq 0$ for all $t \in [0, \infty)$,*

2. *$f(t)$ is differentiable on $[0, \infty)$ and there exists a constant M such that*

$$\dot{f}(t) \geq M, \quad \forall t \geq 0 \tag{A.28}$$

3. *$f \in \mathcal{L}_1$.*

Then we have

$$\lim_{t \to \infty} f(t) = 0 \tag{A.29}$$

Proof. See Liu and Krstić (2001). □

A.8 Leibniz' Differentiation Rule

Lemma A.8. *Let $f(x,t)$ be a function such that both $f(x,t)$ and its partial derivative $\dfrac{\partial}{\partial x} f(x,t)$ are continuous in t and x in some region of the (x,t)-plane, including $a(x) \leq t \leq b(x)$, $x_0 \leq x \leq x_1$. Also suppose that the functions $a(x)$ and $b(x)$ are both continuous and both have continuous derivatives for $x_0 \leq x \leq x_1$. Then for $x_0 \leq x \leq x_1$:*

$$\begin{aligned} \frac{d}{dx} \int_{a(x)}^{b(x)} f(x,t)dt =& f(x, b(x)) \frac{d}{dx} b(x) - f(x, a(x)) \frac{d}{dx} a(x) \\ &+ \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t) dx. \end{aligned} \tag{A.30}$$

Proof. See e.g. Protter and Morrey (2012). □

Appendix B

Additional Material

B.1 Gradient Method

Every on-line adaptive control scheme need some form of adaptive law that provides an estimate of the plant or controller parameters. Most of these adaptive laws are derived by minimizing a cost function with respect to the estimated parameters (Ioannou and Sun, 2012, Appendix B). This section provides a brief overview of the *gradient method*.

B.1.1 Gradient Method with Normalization

Following the notation used in Chapter 3, consider the linear parametric model

$$\vartheta = \psi^T \theta \tag{B.1}$$

where $\vartheta \in \mathbb{R}$ and $\psi \in \mathbb{R}^n$ are measured and $\theta \in \mathbb{R}^n$ is the unknown parameter vector to be estimated. Next, consider the normalized quadratic *cost function*

$$J(\hat{\theta})) = \frac{1}{2} \frac{\left(\vartheta - \psi^T \hat{\theta}\right)^2}{1 + \psi^T \psi} \tag{B.2}$$

An estimate of θ is found as the argument $\hat{\theta}$ that minimizes the cost function. That is,

$$\hat{\theta} = \arg\min \left(J(\hat{\theta})\right) \tag{B.3}$$

The gradient method is a line search method that search for a solution to (B.3) in the direction of the *steepest descent* (Nocedal and Wright, 2006, Section 2.2), that is

$$\dot{\hat{\theta}} = -\Gamma \nabla J(\hat{\theta}). \tag{B.4}$$

where Γ is a scaling matrix, specifying the rate of convergence, and $\nabla J(\hat{\theta})$ is the gradient of $J(\hat{\theta})$. Defining the normalized estimation error

$$\hat{e} = \frac{\left(\vartheta - \psi^T \hat{\theta}\right)}{1 + \psi^T \psi}, \tag{B.5}$$

we then have

$$\nabla J(\hat{\theta}) = -\frac{\left(\vartheta - \psi^T \hat{\theta}\right)}{1 + \psi^T \psi} \psi = -\hat{e}\psi \tag{B.6}$$

which gives the gradient adaptive law

$$\dot{\hat{\theta}} = -\Gamma \hat{e} \psi. \tag{B.7}$$

B.1.2 Gradient Projection

If the estimates are constrained to be within some bounds

$$\underline{\theta} \leq \hat{\theta} \leq \bar{\theta}, \tag{B.8}$$

we have the *constrained optimization problem*

$$\begin{aligned} \hat{\theta} &= \arg\min \left(J(\hat{\theta})\right) \\ \text{s.t. } & \underline{\theta} \leq \hat{\theta} \leq \bar{\theta} \end{aligned} \tag{B.9}$$

The solution to this optimization problem follows from the gradient projection method, where the estimate is forced to be within the set (B.8) by using the adaptive law (B.7) together with the projection operator Proj given component-wise as

$$\dot{\hat{\theta}}_i = \mathrm{Proj}_{a,b}(-\Gamma\hat{e}\psi, \hat{\theta}) = \begin{cases} 0 & \text{if } \hat{\theta}_i = a_i \text{ and } (-\Gamma\hat{e}\psi)_i < 0 \\ 0 & \text{if } \hat{\theta}_i = b_i \text{ and } (-\Gamma\hat{e}\psi)_i > 0 \\ (-\Gamma\hat{e}\psi)_i & \text{otherwise} \end{cases} \quad \text{for } i \in [1, n]. \tag{B.10}$$

where $(-\Gamma\hat{e}\psi)_i$ and $\hat{\theta}_i$ denotes the i-th component of $(-\Gamma\hat{e}\psi)$ and $\hat{\theta}$ respectively. From (Ioannou and Sun, 2012, Theorem 4.4.1): The gradient adaptive law with projection, retain all their properties that are established in the absence of projection and in addition guarantee (B.8).

B.1.3 Gradient Method for Bilinear Parametric Models

As an alternative to the linear parametric model (B.1), consider the *bilinear parametric model*[1]

$$\vartheta = k\left(\theta^T m + w\right) \tag{B.11}$$

[1] The parametric model (4.37) can be written on the form (B.11) by defining $\vartheta = y_1 - a$

where $\vartheta \in \mathbb{R}$, $m \in \mathbb{R}^n$ and $w \in \mathbb{R}$ are measured and $k \in \mathbb{R}$ and $\theta \in \mathbb{R}^n$ are unknown parameters. Furthermore it is assumed that $m \in \mathcal{L}_\infty$. Let the estimation error be given by

$$\hat{e} = \vartheta - \hat{k}\left(\hat{\theta}^T m + w\right) \tag{B.12}$$

where $\hat{k}$ and $\hat{\theta}$ are estimates of the unknown parameters. As in the case of the linear model, consider the quadratic cost function

$$J(\hat{k}, \hat{\theta}) = \frac{1}{2}\hat{e}^2 = \frac{\left(\vartheta - \hat{k}\left(\hat{\theta}^T m + w\right)\right)^2}{2\left(1 + w^2\right)} \tag{B.13}$$

Defining $\xi = \hat{\theta}^T m + w$, this can be written

$$J(\hat{k}, \hat{\theta}) = \frac{\left(\vartheta - k\hat{\theta}^T m - \hat{k}\xi + k\xi - kw\right)^2}{2\left(1 + w^2\right)} \tag{B.14}$$

Treating ξ as an independent variable of time (Ioannou and Sun, 2012, Page 211), the gradient of (B.14) with respect to $\hat{k}$ and $\hat{\theta}$ can be found as

$$\nabla_\theta J(\hat{k}, \hat{\theta}) = -\hat{e}\xi \tag{B.15a}$$

$$\nabla_k J(\hat{k}, \hat{\theta}) = -k\hat{e}m, \tag{B.15b}$$

yielding the adaptive laws

$$\dot{\hat{\theta}} = -\Gamma_1 \hat{e}\xi \tag{B.16a}$$

$$\dot{\hat{k}} = -\gamma_2 k\hat{e}m. \tag{B.16b}$$

B.2 Separation of Variables

The method of separation of variables is a group of methods for solving ODEs or PDEs (Renze, 2017). Here, we will present a method for solving ODEs on the form

$$\frac{d}{dx}y(x) = g(x)h(y(x)). \tag{B.17}$$

Rearranging the terms and integrating with respect to x from $x = a$ to $x = b$ give

$$\int_a^b \frac{1}{h(y(x))}\frac{dy(x)}{dx}dx = \int_a^b g(x)dx \tag{B.18}$$

If $g(x) = c\,\forall x$ for a constant c and $h(y(x)) = y(x)$, this can be solved explicitly to yield

$$\int_a^b \frac{1}{y(x)}\frac{d}{dx}y(x)dx = \int_a^b c dx$$

$$\int_{y(a)}^{y(b)} \frac{1}{y(x)} dy = \int_a^b c dx$$
$$\ln\left(\frac{y(b)}{y(a)}\right) = c(b-a)$$
$$y(b) = y(a)e^{c(b-a)}. \quad \text{(B.19)}$$

B.3 Integral Equation Types

An integral equation is an equation in which the unknown function $y(t)$ appears inside an integral (Kreyszig, 2011, Page 236). As an example of an integral equation, consider

$$\phi(x) = f(x) + \int_0^x K(x,y)\phi(y)dy \quad \text{(B.20)}$$

Integral equations can be categorized by three properties, creating in total eight different types of equations (Arfken and Weber, 2005, Chapter 16):

I) If both integration limits are fixed, the equation is a *Fredholm equation*. If one limit is variable, the equation is a *Volterra equation*. Equation (B.20) is therefore an example of a Volterra equation.

II) If the unknown function appears only inside the integral, it is of *first kind*. If the unknown function appear both inside and outside the integral, it is of *second kind*. Equation (B.20) is therefore of second kind.

III) If f in (B.20) is identically zero, the equation is termed *homogeneous*, otherwise *inhomogeneous*. Equation (B.20) is therefore inhomogeneous.

Appendix C

Conference Papers

The following submitted conference papers are included:

1. *Adaptive Set-Point Regulation of Linear* 2×2 *Hyperbolic Systems with Uncertain Affine Boundary Condition using Collocated Sensing and Control.* Based on the work in Chapter 3. Submitted to the *2017 Asian Control Conference.*

2. *Estimation of an Uncertain Bilinear Boundary Condition in Linear* 2×2 *Hyperbolic Systems with Application to Drilling.* Based on the estimation part in Chapter 4[1]. Submitted to the *17th International Conference on Control, Automation and Systems.*

[1]A more comprehensive journal paper is planned for the closed loop control part of Chapter 4.

Adaptive Set-Point Regulation of Linear 2×2 Hyperbolic Systems with Uncertain Affine Boundary Condition using Collocated Sensing and Control

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Abstract

In this paper, an adaptive control law that stabilizes a 2×2 linear hyperbolic system and achieves set-point regulation is derived. Sensing is restricted to be collocated with the control and anti-collocated with two uncertain parameters in an affine boundary condition. Proof of L_2-boundedness for all signals in the closed loop is given, along with convergence to the set-point in the sense of an appropriate objective. The theory is demonstrated in a simulation.

