Viscous Phenomena and Entropy Production in the Early Universe

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Abstract

This thesis looks at some phenomena in the early universe — the stage of the Universe from when it was populated by all the particles in the Standard Model, until the last positrons disappeared. More specifically from when the Universe was $10^{-12}$ seconds old until a few minutes after the big bang.

The first paper addresses the degrees of freedom related to the elementary particles, and show the evolution of these as the universe expands and cools. As the temperature decreases the particles will go from relativistic velocities to semi- and non-relativistic velocities, before finally disappearing. The temperature at which this happens depends on the particles masses. One important difference between relativistic and non-relativistic particles is that they cool at different rates ($T \propto a^{-1}$ vs. $T \propto a^{-2}$). If we have a mixture of both types a bulk viscous effect will arise, resulting a heat transfer between the two components. My second and third paper discuss these phenomena.

Bulk viscosity and entropy production are at their highest at the end of the lepton era, just before the neutrinos decouple at $T = 10^{10}$ K. At this time the neutrinos have a very long mean free path, resulting in large momentum transfers and heat exchange. Many previous works have concentrated their work on just the lepton era ($T = 10^{12}$ K $\rightarrow$ $10^{10}$ K), a time where most of the Universe consisted of electrons, positrons, neutrinos, and photons. At higher temperatures, hadrons and eventually quarks and gluons make up a significant contribution to the particle soup. I have made a model universe where all particles except the leptons and photon are excluded. I can thus include the heavier cousins of the electron — the muon and tau. By doing this, we get a more qualitative picture of what happens as particle species goes from relativistic velocities to semi- and non-relativistic temperatures and finally disappears one by one.
Preface

This thesis is submitted to the Norwegian University of Science and Technology (NTNU) as a partial fulfillment of the requirements for the degree of Philosophiae Doctor. It is the result of six years of research at the Department of Physics at NTNU under the supervision of professors Kåre Olaussen and Iver H. Brevik.

The first part of my thesis gives a short introduction to the field of cosmology and more importantly to the subjects of viscosity and entropy in the early Universe. This part is also meant to motivate and elucidate my three papers which make up the second part of this thesis. I have tried to make this introduction fairly basic, making it accessible to people new to the field or from a related field.

Lars Husdal
Trondheim, March 2017
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First, I would like to thank professor Kåre Olaussen for accepting me as his student, and your guidance and fruitful discussions during the majority of my stay here at NTNU. Secondly, thanks to professors Iver Brevik and Jens O. Andersen for helping me in my final stages of my Ph.D., their help is highly appreciated, and really accelerated my publication rate. I would also thank professor Jan Myrheim for his help.

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List of Papers

Paper I
Lars Husdal:
*On Effective Degrees of Freedom in the Early Universe*,
Galaxies 4 (2016) no. 78 [1].

Paper II
Lars Husdal:
*Viscosity in a Lepton-Photon Universe*,
Astrophysics and Space Science 361 (2016) no. 8 [2].

Paper III
Lars Husdal and Iver Brevik:
*Entropy in a Lepton-Photon Universe*,
Astrophysics and Space Science 362 (2017) no. 2 [3].
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Part I

Thesis Part
1 Introduction

In 1964, the two radio astronomers Arno Penzias and Robert Wilson accidentally discovered the cosmic microwave background (CMB) radiation — a clear evidence that the Universe began as a hot dense ball which has been expanding ever since. Further studies of the CMB have shown us that the early universe was in a state of almost perfect thermal equilibrium, being both isotropic and homogeneous. This is what we normally would say was a state of maximum entropy. However, as we clearly know, that was not the case, and the total entropy is continuing to increase as time goes by (the arrow of time is by many physicists related to the increase of entropy. So how could a state which is in perfect thermal equilibrium still increase its entropy? One reason has to do with the fact that a non-relativistic gas cools down at a different rate than a relativistic gas, and if we have a mixture of the two phases, entropy will increase. During the early universe era, all the massive particles of the Standard Model went from being relativistic to non-relativistic to essentially disappearing completely. One would thus assume that an increase in the total entropy would occur. Before heading into the physics, a short history of the Universe from the Big Bang to today is in order.

