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A Soft-Limit Theorem for the On-Shell Scattering Amplitudes Describing Processes Involving Massive Nambu-Goldstone Bosons

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I would like to dedicate this thesis to my parents (Bjørnar & Jannicke), grandparents (Frode & Marit and Per-Gunnar & Solbjørg), aunt (Turid) and sister (Katrine). Thank you for your unwavering support and encouragement during my studies.

Problem Description

Massive Nambu-Goldstone bosons are particles with a gap that is protected by symmetry and can be computed exactly with the help thereof. The goal of this project will be to investigate their further properties using Quantum Field Theory, and compare them to those of ordinary, gapless Nambu-Goldstone bosons: low-energy theorems, interactions with other modes etc. This is a follow-up project on the work done in Watanabe, Brauner and Murayama, Phys. Rev. Lett. 111, 021601, 2013. Required background: Classical and (basic) Quantum Field Theory, basics of the theory of Lie groups and Lie algebras.

Abstract

In this thesis we consider systems where both Lorentz invariance and a continuous symmetry group G is explicitly broken by coupling a chemical potential μ to one of the generators Q of G. In such systems it is well known that there exist two different types of massive Goldstone bosons, denoted here as mNGBs and pNGBs. It is also well known that each mNGB corresponds to the explicit breaking of a pair of generators that we can combine into a linear combination $Q_{\pm\sigma}$. Furthermore, $Q_{\pm\sigma}$ satisfies the relation $[Q, Q_{\pm\sigma}] = \pm q_\sigma Q_{\pm\sigma}$, where q_σ are the roots of the Cartan subalgebra. In [1] it is shown that this commutator relation implies that the mass of the mNGBs are given by $q_\sigma\mu$. This formula is derived using only symmetry, which means that the mass of the mNGBs is not subjected to loop corrections. Interestingly, not much is known about the scattering amplitudes for processes involving mNGBs and pNGBs. The goal of this thesis is to investigate such scattering amplitudes and try to make some exact statements about them.

We first showed that the Noether currents $J^\mu_{\pm\sigma}$ corresponding to $Q_{\pm\sigma}$ satisfy the covariant conservation laws $D_\mu J^\mu_{\pm\sigma} = \partial_\mu J^\mu_{\pm\sigma} \pm iq_\sigma\mu J^0_{\pm\sigma} = 0$. Next we defined the current element $\langle f|J^\mu_{\pm\sigma}|i\rangle$, representing a mNGB destroyed/created by $J^\mu_{\pm\sigma}$ interacting with an initial state $|i\rangle$ and a final state $|f\rangle$. By applying the covariant derivative to $\langle f|J^\mu_{\pm\sigma}|i\rangle$ and using the corresponding conservation law we managed to show that any scattering amplitude describing a process involving at least one mNGB vanishes in the soft limit of the mNGB. We proved this statement first perturbatively and then non-perturbatively, assuming a non-composite field. In the future, we would like to generalise our non-perturbative proof to be valid for composite fields as well.

In addition, we wanted to show that a scattering amplitude describing a process involving a pNGB does not vanish in the soft limit of the pNGB. To show this we studied the complete breaking of an $O(3)$ symmetry using an already known effective field theory approach. However, we considered $2 \rightarrow 2$ scattering and found to our surprise that the scattering amplitude vanished. We have not managed to explain this behaviour, although some solutions might be:

- The model has a hidden discrete symmetry, that forces the scattering amplitude to vanish in the soft limit.

- We have considered only the simplest possible scattering event, namely $2 \rightarrow 2$. Perhaps this is too simple, and that we instead should consider $3 \rightarrow 3$ scattering?

- pNGBs also interact weakly.

Sammendrag

I denne avhandlingen ser vi på systemer hvor både Lorentz invarians og en kontinuerlig symmetri gruppe G blir eksplisitt brutt ved å koble et kjemisk potensial μ til en av generatorene Q av G. I slike modeller er det et veletablert faktum at det eksisterer to typer massive Goldstone bosoner, som vi kaller mNGBer og pNGBer. I tillegg er det velkjent at ethvert mNGB korresponderer til ett par med eksplisitt brutte generatorer. Disse to generatorene kan brukes til å konstruere en lineær kombinasjon $Q_{\pm\sigma}$ som tilfredsstiller $[Q, Q_{\pm\sigma}] = \pm q_\sigma Q_{\pm\sigma}$, hvor q_σ er røttene til den Cartanske underalgebraen. I [1] vises det at denne kommutatorrelasjonen impliserer at massen til mNGBer er $q_\sigma \mu$. Da massene til mNGBene kun ble bestemt vha symmetriegenskaper følger det at massen ikke mottar kvantekorreksjoner. Overaskende nok, vet man ikke mye om spredningsamplituden for kollisjoner som involverer mNGBer og pNGBer. Målet med denne avhandlingen er å se om vi kan si noe generelt om disse spredningsamplitudene.

Vi har vist at Noether strømmene $J_{\pm\sigma}^\mu$, som korresponderer til $Q_{\pm\sigma}$, oppfyller bevaringslovene $D_\mu J_{\pm\sigma}^\mu = \partial_\mu J_{\pm\sigma}^\mu \pm i q_\sigma \mu J_{\pm\sigma}^0 = 0$. Vi definerte matrisen $\langle f|J_{\pm\sigma}^\mu|i\rangle$ som representerer et mNGB som vekselvirker med en begynnelsestilstand $|i\rangle$ og en sluttilstand $|f\rangle$. Ved å da bruke bevaringslovene $D_\mu J_{\pm\sigma}^\mu = 0$ beviste vi at en spredningsamplitude hvor minst et mNGB vekselvirker alltid forsvinner i grensa hvor 3-impulsen til det mNGBet forsvinner. Dette ble bevist først ved å bruke perturbasjonsteori og deretter bevist eksakt. Det nevnes at beviset kun er gyldig for et ikke-sammensatt felt. I fremtiden håper vi å kunne kvitte oss med denne antagelsen.

I tillegg, ønsket vi å vise at spredningsamplituden hvor minst et pNGB vekselvirker ikke nødvendigvis forsvinner i grensa hvor 3-impulsen til pNGBet forsvinner. Vi forsøkte å gi indikasjon på dette ved å studere en modell hvor en $O(3)$ symmetri ble fullstendig brutt. For å beskrive denne situasjonen ble vi nødt til å bruke en velkjent effektiv feltteori. Som et konkret eksempel så vi på spredningen av to og to partikler og oppdaget at spredningsamplituden forsvant i grensa hvor 3-impulsen til pNGBet forsvinner. Vi har enda ikke forstått grunnen til at spredningsamplituden forsvant, men noen forklaringer kan være:

- Modellen kan ha en diskret symmetri som krever at spredningsamplituden forsvinner for små 3-impulser.

- Spredningen av to og to partikler er ikke komplisert nok til å verifisere at spredningsamplituden ikke forsvinner. Det kan derfor være lurt å studere spredningen av tre og tre partikler istedet.

- Spredningsamplituden, som beskriver en kollisjon hvor minst et pNGB vekselvirker, forsvinner også for små 3-impulser.

Preface

This final version of my thesis represent two semesters of work. To be concrete part I and II of the main body as well as the appendix is more or less identical[1] to my specialisation project that I wrote in the autumn semester of 2016, see [2]. Meanwhile part III represents the work done this semester. I include all three parts in this thesis because I found it convenient to include all of the equations and explanations in one paper.

In this thesis we investigate some general properties of the two different types of massive Goldstone bosons potentially present in systems where both Lorentz invariance and a continuous symmetry is explicitly broken by a chemical potential. Of particular interest is the soft-limit[2] on-shell scattering amplitude for a process involving at least one massive Goldstone boson. In the specialisation project we conjectured a theorem[3] which we managed to prove this semester. The proof will form the foundation of an article that will be written during the upcoming summer.

I am indebted to my supervisors Jens O. Andersen (NTNU) and Tomas Brauner (UiS), for providing me with a project that I have enjoyed very much working on and for reading through my various drafts. In particular I would like to thank Tomas Brauner for both his time and patience in explaining and discussing various aspects of the project with me.

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[1]Except for some typographical errors that have been corrected and some physical interpretations that have been added/improved in chapters 3 and 4.

[2]By soft limit we mean that the 3-momentum of the NGB, pNGB or mNGB approaches zero.

[3]The theorem can be found on page 60

Contents

List of Tables

List of Figures

Notation and Conventions

In this section we include a list of notation and conventions that are used in this thesis. It is meant to be used as a quick reference when reading through the text.

- $\mathscr{A}_{tree}$: on-shell tree-level scattering amplitude.

- $\mathscr{A}_{\text{off}}$: off-shell tree-level scattering amplitude.

- $\mathscr{L}$: Lagrangian (density).

- $\mathscr{U}$: potential energy (density) or equivalently the static part of $\mathscr{L}$.

- SSB: spontaneous symmetry breaking.

- vev: vacuum expectation value.

- T_i: generator of a symmetry group.

- J_i^μ: a Noether current corresponding to a generator T_i.

- Soft limit: the limit of zero 3-momentum.

- NGB: a massless Nambu-Goldstone boson (of type-I or type-II).

- NGB of type-I: a massless Nambu-Goldstone boson whose dispersion relation is an odd power of momentum in the soft limit.

- NGB of type-II: a massless Nambu-Goldstone boson whose dispersion relation is an even power of momentum in the soft limit.

- mNGB: massive Nambu-Goldstone boson with a fixed gap.

- pNGB: pseudo Nambu-Goldstone boson with an unfixed gap.

- $|i\rangle$ & $|f\rangle$: an arbitrary initial and final state.

- $\langle f|J_\pm^\mu|i\rangle$: a current element describing the interaction between a massive Goldstone boson destroyed/created by $J_\pm^\mu$ and an arbitrary initial and final state.

- The Fourier transformation is defined as $f(x) = \int \frac{d^4k}{(2\pi)^4} f(k) e^{-ikx}$.

- We use the Minkowski metric $\eta_{\mu\nu} = \text{diag}\,(1, -1, -1, -1)$.

- For repeated indices we use the Einstein summation convention.

- w.r.t.: with respect to.

- d.o.f.: degrees of freedom.

- Boldfaced letters such as $\mathbf{p}$ denote 3-vectors.

- MC form: Maurer-Cartan form.

- f_{abc}: structure constants.

- σ_i and τ_i represent the Pauli matrices.

- $\mathrm{Tr}\{A\}$: the trace of the matrix A.

Introduction

In this chapter we give a brief historic overview of the study of Nambu-Goldstone bosons. In particular we will focus on the four types of Goldstone bosons potentially present in systems where a chemical potential is responsible for the explicit breaking of both Lorentz invariance and a continuous symmetry group G. In addition, we describe the process known as spontaneous symmetry breaking and provide one formulation of Goldstone's theorem. Finally we discuss the structure of the thesis, emphasising the logical flow of the text.

1.1   A Brief History of the Nambu-Goldstone Bosons

The first clue for the existence of massless Nambu-Goldstone bosons (NGBs) was discovered by Nambu in 1960. In his paper [3] he was investigating superconductivity by applying field theory to the BCS-Bogoliubov model. To be precise he used the Hartree-Fock method to determine an integral equation that described the self-energy of an electron in an electron gas interacting with phonons and a Coulomb potential. The homogenous solutions of the resulting equation turned out to describe collective excitations of pairs of massless quasiparticles, today known as massless NGBs.

One year later, inspired by Nambu's findings, Goldstone wanted to investigate these massless quasiparticles further. The results of his investigation are found in [4], where he conjectured that if a theory originally invariant under a symmetry group G had a ground state invariant under a corresponding subgroup $H \subset G$, NGBs are born. This conjecture is today known as Goldstone's theorem and was proved in 1962 by Goldstone, Salam and Weinberg in [5]. However, their proof only holds for Lorentz invariant systems leading to a large number of papers attempting to generalise the theorem. An early review can be found in [6].

When attempting to generalise Goldstone's theorem to non-Lorentz invariant systems[1] it was found that predicting the number of NGBs was quite subtle, we call this the counting problem. The first notable breakthrough in solving the counting problem was made by Nielsen and Chadha in 1975, see [7]. The key idea was that there actually exist two types of massless NGBs in non-Lorentz invariant systems exhibiting spontaneous symmetry breaking (SSB). The two types of NGBs can be most easily distinguished by considering their dispersion relations in the limit of zero 3-momentum (soft limit). They found that the soft-limit dispersion relations of type-I and type-II NGBs depend on odd and even powers of the norm of the 3-momentum

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[1] To be concrete: we consider systems where Lorentz invariance is broken explicitly by a chemical potential.

respectively. This important fact led to a successful proof of what is called the Nielsen and Chadha counting rule.

So far we have introduced the two types of massless NGBs, present in systems where Lorentz invariance is broken by the introduction of a chemical potential, whose properties are well known. Let us now introduce the main subject of this thesis namely the massive NGBs who are born when a chemical potential explicitly breaks both Lorentz invariance and a continuous symmetry. These special massive NGBs were first discovered by Nicolis and Piazza in 2013 in [8]. In their paper Nicolis and Piazza investigated theories exhibiting SSB in Lorentz invariant systems at finite charge density. Based on Goldstone's theorem one would expect to find only massless excitations, however this did not happen! Instead they found massive Goldstone bosons whose masses were fixed by the symmetry algebra and the chemical potential μ.[2] The same year Watanabe, Brauner and Murayama generalised these mNGBs in [9] and showed that the mNGBs exist also in non-relativistic systems such as (anti)ferromagnets subjected to an external magnetic field.

Furthermore, Nicolis, Penco, Piazza and Rosen discovered in [1] that there also exist massive NGBs whose masses are not fixed by the symmetry algebra.[3] They found that these second types of massive NGBs had masses that depend on the coupling constants g_i of the model.

To be concise the different types of NGBs that we will come across in this thesis are given in table 1.1, borrowed from my specialisation project [2].

Table 1.1: The table displays the different types of Goldstone bosons we come across in this thesis. Each Goldstone boson is characterised by their mass and one additional characteristic property. E is the energy of the particle, $\mathbf{p}$ is the 3-momentum, n is an integer, q_σ is a root of the Cartan subalgebra and $f(g_i)$ is a function depending on the coupling constants g_i of the theory.

Type of Goldstone boson	Mass	Characteristic property
NGB of type-I	0	$E = \|\mathbf{p}\|^{2n+1}$
NGB of type-II	0	$E = \|\mathbf{p}\|^{2n}$
mNGB	μq_σ	Mass fixed by the symmetry algebra.
pNGB	$f(g_i)\mu$	Mass not fixed by the symmetry algebra.

Finally let us address a particularity of scattering amplitudes describing processes involving massless NGBs, that we are very interested in. Consider that we are given a Lorentz invariant theory and are asked to compute the scattering amplitude for a scattering process involving at least one massless NGB. Weinberg proved in [10] that such an amplitude always[4] vanishes in the soft limit of the NGB. In this thesis we want to show that the same soft-limit theorem holds for mNGBs but does not hold for pNGBs.

1.2 Spontaneous Symmetry Breaking and Goldstone's Theorem

Given the importance of the process we have referred to as SSB let us give a general overview. Consider a theory specified by a Lagrangian density $\mathscr{L}(\phi)$ with a continuous symmetry group

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[2]We will call these particles massive Goldstone bosons (mNGBs).

[3]We will call these particles pseudo Goldstone bosons (pNGBs).

[4]Unless the theory contains a kinematic singularity.

G. If the ground state of the system is also invariant under G, we are not dealing with spontaneous symmetry breaking and no NGBs are born. However consider the case where the ground state of the system is not invariant under G, but is invariant under some subgroup $H \subset G$. Whenever this is the case we say that the symmetry G has been spontaneously broken down to the subgroup H. The generators of the coset space G/H are called broken because they do not annihilate the ground state. Goldstone's theorem tell us that there is a one-to-one correspondence between the broken generators and the NGBs in Lorentz-invariant systems. One possible formulation of the theorem is given below. It is proved in App. B.

Theorem 1: Goldstone's Theorem

Let G be the symmetry group of the Lagrangian with n_g generators $\{T_1, T_2, \ldots, T_{n_G}\}$. Also let $H \subset G$ be the subgroup that leaves the ground state invariant containing $n_H < n_G$ generators $\{T_1, T_2, \ldots T_{n_H}\}$ and consider the SSB of G down to H. For each generator in the set $\{T_1, T_2, \ldots, T_{n_G}\}$ that does not annihilate the ground state there exists exactly one NGB. Thus the total number of NGBs is given by the dimension of the coset space, $\dim G/H = n_G - n_H$.

1.3 Structure of the Thesis

The main body of this thesis consists primarily of three parts. Part I and II represent the work done in the specialisation project the previous semester, while part III represents the work done this semester.

In part I, we consider specific models containing massless NGBs and mNGBs. The goal of part I is to investigate the scattering amplitude corresponding to a collision involving mNGBs to form the foundation of our conjecture found on page. 60.

In part II, we consider a model containing both mNGBs and pNGBs. Our goal was to calculate a scattering amplitude corresponding to a collision involving pNGBs to see if we could verify that the pNGBs do not satisfy the soft-limit theorem. However, to our surprise we found that the scattering amplitude vanished. We have three possible explanations for this

- The model has a hidden symmetry, that forces the scattering amplitude to vanish in the soft limit.

- We have considered only the simplest possible scattering event, namely $2 \to 2$. Perhaps this is too simple, and that we instead should consider $3 \to 3$ scattering?

- pNGBs also interact weakly.

Given more time we would of course address each of these issues in detail.

In part III, we begin by proving Weinberg's soft-limit theorem because it motivates us to determine a new covariant conservation law. Armed with this we look at two concrete examples, and manage to find a procedure for determining a formula for the scattering amplitude from which it is trivial to see that it vanishes in the soft limit. We then prove our theorem, first perturbatively and then non-pertubatively.

Part I

The Soft-Limit Scattering Amplitude of mNGBs

Chapter 2

The Abelian Model

2.1 The Abelian Model without Chemical Potential

In this section we study one of the simplest models exhibiting SSB, namely the Abelian model with a complex scalar field. As a first step we determine the mass spectrum of the theory and show that the existence of the NGB is in accordance with Goldstone's theorem.

Having derived the mass spectrum, we want to investigate scattering amplitudes involving NGBs and verify that the on-shell tree-level amplitude[1] vanishes in the soft limit as mentioned in section 1.1.

Finally, we want to demonstrate that the physical observables are independent of the parametrisation of the field and that some parametrisations provide a more intuitive picture than others.

2.1.1 The Mass Spectrum

The Lagrangian for the complex field ϕ is given by

$$\mathscr{L} = \partial_\mu \phi^* \partial^\mu \phi + m^2 \phi^* \phi - \lambda \left(\phi^* \phi \right)^2 , \tag{2.1}$$

where m and λ are the mass and coupling constant of the theory respectively. Before we can calculate the mass of the fields as done in Eq. (A.14) we need to check if the field's vacuum expectation value (vev) is zero. To do this we recall that the vev is defined as the field configuration that minimises the potential $\mathscr{U}$. The potential is determined from the static part of the Lagrangian and is given by,

$$\mathscr{U} = -m^2 \phi^* \phi + \lambda \left(\phi^* \phi \right)^2 . \tag{2.2}$$

The vev is found by differentiating $\mathscr{U}$ w.r.t. the field ϕ^* and setting the derivative equal to zero,

$$0 = \frac{\partial \mathscr{U}}{\partial \phi^*} = \left[2\lambda \left(\phi^* \phi \right)^2 - m^2 \right] \phi. \tag{2.3}$$

The solutions are

$$\phi = 0 \tag{2.4}$$

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[1]In this paper we will only calculate on-shell tree-level scattering amplitudes. Therefore if we use the word amplitude this should be taken to mean on-shell tree-level scattering amplitude.

which is a local maximum,[2] and

$$\phi^* \phi \equiv |\phi|^2 = \frac{m^2}{2\lambda} \equiv \frac{v^2}{2}, \tag{2.5}$$

which corresponds to a local minimum. The ground state[3] is

$$\phi_0 = \frac{v}{\sqrt{2}} e^{i\theta} \tag{2.6}$$

where θ is an arbitrary phase. There is thus an infinite set of possible nonzero vevs each specified by a unique value of θ. This degeneracy of possible vevs is a general characteristic of all theories exhibiting SSB.

Note that the Lagrangian in Eq. (2.1) is not expanded around its ground state in Eq. (2.6), instead it is expanded around its unstable maximum $\phi = 0$. To find the true mass spectrum of the theory it is necessary to parametrise the field ϕ around the ground state ϕ_0. We choose to write the field ϕ in terms of two real fields π_1 and π_2 fluctuating around the ground state corresponding to $\theta = 0$. This gives,

$$\phi = \frac{1}{\sqrt{2}} \left(v + \pi_1 + i\pi_2 \right). \tag{2.7}$$

If we insert this into Eq. (2.1), and drop constant terms, we obtain

$$\begin{aligned}
\mathscr{L} = {}&\frac{1}{2}\partial_\mu \pi_1 \partial^\mu \pi_1 - m^2 \pi_1^2 + \frac{1}{2}\partial_\mu \pi_2 \partial^\mu \pi_2 \\
&- \frac{\lambda}{4}\pi_1^4 - \frac{\lambda}{2}\pi_1^2 \pi_2^2 - \lambda v \pi_1^3 - \lambda v \pi_1 \pi_2^2 - \frac{\lambda}{4}\pi_2^4.
\end{aligned} \tag{2.8}$$

We can now use Eq. (A.14) to determine the mass spectrum. The potential is

$$\mathscr{U} = m^2 \pi_1^2 + \frac{\lambda}{4}\pi_1^4 + \frac{\lambda}{2}\pi_1^2 \pi_2^2 + \lambda v \pi_1^3 + \lambda v \pi_1 \pi_2^2 + \frac{\lambda}{4}\pi_2^4 \tag{2.9}$$

and evaluating its second derivative with respect to the fields π_1 and π_2 gives

$$m_{\pi_1}^2 = \left. \frac{\partial^2 \mathscr{U}}{\partial \pi_1^2} \right|_{\pi_1 = \pi_2 = 0} = 2m^2 \tag{2.10}$$

and

$$m_{\pi_2}^2 = \left. \frac{\partial^2 \mathscr{U}}{\partial \pi_2^2} \right|_{\pi_1 = \pi_2 = 0} = 0. \tag{2.11}$$

If we apply Goldstone's theorem we can interpret the massless particle as a NGB: The Lagrangian in Eq. (2.1) is invariant under $U(1)$ transformations. However, the ground state $\phi_0 = \frac{v}{\sqrt{2}}$ is not because a $U(1)$ transformation of the ground state results in a different ground state $\phi_0 = \frac{v}{\sqrt{2}} e^{i\alpha}$, where α is some phase factor. Hence the ground state does not share the $U(1)$ symmetry of the Lagrangian. Since the group $U(1)$ only has one generator, which breaks when we choose a ground state, Goldstone's theorem tells us that there should only be one NGB.

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[2]In order to show that this is indeed a maximum one simply checks that the second derivative is negative.
[3]In this paper the words ground state and vev mean the same thing.

2.1.2 Scattering Amplitudes

Feynman Rules

Before we can compute scattering amplitudes it is necessary to determine the Feynman rules of the theory. That is we need the propagators for π_1 and π_2 and the interaction vertices. We use the method presented in appendix A.3. The propagators follow from the quadratic part of Eq. (2.8). In particular the π_1-propagator D_{π_1} can be determined from

$$
\begin{aligned}
\mathscr{L} &= \frac{1}{2}\partial_\mu\pi_1\partial^\mu\pi_1 - m^2\pi_1^2 \\
&= -\frac{1}{2}\pi_1\left(\Box + 2m^2\right)\pi_1 \\
&= \frac{1}{2}\pi_1 D_{\pi_1}^{-1}\pi_1,
\end{aligned}
\tag{2.12}
$$

where we in the second line performed one partial integration and in the third identified the term $-\left(\Box + 2m^2\right)$ as the inverse propagator $D_{\pi_1}^{-1}$. The propagator D_{π_1} is a Green's function and therefore satisfies the equation

$$
-\left(\Box + 2m^2\right)D_{\pi_1}(x - x') = \delta^4(x - x').
\tag{2.13}
$$

Performing a Fourier transformation[4] yields

$$
-(-k^2 + 2m^2)D_{\pi_1}(k) = 1.
\tag{2.14}
$$

Thus we find

$$
iD_{\pi_1}(k) = \frac{i}{k^2 - 2m^2 + i\epsilon},
\tag{2.15}
$$

where $i\epsilon$ is the infinitesimal Feynman prescription, often introduced to avoid the pole at $k_0^2 = 2m^2$. For the π_2-propagator we take the limit $m \to 0$ giving,

$$
iD_{\pi_2}(k) = \frac{i}{k^2 + i\epsilon}.
\tag{2.16}
$$

The part of the Lagrangian containing the particle interactions is,

$$
\mathscr{L}_I = -\frac{\lambda}{4}\pi_1^4 - \frac{\lambda}{2}\pi_1^2\pi_2^2 - \lambda v\pi_1^3 - \lambda v\pi_1\pi_2^2 - \frac{\lambda}{4}\pi_2^4.
\tag{2.17}
$$

By following the procedure described in appendix A.3 we can calculate the Feynman vertices. The complete set of Feynman rules are given in Fig. 2.1.

Having obtained the Feynman rules we can now compute our first on-shell tree-level scattering amplitude involving NGBs.

Pure NGB-Scattering

The first amplitude $i\mathscr{A}_{tree}$ we will calculate corresponds to the process only involving NGBs,[5]

$$
\pi_2(p_1)\pi_2(p_2) \to \pi_2(p_3)\pi_2(p_4).
$$

---

[4]Our convention for the Fourier transformation is mentioned under "Notation and Conventions".

[5]The 4-momentum of each particle is indicated in the bracket.
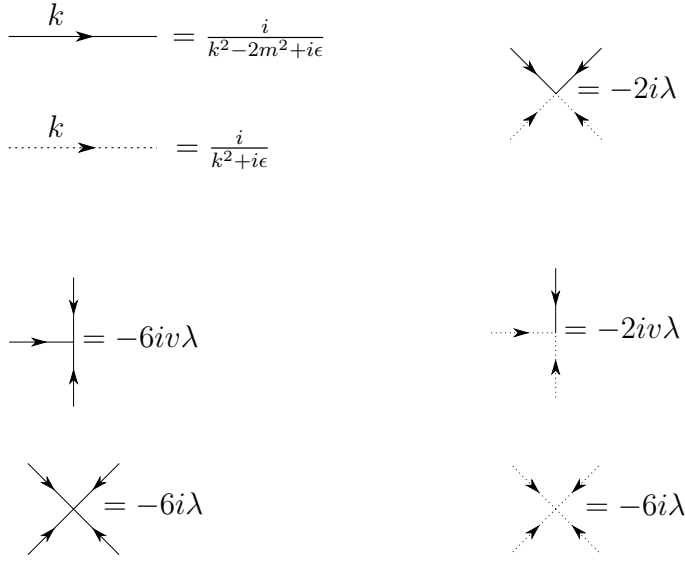
Figure 2.1: Feynman rules for the Abelian model without a chemical potential. The massless field is represented as a dashed line while the massive field is represented as a thick line.

There are four diagrams contributing to this amplitude at tree level, and they are shown in Fig. 2.2.
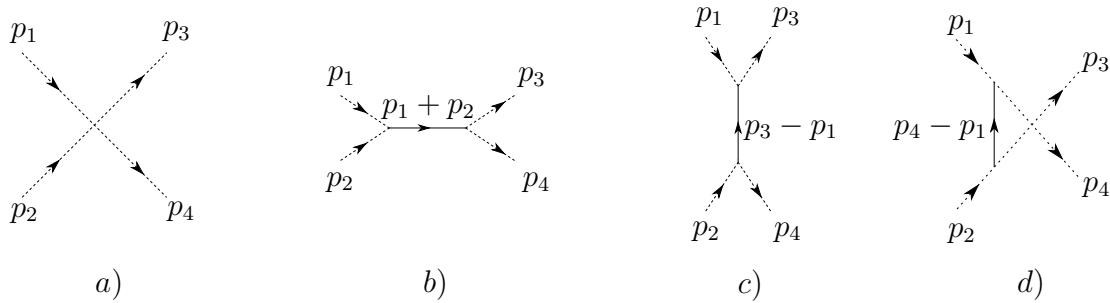
Figure 2.2: The four diagrams contributing to the scattering amplitude for the process $\pi_2(p_1)\pi_2(p_2) \rightarrow \pi_2(p_3)\pi_2(p_4)$ at tree level.

Using the Feynman rules in Fig. 2.1, we can determine the scattering amplitude for each diagram in Fig. 2.2

$$i\mathscr{A}_a = -6i\lambda, \tag{2.18}$$

$$i\mathscr{A}_b = (-2iv\lambda)^2 \frac{i}{(p_1 + p_2)^2 - 2m^2}, \tag{2.19}$$

$$i\mathscr{A}_c = (-2iv\lambda)^2 \frac{i}{(p_1 - p_3)^2 - 2m^2}, \tag{2.20}$$

and

$$i\mathscr{A}_d = (-2iv\lambda)^2 \frac{i}{(p_1 - p_4)^2 - 2m^2}. \tag{2.21}$$

The complete amplitude $i\mathscr{A}_{tree}$ is then given by the sum of the individual amplitudes. Let us assume that the external particles are on shell such that

$$p_i^2 = m_{\pi_2}^2 = 0, \quad (i = 1, 2, 3, 4)$$

and define the Mandelstam variables s, t and u as

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = (p_3 + p_4)^4 = 2p_1 p_2 = 2p_3 p_4, \\
t &= (p_1 - p_3)^2 = (p_4 - p_2)^2 = -2p_1 p_3 = -2p_2 p_4, \\
u &= (p_1 - p_4)^2 = (p_3 - p_2)^2 = -2p_1 p_4 = -2p_2 p_3.
\end{aligned}
\tag{2.22}
$$

Note that the Mandelstam variables vanish when either of the external 4-momenta p_1, p_2, p_3 or p_4 vanishes. The total amplitude is then,

$$
\begin{aligned}
i\mathscr{A}_{tree} &= i\mathscr{A}_a + i\mathscr{A}_b + i\mathscr{A}_c + i\mathscr{A}_d \\
&= -6i\lambda + i\left(-2iv\lambda\right)^2 \left(\frac{1}{s - 2m^2} + \frac{1}{t - 2m^2} + \frac{1}{u - 2m^2} \right).
\end{aligned}
\tag{2.23}
$$

Let us investigate what happens to the amplitude when one of the particles has vanishing momentum such that all the Mandelstam variables approach zero

$$
\begin{aligned}
i\mathscr{A}_{tree} \xrightarrow[s,\,t,\,u \to 0]{} &-6i\lambda + i\left(-2iv\lambda\right)^2 \left(\frac{-3}{2m^2} \right) \\
&= -6i\lambda + 6i\lambda \\
&= 0.
\end{aligned}
\tag{2.24}
$$

Thus if either of the momenta of the NGBs vanishes, then the tree level amplitude vanishes as well. Note that if the NGBs were not on shell, we would not get the cancellation because $p_i^2 \neq 0$ for some i.

Mixed scattering

Let us show that a soft-limit NGB also gives a vanishing tree-level amplitude for the process

$$\pi_1(p_1)\pi_2(p_2) \to \pi_1(p_1')\pi_2(p_2').$$

The contributing Feynman diagrams are given in Fig. 2.3.
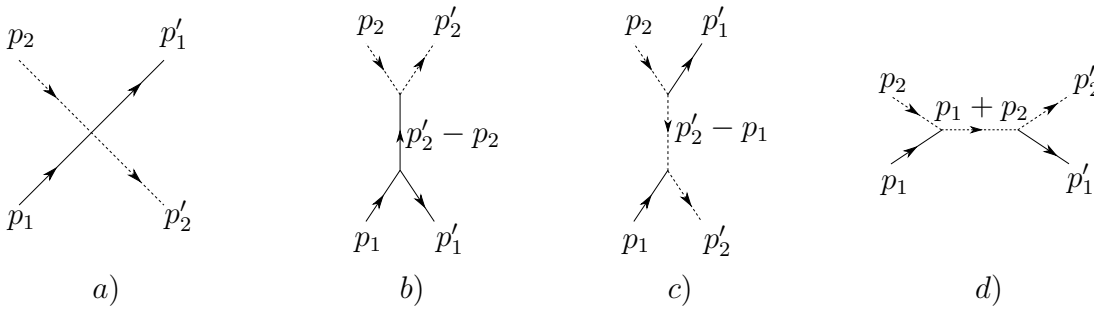
Figure 2.3: The four diagrams contributing to the scattering scattering amplitude for the process $\pi_1(p_1)\pi_2(p_2) \to \pi_1(p_1')\pi_2(p_2')$ at tree level.

The corresponding Mandelstam variables are

$$
\begin{aligned}
t &= (p_2 - p_2')^2, \\
u &= (p_1 - p_2')^2, \\
s &= (p_1 + p_2)^2.
\end{aligned}
\tag{2.25}
$$

Using the Feynman rules, we construct each amplitude

$$
i\mathscr{A}_a = -2i\lambda,
\tag{2.26}
$$

$$
i\mathscr{A}_b = \frac{-12i\lambda^2 v^2}{t - 2m^2} = \frac{6i\lambda m^2}{p_2 p_2' + m^2},
\tag{2.27}
$$

$$
i\mathscr{A}_c = -\frac{4i\lambda^2 v^2}{u} = \frac{-4i\lambda m^2}{p_1' \left(p_1' - 2p_2\right)},
\tag{2.28}
$$

$$
i\mathscr{A}_d = -\frac{4i\lambda^2 v^2}{s} = \frac{-4i\lambda m^2}{p_1 \left(p_1 + 2p_2\right)}.
\tag{2.29}
$$

The complete tree-level amplitude thus reads,

$$
\begin{aligned}
i\mathscr{A}_{tree} &= -2i\lambda + \frac{6i\lambda m^2}{p_2 p_2' + m^2} - \frac{4i\lambda m^2}{p_1' \left(p_1' - 2p_2\right)} - \frac{4i\lambda m^2}{p_1 \left(p_1 + 2p_2\right)} \\
&= -2i\lambda - 4i\lambda^2 v^2 \left(\frac{1}{s} + \frac{1}{u}\right) - 12i\lambda^2 v^2 \left(\frac{1}{t - 2m^2}\right).
\end{aligned}
\tag{2.30}
$$

Let us first investigate the soft limit $p_2 \to 0$, which gives

$$
i\mathscr{A}_{tree} \xrightarrow[p_2 \to 0]{} -2i\lambda + \frac{6i\lambda m^2}{m^2} - \frac{4i\lambda m^2}{p_1'^2} - \frac{4i\lambda m^2}{p_1^2}.
\tag{2.31}
$$

Using that $p_1'^2 = p_1^2 = 2m^2$, we obtain

$$
i\mathscr{A}_{tree} \xrightarrow[p_2 \to 0]{} -2i\lambda + 6i\lambda - 2i\lambda - 2i\lambda = 0.
\tag{2.32}
$$

We then investigate the limit $p_2' \to 0$. However, before we take the limit, we rewrite the amplitude as

$$
i\mathscr{A}_{tree} = -2i\lambda + \frac{6i\lambda m^2}{p_2 p_2' + m^2} - \frac{4i\lambda m^2}{p_1^2 - 2p_1 p_2'} - \frac{4i\lambda m^2}{p_1'^2 + 2p_1' p_2'}.
\tag{2.33}
$$

Now we let $p_2' \to 0$ and obtain

$$
\begin{aligned}
i\mathscr{A}_{tree} \xrightarrow[p_2' \to 0]{} &-2i\lambda + 6i\lambda - \frac{4i\lambda m^2}{2m^2} - \frac{4i\lambda m^2}{2m^2} \\
&= -2i\lambda + 6i\lambda - 2i\lambda - 2i\lambda = 0.
\end{aligned}
\tag{2.34}
$$

Once again we have shown that a scattering process involving on-shell NGBs, has a vanishing tree-level amplitude in the limit where the momentum of a single NGB approaches zero. This is in fact general behaviour and is proved in chapter 6 as well as in [10].

2.1.3 Alternative Parametrisation

We now illustrate that there exists a parametrisation providing a more intuitive picture of why SSB leads to NGBs. We expect that the mass spectrum and the amplitudes should be identical to the ones previously obtained because physical observables should not depend on the field parametrisation. The parametrisation we will use is similar to standard polar coordinates and is given by

$$
\begin{aligned}
\phi &= \frac{1}{\sqrt{2}} \left(v + \rho \right) e^{\frac{i\theta}{v}}, \\
\phi^* &= \frac{1}{\sqrt{2}} \left(v + \rho \right) e^{-\frac{i\theta}{v}}
\end{aligned}
\tag{2.35}
$$

where θ and ρ are real fields and v is the vev in Eq. (2.6). The reason for that this parametrisation is more intuitive can be explained by noting that a constant θ can be removed from the Lagrangian by applying a $U(1)$ transformation on the field ϕ. This means that every interaction term involving θ must depend on an even power[6] of the derivatives $\partial_\mu \theta$, which implies that the interactions are proportional to products of momenta. Thus in the soft limit of the momentum the NGB usually[7] interacts weakly.

If we express the Lagrangian in Eq. (2.1) in terms of the two real fields, we get

$$
\begin{aligned}
\mathscr{L} &= \left[\frac{1}{2} \partial_\mu \rho \partial^\mu \rho - m^2 \rho^2 \right] + \left[\frac{1}{2} \partial_\mu \theta \partial^\mu \theta \right] \\
&+ \frac{1}{v} \rho \partial_\mu \theta \partial^\mu \theta + \frac{1}{2v^2} \rho^2 \partial_\mu \theta \partial^\mu \theta - \frac{\lambda}{4} \rho^4 - \lambda v \rho^3.
\end{aligned}
\tag{2.36}
$$

Mass spectrum

Next, we calculate the mass spectrum from the potential

$$
\mathscr{U} = m^2 \rho^2 + \frac{\lambda}{4} \rho^4 + \lambda v \rho^3.
\tag{2.37}
$$

Differentiating twice yields

$$
\begin{aligned}
m_\rho^2 &= \left. \frac{\partial^2 \mathscr{U}}{\partial \rho^2} \right|_{\rho=\theta=0} = 2m^2, \\
m_\theta^2 &= \left. \frac{\partial^2 \mathscr{U}}{\partial \theta^2} \right|_{\rho=\theta=0} = 0.
\end{aligned}
\tag{2.38}
$$

As expected the mass spectrum is identical to the one determined in Eqs. (2.10) and (2.11) and we identify θ as the NGB. We can understand the existence of the NGB geometrically by plotting the potential in (ϕ, ϕ^*) space.

---

[6]The power must be even because the Lagrangian must be a scalar, such that all the Lorentz indices are contracted.