I. Introduction

A. Background

Linear 2×2 hyperbolic partial differential equations can be used to describe many real-world problems and has attracted considerable research interest in later years. This paper considers adaptive set-point regulation of such systems by using the method of infinite-dimensional backstepping for PDEs. The method in its current form was first introduced for parabolic PDEs in [1], [2], [3], where the gain kernel was expressed as a solution to a well-posed PDE.

The first result using backstepping applied on hyperbolic PDEs was for first order systems in [4]. The method was later extended for second order hyperbolic systems in [5], and for two coupled first order hyperbolic systems in [6]. The results in the latter were used in [7] for disturbance attenuation in managed pressure drilling which is similar to the problem considered in this paper.

While many results exist in the field of adaptive control for parabolic PDEs [8], adaptive control of hyperbolic PDEs is relatively new. Adaptive observers for $n+1$ hyperbolic systems using non-collocated sensing can be found in [9] using swapping filers, and in [10] using a Lyapunov approach. The extension to general $m+n$ systems is given in [11]. An adaptive observer for 2×2 systems using only collocated sensing and control is developed in [12]. Adaptive stabilization of the same type of systems, but without the additive boundary condition is considered in [13] and without the multiplicative boundary condition in [7]. Stabilization of the system in [12] with both multiplicative and additive boundary parameters, i.e. an affine boundary condition, has to the best of our knowledge not previously been addressed.

B. Notation

For a signal $z(x,t)$ defined for $0 \leq x \leq 1$, $t \geq 0$, $||z||$ denotes the L_2-norm ,$i.e.||z|| = \sqrt{\int_0^1 z^2(x,t)dx}$. For a time-varying, real signal $f(t)$, the following vector spaces are used: $f \in \mathscr{L}_p \leftrightarrow (\int_0^\infty |f(t)|^p dt)^{\frac{1}{p}} < \infty$ for $p \geq 1$ with the particular case $f \in \mathscr{L}_\infty \leftrightarrow \sup_{t \geq 0} |f(t)| < \infty$.

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C. Problem Statement

Consider the linear 2×2 first-order hyperbolic system

$$u_t(x,t) + \lambda u_x(x,t) = c_1(x)v(x,t) \tag{1a}$$
$$v_t(x,t) - \mu v_x(x,t) = c_2(x)u(x,t) \tag{1b}$$
$$u(0,t) = \theta_1 v(0,t) + \theta_2 \tag{1c}$$
$$v(1,t) = U(t) \tag{1d}$$

defined for $x \in [0,1], t \geq 0$, where u, v are the system states, $\lambda, \mu > 0$ and $c_1(x), c_2(x) \in C([0,1])$ are known, while

$$\theta_i \in \left[\underline{\theta}_i, \bar{\theta}_i\right] \subset \mathbb{R} \tag{2}$$

for $i \in \{1,2\}$, are unknown boundary parameters with known bounds $\underline{\theta}_1 \leq \bar{\theta}_1$, $\underline{\theta}_2 \leq \bar{\theta}_2$. The objective is to design a control input $U(t)$ such that the system (1) is adaptively stabilized in the L_2-sense, and such that the objective

$$\lim_{t\to\infty} \int_t^{t+T} |rv(0,\tau) - u(0,\tau)|\, d\tau = 0 \tag{3}$$

with $r \notin \left[\underline{\theta}_1, \bar{\theta}_1\right]$, is achieved for some arbitrary $T > 0$. Sensing is restricted to the boundary collocated with actuation, that is,

$$y(t) = u(1,t) \tag{4}$$

is the only available measurement. It is assumed that the initial conditions $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ satisfy $u_0, v_0 \in L_2([0,1])$.

The motivation for stabilizing (1) and achieving (3) comes from the Kick and Loss problem in managed pressure drilling, where the goal is to detect and attenuate any flow between the well and the reservoir by regulating the down-hole pressure to balance the reservoir pressure. It can be shown that this objective is equivalent to (3). The challenge of this problem comes from the fact that the reservoir pressure is unknown and so the system needs to track a reference signal dependent on the time-varying estimates of the unknown boundary parameters θ_1, θ_2 and unmeasured system states u, v.

D. Paper Structure

This paper is organized as follows: In Section II, an observer is presented that estimates the unknown boundary parameters and system states from sensing (4). This observer is formally stated in Theorem 1. Section III contains the main contribution, which is the design of a closed loop adaptive control law that achieves (3) by adaptively tracking a boundary-parameter-estimate-dependent reference signal. The result is formally stated in Theorem 2. An illustrative simulation is given in Section V demonstrating the performance of the controller. Some concluding remarks are offered in Section VI.

II. Observer Design

In this section, the observer from [12] will be presented together with some additional properties needed for solving the adaptive control problem, that were not proven in [12].

A. Observer Equations

Consider the observer

$$\hat{u}_t(x,t)+\lambda\hat{u}_x(x,t) = c_1(x)\hat{v}(x,t) + P_1(x,t)\left(y(t)-\hat{u}(1,t)\right) \tag{5a}$$

$$\hat{v}_t(x,t)-\mu\hat{v}_x(x,t) = c_2(x)\hat{u}(x,t) + P_2(x,t)\left(y(t)-\hat{u}(1,t)\right) \tag{5b}$$

$$\hat{u}(0,t) = \hat{\theta}_1(t)\hat{v}(0,t)+\hat{\theta}_2(t) \tag{5c}$$

$$\hat{v}(1,t) = U(t) \tag{5d}$$

where $\hat{u},\hat{v}$ are estimates of the system states with initial conditions $\hat{u}(x,0)=\hat{u}_0(x)$, $\hat{v}(x,0)=\hat{v}_0(x)$ satisfying $\hat{u}_0$, $\hat{v}_0 \in L_2([0,1])$. The parameters $\hat{\theta}_1$, $\hat{\theta}_2$ are estimates of the boundary parameters θ_1, θ_2 respectively, and P_1, P_2 are output injection gains to be specified.

Subtracting (5) from (1) gives the state estimation error dynamics

$$\tilde{u}_t(x,t)+\lambda\tilde{u}_x(x,t) = c_1(x)\tilde{v}(x,t)-P_1(x,t)\tilde{u}(1,t) \tag{6a}$$

$$\tilde{v}_t(x,t)-\mu\tilde{v}_x(x,t) = c_2(x)\tilde{u}(x,t)-P_2(x,t)\tilde{u}(1,t) \tag{6b}$$

$$\tilde{u}(0,t) = \hat{\theta}_1(t)\tilde{v}(0,t) + \tilde{\theta}_1(t)v(0,t)+\tilde{\theta}_2(t) \tag{6c}$$

$$\tilde{v}(1,t) = 0 \tag{6d}$$

where $\tilde{u}=u-\hat{u}$, $\tilde{v}=v-\hat{v}$, $\tilde{\theta}_1=\theta_1-\hat{\theta}_1$ and $\tilde{\theta}_2=\theta_2-\hat{\theta}_2$.

B. Decoupling the Observer Error Dynamics

The invertible backstepping transformation

$$\tilde{u}(x,t) = \alpha(x,t)+\int_x^1 P^u(x,\xi,t)\alpha(\xi,t)d\xi \tag{7a}$$

$$\tilde{v}(x,t) = \beta(x,t)+\int_x^1 P^v(x,\xi,t)\alpha(\xi,t)d\xi \tag{7b}$$

is used in Lemma 1 in [12] to transform the estimation error dynamics (6) into a target system in terms of the states (α,β) that facilitates the design of an adaptive law and provides the injection gains P_1,P_2 in (5).

Let $d_\alpha=\frac{1}{\lambda}$, $d_\beta=\frac{1}{\mu}$ and $t_F=d_\alpha+d_\beta$. It is shown in [12] that the β-subsystem of the target system is independent of α and we have $\beta\equiv 0$ for $t>d_\beta$ and the target system is reduced to

$$\alpha_t(x,t)+\lambda\alpha_x(x,t) = 0 \tag{8a}$$

$$\alpha(0,t) = \tilde{\theta}_1(t)v(0,t)+\tilde{\theta}_2(t). \tag{8b}$$

C. Adaptive Law

The following lemma, which is a modification of Lemma 2 in [12], provides a linear parametric model that can be used for designing parameter update laws.

Lemma 1: For $t>t_F$, the signals

$$\vartheta(t) = y(t)-\hat{u}(1,t)+\hat{\theta}_1(t-d_\alpha)\bar{v}(t)+\hat{\theta}_2(t-d_\alpha) \tag{9a}$$

$$\bar{v}(t) = \hat{v}(0,t-d_\alpha) + \int_0^1 P^v(0,\xi,-t-d_\alpha)y(t-\frac{\xi}{\lambda})d\xi$$

$$-\int_0^1 P^v(0,\xi,-t-d_\alpha)\hat{u}(1,t-\frac{\xi}{\lambda})d\xi \tag{9b}$$

are related to the unknown parameters through the linear parametric model

$$\vartheta(t) = \psi^T(t)\theta \tag{10}$$

where

$$\psi(t) = \begin{bmatrix}\bar{v}(t) & 1\end{bmatrix}^T, \quad \theta = \begin{bmatrix}\theta_1 & \theta_2.\end{bmatrix}^T \tag{11}$$

In addition, the relationship $\bar{v}(t) = v(0,t-d_\alpha)$, holds for $t > t_F$.

Proof: See [12] ■

The linear relationship (10) facilitates for the application of any standard identification law. We use the gradient method with normalization and projection. State estimates can be generated by combining the resulting parameter estimates with the observer (5). The adaptive law will be restated here together with some properties needed for adaptive control design. This is a modification of [13, Theorem 3] with the additive boundary parameter θ_2 included.