1.1 Chronology of the Universe

According to the latest results from the Planck satellite and other experiments, the age of the universe is $13.799 \pm 0.021$ billion years old [4]. Throughout this time its overall characteristics have changed significantly. There are several ways to categorize the different phases, and it all comes down to context. We will later see that one option is to distinguish them according to how the geometry of the Universe evolves due to the dominating energy contributor. This gives us radiation, matter, and dark energy dominated eras, we being in the latter era now. An illustration of these eras are given later in Figure 4.1. This is important when it comes to the geometry of the Universe as a whole.

If we are more interested in what is going on the smaller scales, a better way would be to divide the evolution into three main different phases, namely the very early universe, the early universe, and structure formation (and if we want, we can add another phase for the future and fate of the Universe). All three phases have distinct characteristics.

The first phase began at the earliest time we can imagine, namely the Big Bang. During this period the four forces we know today separated out one by one from what
we think was one unified force. The other main point here is that the theories we have
for this period are, to some extent, speculative. By that I mean that temperatures and
energies during this phase were higher than what we can produce in any accelerators
today — it involves physics beyond the Standard Model.

The next phase is the early universe and begins at temperatures “low” enough to
be recreated in experiments here on Earth (e.g. the LHC). We are now talking about
Standard Model physics. During this phase, the Universe went from being populated
by all the known particles in the Standard Model of particle physics, to essentially
being dominated by the photons and neutrinos.

The third and current phase is that of structure formation. This phase started with
the photon decoupling which made the Universe transparent and is the theoretical
limit for how far back in time we can see using observational astronomy. During this
period matter cooled down and started to clump together, thus starting to form stars,
galaxies, and other structures — hence the name.

Each of these three phases consists of shorter periods, which we call eras, or epochs
and deserves a closer look. For the early universe, I will distinguish these eras by
number contribution (other sources might use mass domination).

1.1.1 The very early universe

Planck era ($0 \text{ s} \rightarrow 10^{-43} \text{ s}$). According to the classical Big Bang theory, the
Universe began as a singularity with infinite density and temperature. Quantum
effects, however, are not taken into account. Our understanding of quantum theory
only makes sense in a certain range, that below the Planck scale. For time this is
called the Planck time and is defined as $\sqrt{\hbar G/c^5} = 5.39 \times 10^{-44} \text{ s}$
Everything before this is what we call the Planck era. Although we know very little about this era, we
believe that the gravitational force will be as strong as the other fundamental forces
and they will behave as one unified force. A theory describing this era should unite
quantum field theory (QFT) and general relativity (GR). Until such a theory comes
about, we are not able to make any predictions about what was happening in that
era.

The first step towards such a theory is to find a quantum description of gravity.
General Relativity is formulated using classical physics, while the other three forces
are formulated using quantum mechanics. Coupling together a classical and a quan-
tum mechanical system might lead to trouble, which is the case here [5]. We say that
it is not renormalizable. One popular theory for this quantization is loop quantum
gravity. If a theory also unifies gravity with the other three forces it is also a so-called
theory of everything (ToE). String theory, for example, is one such theory.

Grand unification era ($10^{-43} \text{ s} \rightarrow 10^{-36} \text{ s}$). As the Universe cooled down, gravity
splits out as a separate force, while the other three forces: the strong, weak and
electromagnetic forces (collectively called the gauge forces) are still united as one
1.1 Chronology of the Universe

Theories trying to explain and unify the strong and electroweak forces are called Grand Unified Theories. There are many grand unification models. The simplest of these uses SU(5) and was proposed by Howard Georgi and Sheldon Glashow in 1974 [6]. Common for all models is the inclusion of some heavy bosons with masses around $10^{16}$ GeV/$c^2$. These are like analogies to the W and Z particles, but couples quarks to leptons. In the Georgi-Glashow model, there are 12 of these, named X and Y bosons, respectively. When the temperature decreases below this at around $10^{-36}$ s it will result in a symmetry breaking splitting the strong force from the electroweak force. In some theories baryogenesis is caused by the decay of these heavy bosons (X, Y, or something equivalent), as they may violate baryon number [7].