[7]Although it is usually true that the scattering amplitude involving NGBs vanish in the soft limit it is not always true. In fact, there may be kinematic singularities which lead to that the amplitude does not vanish. We will see an example of this in chapter 3.
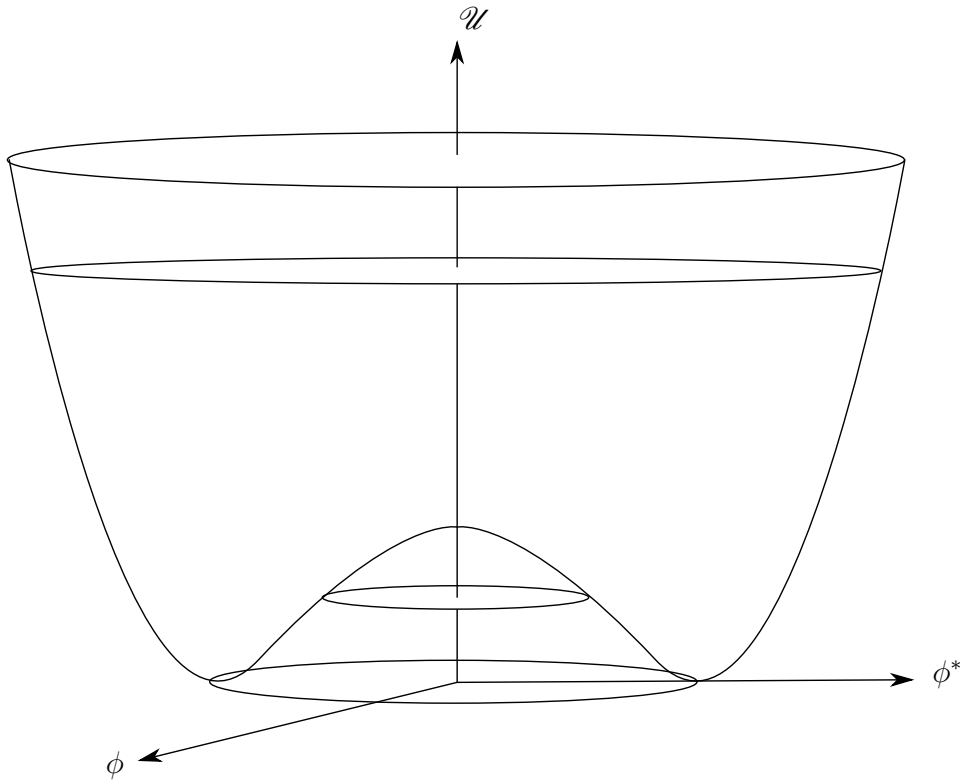
Figure 2.4: The potential $\mathscr{U}$ plotted in (ϕ, ϕ^*) space.

From Fig. 2.4, it is clear that radial excitations cost energy resulting in a massive radial field ρ. However, due to the cylindrical symmetry of the potential an excitation in the azimuthal direction gives a massless angular field θ.

Feynman Rules

Next, we determine the new Feynman rules starting with the propagators. The radial propagator D_ρ comes from the term,

$$
\begin{aligned}
\mathscr{L} &= \frac{1}{2}\partial_\mu\rho\partial^\mu\rho - m^2\rho^2 \\
&= -\frac{1}{2}\rho\left(\Box + 2m^2\right)\rho \\
&= \frac{1}{2}\rho D_\rho^{-1}\rho,
\end{aligned}
\tag{2.39}
$$

and satisfies

$$
-\left(\Box + 2m^2\right)D_\rho(x,x') = \delta^4(x - x').
\tag{2.40}
$$

Performing first a Fourier transformation and then solving the algebraic equation for D_ρ we get,

$$
iD_\rho(k) = \frac{i}{k^2 - 2m^2 + i\epsilon}.
\tag{2.41}
$$

The angular field propagator D_θ is the massless limit of D_ρ

$$
iD_\theta(k) = \frac{i}{k^2 + i\epsilon}.
\tag{2.42}
$$

The interaction vertices are a bit different in this case, because some of the interactions depend on derivatives. We use the method demonstrated in App. A.3.1 to calculate the Feynman rules for the Lagrangian in Eq. (2.36) and the result is shown in Fig. (2.5).
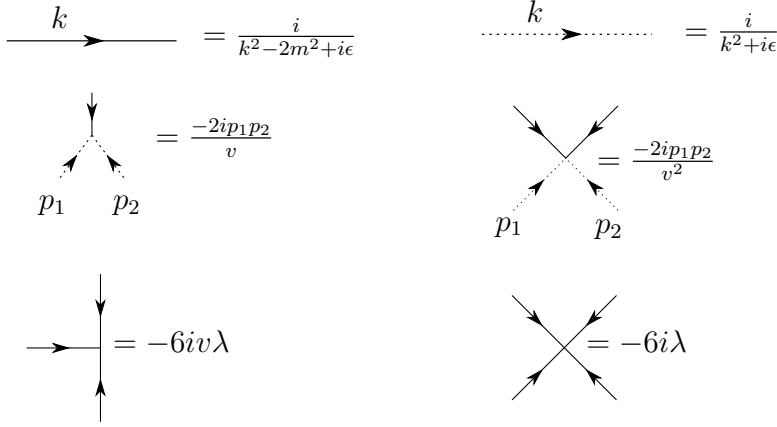
Figure 2.5: Feynman rules for the polar coordinate parametrisation.

Pure NG-scattering

The first process we will look at is the scattering amplitude for

$$\theta(p_1)\theta(p_2) \to \theta(p_3)\theta(p_4),$$

containing only NGBs. There are three tree-level diagrams contributing to the scattering amplitude and they are shown in Fig. (2.6).
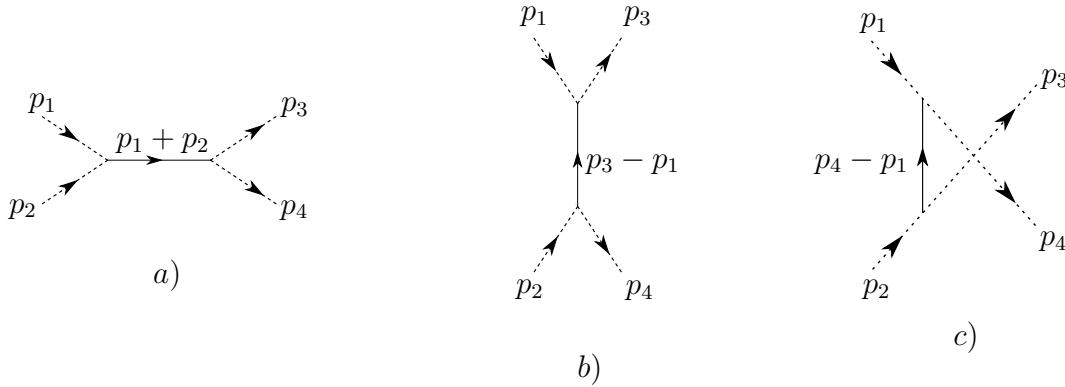
Figure 2.6: Tree-level Feynman diagrams contributing to $\theta(p_1)\theta(p_2) \to \theta(p_3)\theta(p_4)$.

The amplitudes corresponding to the diagrams are

$$i\mathscr{A}_a = -\frac{2i}{v^2}\frac{(p_1p_2)(p_3p_4)}{p_1p_2 - m^2}, \tag{2.43}$$

$$i\mathscr{A}_b = \frac{2i}{v^2}\frac{(p_1p_3)(p_2p_4)}{p_1p_3 + m^2}, \tag{2.44}$$

and

$$i\mathscr{A}_c = \frac{2i}{v^2} \frac{(p_1 p_4)(p_2 p_3)}{p_1 p_4 + m^2}. \tag{2.45}$$

The complete tree-level amplitude becomes,

$$i\mathscr{A}_{tree} = \frac{2i}{v^2} \left[\frac{-(p_1 p_2)(p_3 p_4)}{p_1 p_2 - m^2} + \frac{(p_1 p_3)(p_2 p_4)}{p_1 p_3 + m^2} + \frac{(p_1 p_4)(p_2 p_3)}{p_1 p_4 + m^2} \right] \tag{2.46}$$

and it obviously vanishes when one of the Goldstone momenta p_1, p_2, p_3 or p_4 approaches zero.

In order to verify that Eq. (2.46) is identical to Eq. (2.23) we have to rewrite it using the Mandelstam variables. The Mandelstam variables corresponding to Fig. 2.6 are

$$\begin{aligned}
s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 = 2p_1 p_2 = 2p_3 p_4, \\
t &= (p_1 - p_3)^2 = (p_4 - p_2)^2 = -2p_1 p_3 = -2p_2 p_4, \\
u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 = -2p_1 p_4 = -2p_2 p_3,
\end{aligned} \tag{2.47}$$

and satisfy $s + t + u = 0$ for the scattering of massless particles. Eq. (2.46) becomes,

$$\begin{aligned}
i\mathscr{A}_{tree} &= \frac{2i}{v^2} \left(\frac{-s^2/4}{s/2 - m^2} + \frac{t^2/4}{-t/2 + m^2} + \frac{u^2/4}{-u/2 + m^2} \right) \\
&= -\frac{i}{v^2} \left(\frac{s^2}{s - 2m^2} + \frac{t^2}{t - 2m^2} + \frac{u^2}{u - 2m^2} \right).
\end{aligned} \tag{2.48}$$

If we rewrite the Mandelstam variables as

$$\begin{aligned}
s^2 &= (s - 2m^2)^2 + 4m^2(s - 2m^2) + 4m^4, \\
t^2 &= (t - 2m^2)^2 + 4m^2(t - 2m^2) + 4m^4, \\
u^2 &= (u - 2m^2)^2 + 4m^2(u - 2m^2) + 4m^4,
\end{aligned} \tag{2.49}$$

we may express the amplitude as

$$\begin{aligned}
i\mathscr{A}_{tree} &= -\frac{i}{v^2} \left[\left(s + 2m^2 + \frac{4m^4}{s - 2m^2} \right) + \left(t + 2m^2 + \frac{4m^4}{t - 2m^2} \right) + \left(u + 2m^2 + \frac{4m^4}{u - 2m^2} \right) \right] \\
&= -\frac{i}{v^2} \left[(s + t + u) + 6m^2 + 4m^4 \left(\frac{1}{s - 2m^2} + \frac{1}{t - 2m^2} + \frac{1}{u - 2m^2} \right) \right] \\
&= -6i\lambda - 4i\lambda^2 v^2 \left(\frac{1}{s - 2m^2} + \frac{1}{t - 2m^2} + \frac{1}{u - 2m^2} \right),
\end{aligned} \tag{2.50}$$

which shows that the amplitudes in Eqs. (2.23) and (2.50) are identical.

Mixed scattering

The final scattering amplitude we will compute is for the mixed scattering

$$\rho(p_1)\theta(p_2) \to \rho(p_3)\theta(p_4).$$

The contributing Feynman diagrams are shown in Fig 2.7
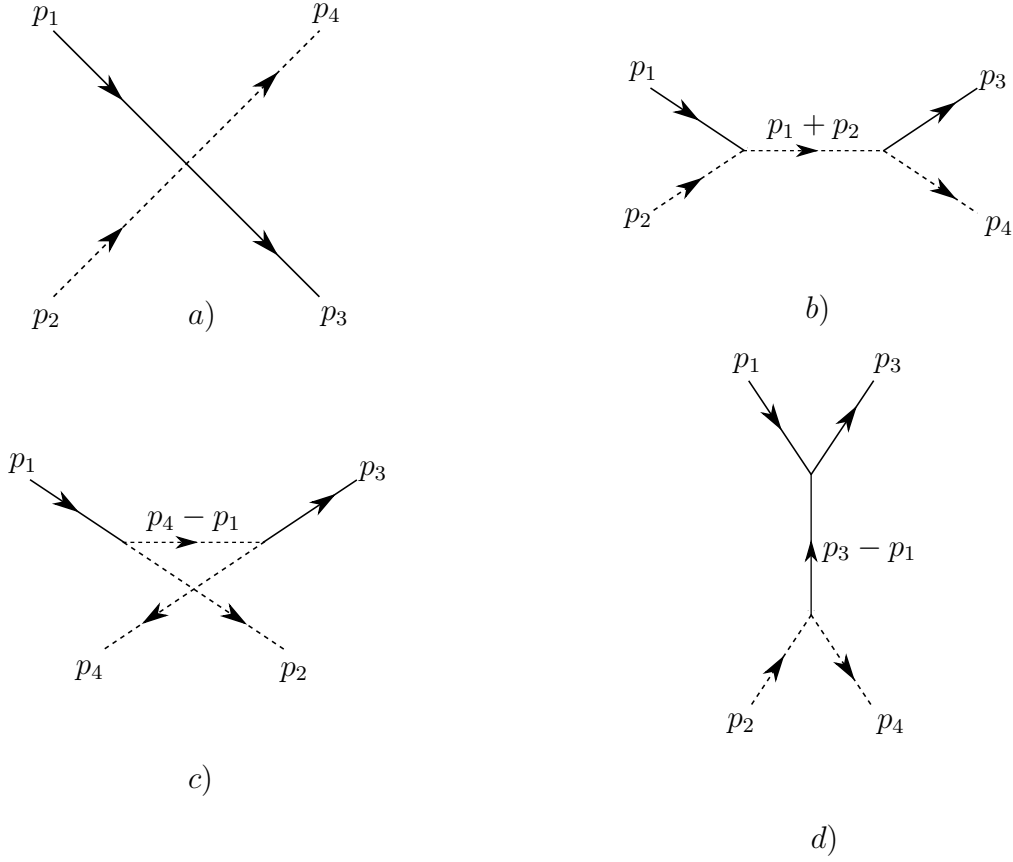
Figure 2.7: Feynman diagrams contributing to $\rho(p_1)\theta(p_2) \rightarrow \rho(p_3)\theta(p_4)$.

and the corresponding Mandelstam variables are

$$
\begin{aligned}
s &= (p_1 + p_2)^2 = p_1^2 + 2p_1p_2 = 2m^2 + 2p_1p_2, \\
u &= (p_1 - p_4)^2 = 2m^2 - 2p_1p_4 = (p_2 - p_3)^2 = 2m^2 - 2p_2p_3, \\
t &= (p_1 - p_3)^2 = 4m^2 - 2p_1p_3 = (p_2 - p_4)^2 = -2p_2p_4.
\end{aligned}
\tag{2.51}
$$

The corresponding amplitudes read

$$
\begin{aligned}
i\mathscr{A}_a &= \frac{+2ip_2p_4}{v^2}, \\
i\mathscr{A}_b &= \left(\frac{2i}{v}\right)^2 p_2(p_1 + p_2)\frac{i}{s}p_4(p_3 + p_4), \\
i\mathscr{A}_c &= \left(\frac{2i}{v}\right)^2 p_4(p_1 - p_4)\frac{i}{u}p_2(p_3 - p_2), \\
i\mathscr{A}_d &= -6i\lambda v\frac{i}{t - 2m^2}\left(+\frac{2i}{v}p_2p_4\right).
\end{aligned}
\tag{2.52}
$$

Summing up the amplitudes in Eq. (2.52) and using the Mandelstam variables gives

$$i\mathscr{A}_{tree} = -\frac{it}{v^2} - \frac{4i}{v^2 s}\left(\frac{s - 2m^2}{2}\right)^2 - \frac{4i}{v^2 u}\left(\frac{2m^2 - u}{2}\right)^2 - 6i\lambda v\frac{1}{t - 2m^2}\frac{t}{v}. \tag{2.53}$$

We can simplify this expression to give

$$i\mathscr{A}_{tree} = -2i\lambda - 4iv^2\lambda^2\left(\frac{1}{s} + \frac{1}{u}\right) - 12iv^2\lambda^2\frac{1}{t - 2m^2}, \tag{2.54}$$

which is identical to Eq. (2.30). The fact that the expressions for the mass spectrum and the on-shell scattering amplitudes were identical in both parametrisations show that the results are parametrisation independent.

2.2 Abelian Model with a Chemical Potential

Now we want to see what happens to the NGB if we consider the Lagrangian[8]

$$\mathscr{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - \lambda\left(\phi^*\phi\right)^2, \tag{2.55}$$

at finite density. To do this we must first find a way to introduce a chemical potential μ. We will see that the chemical potential acts as a constant temporal gauge field, see e.g. [11].

2.2.1 The Chemical Potential as a Gauge Field

From statistical physics [12] we know that adding a chemical potential to a theory is equivalent to working in the grand canonical ensemble where both the energy and the particle number of the system are allowed to fluctuate. Mathematically a system described by a Hamiltonian density $\mathscr{H}$ in the canonical ensemble can be promoted to an equivalent system in the grand canonical ensemble by the replacement

$$\mathscr{H} \to \mathcal{K} = \mathscr{H} - \mu j^0, \tag{2.56}$$

where μ is the chemical potential and j^0 is the conserved Noether charge density. We will now demonstrate that in the Lagrangian formalism the replacement in Eq. (2.56) is equivalent to the replacement

$$\partial_\alpha \to D_\alpha = \partial_\alpha - i\mu\delta_{0\alpha}. \tag{2.57}$$

The Hamiltonian density $\mathscr{H}$ corresponding to the Lagrangian density $\mathscr{L}$ in Eq. (2.55) is obtained by performing the Legendre transformation[9]

$$\mathscr{H} = \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathscr{L}, \tag{2.58}$$

where π and π^* are the conjugate momenta of the fields ϕ and ϕ^* respectively. Using Eq. (2.55) we find that the conjugate momenta are

$$\pi = \frac{\partial\mathscr{L}}{\partial\dot{\phi}} = \dot{\phi}^* \tag{2.59}$$

---

[8]This is the same Lagrangian as in Eq. (2.1) except that we changed the sign in front of m^2.
[9]The dots indicate differentiation w.r.t. the time variable t.

and

$$\pi^* = \frac{\partial \mathscr{L}}{\partial \dot{\phi}^*} = \dot{\phi}. \tag{2.60}$$

Inserting Eqs. (2.59) and (2.60) into Eq. (2.58) yields the Hamiltonian density

$$\mathscr{H} = \pi\pi^* + \nabla\phi \cdot \nabla\phi^* + m^2\phi^*\phi + \lambda\left(\phi^*\phi\right)^2. \tag{2.61}$$

The Lagrangian in Eq. (2.55) is invariant under $U(1)$ transformations. We can calculate the corresponding conserved Noether current by considering infinitesimal transformations of the form

$$\begin{aligned}
\phi &\to e^{-i\theta}\phi = (1 - i\theta)\phi, \\
\phi^* &\to e^{i\theta}\phi^* = (1 + i\theta)\phi^*.
\end{aligned} \tag{2.62}$$

The Noether current in Eq. (A.8) becomes,[10]

$$\begin{aligned}
j^\mu &= \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi\right)}\delta\phi + \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi^*\right)}\delta\phi^* \\
&= \left(\partial^\mu\phi^*\right)(-i\phi) + \left(\partial^\mu\phi\right)(i\phi^*) \\
&= i\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right).
\end{aligned} \tag{2.63}$$

Inserting Eq. (2.63) into Eq. (2.56) we get

$$\mathcal{K} = \pi\pi^* + \nabla\phi \cdot \nabla\phi^* + m^2\phi^*\phi + \lambda\left(\phi^*\phi\right)^2 - i\mu\left(\phi^*\pi^* - \phi\pi\right). \tag{2.64}$$

Now, by performing an inverse Legendre transform, we can calculate the Lagrangian $\mathscr{L}_\mathcal{K}$ corresponding to the Hamiltonian $\mathcal{K}$

$$\mathscr{L}_\mathcal{K} = \pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{K}. \tag{2.65}$$

However, because of the replacement in Eq. (2.64) the conjugate momenta in Eqs. (2.59) and (2.60) are no longer the conjugate momenta π and π^* present in Eq. (2.65). Using thus Hamilton's equations we get

$$\dot{\phi} = \frac{\partial\mathcal{K}}{\partial\pi} = \pi^* + i\mu\phi \implies \pi^* = \dot{\phi} - i\mu\phi \tag{2.66}$$

and

$$\dot{\phi}^* = \frac{\partial\mathcal{K}}{\partial\pi^*} = \pi - i\mu\phi^* \implies \pi = \dot{\phi}^* + i\mu\phi^*. \tag{2.67}$$

By inserting Eqs. (2.66) and (2.67) into Eq. (2.65) we find that

$$\mathscr{L}_\mathcal{K} = \partial_\mu\phi^*\partial^\mu\phi + i\mu\left(\phi^*\partial_0\phi - \phi\partial_0\phi^*\right) + \left(\mu^2 - m^2\right)\phi^*\phi - \lambda\left(\phi^*\phi\right)^2 \tag{2.68}$$

which is equivalent to

$$\mathscr{L}_\mathcal{K} = D_\mu\phi^* D^\mu\phi - m^2\phi^*\phi - \lambda\left(\phi^*\phi\right)^2 \tag{2.69}$$

if we define

$$D_\alpha = \partial_\alpha - i\mu\delta_{0\alpha}, \tag{2.70}$$

which is the replacement rule we wanted to derive.

---

[10]We follow the standard convention of factoring out the infinitesimal parameter θ.

2.2.2 Spontaneous Symmetry Breaking due to a Chemical Potential

Promoting the derivative in Eq. (2.55) to the covariant derivative gives the Lagrangian,

$$\mathscr{L} = D_\mu \phi^* D^\mu \phi - m^2 \phi^* \phi - \lambda \left(\phi^* \phi \right)^2. \tag{2.71}$$

Using Eq. (2.70) the Lagrangian becomes

$$\mathscr{L} = \partial_\mu \phi^* \partial^\mu \phi + i\mu \left(\phi^* \partial_0 \phi - \phi \partial_0 \phi^* \right) + \left(\mu^2 - m^2 \right) \phi^* \phi - \lambda \left(\phi^* \phi \right)^2, \tag{2.72}$$

which reduces to Eq. (2.55) in the limit $\mu \to 0$. We see that the introduction of a chemical potential has led to that space and time is no longer treated on equal footing, which means that the Lagrangian is no longer Lorentz invariant or diagonal. This means that the dispersion relation is no longer $E^2 = m^2 + \mathbf{p}^2$, and that we can not determine the mass spectrum by simply taking the second derivative of the static term of the Lagrangian. In fact, we have to first calculate the new dispersion relation and then evaluate the limit $\mathbf{p} \to 0$ to obtain the new mass spectrum.

Now in order to demonstrate that the system indeed exhibits SSB we should determine the vev. Since, the ground state is time independent the term coupled to the charge density $j_0 = i \left(\phi^* \partial_0 \phi - \phi \partial_0 \phi^* \right)$ will drop out and hence we define the classical potential as

$$\mathscr{U} = \left(\mu^2 - m^2 \right) |\phi|^2 - \lambda |\phi|^4. \tag{2.73}$$

The minimum of the potential can be found by solving the equation,

$$\frac{\partial \mathscr{U}}{\partial |\phi|} = 2 \left(\mu^2 - m^2 - 2\lambda |\phi|^2 \right) |\phi| = 0. \tag{2.74}$$

One solution is $|\phi| = 0$, which is a local maximum for $\mu > m$ and a local minimum for $\mu < m$. Another solution exists if and only if $\mu > m$ and it is given by,

$$|\phi|^2 = \frac{\mu^2 - m^2}{2\lambda} \equiv \frac{v^2}{2}. \tag{2.75}$$

Hence there exists an infinite number of nonzero vevs when $\mu > m$ given by,

$$\phi_0 = \frac{v}{\sqrt{2}} e^{i\theta} \tag{2.76}$$

where θ is some arbitrary phase. Eq. (2.75) implies that there can only be SSB if $\mu > m$, as shown in Fig. (2.8).

$\mu < m$ $\mu > m$

(No SSB)   (SSB)
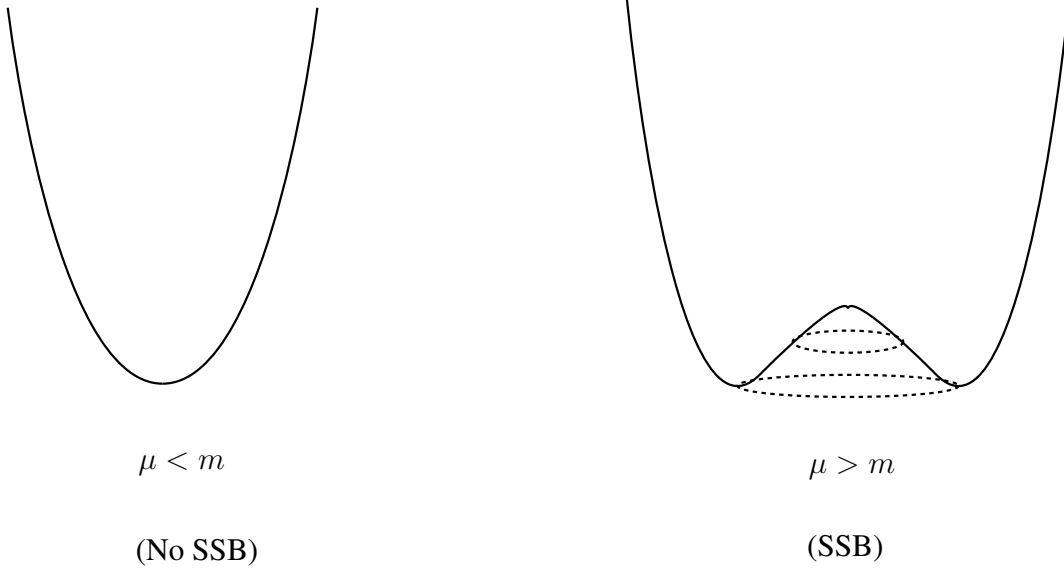
Figure 2.8: SSB is only possible when $\mu > m$.

The reason for this is that when $\mu < m$ the only solution to Eq. (2.74) is $\phi = 0$. In the case $\mu > m$ there is also a nonzero solution to Eq. (2.74). We note that in this case the chemical potential is responsible for the explicit breaking of Lorentz invariance and the SSB of the $U(1)$ symmetry.[11]

Mass Spectrum

As before in order to obtain the mass spectrum we could use either of the two parametrisations

$$
\begin{aligned}
i) &\quad \phi = \frac{1}{\sqrt{2}} \left(v + \pi_1 + i\pi_2 \right), \\
ii) &\quad \phi = \frac{1}{\sqrt{2}} \left(v + \rho \right) e^{\frac{i\theta}{v}}.
\end{aligned}
\tag{2.77}
$$

Using the first parametrisation the Lagrangian becomes

$$
\begin{aligned}
\mathscr{L} = &\frac{1}{2} \left(\partial_\mu \pi_1 \partial^\mu \pi_1 + \partial_\mu \pi_2 \partial^\mu \pi_2 \right) + \mu(-v\partial_0 \pi_2 + \pi_2 \partial_0 \pi_1 - \pi_1 \partial_0 \pi_2) \\
&+ \frac{(\mu^2 - m^2)}{2} \left(v^2 + 2v\pi_1 + \pi_1^2 + \pi_2^2 \right) \\
&- \frac{\lambda}{4} (v^4 + 4v^3 \pi_1 + 2v^2 \pi_2^2 + 6v^2 \pi_1^2 + 4v\pi_1^3 + 4v\pi_1 \pi_2^2 + \pi_1^4 + 2\pi_1^2 \pi_2^2 + \pi_2^4).
\end{aligned}
\tag{2.78}
$$

---

[11]While the replacement $\partial_\alpha \rightarrow \partial_\alpha + i\mu\delta_{0\alpha}$ results in that the chemical potential explicitly breaks Lorentz invariance it is worth noting that there exists an equivalent picture where Lorentz invariance undergoes SSB. In this picture we let the ground state be time dependent $\phi_0(t) = \frac{v}{\sqrt{2}} e^{-i\mu t}$ and we start from the original Lorentz invariant Lagrangian. When we expand the field about this ground state the chemical potential is not present in the Lagrangian at all, which means that in this picture Lorentz invariance is spontaneously broken! The situation where the state is evolving in time is called spontaneous symmetry probing and it is described in [13].

The dispersion relations come from the quadratic part of the Lagrangian

$$\mathscr{L}^{(2)} = \frac{1}{2}\left(\partial_\mu \pi_1 \partial^\mu \pi_1 + \partial_\mu \pi_2 \partial^\mu \pi_2\right) - \mu(\pi_1 \partial_0 \pi_2 - \pi_2 \partial_0 \pi_1)$$
$$+ \frac{\mu^2 - m^2}{2}(\pi_1^2 + \pi_2^2) - \frac{\lambda v^2}{4}(6\pi_1^2 + 2\pi_2^2). \tag{2.79}$$

Using $\lambda v^2 = \mu^2 - m^2$ yields,

$$\mathscr{L}^{(2)} = \frac{1}{2}\left(\partial_\mu \pi_1 \partial^\mu \pi_1 + \partial_\mu \pi_2 \partial^\mu \pi_2\right) - \mu(\pi_1 \partial_0 \pi_2 - \pi_2 \partial_0 \pi_1) + \frac{\mu^2 - m^2}{2}(-2\pi_1^2). \tag{2.80}$$

In matrix form this equation reads

$$\mathscr{L}^{(2)} = \left(\begin{array}{cc} \pi_1 & \pi_2 \end{array}\right) \left(\begin{array}{cc} -\frac{1}{2}\partial_\mu \partial^\mu - (\mu^2 - m^2) & -\mu\partial_0 \\ \mu\partial_0 & -\frac{1}{2}\partial_\mu \partial^\mu \end{array}\right) \left(\begin{array}{c} \pi_1 \\ \pi_2 \end{array}\right) \tag{2.81}$$

and the matrix sandwiched between the two doublets is the inverse propagator D^{-1}. Performing a Fourier transformation yields,

$$D^{-1} = \left(\begin{array}{cc} \frac{1}{2}p^2 - (\mu^2 - m^2) & i\mu\omega \\ -i\mu\omega & \frac{1}{2}p^2 \end{array}\right). \tag{2.82}$$

The dispersion relations are determined by requiring that the matrix D^{-1} is singular, such that

$$0 = \det D^{-1} = \left(\frac{1}{2}p^2 - \mu^2 + m^2\right)\frac{1}{2}p^2 + (i\mu\omega)^2. \tag{2.83}$$

Now we use that $p^2 = \omega^2 - \mathbf{p}^2$ where ω and $\mathbf{p}$ are the energy and 3-momentum respectively, to obtain the equation

$$0 = \frac{1}{4}\left(\omega^2 - \mathbf{p}^2 - 2\mu^2 + 2m^2\right)\left(\omega^2 - \mathbf{p}^2\right) - \mu^2\omega^2$$
$$= \frac{1}{4}\left[\omega^4 - \omega^2\mathbf{p}^2 + \omega^2\left(-\mathbf{p}^2 - 2\mu^2 + 2m^2\right) + \left(-\mathbf{p}^2 - 2\mu^2 + 2m^2\right)\left(-\mathbf{p}^2\right)\right] - \mu^2\omega^2. \tag{2.84}$$

Multiplying by four and collecting equal powers of ω yields

$$0 = (\omega^2)^2 + \omega^2\left(-2\mathbf{p}^2 - 6\mu^2 + 2m^2\right) + \mathbf{p}^2\left(\mathbf{p}^2 + 2\mu^2 - 2m^2\right),$$

and the dispersion relations take the form

$$\omega_\pm^2 = \mathbf{p}^2 + 3\mu^2 - m^2 \pm \left(3\mu^2 - m^2\right)\sqrt{1 + \left(\frac{2\mu|\mathbf{p}|}{3\mu^2 - m^2}\right)^2}. \tag{2.85}$$

The mass spectrum is then obtained by taking the limit $\mathbf{p} \to 0$ of Eq. (2.85) which gives

$$m_{\pi_1}^2 = 6\mu^2 - 2m^2,$$
$$m_{\pi_2}^2 = 0. \tag{2.86}$$

As a check of consistency note that Eqs. (2.85) and (2.86) both reduce to the familiar expressions in Eqs. (2.10) and (2.11) in the limit $\mu \to 0$ and when $m^2 < 0$. In order to characterise the NGB it is useful to consider small 3-momentum $\mathbf{p}$ such that we can expand the square root to obtain

$$\omega_-^2 = \left(\frac{\mu^2 - m^2}{3\mu^2 - m^2} \right) \mathbf{p}^2 + \mathcal{O}(\mathbf{p}^4).$$

The dispersion relation thus looks like that of a phonon for small $\mathbf{p}$,

$$\omega_- = \sqrt{\frac{\mu^2 - m^2}{3\mu^2 - m^2}} |\mathbf{p}|. \tag{2.87}$$

In the literature [7] this NGB is characterised as type-I because $i)$ it is linear in the 3-momentum, and $ii)$ only one generator was spontaneously broken. Thus by introducing a chemical potential, the dispersion relation of the NGB changed from $\omega = \sqrt{m^2 + \mathbf{p}^2}$ to $\omega_- = \sqrt{\frac{\mu^2 - m^2}{3\mu^2 - m^2}} |\mathbf{p}|$.

Non-Abelian Models

In the spirit of the previous chapter we will first introduce an $SO(3)$ model without a chemical potential and then add a chemical potential to see how the NGBs change. We will find that, whereas the introduction of a chemical potential in the Abelian model did not change the mass of the NGBs, a chemical potential in a non-Abelian model will. In fact, we obtain a mNGB as mentioned in the introduction. We then try to give indication for the existence of a soft-limit theorem for mNGB which is similar to the soft-limit theorem for NGBs discussed in section 2.1.2.

3.1 The Non-Abelian Linear SO(3) Model without a Chemical Potential

We consider a vector field $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ and the Lagrangian

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \vec{\phi}\right)^2 + \frac{1}{2}m^2\vec{\phi}^2 - \frac{\lambda}{4}\left(\vec{\phi}^2\right)^2, \tag{3.1}$$

which is invariant under $SO(3)$ transformations.

In order to show that this model exhibits SSB, we must find at least one nonzero vev.

Spontaneous Symmetry Breaking of the Model

As usual we start by looking at the potential,

$$\mathscr{U} = -\frac{1}{2}m^2\vec{\phi}^2 + \frac{\lambda}{4}\left(\vec{\phi}^2\right)^2 = -\frac{1}{2}m^2|\vec{\phi}|^2 + \frac{\lambda}{4}|\vec{\phi}|^4 \tag{3.2}$$

and then determine the field configuration that minimises the energy by solving the equation

$$\frac{\partial \mathscr{U}}{\partial|\phi|} = \left(-m^2 + \lambda|\vec{\phi}|^2\right)|\vec{\phi}| = 0.$$

This implies that there are an infinite number of nonzero vevs which all lie on the sphere

$$\vec{\phi}^2 = \frac{m^2}{\lambda} \equiv v^2. \tag{3.3}$$

Mass Spectrum

We choose the ground state

$$\vec{\phi}_0 = (\phi_{01}, \phi_{02}, \phi_{03}) = (0, 0, v), \qquad (3.4)$$

and introduce a quantum field $\eta(x)$ which fluctuates around the classical field $\phi_{03} = v$. I.e the field parametrisation reads,

$$\phi_3 = v + \eta(x). \qquad (3.5)$$

In order to determine the number of Goldstone modes, we substitute the parametrisation in Eq. (3.5) into the potential in Eq. (3.2). In the following we drop the constants in the potential and introduce for brevity $\Phi^2 = \phi_1^2 + \phi_2^2$.

$$
\begin{aligned}
\mathscr{U} &= -\frac{1}{2}m^2(\phi_1^2 + \phi_2^2 + \phi_3^2) + \frac{\lambda}{4}(\phi_1^2 + \phi_2^2 + \phi_3^2)^2 \\
&= -\frac{1}{2}m^2(\Phi^2 + \eta^2 + 2v\eta) + \frac{\lambda}{4}(\Phi^2 + v^2 + \eta^2 + 2v\eta)^2 \\
&= -\frac{1}{2}m^2(\Phi^2 + \eta^2 + 2v\eta) + \frac{\lambda}{4}(\phi^4 + 2v^2\Phi^2 + 2\Phi^2\eta^2 + 4v\Phi^2\eta \\
&\qquad\qquad\qquad\qquad\qquad + 6v^2\eta^2 + 4v^3\eta + \eta^4 + 4v\eta^3).
\end{aligned}
\qquad (3.6)
$$

Due to the presence of the Φ^2-terms in Eq. (3.6) we see that the ground state is invariant under the $SO(2)$ subgroup of $SO(3)$.

Now, we evaluate the masses of the particles by calculating the relevant second derivatives evaluated in the ground state configuration $\phi_1 = \phi_2 = \eta = 0$. Since the fields ϕ_1 and ϕ_2 have identical couplings in the potential, their masses will be identical and it is only necessary to evaluate one of the second derivatives

$$m_{\phi_1}^2 = \left.\frac{\partial^2 \mathscr{U}}{\partial \phi_1^2}\right|_{\phi_1=\phi_2=\eta=0} = -m^2 + v^2\lambda = -m^2 + m^2 = 0. \qquad (3.7)$$

This reveals that there are two Goldstone bosons,

$$m_{\phi_1}^2 = m_{\phi_2}^2 = 0. \qquad (3.8)$$

The mass of the η excitation is

$$m_\eta^2 = \left.\frac{\partial^2 \mathscr{U}}{\partial \eta^2}\right|_{\phi_1=\phi_2=\eta=0} = -m^2 + 3\lambda v^2 = 2m^2. \qquad (3.9)$$

We can interpret Eq. (3.8) and Eq. (3.9) as follows: The minimum of the potential is on a sphere in field space given by Eq. (3.3). Thus the two NGBs correspond to azimuthal and polar excitations, while the massive particle corresponds to radial excitations. For an $SO(N)$ invariant theory, we would find $N-1$ NGBs because the angular element $d\Omega$ always contain $N-1$ angles in N dimensions.

However, this intuitive argument only holds for $SO(N)$ theories. In order to apply Goldstone's theorem it is necessary to evaluate the number of broken generators.

The Broken Generators

The generators of the Lie algebra $so(3)$ are the three matrices,

$$T_1 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad T_3 = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.10)$$

In order to make sense of the term broken generator, we will consider infinitesimal $SO(3)$ transformations of the form[1]

$$\begin{aligned} \vec{\phi}_0 \to \vec{\phi}_0' = e^{i\theta^a T_a} \vec{\phi}_0 &= \left[1 + i\theta^a T_a + \mathcal{O}(\theta^2) \right] \vec{\phi}_0 \\ &= \left[1 + i\theta^1 T_1 + i\theta^2 T_2 + i\theta^3 T_3 + \mathcal{O}(\theta^2) \right] \vec{\phi}_0. \end{aligned} \quad (3.11)$$

This expression reveals that the ground state is invariant under an infinitesimal transformation if and only if

$$T_i \vec{\phi}_0 = 0. \quad (3.12)$$

If this expression is not satisfied then we are dealing with a broken generator. It is called broken because the ground state is not invariant under the symmetry, unlike the Lagrangian.

Using this definition it is easy to check which of the $so(3)$ generators that are broken by the ground state:

$$\begin{aligned} T_1 \vec{\phi}_0 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} &= -i \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}, \quad \text{broken} \to \text{NGB} \\[2mm] T_2 \vec{\phi}_0 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} &= i \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}, \quad \text{broken} \to \text{NGB} \\[2mm] T_3 \vec{\phi}_0 = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{unbroken} \to \text{Massive.} \end{aligned} \quad (3.13)$$

Note that the vectors $T_1 \vec{\phi}_0$ and $T_2 \vec{\phi}_0$ point in the directions of the ϕ_2 and ϕ_1 components of the field. This is not a coincidence. One can always identify the Goldstone components of the field by acting with the broken generators on the ground state.

The calculation also confirms that the ground state is invariant under the $SO(2)$ subgroup spanned by the generator T_3. To apply Goldstone's theorem we note that the group $SO(3)$ has three generators T_1, T_2 and T_3, while the subgroup $SO(2)$ has one generator T_3. Goldstone's theorem predicts that the number of NGBs should be equal to the dimension of the coset space $SO(3)/SO(2)$, which in this case is two. So Goldstone's theorem is satisfied.

---

[1]Here an implicit sum over $a = 1, 2, 3$ is implied.