Theorem 1 (Modified from Theorem 3 in [13]): Consider the adaptive law

$$\dot{\hat{\theta}}(t) = \begin{cases}\mathrm{Proj}_{\underline{\theta},\bar{\theta}}\left(\Gamma\frac{\vartheta(t)-\psi^T(t)\hat{\theta}(t)}{1+\psi^T(t)\psi(t)}\psi(t), \hat{\theta}(t)\right) & \text{for } t > t_F \\ 0 & \text{otherwise}\end{cases} \tag{12}$$

for some adaptation gain $\Gamma = \Gamma^T > 0$, where $\hat{\theta}(t) = [\hat{\theta}_1(t), \hat{\theta}_2(t)]^T$, $\underline{\theta} = [\underline{\theta}_1, \underline{\theta}_2]^T$ and $\bar{\theta} = [\bar{\theta}_1, \bar{\theta}_2]^T$, with ϑ, ψ generated using Lemma 1, and Proj is the projection operator with $a = [a_1, a_2]$, $b = [b_1, b_2]$, $\omega = [\omega_1, \omega_2]$, $\omega = [\omega_1, \omega_2]$ given component-wise by

$$\mathrm{Proj}_{a,b}(\tau,\omega) = \begin{cases}0 & \text{if } \omega_i = a_i \text{ and } \tau_i < 0\\ 0 & \text{if } \omega_i = b_i \text{ and } \tau_i > 0\\ \tau_i & \text{otherwise}\end{cases}$$

$$\text{for } i = 1,2. \tag{13}$$

Suppose system (1) and observer (5) have a unique solution $u, v, \hat{u}, \hat{v} \in L_2([0,1])\ \forall t \geq 0$ and the initial estimates $\hat{\theta}_0 = \hat{\theta}(0)$ are within the bounds (2), then the adaptive law (12) has the following properties:

1) $\hat{\theta}(t) \in [\underline{\theta}, \bar{\theta}], \quad \forall t > 0$
2) $\dot{\hat{\theta}}_1, \dot{\hat{\theta}}_2, \in \mathscr{L}_\infty \cap \mathscr{L}_2$.
3) $D = \frac{\tilde{\theta}_1(\cdot)v(0,\cdot)+\tilde{\theta}_2(\cdot)}{\sqrt{2+v^2(0,\cdot)}} \in \mathscr{L}_2$.
4) If $\bar{v} \in \mathscr{L}_\infty$, then $\hat{u}(x,\cdot), \hat{v}(x,\cdot) \in \mathscr{L}_\infty$ for all $x \in [0,1]$.

Proof: From [14, Theorem 4.4.1], the gradient adaptive law (12) with the projection retain all properties that are established in the absence of projection. Therefore, in proving properties 1 through 4 the unprojected adaptive law

$$\dot{\hat{\theta}}(t) = \Gamma\frac{\vartheta(t)-\psi^T(t)\hat{\theta}(t)}{1+\psi^T(t)\psi(t)}\psi(t) \tag{14}$$

will be considered. Furthermore, the projection operator will guarantee that the estimates θ_1, θ_2 remain within the bounds (2) for all $t > 0$.

Inserting the parametric model (10) into the right hand side of (14) and using $\tilde{\theta} = \theta - \hat{\theta}$ for $t > t_F$ give

$$\dot{\hat{\theta}}(t) = \Gamma\frac{\psi^T(t)\tilde{\theta}(t)}{1+\psi^T(t)\psi(t)}\psi(t). \tag{15}$$

Forming the Lyapunov function

$$V_0 = \frac{1}{2}\tilde{\theta}^T(t)\Gamma^{-1}\tilde{\theta}(t), \tag{16}$$

differentiating with respect to time and inserting (15) give

$$\dot{V}_0 = -\tilde{\theta}^T(t)\Gamma^{-1}\dot{\hat{\theta}}(t) = -\frac{\left(\psi^T(t)\tilde{\theta}(t)\right)^2}{1+\psi^T(t)\psi(t)} \leq 0. \tag{17}$$

Hence $V_0 \in \mathscr{L}_\infty$. Integrating (17) from $t=0$ to $t=\infty$, using that $V_0 \geq 0$ is a non-increasing function of time, gives

$$\frac{\psi^T(\tau)\tilde{\theta}(\tau)}{\sqrt{1+\psi^T(\tau)\psi(\tau)}} \in \mathscr{L}_2 \cap \mathscr{L}_\infty. \tag{18}$$

Substituting (18) into (15) gives Property 2.

Let $\gamma_{min}, \gamma_{max}$ be the smallest and largest eigenvalue of Γ, respectively. Starting from (17), a lower bound for $\dot{V}_0$ can be found as follows:

$$\begin{aligned}\dot{V}_0 = &-\frac{\left(\psi^T(t)\tilde{\theta}(t)\right)^2}{1+\psi^T(t)\psi(t)} = -\tilde{\theta}^T(t)\frac{\psi(t)\psi^T(t)}{1+\psi^T(t)\psi(t)}\tilde{\theta}(t) \\ \geq &-\tilde{\theta}^T(t)\tilde{\theta}(t) \geq -2\gamma_{max}\frac{1}{2}\tilde{\theta}^T(t)\Gamma^{-1}\tilde{\theta}(t) \geq -2\gamma_{max}V_0.\end{aligned} \tag{19}$$

A lower bound for $V_0(t)$ can now be found as

$$V_0(t) \geq e^{-2d_\alpha\gamma_{max}}V_0(t-d_\alpha), \tag{20}$$

for $t > d_\alpha$, meaning that the decay rate of V_0 is at maximum exponential. The following lower bound can then be obtained:

$$\tilde{\theta}^T(t)\tilde{\theta}(t) \geq \frac{\gamma_{min}}{\gamma_{max}}e^{-2d_\alpha\gamma_{max}}\tilde{\theta}^T(t-d_\alpha)\tilde{\theta}(t-d_\alpha). \tag{21}$$

Substituting (21) into (17), integrating from $t=0$ to $t=\infty$, and inserting (11) give property 3.

From (8), $\hat{\theta}_1, \hat{\theta}_2 \in \mathscr{L}_\infty$ and the assumption that $\bar{v}$ is bounded, one gets $\alpha(x,\cdot) \in \mathscr{L}_\infty$ for all $x \in [0,1]$. Boundedness of the observer kernels gives property 4.

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III. Closed Loop Adaptive Control

The main result from this section will be a control law $U(t)$ that, together with Theorem 1, adaptively stabilizes (1) in the L_2-sense and achieves (3). The adaptive control design follows similar steps as those in [13], so we start by restating some of the operators from [13], before the main theorem is presented. The stability proof is deferred to Section IV.

A. Backstepping Operators

Consider the operators from [13]

$$\mathscr{K}, \mathscr{K}_0 : L_2([0,1]) \times L_2([0,1]) \to L_2([0,1]) \tag{22}$$

given as

$$\mathscr{K}[a,b](x) = b(x) - \mathscr{K}_0[a,b](x) \tag{23a}$$

$$\mathscr{K}_0[a,b](x) = \int_0^x K^u(x,\xi)a(\xi)d\xi + \int_0^x K^v(x,\xi)b(\xi)d\xi \tag{23b}$$

where $a(x)$, $b(x)$ are two signals defined for $x \in [0,1]$ and (K^u, K^v) is the unique solution to a time-invariant system of PDEs given in [13]. Consider also the operator

$$\mathscr{G}[t], \mathscr{G}_0[t] : L_2([0,1]) \to L_2([0,1]), \tag{24}$$

from [13] given as

$$\mathscr{G}[a;t](x) = a(x) - \mathscr{G}_0[a;t](x) \tag{25a}$$

$$\mathscr{G}_0[a;t](x) = \frac{1}{\mu}\int_0^x g(x-\xi,t)a(\xi)d\xi \tag{25b}$$

where g is the on-line solution to the Volterra equation

$$g(x,t) = -\mathscr{G}[\hat{\theta}_1 H](x,t) \tag{26}$$

where

$$H(x) = -\lambda K^u(x,0). \tag{27}$$

Since kernels K^u, K^v are time-invariant, they can be calculated off-line, while g is time-dependent and must be calculated on-line.

B. Main Result

Theorem 2: Consider the system (1), the observer (5) and the adaptive law (12). The control law

$$\begin{aligned} U(t) =& \mathscr{K}_0[\hat{u},\hat{v}](1,t) + \mathscr{G}_0[\mathscr{K}[\hat{u},\hat{v}];t](1,t) \\ &+ \frac{\hat{\theta}_2(t)}{r-\hat{\theta}_1(t)} - \frac{1}{\mu}\int_0^1 \mathscr{G}[H;t](\xi,t)d\xi\,\hat{\theta}_2(t) \end{aligned} \tag{28}$$

where $\mathscr{K}, \mathscr{K}_0, \mathscr{G}, \mathscr{G}_0$ are the operators defined in (23) and (25), H is defined in (27), $r \notin [\underline{\theta}_1, \bar{\theta}_1]$, and $\hat{\theta}_1, \hat{\theta}_2$ are generated from the adaptive law (12), guarantees (3). Moreover, all signals in the closed loop system are bounded.

Remark 1: It should be noted that for $\theta_2 = 0$ and $\hat{\theta}_2 \equiv 0$, the control law (28) reduces to the control law presented in [12, Theorem 4]. In that case, the only solution satisfying (3) with $r \neq \theta_1$ is $u(0,t), v(0,t) \to 0$.