**Inflationary era ($10^{-36}$ s $\rightarrow$ $10^{-32}$ s).** After the grand unification era, the Universe is thought to have gone through a phase of rapid exponential expansion called inflation. The linear size of the Universe is thought to have increased by a factor of at least $10^{26}$ and volume by $10^{78}$ [8]. The actual mechanism behind inflation is speculative but is thought to have started around the time of the GUT transition by a scalar field called the inflaton (field). This field is thought to be quite similar to dark energy or the Higgs field, but involving much higher energies.

The idea of an early inflationary epoch was proposed by a number of physicists around 1980. Among them was Starobinsky who looked at non-singular cosmological models, and saw the importance of quantum corrections to Einsteins Field Equations [9, 10]. Guth is often credited for seeing the importance of inflation, and in his original theory, the inflaton field would be in what we call a false vacuum — a kind of positive energy (as opposed to the negative energy of gravity), which according to general relativity would accelerate the expansion of space at an exponential rate [11]. More modern versions (e.g. Linde [12]) of inflation have abandoned the false vacuum part, but rather assuming that the inflaton field starts at a high energy state, from which it slow rolls down a potential well [13]. While the value of the inflaton field was dropping very slowly, the particle content in the Universe would cool down and dilute rapidly. As the potential start to drop more quickly, the inflation process stops and a reheating process starts. In this process the potential energy which is released results in the creation of particles, in a process we call reheating. In practice, everything in our observable universe results from this process. This era is thought to have ended at around $10^{-32}$ seconds.

Inflation theory has a lot of supporters as it solves several big questions in cosmology, the three biggest ones being the flatness problem, the horizon problem, and the magnetic monopole problem. *The flatness problem* has to do with the curvature of the Universe being so small. Without inflation, this would require the energy density in the Universe to be very fine-tuned to a special value (the critical density). If there was an inflationary era then the Universe we observe today is just a very small fraction of a bigger universe, which might very well be curved. *The horizon problem* has to do with why the horizon on the opposite sides of the Universe (relative to an
1 Introduction

observer) have the same temperature. Classical theory says that they should never have been in contact with each other (causally separated). If the whole Universe went through an inflationary period, the whole observable universe was all very close and causally connected before inflation separated them with superluminal speeds. The last problem has to do with magnetic monopoles, which are a type of hypothetical particles with magnetic charge (just a north or south pole). These particles would make the connection between electricity and magnetism more symmetric. No magnetic monopoles have ever been observed, but they might be too massive to be produced from the potential energy of the inflaton field. In such case, the reason we haven’t seen any magnetic monopoles isn’t necessary because they don’t exist, but because they are so diluted we just haven’t seen them yet.

Cosmic inflation is the simplest theory which solves all the aforementioned problems. According to the theory the structure we see in the Universe today originate from quantum fluctuations in the inflaton field which would grow to macroscopic sizes during the exponential growth. The effects of the primordial fluctuations were studied by among others: Mukhanov and Chibisov [14, 15], and Hawking [16]

**Electroweak era** ($10^{-32} \, \text{s} \rightarrow 10^{-11} \, \text{s}$). After the symmetry breaking at the GUT scale, the strong force separated out from the what we call the electroweak force. The Universe is filled with hot quark-gluon plasma, leptons, and the gauge bosons of the electromagnetic and weak forces. In the electroweak era, these bosons were all massless and named the $W_1$, $W_2$, $W_3$, and $B$ bosons. The electroweak era lasted quite long (if we look at it from a logarithmic point of view). Then at around 100 GeV, when the Universe was around $10^{-11} \, \text{s}$ old, the Higgs mechanism caused a spontaneous symmetry breaking of the electroweak force. The four aforementioned bosons split into the three massive $W^+$, $W^-$, and $Z^0$ particles; and the massless photon.