3.2   The Non-Abelian Linear SO(3) Model with a Chemical Potential

Now that we have established that the SSB $SO(3) \rightarrow SO(2)$ result in two NGBs, let us see what happens when we add a chemical potential to the theory. The relation between the chemical potential and the covariant derivative is determined in exactly the same way as in section 2.2.1. The result is the replacement rule

$$\partial_\alpha \rightarrow D_\alpha \vec{\phi} = (\partial_\alpha - i\mu T_3 \delta_{0\alpha})\vec{\phi}, \tag{3.14}$$

when we choose to couple the chemical potential to the generator T_3. Replacing ∂_μ by D_μ in Eq. (3.1) gives

$$\mathscr{L} = \frac{1}{2}\left(D_\mu \vec{\phi}\right)^2 + \frac{1}{2}m^2\vec{\phi}^2 - \frac{\lambda}{4}\left(\vec{\phi}^2\right)^2. \tag{3.15}$$

Explicit Symmetry Breaking by the Chemical Potential

In order to show the effect of the chemical potential it is necessary to express the covariant derivative in vector form,

$$D_0\vec{\phi} = (\partial_0 - i\mu T_3)\,\vec{\phi} = \begin{pmatrix} \partial_0\phi_1 - \mu\phi_2 \\ \partial_0\phi_2 + \mu\phi_1 \\ \partial_0\phi_3 \end{pmatrix}. \tag{3.16}$$

The kinetic term of the Lagrangian becomes

$$\left(D_\mu\vec{\phi}\right)^2 = \left(D_\mu\vec{\phi}\right)\left(D^\mu\vec{\phi}\right) = \left(D_0\vec{\phi}\right)\left(D_0\vec{\phi}\right) - \left(\nabla\vec{\phi}\right)^2$$

$$= \begin{pmatrix} \partial_0\phi_1 - \mu\phi_2 & \partial_0\phi_2 + \mu\phi_1 & \partial_0\phi_3 \end{pmatrix} \begin{pmatrix} \partial_0\phi_1 - \mu\phi_2 \\ \partial_0\phi_2 + \mu\phi_1 \\ \partial_0\phi_3 \end{pmatrix} - \left(\nabla\vec{\phi}\right)^2 \tag{3.17}$$

$$= \left(\partial_\mu\vec{\phi}\right)^2 + \mu^2\left(\phi_1^2 + \phi_2^2\right) - 2\mu\left(\phi_2\partial_0\phi_2 - \phi_1\partial_0\phi_2\right).$$

Substituting Eq. (3.17) into the Lagrangian in Eq. (3.15) yields,

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu\vec{\phi}\right)^2 + \left(\phi_1^2 + \phi_2^2\right)\left(\frac{\mu^2 + m^2}{2}\right) + \frac{1}{2}m^2\phi_3^2$$

$$- \mu\left(\phi_2\partial_0\phi_1 - \phi_1\partial_0\phi_2\right) - \frac{\lambda}{4}\left(\phi_1^2 + \phi_2^2 + \phi_3^2\right)^2. \tag{3.18}$$

By studying the Lagrangian in Eq. (3.18) we see that the terms involving the chemical potential explicitly break the $SO(3)$ symmetry. That is by introducing the chemical potential the $SO(3)$ symmetry of the Lagrangian is lost. The new symmetry of the Lagrangian is in fact $SO(2)$. Furthermore, we note that the chemical potential also breaks Lorentz invariance. Thus for the Lagrangian in Eq. (3.18) Goldstone's theorem, in the form given in section 1.2, is not valid.

The Vacuum Expectation Value

For convenience we define $|\Phi|^2 = \phi_1^2 + \phi_2^2$, and identify the potential as

$$\mathscr{U} = -\frac{(\mu^2 + m^2)}{2}|\Phi|^2 - \frac{1}{2}m^2\phi_3^2 + \frac{\lambda}{4}\left(|\Phi|^2 + \phi_3^2\right)^2. \tag{3.19}$$

To determine the vevs we need the derivatives w.r.t. $|\Phi|$ and ϕ_3

$$\frac{\partial\mathscr{U}}{\partial|\Phi|} = 0 = |\Phi|\left(-\mu^2 - m^2 + \lambda|\Phi|^2 + \lambda\phi_3^2\right),$$

$$\frac{\partial\mathscr{U}}{\partial\phi_3} = 0 = \phi_3\left(-m^2 + \lambda|\Phi|^2 + \lambda\phi_3^2\right).$$

The only solutions that correspond to stable minima is

$$|\Phi|^2 = \frac{\mu^2 + m^2}{\lambda} \equiv v^2 \quad \text{and} \quad \phi_3 = 0. \tag{3.20}$$

Thus the potential takes its minimum value on a circle in the plane $\phi_3 = 0$ and we have a set of nonzero vevs.

Mass Spectrum

As expected from chapter 2 there are an infinite number of available ground states, all of which lie on the circle in Eq. (3.20). In order to proceed we need to pick one particular ground state and parametrise the field accordingly. The ground state we choose is

$$\vec{\phi}_0 = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \tag{3.21}$$

and the corresponding parametrisation is,

$$\vec{\phi} = \begin{pmatrix} v + \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \tag{3.22}$$

By substituting Eq. (3.22) into Eq. (3.18) we get,

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu\psi_1\right)^2 + \frac{1}{2}\left(\partial_\mu\psi_2\right)^2 + \frac{1}{2}\left(\partial_\mu\psi_3\right)^2 + \left(\frac{\mu^2 + m^2}{2}\right)\left[\left(\psi_1 + v\right)^2 + \psi_2^2\right]$$
$$+ \frac{m^2}{2}\psi_3^2 - \mu\left(\psi_2\partial_0\psi_1 - \left(\psi_1 + v\right)\partial_0\psi_2\right) - \frac{\lambda}{4}\left(\left(\psi_1 + v\right)^2 + \psi_2^2 + \psi_3^2\right)^2. \tag{3.23}$$

Extracting the quadratic part $\mathscr{L}^{(2)}$ yields

$$\mathscr{L}^{(2)} = \frac{1}{2}\partial_\mu\psi_1\partial^\mu\psi_1 + \frac{1}{2}\partial_\mu\psi_2\partial^\mu\psi_2 + \frac{1}{2}\partial_\mu\psi_3\partial^\mu\psi_3 + \left(\frac{\mu^2 + m^2}{2}\right)\left(\psi_1^2 + \psi_2^2\right)$$
$$+ \frac{m^2}{2}\psi_3^2 - \mu\left(\psi_2\partial_0\psi_1 - \psi_1\partial_0\psi_2\right) - \lambda\left(\frac{3}{2}v^2\psi_1^2 + \frac{1}{2}v^2\psi_2^2 + \frac{1}{2}v^2\psi_3^2\right), \tag{3.24}$$

which can be written in matrix form as

$$
\mathscr{L}^{(2)} = \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 \end{pmatrix} \begin{pmatrix} -\frac{1}{2}\partial_\mu\partial^\mu - \left(\mu^2 + m^2\right) & \mu\partial_0 & 0 \\ -\mu\partial_0 & -\frac{1}{2}\partial_\mu\partial^\mu & 0 \\ 0 & 0 & -\frac{1}{2}\partial_\mu\partial^\mu - \frac{\mu^2}{2} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}.
$$
(3.25)

As in Eq. (2.81) the matrix sandwiched between the two vectors is the inverse propagator D^{-1}. Performing a Fourier transformation gives

$$
D^{-1} = \begin{pmatrix} \frac{1}{2}\left(\omega^2 - \mathbf{p}^2\right) - \left(\mu^2 + m^2\right) & -i\mu\omega & 0 \\ i\mu\omega & \frac{1}{2}\left(\omega^2 - \mathbf{p}^2\right) & 0 \\ 0 & 0 & \frac{1}{2}\left(\omega^2 - \mathbf{p}^2\right) - \frac{\mu^2}{2} \end{pmatrix}
$$
(3.26)

and the dispersion relations can then be determined by requiring $\det D^{-1} = 0$. However, to find the mass spectrum we also set $\mathbf{p} = 0$. Thus the characteristic equation reads,

$$
0 = \frac{1}{2}\left(\omega^2 - \mu^2\right)\left[\frac{1}{2}\omega^2\left(\frac{1}{2}\omega^2 - \mu^2 - m^2\right) - \mu^2\omega^2\right].
$$
(3.27)

By factorising and completing the square, the mass spectrum reads

$$
\begin{aligned}
m_{\psi_1} &= \sqrt{6\mu^2 + 2m^2}, \\
m_{\psi_2} &= 0, \\
m_{\psi_3} &= \mu.
\end{aligned}
$$
(3.28)

By comparing Eqs. (3.28), (3.8) and (3.9) we note that one of the previously massless NGBs has become a mNGB with mass μ. In section 3.3 we will investigate its scattering amplitude.

Physical Interpretation

Let us try to understand the emergence of the mNGB physically. By including a small chemical potential the symmetry of the Lagrangian is explicitly broken as $SO(3) \to SO(2)$. In this case, there is one exact symmetry direction (generated by T_3) and one direction were the symmetry is approximate (generated by a linear combination of T_1 and T_2). Thus excitations in the exact symmetry direction give the NGB, while excitations in the approximate symmetry direction give the mNGB.[2] This interpretation is also consistent in the limit $\mu \to 0$. In this limit the direction of approximate symmetry becomes a direction of exact symmetry. Hence, the mNGB becomes a massless NGB.

The Nonlinear Sigma Model

In order to investigate the scattering amplitudes of the NGB and the mNGB we will introduce the nonlinear sigma model. The nonlinear sigma model is useful because it describes only the Goldstone bosons of the theory [14], as we now show.

For our case the nonlinear sigma model reads,

$$
\mathscr{L} = \frac{1}{2}\left(D_\mu\vec{\chi}\right)^2 = \frac{1}{2}\left(D_0\vec{\chi}\right)^2 - \frac{1}{2}\left(\partial_i\vec{\chi}\right)^2
$$
(3.29)

---

[2]In fact in chapter 9 we show that a mNGB always corresponds to the explicit breaking of two generators.

where $\vec{\chi} = (\chi_1, \chi_2, \chi_3)$ is a vector field. Since there are only two NGBs, but three degrees of freedom (d.o.f.) in $\vec{\chi}$ we need a constraint. In order to correctly impose a constraint on the field $\vec{\chi}$ it is useful to imagine that we turn on the chemical potential in an adiabatic manner. Since the NGBs live in the coset space $SO(3)/SO(2)$, which has the geometry of the unit sphere S^2, we impose the spherically symmetric constraint[3]

$$\chi_1^2 + \chi_2^2 + \chi_3^2 = v^2.$$

Now we adiabatically turn on the chemical potential, and observe how the field responds.[4]

We choose the ground state to be $\vec{\chi}_0 = (v, 0, 0)$ and treat χ_2 and χ_3 as independent d.o.f. The temporal part of the nonlinear sigma model is,

$$(D_0\vec{\chi})^2 = (\partial_0\chi_1)^2 + (\partial_0\chi_2)^2 + (\partial_0\chi_3)^2 + \mu^2 \left(\chi_1^2 + \chi_2^2\right) - 2\mu \left(\chi_2\partial_0\chi_1 - \chi_1\partial_0\chi_2\right). \quad (3.30)$$

If we insert this into Eq. (3.29), we obtain

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2} \left(\partial_\mu\chi_2\right)^2 + \frac{1}{2} \left(\partial_\mu\chi_3\right)^2 - \frac{\mu^2}{2}\chi_3^2 + \frac{\mu}{2v} \left(\chi_2^2\partial_0\chi_2 - \chi_3^2\partial_0\chi_2\right) + \frac{\mu}{v}\chi_2\chi_3\partial_0\chi_3 \\
&\quad + \frac{1}{2v^2} \left(\chi_2^2\partial_\mu\chi_2\partial^\mu\chi_2 + \chi_3^2\partial_\mu\chi_3\partial^\mu\chi_3 + 2\chi_2\chi_3\partial_\mu\chi_2\partial^\mu\chi_3\right) + \mathcal{O}\left(\chi^5\right).
\end{aligned} \quad (3.31)$$

The mass spectrum can be obtained from Eq. (3.31) by considering its quadratic part

$$\begin{aligned}
\mathcal{L}^{(2)} &= \frac{1}{2}\partial_\mu\chi_2\partial^\mu\chi_2 + \frac{1}{2}\partial_\mu\chi_3\partial^\mu\chi_3 - \frac{\mu^2}{2}\chi_3^2 \\
&= \left(\begin{array}{cc} \chi_2 & \chi_3 \end{array}\right) \left(\begin{array}{cc} -\frac{1}{2}\partial_\mu\partial^\mu & 0 \\ 0 & -\frac{1}{2}\partial_\mu\partial^\mu - \frac{\mu^2}{2} \end{array}\right) \left(\begin{array}{c} \chi_2 \\ \chi_3 \end{array}\right).
\end{aligned} \quad (3.32)$$

We denote the matrix as D^{-1} and perform a Fourier transformation

$$D^{-1} = \left(\begin{array}{cc} \frac{1}{2}\left(\omega^2 - \mathbf{p}^2\right) & 0 \\ 0 & \frac{1}{2}\left(\omega^2 - \mathbf{p}^2\right) - \frac{\mu^2}{2} \end{array}\right). \quad (3.33)$$

The dispersion relation follows from requiring that the matrix in Eq. (3.33) is singular,

$$0 = \det D^{-1} = \frac{1}{4}\left(\omega^2 - \mathbf{p}^2\right)\left(\omega^2 - \mathbf{p}^2 - \mu^2\right).$$

Thus the dispersion relations become

$$\begin{aligned}
\omega_{\chi_2}(\mathbf{p}) &= |\mathbf{p}|, \\
\omega_{\chi_3}(\mathbf{p}) &= \sqrt{\mu^2 + \mathbf{p}^2},
\end{aligned} \quad (3.34)$$

---

[3]For some problems it might be useful to express χ_i in terms of spherical coordinates, such that the constraint $\chi_1^2 + \chi_2^2 + \chi_3^2 = v^2$ fixes the radius. The massive and the massless NGB then corresponds to the angular fields θ and ϕ.

[4]The situation is analogues to an antiferromagnet placed in an external field: Before we turn on the field there is no preferred direction for the spins to align, however as the field is turned on the spins will align perpendicular to the field. More on this in chapter 9.

and the mass spectrum can be determined by taking the limit $\mathbf{p} \to 0$,

$$
\begin{aligned}
\omega_{\chi_2}(0) &= m_{\chi_2} = 0, \\
\omega_{\chi_3}(0) &= m_{\chi_3} = \mu.
\end{aligned}
\tag{3.35}
$$

Since the mass spectrum is the same as before, we conclude that the nonlinear sigma model contains only the Goldstone modes of the theory. In addition, we note that when $\mu = 0$ both the NGBs are of type-I. This, combined with the Abelian example of the last chapter, indicate that one NGB of type-I is born for each generator that is spontaneously broken. This is consistent with the existing literature, see e.g. [7].

3.3 Scattering Amplitudes in the Linear SO(3) Model with a Chemical Potential

In this section we use the nonlinear sigma model in Eq. (3.31) to investigate the soft limit of a scattering amplitude describing a process involving the mNGB. Let us denote by π the NGB and by G the mNGB. The scattering amplitude we will calculate corresponds to the process

$$
\pi(k)G(p) \to \pi(k')G(p').
\tag{3.36}
$$

However, before we can calculate the scattering amplitude we need the Feynman rules.

Feynman Rules

With the new notation and using the chain rule the Lagrangian in Eq. (3.31) can be rewritten as

$$
\begin{aligned}
\mathscr{L} ={}& \frac{1}{2}\left(\partial_\mu \pi\right)^2 + \frac{1}{2}\left(\partial_\mu G\right)^2 - \frac{\mu^2}{2}G^2 \\
&+ \frac{\mu}{2v}\left[\frac{1}{3}\frac{d}{dt}\left(\pi^3\right) - G^2\dot{\pi} + \pi\frac{d}{dt}(G^2)\right] \\
&+ \frac{1}{2v^2}\left[\pi^2\left(\partial_\mu \pi\right)^2 + G^2\left(\partial_\mu G\right)^2 + 2\pi G\partial_\mu \pi \partial^\mu G\right].
\end{aligned}
\tag{3.37}
$$

Dropping the total derivatives[5] and performing a partial integration w.r.t. time we can simplify Eq. (3.37) to

$$
\begin{aligned}
\mathscr{L} ={}& \frac{1}{2}\left(\partial_\mu \pi\right)^2 + \frac{1}{2}\left(\partial_\mu G\right)^2 - \frac{\mu^2}{2}G^2 - \frac{\mu}{v}G^2\dot{\pi} \\
&+ \frac{1}{2v^2}\left[\pi^2\left(\partial_\mu \pi\right)^2 + G^2\left(\partial_\mu G\right)^2 + 2\pi G\partial_\mu \pi \partial^\mu G\right].
\end{aligned}
\tag{3.38}
$$

The Feynman rules corresponding to Eq. (3.38), that we will need, are shown in Fig 3.1.

---

[5]Alternatively we could have kept the corresponding Feynman vertices, but in the end they turn out to be zero due to conservation of momentum.
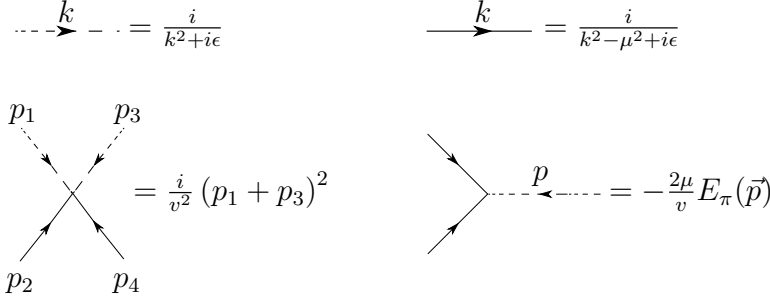
Figure 3.1: The Feynman rules in the nonlinear sigma model needed to calculate the amplitude for $\pi(k)G(p) \to \pi(k')G(p')$. The NGB is represented as a dashed line and the mNGB as a thick line.

The Scattering Amplitude

Using the Feynman rules we can calculate the scattering amplitude of the process in Eq (3.36). There are three diagrams contributing, as shown in Fig. 3.2.
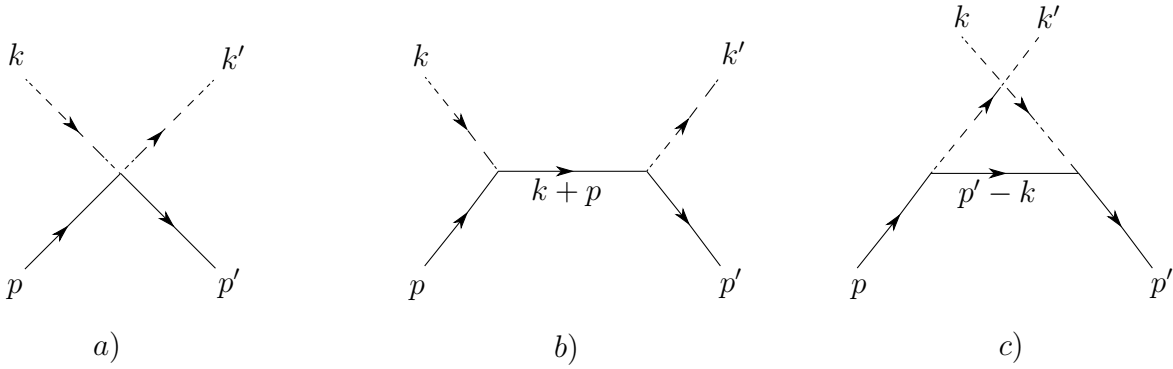
Figure 3.2: Feynman diagrams contributing to $\pi(k)G(p) \to \pi(k')G(p')$.

Summing up the diagrams give[6]

$$-i\mathscr{A}_{tree} = \frac{i}{v^2}\left(k - k'\right)^2 + \left(\frac{-2\mu}{v}\right)(k_0)\frac{i}{(k+p)^2 - \mu^2}\left(-\frac{2\mu}{v}\right)(-k_0')$$
$$+ \left(\frac{2\mu}{v}\right)(k_0')\frac{i}{(k-p')^2 - \mu^2}\left(-\frac{2\mu}{v}\right)(k_0) \tag{3.39}$$

If we then use the on-shell conditions $k^2 = k'^2 = 0$ we obtain

$$-i\mathscr{A}_{tree} = -\frac{2i}{v^2}kk' - \frac{4i\mu^2}{v^2}\frac{k_0k_0'}{(k+p)^2 - \mu^2} - \frac{4i\mu^2}{v^2}\frac{k_0k_0'}{(k-p')^2 - \mu^2}. \tag{3.40}$$

Thus the total amplitude is

$$\mathscr{A}_{tree} = \frac{2}{v^2}kk' + \frac{4\mu^2}{v^2}k_0k_0'\left(\frac{1}{(p+k)^2 - \mu^2} + \frac{1}{(k-p')^2 - \mu^2}\right). \tag{3.41}$$

---

[6]In this case we let the sum of the diagrams equal $-i\mathscr{A}_{tree}$ because then the negative signs cancel in $\mathscr{A}_{tree}$.

Soft Limits

Having obtained the amplitude we can investigate the soft limits of both the mNGB and the NGB. The first soft limit we will investigate is that of the NGB where $k \to 0$. Based on the calculation for the Abelian model, we might expect that this limit vanishes and thus the particle to interact weakly. The soft-limit amplitude becomes,

$$\mathscr{A}_{tree} \xrightarrow[k\to0]{} \frac{4\mu^2}{v^2} \left(\frac{k_0 k_0'}{2pk} - \frac{k_0 k_0'}{2p'k} \right). \tag{3.42}$$

In order to check if the amplitude vanishes, we need the following expressions

$$\begin{aligned}
pk &= p_0 k_0 - |\mathbf{p}||\mathbf{k}|\cos(\theta) \\
&= |\mathbf{k}|\big[p_0 - |\mathbf{p}|\cos(\theta)\big], \\
p'k &= p_0' k_0 - |\mathbf{p}'||\mathbf{k}|\cos(\phi) \\
&= |\mathbf{k}|\big[p_0' - |\mathbf{p}'|\cos(\phi)\big],
\end{aligned} \tag{3.43}$$

where we used the dispersion relations in Eq. (3.34). The angles θ and ϕ are defined in Fig. (3.3).
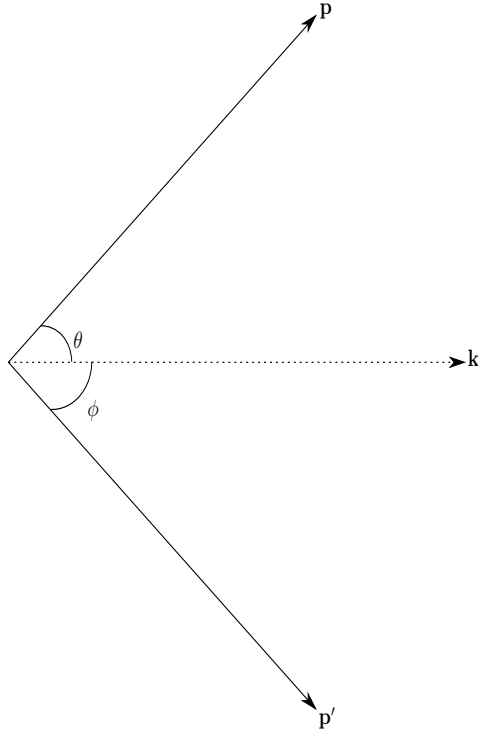
Figure 3.3: Definition of the angles used in Eq. (3.43). The 4-momenta p and p' represent the incoming and outgoing mNGB respectively. The 4-momentum k labels the incoming NGB.

Inserting Eq. (3.43) into Eq. (3.42) gives

$$\mathscr{A}_{tree} \xrightarrow[k\to0]{} \frac{2\mu^2}{v^2} \left(\frac{k_0'}{p_0 - |\mathbf{p}|\cos(\theta)} - \frac{k_0'}{p_0' - |\mathbf{p}'|\cos(\phi)} \right). \tag{3.44}$$

Thus the amplitude does not always vanish in soft limit of the NGB $k \to 0$. Let us determine the kinematic constraint that ensures that the amplitude vanishes. If the amplitude is to vanish then

$$p_0 - |\mathbf{p}| \cos(\theta) = p_0' - |\mathbf{p}'| \cos(\theta) \tag{3.45}$$

must hold. If we use Eq. (3.43) we can write

$$\frac{kp}{|\mathbf{k}|} = \frac{kp'}{|\mathbf{k}|} \tag{3.46}$$

or

$$\frac{k}{|\mathbf{k}|} \left(p - p' \right) = 0. \tag{3.47}$$

Using conservation of 4-momenta gives

$$k \left(p - p' \right) = k \left(k' - k \right) = kk' = k_0 k_0' - |\mathbf{k}||\mathbf{k}'| \cos \alpha = |\mathbf{k}||\mathbf{k}'| \left[1 - \cos \alpha \right], \tag{3.48}$$

where α is the angle between the momenta $\mathbf{k}$ and $\mathbf{k}'$. From Eq. (3.48) we see that the scattering amplitude vanishes only if the NGB undergoes forward scattering $\alpha = 0$ in the collision. More interestingly, we see that our naive expectation was wrong and that the scattering amplitude is nonzero when $\mathbf{k}$ and $\mathbf{k}'$ are not collinear! This is an example of a *kinematic singularity*. Kinematic singularities are further discussed in [10] where it is proved that they usually arise because of cubic interaction terms in the Lagrangian.

Finally we investigate the soft limit of the mNGB $p \to \left(\mu, \vec{0} \right)$. The amplitude in Eq. (3.41) becomes

$$\mathscr{A}_{tree} \xrightarrow[p \to (\mu, \vec{0})]{} \frac{2}{v^2} kk' + \frac{4\mu^2}{v^2} k_0 k_0' \left(\frac{1}{2\mu k_0} - \frac{1}{2\mu k_0'} \right) = \frac{2}{v^2} kk' + \frac{2\mu}{v^2} \left(k_0' - k_0 \right). \tag{3.49}$$

Using conservation of 4-momenta, we can write

$$-2kk' = (k - k')^2 = (p' - p)^2 = 2\mu^2 - 2pp' \xrightarrow[p \to (\mu, \vec{0})]{} 2\mu^2 - 2\mu p_0'$$
$$= 2\mu^2 - 2\mu(k_0 + p_0 - k_0') = 2\mu^2 - 2\mu^2 - 2\mu(k_0 - k_0') = -2\mu(k_0 - k_0'). \tag{3.50}$$

Inserting Eq. (3.50) into Eq. (3.49) we obtain

$$\mathscr{A}_{tree} \to \frac{2\mu}{v^2} \left(k_0 - k_0' \right) + \frac{2\mu}{v^2} \left(k_0' - k_0 \right) = 0. \tag{3.51}$$

This gives us a first indication that mNGBs also have a vanishing scattering amplitude in the soft limit. However, it does not exclude the possibility that there exist systems with kinematic singularities.

A Higgs-Like Model

In this section we show another case where Goldstone's theorem does not apply due to a chemical potential. Furthermore, the model indicates that there exist two different types of massless NGBs as we mentioned in the introduction. They are characterised by their dispersion relations in the soft limit and are denoted by type-I and type-II. The Lagrangian we will consider is similar to the Higgs model and is given by

$$\mathscr{L} = D_\mu \phi^\dagger D^\mu \phi - m^2 \phi^\dagger \phi - \lambda \left(\phi^\dagger \phi \right)^2 \tag{4.1}$$

where ϕ is a complex doublet and $D_\mu = \partial_\mu - i\mu \delta_{0\mu}$.

4.1   Symmetries of the Model

We begin by determining the most important symmetries of the Lagrangian in the cases $\mu = 0$ and $\mu \neq 0$.

Zero Chemical Potential

In the case $\mu = 0$ the Lagrangian is invariant under $SO(4)$.[1] However, to make the symmetry breaking pattern more explicit we will use the fact that $SO(4)$ is locally isomorphic to $SU(2) \times SU(2)$, that is

$$SO(4) \simeq SU(2) \times SU(2). \tag{4.2}$$

Nonzero Chemical Potential

To show that a nonzero chemical potential breaks Lorentz invariance and $SO(4)$ let us expand Eq. (4.1)

$$\begin{aligned}
\mathscr{L} &= (\partial_0 + i\mu) \, \phi^\dagger \, (\partial_0 - i\mu) \, \phi - m^2 \phi^\dagger \phi - \lambda \left(\phi^\dagger \phi \right)^2 \\
&= \partial_\mu \phi^\dagger \partial^\mu \phi + i\mu \left(\phi^\dagger \partial_0 \phi - \partial_0 \phi^\dagger \phi \right) + \phi^\dagger \phi \left(\mu^2 - m^2 \right) - \lambda \left(\phi^\dagger \phi \right)^2 .
\end{aligned} \tag{4.3}$$

---

[1]This can be shown explicitly by writing $\phi = \begin{pmatrix} \pi_1 + i\pi_2 \\ \pi_3 + i\pi_4 \end{pmatrix}$, where π_1, π_2, π_3 and π_4 are real fields. However, it is not necessary because the doublets always occur in the products $\phi^* \phi$.

Thus, symmetries are lost due to the term $\mu j^0 \equiv i\mu \left(\phi^\dagger \partial_0 \phi - \partial_0 \phi^\dagger \phi \right)$. The Lagrangian in Eq. (4.3) is invariant under global $SU(2)$ and $U(1)$ transformations, i.e the symmetry group is

$$SU(2) \times U(1). \tag{4.4}$$

Hence by introducing a chemical potential we have the following explicit symmetry breaking pattern,

$$SU(2) \times SU(2) \xrightarrow{\mu} SU(2) \times U(1). \tag{4.5}$$

4.2 Vacuum Expectation Value and Symmetry Breaking

We will now show that for $\mu > m$ the symmetry is spontaneously broken by a nonzero vev. The vev is determined from the static part of the Lagrangian

$$
\begin{aligned}
\mathscr{U} &= -\phi^\dagger \phi \left(\mu^2 - m^2 \right) + \lambda \left(\phi^\dagger \phi \right)^2 \\
&= - \left(|\phi_1|^2 + |\phi_2|^2 \right) \left(\mu^2 - m^2 \right) + \lambda \left(|\phi_1|^2 + |\phi_2|^2 \right)^2 .
\end{aligned}
\tag{4.6}
$$

Since $|\phi_1|^2$ and $|\phi_2|^2$ appear identically in the equation above, we need only calculate the derivative with respect to one of them to find the vev. This gives

$$0 = \frac{\partial \mathscr{U}}{\partial |\phi_1|} = 2|\phi_1| \left[m^2 - \mu^2 + 2\lambda \left(|\phi_1|^2 + |\phi_2|^2 \right) \right], \tag{4.7}$$

hence the vev is

$$|\phi_1|^2 + |\phi_2|^2 = \phi^\dagger \phi = \frac{\mu^2 - m^2}{2\lambda} \equiv \frac{v^2}{2}. \tag{4.8}$$

Note that Eq. (4.8) only has a real solution when $\mu > m$. In the case $\mu < m$ the vev is zero and there is no SSB, just like in chapter 2.

4.3 The Broken Generators

We now choose the ground state as

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{4.9}$$

and try to determine which generators it break. One set of generators for $SU(2) \times U(1)$ is the set $\frac{1}{2} (\sigma_1, \sigma_2, \sigma_3, I)$, in which case all of the generators are broken and one might expect four NGBs. However, by applying a general $SU(2) \times U(1)$ transformation we can show that there

is a more interesting second set of generators. Consider the infinitesimal transformation,

$$
\begin{aligned}
\phi_0 \rightarrow \phi' &= e^{\frac{i\alpha^a \sigma_a}{2}} e^{\frac{i\beta}{2}} \phi_0 \\
&= \left(1 + \frac{i\alpha^a \sigma_a}{2}\right)\left(1 + \frac{i\beta}{2}\right)\phi_0 \\
&= \left(1 + \frac{i\beta}{2} + \frac{i\alpha^a \sigma_a}{2}\right)\phi_0 \\
&= \left(1 + \frac{i\beta}{2} + \frac{i\alpha^1}{2}\sigma_1 + \frac{i\alpha^2}{2}\sigma_2 + \frac{i\alpha^3}{2}\sigma_3\right)\phi_0 \\
&= \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0 \\ v \end{pmatrix} + \begin{pmatrix} \frac{v}{2}\left(i\alpha^1 + \alpha^2\right) \\ \frac{iv}{2}\left(\beta - \alpha^3\right) \end{pmatrix}\right].
\end{aligned}
\tag{4.10}
$$

Here σ_a are the three Pauli matrices while α^a and β are infinitesimal parameters. From Eq. (4.10) it is clear that in order for the ground state to be invariant under an $SU(2) \times U(1)$ transformation we must require,

$$
\alpha^1 = \alpha^2 = 0 \quad \text{and} \quad \beta = \alpha^3.
\tag{4.11}
$$

Eq. (4.11) corresponds to a generator of the form $\frac{1}{2}(1 + \sigma_3)$. A different set of independent generators for $SU(2) \times U(1)$ is therefore

$$
\frac{1}{2}\left(\sigma_1, \sigma_2, 1 - \sigma_3, 1 + \sigma_3\right).
\tag{4.12}
$$

This set consists of 3 broken and 1 unbroken generator and the symmetry breaking pattern is therefore $SU(2) \times U(1) \rightarrow U(1)'$.[2] If we were to apply Goldstone's theorem we would therefore expect 3 massless NGBs. However, we will see that in fact we only get two gapless bosons. The reason for this is related to the fact that the theory is not Lorentz invariant and therefore the number of broken generators is only an upper limit for the possible number of NGBs. In order to get the correct number we could have used the counting rule presented in [15].

4.4 Dispersion Relations

In order to determine the dispersion relations we could use either of two the parametrisations

$$
\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_1 + i\psi_2 \\ v + \psi_3 + i\psi_4 \end{pmatrix} \quad \text{or} \quad \phi = \frac{1}{\sqrt{2}}e^{\frac{i\tilde{\sigma}\tilde{\pi}}{v}}\begin{pmatrix} 0 \\ v + \psi \end{pmatrix}.
\tag{4.13}
$$

We choose the left parametrisation. Inserting Eq. (4.13) into the Lagrangian in Eq. (4.3) and keeping only the quadratic terms yield

$$
\begin{aligned}
\mathscr{L}^{(2)} &= \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix}\begin{pmatrix} -\frac{1}{2}\partial_\mu \partial^\mu & -\mu\partial_0 \\ \mu\partial_0 & -\frac{1}{2}\partial_\mu \partial^\mu \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\
&\quad + \begin{pmatrix} \psi_3 & \psi_4 \end{pmatrix}\begin{pmatrix} -\frac{1}{2}\partial_\mu \partial^\mu - \left(\mu^2 - m^2\right) & -\mu\partial_0 \\ \mu\partial_0 & -\frac{1}{2}\partial_\mu \partial^\mu \end{pmatrix}\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}.
\end{aligned}
\tag{4.14}
$$

---

[2]The prime means that the unbroken $U(1)'$ is different from the original $U(1)$.

Denoting the matrices in Eq. (4.14) as D_1^{-1} and D_2^{-1} respectively and then performing a Fourier transformation we obtain

$$D_1^{-1} = \begin{pmatrix} \frac{1}{2}\left(\omega^2 - \mathbf{p}^2\right) & i\mu\omega \\ -i\mu\omega & \frac{1}{2}\left(\omega^2 - \mathbf{p}^2\right) \end{pmatrix},$$

$$D_2^{-1} = \begin{pmatrix} \frac{1}{2}\left(\omega^2 - \mathbf{p}^2\right) - \left(\mu^2 - m^2\right) & i\mu\omega \\ -i\mu\omega & \frac{1}{2}\left(\omega^2 - \mathbf{p}^2\right) \end{pmatrix}.$$

(4.15)

The second of these matrices is identical to the one found in the Abelian case displayed in Eq. (2.82). In order to determine the dispersion relations we require that the determinants vanish. The dispersion relations associated with D_1^{-1} are then

$$\omega_\pm = \sqrt{\mathbf{p}^2 + \mu^2} \pm \mu.$$

(4.16)

In the soft limit $\mathbf{p} \to 0$ the dispersion relations become

$$\omega_+ = 2\mu,$$
$$\omega_- = 0.$$

(4.17)

Note that the first particle is a mNGB, while the second is massless NGB. The massless NGB depends on the 3-momentum in a quadratic manner and is therefore characterised as type-II, as we mentioned in the introduction.

The dispersion relations associated with D_2^{-1} are

$$\Omega_\pm^2 = \left(\mathbf{p}^2 + 3\mu^2 - m^2\right) \pm \left(3\mu^2 - m^2\right)\sqrt{1 + \frac{4\mu^2\mathbf{p}^2}{\left(3\mu^2 - m^2\right)^2}}.$$

(4.18)

which in the soft limit becomes

$$\Omega_+ = \sqrt{6\mu^2 - 2m^2},$$
$$\Omega_- = \left(\sqrt{\frac{\mu^2 - m^2}{3\mu^2 - m^2}}\right)|\mathbf{p}|.$$

(4.19)

The first particle in Eq. (4.19) is a massive particle and the second is a NGB of type-I.

A Simplified Physical Interpretation

In this model the complete symmetry breaking pattern is

$$SU(2) \times SU(2) \overset{\mu}{\to} SU(2) \times U(1) \to U(1)'.$$

(4.20)

The mNGB come from the explicit breaking of the two generators of $SU(2)$. This leaves us with four generators that may potentially be spontaneously broken. Now by then choosing a ground state we spontaneously break three generators in the last step. Our previous examples tell us that the type-I NGB corresponds to only one spontaneously broken generator. Thus we are forced to conclude that the existence of one type-II NGB requires two spontaneously broken generators, consistent with the existing literature.

Part II

The Soft-Limit Scattering Amplitude of pNGBs

Chapter 5

The Complete Breaking of an $O(3)$ Symmetry

Up to now we have only considered examples with gapless NGBs and mNGBs with a fixed gap. However, recent research [1] suggests that there exists a second kind of massive Goldstone boson which we call a pseudo Goldstone boson (pNGB). The pNGB is different from the other bosons because its gap depends on the free parameters of the theory and may therefore be adjustable. Next we present a concrete example of a theory containing one NGB, mNGB and pNGB. We begin by calculating the mass spectrum of the theory and finish by calculating the soft-limit behaviour of the on-shell tree-level scattering amplitude involving both the mNGB and the pNGB.

The example we will be considering is the case when three of the generators of $O(3)$ are broken. This situation can not be described using a vector field $\vec{\phi}$ as we used in chapter 3. However, we can construct an effective low-energy Lagrangian by following the procedure developed in [16] called the coset construction.