To improve readability, the control law $U(t)$ is decomposed into two parts

$$U(t) = U_1(t) + U_2(t) \tag{29}$$

where U_1 decouples the observer dynamics and eliminates boundary terms, bringing the system into an equivalent target system for which stability analysis is easier, while U_2 implements reference tracking so that the objective (3) is achieved. The terms are given in Lemma 2 and 4 in the next to sub-sections, respectively, while proof of Theorem 2 is deferred to Section IV

C. Decoupling the Observer Dynamics

Lemma 2: Consider the observer (5), and the operators $\mathscr{K}$, $\mathscr{K}_0$ from (23) and $\mathscr{G}, \mathscr{G}_0$ from (25). The transformation

$$w(x,t) = \hat{u}(x,t) \tag{30a}$$

$$z(x,t) = \mathscr{K}[\hat{u},\hat{v}](x,t) \tag{30b}$$

$$\zeta(x,t) = \mathscr{G}[z;t](x,t), \tag{30c}$$

and the control law (29) with

$$U_1(t) = \mathscr{K}_0[\hat{u},\hat{v}](1,t) + \mathscr{G}_0[z;t](1,t), \tag{31}$$

map (5) into the target system

$$\begin{aligned}
w_t(x,t) + \lambda w_x(x,t) =& c_1(x)z(x,t) + P_1(x,t)\alpha(1,t) \\
&+ \int_0^x \kappa_1(x,\xi)w(\xi,t)d\xi \\
&+ \int_0^x \kappa_2(x,\xi)z(\xi,t)d\xi && \text{(32a)} \\
\zeta_t(x,t) - \mu\zeta_x(x,t) =& \int_0^x B(x,\xi,t)\zeta(\xi,t)d\xi + H_1(x,t)\hat{\theta}_2(t) \\
&+ \Omega_1(x,t)\alpha(1,t) && \text{(32b)} \\
w(0,t) =& \hat{\theta}_1(t)\zeta(0,t) + \hat{\theta}_2(t) && \text{(32c)} \\
\zeta(1,t) =& U_2(t) && \text{(32d)}
\end{aligned}$$

where $\Omega(x) = \mathscr{K}[P_1,P_2](x)$, H is defined in (27), $\Omega_1(x,t) = \mathscr{G}[\Omega;t](x,t)$, $H_1(x,t) = \mathscr{G}[H;t](x,t)$, κ_1, κ_2 can be found as the solution to a 2×2 Volterra equation (see [13]). α is defined in (7), $\hat{\theta}_1, \hat{\theta}_2$ is obtained from (12) in Theorem 1, U_2 is the control signal to be designed, and B is defined in [13], and has the property $||B||^2 \in \mathscr{L}_1 \cap \mathscr{L}_\infty$.
The proof is similar to the proof of Lemma 7 and 8 in [13] and is therefore omitted.

The significance of Lemma 2 is that subsystem (32b) is independent of w. If $\zeta, \alpha, \hat{\theta}_1, \hat{\theta}_2$ are bounded, it can be noted from the transport equation (32a) and boundary condition (32c) that w will be bounded as well. Furthermore, $w(0,t)$ is uniquely determined by $\hat{\theta}_1, \hat{\theta}_2, \zeta$ in (32c). The problem of stabilizing (1) in the sense of (3) is therefore reduced to stabilizing ζ and α in the sense of some appropriate objective.

D. Reference Signal and Tracking

Stabilization of (1) in the sense of (3) can be transformed into a tracking problem for the ζ-system (32b). Specifically, an equivalent objective is for the ζ-system to track a time-varying reference signal $\zeta^*(t)$ selected as

$$\zeta^*(t) = \frac{\hat{\theta}_2(t)}{r - \hat{\theta}_1(t)} \tag{33}$$

where $\hat{\theta}_1, \hat{\theta}_2$ are generated using the adaptive law (12) in Theorem 1. The following lemma motivates the use of this reference signal.

Lemma 3: Consider the reference signal (33). If, for some $T > 0$,

$$\int_t^{t+T} \left|\zeta(0,\tau) - \zeta^*(\tau - d_\beta)\right| d\tau \to 0 \tag{34}$$

and $r \notin \left[\underline{\theta}_1, \bar{\theta}_1\right]$, then

$$\int_t^{t+T} |r\hat{v}(0,\tau) - \hat{u}(0,\tau)|\, d\tau \to 0. \tag{35}$$

If in addition

$$\int_t^{t+T} |\alpha(0,\tau)|\, dt \to 0 \tag{36}$$

and

$$||\alpha|| \to 0 \tag{37}$$

then the objective (3) is satisfied.

Proof: Starting with the integrand of (34), using transformation (30) evaluated at $x = 0$, rearranging and inserting the boundary condition (5c) give

$$\left|\zeta(0,\tau) - \zeta^*(\tau - d_\beta)\right| = \frac{|\hat{v}(0,\tau)r - \hat{u}(0,t)|}{\left|r - \hat{\theta}_1(t)\right|}. \tag{38}$$

Since $\hat{\theta}_1(t)$ is generated using projection, implying $\hat{\theta}_1 \in \left[\underline{\theta}_1, \bar{\theta}_1\right]$ and since, by assumption, $r \notin \left[\underline{\theta}_1, \bar{\theta}_1\right]$, there exists a $\delta > 0$ such that

$$\left|\zeta(0,\tau) - \zeta^*(\tau - d_\beta)\right| \geq \frac{1}{\delta}\left|\hat{v}(0,\tau)r - \hat{u}(0,t)\right| \tag{39}$$

From Theorem 1, we have $\dot{\hat{\theta}}_i \in \mathscr{L}_2$ for $i \in \{1,2\}$, implying $\zeta^*(t) \to \zeta^*(t - d_\beta)$. Integrating both sides from $\tau = t$ to $\tau = t + T$, it can be seen that (34) implies (36) and the first part of the proof is complete.

For the second part; from the backstepping transformation (7), the fact that $\beta \equiv 0$ for $t > t_\beta$, boundedness of the observer kernels, and using (36) and Cauchy-Schwarz' inequality, one obtains

$$|\tilde{u}(0,t)| \leq |\alpha(0,t)| + h||\alpha||, \quad \tilde{v}(0,t) \leq h||\alpha||. \tag{40}$$

Next, starting with (3), substituting $u = \tilde{u} + \hat{u}$ and $v = \tilde{v} + \hat{v}$ and inserting (40) give

$$\begin{aligned}\int_t^{t+T} |rv(0,\tau) - u(0,\tau)|\, d\tau \leq &\int_t^{t+T} |r\hat{v}(0,\tau) - \hat{u}(0,\tau)|\, d\tau \\ &+ \int_t^{t+T} |\alpha(0,\tau)| + ||\alpha|| d\tau.\end{aligned} \tag{41}$$

Finally, from (35)–(37), the right hand side will converge to zero asymptotically and the objective (3) follows. ■

The problem of stabilizing (1) and achieving (3) is now transformed to the problem of finding a controller that achieves (34), (36) and (37). A time delayed version of the signal (33) can be modeled as the simple transport equation

$$\phi_t(x,t) - \mu\phi_x(x,t) = 0 \tag{42a}$$

$$\phi(1,t) = \zeta^*(t). \tag{42b}$$

Lemma 4: Consider system (32b) and (32d) and (42). The linear transformation

$$\eta(x,t) = \zeta(x,t) - \phi(x,t) + H_2(x,t)\hat{\theta}_2(t) \tag{43}$$

and control law

$$U_2(t) = \zeta^*(t) - H_2(1,t)\hat{\theta}_2(t), \tag{44}$$

map system (32b), (32d) and (42) into the target system

$$\begin{aligned}\eta_t(x,t) - \mu\eta_x(x,t) = &H_2(x,t)\dot{\hat{\theta}}_2(t) + \left(\frac{\partial}{\partial t}H_2(x,t)\right)\hat{\theta}_2(t) \\ &+ \Omega_1(x,t)\alpha(1,t) + \int_0^x B(x,\xi,t) \\ &\times \left(\eta(x,t) - H_2(\xi,t)\hat{\theta}_2(t) + \phi(\xi,t)\right) d\xi\end{aligned} \tag{45a}$$

$$\eta(1,t) = 0 \tag{45b}$$

where

$$H_2(x,t) = \frac{1}{\mu}\int_0^x H_1(\xi,t)\, d\xi \tag{46}$$

has the property $\left|\frac{\partial}{\partial t}H_2(x,t)\right| \in \mathscr{L}_2 \cap \mathscr{L}_\infty$. The reference signal ζ^* is generated from (33) and $\dot{\hat{\theta}}_2$ from the adaptive law (12).

Proof: Differentiating (43) with respect to time and space and inserting the dynamics (32b) and (42a) give (45a). Evaluating (43) at $x = 1$, and inserting (32d), (42b) and (44) give (45b). The last property can be seen from inserting the definitions of H_1 and H_2 from Lemma 2, using that $|g|$ is bounded by $|\theta_1|$, boundedness of H (property of the operator) and from Theorem 1 that $\hat{\theta}_1 \in \mathscr{L}_2 \cap \mathscr{L}_\infty$. ■

IV. Stability Proof

Stabilization of (1) and achieving (3) is equivalent to stabilizing the coupled system

$$\alpha_t(x,t) + \lambda \alpha_x(x,t) = 0 \tag{47a}$$

$$\begin{aligned}\eta_t(x,t) - \mu \eta_x(x,t) =& H_2(x,t)\dot{\hat{\theta}}_2(t) + \frac{\partial}{\partial t}H_2(x,t)\hat{\theta}_2(t)\\ &+ \Omega_1(x,t)\alpha(1,t) + \int_0^x B(x,\xi,t)\\ &\times \left(\eta(x,t) - H_2(\xi,t)\hat{\theta}_2(t) + \phi(\xi,t)\right) d\xi\end{aligned} \tag{47b}$$

$$\alpha(0,t) = \tilde{\theta}_1(t)v(0,t) + \tilde{\theta}_2(t) \tag{47c}$$

$$\eta(1,t) = 0 \tag{47d}$$

and achieving

$$\lim_{t\to\infty}\int_t^{t+T} |\eta(0,\tau)|\, d\tau = 0, \quad \lim_{t\to\infty}\int_t^{t+T} |\alpha(0,\tau)|\, d\tau = 0 \tag{48}$$

where the state $v(0,t)$ in boundary condition (47d) is related to ζ^* and α through

$$v(0,t) = \eta(0,t) + \zeta^*(t - d_\beta) + \int_0^1 P^v(0,\xi,t)\alpha(\xi,t)d\xi \tag{49}$$

as can be seen from using $\beta \equiv 0$ for $t > d_\beta$ in (7b), the transformations (30) and (43), and the definition of ζ^* in (33).