5.1 Definitions in the Effective Field Theory

The Lagrangian we will consider has the form

$$\mathscr{L} = \frac{1}{2}g_{ab}(\pi)D_\mu\pi^a D^\mu\pi^b. \tag{5.1}$$

There are in total three Goldstone bosons π_a, one for each broken generator T_a. The metric $g_{ab}(\pi)$ is in general too complicated to be calculated in closed form. However, we can make good progress by first introducing $U(\pi) = e^{i\pi^a T_a}$ and then defining the Maurer-Cartan form (MC form) $\omega_b^a(\pi)$ implicitly as[1]

$$- iU(\pi)^{-1}\partial_\mu U(\pi) \equiv T_a\omega_b^a(\pi)\partial_\mu\pi^b. \tag{5.2}$$

The metric is of the form

$$g_{ab}(\pi) = g_{cd}(0)\omega_a^c(\pi)\,\omega_b^d(\pi) \tag{5.3}$$

---

[1]The reason for us to define $U(\pi)$ and the MC form in this way is that it enables us to calculate the MC form order by order in a series expansion in π^a. We will see this soon enough.

where $g_{cd}(0) = \mathrm{diag}\,(g_1, g_2, g_3)$ can be chosen to be a diagonal matrix containing the coupling constants of the theory.[2]

The covariant derivative D_μ can be specified by performing an infinitesimal $O(3)$ transformation on the NGB fields $U(\pi)$. The transformation gives,

$$e^{i\pi^a T_a} \rightarrow e^{i\epsilon^a T_a} e^{i\pi^a T_a} \equiv e^{i\pi'^a T_a} \tag{5.4}$$

where ϵ^a are three infinitesimals and $\pi'^a = \pi^a + \epsilon^b h_b^a(\pi)$. The nonlinear function $h_b^a(\pi)$ specifies the covariant derivative as

$$D_\mu \pi^a \equiv \partial_\mu \pi^a - A_\mu^e h_e^a. \tag{5.5}$$

The gauge field is $A_\mu^e = (\mu^e, \vec{0}) = (\mu, \vec{0})$.

With these definitions we will first attempt to construct the Lagrangian up to second order, and hence determine the mass spectrum. As a first step we calculate the functions ω and $\omega_a^c h_b^a$, because these are the ones that we need to construct the Lagrangian.

5.1.1 Calculating ω

To calculate ω we apply the formula

$$e^{-A} \partial_\mu e^A = \int_0^1 dt\, e^{-tA} \left(\partial_\mu A \right) e^{tA} \tag{5.6}$$

to the left side of the definition in (5.2) giving

$$-iU^{-1} \partial_\mu U = -i \int_0^1 dt \big(e^{-it\pi^a T_a} \left(iT_a \partial_\mu \pi^a \right) e^{it\pi^a T_a} \big). \tag{5.7}$$

We can evaluate the RHS of this equation by using the Baker–Campbell–Hausdorff formula $e^{-B} A e^B = A + \big[A, B\big] + \frac{1}{2!}\big[[A, B], B\big] + \ldots$, which gives

$$-iU^{-1} \partial_\mu U = \int_0^1 dt \Big(T_a \partial_\mu \pi^a - it\pi^a \partial_\mu \pi^b \big[T_a, T_b\big] - \frac{t^2}{2} \pi^a \pi^b \partial_\mu \pi^c \big[T_a, [T_b, T_c]\big] + \mathcal{O}(\pi^3) \Big). \tag{5.8}$$

Then by performing the integration we obtain

$$-iU^{-1} \partial_\mu U = T_a \partial_\mu \pi^a - \frac{i}{2} \pi^a \partial_\mu \pi^b \big[T_a, T_b\big] - \frac{1}{6} \pi^a \pi^b \partial_\mu \pi^c \big[T_a, [T_b, T_c]\big] + \mathcal{O}(\pi^3). \tag{5.9}$$

To proceed we recall that the structure constants f_{abc} of $O(3)$ is the Levi-Civita tensor ϵ_{abc} and evaluate the commutators

$$\big[T_a, T_b\big] = i\epsilon_{abc} T_c \tag{5.10}$$

and

$$\big[T_a, [T_b, T_c]\big] = \big[T_a, i\epsilon_{bcd} T_d\big] = i\epsilon_{bcd} i\epsilon_{ade} T_e = -\epsilon_{dbc}\epsilon_{dea} T_e = -\left(\delta_{be}\delta_{ca} - \delta_{ba}\delta_{ce}\right) T_e, \tag{5.11}$$

---

[2]In principle any choice of g_{cd} is allowed as long as it is consistent with the unbroken discrete symmetry $Z_2 \times Z_2 \times Z_2$.

where we in the final equality used that $\epsilon_{dbc}\epsilon_{dea} = \delta_{be}\delta_{ca} - \delta_{ba}\delta_{ce}$. Inserting the commutators into (5.9) we obtain

$$-iU^{-1}\partial_\mu U = T_a\partial_\mu\pi_a + \frac{1}{2}\pi^a\partial_\mu\pi^b\epsilon_{abc}T_c + \frac{1}{6}\pi^a\pi^b\partial_\mu\pi^c\left(\delta_{be}\delta_{ca} - \delta_{ba}\delta_{ce}\right)T_e + \mathcal{O}(\pi^3). \quad (5.12)$$

which can be simplified as follows,

$$\begin{aligned}
-iU^{-1}\partial_\mu U &= T_a\partial_\mu\pi_a + \frac{1}{2}\pi^a\partial_\mu\pi^b\epsilon_{abc}T_c + \frac{1}{6}\left(\pi^c\pi^e\partial_\mu\pi^c T_e - \pi^a\pi^a\partial_\mu\pi^e T_e\right) + \mathcal{O}(\pi^3) \\
&= T_a\partial_\mu\pi^b\delta_{ab} + \frac{1}{2}\pi^c\partial_\mu\pi^b\epsilon_{cba}T_a + \frac{1}{6}\left(\pi^b\pi^a\partial_\mu\pi^b T_a - \vec{\pi}^2\partial_\mu\pi^b T_a\delta_{ab}\right) + \mathcal{O}(\pi^3) \\
&= T_a\partial_\mu\pi^b\left(\delta_{ab} + \frac{1}{2}\pi^c\epsilon_{cba} + \frac{1}{6}\left(\pi^a\pi^b - \vec{\pi}^2\delta_{ab}\right)\right) + \mathcal{O}(\pi^3) \\
&= T_a\partial_\mu\pi^b\left(\delta_{ab} - \frac{1}{2}\pi^c\epsilon_{abc} + \frac{1}{6}\left(\pi^a\pi^b - \vec{\pi}^2\delta_{ab}\right)\right) + \mathcal{O}(\pi^3).
\end{aligned}$$
$$(5.13)$$

Thus by using Eq. (5.2) we can read off the MC form ω up to second order in the fields π^a

$$\omega_b^a = \delta_{ab} - \frac{1}{2}\epsilon_{abc}\pi^c + \frac{1}{6}\left(\pi_a\pi_b - \delta_{ab}\vec{\pi}^2\right) + \mathcal{O}\left(\pi^3\right). \quad (5.14)$$

5.1.2 Calculating $\omega_a^c h_b^a$

To calculate $\omega_a^c h_b^a$ it is convenient to start from the definition given by Eq. (5.4). Multiplication from the left by $e^{-i\pi^a T_a}$ gives

$$e^{-i\pi^a T_a}e^{i\epsilon^a T_a}e^{i\pi^a T_a} = e^{-i\pi^a T_a}e^{i\pi'^a T_a}. \quad (5.15)$$

If we expand the exponential containing the infinitesimal ϵ^a on the LHS and add and subtract 1 on the RHS we get

$$e^{-i\pi^a T_a}\left(1 + i\epsilon^a T_a\right)e^{i\pi^a T_a} = 1 + e^{-i\pi^a T_a}\left(e^{i\pi'^a T_a} - e^{i\pi^a T_a}\right), \quad (5.16)$$

which can be simplified to

$$1 + ie^{-i\pi^a T_a}\epsilon^a T_a e^{i\pi^a T_a} = 1 + e^{-i\pi^a T_a}\left(e^{i\pi^a T_a + i\epsilon^b h_b^a T_a} - e^{i\pi^a T_a}\right). \quad (5.17)$$

Applying the Baker–Campbell–Hausdorff formula to the second term on the LHS gives,

$$e^{-i\pi^a T_a}\epsilon^a T_a e^{i\pi^a T_a} = \epsilon^a T_a + \pi^a\epsilon^b f_{ab}^c T_c + \frac{1}{2}\pi^a\epsilon^b f_{ab}^e\pi^d f_{de}^c T_c + \mathcal{O}(\pi^2). \quad (5.18)$$

Inserting this into Eq. (5.17) and then expanding the exponential $e^{i\epsilon^b h_b^a T_a}$ on the RHS, yields

$$\begin{aligned}
1 + i\left(\epsilon^a T_a + \pi^a\epsilon^b f_{ab}^c T_c + \frac{1}{2}\pi^a\epsilon^b f_{ab}^e\pi^d f_{de}^c T_c\right) &= 1 + e^{-i\pi^a T_a}\left[\left(1 + i\epsilon^b h_b^a T_a\right)e^{i\pi^a T_a} - e^{-i\pi^a T_a}\right] \\
&= 1 + i\epsilon^b h_b^a e^{-i\pi^a T_a}T_a e^{i\pi^a T_a} \\
&= 1 + \epsilon^b h_b^a e^{-i\pi^a T_a}\frac{\partial}{\partial\pi^a}e^{i\pi^a T_a}.
\end{aligned}$$
$$(5.19)$$

To proceed we exploit the relation

$$\omega_a^i T_i = -iU^{-1} \frac{\partial}{\partial \pi^a} U, \qquad (5.20)$$

giving

$$i \left(\epsilon^a T_a + \pi^a \epsilon^b f_{ab}^c T_c + \frac{1}{2} \pi^a \epsilon^b f_{ab}^e \pi^d f_{de}^c T_c \right) = i\epsilon^b h_b^a T_c \omega_a^c. \qquad (5.21)$$

To extract $\omega_a^c h_b^a$ we equate the coefficients for the term $i\epsilon^b T_c$

$$i\epsilon^b T_c \left(\delta_{cb} + \pi^a f_{ab}^c + \frac{1}{2} \pi^a \pi^d f_{ab}^e f_{de}^c \right) = i\epsilon^b T_c \left(h_b^a \omega_a^c \right). \qquad (5.22)$$

Thus we find

$$\omega_a^c h_b^a = \delta_{cb} + \pi^a f_{ab}^c + \frac{1}{2} \pi^a \pi^d f_{ab}^e f_{de}^c + \mathcal{O}(\pi^3). \qquad (5.23)$$

The product of Levi-Civita tensors in $\omega_a^c h_b^a$ can be expressed in terms of Kronecker delta functions as

$$\epsilon_{eab} \epsilon_{ecd} = \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}. \qquad (5.24)$$

The matrix $\omega_a^c h_b^a$ then takes the form

$$\omega_a^c h_b^a = \delta_{bc} + \pi^a \epsilon_{cab} + \frac{1}{2} \left(\pi^c \pi^b - \pi^a \pi^a \delta_{bc} \right).$$

For personal preference we rename the indices and obtain,

$$\omega_c^a h_b^c = \delta_{ab} - \epsilon_{abc} \pi^c + \frac{1}{2} \left(\pi^a \pi^b - \vec{\pi}^2 \delta_{ab} \right). \qquad (5.25)$$

5.1.3 The Lagrangian to Second Order

The first step in determining the mass spectrum of the theory is to find the quadratic part of the Lagrangian. The starting point is Eq. (5.1), and we proceed by inserting the covariant derivative displayed in Eq. (5.5). This gives

$$
\begin{aligned}
\mathscr{L} &= \frac{1}{2} g_{cd}(0) \omega_a^c \omega_b^d D_\mu \pi^a D^\mu \pi^b \\
&= \frac{1}{2} g_{cd} \omega_a^c \omega_b^d \left(\partial_\mu \pi^a - A_\mu^e h_e^a \right) \left(\partial^\mu \pi^b - A^{f\mu} h_f^b \right) \\
&= \frac{1}{2} g_{cd} \omega_a^c \omega_b^d \partial_\mu \pi^a \partial^\mu \pi^b - \frac{1}{2} g_{cd} \omega_a^c \omega_b^d \partial_\mu \pi^a A^{f\mu} h_f^b \\
&\quad - \frac{1}{2} g_{cd} \omega_a^c h_e^a \omega_b^d \partial^\mu \pi^b A_\mu^e + \frac{1}{2} g_{cd} \omega_a^c h_e^a \omega_b^d h_f^b A_\mu^e A^{f\mu}.
\end{aligned}
\qquad (5.26)
$$

Now, if we substitute Eqs. (5.14) and (5.25) into Eq. (5.26) and only keep second order terms we obtain, after the algebraic dust settles,

$$
\begin{aligned}
\mathscr{L} &= \frac{1}{2} \left[g_1 \left(\partial_\mu \pi_1 \right)^2 + g_2 \left(\partial_\mu \pi_2 \right)^2 + g_3 \left(\partial_\mu \pi_3 \right)^2 \right] - \frac{1}{2} g_3 \mu \left(\pi_1 \dot{\pi}_2 - \dot{\pi}_1 \pi_2 \right) \\
&\quad + \mu (g_2 \pi_1 \dot{\pi}_2 - g_1 \dot{\pi}_1 \pi_2) + \pi_1^2 \left[\frac{1}{2} \mu^2 \left(g_2 - g_3 \right) \right] + \pi_2^2 \left[\frac{1}{2} \mu^2 \left(g_1 - g_3 \right) \right].
\end{aligned}
\qquad (5.27)
$$

If we use integration by parts we can write

$$\pi_1 \dot{\pi}_2 = \frac{1}{2}\pi_1 \dot{\pi}_2 - \frac{1}{2}\dot{\pi}_1 \pi_2 = \frac{1}{2}\left(\pi_1 \dot{\pi}_2 - \dot{\pi}_1 \pi_2\right),$$
$$-\dot{\pi}_1 \pi_2 = \frac{1}{2}\left(\pi_1 \dot{\pi}_2 - \dot{\pi}_1 \pi_2\right) \tag{5.28}$$

such that the Lagrangian in Eq. (5.27) becomes

$$\begin{aligned}
\mathscr{L} = &\frac{1}{2}\left[g_1 \left(\partial_\mu \pi_1\right)^2 + g_2 \left(\partial_\mu \pi_2\right)^2 + g_3 \left(\partial_\mu \pi_3\right)^2\right] \\
&- \frac{1}{2}g_3 \mu \left(\pi_1 \dot{\pi}_2 - \dot{\pi}_1 \pi_2\right) + \frac{1}{2}g_2 \mu \left(\pi_1 \dot{\pi}_2 - \dot{\pi}_1 \pi_2\right) \\
&+ \frac{1}{2}g_1 \mu \left(\pi_1 \dot{\pi}_2 - \dot{\pi}_1 \pi_2\right) \\
&- \frac{1}{2}\mu^2 \left(g_3 - g_2\right)\pi_1^2 - \frac{1}{2}\mu^2 \left(g_3 - g_1\right)\pi_2^2.
\end{aligned} \tag{5.29}$$

More compactly we can write

$$\begin{aligned}
\mathscr{L} = &\frac{1}{2}\left[g_1 \left(\partial_\mu \pi_1\right)^2 + g_2 \left(\partial_\mu \pi_2\right)^2 + g_3 \left(\partial_\mu \pi_3\right)^2\right] \\
&+ \frac{1}{2}\mu \left(g_1 + g_2 - g_3\right)\left(\pi_1 \dot{\pi}_2 - \dot{\pi}_1 \pi_2\right) \\
&- \frac{1}{2}\mu^2 \left(g_3 - g_2\right)\pi_1^2 - \frac{1}{2}\mu^2 \left(g_3 - g_1\right)\pi_2^2.
\end{aligned} \tag{5.30}$$

5.1.4 The Mass Spectrum

From Eq. (5.30) we can extract the inverse propagator D^{-1}

$$D^{-1} = \begin{pmatrix} -g_1 \partial_\mu \partial^\mu - \mu^2 \left(g_3 - g_2\right) & \mu \left(g_1 + g_2 - g_3\right)\partial_0 & 0 \\ -\mu \left(g_1 + g_2 - g_3\right)\partial_0 & -g_2 \partial_\mu \partial^\mu - \mu^2 \left(g_3 - g_1\right) & 0 \\ 0 & 0 & -g_3 \partial_\mu \partial^\mu \end{pmatrix}, \tag{5.31}$$

and perform a Fourier transformation to obtain

$$D^{-1} = \begin{pmatrix} g_1 \left(\omega^2 - \mathbf{p}^2\right) - \mu^2 \left(g_3 - g_2\right) & -\mu \left(g_1 + g_2 - g_3\right)i\omega & 0 \\ \mu \left(g_1 + g_2 - g_3\right)i\omega & g_2 \left(\omega^2 - \mathbf{p}^2\right) - \mu^2 \left(g_3 - g_1\right) & 0 \\ 0 & 0 & g_3 \left(\omega^2 - \mathbf{p}^2\right) \end{pmatrix}. \tag{5.32}$$

The mass spectrum follows from that the matrix D^{-1} should be singular, giving the characteristic equation:

$$\begin{aligned}
0 = g_3 \left(\omega^2 - \mathbf{p}^2\right) \Big\{ &\left[g_1 \left(\omega^2 - \mathbf{p}^2\right) - \mu^2 \left(g_3 - g_2\right)\right] \\
&\times \left[g_2 \left(\omega^2 - \mathbf{p}^2\right) - \mu^2 \left(g_3 - g_1\right)\right] - \mu^2 \omega^2 \left(g_1 + g_2 - g_3\right)^2 \Big\}.
\end{aligned} \tag{5.33}$$

Solving Eq. (5.33) gives the following dispersion relations

$$\omega_1 = \omega_- = \sqrt{\mathbf{p}^2 + \mu^2 + \mu^2 \frac{g_3 \left(g_3 - g_1 - g_2\right)}{2 g_1 g_2} \left(1 - \Omega\right)},$$

$$\omega_2 = \omega_+ = \sqrt{\mathbf{p}^2 + \mu^2 + \mu^2 \frac{g_3 \left(g_3 - g_1 - g_2\right)}{2 g_1 g_2} \left(1 + \Omega\right)}, \tag{5.34}$$

$$\omega_3 = |\mathbf{p}|$$

with

$$\Omega = \sqrt{1 + \frac{4 \mathbf{p}^2 g_1 g_2}{\mu^2 g_3^2}}. \tag{5.35}$$

By taking the limit $\mathbf{p} \to 0$ we obtain the mass spectrum:

$$m_1 = \mu,$$

$$m_2 = \sqrt{\frac{\left(g_1 - g_3\right)\left(g_2 - g_3\right)}{g_1 g_2}} \mu, \tag{5.36}$$

$$m_3 = 0.$$

Eq. (5.36) reveals that the theory contains one NGB and two massive modes. One of the massive modes has a gap exactly equal to μ and is a mNGB. We expect the scattering amplitude involving the mNGB to vanish in the soft limit. This expectation is based on the example in chapter 3. The gap of the second massive mode is model dependent, and we expect the mass not to be protected from higher order corrections by a symmetry. This is therefore a pNGB and we expect the soft-limit scattering amplitude to be nonzero.

5.1.5 Noether Charges

Now, we will calculate the Noether charge densities for the three $O(3)$ generators[3] as a series expansion up to first order in the fields π^a. The Noether charge density is given by

$$j^0 = \frac{\partial \mathscr{L}}{\partial \left(\dot{\pi}_a\right)} \delta \pi_a$$

$$= \left(g_1 \dot{\pi}_1 - \frac{1}{2}\mu \left(g_1 + g_2 - g_3\right) \pi_2\right) \delta \pi_1 + \left(g_2 \dot{\pi}_2 + \frac{1}{2}\mu \left(g_1 + g_2 - g_3\right) \pi_1\right) \delta \pi_2 + g_3 \dot{\pi}_3 \delta \pi_3 \tag{5.37}$$

and our task is then mainly to calculate the variations $\delta \pi_a$. We begin by looking at an infinitesimal T_1 transformation on the form

$$e^{i\pi^a T_a} \to e^{i\epsilon T_1} e^{i\pi^a T_a} \tag{5.38}$$

and expand the exponentials

$$\left(1 + i\pi^a T_a\right) \to \left(1 + i\epsilon T_1\right)\left(1 + i\pi^a T_a\right) = 1 + i\pi^a T_a + i\epsilon T_1 - \epsilon T_1 \pi^a T_a. \tag{5.39}$$

---

[3]We consider only the generators of the continuous symmetry.

Expanding the brackets on the LHS and writing out the summations give

$$i\left(\pi_1 T_1 + \pi_2 T_2 + \pi_3 T_3\right) \rightarrow i\left(\pi_1 T_1 + \pi_2 T_2 + \pi_3 T_3\right) + i\epsilon T_1 - \epsilon\pi_1 T_1^2 - \epsilon\pi_2 T_1 T_2 - \epsilon\pi_3 T_1 T_3. \tag{5.40}$$

Performing all matrix multiplications we end up with

$$\begin{pmatrix} 0 & \pi_3 & -\pi_2 \\ -\pi_3 & 0 & \pi_1 \\ \pi_2 & -\pi_1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & \pi_3 & -\pi_2 \\ -\pi_3 + \epsilon\pi_2 & 0 & \pi_1 + \epsilon \\ \pi_2 + \epsilon\pi_3 & -\pi_1 - \epsilon & 0 \end{pmatrix} \tag{5.41}$$

the variations of the fields then read

$$\begin{aligned} \delta\pi_1 &= \epsilon, \\ \delta\pi_2 &= \epsilon\pi_3, \\ \delta\pi_3 &= -\epsilon\pi_2. \end{aligned} \tag{5.42}$$

The Noether charge density for the T_1 transformation then takes the form,

$$j_{T_1}^0 = g_1\dot{\pi}_1 - \frac{1}{2}\mu\left(g_1 + g_2 - g_3\right)\pi_2. \tag{5.43}$$

The calculation for T_2 and T_3 are almost identical and hence omitted. The Noether charges are thus

$$\begin{aligned} j_{T_1}^0 &= g_1\dot{\pi}_1 - \frac{1}{2}\mu\left(g_1 + g_2 - g_3\right)\pi_2, \\ j_{T_2}^0 &= g_2\dot{\pi}_2 + \frac{1}{2}\mu\left(g_1 + g_2 - g_3\right)\pi_1, \\ j_{T_3}^0 &= g_3\dot{\pi}_3. \end{aligned} \tag{5.44}$$

Note that the currents are all linear in the fields. This is actually one of the hallmarks of SSB: whenever a generator is spontaneously broken, the corresponding NGB will appear linearly in the corresponding Noether current. In fact, these linear terms are the most interesting piece of the current in the case of broken symmetry, because they determine how the current operator couples to the NGB state. The coupling is described by the matrix element $\langle 0|j^0|\pi_i\rangle$ which is interpreted as the amplitude to generate the NGB from the vacuum by the current.

5.2 Scattering Amplitudes

In this section we try to show that one can distinguish massive modes and pseudo modes by the soft-limit behaviour of the scattering amplitudes for $2 \rightarrow 2$ scattering. Specifically we try to show that the scattering amplitude for the massive mode vanishes in the soft limit, while the scattering amplitude for the pseudo mode does not vanish in the soft limit.

However, before we can calculate scattering amplitudes in this theory we need the Lagrangian in Eq. (5.30) up to fourth order in the fields. Once we know this we can calculate the relevant Feynman rules and propagators. In this case we observe from Eq. (5.32) that the propagator is energy dependent and therefore not diagonalisable. Therefore we must use the same method as in [17] to calculate the scattering amplitude. That is we must first construct the transition vacuum amplitudes and from these construct the total scattering amplitude.[4]

---

[4]Please note that in this section we change the notation for the fields as $\pi_i \rightarrow \phi_i$.

5.2.1 The Lagrangian

The calculation of the Lagrangian to fourth order is identical to the calculation in section 5.1.3 except that one needs Eqs. (5.14) and (5.25) to fourth order. The calculations were performed using Mathematica and the result is

$$\mathscr{L} = \mathscr{L}^{(2)} + \mathscr{L}^{(3)} + \mathscr{L}^{(4)} \tag{5.45}$$

where

$$
\begin{aligned}
\mathscr{L}^{(2)} = \frac{1}{2} &\left[g_1 \left(\partial_\mu \phi_1 \right)^2 + g_2 \left(\partial_\mu \phi_2 \right)^2 + g_3 \left(\partial_\mu \phi_3 \right)^2 \right] \\
&+ \frac{1}{2} \mu \left(g_1 + g_2 - g_3 \right) \left(\phi_1 \dot{\pi}_2 - \dot{\pi}_1 \phi_2 \right) \\
&- \frac{1}{2} \mu^2 \left(g_3 - g_2 \right) \phi_1^2 - \frac{1}{2} \mu^2 \left(g_3 - g_1 \right) \phi_2^2,
\end{aligned}
\tag{5.46}
$$

$$
\begin{aligned}
\mathscr{L}^{(3)} = &\left[\frac{\mu \left(g_2 - 3g_1 + 3g_3 \right)}{4} \phi_2^2 \dot{\phi}_3 \right] \\
&+ \left[\frac{\mu^2 \left(g_1 - g_2 \right)}{2} \phi_1 \phi_2 \phi_3 - \frac{\left(g_1 - g_2 \right)}{2} \phi_3 \partial_\mu \phi_1 \partial^\mu \phi_2 \right. \\
&\left. + \frac{\left(g_1 - g_3 \right)}{2} \phi_2 \partial_\mu \phi_1 \partial^\mu \phi_3 - \frac{\left(g_2 - g_3 \right)}{2} \phi_1 \partial_\mu \phi_2 \partial^\mu \phi_3 \right] \\
&+ \left[\frac{\mu \left(g_1 - 3g_2 + 3g_3 \right)}{4} \phi_1^2 \dot{\phi}_3 \right],
\end{aligned}
\tag{5.47}
$$

and

$$
\begin{aligned}
\mathscr{L}^{(4)} = &\left[\mu^2 \left(\frac{g_2}{8} - \frac{g_1}{6} + \frac{g_3}{24} \right) \phi_2^2 \phi_3^2 + \left(\frac{g_1}{8} - \frac{g_2}{6} \right) \left(\partial_\mu \phi_2 \right)^2 \phi_3^2 \right. \\
&\left. + \left(\frac{g_2}{6} - \frac{g_1}{4} + \frac{g_3}{6} \right) \phi_2 \left(\partial_\mu \phi_2 \partial^\mu \phi_3 \right) \phi_3 + \left(\frac{g_1}{8} - \frac{g_3}{6} \right) \phi_2^2 \left(\partial_\mu \phi_3 \right)^2 \right] \\
&+ \left[\mu^2 \left(\frac{g_1}{8} - \frac{g_2}{6} + \frac{g_3}{24} \right) \phi_1^2 \phi_3^2 + \left(\frac{g_2}{8} - \frac{g_1}{6} \right) \phi_3^2 \left(\partial_\mu \phi_1 \right)^2 \right. \\
&\left. + \left(\frac{g_1}{6} - \frac{g_2}{4} + \frac{g_3}{6} \right) \phi_1 \left(\partial_\mu \phi_1 \partial^\mu \phi_3 \right) \phi_3 + \left(\frac{g_2}{8} - \frac{g_3}{6} \right) \phi_1^2 \left(\partial_\mu \phi_3 \right)^2 \right] \\
&+ \left[\mu \left(\frac{g_1}{3} - \frac{g_2}{4} - \frac{g_3}{24} \right) \dot{\phi}_1 \phi_2 \phi_3^2 + \mu \left(\frac{g_1}{4} - \frac{g_2}{3} + \frac{g_3}{24} \right) \phi_1 \dot{\phi}_2 \phi_3^2 \right. \\
&\left. + 5\mu \left(\frac{g_2 - g_1}{12} \right) \phi_1 \phi_2 \dot{\phi}_3 \phi_3 \right]
\end{aligned}
\tag{5.48}
$$

Each square bracket in $\mathscr{L}^{(3)}$ and $\mathscr{L}^{(4)}$ represents a Feynman vertex. The procedure for calculating the Feynman rules is the same as always, however the terms are a bit more complicated than what we are used to. The calculation is therefore shown in App. C.1, and the results are shown in Fig. 5.1

$$p_{2-} \blacktriangleright\!-\!\dashv \quad = \frac{\mu\left[g_2 - 3(g_1 - g_3)\right]}{2} E_3(p_3)$$

with p_3 (dotted, incoming top) and p_1 (incoming bottom).

$$p_3\!\cdots\!\blacktriangleright\!\dashv \quad = \frac{\mu\left[g_1 - 3(g_2 - g_3)\right]}{2} E_3(p_3)$$

with p_2 (top) and p_1 (bottom).

$$p_3\!\cdots\!\blacktriangleright\!\dashv \quad = \frac{i\mu^2(g_1 - g_2)}{2} + \frac{i(g_1 - g_2)}{2}(p_1 p_2) + \frac{i(g_3 - g_1)}{2}(p_1 p_3) + \frac{i(g_2 - g_3)}{2}(p_2 p_3)$$

with p_1 (top) and p_2 (bottom).

$$= i\mu^2\left(\frac{g_2}{2} - \frac{2g_1}{3} + \frac{g_3}{6}\right) + i\left(\frac{2g_2}{3} - \frac{g_1}{2}\right)p_1 p_2 - i\left(\frac{g_1}{4} - \frac{g_2}{6} - \frac{g_3}{6}\right)(p_1 + p_2)^2 + i\left(\frac{2g_3}{3} - \frac{g_1}{2}\right)p_3 p_4$$

with legs p_3, p_2 (top), p_1, p_4 (bottom).

$$= i\mu^2\left(\frac{g_1}{2} - \frac{2g_2}{3} + i\frac{g_3}{6}\right) + i\left(\frac{2g_1}{3} - \frac{g_2}{2}\right)p_1 p_2 + i\left(\frac{g_1}{6} - \frac{g_2}{4} + \frac{g_3}{6}\right)(p_1 + p_2)^2 + i\left(\frac{2g_3}{3} - \frac{g_2}{2}\right)p_3 p_4$$

with legs p_1, p_4 (top), p_3, p_2 (bottom).

$$= \mu E_1(p_1)\left[\frac{13g_1 - 11g_2 - g_3}{12}\right] + \mu E_2(p_2)\left[\frac{11g_1 - 13g_2 + g_3}{12}\right]$$

with legs p_1, p_4 (top), p_3, p_2 (bottom).

Figure 5.1: The Feynman rules, that we need, for computing scattering amplitudes when the $O(3)$ symmetry is completely broken. The fields ϕ_1, ϕ_2 and ϕ_3 are represented by the solid line, the dashed line and the dotted line respectively.

5.2.2 The Propagator

The propagator D is the inverse of Eq. (5.32). The fact that the inverse propagator

$$D^{-1} = \begin{pmatrix} g_1\left(\omega^2 - \mathbf{p}^2\right) - \mu^2\left(g_3 - g_2\right) & -\mu\left(g_1 + g_2 - g_3\right)i\omega & 0 \\ \mu\left(g_1 + g_2 - g_3\right)i\omega & g_2\left(\omega^2 - \mathbf{p}^2\right) - \mu^2\left(g_3 - g_1\right) & 0 \\ 0 & 0 & g_3\left(\omega^2 - \mathbf{p}^2\right) \end{pmatrix} \tag{5.49}$$

is in block diagonal form makes the task of inverting it almost trivial. We simply define the submatrix

$$A^{-1} = \begin{pmatrix} g_1\left(\omega^2 - \mathbf{p}^2\right) - \mu^2\left(g_3 - g_2\right) & -\mu\left(g_1 + g_2 - g_3\right)i\omega \\ \mu\left(g_1 + g_2 - g_3\right)i\omega & g_2\left(\omega^2 - \mathbf{p}^2\right) - \mu^2\left(g_3 - g_1\right) \end{pmatrix} \tag{5.50}$$

whose inverse is

$$A = \frac{1}{\det A^{-1}} \begin{pmatrix} g_2 \left(\omega^2 - \mathbf{p}^2\right) - \mu^2 \left(g_3 - g_1\right) & \mu \left(g_1 + g_2 - g_3\right) i\omega \\ -\mu \left(g_1 + g_2 - g_3\right) i\omega & g_1 \left(\omega^2 - \mathbf{p}^2\right) - \mu^2 \left(g_3 - g_2\right) \end{pmatrix} \tag{5.51}$$

with

$$\det A^{-1} = g_1 g_2 \left(\omega^2 - \omega_+^2\right) \left(\omega^2 - \omega_-^2\right). \tag{5.52}$$

Hence the propagator is

$$D = \begin{pmatrix} A & 0 \\ 0 & \frac{1}{g_3\left(\omega^2 - \mathbf{p}^2\right)} \end{pmatrix}. \tag{5.53}$$

5.2.3 Vacuum Transition Amplitudes

By following the procedure described in [17] we will use the propagator in Eq. (5.53) to extract the vacuum transition amplitudes for the fields ϕ_1, ϕ_2 and ϕ_3. To be specific we will employ Eq. (22) in [17] which reads

$$D^{AB}\left(\mathbf{p}, \omega\right) = (2\pi)^3 \sum_n \left[\frac{\langle 0|A\left(0\right)|n, \mathbf{p}\rangle \langle n, \mathbf{p}|B\left(0\right)|0\rangle}{\omega - E_n\left(\mathbf{p}\right) + i\epsilon} - \frac{\langle 0|B\left(0\right)|n, -\mathbf{p}\rangle \langle n, -\mathbf{p}|A\left(0\right)|0\rangle}{\omega + E_n\left(\mathbf{p}\right) - i\epsilon} \right]. \tag{5.54}$$

It is called the Kallen-Lehmann representation of the time-ordered Green's function $D^{AB} = -i\langle 0|T\left[A(x)B(y)\right]|\rangle$. In this notation $A(x)$ and $B(x)$ are generic bosonic operators and $E_n(\mathbf{p})$ is the energy of the state $|n, \mathbf{p}\rangle$ which is normalised as $\langle m, \mathbf{p}_1|n, \mathbf{p}_2\rangle = \delta_{mn}\delta^3(\mathbf{p}_1 - \mathbf{p}_2)$. In order to use Eq. (5.54) we first need to determine how the propagator in Eq. (5.51) behaves close to the two poles $\omega = \omega_+$ and $\omega = \omega_-$.

Behaviour Close to the Poles

In order to simplify the expressions we define

$$\Delta \equiv \omega_+^2 - \omega_-^2 = \mu^2 \frac{g_3\left(g_3 - g_1 - g_2\right)}{g_1 g_2}\Omega. \tag{5.55}$$

If we write $\omega = \omega_\pm \mp i\epsilon$ the determinant in Eq. (5.51) takes the form

$$\det A^{-1} = \pm 2g_1 g_2 \omega_\pm \Delta \left(\omega - \omega_\pm\right) \tag{5.56}$$

and the propagator reads

$$A^\pm = \frac{1}{\det A^{-1}} \begin{pmatrix} g_2 \left(\omega_\pm^2 - \mathbf{p}^2\right) - \mu^2 \left(g_3 - g_1\right) & i\mu \left(g_1 + g_2 - g_3\right) \omega_\pm \\ -i\mu \left(g_1 + g_2 - g_3\right) \omega_\pm & g_1 \left(\omega_\pm^2 - \mathbf{p}^2\right) - \mu^2 \left(g_3 - g_2\right) \end{pmatrix}. \tag{5.57}$$

We can rewrite the second diagonal term as

$$\begin{aligned} g_1 \left(\omega_\pm^2 - \mathbf{p}^2\right) - \mu^2 \left(g_3 - g_2\right) &= g_1 \left[\mu^2 + \mu^2 \frac{g_3\left(g_3 - g_1 - g_2\right)}{2g_1 g_2}\left(1 \pm \Omega\right)\right] - \mu^2 \left(g_3 - g_1\right) \\ &= \mu^2 \left(g_1 + g_2 - g_3\right) + \frac{\mu^2 g_3\left(g_3 - g_1 - g_2\right)}{2g_2}\left(1 \pm \Omega\right) \\ &= \mu^2 \left(g_1 + g_2 - g_3\right)\left(1 - \frac{g_3}{2g_2}\left(1 \pm \Omega\right)\right). \end{aligned} \tag{5.58}$$

By replacing $g_1 \leftrightarrow g_2$ we get a similar expression for the first diagonal term

$$g_2\left(\omega_\pm^2 - \mathbf{p}^2\right) - \mu^2\left(g_3 - g_1\right) = \mu^2\left(g_1 + g_2 - g_3\right)\left(1 - \frac{g_3}{2g_1}\left(1 \pm \Omega\right)\right). \qquad (5.59)$$

Eq. (5.57) then becomes

$$\begin{aligned}
A^\pm &= \frac{\mu^2\left(g_1 + g_2 - g_3\right)}{\pm 2g_1 g_2 \omega_\pm \Delta\left(\omega - \omega_\pm\right)}\begin{pmatrix} 1 - \frac{g_3}{2g_1}\left(1 \pm \Omega\right) & \frac{i\omega_\pm}{\mu} \\ -\frac{i\omega_\pm}{\mu} & 1 - \frac{g_3}{2g_2}\left(1 \pm \Omega\right) \end{pmatrix} \\
&= \frac{\pm 1}{2\omega_\pm\left(\omega - \omega_\pm\right)g_3\Omega}\begin{pmatrix} \frac{g_3}{2g_1}\left(1 \pm \Omega\right) - 1 & -\frac{i\omega_\pm}{\mu} \\ +\frac{i\omega_\pm}{\mu} & \frac{g_3}{2g_2}\left(1 \pm \Omega\right) - 1 \end{pmatrix}.
\end{aligned} \qquad (5.60)$$

Extracting the Transition Amplitudes

If we now let $|1\rangle$, $|2\rangle$ and $|3\rangle$ be states with energy ω_+, ω_- and ω_3 respectively we can use Eq. (5.60) to extract the transition amplitudes. By using the Kallen-Lehmann representation we can write[5]

$$\begin{aligned}
\langle 0|\phi_1\phi_1^\dagger|0\rangle &= \frac{1}{\left(2\pi\right)^3}\mathrm{Res}_{\omega\to\omega_+}A_{11}^+, \\
\langle 0|\phi_2\phi_2^\dagger|0\rangle &= \frac{1}{\left(2\pi\right)^3}\mathrm{Res}_{\omega\to\omega_+}A_{22}^+,
\end{aligned} \qquad (5.61)$$

which after inserting a complete set of states $|1\rangle\langle 1| = 1$ on the LHS becomes

$$\begin{aligned}
\langle 0|\phi_1|1\rangle &= e^{i\theta_1}\sqrt{\frac{\frac{g_3}{2g_1}\left(1 + \Omega\right) - 1}{\left(2\pi\right)^3 2\omega_+ g_3\Omega}}, \\
\langle 0|\phi_2|1\rangle &= e^{i\theta_2}\sqrt{\frac{\frac{g_3}{2g_2}\left(1 + \Omega\right) - 1}{\left(2\pi\right)^3 2\omega_+ g_3\Omega}}.
\end{aligned} \qquad (5.62)$$

Here θ_1 and θ_2 are the two phase factors. By choosing $\theta_1 = 0$ we can determine θ_2 by requiring that

$$\langle 0|\phi_1\phi_2^\dagger|0\rangle = \frac{1}{\left(2\pi\right)^3}\mathrm{Res}_{\omega\to\omega_+}\left(\omega - \omega_+\right)A_{12}^+. \qquad (5.63)$$

This results in that $\theta_2 = \frac{\pi}{2}$. The transition amplitudes $\langle 0|\phi_1|2\rangle$ and $\langle 0|\phi_2|2\rangle$ is obtained by performing a similar calculation where we use A^- instead of A^+.

The final transition amplitude $\langle 0|\phi_3|3\rangle$ can be determined by considering the $(3,3)$ element in the propagator

$$D_{33} = \frac{1}{g_3(\omega - \omega_3)(\omega + \omega_3)} \qquad (5.64)$$

close to the pole $\omega = \omega_3 - i\epsilon$. Specifically we can use the Kallen-Lehmann representation to write

$$\langle 0|\phi_3\phi_3^\dagger|0\rangle = \frac{1}{\left(2\pi\right)^3}\mathrm{Res}_{\omega\to\omega_3}D_{33} = \frac{1}{2\left(2\pi\right)^3 g_3\omega_3}. \qquad (5.65)$$

---

[5]Where $\mathrm{Res}_{x\to x_0}f(x)$ denotes the complex residue of $f(x)$ at the point x_0.