Proof: [Proof of Theorem 2] Consider the Lyapunov function candidate

$$V_3 = a_1 V_1 + V_2 \tag{50}$$

where $a_1 > 0$ is a constant to be decided, and

$$V_1 = \lambda^{-1}\int_0^1 e^{-\delta x}\alpha^2(x,t)dx, \;\; V_2 = \mu^{-1}\int_0^1 e^{kx}\eta^2(x,t)dx \tag{51}$$

where α, η are the system states in the coupled system (47), H_2 and Ω_1 are defined in Lemma 2, and $\hat{\theta}_1, \hat{\theta}_2$ are obtained from the adaptive law (12) in Theorem 1. It can then be shown that $\dot{V}_3$ satisfies

$$\begin{aligned}\dot{V}_3 \le& a_1\left(-e^{-\delta}\alpha^2(1,t) + \left(\tilde{\theta}_1(t)v(0,t) + \tilde{\theta}_2(t)\right)^2 - \delta\lambda V_1\right)\\ &- \eta^2(0,t) - \mu k V_2 + \bar{\Omega}_1 V_2 + \frac{\bar{\Omega}_1}{\mu k}\left(e^k - 1\right)\alpha^2(1,t)\\ &+ 2V_2 + \left(\bar{H}_2\bar{\theta}_2\right)^2 e^k \mu^{-1}||B||^2 + \left(\bar{\zeta}^*\right)^2 e^k \mu^{-1}||B||^2\\ &+ \left(1 + e^k||B||^2\right)V_2 + \bar{H}_2 V_2 + \frac{\bar{H}_2}{\mu k}\left(e^k - 1\right)\dot{\hat{\theta}}_2^2(t)\\ &+ \bar{\theta}_2 h_3 V_2 + \frac{\bar{\theta}_2 c}{\mu k}\left(e^k - 1\right)\dot{\hat{\theta}}_1^2(t).\end{aligned} \tag{52}$$

Selecting $k > \frac{1}{\mu}\left(3+\bar{\Omega}_1+\bar{H}_2+\bar{\theta}_2 c\right)$, $a_1 = e^{\delta}\frac{\bar{\Omega}_1}{k\mu}(e^k-1)$ and $\delta=1$, yield

$$\begin{aligned}\dot{V}_3 \leq & a_1\left(\tilde{\theta}_1(t)v(0,t)+\tilde{\theta}_2(t)\right)^2-\eta^2(0,t)-h_4V_3\\ &+e^k||B||^2V_2+\left(\left(\bar{H}_2\bar{\theta}_2\right)^2+\left(\bar{\zeta}^*\right)^2\right)e^k\mu^{-1}||B||^2\\ &+\frac{\bar{H}_2}{\mu k}\left(e^k-1\right)\dot{\hat{\theta}}_2^2(t)+\frac{\bar{\theta}_2 c}{\mu k}\left(e^k-1\right)\dot{\hat{\theta}}_1^2(t).\end{aligned} \tag{53}$$

The first term can be rewritten on the form considered in property 3 of Theorem 1 by dividing and multiplying by $\left(2+v^2(0,t)\right)$. Using relation (49) gives

$$\dot{V}_3 \leq -h_4V_3+l_1(t)V_3(t)+l_2(t)-\left(1-h_5D^2(t)\right)\eta^2(0,t) \tag{54}$$

for some constants $h_4, h_5 > 0$, $D(t)$ defined in Property 2 of Theorem 1, and $l_1(t), l_2(t) \geq 0$ real valued functions given by

$$l_1(t) = 2a_1D^2(t)(\bar{P}^v)^2e^{\delta}\lambda+e^k||B||^2 \tag{55a}$$

$$\begin{aligned}l_2(t) = &\left(\left(\bar{H}_2\bar{\theta}_2\right)^2+\left(\bar{\zeta}^*\right)^2\right)e^k\mu^{-1}||B||^2\\ &+a_1D^2(t)\left(2+4(\bar{\zeta}^*)^2\right)\\ &+\left(\frac{\bar{\theta}_2 c}{\mu k}\left(e^k-1\right)+\frac{\bar{H}_2}{\mu k}\left(e^k-1\right)\right)\dot{\hat{\theta}}_1^2(t).\end{aligned} \tag{55b}$$

From Theorem 2 property 1, we have that $\hat{\theta} \in \mathscr{L}_\infty$, which together with property 2 and 3, and $||B||^2 \in \mathscr{L}_1 \cap \mathscr{L}_\infty$ from Lemma 2 give $l_1, l_2 \in \mathscr{L}_1 \cap \mathscr{L}_\infty$ and since all terms are squared in (55) that $l_1(t), l_2(t) \geq 0$ for all $t > 0$. Furthermore, it can be shown that $\dot{V}_0(t) \leq -h_5D(t)$ for $V_0 \geq 0$ defined in (14). Lastly, from (50), we have that $V_3 \geq 0$. Lemma 8 in [15] can now be applied, yielding $V_3 \in \mathscr{L}_1 \cap \mathscr{L}_\infty$. It follows that $||\alpha||, ||\eta|| \in \mathscr{L}_\infty$, and from the invertibility of the transforms (7), (30), (30) and (43) that

$$||\hat{u}||, ||\hat{v}||, ||u||, ||v|| \in \mathscr{L}_\infty \tag{56}$$

By using the same backstepping transformation considered in [13], boundedness point-wise i space can be proven, that is

$$u(x,\cdot), v(x,\cdot) \in \mathscr{L}_\infty, \quad \forall x \in [0,1] \tag{57}$$

leaving all signals in the closed loop bounded.

With v bounded, it follows from Theorem 1 property 4 that $\hat{u}, \hat{v}$ are bounded point-wise in space. Now, since $\hat{\theta}_1, \hat{\theta}_2 \in \mathscr{L}_\infty$ from Theorem 1 and $V_3, l_1, l_2 \in \mathscr{L}_\infty$, the right hand side of (54) is bounded, implying $\dot{V}_3 \in \mathscr{L}_\infty$. This result, together with $V_3 \in \mathscr{L}_1 \cap \mathscr{L}_\infty$ give, by Barbalat's Lemma (see [16, Corollary A.7]) $V_3 \to 0$ and $||\alpha||, ||\eta|| \to 0$.

Consider the Lyapunov function candidate

$$V_4 = ||\eta||^2 = \int_0^1 \eta^2(x,t)dx. \tag{58}$$

Differentiating with respect to time, and then integrating from t to $t+T$ gives

$$\begin{aligned}V_4(t+T)-V_4(t) = &-\mu\int_t^{t+T}\eta^2(0,\tau)d\tau\\ &+2\int_t^{t+T}\int_0^1\eta(x,\tau)\int_0^x B(x,\xi,\tau)\\ &\times\left(\eta(\xi,\tau)-H_2(\xi)\hat{\theta}_2(\tau)+\phi(\xi,\tau)\right)d\xi\,dx\,d\tau\end{aligned}$$

$$+2\int_t^{t+T}\int_0^1 \eta(x,\tau)\Omega_1(x,\tau)\,dx\,\alpha(1,\tau)\,d\tau$$
$$+2\int_t^{t+T}\int_0^1 \eta(x,\tau)H_2(x)dx\,\dot{\hat{\theta}}_2(\tau)\,d\tau. \tag{59}$$

Since $||\eta||, V_4 \to 0$ and $\int_t^{t+T}\eta^2(0,\tau)d\tau, V_4 \geq 0$, all terms on the right hand side of (59) converge to zero, and the left hand side is bounded from below. Furthermore, equation (47a) is a simple transport equation and we have $||\alpha|| \to 0$. By the squeeze theorem, it then follows that

$$\int_t^{t+T}|\eta(0,\tau)|d\tau \to 0, \quad \int_t^{t+T}|\alpha(0,\tau)|d\tau \to 0. \tag{60}$$

Inserting transformation (43) and the reference signal (33) into (60) yield (34) where the explicit solution to (42) has been inserted. By Lemma 3, the objective (3) is satisfied.

Inserting (31) and (44) from Lemma 2 and 4 respectively, together with the operators (23) and (25), into (29) give (28). ■

V. Simulation

The system was implemented in MATLAB with the adaptive observer of Theorem 1 and the controller of Theorem 2. The system parameters were chosen as $\lambda = \mu = 3$, $c_1(x) = 3e^{-2x}$, $c_2(x) = 3e^{2x}$, $\theta_1 = \frac{1}{10}$, $\theta_2 = \frac{1}{3}$ and $r = -1$. The adaptation gain was chosen as $\gamma_1 = \gamma_2 = 5$. The system is open loop ($U(t) \equiv 0$) unstable.

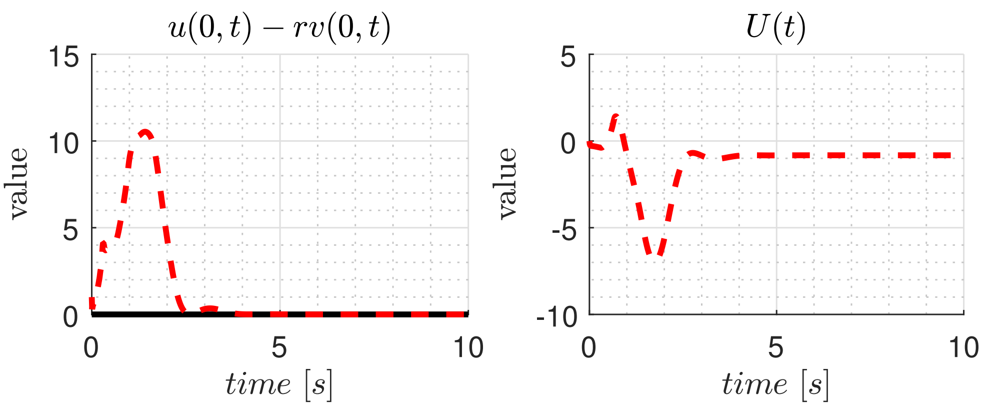

Fig. 1. Left: Control objective. Right: Actuation signal.

Figure 1 shows that the control objective is achieved and that the control signal converges to a steady state value.

VI. Concluding Remarks

We have combined an adaptive observer estimating the system states and unknown affine boundary parameters of a 2×2 linear hyperbolic system with a control law that stabilizes the system in the L_2-sense and achieves the control objective by adaptively tracking a boundary-parameter-estimate-dependent reference signal. Proofs of convergence, L_2- and point-wise boundedness was also given. The theory was demonstrated in a simulation.

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Estimation of an Uncertain Bilinear Boundary Condition in Linear 2×2 Hyperbolic Systems with Application to Drilling

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Abstract: In this paper, unknown parameters appearing in a special bilinear boundary condition are estimated by using swapping design filters to bring a system of 2×2 linear hyperbolic equations to static form. Estimates of the unknown parameters can then be generated by using any standard parameter identification law, and state estimates can be generated from the static relationship and parameter estimates. Sensing is allowed at both boundaries, and the measurement collocated with the uncertain parameters is allowed to be an arbitrary linear combination of the system states. Proof of boundedness of the adaptive law and conditions for parameter convergence are given. The theory is applied to the kick and and loss detection problem in manged pressure drilling and demonstrated in a simulation.