If we now insert the identity $1 = |3\rangle \langle 3|$ between ϕ_3 and $\phi_3^\dagger$ and take the square root we find that $\langle 0|\phi_3|3\rangle = \sqrt{\frac{1}{2(2\pi)^3 \omega_3}}$. To summarise we thus have the following transition amplitudes

$$
\begin{aligned}
\langle 0|\phi_1|1\rangle &= \sqrt{\frac{\frac{g_3}{2g_1}(1+\Omega)-1}{(2\pi)^3 2\omega_+ g_3 \Omega}}, \\
\langle 0|\phi_2|1\rangle &= i\sqrt{\frac{\frac{g_3}{2g_2}(1+\Omega)-1}{(2\pi)^3 2\omega_+ g_3 \Omega}}, \\
\langle 0|\phi_1|2\rangle &= \sqrt{\frac{\frac{g_3}{2g_1}(\Omega-1)+1}{(2\pi)^3 2\omega_- g_3 \Omega}}, \\
\langle 0|\phi_2|2\rangle &= -i\sqrt{\frac{\frac{g_3}{2g_2}(\Omega-1)+1}{(2\pi)^3 2\omega_- g_3 \Omega}}, \\
\langle 0|\phi_3|3\rangle &= \sqrt{\frac{1}{2(2\pi)^3 \omega_3}}.
\end{aligned}
\tag{5.66}
$$

Thus particles with the energies ω_+, ω_- and ω_3 are described by the states

$$
\begin{aligned}
\Psi_+ &= \begin{pmatrix} \langle 0|\phi_1|1\rangle \\ \langle 0|\phi_2|1\rangle \end{pmatrix}, \\
\Psi_- &= \begin{pmatrix} \langle 0|\phi_1|2\rangle \\ \langle 0|\phi_2|2\rangle \end{pmatrix},
\end{aligned}
\tag{5.67}
$$

and

$$
\Psi_3 = \langle 0|\phi_3|3\rangle. \tag{5.68}
$$

respectively. Now we have all the ingredients necessary to compute the scattering amplitudes we are interested in.

5.2.4 Scattering Amplitudes

We want to compute the soft-limit behaviour of the two scattering amplitudes

$$
\Psi_\pm \Psi_3 \to \Psi_\pm \Psi_3. \tag{5.69}
$$

The computation of the amplitudes is now slightly more complicated than before because the propagator in Eq. (5.53) is energy dependent and can therefore not be diagonalised. This results in that the fields ϕ_1 and ϕ_2 mix. To be concrete the Feynman diagrams contributing to the scattering events are shown in Fig. 5.2. Note that each of the Latin indices in the figure can take the values $\{1, 2\}$ giving a total of $4 + 16 + 16 = 36$ Feynman diagrams. To calculate the amplitudes we collect the triple and quadruple vertices in Fig. 5.1 in two matrices V and W respectively. This allows us to calculate the scattering amplitude by computing

$$
i\mathscr{A}_\pm = \Psi_3^\dagger \Psi_\pm^\dagger \{W + VDV + VDV\} \Psi_3 \Psi_\pm \tag{5.70}
$$

where the last term represents the exchange diagram. We have done this calculation in Mathematica. Given the complexity of the analytic expression for the scattering amplitudes it was
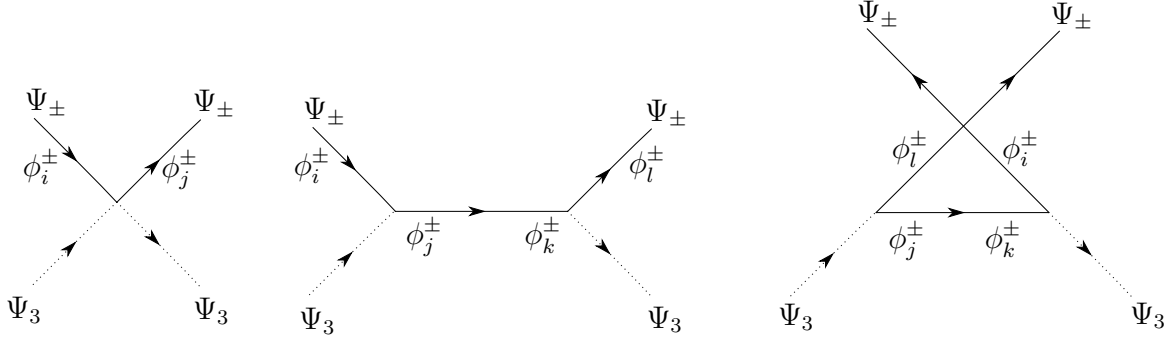
Figure 5.2: The Feynman diagrams contributing to $\Psi_\pm\Psi_3 \to \Psi_\pm\Psi_3$. The Latin indices can take the values $\{1,2\}$. The signs $\pm$ refers to whether or not the field obeys ω_+ or ω_-.

necessary to introduce numerical values for all the parameters of the theory. To study the soft limit of the mNGB and the pNGB we introduced a scale factor z in their 3-momentum and let $z \to 0$. In order to use the script we had to make sure that all of the momenta stayed on shell, when taking the limit. In order to do this we generated the momenta for the two modes $\Psi_\pm$ randomly and then introduced the scale factor. Having done this the last two momenta were determined by requiring momentum conservation. We then, for each value of z, generated a set of on-shell momenta that we used to calculate the amplitude. Our output is the amplitude as a function of the scale factor z. The results are shown in Fig. 5.3 along with the model parameters.
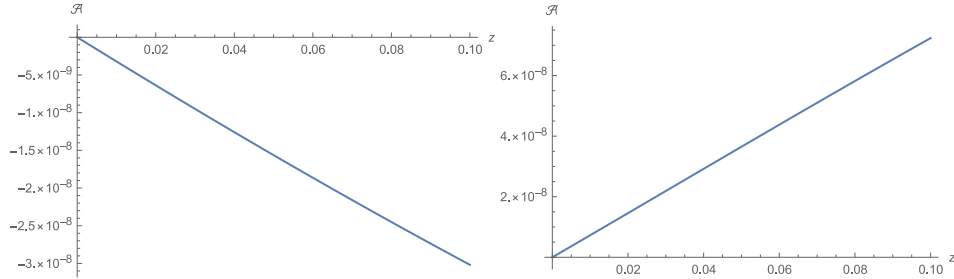
Figure 5.3: The scattering amplitudes as functions of the scale parameter z, with $\mu = 1$ $g_1 = 1$, $g_2 = 2$ and $g_3 = 5$. The left figure corresponds to $\Psi_+ + \Psi_3 \to \Psi_+\Psi_3$. The right figure corresponds to $\Psi_-\Psi_3 \to \Psi_-\Psi_3$.

In the calculation the momenta at $z = 1$ were

$$
\begin{aligned}
k_1 &= (2.6147, -0.741869, 0.0414662, -0.23057), \\
k_2 &= (2.65305, 0.549901, 0.0638119, 0.668636), \\
p_1 &= (32.3189, 19.309, -7.53947, -24.7958), \\
p_2 &= (32.2805, 18.0172, -7.56181, -25.695),
\end{aligned}
\tag{5.71}
$$

for the process $\Psi_-(k_1)\Psi_3(p_1) \to \Psi_-(k_2)\Psi_3(p_2)$ and

$$
\begin{aligned}
p_1 &= (1.30027, 0.847035, 0.573307, -0.224029), \\
p_2 &= (1.26091, 0.657704, -0.333695, 0.630827), \\
k_1 &= (-20.2082, -9.36737, -12.1733, -13.1314), \\
k_2 &= (-20.1688, -9.17804, -11.2663, -13.9863),
\end{aligned}
\tag{5.72}
$$

for the process $\Psi_+(k_1) + \Psi_3(p_1) \to \Psi_+(k_2) + \Psi_3(p_2)$. As expected we found that the scattering amplitude involving the mNGB vanished

$$\mathscr{A}\left(\Psi_-\Psi_3 \to \Psi_-\Psi_3\right) \xrightarrow[z \to 0]{} 0. \tag{5.73}$$

However, we were surprised to see that the scattering amplitude also vanished for the process that involved the pNGB

$$\mathscr{A}\left(\Psi_+\Psi_3 \to \Psi_+\Psi_3\right) \xrightarrow[z \to 0]{} 0. \tag{5.74}$$

This could mean either that there is some hidden symmetry in our model, or that the kinematics of the $2 \to 2$ scattering is too simple to tell the difference between the two modes.

Part III

Proving The Conjectures

6

Weinberg's Soft-Limit Theorem and Covariant Current Conservation

In this chapter we want primarily to reach two goals. Firstly we want to show how Weinberg proves his soft-limit theorem, because it underpins all of the discussions that follow. In short the proof is performed by applying current conservation to a matrix element[1] $\langle f|J^\mu|i\rangle$ which gives us an expression for the corresponding scattering amplitude. In Weinberg's case this is relatively straight forward because the conservation law is simply $\partial_\mu J^\mu = 0$. However, in systems where a symmetry is explicitly broken by a chemical potential the RHS of this conservation law is no longer zero as shown in e.g. [18]. In order to obtain an equation that we can solve for the scattering amplitude we have to look for a new conservation law that takes the form $D_\mu J^\mu = 0$. Hence our second goal is to find such a conservation law. Having justified the conservation law we could in principle begin proving our theorems. However, since it usually is hard to prove something without having looked at specific examples we choose to spend chapters 7 and 8 by revisiting some models that we are already familiar with to see if we can learn how to construct a proof.

6.1   A Collection of the Conjectures

Up to now we have conjectured two soft-limit theorems. Let us now state them explicitly and precisely for future reference:

> **Theorem 2: Weinberg's Soft-Limit Theorem for NGBs**
>
> The on-shell tree-level scattering amplitude corresponding to a process involving at least one massless NGB vanishes in the soft limit of the NGB, unless a kinematic singularity appears.

---

[1]This matrix element describes an initial state $|i\rangle$ interacting with a NGB created/destroyed by the current J^μ to form a final state $|f\rangle$.

> ### Theorem 3: The Soft-Limit Theorem for mNGBs
>
> The on-shell tree-level scattering amplitude corresponding to a process involving at least one mNGB vanishes in the soft limit of the mNGB, unless a kinematic singularity appears.

The first theorem is well known and is proved in [10]. However, the proof will be presented here both for completeness and the fact that the procedure for proving the second theorem follows similar logic.

6.2 Weinberg's Proof - Massless Goldstone Bosons

Proof. In this case we consider the SSB of a Lorentz invariant theory with at least one massless NGB $\pi(x)$, created by a broken current $J^\mu(x)$.[2] From now on we will use the Heisenberg picture, as in [19], where the spacetime dependence of the current can be expressed as $J^\mu(x) = e^{iPx}J^\mu(0)e^{-iPx}$, where P is the 4-momentum operator. The scattering process we will consider is shown in Fig. 6.1, where an initial state $|i\rangle$ and a NGB $\pi(k)$ scatters into a final state $|f\rangle$.
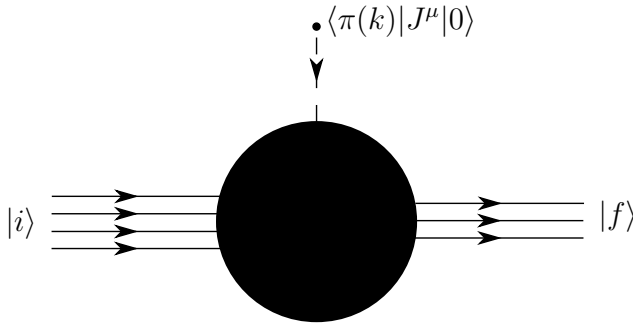
Figure 6.1: The figure represents the pole structure of the matrix element $\langle f|J^\mu(0)|i\rangle$ describing the process $|i\rangle + |\pi(k)\rangle \to |f\rangle$ with corresponding scattering amplitude $\mathscr{A}_{\mathrm{fi}}$. The large black circle represents the scattering amplitude $\mathscr{A}_{\mathrm{fi}}$, the small black dot represents the matrix element $\langle\pi(k)|J^\mu|0\rangle$ responsible for the creation of a massless NGB and the dashed line represents the massless NGB propagator i/k^2. Combining these ingredients we can express the pole structure as $\mathscr{A}_{\mathrm{fi}} \times \langle\pi(k)|J^\mu|0\rangle \times i/k^2$.

Mathematically Fig. (6.1) represents the pole structure of the matrix element

$$\langle f|J^\mu(x)|i\rangle = e^{i(p_f-p_i)x}\langle f|J^\mu(0)|i\rangle = e^{ikx}\langle f|J^\mu(0)|i\rangle, \tag{6.1}$$

where k is the momentum of the NGB.[3] We know that the mass of a particle can always be determined from the pole of the propagator. Since, in this case, the NGB is massless we know that Eq. (6.1) has a pole that looks like i/k^2. In addition, it follows from Lorentz invariance that $\langle\pi(k)|J^\mu(0)|0\rangle = k^\mu F$, where F is a scalar function. The large black circle in Fig. 6.1 represents the scattering amplitude $\mathscr{A}_{\mathrm{fi}}$ corresponding to the process $|i\rangle + |\pi(k)\rangle \to |f\rangle$. Combining

---

[2] By broken current we mean the Noether current corresponding to the spontaneously broken symmetry.

[3] Note that we used momentum conservation, to write $k = p_f - p_i$, where p_f and p_i are the momenta of the states $|f\rangle$ and $|i\rangle$ respectively.

the ingredients as shown in Fig. 6.1 yields the Laurent series

$$\langle f|J^\mu(x)|i\rangle = e^{ikx}\left(N_{\text{fi}}^\mu + \frac{ik^\mu F}{k^2}\mathscr{A}_{\text{fi}}\right),$$ (6.2)

where N_{fi}^μ is the contribution to the matrix element $\langle f|J^\mu|i\rangle$ that does not contain the Goldstone pole.[4] Differentiating both sides of Eq. (6.2) w.r.t. x and using the fact that the current J^μ is a conserved quantity we obtain

$$0 = ik_\mu\left(N_{\text{fi}}^\mu + \frac{ik^\mu F}{k^2}\mathscr{A}_{\text{fi}}\right)e^{ikx},$$ (6.3)

from which it follows that

$$\mathscr{A}_{\text{fi}} = \frac{i}{F}k_\mu N_{\text{fi}}^\mu = \frac{i}{F}\left(k_0 N_{\text{fi}}^0 - \mathbf{k}\cdot\mathbf{N}_{\text{fi}}\right).$$ (6.4)

Eq. (6.4) shows that the scattering amplitude vanishes in the soft limit of the NGB $k^\mu \to 0$ if and only if N_{fi}^μ has no poles. However, if N_{fi}^μ has a pole in the soft limit then the scattering amplitude may not vanish. This is what happened in section 3.3 where we encountered a kinematic singularity. To be more precise, what happened was that $\mathbf{N}_{\text{fi}}$ contained a factor $1/\mathbf{k}$ that cancelled the factor $\mathbf{k}$ in Eq. (6.4) only when the massless NGBs before and after the collision were not collinear. In the collinear case the cancellation did not happen, resulting in an amplitude that vanished in the soft limit. ∎

6.3 The Covariant Conservation Law

In the previous section the proof relies on the fact that the broken current is conserved i.e that $\partial_\mu J^\mu = 0$. In cases with mNGBs the RHS of this conservation law will be nonzero due to the additional term(s) explicitly breaking the symmetry, as mentioned in [18]. To be specific, the would-be conservation law reads $\partial_\mu J^\mu = \delta\mathscr{L}$ where $\delta\mathscr{L}$ is the variation of the Lagrangian under a symmetry transformation. As $\delta\mathscr{L}$ varies from system to system, we are forced to find a different conservation law that takes the same form in all systems. It turns out that we can find such a conservation law by considering a gauge theory, where the gauge field contains the chemical potential. The formulas and conventions we use in this section can be found in any standard textbook on QFT such as [10], [19], [20] and [21].

For simplicity we consider a gauge theory with a Lagrangian that only depends on the fields and their first derivatives $\mathscr{L}(\phi^i, D_\mu\phi^i)$. The i'th component of the covariant derivative is defined as

$$D_\mu\phi^i = \partial_\mu\phi^i - iA_{\mu a}(T_a)^i_j\phi^j$$ (6.5)

where $A_{\mu a} = \mu_a\delta_{\mu 0}$ is the gauge field(s) containing the chemical potential(s) μ_a, and T_a are the generator(s) coupled to the chemical potential(s). In order to derive the conservation law we consider the Lagrangian under the infinitesimal gauge transformations

$$\delta\phi^i = i\theta_a(T_a)^i_j\phi^j,$$
$$\delta A_{\mu a} = \partial_\mu\theta_a + f_{abc}A_{\mu b}\theta_c.$$ (6.6)

---

[4] N_{fi}^μ is called the non-pole contribution to the matrix element $\langle f|J^\mu|i\rangle$. The name can in some cases be misleading given that N_{fi}^μ may still contain poles not related to the Goldstone boson.

By first requiring the variation of the action to be zero we obtain

$$0 = \delta S = \int d^4x \left[\frac{\partial \mathscr{L}}{\partial \phi^i} \delta \phi^i + \frac{\partial \mathscr{L}}{\partial (D_\mu \phi^i)} \delta (D_\mu \phi^i) \right]. \tag{6.7}$$

To proceed we need to find an expression for the variation of the covariant derivative $\delta (D_\mu \phi^i)$. Eq. (6.5) implies that

$$\delta (D_\mu \phi^i) = \partial_\mu (\delta \phi^i) - i(\delta A_{\mu a})(T_a)^i_j \phi^j - i A_{\mu a}(T_a)^i_j (\delta \phi^j), \tag{6.8}$$

and substituting the variations in Eq. (6.6) into Eq. (6.8) yields

$$\delta (D_\mu \phi^i) = i\theta_a (T_a)^i_j \partial_\mu \phi^j - i f_{abc} A_{\mu b} \theta_c (T_a)^i_j \phi^j + A_{\mu a} \theta_b \left[(T_a)^i_j (T_b)^j_k \right] \phi^k. \tag{6.9}$$

Using then the definition of the structure constants

$$\left[T_a, T_b \right] = i f_{abc} T_c \tag{6.10}$$

we can rewrite the term in the square brackets of Eq. (6.9) as

$$\begin{aligned} \left[(T_a)^i_j (T_b)^j_k \right] = (T_a T_b)^i_k &= (T_b T_a)^i_k + i f_{abc} (T_c)^i_k \\ &= (T_b)^i_j (T_a)^j_k + i f_{abc} (T_c)^i_k. \end{aligned} \tag{6.11}$$

If we substitute Eq. (6.11) into Eq. (6.9) and simplify we obtain

$$\delta (D_\mu \phi^i) = i\theta_a (T_a)^i_j \left[\partial_\mu \phi^j - i A_{\mu b}(T_b)^j_k \phi^k \right] = i\theta_a (T_a)^i_j D_\mu \phi^j. \tag{6.12}$$

Now that we have explicit expressions for $\delta \phi^i$ and $\delta (D_\mu \phi^i)$ we can continue working with the variation of the action. Substituting Eqs. (6.6) and (6.12) into Eq. (6.7) gives

$$0 = \delta S = \int d^4x \left[\frac{\partial \mathscr{L}}{\partial \phi^i} \phi^j + \frac{\partial \mathscr{L}}{\partial (D_\mu \phi^i)} D_\mu \phi^j \right] i\theta_a (T_a)^i_j. \tag{6.13}$$

In order to obtain a Noether current, we need to find the analogue of the Euler-Lagrange equations such that we can express $\partial \mathscr{L}/\partial \phi^i$ in terms of the covariant derivative. The equation of motion for ϕ^i is

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial (\partial_\mu \phi^i)} = \frac{\partial \mathscr{L}}{\partial \phi^i} + \frac{\partial \mathscr{L}}{\partial (D_\mu \phi^j)} \frac{\partial (D_\mu \phi^j)}{\partial \phi^i}, \tag{6.14}$$

which can be expressed as

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial (\partial_\mu \phi^i)} = \frac{\partial \mathscr{L}}{\partial \phi^i} - i A_{\mu a}(T_a)^j_i \frac{\partial \mathscr{L}}{\partial (D_\mu \phi^j)} \tag{6.15}$$

by using Eq. (6.5) to calculate $\partial (D_\mu \phi^j)/\partial \phi^i$. From Eq. (6.15) it follows that

$$\frac{\partial \mathscr{L}}{\partial \phi^i} = \partial_\mu \frac{\partial \mathscr{L}}{\partial (\partial_\mu \phi^i)} + i A_{\mu a}(T_a)^j_i \frac{\partial \mathscr{L}}{\partial (D_\mu \phi^j)} \equiv D_\mu \frac{\partial \mathscr{L}}{\partial (D_\mu \phi^i)}, \tag{6.16}$$

where we used

$$\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^i)} = \frac{\partial \mathscr{L}}{\partial(D_\mu \phi^j)} \frac{\partial(D_\mu \phi^j)}{\partial(\partial_\mu \phi^i)} = \frac{\partial \mathscr{L}}{\partial(D_\mu \phi^j)} \delta_i^j = \frac{\partial \mathscr{L}}{\partial(D_\mu \phi^i)}, \tag{6.17}$$

and defined

$$D_\mu x_i \equiv \partial_\mu x_i + i A_{\mu a} (T_a)_i^j x_j. \tag{6.18}$$

Thus we can substitute Eq. (6.16) into Eq. (6.13) to obtain

$$0 = \int d^4 x \left\{ D_\mu \left[\frac{\partial \mathscr{L}}{\partial(D_\mu \phi^i)} \right] \phi^j + \frac{\partial \mathscr{L}}{\partial(D_\mu \phi^i)} D_\mu \phi^j \right\} i\theta_a (T_a)_j^i. \tag{6.19}$$

By then using the product rule we obtain

$$0 = \int d^4 x \, D_\mu \left(\frac{\partial \mathscr{L}}{\partial(D_\mu \phi^i)} \phi^j \right) i\theta_a (T_a)_j^i. \tag{6.20}$$

Since the variations $i\theta_a (T_a)_j^i$ are independent we obtain the conservation law

$$D_\mu J_a^\mu = 0, \tag{6.21}$$

where we identified the Noether current as

$$J_a^\mu = i \frac{\partial \mathscr{L}}{\partial(D_\mu \phi^i)} (T_a)_j^i \phi^j = \frac{\partial \mathscr{L}}{\partial(D_\mu \phi^i)} \delta\phi_a^i. \tag{6.22}$$

Eq. (6.21) is called the covariant conservation law, and will be used extensively in the following discussions. The reason for its usefulness is that the RHS is independent of the variation of the Lagrangian $\delta\mathscr{L}$. As we will see this enables us to write down equations that we can solve for the scattering amplitude in systems where the chemical potential explicly breaks a symmetry, analogues to how we used the conservation law $\partial_\mu J^\mu = 0$ in the previous section.

In order to actually use Eq. (6.21) we should figure out how the covariant derivative acts on a tensor product representation such as the Noether current. The current has the same tensor structure as the quantity $x_a = x_i (T_a)_j^i \phi^j$, and we should therefore calculate $D_\mu x_a$. By first using the product rule

$$D_\mu x_a = [D_\mu x_i](T_a)_j^i \phi^j + x_i (T_a)_j^i [D_\mu \phi^j] \tag{6.23}$$

and then replacing the square brackets by Eqs. (6.5) and (6.18) respectively, we find that

$$D_\mu x_a = \partial_\mu x_a + i A_{\mu b} x_i \left\{ (T_b)_j^i (T_a)_k^j - (T_a)_j^i (T_b)_k^j \right\} \phi^k. \tag{6.24}$$

The curly brackets in Eq. (6.24) is given by

$$(T_b)_j^i (T_a)_k^j - (T_a)_j^i (T_b)_k^j = [T_b, T_a]_k^i = i f_{bac} (T_c)_k^i. \tag{6.25}$$

Substituting Eq. (6.25) into Eq. (6.24) and performing the permutation $b \leftrightarrow a$ yields

$$D_\mu x_a = \partial_\mu x_a + f_{abc} A_{\mu b} x_i (T_c)_k^i \phi^k = \partial_\mu x_a + f_{abc} A_{\mu b} x_c. \tag{6.26}$$

Thus the covariant conservation law takes the explicit form

$$\partial_\mu J_a^\mu + f_{abc} A_{\mu b} J_c^\mu = 0. \tag{6.27}$$

64

Chapter 7

Revisiting the Nonlinear Sigma Model for $SO(3)$

A successful proof of theorem 3 requires that we find a formula for the scattering amplitude which does not rely on explicitly calculating the corresponding Feynman diagrams. In other words we want to obtain a formula similar to that of Eq. (6.4) where we expressed the amplitude only in terms of the non-pole contributions of the matrix element $\langle f|J^\mu|i\rangle$. To find a way of doing this it is useful to first discuss a simple concrete model. In this context simple means a model that contains mNGBs with relativistic propagators, but does not allow for the complications of mixing discussed in Chapter 5. An ideal candidate is of course the nonlinear sigma model from section 3.2 containing a mNGB, a NGB and rotational symmetry. The linear and nonlinear sigma models are both commonly used when discussing symmetry breaking and Goldstone bosons, see e.g. [8], [9], [2], and [16]. However, in this chapter we will use the nonlinear sigma model in quite a different way than what has been done earlier.

7.1 The Model

The Lagrangian is

$$\mathscr{L} = \frac{1}{2}(D_\mu \vec{\chi})^2 \tag{7.1}$$

where the magnitude of the field $\vec{\chi}$ is constrained such that $\vec{\chi} = (\sqrt{v^2 - \pi^2 - G^2}, \pi, G)$ is expressed in terms of the vev v, the NGB π and the mNGB G. The covariant derivative is $D_\mu = \partial_\mu - i\mu T_3 \delta_{\mu 0}$, where the generator is given in Eq. (3.10).

In order to obtain the desired formula for the scattering amplitude we need to calculate matrix elements of the form $\langle f|J_a^\mu|i\rangle$ where J_a^μ is the Noether current corresponding to the generator T_a that excites mNGBs. If we choose the initial state $|i\rangle$ to contain one NGB and the final state $|f\rangle$ to contain both a mNGB and a NGB the matrix element $\langle f|J_a^\mu|i\rangle$ represents the process $\pi(k) + G(p) \to \pi(k') + G(p')$. In Feynman's language $\langle f|J_a^\mu|i\rangle$ can be expressed diagrammatically, as shown in Fig. 7.1. To compute these diagrams it is clear that we need to find Feynman rules involving the Noether currents J^μ. The method for calculating these is actually pretty similar to the method we have used to calculate Feynman rules so far. The reason being that the matrix element $\langle f|J_a^\mu|i\rangle$ basically has the same structure as the S-matrix $\langle f|S|i\rangle$. Hence we can proceed in the usual way, see [20], by expanding the Noether current in terms
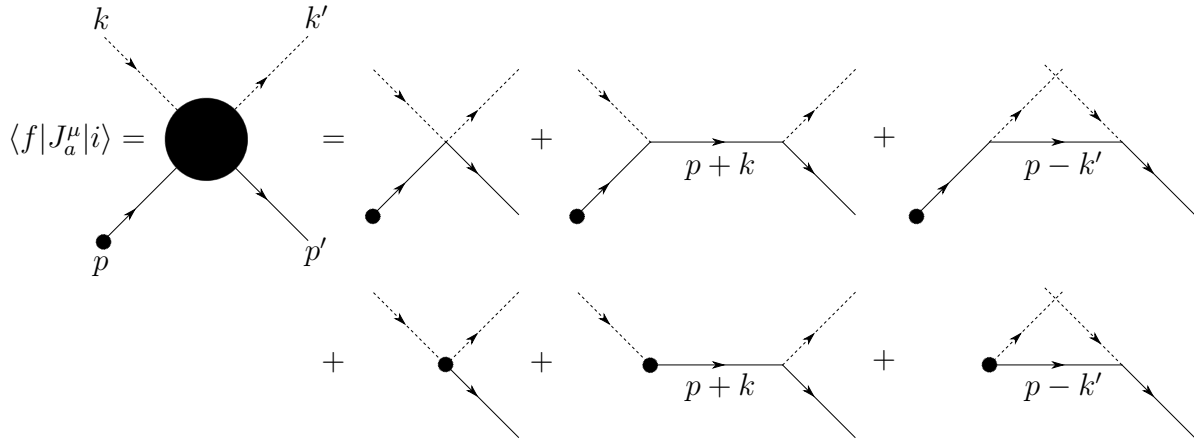
Figure 7.1: The figure shows the Feynman diagrams that represent the matrix element $\langle f|J_a^\mu|i\rangle$ at tree level. The incoming NGB and mNGB and the outgoing NGB and mNGB have momenta k, p, k' and p' respectively. To avoid clutter the external momenta are only labelled in the first diagram. The small black dot represents the current J_a^μ. The diagram describes an initial state containing a NGB interacting with a mNGB created by the Noether current to form a final state consisting of a NGB and a mNGB. The diagrams in the first and second line represent the pole and non-pole contributions respectively.

of the fields[1], wedging the current between the initial and final states that we have expressed in terms of creation and annihilation operators and finally evaluating the Wick contractions to obtain the Feynman rules. The analogy between $\langle f|J_a^\mu|i\rangle$ and $\langle f|S|i\rangle$ tell us that the Feynman rules we desire can be evaluated by treating the Noether current in the same way that we usually treat the Lagrangian. To be concrete each term in the Noether current corresponds to a current-interaction, and the Feynman rule can be evaluated by computing functional derivatives w.r.t. the corresponding fields.

7.2 The Noether Currents

Let us first calculate the Noether currents using Eq. (6.22), which we derived in the previous chapter. The general expression is[2]

$$J_a^\mu = -\frac{\partial \mathscr{L}}{\partial (D_\mu \chi^i)} i(T_a)_k^i \chi^k. \tag{7.2}$$

A useful representation for the generators is the adjoint representation in which $(T_a)_k^i = i f_{ak}^i$,[3] see [19]. Since we are dealing with the group $SO(3)$ the structure constants are given in terms of the Levi-Civita tensor $f_{ak}^i = \epsilon_{ak}^i$. Substituting the Lagrangian in Eq. (7.1) into Eq. (7.2) yields

$$J_a^\mu = (D^\mu \vec{\chi})_i \epsilon_{ak}^i \chi^k. \tag{7.3}$$

---

[1]This step is analogues to Dyson's expansion, where the S-matrix is written as a perturbation series in terms of the fields.

[2]Note that we chose to add a minus sign, without any physical consequences.

[3]There is no distinction between upper and lower indices in the structure constants in this thesis.

Thus the first component of J_a^μ is

$$
\begin{aligned}
J_1^\mu &= (D^\mu\vec{\chi})_3\chi_2 - (D^\mu\vec{\chi})_2\chi_3 \\
&= (\partial^\mu\chi_3)\chi_2 - (\partial^\mu\chi_2 + \mu\delta^{\mu 0}\chi_1)\chi_3 \\
&= \pi\partial^\mu G - G\partial^\mu\pi - \mu\delta^{\mu 0}G\sqrt{v^2 - \pi^2 - G^2}.
\end{aligned}
\tag{7.4}
$$

The second and third components of J_a^μ can be calculated in exactly the same way, giving

$$
\begin{aligned}
J_2^\mu &= -\delta^{\mu 0}\mu\pi G - \frac{(v^2 - \pi^2)\partial^\mu G + \pi G\partial^\mu\pi}{\sqrt{v^2 - \pi^2 - G^2}}, \\
J_3^\mu &= \delta^{\mu 0}\mu(v^2 - G^2) + \frac{\pi G\partial^\mu G + (v^2 - G^2)\partial^\mu\pi}{\sqrt{v^2 - \pi^2 - G^2}}.
\end{aligned}
\tag{7.5}
$$

Before we proceed, let us have a look at Fig. 7.1. From the second line we see that we are going to need Feynman rules both bilinear and trilinear in the fields. As discussed the correct method to obtain the Feynman rules is to calculate variational derivatives of the Noether currents. However, in their present forms this will not help us much because of the square roots. In order to obtain all of the necessary Feynman rules we are forced to expand the Noether currents up to third order in the fields. We expand the square roots as

$$
\begin{aligned}
\sqrt{v^2 - \pi^2 - G^2} &= v\sqrt{1 - \frac{\pi^2 + G^2}{v^2}} = v\left[1 - \frac{\pi^2 + G^2}{2v^2} + \mathcal{O}(\chi_i^4)\right], \\
\frac{1}{\sqrt{v^2 - \pi^2 - G^2}} &= \frac{1}{v\sqrt{1 - \frac{\pi^2 + G^2}{v^2}}} = \frac{1}{v}\left[1 + \frac{\pi^2 + G^2}{2v^2} + \mathcal{O}(\chi_i^4)\right].
\end{aligned}
\tag{7.6}
$$

If we substitute this into the Noether currents we obtain the following three expansions

$$
\begin{aligned}
J_1^\mu &= \pi\partial^\mu G - G\partial^\mu\pi + \delta^{\mu 0}\left(\mu G\frac{\pi^2 + G^2}{2v} - \mu v G\right) + \mathcal{O}(\chi_i^4), \\
J_2^\mu &= -v\partial^\mu G - \delta^{\mu 0}\mu\pi G - \frac{1}{2v}\partial^\mu G(G^2 - \pi^2) - \frac{1}{v}\pi G\partial^\mu\pi + \mathcal{O}(\chi_i^4), \\
J_3^\mu &= v\partial^\mu\pi + \delta^{\mu 0}\mu(v^2 - G^2) + \frac{1}{v}\pi G\partial^\mu G - \frac{1}{2v}\partial^\mu\pi(G^2 - \pi^2) + \mathcal{O}(\chi_i^4).
\end{aligned}
\tag{7.7}
$$

Note that the mNGB is only present linearly in J_1^μ and J_2^μ, meaning that only these currents excite mNGBs. It follows that $\langle f|J_{1,2}^\mu|i\rangle$ will contain both pole and non-pole contributions while $\langle f|J_3^\mu|i\rangle$ will only contain non-pole contributions, see Fig. 7.1. Since the scattering amplitude is connected to the pole pieces, it is only necessary to calculate $\langle f|J_{1,2}^\mu|i\rangle$.

7.3 The Matrix Element $\langle f|J_2^\mu|i\rangle$

7.3.1 The Feynman Current Rules from J_2^μ

In order to compute $\langle f|J_2^\mu|i\rangle$ we see from Fig. 7.1 that we only need the following terms from J_2^μ:

$$
-v\partial^\mu G - \delta^{\mu 0}\mu\pi G - \left(\frac{1}{v}\pi G\partial^\mu\pi - \frac{1}{2v}\pi^2\partial^\mu G\right).
\tag{7.8}
$$

Of these only the third is nontrivial, so let us demonstrate how to calculate the corresponding Feynman current rule. By performing a Fourier transformation we obtain[4]

$$\left(\frac{1}{2v}\pi^2\partial^\mu G - \frac{1}{v}\pi G\partial^\mu\pi\right) \rightarrow \int \frac{i}{v}\pi(p_1)G(p_3)\pi(p_2)p_2 - \frac{i}{2v}\pi(p_1)\pi(p_2)G(p_3)p_3. \tag{7.9}$$

The Feynman current rule $V^\mu(k, k', p)$ is then obtained by calculating the three variational derivatives

$$V^\mu(k, k', p) = \int \frac{\delta^3}{\delta\pi(k)\delta\pi(k')\delta G(p)}\left(\frac{i}{v}\pi(p_1)G(p_3)\pi(p_2)p_2 - \frac{i}{2v}\pi(p_1)\pi(p_2)G(p_3)p_3\right)$$

$$= \frac{i}{v}(k + k' - p)^\mu. \tag{7.10}$$

All of the Feynman current rules, that we are going to need, associated with J_2^μ are shown in Fig. 7.2.

Figure 7.2: The figure shows the Feynman current rules that will be used to evaluate the matrix element $\langle f|J_2^\mu|i\rangle$. The solid and dashed line represents G and π respectively.

7.3.2 Calculating the Matrix Element

Now we can calculate the matrix element $\langle f|J_2^\mu|i\rangle$ shown in Fig. 7.1 when $a = 2$, by using the Feynman rules in Figs. 3.1 and 7.2. This gives

$$\langle f|J_2^\mu|i\rangle = -ivp^\mu\frac{i}{p^2 - \mu^2}(-i\mathscr{A}_{\text{off}}) + N_2^\mu \tag{7.11}$$

where $\mathscr{A}_{\text{off}}$ is the off-shell[5] scattering amplitude calculated in Eq. (3.39) and N_2^μ are the non-pole contributions

$$N_2^\mu = \frac{i}{v}(p' + k - k')^\mu + \delta^{\mu 0}\frac{2\mu^2}{v}\left\{\frac{ik_0}{(p - k')^2 - \mu^2} - \frac{ik_0'}{(p + k)^2 - \mu^2}\right\}. \tag{7.12}$$

---

[4]In this case we use the same convention as in App. A.3.1, where we suppress all differentials and exponentials as shown in Eq. (A.17).

[5]In this case "off-shell" refers to the fact, that none of the particles in the collision $\pi(k)+G(p) \rightarrow \pi(k')+G(p')$ are on shell yet.

7.4 The Matrix Element $\langle f | J_1^\mu | i \rangle$

In this section we will calculate $\langle f | J_1^\mu | i \rangle$. As before we need to first determine the Feynman current rules from the corresponding Noether current which in this case is J_1^μ. The rules relevant for our calculation are shown in Fig. 7.3
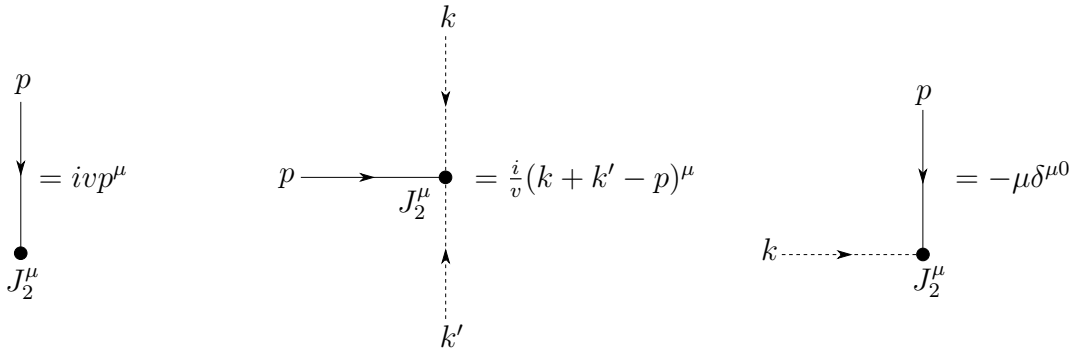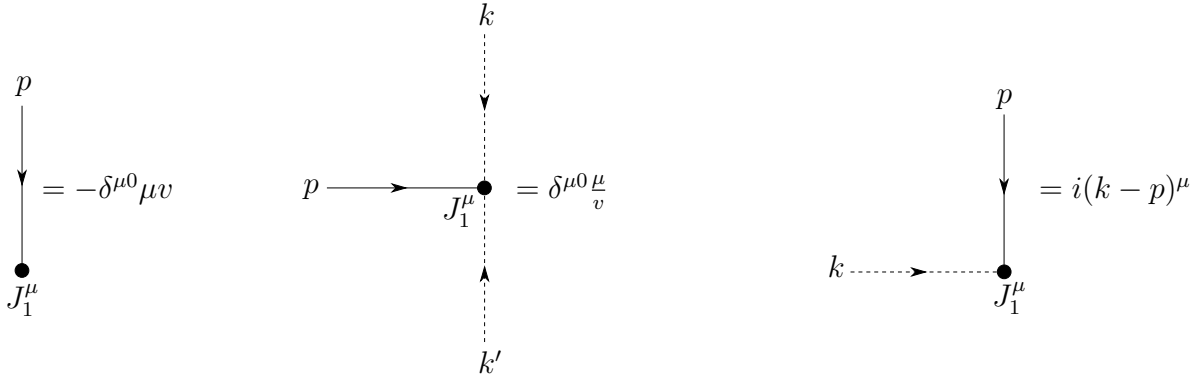

Figure 7.3: The figure shows the Feynman rules that will be used to evaluate the matrix element $\langle f | J_1^\mu | i \rangle$.

and were calculated from the terms:

$$\pi \partial^\mu G - G \partial^\mu \pi + \delta^{\mu 0} \left(\frac{\mu}{2v} G \pi^2 - \mu v G \right), \tag{7.13}$$

present in J_1^μ.

The Feynman diagrams contributing to $\langle f | J_1^\mu | i \rangle$ are displayed in Fig. 7.1 when $a = 1$. By using the Feynman rules in Fig. 7.3 we obtain

$$\langle f | J_1^\mu | i \rangle = -\delta^{\mu 0} \mu v \frac{i}{p^2 - \mu^2} (-i \mathscr{A}_{\text{off}}) + N_1^\mu, \tag{7.14}$$

where the non-pole contributions are

$$N_1^\mu = \delta^{\mu 0} \frac{\mu}{v} + \frac{2\mu}{v} \left\{ \frac{(p - 2k')^\mu k_0}{(p - k')^2 - \mu^2} - \frac{(2k + p)^\mu k_0'}{(p + k)^2 - \mu^2} \right\}. \tag{7.15}$$

7.5 Current Conservation

Note that the off-shell scattering amplitude $-i\mathscr{A}_{\text{off}}$ is present in both matrix elements $\langle f | J_{1,2}^\mu | i \rangle$. If we had an equation involving both matrix elements we would be able to solve for the amplitude and hence determine the amplitude in terms of the non-pole contributions. The obvious candidate for such an equation is the covariant conservation law that we discussed in section 6.3, where we showed that the Noether currents satisfies the equation $D_\mu J_a^\mu = 0$. If we use the matrix representation for the covariant derivative $D_\mu = \partial_\mu - i\mu T_3 \delta_{\mu 0}$ and collect the Noether currents into a vector $\vec{J}^\mu = (J_1^\mu, J_2^\mu, J_3^\mu)^\mathsf{T}$ we obtain

$$0 = D_\mu \vec{J}^\mu = \begin{pmatrix} \partial_\mu J_1^\mu - \delta_{\mu 0} \mu J_2^\mu \\ \delta_{\mu 0} \mu J_1^\mu + \partial_\mu J_2^\mu \\ \partial_\mu J_3^\mu \end{pmatrix}, \tag{7.16}$$

which is satisfied when the Noether currents satisfy the Euler-Lagrange equations. Eq. (7.16) implies that we can write

$$0 = \mu \delta_{\mu 0} \langle f | J_1^\mu | i \rangle + \partial_\mu \langle f | J_2^\mu | i \rangle = \mu \delta_{\mu 0} \langle f | J_1^\mu | i \rangle + i p_\mu \langle f | J_2^\mu | i \rangle, \qquad (7.17)$$

where p is the momentum of the mNGB, created by J_2^μ. Let us check whether or not this equation is satisfied when the momenta k, k', p and p' are all off shell. By substituting Eqs. (7.11) and (7.14) into the RHS of Eq. (7.17) we obtain

$$
\begin{aligned}
i p_\mu \langle f | J_2^\mu | i \rangle + \mu \delta_{\mu 0} \langle f | J_1^\mu | i \rangle = i p_\mu \Bigg\{ & \frac{v p^\mu}{p^2 - \mu^2} (-i \mathscr{A}_{\text{off}}) + \frac{i}{v} (k - k' + p')^\mu \\
& - \frac{2\mu^2 \delta^{\mu 0} k_0'}{v} \frac{i}{(k+p)^2 - \mu^2} + \frac{2\mu^2 \delta^{\mu 0} k_0}{v} \frac{i}{(p'-k)^2 - \mu^2} \Bigg\} \\
+ \mu \delta_{\mu 0} \Bigg\{ & - \frac{\mu v \delta^{\mu 0} i}{p^2 - \mu^2} (-i \mathscr{A}_{\text{off}}) + \frac{\mu \delta^{\mu 0}}{v} - \frac{2\mu k_0'}{v} \frac{(2k + p)^\mu}{(p+k)^2 - \mu^2} \\
& + \frac{2\mu k_0}{v} \frac{(p - 2k')^\mu}{(p'-k')^2 - \mu^2} \Bigg\},
\end{aligned}
$$
$$(7.18)$$

where the crosses indicate cancellations. If we substitute Eq. (3.39) for the off-shell amplitude into the expression above and use momentum conservation we obtain

$$i p_\mu \langle f | J_2^\mu | i \rangle + \mu \delta_{\mu 0} \langle f | J_1^\mu | i \rangle = \frac{1}{v} (\mu^2 - p'^2) \qquad (7.19)$$

which only vanishes if we use the on-shell condition $p'^2 = \mu^2$. Thus the only requirement necessary for Eq. (7.17) to be satisfied is that p' is on shell. However, note that the other momenta k, k' and p need not be on shell for the conservation law to hold.

By performing a very similar calculation we can also show that $\partial_\mu \langle f | J_1^\mu | i \rangle - \delta_{\mu 0} \mu \langle f | J_2^\mu | i \rangle = 0$ is satisfied only when p', k and k' are all on shell.

Thus it is clear that both $\partial_\mu \langle f | J_2^\mu | i \rangle + \mu \delta_{\mu 0} \langle f | J_1^\mu | i \rangle = 0$ and $\partial_\mu \langle f | J_1^\mu | i \rangle - \delta_{\mu 0} \mu \langle f | J_2^\mu | i \rangle = 0$ are satisfied simultaneously if and only if p', k and k' are on shell. However, it is not yet necessary to impose that p should be on shell as well.

7.6 The Scattering Amplitude

Having verified that

$$i p_\mu \langle f | J_2^\mu | i \rangle + \mu \delta_{\mu 0} \langle f | J_1^\mu | i \rangle = 0 \qquad (7.20)$$

is satisfied when p' is on shell we can finally construct a formula for the scattering amplitude. If we once again substitute Eqs. (7.11) and (7.14) into Eq. (7.20) we get

$$
\begin{aligned}
0 = ip_\mu \langle f|J_2^\mu|i\rangle + \mu\delta_{\mu 0} \langle f|J_1^\mu|i\rangle &= ip_\mu \left\{ \frac{vp^\mu}{p^2 - \mu^2}(-i\mathscr{A}_{\text{off}}) + N_2^\mu \right\} \\
&\quad + \mu\delta_{\mu 0} \left\{ \frac{i\mu v\delta^{\mu 0}}{p^2 - \mu^2}(i\mathscr{A}_{\text{off}}) + N_1^\mu \right\} \\
&= (-i\mathscr{A}_{\text{off}}) \left\{ \frac{iv(p^2 - \mu^2)}{p^2 - \mu^2} \right\} + ip_\mu N_2^\mu + \mu N_1^0 \\
&= iv(-i\mathscr{A}_{\text{off}}) + ip_\mu N_2^\mu + \mu N_1^0.
\end{aligned}
\tag{7.21}
$$

From Eq. (7.21) it follows that the scattering amplitude is given by

$$
-i\mathscr{A}_{\text{off}} = \frac{i}{v}(ip_\mu N_2^\mu + \mu N_1^0),
\tag{7.22}
$$

where p, k and k' are still off shell. This is exactly the kind of formula that we have been looking for, where we note that the scattering amplitude can be expressed in terms of the non-pole contributions of the matrix elements $\langle f|J_{1,2}^\mu|i\rangle$.

7.6.1 Soft Limit

Our conjecture on page 60 states that the scattering amplitude in Eq. (7.22) should vanish when we let p, k, p' and k' be on shell and send the three momentum $\mathbf{p} \to 0$. However, at first glance this does not appear to be the case. Our next step is therefore to determine how the vanishing of the on-shell scattering amplitude in the soft limit follows from the conservation laws. That is, we want to show that $-i\mathscr{A}_{\text{on}} = \frac{i}{v}(ip_\mu N_2^\mu + \mu N_1^0)$ vanishes in the soft limit of the on-shell mNGB $p_\mu \to \mu\delta_{\mu 0}$, when also k, k' and p' are on shell.

In deriving Eq. (7.22) we only used the conservation law $\partial_\mu \langle f|J_2^\mu|i\rangle + \mu\delta_{\mu 0} \langle f|J_1^\mu|i\rangle = 0$. Thus we still have one more conservation law at our disposal

$$
\partial_\mu \langle f|J_1^\mu|i\rangle - \mu\delta_{\mu 0} \langle f|J_2^\mu|i\rangle = 0
\tag{7.23}
$$

which is satisfied if p', k and k' are all on shell. First consider the LHS and replace $\langle f|J_2^\mu|i\rangle$ and $\langle f|J_1^\mu|i\rangle$ by Eqs. (7.11) and (7.14) respectively, as shown below

$$
\begin{aligned}
\partial_\mu \langle f|J_1^\mu|i\rangle - \mu\delta_{\mu 0} \langle f|J_2^\mu|i\rangle &= ip_\mu \langle f|J_1^\mu|i\rangle - \mu\delta_{\mu 0} \langle f|J_2^\mu|i\rangle \\
&= ip_\mu \left\{ \frac{i\mu v\delta^{\mu 0}}{p^2 - \mu^2}(i\mathscr{A}_{\text{off}}) + N_1^\mu \right\} \\
&\quad - \mu\delta_{\mu 0} \left\{ \frac{vp^\mu}{p^2 - \mu^2}(-i\mathscr{A}_{\text{off}}) + N_2^\mu \right\} \\
&= (-i\mathscr{A}_{\text{off}}) \left\{ \frac{\mu v p_0 - \mu v p_0}{p^2 - \mu^2} \right\} + ip_\mu N_1^\mu - \mu N_2^0.
\end{aligned}
\tag{7.24}
$$

Note that the term in the curly brackets vanishes exactly, without the need to impose any on-shell condition on the momenta p, k, p' and k'. If we now use the on-shell conditions $p'^2 = \mu^2$, $k^2 = 0$ and $k'^2 = 0$ we can set the RHS of Eq. (7.24) equal to zero and obtain

$$0 = ip_\mu N_1^\mu - \mu N_2^0 \implies ip_\mu N_1^\mu = \mu N_2^0. \tag{7.25}$$

If we in addition also let the mNGB be on shell and send its 3-momentum to zero such that $p_\mu \to \mu \delta_{\mu 0}$, Eq. (7.25) implies that[6]

$$\mu N_{1,\text{soft}}^0 = -i\mu N_{2,\text{soft}}^0. \tag{7.26}$$

If we now take the on shell soft limit of the mNGB in Eq. (7.22) we obtain

$$
\begin{aligned}
-i\mathscr{A}_{\text{on}} = \frac{i}{v}(ip_\mu N_2^\mu + \mu N_1^0) &\to \frac{i}{v}(i\mu N_{2,\text{soft}}^\mu + \mu N_{1,\text{soft}}^0) \\
&= \frac{i}{v}(i\mu N_{2,\text{soft}}^\mu - i\mu N_{2,\text{soft}}^\mu) = 0.
\end{aligned}
\tag{7.27}
$$

Thus we have demonstrated that by using one of the available conservation laws we can construct a formula for the off-shell scattering amplitude $-i\mathscr{A}_{\text{off}}$, where only p' is on shell. By also using the second available conservation law we managed to show that if p, p', k and k' are all on shell the corresponding on-shell scattering amplitude $-i\mathscr{A}_{\text{on}}$ vanishes in the soft limit of the mNGB. In principle the method discussed in this chapter can be applied to all other symmetry breaking patterns. However, before we attempt such a generalisation it is useful to see how the argument goes for a theory where the mNGB propagator is non-relativistic, such that $p^2 \neq (\text{mass})^2$.

---

[6]The subscripts "soft" indicate that the non-pole contributions are evaluated in the on shell soft limit $p_\mu \to \mu \delta_{\mu 0}$ of the mNGB.

Chapter 8

Revisiting the Higgs-Like Model

In chapter 4 we studied a Higgs-like model with the symmetry breaking pattern

$$SU(2)_L \times SU(2)_R \xrightarrow{\mu} SU(2)_L \times U(1)_R \to U(1)' \tag{8.1}$$

and found that the particle spectrum consists of a Higgs mode, a NGB of type-I, a NGB of type-II and a mNGB with mass 2μ. In this chapter we want to study this model in the same way that we studied the $SO(3)$ model in the previous chapter. That is, our goal is to find a formula for the scattering amplitude $\mathscr{A}_{\text{off}}(\text{mNGB} + \text{NGB} \to \text{mNGB} + \text{NGB})$ expressed in terms of the non-pole contributions of some current element. From this formula we should be able to show that the scattering amplitude vanishes in the soft limit of the mNGB if we require all particles to be on shell. To find such a formula we will have to determine the following

- The Feynman rules,

- The scattering amplitude,

- The Noether currents,

- The Feynman current rules,

- The matrix current element.

The Higgs-like model is one of the standard examples when discussing both massless and massive Goldstone bosons. It is mentioned or discussed in a lot of papers such as [1], [2], [7], [9], [17], [18] and [22]. In this section we use conventions that are similar to the ones found in [9], but in this thesis we focus on the mNGB instead of the massless NGBs.

8.1   The Lagrangian

It might be tempting to start from the Lagrangian in Eq. (4.1). However, this turns out to be inconvenient because we then need to figure out how the fields transform under the symmetry transformations $SU(2)_L \times U(1)_R$. Instead we will use a Lagrangian that is completely equivalent, although it may not look that way at first. The Lagrangian we will consider is given by

$$\mathscr{L} = \frac{1}{2} \text{Tr} \left(D_\mu \Phi^\dagger D^\mu \Phi \right) - \frac{m^2}{2} \text{Tr} \left(\Phi^\dagger \Phi \right) - \frac{\lambda}{4} \left[\text{Tr} \left(\Phi^\dagger \Phi \right) \right]^2. \tag{8.2}$$

Here $\Phi = (\phi^c | \phi)$ is a matrix consisting of a complex doublet ϕ and its charge conjugated field $\phi^c = i\tau_2\phi^*$, where $\vec{\tau}$ is the Pauli matrices and $*$ denotes complex conjugation. The field transforms as $\Phi \to U_L \Phi U_R^\dagger$ under an $SU(2)_L \times SU(2)_R$ transformation, making the $SU(2)_L \times SU(2)_R$ invariance of Eq. (8.2) obvious. The covariant derivative is defined as

$$D_\mu \Phi = \partial_\mu \Phi - iL_\mu \Phi + i\Phi R_\mu, \tag{8.3}$$

where $L_\mu = \vec{L}_\mu \cdot \vec{\tau}$ and $R_\mu = \vec{R}_\mu \cdot \vec{\tau}$ are the generators of $SU(2)_L$ and $SU(2)_R$ respectively. The symmetry breaking pattern in Eq. (8.1) tells us that the mNGBs come from the explicit breaking of $SU(2)_R$ only. This means that if we couple the chemical potential to R_3^μ the generators R_1^μ and R_2^μ break explicitly. Thus since we are interested only in the interactions involving mNGBs we only care about the currents $J_{R_1}^\mu$ and $J_{R_2}^\mu$ meaning that we can set $L_\mu = 0$ in all of our calculations. However, before we begin calculating the Noether currents it is useful to expand $D_\mu \Phi^\dagger D^\mu \Phi$ as

$$D_\mu \Phi^\dagger D^\mu \Phi = \partial_\mu \Phi^\dagger \partial^\mu \Phi + i\partial_\mu \Phi^\dagger \Phi R^\mu - iR_\mu \Phi^\dagger \partial^\mu \Phi + R_\mu \Phi^\dagger \Phi R^\mu \tag{8.4}$$

and substitute this into the Lagrangian to obtain

$$\mathscr{L} = \frac{1}{2} \mathrm{Tr} \left(\partial_\mu \Phi^\dagger \partial^\mu \Phi + i\partial_\mu \Phi^\dagger \Phi R_a^\mu \tau_a - iR_{\mu a}\tau_a \Phi^\dagger \partial^\mu \Phi + R_{\mu a}\tau_a \Phi^\dagger \Phi R_b^\mu \tau_b \right)$$
$$- \frac{m^2}{2} \mathrm{Tr} \left(\Phi^\dagger \Phi \right) - \frac{\lambda}{4} \left[\mathrm{Tr} \left(\Phi^\dagger \Phi \right) \right]^2. \tag{8.5}$$

8.2 The Noether Currents

Having introduced the gauge field $R^\mu = R_c^\mu \tau_c$ it is natural to calculate the Noether currents as $J_{R_c}^\mu = \delta \mathscr{L} / \delta R_c^\mu$.[1] This gives

$$J_{R_c}^\mu = \frac{1}{2} \mathrm{Tr} \left(i\partial^\mu \Phi^\dagger \Phi \tau_c - i\tau_c \Phi^\dagger \partial^\mu \Phi + 2R_c^\mu \Phi^\dagger \Phi \right) \tag{8.6}$$

where we used that $\delta R_a^\mu / \delta R_c^\nu = \delta^{\mu\nu}\delta_{ac}$ and $\{\tau_a, \tau_b\} = 2\delta_{ab}$. Hence, $J_{R_1}^\mu$ is given by[2]

$$J_{R_1}^\mu = \frac{1}{2} \mathrm{Tr} \left(i\partial^\mu \Phi^\dagger \Phi \tau_1 - i\Phi^\dagger \partial^\mu \Phi \tau_1 \right). \tag{8.7}$$

Let us first evaluate $\partial_\mu \Phi^\dagger \Phi \tau_1$ by first performing the matrix multiplication

$$\partial_\mu \Phi^\dagger \Phi \tau_1 = \begin{pmatrix} (\partial_\mu \phi^c)^\dagger \\ (\partial_\mu \phi)^\dagger \end{pmatrix} \begin{pmatrix} \phi^c \mid \phi \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} (\partial_\mu \phi^c)^\dagger \phi & \cdots \\ \cdots & (\partial_\mu \phi)^\dagger \phi^c \end{pmatrix}. \tag{8.8}$$

---

[1] This expression comes from the fact that when we gauge the theory, the replacement $\partial_\mu \to D_\mu$ is equivalent to adding a term $A_\mu J^\mu$ to the Lagrangian. Thus the path integral takes the form

$$Z[J^\mu] = \int \mathscr{D}\phi \exp \left\{ i \int d^4x \, \mathscr{L} + A_\mu J^\mu \right\}$$

and we can obtain the classical Noether current by calculating the variational derivative w.r.t. the gauge field A_μ.

[2] We ignored the term $2R_1^\mu \Phi^\dagger \Phi$ because in the end we will couple the chemical potential to τ_3 only, such that $R_1^\mu = R_2^\mu = 0$.

If we then rewrite the diagonal elements as

$$(\partial_\mu \phi^c)^\dagger \phi = (\partial_\mu i\tau_2 \phi^*)^\dagger \phi = -i(\partial_\mu \phi)^\mathsf{T} \tau_2 \phi,$$
$$(\partial_\mu \phi)^\dagger \phi^c = (\partial_\mu \phi)^\dagger i\tau_2 \phi^* = i(\partial_\mu \phi)^\dagger \tau_2 \phi^*,$$

(8.9)

we obtain

$$\partial_\mu \Phi^\dagger \Phi \tau_1 = \begin{pmatrix} -i(\partial_\mu \phi)^\mathsf{T} \tau_2 \phi & \cdots \\ \cdots & i(\partial_\mu \phi)^\dagger \tau_2 \phi^* \end{pmatrix}.$$

(8.10)

A similar calculation yields

$$\Phi^\dagger \partial^\mu \Phi \tau_1 = \begin{pmatrix} -i\phi^\mathsf{T} \tau_2 \partial^\mu \phi & \cdots \\ \cdots & i\phi^\dagger \tau_2 (\partial^\mu \phi)^* \end{pmatrix}.$$

(8.11)

By substituting Eqs. (8.10) and (8.11) into Eq. (8.7) and evaluating the trace we obtain

$$J_{R_1}^\mu = -\phi^\mathsf{T} \tau_2 \partial^\mu \phi + \phi^\dagger \tau_2 (\partial^\mu \phi)^*.$$

(8.12)

Since the calculation for $J_{R_2}^\mu$ is very similar we omit the calculation and state the result

$$J_{R_2}^\mu = -i\phi^\mathsf{T} \tau_2 \partial^\mu \phi - i\phi^\dagger \tau_2 (\partial^\mu \phi)^*.$$

(8.13)

Before we can proceed, by parametrising the field ϕ, we need to check that our model has a vev.

8.3   Expanding the Lagrangian and Determining the Vev

Recall that we decided to couple the chemical potential to the third Pauli matrix, meaning that $R_\mu = \mu \delta_{\mu 0} \tau_3$. Substituting this into Eq. (8.5) gives

$$\mathscr{L} = \frac{1}{2} \operatorname{Tr} \left(\partial_\mu \Phi^\dagger \partial^\mu \Phi + i\mu \dot{\Phi}^\dagger \Phi \tau_3 - i\mu \Phi^\dagger \dot{\Phi} \tau_3 + (\mu^2 - m^2)\Phi^\dagger \Phi \right) - \frac{\lambda}{4} \left[\operatorname{Tr} \left(\Phi^\dagger \Phi \right) \right]^2. \quad (8.14)$$

By denoting the two components of ϕ as ϕ_1 and ϕ_2 it is straightforward to evaluate all of the quantities in the Lagrangian, this results in

$$\mathscr{L} = \partial_\mu \phi_1^* \partial^\mu \phi_1 + \partial_\mu \phi_2^* \partial^\mu \phi_2 + i\mu(\dot{\phi}_1 \phi_1^* - \dot{\phi}_1^* \phi_1) + i\mu(\dot{\phi}_2 \phi_2^* - \dot{\phi}_2^* \phi_2)$$
$$+ (\mu^2 - m^2)(\phi_1^* \phi_1 + \phi_2^* \phi_2) - \lambda(\phi_1^* \phi_1 + \phi_2^* \phi_2)^2.$$

(8.15)

This is the same expression as in Eq. (4.3). Thus we have shown that the two Lagrangians in Eqs. (4.3) and (8.2) indeed are equivalent. As in chapter 4 we find that the vev is

$$v^2 = \frac{\mu^2 - m^2}{\lambda}$$

(8.16)

and parametrise the field like

$$\phi = \begin{pmatrix} \psi \\ \frac{1}{\sqrt{2}}(v + \psi_3 + i\psi_4) \end{pmatrix}.$$

(8.17)

By substituting Eq. (8.17) into Eq. (8.15) and simplifying we find that

$$
\begin{aligned}
\mathscr{L} = {} & \partial_\mu \psi^* \partial^\mu \psi + i\mu(\psi^* \dot{\psi} - \psi \dot{\psi}^*) \\
& + \frac{1}{2}(\partial_\mu \psi_3)^2 + \frac{1}{2}(\partial_\mu \psi_4)^2 - \mu(\psi_3 \dot{\psi}_4 - \psi_4 \dot{\psi}_3) - \lambda v^2 \psi_3^2 \\
& - \lambda(\psi^* \psi)^2 - 2\lambda v(\psi^* \psi)\psi_3 - \lambda \psi^* \psi(\psi_3^2 + \psi_4^2) - \lambda\left(\frac{\psi_3^4}{4} + \frac{\psi_3^2 \psi_4^2}{2} + \frac{\psi_4^4}{4}\right) \\
& - \lambda v \psi_3^3 - \lambda v \psi_3 \psi_4^2,
\end{aligned}
\tag{8.18}
$$

where we dropped the constant term.

8.4 Dispersion Relations, Propagators and Feynman Rules

Before we can calculate scattering amplitudes involving the mNGB we need to analyse the particle spectrum more closely than what we did in chapter 4. To this end we rewrite Eq. (8.18) in matrix form,

$$
\begin{aligned}
\mathscr{L} = {} & \frac{1}{2} \begin{pmatrix} \psi^* & \psi \end{pmatrix} \begin{pmatrix} -\partial_\mu \partial^\mu + 2i\mu\partial_0 & 0 \\ 0 & -\partial_\mu \partial_\mu - 2i\mu\partial_0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi^* \end{pmatrix} \\
& + \frac{1}{2} \begin{pmatrix} \psi_3 & \psi_4 \end{pmatrix} \begin{pmatrix} -\partial_\mu \partial^\mu - 2\lambda v^2 & -2\mu\partial_0 \\ 2\mu\partial_0 & -\partial_\mu \partial_\mu \end{pmatrix} \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} \\
& - \lambda(\psi^* \psi)^2 - 2\lambda v(\psi^* \psi)\psi_3 - \lambda \psi^* \psi(\psi_3^2 + \psi_4^2) - \lambda\left(\frac{\psi_3^4}{4} + \frac{\psi_3^2 \psi_4^2}{2} + \frac{\psi_4^4}{4}\right) \\
& - \lambda v \psi_3^3 - \lambda v \psi_3 \psi_4^2,
\end{aligned}
\tag{8.19}
$$

and perform a Fourier transformation to obtain the inverse propagators

$$
D_1^{-1} = \begin{pmatrix} \omega^2 - \mathbf{p}^2 + 2\mu\omega & 0 \\ 0 & \omega^2 - \mathbf{p}^2 - 2\mu\omega \end{pmatrix}
\tag{8.20}
$$

and

$$
D_2^{-1} = \begin{pmatrix} \omega^2 - \mathbf{p}^2 - 2\lambda v^2 & 2i\mu\omega \\ -2i\mu\omega & \omega^2 - \mathbf{p}^2 \end{pmatrix}.
\tag{8.21}
$$

Since the inverse propagator in Eq. (8.20) is diagonal in the (ψ^*, ψ) basis the matrix describes the propagation of a particle-antiparticle pair. By using the same convention for charge flow as we use in QED [3] the two equations[4]

$$
\begin{aligned}
\omega^2 + 2\mu\omega - \mathbf{p}^2 &= 0 \\
\omega^2 - 2\mu\omega - \mathbf{p}^2 &= 0
\end{aligned}
\tag{8.22}
$$

determines the dispersion relation for the particle and antiparticle respectively. The solutions are

$$
\begin{aligned}
\omega &= -\mu + \sqrt{\mu^2 + \mathbf{p}^2}, \\
\omega &= \mu + \sqrt{\mu^2 + \mathbf{p}^2}.
\end{aligned}
\tag{8.23}
$$

---

[3] To be explicit: the charge is created at point x by $\psi(x)$ and flows to point y where it is annihilated by $\psi^*(y)$.
[4] Obtained by requiring that $\det D_1^{-1} = 0$.

From the dispersion relations we see that the type-II NGB and the mNGB actually form a particle-antiparticle pair![5] Given that the inverse propagator is diagonal it is trivial to determine the propagators, the results are shown in Fig. 8.1.

There is not much subtlety in the particle spectrum associated with D_2^{-1}, which we calculated in Eq. (4.19). However, since the propagator is not diagonal the fields ψ_3 and ψ_4 exhibit mixing which means we have to be careful when evaluating Feynman diagrams. The propagator for (ψ_3, ψ_4) is found by inverting D_2^{-1} and is equal to

$$iD_2 = \frac{i}{p^2(p^2 - 2\lambda v^2) - 4\mu^2 p_0^2} \begin{pmatrix} p^2 & -2i\mu p_0 \\ 2i\mu p_0 & p^2 - 2\lambda v^2 \end{pmatrix}. \tag{8.24}$$

Finally the relevant interactions follow from the penultimate line in Eq. (8.18). For convenience all of the Feynman rules that we are going to use when calculating scattering amplitudes, are shown in Fig. 8.1.

$= -4i\lambda$

$\psi_3 \quad = -2i\lambda v$

$\dfrac{p}{} = \dfrac{ip^2}{p^2(p^2 - 2\lambda v^2) - 4\mu^2 p_0^2}$

$\dfrac{p}{} = \dfrac{i}{p^2 - 2\mu p_0}$
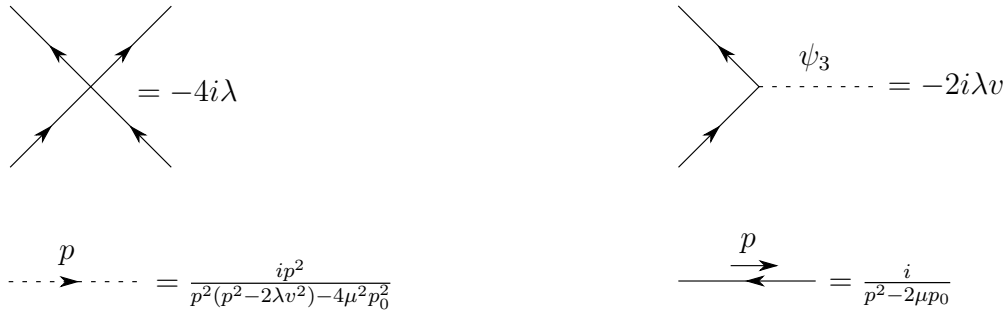
Figure 8.1: The Feynman rules needed for computing particle-antiparticle scattering in the $SU(2) \times U(1)$ model. The solid and dashed line represents the particle-antiparticle pair and the (ψ_3, ψ_4) particle respectively.

8.5   Particle-Antiparticle Scattering

In this section we want to calculate the scattering amplitude for the elastic scattering of the mNGB and the type-II NGB and show that it vanishes in the corresponding soft limits. The Feynman diagrams contributing to the off-shell scattering amplitude is shown in Fig. 8.2.

Using the Feynman rules in Fig. 8.1 the off-shell amplitude is

$$\begin{aligned} -i\mathscr{A}_{\text{off}} = -4i\lambda + (2i\lambda v)^2 \frac{i(p+k)^2}{(p+k)^2\big[(p+k)^2 - 2\lambda v^2\big] - 4\mu^2(p_0 + k_0)^2} \\ + (2i\lambda v)^2 \frac{i(p-p')^2}{(p-p')^2\big[(p-p')^2 - 2\lambda v^2\big] - 4\mu^2(p_0 - p'_0)^2}, \end{aligned} \tag{8.25}$$

where p, p', k and k' are off shell.

---

[5]Of course it is not a real particle-antiparticle pair because their masses are different. It would perhaps be more fitting to denote the mNGB as a pseudo-antiparticle, but for simplicity we will call it an antiparticle.

Figure 8.2: The scattering amplitude for particle-antiparticle scattering.

mNGB Soft Limit

The soft limit of the on-shell mNGB is $p_\mu \to 2\mu\delta_{\mu 0}$. In this limit we have the on-shell relations

$$
\begin{aligned}
(p+k)^2 &\to 2\mu(p_0 - k_0) + 4\mu k_0 = 2\mu(p_0 + k_0) \implies (p+k)^4 \to 4\mu^2(p_0 + k_0)^2, \\
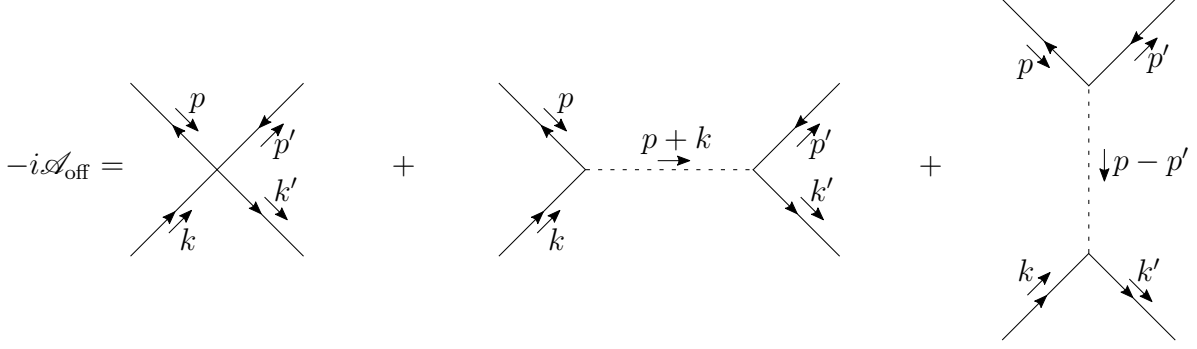(p-p')^2 &\to 2\mu(p_0 + p_0') - 4\mu p_0' = 2\mu(p_0 - p_0') \implies (p-p')^4 \to 4\mu^2(p_0 - p_0')^2,
\end{aligned}
\tag{8.26}
$$

where we used $p^2 = 2\mu p_0$, $p'^2 = 2\mu p_0'$, $k^2 = -2\mu k_0$ and $k'^2 = -2\mu k_0'$ which are consequences of the non-relativistic dispersion relations in Eq. (8.23). By using these relations we see that the two fractions in Eq. (8.25) simplify and we obtain

$$
-i\mathscr{A}_{\text{on}} \to -4i\lambda + 2i\lambda + 2i\lambda = 0,
\tag{8.27}
$$

in the soft limit where $p \to \mu\delta_{\mu 0}$. Thus the scattering amplitude vanishes in the soft limit of the mNGB when p, k, p' and k' are on shell.

NGB Soft Limit

We also expect the on-shell amplitude to vanish in the limit $k_\mu \to 0$. An explicit calculation confirms this,

$$
\begin{aligned}
-i\mathscr{A}_{\text{on}} &\to -4i\lambda + (2i\lambda v)^2 \frac{ip^2}{p^2\left[p^2 - 2\lambda v^2\right] - 4\mu^2 p_0^2} + (2i\lambda v)^2 \frac{ik'^2}{k'^2\left[k'^2 - 2\lambda v^2\right] - 4\mu^2 k_0'^2} \\
&= -4i\lambda + 2i\lambda + 2i\lambda = 0,
\end{aligned}
\tag{8.28}
$$

where we once again used that $p^2 = 2\mu p_0$, $p'^2 = 2\mu p_0'$, $k^2 = -2\mu k_0$ and $k'^2 = -2\mu k_0'$.

8.6 Expanding the Noether Currents

So far we have determined that our model contains a nonzero vev when $\mu > m$, and that it has an on-shell amplitude describing particle-antiparticle scattering that vanishes in the soft limits of the Goldstone bosons. Let us now try to use the same strategy that we used in the previous chapter to obtain a formula for the scattering amplitude in terms of the non-pole contributions of some matrix current element. As a first step we need to express the currents in Eqs. (8.12)

and (8.13) in terms of the fields ψ, ψ_3 and ψ_4. To do this we need to calculate $\phi^\intercal \tau_2 \partial^\mu \phi$ and $\phi^\dagger \tau_2 (\partial^\mu \phi)^*$, by using the second Pauli matrix and parametrising ϕ as in Eq. (8.17). This gives

$$
\begin{aligned}
\phi^\intercal \tau_2 \partial^\mu \phi &= \frac{1}{\sqrt{2}} \Big\{ iv\partial^\mu \psi + \big(\psi \partial^\mu \psi_4 - \psi_4 \partial^\mu \psi\big) + i\big(\psi_3 \partial^\mu \psi - \psi \partial^\mu \psi_3\big) \Big\}, \\
\phi^\dagger \tau_2 (\partial^\mu \phi)^* &= \frac{1}{\sqrt{2}} \Big\{ iv\partial^\mu \psi^* + \big(\psi_4 \partial^\mu \psi^* - \psi^* \partial^\mu \psi_4\big) + i\big(\psi_3 \partial^\mu \psi^* - \psi^* \partial^\mu \psi_3\big) \Big\}.
\end{aligned}
\tag{8.29}
$$

By substituting the expressions in Eq. (8.29) into the Noether currents given by Eqs. (8.12) and (8.13) we see that both of the currents $J_{R_1}^\mu$ and $J_{R_2}^\mu$ have a linear piece that contains the fields ψ and ψ^*. Thus we could continue in exactly the same way as we did in the previous chapter by first calculating $\langle f | J_{R_1}^\mu | i \rangle$ and then $\langle f | J_{R_2}^\mu | i \rangle$. By then using one of the two conservation laws we could find an explicit formula for the scattering amplitude in Eq. (8.25). However, from this formula it is not at all obvious that the amplitude should vanish in the soft limit. In fact, to show that it vanishes we have to use the second available conservation law as well. In principle this is not a problem, but it is hard to generalise this strategy. However, if we could combine the Noether currents into a linear combination we would incorporate the information contained in both conservation laws. This would give us **one** current, satisfying **one** conservation law. If we can calculate the amplitude from this special current, we should obtain a formula from which it should be trivial to see that the amplitude vanishes in the soft limit. We will now demonstrate that the construction of such a current is possible, while we will justify it rigorously in chapter 9.

8.7 The Current Element

We define the current

$$
J_\pm^\mu = \frac{1}{\sqrt{2}} \Big(J_{R_1}^\mu \pm i J_{R_2}^\mu \Big)
\tag{8.30}
$$

and choose to calculate $\langle f | J_-^\mu | i \rangle$, where $| f \rangle$ contains both the mNGB and the type-II NGB while $| i \rangle$ only contains the type-II NGB. The mNGB interacting with $| i \rangle$ is created by the current

$$
J_-^\mu = -iv\partial^\mu \psi - i\big(\psi_3 \partial^\mu \psi - \psi \partial^\mu \psi_3\big) + \big(\psi_4 \partial^\mu \psi - \psi \partial^\mu \psi_4\big),
\tag{8.31}
$$

which satisfies the conservation law[6]

$$
\big(\partial_\mu - 2i\mu \delta_{\mu 0}\big) J_-^\mu = 0.
\tag{8.32}
$$

By treating the current J_-^μ as a Lagrangian we can calculate the Feynman current rules. For convenience we display all of the ingredients that we will need for calculating the current element $\langle f | J_-^\mu | i \rangle$ in Fig. 8.3. By using these ingredients we can draw the Feynman diagrams that contribute to the current element $\langle f | J_-^\mu | i \rangle$ as in Fig. 8.4.

An explicit calculation of the current element gives

$$
\langle f | J_-^\mu | i \rangle = \frac{ivp_\mu}{p^2 - 2\mu p_0}(-i\mathscr{A}_{\text{off}}) + N_-^\mu,
\tag{8.33}
$$

---

[6]This can be checked explicitly by using the Euler-Lagrange equations for the fields ψ, ψ_3 and ψ_4 to express $\partial_\mu \partial^\mu \psi$, $\partial_\mu \partial^\mu \psi_3$ and $\partial_\mu \partial^\mu \psi_4$ in terms of time derivatives and products of fields. However, due to an enormous amount of algebra I used Mathematica and have therefore omitted the calculation.