Keywords: parameter estimation, partial differential equations, hyperbolic systems, linear systems

1. INTRODUCTION

1.1 Background

Linear 2×2 hyperbolic partial differential equations can be used to describe many real-world problems and has attracted considerable research interest in later years. This paper considers state and boundary parameter estimation of such systems by using the method of infinite-dimensional backstepping for PDEs. The infinite-dimensional backstepping method in its current form was first introduced for parabolic PDEs in [13], [15], [16] for parabolic PDEs, where the gain kernel was expressed as a solution to a well-posed PDE. The first result using backstepping applied on hyperbolic PDEs was for first order systems in [11]. The method was later extended for second order hyperbolic systems in [14], and for two coupled first order hyperbolic systems in [18]. The results in the latter were used in [1] for disturbance attenuation in managed pressure drilling which is similar to the problem considered in this paper. While many results exist in the field of adaptive control for parabolic PDEs [17], adaptive control of hyperbolic PDEs is relatively new. Adaptive observers for $n + 1$ hyperbolic systems using non-collocated sensing can be found in [5] using swapping filers, and in [7] using a Lyapunov approach. The extension to general $m + n$ systems is given in [6]. An adaptive observer for 2×2 systems using only collocated sensing and control is developed in [3]. Adaptive stabilization of the same type of system, but without the additive boundary condition is considered in [4] and without the multiplicative boundary condition in [1]. The system in [5] is similar to the system considered in this paper, but with an affine boundary condition. Compared to the linear parametric model, the bilinear form has some desirable properties regarding parameter convergence.

The parameter estimation scheme developed in this paper is demonstrated on the kick and loss detection problem in managed pressure drilling, where the goal is to detect any sudden inflow into the well-bore or outflow into the reservoir by changes in reservoir pressure and the relation between pressure difference and net inflow. Previous results on kick/loss detection and attenuation in MPD have mainly focused on using lumped drilling models. A lumped ODE model is applied on a gas kick detection and mitigation problem in [19] by using a method for switched control of the bottom-hole pressure. Another lumped model for estimation and control of in-/outflux is presented in [8]. Kick handling methods for a first-order approximation to the PDE system is presented in [2] using LMI (Linear Matrix Inequality) based controller design. In/out-flux detection using an infinite dimensional observer is presented in [9]. Detection and handling of kick and loss using a distributed PDE model incorporating a model of the reservoir inflow relation, has to the best of the authors' knowledge not previously been addressed.

1.2 Notation

For a signal $z(x,t)$ defined for $0 \leq x \leq 1$, $t \geq 0$, $||z||$ denotes the L_2-norm , $i.e.||z|| = \sqrt{\int_0^1 z^2(x,t)dx}$. For a time-varying, real signal $f(t)$, the following vector spaces are used: $f \in \mathcal{L}_p \leftrightarrow \left(\int_0^\infty |f(t)|^p dt\right)^{\frac{1}{p}} < \infty$ for $p \geq 1$ with the particular case $f \in \mathcal{L}_\infty \leftrightarrow \sup_{t \geq 0}|f(t)| < \infty$.

1.3 Problem Statement

To model the annular pressure and flow in a well using managed pressure drilling, a modification of the model presented in [12] is used. The reservoir relation is modeled using a *productivity index based inflow* model at the bottom-hole boundary. This gives the following model of the system:

$$p_t(z,t) = -\frac{\beta}{A_1} q_z(z,t) \tag{1a}$$

$$q_t(z,t) = -\frac{A_1}{\rho} p_z(z,t) - \frac{F_1}{\rho} q(z,t) - A_1 g \tag{1b}$$

$$q(0,t) = J\left(p_r - p(0,t)\right) + q_{bit} \tag{1c}$$

$$p(l,t) = p_l(t) \tag{1d}$$

where $z \in [0,l]$ is the spatial independent variable, $t > 0$ is time, l is the well depth, $p(z,t)$ is pressure, $q(z,t)$ is volumetric flow, β is the bulk modulus of the mud, ρ is the density of the mud, A_1 is the cross sectional area of the annulus, F_1 is the friction factor, g is the acceleration of gravity, $J > 0$ is the productivity index, p_r the reservoir pressure and q_{bit} the flow through the drill bit. $p_l(t) = p(l,t)$ and $p_0(t) = p(0,t)$ are measured and $p_l(t) = p(l,t)$ can potentially be manipulated by control, although control design is not considered in this paper.

By using a suitable change of variables, it can be shown (see [1]) that (1) can be transformed into the linear 2×2 first-order hyperbolic system

$$u_t(x,t) + \lambda u_x(x,t) = c_1(x)v(x,t) \tag{2a}$$

$$v_t(x,t) - \mu v_x(x,t) = c_2(x)u(x,t) \tag{2b}$$

$$u(0,t) = rv(0,t) + k(\theta - y_0(t)) \tag{2c}$$

$$v(1,t) = U(t) \tag{2d}$$

defined for $x \in [0,1]$, $t \geq 0$, where u, v are the system states and $\lambda, \mu > 0,\ c_1(x), c_2(x) \in C([0,1])$ are known, while $k, \theta \in \mathbb{R}$, are unknown boundary parameters, but where $\mathrm{sign(k)}$ is known. The measurement collocated with actuation is related to $p_l(t)$ by

$$\begin{aligned} y_1(t) &= u(1,t) \\ &= \frac{1}{2}\left(q(l,t) - q_{bit} + \frac{A_1}{\sqrt{\beta\rho}}(p(l,t) + \rho g l + \frac{F_1}{A_1} q_{bit} l)\right) \\ &\times \exp\left(\frac{lF_1}{2\sqrt{\beta\rho}}\right) \end{aligned} \tag{3}$$

while the measurement anti-collocated with actuation can be found as a linear combination of the system states and is related to $p_0(t)$ by

$$y_0(t) = a_0 u(0,t) + b_0 v(0,t) = \frac{A_1}{\sqrt{\beta\rho}} p_0(t) \tag{4}$$

with $a_0 \neq 0$. Furthermore, $U(t)$ is the top-side control input and it is assumed that the initial conditions $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ satisfy $u_0,\ v_0 \in L_2([0,1])$.

2. OBSERVER DESIGN

In this section, swapping filters for state and parameter estimation are presented. Non-adaptive and adaptive relations between the system states and swapping filters are found, from which an adaptive estimation error driving the on-line parameter updates is generated.

2.1 Filter Design

Consider the input filters

$$\begin{aligned} a_t(x,t) + \lambda a_x(x,t) =& c_1(x)b(x,t) \\ &+ P_1(x)(y_1(t) - a(1,t)) \end{aligned} \tag{5a}$$

$$\begin{aligned} b_t(x,t) - \mu b_x(x,t) =& c_2(x)a(x,t) \\ &+ P_2(x)(y_1(t) - a(1,t)) \end{aligned} \tag{5b}$$

$$a(0,t) = rb(0,t) \tag{5c}$$

$$b(1,t) = U(t) \tag{5d}$$

and parameter filters

$$\begin{aligned} m_t(x,t) + \lambda m_x(x,t) =& c_1(x,t)n(x,t) \\ &- P_1(x)m(1,t) \end{aligned} \tag{6a}$$

$$\begin{aligned} n_t(x,t) - \mu n_x(x,t) =& c_2(x)m(x,t) \\ &- P_2(x)m(1,t) \end{aligned} \tag{6b}$$

$$m(0) = rn(0,t) + 1 \tag{6c}$$

$$n(1) = 0 \tag{6d}$$

and

$$w_t(x,t) + \lambda w_x(x,t) = c_1(x)z(x,t) - P_1(x)w(1,t) \tag{7a}$$

$$z_t(x,t) - \mu z_x(x,t) = c_2(x)w(x,t) - P_2(x)w(1,t) \tag{7b}$$

$$w(0,t) = rz(0,t) - y_0(t) \tag{7c}$$

$$z(1,t) = 0 \tag{7d}$$

where P_1, P_2 are gains to be designed. The input filters model how the control signal $U(t)$ affect the system states u, v, while the parameter filters model the effect of the boundary parameters k and θ on the system states.

2.2 Relationship to the System States

The *non-adaptive* state estimates are defined as

$$\bar{u}(x,t) = a(x,t) + k\left(\theta m(x,t) + w(x,t)\right) \tag{8a}$$

$$\bar{v}(x,t) = b(x,t) + k\left(\theta n(x,t) + z(x,t)\right) \tag{8b}$$

where the last term has the same bilinear form as boundary condition (2c). The non-adaptive state estimates are related to the system states through

$$u(x,t) = \bar{u}(x,t) + e(x,t) \tag{9a}$$

$$v(x,t) = \bar{v}(x,t) + \epsilon(x,t) \tag{9b}$$

where e, ϵ represent the *non-adaptive* estimation error.

Lemma 1: The error terms e and ϵ in (9) have the dynamics

$$e_t(x,t) + \lambda e_x(x,t) = c_1(x)\epsilon(x,t) - P_1(x)e(1,t) \tag{10a}$$

$$\epsilon_t(x,t) - \mu\epsilon_x(x,t) = c_2(x)e(x,t) - P_2(x)e(1,t) \tag{10b}$$

$$e(0,t) = r\epsilon(0,t) \tag{10c}$$

$$\epsilon(1,t) = 0. \tag{10d}$$

Proof: Inserting the static estimates in (8) into (9), rearranging, differentiating w.r.t. time and space and inserting the system dynamics (2a) and (2b) and filter dynamics (5a), (5b), (6a), (6b), (7a) and (7b), yield (10a) and (10b). The boundary condition in (10c) follows from evaluating (9a) at $x = 0$ and inserting (2c), (5c), (6c) and (7c). Similarly, the boundary condition in (10d) follows from evaluating (9b) at $x = 1$ and inserting (2d), (5d), (6d) and (7d). ■

If the error terms e, ϵ in (9) go to zero in finite time, then (8) is a static representation of the system states. Stability of the error system is addressed in the next section by first transforming (9) into an equivalent target system.