[7]In chapter 9 we will justify this conservation law rigorously.

where

$$N_-^\mu = \frac{i(p+2k)^\mu \big[(p+k)^2 + 2\mu(p_0 + k_0)\big](2i\lambda v)}{(p+k)^2\big[(p+k)^2 - 2\lambda v^2\big] - 4\mu^2(p_0 + k_0)^2}$$
$$+ \frac{i(p-2p')^\mu \big[(p-p')^2 + 2\mu(p_0 - p'_0)\big](2i\lambda v)}{(p-p'^2)\big[(p-p')^2 - 2\lambda v^2\big] - 4\mu^2(p_0 - p'_0)^2}.$$

(8.34)
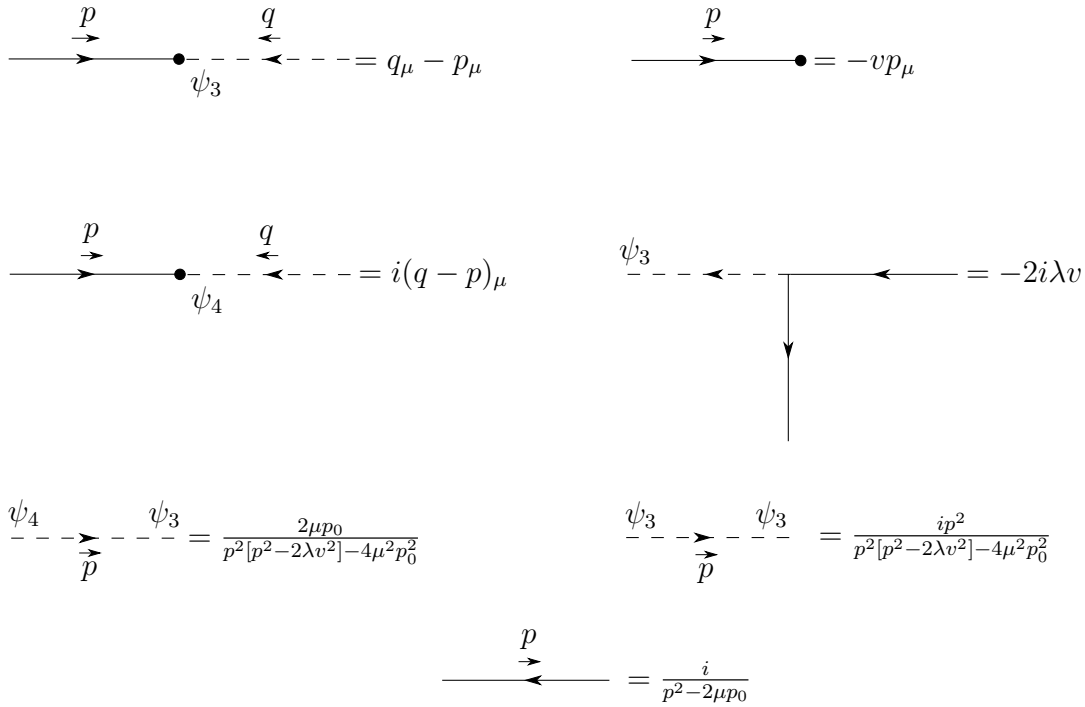
Figure 8.3: The Feynman rules we will use to calculate $\langle f | J_-^\mu | i \rangle$. The black dot represents the current J_-^μ responsible for the creation of the mNGB.
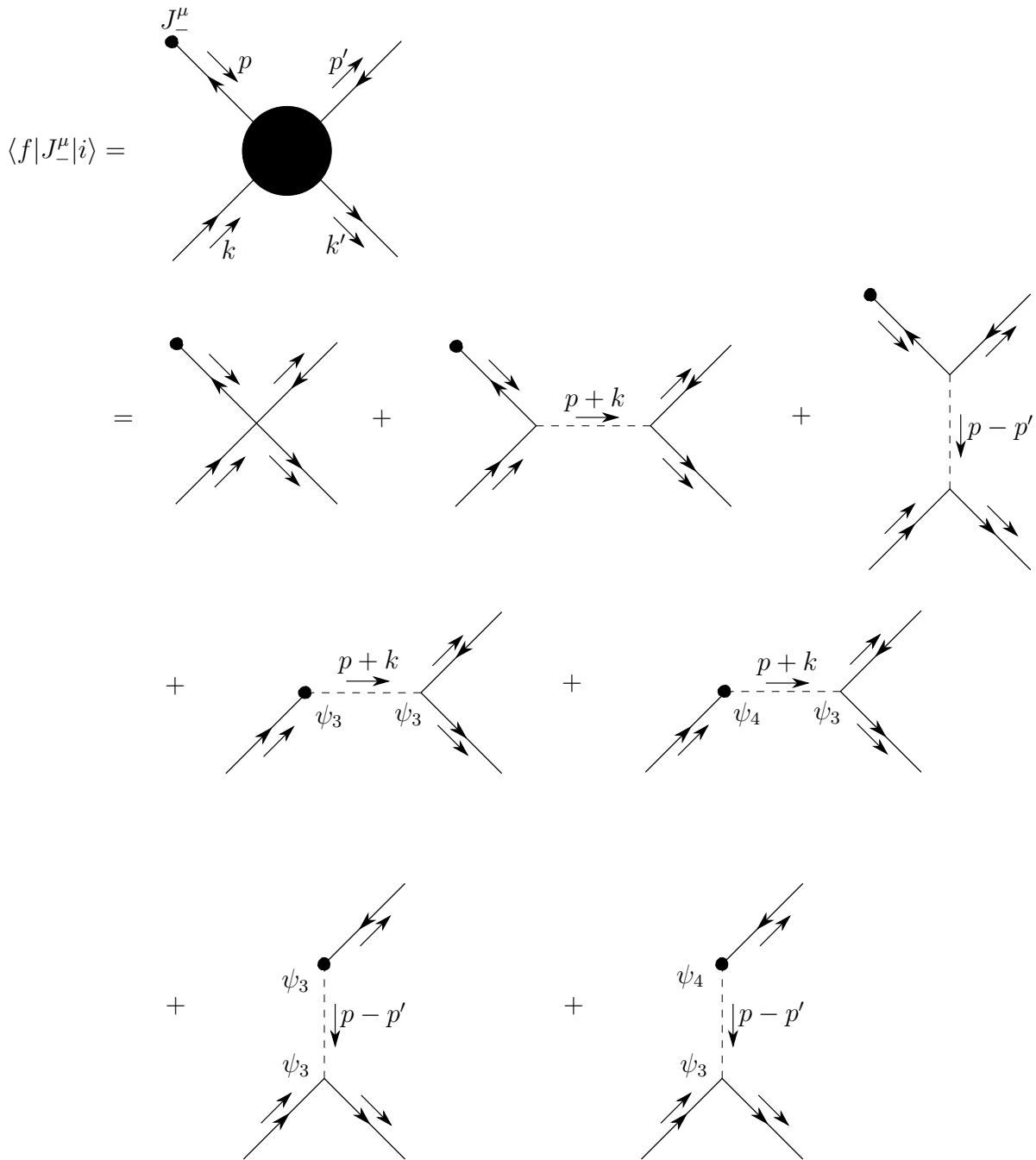
Figure 8.4: The figure shows the Feynman diagrams that represent the matrix element $\langle f | J_-^\mu | i \rangle$ at tree level. The incoming NGB and mNGB and the outgoing NGB and mNGB have momenta k, p, k' and p' respectively. To avoid clutter the external momenta are only labelled in the first diagram. The small black dot represents the current J_-^μ. The diagrams describe an initial state with a NGB interacting with a mNGB created by the Noether current to form a final state consisting of a NGB and a mNGB. Since in this case the fields ψ_3 and ψ_4 mix we get four non-pole contributions.

8.8 Current Conservation

Now, we will investigate the conservation law

$$\partial_\mu \langle f|J_-^\mu|i\rangle - 2i\mu \langle f|J_-^0|i\rangle = 0. \tag{8.35}$$

That is we want to see which of the momenta p, p', k and k' have to be on shell for the conservation law to be satisfied. To this end we substitute Eq. (8.33) into the LHS of of Eq. (8.35),

$$ip_\mu \langle f|J_-^\mu|i\rangle - 2i\mu \langle f|J_-^0|i\rangle = ip_\mu \left\{ \frac{ivp^\mu(-i\mathscr{A}_{\text{off}})}{p^2 - 2\mu p_0} - \frac{i(p+2k)^\mu(-2i\lambda v)\big[(p+k)^2 + 2\mu(p_0+k_0)\big]}{(p+k)^2\big[(p+k)^2 - 2\lambda v^2\big] - 4\mu^2(p_0+k_0)^2} \right.$$

$$\left. + \frac{i(2p'-p)^\mu(-2i\lambda v)\big[(p-p')^2 + 2\mu(p_0-p_0')\big]}{(p-p')^2\big[(p-p')^2 - 2\lambda v^2\big] - 4\mu^2(p_0-p_0')^2} \right\}$$

$$- 2i\mu \left\{ \frac{ivp_0(-i\mathscr{A}_{\text{off}})}{p^2 - 2\mu p_0} - \frac{i(p+2k)_0(-2i\lambda v)\big[(p+k)^2 + 2\mu(p_0+k_0)\big]}{(p+k)^2\big[(p+k)^2 - 2\lambda v^2\big] - 4\mu^2(p_0+k_0)^2} \right.$$

$$\left. + \frac{i(2p'-p)_0(-2i\lambda v)\big[(p-p')^2 + 2\mu(p_0-p_0')\big]}{(p-p')^2\big[(p-p')^2 - 2\lambda v^2\big] - 4\mu^2(p_0-p_0')^2} \right\}.$$

If we collect the the terms with the same denominator we obtain

$$- i\left\{ (-i\mathscr{A}_{\text{off}})(-iv) + \frac{i\big[(p+k)^2 + 2\mu(p_0+k_0)\big](-2i\lambda v)}{(p+k)^2\big[(p+k)^2 - 2\lambda v^2\big] - 4\mu^2(p_0+k_0)^2} \left\{ (p_\mu - 2\mu\delta_{\mu 0})(2k^\mu + p^\mu) \right\} \right.$$

$$\left. + \frac{i\big[(p-p')^2 + 2\mu(p_0-p_0')\big](-2i\lambda v)}{(p-p')^2\big[(p-p')^2 - 2\lambda v^2\big] - 4\mu^2(p_0-p_0')^2} \left\{ (p_\mu - 2\mu\delta_{\mu 0})(p^\mu - 2p'^\mu) \right\} \right\}.$$

To simplify this we use the following on-shell relations:

$$(p_\mu - 2\mu\delta_{\mu 0})(2k^\mu + p^\mu) = \big[(p_\mu' + k_\mu') - k_\mu\big]\big[k^\mu + (p'^\mu + k'^\mu)\big] - 2\mu(2k_0 + p_0)$$
$$= (p_\mu + k_\mu)^2 - 2\mu(p_0 + k_0) \tag{8.36}$$

and

$$(p_\mu - 2\mu\delta_{\mu 0})(p^\mu - 2p'^\mu) = \big[(k_\mu' - k_\mu) + p_\mu'\big]\big[(k'^\mu - k^\mu) - p'^\mu\big] - 2\mu(p_0 - 2p_0')$$
$$= (p - p')^2 - 2\mu(p_0 - p_0'). \tag{8.37}$$

Note that Eqs. (8.36) and (8.37) were obtained by assuming that p', k and k' were on shell, but p was kept off shell. Using the on-shell relations and Eq. (8.25) we end up with

$$ip_\mu \langle f|J_-^\mu|i\rangle - 2i\mu \langle f|J_-^\mu|i\rangle = -i\big(-4\lambda v + 2\lambda v + 2\lambda v\big) = 0. \tag{8.38}$$

We emphasise that the equality only holds provided all momenta except that of the mNGB (created by J_-^μ) are on shell.

8.9 Scattering Amplitude

In this section we use the conservation law we just discussed to derive a formula for the off-shell scattering amplitude $-i\mathscr{A}_{\text{off}}$.[8] Substituting Eq. (8.33) into Eq. (8.35) gives

$$ip_\mu \left\{ \frac{ivp^\mu}{p^2 - 2\mu p_0}(-i\mathscr{A}_{\text{off}}) + N_-^\mu \right\} - 2i\mu\delta_{\mu 0}\left\{ \frac{ivp^\mu}{p^2 - 2\mu p_0}(-i\mathscr{A}_{\text{off}}) + N_-^\mu \right\} = 0. \tag{8.39}$$

[8]To avoid confusion: the subscript in this case indicates that only p is off shell.

Collecting the two terms involving the amplitude gives

$$\frac{(-i\mathscr{A}_{\text{off}})}{p^2 - 2\mu p_0}(p^2 - 2\mu p_0)(-v) + ip_\mu N^\mu_- - 2i\mu N^0_- = 0. \tag{8.40}$$

By solving for the amplitude we obtain the formula

$$-i\mathscr{A}_{\text{off}} = \frac{ip_\mu N^\mu_- - 2i\mu N^0_-}{v}, \tag{8.41}$$

which obviously vanishes when we let p be on shell and send the 3-momentum to zero such that $p_\mu = 2\mu\delta_{\mu 0}$. Showing that our formula reproduces the amplitude in Eq. (8.25) is redundant because the calculation in this section is equivalent to the calculation in the previous section.

Generalisations

Having looked at the structure of the concrete examples in chapters 7 and 8 it is now much easier to make some general remarks and finally prove theorem 3. We begin by investigating some of the well known exact statements about mNGBs discussed in [1] and [9]. These well established statements combined with what we learnt from the previous two chapters allow us to formulate a proof of theorem 3 in the perturbative picture.

Following this discussion we want to make some statements that do not rely on perturbation theory. We begin by determining the exact form of the dispersion relation for mNGBs and finish by proving theorem 3 non-perturbatively.

9.1 Some Well-Known Results

In this section we follow the main argument of [9] closely, in order to understand the results we have discussed so far at a deeper level. Let us consider a many-body system, initially at zero chemical potential μ, with Hamiltonian $\mathscr{H}$ whose symmetry group is denoted by G. The Hamiltonian describing the system at nonzero μ is thus $\tilde{\mathscr{H}} = \mathscr{H} - \mu Q$, where Q is one of the generators of G. The symmetry group of $\tilde{\mathscr{H}}$ is denoted by $\tilde{G}$, and we define the vacuum state $|0\rangle$ by

$$\tilde{\mathscr{H}} |0\rangle \equiv 0. \tag{9.1}$$

Given that Q_i, the generators of G, is a symmetry of the Hamiltonian $\mathscr{H}$ it follows from standard quantum mechanics that $[\mathscr{H}, Q_i] = 0$, which means that

$$Q_i(t) = \int d^3x \, e^{i(\mathscr{H}t - \mathbf{P}\mathbf{x})} J_i^0(0) e^{-i(\mathscr{H}t - \mathbf{P}\mathbf{x})} \tag{9.2}$$

is time independent.[1] In our calculations up to this chapter we have made in particular two important well known observations:

1. Massless NGBs are born when the generators of $\tilde{G}$ are broken spontaneously. A type-I NGB corresponds to one spontaneously broken generator, and a type-II NGB corresponds to two spontaneously broken generators.

---

[1] Eq. (9.2) describes the time-evolution of an operator Q in the Heisenberg picture. $\mathbf{P}$ and $\mathbf{x}$ are the 3-momentum operator and position respectively.

2. mNGBs are born when the generators of G are broken explicitly in the step $G \xrightarrow{\mu} \tilde{G}$. So far we have noticed that a pair of explicitly broken generators always seem to give exactly one mNGB.

The second statement is a result of the Cartan decomposition. In fact the explicitly broken generators can be grouped together in pairs of linear combinations $Q_{\pm\sigma}$ that satisfy

$$[Q, Q_{\pm\sigma}] = \pm q_\sigma Q_{\pm\sigma}, \tag{9.3}$$

where σ labels each pair and q_σ is a positive real number. By defining an order parameter $\lambda_\sigma = \langle 0|[Q_{+\sigma}(t), J^0_{-\sigma}(0)]|0\rangle$, using Eq. (9.2), inserting a complete set of eigenstates $1 = \sum_n |n, \mathbf{p}\rangle \langle n, \mathbf{p}|$ and performing the spatial integration the authors of [9] obtain

$$\begin{aligned}
\lambda_\sigma = \sum_n e^{-i[E_n(0) - \mu q_\sigma]t} \left| \langle 0|J^0_{+\sigma}|n, 0\rangle \right|^2 \\
- \sum_n e^{i[E_n(0) + \mu q_\sigma]t} \left| \langle 0|J^0_{-\sigma}|n, 0\rangle \right|^2.
\end{aligned} \tag{9.4}$$

Note that $\lambda_\sigma = \langle 0|[Q_{+\sigma}(t), J^0_{-\sigma}(0)]|0\rangle$ is time independent, because $Q_{+\sigma}(t)$ commutes with the Hamiltonian. If we assume that the mass of the n'th excitation is $E_n(0) \geq 0$ it follows that $\langle 0|J^0_{-\sigma}|n, 0\rangle = 0$ for all n. There are then two things that can happen: Either $\langle 0|J^0_{+\sigma}|n, 0\rangle = 0$ resulting in no mNGBs or $\langle 0|J^0_{+\sigma}|n, 0\rangle \neq 0$ from which it follows that $\tilde{\mathscr{H}}|n, 0\rangle = E_n(0)|n, 0\rangle = \mu q_\sigma |n, 0\rangle$. Thus if $\langle 0|J^0_{+\sigma}|n, 0\rangle \neq 0$ the state $|n, \mathbf{p}\rangle$ is a mNGB with mass μq_σ.

Furthermore, if $\langle 0|J^0_{-\sigma}|n, 0\rangle = 0$ and $\langle 0|J^0_{+\sigma}|n, 0\rangle \neq 0$ it follows from the definition of λ_σ that $Q_{+\sigma}|0\rangle = 0$ and $Q_{-\sigma}|0\rangle \neq 0$ respectively. So the operator $Q_{+\sigma}$ annihilates the vacuum while $Q_{-\sigma}$ excites a particle. To find the mass of the particle excited by $Q_{-\sigma}$, consider the commutator

$$[\tilde{\mathscr{H}}, Q_{-\sigma}]|0\rangle = [\mathscr{H}, Q_{-\sigma}]|0\rangle - \mu[Q, Q_{-\sigma}]|0\rangle. \tag{9.5}$$

Using the facts $[\mathscr{H}, Q_{-\sigma}] = 0$, $\tilde{\mathscr{H}}|0\rangle = 0$ and Eq. (9.3) we obtain

$$\tilde{\mathscr{H}} Q_{-\sigma}|0\rangle = \mu q_\sigma Q_{-\sigma}|0\rangle. \tag{9.6}$$

Hence the state $Q_{-\sigma}|0\rangle$ is a state with energy μq_σ which we identify as a mNGB.

Next we should check that this is consistent with the two models we studied in the two previous chapters.

9.1.1 The Nonlinear Sigma Model

At zero chemical potential the symmetry group is $G = SO(3)$ with the three generators T_1, T_2 and T_3. Choosing a ground state $\phi_0 = (0, 0, v)$ results in that T_1 and T_2 spontaneously break leading to two type-I NGBs.

Coupling a nonzero chemical potential to the generator T_3 leads to the explicit breaking of T_1 and T_2. Thus the new symmetry group $\tilde{G}$ is the $SO(2)$ subgroup of $SO(3)$ generated by T_3. We found a type-I NGB from the SSB of T_3 and a mNGB from the explicit breaking of T_1 and T_2. If we define $T_\pm = 1/\sqrt{2}(T_1 \pm iT_2)$ the commutator in Eq. (9.3) takes the form

$$[T_3, T_\pm] = \pm T_\pm, \tag{9.7}$$

implying that the mass of the mNGB is μ, agreeing with our explicit calculation in chapter 3. With the benefit of hindsight we note that we could also have performed the calculation in chapter 7 by working with the current $J^\mu_\pm = 1/\sqrt{2}(J^\mu_1 + iJ^\mu_2)$ instead of J^μ_1 and J^μ_2 seperately.

Physical Application

As mentioned in chapter 3 this example has the physical interpretation of an antiferromagnet, where the chemical potential can be viewed as an external magnetic field. In the case of no external magnetic field, we assume that the spins are aligned alternately in some arbitrary direction such that the symmetry is spontaneously broken like $SO(3) \to SO(2)$ resulting in two type-I NGBs.

Now, assume that in the ground state the spins are aligned alternately in the z-direction. If we turn on an external magnetic field in the z-direction, the spins in the ground state become unstable. This results in a new ground state that points in a direction orthogonal to the z-axis. Choosing this direction to be $\hat{x}$ results in that the generator T_3 spontaneously breaks giving one type-I NGB. In addition, as a result of applying the field in the z-direction the generators T_1 and T_2 were broken explicitly resulting in the mNGB.

9.1.2 The Higgs-Like Model

In this case we coupled the chemical potential to the generator R_3 leading to the symmetry breaking pattern $SU(2)_L \times SU(2)_R \xrightarrow{\mu} SU(2)_L \times U(1)_R \to U(1)'$. In the first step the generators R_1 and R_2 were explicitly broken giving a mNGB. In the second step three generators were broken spontaneously giving one type-I and one type-II NGB.

If we define the generators $R_\pm = 1/\sqrt{2}\,(R_1 \pm iR_2)$ the commutator in Eq. (9.3) takes the form

$$[R_3, R_\pm] = \pm 2R_\pm \tag{9.8}$$

implying that the mass of the mNGB is 2μ, agreeing with our explicit calculation. Eq. (9.8) also tells us that the mNGB was created by the operator R_-. This is what motivated us to define the current $J_-^\mu = J_{R_1}^\mu - iJ_{R_2}^\mu$ in Eq. (8.30) responsible for the creation of the mNGB.

9.2 Perturbative Proof

In this section we generalise the calculations in Chapters 7 and 8 and attempt to prove theorem 3 in the perturbative picture. Based on the previous section and the examples in the aforementioned chapters we know that there exists a current J_-^μ responsible for the creation of a mNGB with momentum p. If we define an initial state $|i\rangle$ and a final state $|f\rangle$ we can parametrise the matrix element $\langle f|J_-^\mu|i\rangle$ as

$$\langle f|J_-^\mu|i\rangle = (J_-^\mu)^{(1)} \times P \times (-i\mathscr{A}_{\text{off}}) + N_-^\mu, \tag{9.9}$$

which is very similar to the parametrisation used in [10] for massless NGBs. Here $(J_-^\mu)^{(1)}$ is the linear Feynman current rule,[2] P is the propagator of the mNGB, $\mathscr{A}_{\text{off}}$ is the scattering amplitude with p off shell and N_-^μ are the non-pole contributions of $\langle f|J_-^\mu|i\rangle$.

9.2.1 Conservation Law

To begin with we need to determine the conservation law that J_-^μ satisfies. From our discussion in section 9.1 we recall that a mNGB always corresponds to two explicitly broken generators

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[2]In our concrete examples $(J_-^\mu)^{(1)}$ can be found in Figs. 7.2 and 8.3.

denoted by e.g. Q_1 and Q_2. We also emphasised that we can always construct the linear combination

$$Q_\pm = Q_1 \pm iQ_2, \tag{9.10}$$

which satisfies

$$[Q, Q_\pm] = \pm qQ_\pm. \tag{9.11}$$

Here Q is the generator coupled to the chemical potential μ and q is a positive real number. By substituting Eq. (9.10) into Eq. (9.11) we obtain two equations:

$$
\begin{aligned}
[Q, Q_+] = qQ_+ &\implies [Q, Q_1] + i[Q, Q_2] = qQ_1 + iqQ_2, \\
[Q, Q_-] = -qQ_- &\implies [Q, Q_1] - i[Q, Q_2] = -qQ_1 + iqQ_2.
\end{aligned}
\tag{9.12}
$$

By adding and subtracting these equations we obtain

$$
\begin{aligned}
[Q, Q_1] &= iqQ_2, \\
[Q, Q_2] &= -iqQ_1,
\end{aligned}
\tag{9.13}
$$

respectively. In order to proceed we have to identify the structure constants of the coset space, where the mNGBs live. From Eq. (9.13) we see that the structure constants are

$$f_{a12} = q, \quad f_{a21} = -q, \tag{9.14}$$

where a labels the explicitly unbroken generator Q. Using Eq. (6.27) we can write down the conservation laws for the broken currents J_1^μ and J_2^μ

$$
\begin{aligned}
D_\mu J_1^\mu &= \partial_\mu J_1^\mu + f_{1a2} A_{\mu a} J_2^\mu = \partial_\mu J_1^\mu - \mu q J_2^0, \\
D_\mu J_2^\mu &= \partial_\mu J_2^\mu + f_{2b1} A_{\mu b} J_1^\mu = \partial_\mu J_2^\mu + \mu q J_1^0,
\end{aligned}
\tag{9.15}
$$

where we used that $A_{\mu a} = \mu_a \delta_{\mu 0} = \mu \delta_{\mu 0}$. If we then define the currents $J_\pm^\mu = J_1^\mu \pm iJ_2^\mu$ and use the conservation laws in Eq. (9.15) we obtain

$$
\begin{aligned}
D_\mu J_+^\mu &= \partial_\mu J_+^\mu + i\mu q J_+^0 = 0, \\
D_\mu J_-^\mu &= \partial_\mu J_-^\mu - i\mu q J_-^0 = 0.
\end{aligned}
\tag{9.16}
$$

Note that the conservation laws in Eq. (9.16) only holds when all momenta except that of the mNGB (destroyed/created by $J_\pm^\mu$) is on shell.

9.2.2 The Connection Between the Propagator and the Noether Current

In section 6.3 we derived the equation of motion for a general gauge theory,

$$D_\mu \frac{\partial \mathscr{L}}{\partial(D_\mu \phi^i)} = \frac{\partial \mathscr{L}}{\partial \phi^i}. \tag{9.17}$$

For mNGBs the gauge field is given by the chemical potential $A_\mu = \mu \delta_{\mu 0}$ and the fields ϕ^i transform in general like

$$\delta\phi_a^i = C_a^i + P_a(\phi^i) \tag{9.18}$$

where C_a^i is a constant and $P_a(\phi^i)$ is a polynomial in the fields ϕ^i. Therefore to linear order the mNGB has a shift symmetry, meaning that $\delta\mathcal{L}/\delta\phi^i = 0$ such that the linearised equation of motion takes the form

$$D_\mu \frac{\partial\mathcal{L}}{\partial(D_\mu\phi^i)} = 0. \tag{9.19}$$

In earlier chapters we have used that the inverse propagator is obtained from the terms quadratic in the Lagrangian which is equivalent to saying that the inverse propagator is the Green's function corresponding to the linearised equations of motion. Thus the inverse propagator P^{-1} is the Green's function to the operator contained Eq. (9.19).

The Noether currents are defined as $J_a^\mu = \frac{\delta\mathcal{L}}{\partial(D_\mu\phi^i)}\delta\phi_a^i$ which to linear order is

$$J_{\mathsf{Lin},a}^\mu = \frac{\delta\mathcal{L}}{\partial(D_\mu\phi^i)}C_a^i. \tag{9.20}$$

Since the current is conserved we can write

$$D_\mu J_{\mathsf{Lin},a}^\mu = D_\mu \frac{\partial\mathcal{L}}{\partial(D_\mu\phi^i)}C_a^i = 0. \tag{9.21}$$

Note that Eq. (9.19) and Eq. (9.21) are identical up to a constant.

Furthermore we can parametrise the linear Feynman current rule as

$$(J_-^\mu)^{(1)} = ip^\mu A_- + i\delta^{\mu 0}B_-, \tag{9.22}$$

where both A_- and B_- are constants. By performing an inverse Fourier transformation we get the linear piece of the Noether current

$$J_{\mathsf{Lin},-}^\mu = A_-\partial^\mu\phi + i\delta^{\mu 0}B_-\phi,\,^3 \tag{9.23}$$

where $J_{\mathsf{Lin},-}^\mu$ is a linear combination of the currents $J_{\mathsf{Lin},a}^\mu$. Acting on Eq. (9.23) with the covariant derivative $D_\mu = \partial_\mu - i\mu q\delta_{\mu 0}$ we obtain the equation

$$0 = A_-\partial_\mu\partial^\mu\phi + iB_-\partial_0\phi - i\mu qA_-\partial_0\phi + \mu qB_-\phi. \tag{9.24}$$

Performing a Fourier transformation gives

$$0 = \big\{ -A_-p^2 - B_-p_0 + \mu qA_-p_0 + \mu qB_- \big\}\phi. \tag{9.25}$$

Hence, because Eq. (9.19) and Eq. (9.21) are identical up to a constant we can read off the inverse propagator as

$$P^{-1} = -A_-p^2 - B_-p_0 + \mu qA_-p_0 + \mu qB_-. \tag{9.26}$$

---

[3] The attentive reader might wonder why there is not a minus sign in front of A_-. In our convention the Fourier transform is given by $\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx}\phi(k)$, when the momentum flows into the vertex. However, since we are assuming that the mNGB is created by the current J_-^μ the momentum should flow out of the vertex resulting in the change of sign.

9.2.3 Proving the Theorem

Proof. We now have all of the ingredients needed to prove theorem 3 perturbatively. By using Eqs. (9.9) and (9.16) we can write

$$0 = D_\mu \langle f | J_-^\mu | i \rangle = D_\mu \left\{ e^{ipx} \left[(J_-^\mu)^{(1)} \times P \times (-i\mathscr{A}_{\text{off}}) + N_-^\mu \right] \right\}. \tag{9.27}$$

Factoring out the exponential gives the equation

$$0 = ip_\mu \left[(J_-^\mu)^{(1)} P(-i\mathscr{A}_{\text{off}}) + N_-^\mu \right] - i\mu q \left[(J_-^0)^{(1)} P(-i\mathscr{A}_{\text{off}}) + N_-^0 \right], \tag{9.28}$$

and if we then collect the terms involving the scattering amplitude we obtain

$$0 = (-i\mathscr{A}_{\text{off}}) P (ip_\mu - i\mu q \delta_{\mu 0}) (J_-^\mu)^{(1)} (-i\mathscr{A}_{\text{off}}) + ip_\mu N_-^\mu - i\mu q N_-^0. \tag{9.29}$$

By replacing $(J_-^\mu)^{(1)}$ with Eq. (9.22) we obtain

$$\begin{aligned} 0 &= (-i\mathscr{A}_{\text{off}}) P \left[-A_- p^2 - B_- p_0 + \mu q A_- p_0 + \mu q B_- \right] + ip_\mu N_-^\mu - i\mu q N_-^0 \\ &= (-i\mathscr{A}_{\text{off}}) + ip_\mu N_-^\mu - i\mu q N_-^0, \end{aligned} \tag{9.30}$$

where we in the penultimate line used Eq. (9.26). By solving for the scattering amplitude we obtain

$$-i\mathscr{A}_{\text{off}} = i\mu q N_-^0 - ip_\mu N_-^\mu \tag{9.31}$$

which vanishes when we let p be on shell and evaluate the soft limit such that $p_\mu = \delta_{\mu 0} \mu q$. ∎

Hence consider that we are given a scattering process $|i\rangle + \text{mNGB} \rightarrow |f\rangle$ with all momenta on shell and the corresponding on-shell scattering amplitude $-i\mathscr{A}_{on}$. We can say that $-i\mathscr{A}_{on}$ vanishes in the soft limit of the mNGB, unless N_-^μ is at the same time singular. N_-^μ being singular in the soft limit, would suggest the presence of a kinematic singularity.

9.3 The Dispersion Relation

Before we attempt to prove theorem 3 non-perturbatively, let us see if we can say something general about the dispersion relation for mNGBs. This discussion is motivated by attempting to generalise section IV in [18]. If we consider a theory with rotational invariance and assume that the current J_+^μ acts as an annihilation operator for mNGBs we can write,

$$\langle 0 | J_+^\mu(x) | \pi(p) \rangle = e^{-ipx} \left\{ ip^\mu F(|\mathbf{p}|) + i\delta^{\mu 0} G(|\mathbf{p}|) \right\}. \tag{9.32}$$

Here the functions $F(|\mathbf{p}|)$ and $G(|\mathbf{p}|)$ are scalars and p is the on-shell momentum of the mNGB $|\pi(p)\rangle$. In this case the term $i\delta^{\mu 0} G(|\mathbf{p}|)$ takes into account that our theory is in general not Lorentz invariant. Using the fact that the current J_+^μ is conserved we can write

$$\begin{aligned} 0 = D_\mu \langle 0 | J_+^\mu(x) | \pi(p) \rangle &= \partial_\mu \langle 0 | J_+^\mu(x) | \pi(p) \rangle + i\mu q \langle 0 | J_+^0(x) | \pi(p) \rangle \\ &= p^2 F(|\mathbf{p}|) + p_0 G(|\mathbf{p}|) - \mu q p_0 F(|\mathbf{p}|) - \mu q G(|\mathbf{p}|). \end{aligned} \tag{9.33}$$

To avoid clutter let us suppress the arguments of the functions $F(|\mathbf{p}|)$ and $G(|\mathbf{p}|)$. Furthermore by writing $p^2 = \omega^2 - \mathbf{p}^2$ we obtain the quadratic equation

$$0 = F\omega^2 + (G - \mu q F)\omega - (\mu q G + \mathbf{p}^2 F). \tag{9.34}$$

By taking the limit $\mathbf{p} \to 0$ such that $\omega(0) = m$, where m is the mass of the mNGB we obtain

$$0 = F(0)m^2 + [G(0) - \mu q F(0)]m - \mu q G(0). \tag{9.35}$$

The solutions to this equation are

$$m_+ = \mu q \quad \& \quad m_- = -\frac{G(0)}{F(0)}. \tag{9.36}$$

The first solution tells us that the mass of the mNGB is exactly given by μq, as we have already shown using the symmetry algebra. If we assume that both $F(0)$ and $G(0)$ are positive then the second solution is unphysical.

We can also try to say something about the form of the dispersion relation of a mNGB by solving Eq. (9.34) exactly. The solution is

$$\omega_\pm = \frac{-G + \mu q F \pm (G + \mu q F)\sqrt{1 + \frac{4F^2}{(G+\mu q F)^2}\mathbf{p}^2}}{2F} \tag{9.37}$$

and we see that the dispersion relation of the mNGB is fully determined by the functions F and G.

9.4 Non-Perturbative Proof

In this section we prove theorem 3 non-perturbatively, following Weinberg's strategy described in chapter 6. The idea is to view $\langle f|J_-^\mu|i\rangle$ as an amputated Green's function where we have appended an external mNGB propagator with off-shell momentum p. The momenta of all the other particles are kept on shell such that the current satisfies the covariant conservation law in Eq. (9.16). The situation we will consider is illustrated in Fig. 9.1, where an initial state $|i\rangle$ interacts with an off-shell mNGB created by J_-^μ to form a final state $|f\rangle$.

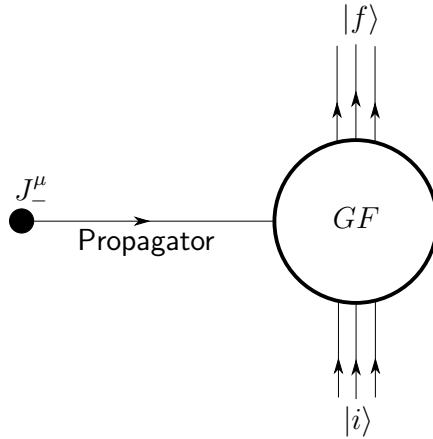**Figure 9.1:** The figure shows the pole structure of the current element $\langle f|J_-^\mu|i\rangle$. It consists of the Noether current (small black circle), a mNGB propagator and an amputated Green's function (GF).

Note that our proof is not valid for the pNGBs discussed in chapter 5 because their propagator (and hence their dispersion relation) takes a different form than the mNGBs.

Proof. Let us start by trying to extract the mNGB propagator contained in the Källen-Lehmann representation $D^{\phi\phi} = -i\langle 0|T\{\phi(x)\phi(y)\}|0\rangle$. In momentum space the Källen-Lehmann propagator takes the form

$$D^{\phi\phi}(\mathbf{p},\omega) = (2\pi)^3 \sum_n \left[\frac{\langle 0|\phi(0)|n,\mathbf{p}\rangle \langle n,\mathbf{p}|\phi(0)|0\rangle}{\omega - E_n(\mathbf{p}) + i\epsilon} \right.$$
$$\left. - \frac{\langle 0|\phi(0)|n,-\mathbf{p}\rangle \langle n,-\mathbf{p}|\phi(0)|0\rangle}{\omega + E_n(\mathbf{p}) - i\epsilon} \right]. \tag{9.38}$$

We used this expression also in chapter 5, and the formula is taken from [17]. Let us for simplicity assume that the field ϕ is not composite, such that there is a one to one correspondence between the number of fields and the number of mNGBs.[4] The pole of the current element $\langle f|J_-^\mu|i\rangle$ comes from the pole of the propagator in Eq. (9.38). For the particular mNGB created by J_-^μ there is a pole at $p_0 = E_m(\mathbf{p})$ where $E_m(\mathbf{p})$ is the on-shell dispersion relation of the mNGB. Therefore the pole structure that will appear in $\langle f|J_-^\mu|i\rangle$ is

$$D^{\phi\phi} \xrightarrow{pole} \frac{\langle 0|\phi(0)|m,\mathbf{p}\rangle \langle m,\mathbf{p}|\phi(0)|0\rangle}{p_0 - E_m(\mathbf{p})}. \tag{9.39}$$

The remaining terms in Eq. (9.38) will give a contribution to the non-pole contributions N_-^μ of $\langle f|J_-^\mu|i\rangle$. Now, we can proceed in exactly the same manner as we did in chapter 6. By attaching $\langle m,\mathbf{p}|\phi(0)|0\rangle$ to the Noether current J^μ and $\langle 0|\phi(0)|m,\mathbf{p}\rangle$ to the Green's function (GF) displayed in Fig. 9.1 we obtain the current element $\langle m,\mathbf{p}|J_-^\mu|0\rangle$ and the scattering amplitude[5] $-i\mathscr{A}_{\text{off}}$ respectively. Combining the ingredients as shown in Fig. 9.1 gives a Laurent expansion of the current element,

$$\langle f|J_-^\mu(0)|i\rangle = \langle m,\mathbf{p}|J_-^\mu|0\rangle \times \frac{i}{p_0 - E_m(\mathbf{p})} \times -i\mathscr{A}_{\text{off}} + N_-^\mu. \tag{9.40}$$

In the Heisenberg picture we can immediately write down this expression as a function of spacetime x,

$$\langle f|J_-^\mu(x)|i\rangle = e^{ipx}\left[\langle m,\mathbf{p}|J_-^\mu|0\rangle \frac{i}{p_0 - E_m(\mathbf{p})}(-i\mathscr{A}_{\text{off}}) + N_-^\mu \right]. \tag{9.41}$$

Now, if we assume rotational invariance we can parametrise $\langle m,\mathbf{p}|J_-^\mu|0\rangle$ in terms of two scalar functions $F(|\mathbf{p}|)$ and $G(|\mathbf{p}|)$ to obtain[6]

$$\langle m,\mathbf{p}|J_-^\mu|0\rangle = ip^\mu F(|\mathbf{p}|) + i\delta^{\mu 0}G(|\mathbf{p}|). \tag{9.42}$$

By first substituting Eq. (9.42) into Eq. (9.41) and then acting with the covariant derivative $D_\mu = \partial_\mu - i\mu q\delta_{\mu 0}$ on $\langle f|J_-^\mu|i\rangle$ we obtain

$$0 = ip_\mu\left\{ (ip^\mu F + i\delta^{\mu 0}G)\frac{\mathscr{A}_{\text{off}}}{p_0 - E_m(\mathbf{p})} + N_-^\mu \right\} - i\mu q\left\{ (ip_0 F + iG)\frac{\mathscr{A}_{\text{off}}}{p_0 - E_m(\mathbf{p})} + N_-^0 \right\}. \tag{9.43}$$

---

[4]Equivalently we can say that we ignore the case were the fields mix.

[5]The subscript indicates that p still is off shell

[6]To avoid clutter we suppress the momentum dependence and write $F \equiv F(|\mathbf{p}|)$ and $G \equiv G(|\mathbf{p}|)$ in the following.