2.3 Error Dynamics Analysis

To facilitate the analysis, consider the operators

$$\mathcal{P}_1, \mathcal{P}_2 : L^2([0,1]) \times L^2([0,1]) \to L^2([0,1]) \tag{11}$$

given as

$$\begin{aligned}\mathcal{P}_1[a,b](x) =& a(x) + \int_x^1 P^{uu}(x,\xi)a(\xi)d\xi \\ &+ \int_x^1 P^{uv}(x,\xi)b(\xi)d\xi \end{aligned}\tag{12a}$$

$$\begin{aligned}\mathcal{P}_2[a,b](x) =& b(x) + \int_x^1 P^{vu}(x,\xi)a(\xi)d\xi \\ &+ \int_x^1 P^{vv}(x,\xi)b(\xi)d\xi \end{aligned}\tag{12b}$$

where $a(x), b(x)$ are two signals defined for $x \in [0,1]$ and $(P^{uu}, P^{uv}, P^{vu}, P^{vv})$ is the solution to

$$\lambda P_x^{uu}(x,\xi) + \lambda P_\xi^{uu}(x,\xi) = c_1(x)P^{vu}(x,\xi) \tag{13a}$$
$$\lambda P_x^{uv}(x,\xi) - \mu P_\xi^{uv}(x,\xi) = c_1(x)P^{vv}(x,\xi) \tag{13b}$$
$$\mu P_x^{vu}(x,\xi) - \lambda P_\xi^{vu}(x,\xi) = -c_2(x)P^{uu}(x,\xi) \tag{13c}$$
$$\mu P_x^{vv}(x,\xi) + \mu P_\xi^{vv}(x,\xi) = -c_2(x)P^{uv}(x,\xi) \tag{13d}$$
$$P^{uv}(x,x)\lambda + P^{uv}(x,x)\mu = -c_1(x) \tag{13e}$$
$$P^{vu}(x,x)\lambda + P^{vu}(x,x)\mu = c_2(x) \tag{13f}$$
$$P^{uu}(0,\xi) = rP^{vu}(0,\xi) \tag{13g}$$
$$P^{uv}(0,\xi) = rP^{vv}(0,\xi). \tag{13h}$$

It is shown i [18] that (13) has a bounded, continuous and unique solution. Furthermore, it is shown that the mapping $(a,b) \to (\bar{a},\bar{b})$ given by

$$a(x) = \mathcal{P}_1[\bar{a},\bar{b}](x) \tag{14a}$$
$$b(x) = \mathcal{P}_2[\bar{a},\bar{b}](x) \tag{14b}$$

is invertible.

Using the operators defined in (12), the non-adaptive error system can be transformed into an equivalent target system for which the stability analysis is easier. The backstepping transformation and corresponding target system used in the next lemma was first used in [18].

Lemma 2: Let $d_\alpha = \frac{1}{\lambda}$, $d_\beta = \frac{1}{\mu}$ and consider the non-adaptive error system (10). If the injection terms are selected as

$$P_1(x) = \lambda P^{uu}(x,1) \tag{15a}$$
$$P_2(x) = \lambda P^{vu}(x,1), \tag{15b}$$

then the error terms e, ϵ will tend to zero in a finite time given by $t_F = d_\alpha + d_\beta$, and (8) is a static representation of the system states u, v.

Proof: Consider the transformation

$$e(x,t) = \mathcal{P}_1[\alpha,\beta](x,t) \tag{16a}$$
$$\epsilon(x,t) = \mathcal{P}_2[\alpha,\beta](x,t) \tag{16b}$$

where $\mathcal{P}_1, \mathcal{P}_2$ are defined in (12). It is shown in [18] that the transformation maps the non-adaptive error system (10) into the target system

$$\alpha_t(x,t) + \lambda\hat{\alpha}_x(x,t) = 0 \tag{17a}$$
$$\beta_t(x,t) - \mu\hat{\beta}_x(x,t) = 0 \tag{17b}$$
$$\alpha(0,t) = r\beta(0,t) \tag{17c}$$
$$\beta(1,t) = 0. \tag{17d}$$

The subsystem consisting of (17b) and (17d) is a simple transport equation and will be zero $\beta \equiv 0$ for all $t > d_\beta$, reducing the boundary condition (17c) to $\alpha(0,t) = 0$ and we have $\alpha \equiv 0$ for another $t \geq d_\alpha$. From the invertibility of transformation (16), $e, \epsilon \equiv 0$ for all $t \geq d_\alpha + d_\beta$ follows and the relation (9) is reduced to

$$u(x,t) = \bar{u}(x,t) \tag{18a}$$
$$v(x,t) = \bar{v}(x,t) \tag{18b}$$

for all $t \geq d_\alpha + d_\beta$. ■

2.4 Adaptive Law

Before presenting the adaptive law and the main result of this section, an equivalent set of filter systems will be derived using a backstepping transformation. This equivalent set will be used to prove properties of the adaptive law.

Lemma 3: If P_1, P_2 are selected according to (15), the transformation

$$m(x,t) = \mathcal{P}_1[\check{m},\check{n}](x,t) \tag{19a}$$
$$n(x,t) = \mathcal{P}_2[\check{m},\check{n}](x,t) \tag{19b}$$

map the filters (6) into the target system

$$\check{m}_t(x,t) + \lambda\check{m}_x(x,t) = 0 \tag{20a}$$
$$\check{n}_t(x,t) - \mu\check{n}_x(x,t) = 0 \tag{20b}$$
$$\check{m}(0,t) = r\check{n}(0,t) + 1 \tag{20c}$$
$$\check{n}(1,t) = 0, \tag{20d}$$

and the transformation

$$w(x,t) = \mathcal{P}_1[\check{w},\check{z}](x,t) \tag{21a}$$

$$z(x,t) = \mathcal{P}_2[\check{w}, \check{z}](x,t) \tag{21b}$$

map the filters (7) into the target system

$$\check{w}_t(x,t) + \lambda \check{w}_x(x,t) = 0 \tag{22a}$$
$$\check{z}_t(x,t) - \mu \check{z}_x(x,t) = 0 \tag{22b}$$
$$\check{w}(0,t) = r\check{z}(0,t) - y_0(t) \tag{22c}$$
$$\check{z}(1,t) = 0 \tag{22d}$$

with $\mathcal{P}_1, \mathcal{P}_2$ defined in (12).

Proof: Equations (20a) and (20b) follow from differentiating (19) and inserting (6a) and (6b). Similarly, (22a) and (22b) follow from differentiating (21) and inserting (7a) and (7b). The boundary conditions in (20c) and (22c) are obtained by evaluating (19) and (21) at $x = 0$ and using (13). The boundary conditions in (20d) and (22d) follow trivially from evaluating (19) and (21) at $x = 1$ and inserting (6d) and (21). ■

Using that $e(1,t) = 0$ for all $t > t_F$ from Lemma 1 and inserting (3), the static relationship (9) evaluated at $x = 1$ can be written on the bilinear form

$$y_1(t) - a(1,t) = k(\theta m(1,t) + w(1,t)). \tag{23}$$

Motivated by this bilinear form of the static relationship, the following *adaptive* state estimates are generated:

$$\hat{u}(x,t) = a(x,t) + \hat{k}(t)\left(\hat{\theta}(t)m(x,t) + w(x,t)\right) \tag{24a}$$
$$\hat{v}(x,t) = b(x,t) + \hat{k}(t)\left(\hat{\theta}(t)n(x,t) + z(x,t)\right). \tag{24b}$$

The adaptive state estimates are related to the system states through

$$u(x,t) = \hat{u}(x,t) + \hat{e}(x,t) \tag{25a}$$
$$v(x,t) = \hat{v}(x,t) + \hat{\epsilon}(x,t) \tag{25b}$$

where $\hat{e}, \hat{\epsilon}$ represent the *adaptive* estimation error.

Evaluating (25a) at $x = 1$, inserting (3) and rearranging then give

$$\hat{e}(1,t) = y_1(t) - a(1,t) - \hat{k}(t)\left(\hat{\theta}(t)m(x,t) + w(x,t)\right) \tag{26}$$

Assuming the sign of k is known, the *gradient method for bilinear parametric models* in [10, Theorem 4.52] can be used to minimize a cost function based on the square error $\hat{e}^2(1,t)$ and thereby forming an adaptive law for the parameter estimates $\hat{\theta}, \hat{k}$. The following theorem presents the main result on parameter estimation.

Theorem 1: Consider the adaptive law

$$\dot{\hat{\theta}}(t) = \begin{cases} \gamma_1 \mathrm{sign}(k) \dfrac{\hat{e}(1,t)}{1 + w^2(1,t)} m(1,t) & t \geq t_F \\ 0 & \text{otherwise} \end{cases} \tag{27a}$$

$$\dot{\hat{k}}(t) = \begin{cases} \gamma_2 \Xi(t) \dfrac{\hat{e}(1,t)}{1 + w^2(1,t)} & t \geq t_F \\ 0 & \text{otherwise} \end{cases} \tag{27b}$$

for some adaptation gain $\gamma_1, \gamma_2 > 0$ where $\Xi(t) = \hat{\theta}(t)m(1,t) + w(1,t)$, $m(1,t)$ and $w(1,t)$ are filters given in (6) and (7), $\hat{e}(1,t)$ is the adaptive estimation error (26). Suppose system (2) has a unique solution $u, v \in L_2([0,1])$ $\forall t \geq 0$ and sign(k) is known, then the adaptive law (27) has the following properties:

1. $\hat{\theta}$, $\hat{k}$, $\in \mathcal{L}_\infty$.
2. $\dot{\hat{\theta}}$, $\dot{\hat{k}}$, $\in \mathcal{L}_\infty \cap \mathcal{L}_2$.
3. $\hat{\theta}(t) \to \hat{\theta}(t - d_\beta)$ and $\hat{k}(t) \to \hat{k}(t - d_\beta)$.
4. $\frac{\tilde{k}(\theta - y_0) + \hat{k}\tilde{\theta}}{\sqrt{1 + w^2(1,\cdot)}} \in \mathcal{L}_2$ where $\tilde{\theta} = \theta - \hat{\theta}$ and $\tilde{k} = k - \hat{k}$.
5. If $w(1,\cdot) \in \mathcal{L}_\infty$ and $\hat{\theta}m(1,\cdot) + w(1,\cdot) \in \mathcal{L}_2$, then $\hat{\theta}$ converges to θ and $\hat{k}$ converges to some constant.