Collecting the terms involving the scattering amplitude yields

$$0 = \frac{\mathscr{A}_{\text{off}}}{p_0 - E_m(\mathbf{p})} \Big\{ -p^2 F - p_0 G + \mu q p_0 F + \mu q G \Big\} + i p_\mu N_-^\mu - i \mu q N_-^0. \tag{9.44}$$

In order to proceed we focus on the expression in the curly brackets in Eq. (9.44). By first writing $p^2 = p_0^2 - \mathbf{p}^2$ and then taking the soft limit $\mathbf{p} \to 0$ we get

$$\begin{aligned} -p^2 F - p_0 G + \mu q p_0 F + \mu q G &= -p_0^2 F + \mathbf{p}^2 F - p_0 G + \mu q p_0 F + \mu q G \\ &\xrightarrow{\text{soft}} -p_0^2 F(0) - p_0 G(0) + \mu q \big[p_0 F(0) + G(0) \big] \\ &= -\big[p_0 F(0) + G(0) \big] \big[p_0 - E_m(0) \big]. \end{aligned} \tag{9.45}$$

In the penultimate line we identified the mass of the mNGB as $E_m(0) = \mu q$. Thus, in the soft limit the factor $p_0 - E_m(0)$ in Eq. (9.45) cancels the denominator in Eq. (9.44). This results in the expression

$$0 = -\mathscr{A}_{\text{off}}^{\text{soft}} \big[p_0 F(0) + G(0) \big] + i p_0 N_-^{\mu, \text{soft}} - i \mu q N_-^{0, \text{soft}}, \tag{9.46}$$

where the superscript "soft" indicates that the given quantity is being evaluated in the soft limit of the mNGB $\mathbf{p} \to 0$. By solving for the soft-limit scattering amplitude we find

$$-\mathscr{A}_{\text{off}}^{\text{soft}} = \frac{i \mu q N_-^{0, \text{soft}} - i p_0 N_-^{\mu, \text{soft}}}{G(0) - p_0 F(0)} \tag{9.47}$$

which vanishes when we let p go on shell such that $p_0 \to \mu q$ in the soft limit. ∎

Let us comment briefly on the case $F(0) = G(0) = 0$, where Eq. (9.47) appears to be singular. According to section 9.1 and [9] mNGBs satisfy both $\langle 0 | J_{+\sigma}^\mu | m, 0 \rangle \neq 0$ and $\langle 0 | J_{-\sigma}^\mu | m, 0 \rangle = 0$. This means that $J_{+\sigma}^\mu$ and $J_{-\sigma}^\mu$ destroys and creates mNGBs respectively. In our proof, see Eq. (9.42), we used that $\langle m, 0 | J_-^0 | 0 \rangle = i p_0 F(0) + i G(0) = \big(\langle 0 | J_+^0 | m, 0 \rangle \big)^* \neq 0$. Hence, we see that mNGBs require that $i p_0 F(0) + i G(0) \neq 0$ and we do not have to worry about the case $F(0) = G(0) = 0$.[7]

As in section 9.2 we once again note that if N_-^μ is singular in the soft limit of the mNGB the on-shell scattering amplitude may not vanish due to a kinematic singularity. However, if N_-^μ is not singular in the soft limit we can with confidence say that the on-shell scattering amplitude vanishes in soft limit of the mNGB. Thus we have proved theorem 3.

---

[7]However, it is interesting to note that according to [9] the condition $F(0) = G(0) = 0$ is exactly satisfied by the pNGBs we discussed in chapter 5.

Chapter 10

Discussion, Conclusion and Outlook

As mentioned in both the preface and introduction, the main body of this thesis consists of parts I, II and III. Part I and II are more or less identical to last semesters specialisation project found in [2], while Part III represents the work done this semester. All parts have been included for completeness.

10.1 Part I

We began by considering an Abelian model with one complex field. This model had a nonzero vev, thus triggering SSB. In accordance with Goldstone's theorem this gave us one massless NGB π_2 and a massive mode π_1. Then by computing the scattering amplitudes for the processes $\pi_1\pi_2 \to \pi_1\pi_2$ and $\pi_2\pi_2 \to \pi_2\pi_2$ we were able to show that the amplitudes vanished in the soft limit of the NGB, in accordance with theorem 2. The introduction of a chemical potential broke Lorentz invariance and modified the dispersion relation of the NGB. However, the NGB remained massless because a **pair** of generators were not broken explicitly.

In order to discuss a more complicated example we next looked at the non-Abelian linear $SO(3)$ model. At zero chemical potential the SSB pattern is $SO(3) \to SO(2)$ which in accordance with Goldstone's theorem gave us two NGBs (one for each broken generator). Coupling a chemical potential to the generator T_3 we found that the new symmetry breaking pattern is $SO(3) \xrightarrow{\mu} SO(2) \to \emptyset$. An explicit calculation gave us one mNGB G and a NGB of type-I π. Once again we can understand this particle spectrum by looking at the generators. The mNGB corresponds to the explicitly broken pair T_1 and T_2 while the spontaneous breaking of T_3 gave us the NGB of type-I. In order to give indication for our conjecture on page 60 we calculated the scattering amplitude for the process $\pi G \to \pi G$, using the nonlinear sigma model. We found that in the the soft limit of the mNGB the amplitude vanished. However, in the soft limit of the type-I NGB we found a kinematic singularity. The reason for the existence of this singularity is explained in [10].

Next we had a very brief look at a model invariant under $SU(2)_L \times SU(2)_R$ transformations. By introducing a chemical potential we obtained the symmetry breaking pattern $SU(2)_L \times SU(2)_R \xrightarrow{\mu} SU(2)_L \times U(1)_R \to U(1)'$. Again we found one mNGB corresponding to the explicit breaking of the pair R_1 and R_2. In addition we found NGBs of both types. This is consistent with literature such as [9] which tells us that type-I NGBs and type-II NGBs correspond to the spontaneous breaking of one and two generators respectively.

10.2   Part II

In part II we considered the complete breaking of an $O(3)$ symmetry. To describe this phenomenon we had to use a well known effective field theory approach and ended up with a mNGB Ψ_-, a pNGB Ψ_+ and a NGB of type-I Ψ_3. We tried to calculate the scattering amplitudes for the two processes $\Psi_-\Psi_3 \to \Psi_-\Psi_3$ and $\Psi_+\Psi_3 \to \Psi_+\Psi_3$ with corresponding on-shell scattering amplitudes $\mathscr{A}_-$ and $\mathscr{A}_+$ respectively. We found that $\mathscr{A}_-$ vanished in the soft limit of the mNGB, giving further support to our conjecture on page 60. However, we were surprised to find that the amplitude $\mathscr{A}_+$ vanished in the soft limit of the pNGB. There could be several reasons for this:

- The model has a hidden discrete symmetry, that forces the scattering amplitude to vanish in the soft limit.

- We have considered only the simplest possible scattering event, namely $2 \to 2$. Perhaps this is too simple, and that we instead should consider $3 \to 3$ scattering?

- pNGBs also interact weakly.

Given more time we would of course address each of these issues in detail.

10.3   Part III

In part III we began by going through Weinberg's proof of theorem 2. In short the proof goes like this: Define a current element $\langle f|J^\mu(0)|i\rangle$ describing an initial state $|i\rangle$ interacting with a NGB created by J^μ to form a final state $|f\rangle$. By assuming Lorentz invariance we can parametrise the current element as $\langle f|J^\mu(0)|i\rangle = N_{\mathrm{fi}}^\mu + \frac{ik^\mu F}{k^2}\mathscr{A}_{\mathrm{fi}}$, where k is the momentum of the NGB, $\mathscr{A}_{\mathrm{fi}}$ is the scattering amplitude corresponding to the process $|i\rangle + \mathrm{NGB} \to |f\rangle$ and N_{fi}^μ are the non-pole contributions of the current element. If we then apply current conservation $\partial_\mu \langle f|J^\mu|i\rangle = 0$ we obtain a formula for the scattering amplitude in terms of the non-pole contributions N_{fi}^μ. From this formula it is trivial to see that the scattering amplitude vanishes in the soft limit $k \to 0$ unless N_{fi}^μ is also singular in the same limit.

However, in systems where a symmetry is explicitly broken by a chemical potential the conservation law $\partial_\mu \langle f|J^\mu|i\rangle = 0$ does not hold. Hence we realised that if we were ever going to be able to prove theorem 3 we needed a new conservation law. We were able to derive such a conservation law by viewing the chemical potential as a gauge field $A_\mu = \mu\delta_{\mu 0}$ and then requiring the variation of the action to be zero under infinitesimal gauge transformations. The new conservation law takes the form $D_\mu J_a^\mu = 0$.

Having obtained this new covariant conservation law we revisited the nonlinear $SO(3)$ model and tried to perform Weinberg's argument perturbatively. We found that we indeed could obtain a formula for the scattering amplitude if we used one of the two available conservation laws $D_\mu J_1^\mu = 0$ and $D_\mu J_2^\mu = 0$. However, from the formula it was not trivial to see that the amplitude vanished in the soft limit of the mNGB. To show that the amplitude vanished we also had to use the second conservation law. In principle this is not a problem, but it is hard to generalise such a procedure.

In order to get a better understanding of how we could generalise to other symmetry breaking patterns, we revisited the model invariant under $SU(2)_L \times SU(2)_R$ transformations. Based

on the previous example we realised that it may beneficial to combine the two Noether currents into a linear combination $J_\pm^\mu = \frac{1}{\sqrt{2}}(J_1^\mu \pm i J_2^\mu)$. This results in that we can incorporate the information contained in $D_\mu J_1^\mu$ and $D_\mu J_2^\mu$ separately into just one conservation law $D_\mu J_-^\mu = \partial_- J_-^\mu - 2i\mu J_-^0$. If we then proceed in the same manner as in the previous example we obtain a formula for the scattering amplitude expressed in terms of the non-pole contributions of $\langle f|J_-^\mu|i\rangle$. However, in this case we can see directly from the formula that the soft-limit on-shell amplitude vanishes. In addition, it turns out that this procedure is easier to generalise!

Having calculated two concrete examples we began reviewing [9] in order to explain some of our results so far. First of all the Cartan subalgebra tells us that each pair of explicitly broken generators correspond to one mNGB. Furthermore, each pair of explicitly broken generators can be combined into pairs $Q_{\pm\sigma}$ that satisfy the commutator $[Q, Q_{\pm\sigma}] = \pm q_\sigma Q_{\pm\sigma}$.[1] As a consequence of the commutator relation the mass of the mNGB is $q_\sigma \mu$. By using these well established facts, we have demonstrated that we can always construct pairs of currents $J_{\pm\sigma}^\mu$ that satisfy the conservation law $D_\mu J_{\pm\sigma}^\mu = \partial_\mu J_{\pm\sigma}^\mu \pm iq\mu J_{\pm\sigma}^0 = 0$. Such a construction results in that $J_{+\sigma}^\mu$ and $J_{-\sigma}^\mu$ destroys and creates mNGBs respectively. This allowed us to prove theorem 3 perturbatively, by acting with the covariant derivative D_μ on the current element $\langle f|J_{\pm\sigma}^\mu|i\rangle$. We were also able to prove the theorem in the non-perturbative picture. We did this by viewing the current element $\langle f|J_-^\mu|i\rangle$ as an amputated Green's function with one appended mNGB propagator plus non-pole contributions. By then acting with the covariant derivative on $\langle f|J_-^\mu|i\rangle$ and using the corresponding conservation law we successfully proved theorem 3 for the case of a non-composite field.

10.4   Outlook

Finally let us address some of the subjects of future work. First of all, we would like to investigate the complete breaking of the $O(3)$ symmetry further. As mentioned we could do this by considering $3 \to 3$ scattering, and check whether or not the corresponding scattering amplitude vanishes in the soft limit of the pNGB. If the scattering amplitude does not vanish in the soft limit of the pNGB, we have proved that the pNGBs indeed interact strongly.

In addition, we would also like to generalise our non-perturbative proof of theorem 3 to a composite field. Generalising in this way requires us to include more than just one pole-term from the Källen-Lehmann propagator in Eq. (9.38). Other than that the proof should be similar in both structure and logic.

Another interesting aspect of scattering amplitudes involving mNGBs that we have merely mentioned in this thesis is the existence of kinematic singularities. Both of our formulas in Eqs. (9.31) and (9.47) suggest that kinematic singularities may appear when the non-pole contributions N_-^μ of $\langle f|J_-^\mu|i\rangle$ are singular in the soft limit of the mNGB. If we could find a concrete example exhibiting such kinematic singularities, we may be able to find a deeper physical meaning for their existence.

Finally it would be interesting to investigate if there exists a classification of mNGBs that are similar to the classification for NGBs, as suggested in [18]. That is, perhaps the mNGBs also can be classified as type-I and type-II? A starting point for this discussion could be to solve Eq. (9.34) for the cases $G/F - \mu q \gg 1$ and $G/F - \mu q \ll 1$ in the limit of vanishing 3-momentum $\mathbf{p} \to 0$. This may result in that we can distinguish type-I and type-II mNGBs by the behaviour of the functions F and G in the limit $\mathbf{p} \to 0$.

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[1]We have not included the proves of these statements in this thesis, because they are well known.

Bibliography

[1] A. Nicolis, R. Penco, F. Piazza, and R. A. Rosen. More on gapped Goldstones at finite density: More gapped Goldstones. *JHEP*, 2013(11):55, 2013.

[2] M. F. Jakobsen. Soft-limit behaviour of the on-shell tree-level scattering amplitude for both massive and pseudo Goldstone bosons. *Norwegian University of Science and Technology*, Specialisation project, 2016.

[3] Y. Nambu. Quasi-particles and gauge invariance in the theory of superconductivity. *Phys. Rev.*, 117(3):648–663, 1960.

[4] J. Goldstone. Field theories with "superconductor" solutions. *Il Nuovo Cimento (1955-1965)*, 19(1):154–164, 1961.

[5] J. Goldstone, A. Salam, and S. Weinberg. Broken symmetries. *Phys. Rev.*, 127(3):965–970, 1962.

[6] G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble. Broken symmetries and the Goldstone theorem. *Adv. Part. Phys.*, 2:567–708, 1968.

[7] H.B. Nielsen and S. Chadha. On how to count Goldstone bosons. *Nuclear Physics B*, 105(3):445–453, 1976.

[8] A. Nicolis and F. Piazza. Implications of relativity on nonrelativistic Goldstone theorems: Gapped excitations at finite charge density. *Phys. Rev. Lett.*, 110(1):011602, 2013.

[9] H. Watanabe, T. Brauner, and H. Murayama. Massive Nambu-Goldstone bosons. *Phys. Rev. Lett.*, 111(2):021601, 2013.

[10] S. Weinberg. *The Quantum Theory of Fields (Vol. 2)*. Cambridge University Press, 1996.

[11] J. I. Kapusta. Bose-Einstein condensation, spontaneous symmetry breaking, and gauge theories. *Phys. Rev. D*, 24(2):426–439, 1981.

[12] J.O. Andersen. *Introduction to Statistical Mechanics*. Akademika forlag, 2012.

[13] A. Nicolis and F. Piazza. Spontaneous symmetry probing. *JHEP*, 6:25, 2012.

[14] V. P. Nair. *Quantum Field Theory - A Modern Perspective*. Springer, 2004.

[15] H. Watanabe and T. Brauner. On the number of Nambu-Goldstone bosons and its relation to charge densities. *Phys. Rev. D*, 84:125013, 2011.

[16] H. Leutwyler. Nonrelativistic effective lagrangians. *Phys. Rev. D*, 49:3033–3043, 1994.

[17] T. Brauner. Spontaneous symmetry breaking in the linear sigma model at finite chemical potential: One-loop corrections. *Phys. Rev. D*, 74:085010, 2006.

[18] T. Brauner. Goldstone bosons in presence of charge density. *Phys. Rev. D*, 75:105014, 2007.

[19] M.E. Peskin and D.V. Schroeder. *An Introduction to Quantum Field Theory*. Westview Press, 2016.

[20] S. Blundell and T. Lancaster. *Quantum Field Theory for the Gifted Amateur*. Oxford University Press, 2014.

[21] M. Srednicki. *Quantum Field Theory*. Cambridge University Press, 2007.

[22] T. Brauner. Spontaneous symmetry breaking and Nambu-Goldstone bosons in quantum many-body systems. *Symmetry*, 2:609–657, 2010.

Appendices

Field Theory in a Nutshell

A.1 The Euler-Lagrange Equations

The fields $\phi_a(x^\mu)$ are functions that map points in spacetime x^μ into numbers in field space. In quantum field theory particles are viewed as excitations in the field ϕ_a. The dynamics of the field ϕ_a is governed by the Euler-Lagrange equations which can be derived from Hamilton's principle. If we define the action S

$$S = \int_\Omega d^4x \mathscr{L}(\phi_a, \partial_\mu \phi_a) \tag{A.1}$$

where Ω is the spacetime volume where the field is nonzero, and $\mathscr{L}$ is the Lagrangian then Hamilton's principle states that

$$\frac{\delta S}{\delta \phi_a} = 0. \tag{A.2}$$

Thus if we consider variations of the fields $\delta\phi_a$ that vanish on the boundary $\partial\Omega$ we find that

$$
\begin{aligned}
0 = \delta S &= \frac{\delta S}{\delta \phi_a} \delta \phi_a \\
&= \int_\Omega d^4x \, \frac{\partial \mathscr{L}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \partial_\mu \delta \phi_a \\
&= \int_\Omega d^4x \, \frac{\partial \mathscr{L}}{\partial \phi_a} \delta \phi_a + \left[\delta \phi_a \left(\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \right) \right]_{\partial\Omega} - \int_\Omega d^4x \, \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \right) \delta \phi_a \\
&= \int_\Omega d^4x \left[\frac{\partial \mathscr{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \right] \delta \phi_a,
\end{aligned}
\tag{A.3}
$$

where we in the penultimate line performed a partial integration. Thus, since the variations $\delta\phi_a$ are arbitrary, the Euler-Lagrange equations are

$$\frac{\partial \mathscr{L}}{\partial \phi_a} = \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} \right). \tag{A.4}$$

A.2 Noether's Theorem

We say that a transformation leading to a variation $\delta\phi_a$ is a symmetry if the corresponding change in the Lagrangian is a surface term $\delta\mathscr{L} = \partial_\mu F^\mu$. From this we can derive Emmy Noether's famous theorem by considering what happens to the Lagrangian under a symmetry transformation

$$\delta\mathscr{L} = \partial_\mu F^\mu = \frac{\partial\mathscr{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu\delta\phi_a. \tag{A.5}$$

If we now use the Euler-Lagrange equation in the first term we get

$$\partial_\mu F^\mu = \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\right)\delta\phi_a + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu\delta\phi_a. \tag{A.6}$$

Hence by using the product rule

$$0 = \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a - F^\mu\right) \equiv \partial_\mu j^\mu \tag{A.7}$$

and we find that the conserved Noether current is

$$j^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a - F^\mu. \tag{A.8}$$

From the continuity equation (A.7) we can show that j^0 corresponds to a conserved charge density

$$0 = \partial_\mu j^\mu = \partial_0 j_0 - \nabla\cdot\vec{j}. \tag{A.9}$$

If we integrate both sides of this equation we get

$$\frac{d}{dt}\int_V d^3x\, j_0 = \int_V d^3x\, \nabla\cdot\vec{j} = \int_{\partial V} d\vec{S}\cdot\vec{j} \tag{A.10}$$

by applying the divergence theorem. Thus if there is no flux of $\vec{j}$ through the surface ∂V at infinity then

$$Q = \int_V d^3x\, j_0 \tag{A.11}$$

is a conserved charge.

A.3 Masses and Interactions

The Lagrangian $\mathscr{L}$ for n particles has the following general structure

$$\mathscr{L} = \mathscr{L}_{free}^{(1)} + \mathscr{L}_{free}^{(2)} + \cdots + \mathscr{L}_{free}^{(n)} + \mathscr{L}_I \tag{A.12}$$

The superscript $i = 1, \ldots, n$ labels each particle. The term $\mathscr{L}_{free}^i$ represents the free Lagrangian for the particle i. It is related to the propagator of the particle in the sense that if it is substituted into the Euler-Lagrange equation the free wave equation for the particle emerges. If we are

dealing with a Lorentz invariant theory, such that the terms $\mathscr{L}_{free}^{(i)}$ are not coupled and the field i has a zero vacuum expectation value one can also determine the mass from the free Lagrangian $\mathscr{L}_{free}^{(i)}$. To be specific we can write the free Lagrangian as the difference between a kinetic term $\mathscr{T}$ and a potential term $\mathscr{U}$ analogous to classical mechanics

$$\mathscr{L}_{free}^{(i)} = \mathscr{T}^i - \mathscr{U}^i. \tag{A.13}$$

If we do this, and the propagator is diagonal, we can calculate the mass of particle i by

$$m_i = \left.\frac{\partial^2 \mathscr{U}}{\partial \phi_i^2}\right|_{\phi=0}. \tag{A.14}$$

The term $\mathscr{L}_I$ contains all of the terms that describe how the different fields ϕ_i interact with each other. We can determine the vertex describing the interaction between the fields by calculating variational derivatives after having performed a Fourier transformation. Let us illustrate this by an example.

A.3.1   An Example of the Calculation of a Feynman Vertex

In this paper we frequently come across terms in the Lagrangian that depend on time-derivatives, spacetime-derivatives and no derivatives. To demonstrate the method for determining the Feynman vertices in the paper, we consider the example:

$$\mathscr{L} = \dot{\phi}_1 \partial_\mu \phi_2(x) \partial^\mu \phi_3(x) \phi_3(x). \tag{A.15}$$

The first step is to perform a Fourier transformation. In our convention the Fourier transformation of a field $\phi_i(x)$ is

$$\phi_i(x) = \int \frac{d^4k}{(2\pi)^4} \phi(k) e^{-i(E_i t - \mathbf{p}\mathbf{x})}. \tag{A.16}$$

In addition, the notation $\phi_i(k_j)$ means that the field ϕ_i has four momentum k_j. The field ϕ_i has the dispersion relation $E_i(\mathbf{k}_i)$. Performing a Fourier transformation gives,

$$\mathscr{L} = -i \int \frac{d^4k_1 d^4k_2 d^4k_3 d^4k_4}{(2\pi)^{16}} e^{-ix(k_1+k_2+k_3+k_4)} E_1(\mathbf{k}_1)(k_2 k_3) \left[\phi_1(k_1)\phi_2(k_2)\phi_3(k_3)\phi_3(k_4)\right].$$

For notational simplicity we introduce the notation

$$\int \frac{d^4k_1 d^4k_2 d^4k_3 d^4k_4}{(2\pi)^{16}} e^{-ix(k_1+k_2+k_3+k_4)} \equiv \int . \tag{A.17}$$

The vertex $V(p_1, p_2, p_3, p_4)$ with inflowing momenta can then be calculated by

$$iV(p_1, p_2, p_3, p_4) = \frac{i\delta^4 \mathscr{L}}{\delta\phi_1(p_1)\delta\phi_2(p_2)\delta\phi_3(p_3)\delta\phi_3(p_4)}. \tag{A.18}$$

Let us demonstrate this calculation:

$$
\begin{aligned}
iV(p_1, p_2, p_3, p_4) &= \int E_1(\mathbf{k}_1)(k_2 k_4) \frac{i\delta^4}{\delta\phi_1(p_1)\delta\phi_2(p_2)\delta\phi_3(p_3)\delta\phi_3(p_4)} \Big[\phi_1(k_1)\phi_2(k_2)\phi_3(k_3)\phi_3(k_4)\Big] \\
&= \int E_1(\mathbf{k}_1)(k_2 k_3) \frac{\delta^3}{\delta\phi_1(p_1)\delta\phi_2(p_2)\delta\phi_3(p_3)} \Big[\phi_1(k_1)\phi_2(k_2)\delta(k_3 - p_4)\phi_3(k_4) \\
&\qquad\qquad\qquad +\phi_1(k_1)\phi_2(k_2)\phi_3(k_3)\delta(k_4 - p_4)\Big] \\
&= E_1(\mathbf{k}_1)(k_2 k_3)\Big[\delta(k_1 - p_1)\delta(k_2 - p_2)\delta(k_3 - p_4)\delta(k_4 - p_3) \\
&\qquad\qquad +\delta(k_1 - p_1)\delta(k_2 - p_2)\delta(k_3 - p_3)\delta(k_4 - p_4)\Big] \\
&= E_1(\mathbf{p}_1)\Big[(p_2 p_4) + (p_2 p_3)\Big].
\end{aligned}
\tag{A.19}
$$

Appendix B

A Simple Proof of Goldstone's Theorem

We have on several occasions referred to Goldstone's theorem througout the text. Here we give one of the many possible proofs.

Proof. Noether's theorem guarantees that for each continuous symmetry there exists one conserved charge Q. In the Heisenberg picture, this means that Q commutes with the Hamiltonian $\mathscr{H}$ such that $[\mathscr{H}, Q] = 0$. Furthermore, we denote the ground state as $|0\rangle$ and subtract a possible vacuum energy such that $\mathscr{H}|0\rangle = 0$.

By acting on the ground state with an infinitesimal symmetry $U(\theta)$ we can write

$$U(\theta)|0\rangle = e^{i\theta Q}|0\rangle = (1 + i\theta Q)|0\rangle. \tag{B.1}$$

From this it is clear that if $Q|0\rangle \neq 0$ the system exhibits SSB. We can determine the energy of the state $Q|0\rangle$ by acting on it with the Hamiltonian, we find that

$$\mathscr{H}Q|0\rangle = (\mathscr{H}Q - Q\mathscr{H})|0\rangle = [\mathscr{H}, Q]|0\rangle = 0, \tag{B.2}$$

where we in the last step used that $[\mathscr{H}, Q] = 0$. Using Eq. (A.11) we can write $Q = \int_V d^3x\, j^0$. We now define the state $|\mathbf{k}\rangle = \int_V d^3x\, e^{i\mathbf{k}\mathbf{x}} j^0 |0\rangle$, which is an eigenstate of the momentum operator $\mathbf{P}$, such that $\mathbf{P}|\mathbf{k}\rangle = \mathbf{k}|\mathbf{p}\rangle$. Note that in the limit $\mathbf{k} \to 0$ the state $\mathbf{k}$ reduces to the state $Q|0\rangle$ which has zero energy. If our theory is Lorentz invariant the dispersion relation $E = \sqrt{\mathbf{p}^2 + m^2}$ tells us that the mass m of the state $|\mathbf{0}\rangle$ is zero. We have thus shown that for each broken continuous symmetry we can find a state with no mass. This state is the NGB. $\blacksquare$

Note that in systems without Lorentz invariance the mass of the state $|\mathbf{k}\rangle$ only vanishes if the dispersion relation is gapless. If there is a gap the state will be massive. Finally we mention that in systems without Lorentz invariance it is not true that each broken generator corresponds to exactly one Goldstone boson. This is what makes the counting of Goldstone bosons nontrivial in systems without Lorentz invariance.

The Complete Breaking of an $O(3)$ Symmetry

C.1 The Feynman Rules

The Feynman rules for the model considered in chapter 5 is slightly more complicated than the other cases. We therefore include their calculation in this appendix. For notational simplicity we once again absorb all differentials and exponentials into the integral symbol $\int$.

C.1.1 Third Order

The first term in Eq. (5.47) is

$$\mathscr{L}_1^{(3)} = \frac{\mu(g_2 - 3g_1 + 3g_3)}{4}\phi_2^2\dot{\phi}_3 \equiv A\phi_2^2\dot{\phi}_3. \tag{C.1}$$

By performing a Fourier transformation we get

$$\mathscr{L}_1^{(3)} = -iA\int \phi_3(k_1)\phi_2(k_2)\phi_2(k_3)E_3(k_1). \tag{C.2}$$

The corresponding Feynman rule is thus

$$
\begin{aligned}
iV(p_1, p_2, p_3) &= \frac{i\delta^3 \mathscr{L}_1^{(3)}}{\delta\phi_2(p_1)\delta\phi_2(p_2)\delta\phi_3(p_3)} \\
&= A\int \Big[\delta(k_1 - p_3)\delta(k_2 - p_2)\delta(k_3 - p_1) + \delta(k_1 - p_3)\delta(k_2 - p_1)\delta(k_3 - p_2)\Big]E_3(k_1) \\
&= 2AE_3(p_3) \\
&= \frac{\mu(g_2 - 3(g_1 - g_3))}{2}E_3(p_3).
\end{aligned}
\tag{C.3}
$$

The second term in Eq. (5.47) is

$$
\begin{aligned}
\mathscr{L}_2^3 &= \frac{\mu^2(g_1 - g_2)}{2}\phi_1\phi_2\phi_3 - \frac{(g_1 - g_2)}{2}\phi_3(\partial_\mu\phi_1\partial^\mu\phi_2) \\
&\quad + \frac{(g_1 - g_3)}{2}\phi_2(\partial_\mu\phi_1\partial^\mu\phi_3) - \frac{(g_2 - g_3)}{2}\phi_1(\partial_\mu\phi_2\partial^\mu\phi_3).
\end{aligned}
\tag{C.4}
$$

By performing a Fourier transformation we get

$$
\begin{aligned}
\mathscr{L}_2^3 = \int & \frac{\mu^2(g_1 - g_2)}{2} \phi_1(k_1)\phi_2(k_2)\phi_3(k_3) \\
& + \frac{(g_1 - g_2)}{2} \phi_1(k_1)\phi_2(k_2)\phi_3(k_3)k_{1\mu}k_2^\mu \\
& + (\frac{g_3 - g_1}{2})\phi_1(k_1)\phi_2(k_2)\phi_3(k_3)k_{1\mu}k_3^\mu \\
& + (\frac{g_2 - g_3}{2})\phi_1(k_1)\phi_2(k_2)\phi_3(k_3)k_{2\mu}k_3^\mu \\
\equiv \int & \phi_1(k_1)\phi_2(k_2)\phi_3(k_3)\Big[A + B(k_1 k_2) + C(k_1 k_3) + D(k_2 k_3)\Big]
\end{aligned}
\tag{C.5}
$$

The corresponding Feynman rule is thus

$$
\begin{aligned}
iV(p_1, p_2, p_3) &= \frac{i\delta^3 \mathscr{L}_2^{(3)}}{\delta\phi_1(p_1)\delta\phi_2(p_2)\delta\phi_3(p_3)} \\
&= i\int \Big[A + B(k_1 k_2) + C(k_1 k_3) + D(k_2 k_3)\Big]\delta(k_1 - p_1)\delta(k_2 - p_2)\delta(k_3 - p_3)) \\
&= iA + iB(p_1 p_2) + iC(p_1 p_3) + iD(p_2 p_3) \\
&= \frac{i\mu^2(g_1 - g_2)}{2} + \frac{i(g_1 - g_2)}{2}(p_1 p_2) + \frac{i(g_3 - g_1)}{2}(p_1 p_3) + \frac{i(g_2 - g_3)}{2}(p_2 p_3).
\end{aligned}
\tag{C.6}
$$

The third term in Eq. (5.47) is

$$
\mathscr{L}_2^3 = \frac{\mu(g_1 - 3g_2 + 3g_3)}{4}\phi_1^2\dot{\phi}_3 \equiv A\phi_1^2\dot{\phi}_3.
\tag{C.7}
$$

By performing a Fourier transformation we get

$$
\mathscr{L}_2^3 = \int -iE_3(k_3)A\phi_1(k_1)\phi_1(k_2)\phi_3(k_3)
\tag{C.8}
$$

The corresponding Feynman rule is thus

$$
\begin{aligned}
iV(p_1, p_2, p_3) &= \frac{i\delta^3 \mathscr{L}_2^{(3)}}{\delta\phi_2(p_1)\delta\phi_2(p_2)\delta\phi_3(p_3)} \\
&= 2AE_3(p_3) \\
&= \frac{\mu(g_1 - 3g_2 + 3g_3)}{2}E_3(p_3).
\end{aligned}
\tag{C.9}
$$

This concludes the Feynman rules at third order.

C.1.2 Fourth Order

The first term in Eq. (5.48) is

$$
\begin{aligned}
\mathscr{L}_1^{(4)} = \mu^2(&\frac{g_2}{8} - \frac{g_1}{6} + \frac{g_3}{24})\phi_2^2\phi_3^2 + (\frac{g_1}{8} - \frac{g_2}{6})(\partial_\mu\phi_2)^2\phi_3^2 \\
& + (\frac{g_2}{6} - \frac{g_1}{4} + \frac{g_3}{6})\phi_2(\partial_\mu\phi_2\partial^\mu\phi_3)\phi_3 + (\frac{g_1}{8} - \frac{g_3}{6})\phi_2^2(\partial_\mu\phi_3)^2
\end{aligned}
\tag{C.10}
$$

By performing a Fourier transformation we get

$$
\begin{aligned}
\mathscr{L}_2^4 = \int \mu^2 & (\frac{g_2}{8} - \frac{g_1}{6} + \frac{g_3}{24})\phi_2(k_1)\phi_2(k_2)\phi_3(k_3)\phi_3(k_4) \\
& - (\frac{g_1}{8} - \frac{g_2}{6})\phi_2(k_1)\phi_2(k_2)\phi_3(k_3)\phi_3(k_4)k_1k_2 \\
& - (\frac{g_2}{6} - \frac{g_1}{4} + \frac{g_3}{6})\phi_2(k_1)\phi_2(k_2)\phi_3(k_3)\phi_3(k_4)k_2k_3 \\
& - (\frac{g_1}{8} - \frac{g_3}{6})\phi_2(k_1)\phi_2(k_2)\phi_3(k_3)\phi_3(k_4)k_3k_4 \\
\equiv \int & \phi_2(k_1)\phi_2(k_2)\phi_3(k_3)\phi_3(k_4)\Big[A + B(k_1k_2) + C(k_2k_3) + D(k_3k_4)\Big]
\end{aligned}
\tag{C.11}
$$

The corresponding Feynman rule is thus

$$
\begin{aligned}
iV(p_1,p_2,p_3,p_4) &= \frac{i\delta^4 \mathscr{L}_1^{(4)}}{\delta\phi_2(p_1)\delta\phi_2(p_2)\delta\phi_3(p_3)\delta\phi_3(p_4)} \\
&= 4A + 4Bp_1p_2 + C(p_1p_4 + p_2p_4 + p_1p_3 + p_2p_3) + 4Dp_3p_4
\end{aligned}
\tag{C.12}
$$

We can rewrite

$$
p_1p_4 + p_2p_4 + p_1p_3 + p_2p_3 = (p_1 + p_2)(p_3 + p_4) = -(p_1 + p_2)^2
\tag{C.13}
$$

by using momentum conservation $p_1 + p_2 + p_3 + p_4 = 0$. Hence the Feynman rule takes the form

$$
\begin{aligned}
iV(p_1,p_2,p_3,p_4) = i\mu^2(\frac{g_2}{2} - \frac{2g_1}{3} + \frac{g_3}{6}) + i(\frac{2g_2}{3} - \frac{g_1}{2})p_1p_2 \\
- i(\frac{g_1}{4} - \frac{g_2}{6} - \frac{g_3}{6})(p_1 + p_2)^2 + i(\frac{2g_3}{3} - \frac{g_1}{2})p_3p_4.
\end{aligned}
\tag{C.14}
$$

The second term in Eq. (5.48) is

$$
\begin{aligned}
\mathscr{L}_2^{(4)} = \mu^2 & (\frac{g_1}{8} - \frac{g_2}{6} + \frac{g_3}{24})\phi_1^2\phi_3^2 + (\frac{g_2}{8} - \frac{g_1}{6})\phi_3^2(\partial_\mu\phi_1)^2 \\
& + (\frac{g_1}{6} - \frac{g_2}{4} + \frac{g_3}{6})\phi_1(\partial_\mu\phi_1\partial^\mu\phi_3)\phi_3 + (\frac{g_2}{8} - \frac{g_3}{6})\phi_1^2(\partial_\mu\phi_3)^2 \\
\equiv A\phi_1^2\phi_3^2 & + B\phi_3^2(\partial_\mu\phi_1)^2 + C\phi_1(\partial_\mu\phi_1\partial^\mu\phi_3)\phi_3 + D\phi_1^2(\partial_\mu\phi_3)^2.
\end{aligned}
\tag{C.15}
$$

By performing a Fourier transformation we get

$$
\mathscr{L}_2^{(4)} = \int \phi_1(k_1)\phi_1(k_2)\phi_3(k_3)\phi_3(k_4)\Big[A - B(k_1k_2) - C(k_2k_3) - D(k_3k_4)\Big]
\tag{C.16}
$$

The corresponding Feynman rule is thus

$$
\begin{aligned}
iV(p_1,p_2,p_3,p_4) &= \frac{i\delta^4 \mathscr{L}_2^{(4)}}{\delta\phi_1(p_1)\delta\phi_1(p_2)\delta\phi_3(p_3)\delta\phi_3(p_4)} \\
&= 4A - 4B(p_1p_2) - C(p_1p_4 + p_1p_3 + p_2p_4 + p_2p_3) - 4D(p_3p_4) \\
&= i\left[\mu^2\left(\frac{g_1}{2} - \frac{2g_2}{3} + \frac{g_3}{6}\right) + \left(\frac{2g_1}{3} - \frac{g_2}{2}\right)p_1p_2 \right. \\
&\quad \left. + \left(\frac{g_1}{6} - \frac{g_2}{4} + \frac{g_3}{6}\right)(p_1 + p_2)^2 + \left(\frac{2g_3}{3} - \frac{g_2}{2}\right)p_3p_4\right].
\end{aligned}
\tag{C.17}
$$

The second term in Eq. (5.48) is

$$\mathscr{L}_3^{(4)} = \mu(\frac{g_1}{3} - \frac{g_2}{4} - \frac{g_3}{24})\dot{\phi}_1\phi_2\phi_3^2 + \mu(\frac{g_1}{4} - \frac{g_2}{3} + \frac{g_3}{24})\phi_1\dot{\phi}_2\phi_3^2$$
$$+ 5\mu(\frac{g_2 - g_1}{12})\phi_1\phi_2\dot{\phi}_3\phi_3 \qquad\qquad \text{(C.18)}$$
$$\equiv A\dot{\phi}_1\phi_2\phi_3^2 + B\phi_1\dot{\phi}_2\phi_3^2 + C\phi_1\phi_2\dot{\phi}_3\phi_3.$$

By performing a Fourier transformation we get

$$\mathscr{L}_3^{(4)} = -i\int (AE_1(k_1) + BE_2(k_2) + CE_1(k_3))\phi_1(k_1)\phi_2(k_2)\phi_3(k_3)\phi_3(k_4). \qquad \text{(C.19)}$$

The corresponding Feynman rule is thus

$$iV(p_1, p_2, p_3, p_4) = \frac{i\delta^4 \mathscr{L}_3^{(4)}}{\delta\phi_1(p_1)\delta\phi_2(p_2)\delta\phi_3(p_3)\delta\phi_3(p_4)}$$
$$= \Big\{ 2AE_1(p_1) + 2BE_2(p_2) + C\big[E_3(p_3) + E_3(p_4)\big] \Big\}. \qquad \text{(C.20)}$$

Using that $E_3(p_3) + E_3(p_4) = -E_1(p_1) - E_2(p_2)$ we get

$$iV(p_1, p_2, p_3, p_4) = \mu E_1(p_1)\left[\frac{13g_1 - 11g_2 - g_3}{12}\right] + \mu E_2(p_2)\left[\frac{11g_1 - 13g_2 + g_3}{12}\right] \qquad \text{(C.21)}$$