Proof: Consider the Lyapunov function candidate

$$V_0 = |k| \frac{1}{2\gamma_1} \tilde{\theta}^2 + \frac{1}{2\gamma_2} \tilde{k}^2 \tag{28}$$

where $\tilde{\theta} = \theta - \hat{\theta}$ and $\tilde{k} = k - \hat{k}$. Differentiating and inserting the adaptive laws (27) for $t > t_F$ give

$$\dot{V}_0 = |k| \frac{1}{\gamma_1} \tilde{\theta} \dot{\hat{\theta}} + \frac{1}{\gamma_2} \tilde{k} \dot{\hat{k}} = -\frac{\hat{e}^2(1,t)}{1 + w^2(t)} \leq 0 \tag{29}$$

which shows that $V_0, \tilde{\theta}, \tilde{k} \in \mathcal{L}_\infty$, and Property 1 follows.

The transformed filter system $(\check{m}, \check{n})$ in (20) is a simple cascaded transport equation and we have $\check{m} \equiv 1$ and $\check{n} \equiv 0$ for all $x \in [0,1]$ and $t > t_F$. From the invertibility of transformation (19), we have $m(x,\cdot), n(x,\cdot) \in \mathcal{L}_\infty$, which together with Property 1 give

$$\frac{\hat{e}(1,\cdot)}{\sqrt{1 + w^2(1,\cdot)}} \in \mathcal{L}_\infty, \qquad \frac{m(1,\cdot)}{\sqrt{1 + w^2(1,\cdot)}} \in \mathcal{L}_\infty \tag{30}$$

Integrating (29) from $t = 0$ to $t = \infty$ and using that $V_0 \geq 0$ is a non-increasing function of time give

$$\frac{\hat{e}(1,\cdot)}{\sqrt{1 + w^2(\cdot)}} \in \mathcal{L}_2, \tag{31}$$

from which it follows, together with (30) and the adaptive laws (27) that $\dot{\hat{\theta}}, \dot{\hat{k}} \in \mathcal{L}_\infty \cap \mathcal{L}_2$ (Property 2).

The proof of Property 3 follow from Property 2 by using Cauchy-Schwarz' inequality.

Let

$$\Theta(t) = \left[\tilde{k}(t), \quad \sqrt{|k|}\tilde{\theta}(t)\right]^T \tag{32a}$$

$$\Psi(t) = \frac{1}{\sqrt{1 + w^2(1,t)}} \begin{bmatrix} \hat{\theta}m(1,t) + w(1,t) \\ \mathrm{sign}(k)\sqrt{|k|}m(1,t) \end{bmatrix} \tag{32b}$$

$$\Gamma = \mathrm{diag}([\gamma_1, \gamma_2]) \tag{32c}$$

We then have $V_0 = \Theta^T(t)\Gamma^{-1}\Theta(t)$ and

$$\dot{V}_0 = -\hat{e}^2(1,t) = \Theta^T(t)\Psi(t)\Psi^T(t)\Theta(t). \tag{33}$$

From Property 1, a lower bound for $\dot{V}_0$ can be found as follows:

$$\begin{aligned}\dot{V}_0 =& -\hat{e}^2(1,t) = \Theta^T(t)\Psi(t)\Psi^T(t)\Theta(t)\\ \geq& -2h\gamma_{max}\frac{1}{2}\Theta^T(t)\Gamma^{-1}\Theta(t) \geq -2h\gamma_{max}V_0\end{aligned} \tag{34}$$

where $h > 0$ is a constant and γ_{max} the largest eigenvalue or Γ. A lower bound for V_0 can now be found as

$$V_0(t) \geq e^{-2d_\alpha h\gamma_{max}}V_0(t-d_\alpha), \tag{35}$$

which shows that the decay rate of V_0 is at maximum exponential. The following lower bound can then be obtained

$$\tilde{\Theta}^T(t)\tilde{\Theta}(t) \geq \frac{\gamma_{min}}{\gamma_{max}}e^{-2d_\alpha h\gamma_{max}}\tilde{\Theta}^T(t-d_\alpha)\tilde{\Theta}(t-d_\alpha) \tag{36}$$

Substituting the relation (36) into (33), integrating the result from $t = 0$ to $t = \infty$, using that $V_0 \geq 0$ is a non-increasing function of time, and using that $m(1,t) = \check{m}(1,t) = 1$ for all $t > t_F$ and similarly that $w(1,t) = \check{w}(1,t) = y_0(t-d_\alpha)$ for all $t > t_F$ gives, by inserting (32a) and (32b) and rearranging the terms, Property 4.

If the signal $\hat{\theta}(\tau)m(1,\tau) + w(1,\tau)$ is square integrable and by treating the same signal as an external input, it can be seen from inserting (26) into (27a) that (27a) form an exponentially stable system and it follows that $\tilde{\theta} \to 0$ as $t \to \infty$, or equivalently the first part of Property 5. The second part of Property 5 can be seen from (27b) by using Cauchy-Schwarz' inequality. ■

3. SIMULATION

The state and parameter estimator developed in the previous section is now applied to a managed pressure drilling system (MPD) and tested together with two closed loop control methods, the first of witch provides *constant top-side flow*. That is,

$$q_l(t) = q_{bit}, \qquad \forall t > 0. \tag{37}$$

This control method can be shown to stabilize the bottom hole pressure at the reservoir pressure (see [19]). Furthermore, it can be shown that this control law can be implemented as

$$U(t) = -y_1(t)\exp\left(-\frac{lF_1}{\sqrt{\beta\rho}}\right). \tag{38}$$

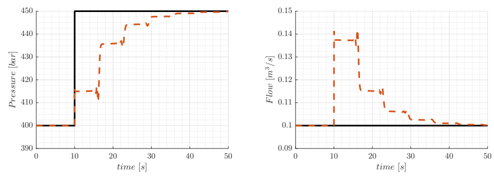
The second method provides *constant top-side pressure*. That is

$$p_l(t) = p_{sp} \tag{39}$$

where p_{sp} is some constant set-point. This control method can be implemented as

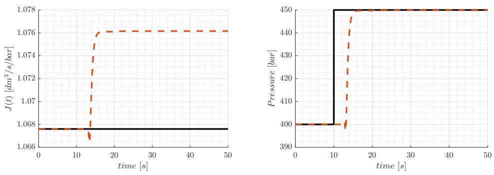
$$U(t) = -\frac{A_1}{\sqrt{\beta\rho}}\left(p_{sp} + \rho gl + \frac{F_1}{A_1}q_{bit}l\right)\exp\left(-\frac{lF_1}{2\sqrt{\beta\rho}}\right)$$

$$+ y_1(t)\exp\left(-\frac{lF_1}{\sqrt{\beta\rho}}\right). \tag{40}$$



(a) Pressure. (b) Flow.

Fig. 1: Bottom-hole pressure and flow (red dashed), and reservoir pressure and drill bit flow (black solid) using the constant flow control method.

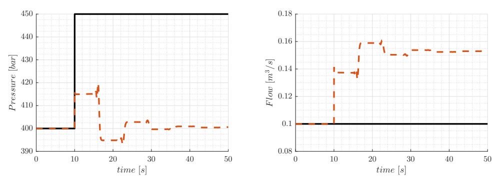


(a) Productivity index. (b) Reservoir pressure.

Fig. 2: Parameter estimates (red dashed) and actual parameters (black solid) using the constant flow control method.

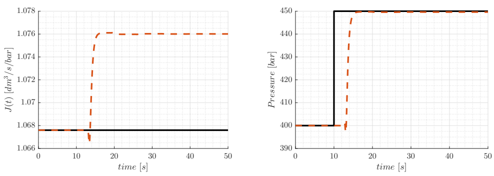
The complete system consisting of the MPD dynamics, swapping filters (5)–(7), state estimates (24), the adaptive law of Theorem 1, and control law (38) or (40) was implemented in MATLAB. The system parameters were chosen as $\beta = 7317\,\mathrm{Pa}$, $\rho = 1250\,\mathrm{kg\,m^{-3}}$, $l = 2500\,\mathrm{m}$, $A_1 = 0.024\,\mathrm{m^2}$, $F_1 = 10$, $g = 9.81\,\mathrm{m\,s^{-2}}$, $q_{bit} = 1/60\mathrm{m^3\,s^{-1}}$, $J = 1.068\times 10^{-8}\,\mathrm{m^3\,s^{-1}\,Pa^{-1}}$. The reservoir pressure was initially set to $p_r(0) = 400\,\mathrm{bar}$ and kept constant until a step to $p_r(t \geq t_0) = 450\,\mathrm{bar}$ occurs at $t_0 = 10\,\mathrm{s}$. The system is at steady state at $t = 0$ with the initial bottom-hole pressure set equal to the reservoir pressure and the bottom-hole flow equal to the drill bit flow. The adaptation gain was selected as $\gamma_1 = \gamma_2 = 5$.

For the constant flow control method, Figs. 1 and 2 show that the reservoir pressure estimate converges to its



(a) Pressure. (b) Flow.

Fig. 3: Bottom-hole pressure and flow (red dashed), and reservoir pressure and drill bit flow (black solid) using the constant pressure control method



(a) Productivity index. (b) Reservoir pressure.

Fig. 4: Parameter estimates (red dashed) and actual parameters (black solid) using the constant pressure control method.

true value and the productivity index to some constant value as the bottom-hole pressure is stabilized at the reservoir pressure. For the constant pressure control method, Figs. 3 and 4 show that the reservoir pressure estimate converges to its true value and the productivity index to some constant value even without convergence in the bottom-hole pressure to the reservoir pressure.

4. CONCLUDING REMARKS

We have designed swapping filters that transform a 2×2 linear hyperbolic system with a bilinear boundary condition into a static form. The gradient method was used to generate parameter and state estimates. Proofs of boundedness of the adaptive law and conditions for parameter convergence was given. The theory was applied to the kick/loss detection problem in managed pressure drilling and demonstrated in simulations.

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