The electroweak energy scale of 100 GeV has been possible to recreate here on Earth for some decades now. The electroweak theory is thus well understood and verified by experiments. However, there is a big gap between the highest energies we can produce in accelerators today (13 TeV at the LHC) and the $10^{10} \, \text{TeV}$ at the end of inflation. Most of the electroweak era is still uncharted territory.

1.1.2 The early universe

**Quark era** ($10^{-11} \, \text{s} \rightarrow 10^{-5} \, \text{s}$). At this time all the four fundamental forces have settled down to their current form. The temperature is not high enough to create the heavier particles like $W$, $Z$, $H$, and $t$, so these will quickly annihilate. We are thus left with (the remaining five) quarks, gluons, leptons, and photons. The temperature is too high for quarks to form hadrons in the form of baryons and mesons. Instead, they are in a state together with gluons which we call a quark-gluon plasma, where they act as free elementary particles.
1.1 Chronology of the Universe

![Figure 1.1](image)

**Figure 1.1:** Relative number densities of the different particle types from $k_B T = 10^6$ to $10^{-2}$ MeV. We can categorize the different eras of the Universe according to the dominant particle species. From left to right (high to low temperature) we have the quark, hadron, lepton, and photon eras. From a number density (or energy density) point of view, the hadron era is very short.
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**Hadron era ($10^{-5}$ s → $10^{-4}$ s).** Somewhere between 150 and 300 MeV, there is an important phase transition. The previously freely roaming quarks and gluons will clump together to form color-neutral hadronic particles like mesons and baryons (see Chapter 2 on particle physics). By number density or (kinetic) energy density (which does not include rest mass) this period does not last very long. The low temperature only allows for a small production of the heavier baryons like protons and neutrons. The lighter pions exist for a while longer. Eventually, all the anti-hadrons will annihilate and we are left with only a very small portion of hadrons.

**Lepton era ($10^{-4}$ s → 100 s).** When the temperature dropped to around 100 MeV the number and energy density of the Universe became dominated by the leptons (neutrinos, electrons, positrons and also some muons). At this time neutrinos were coupled with the charged leptons through the weak interaction and remained in equilibrium. At around $T = 1$ MeV the mean free path of neutrinos became greater than the Hubble distance (the distance to something receding by the speed of light and is defined as $cH^{-1}$), and decoupled directly from the electron-positrons and indirectly from the photons. From here on the neutrinos and the photon-coupled particles cool down independently from each other. Shortly after the neutrino decoupling the last numerous massive particles — the electrons and positrons will annihilate. The photons will be heated by this process, while the neutrinos won’t. Today the cosmic microwave background photons have a temperature of 2.73 K, while the cosmic microwave neutrinos have (according to theory) a temperature of 1.95 K. At around 100 keV most of the electrons and positrons have annihilated and essentially all the energy in the Universe is contained in the photons and neutrinos).

**Photon era (100 s → 380 000 y).** As the photon era begins, the antimatter particles are all but extinct. The excess matter particles (electrons, protons, and neutrons) are outnumbered by the photons by roughly one to one billion. As we enter the photon era, the protons and neutrons start to clump together to form elements such as deuterium, helium, beryllium, and lithium. This happens gradually, but a well-used definition is 3 minutes after the Big Bang [17]. At this time the amount of helium-4 and deuterium was about one ten-thousandth that of the proton [8]. This is the Big Bang nucleosynthesis (BBN) (sub)era, and last until about 20 minutes after the Big Bang (see Figure B.1.) For thousands of years the Universe continued to dilute and accol. Eventually, after being radiation dominated since its formation the Universe became matter dominated, 47 000 years after the Big Bang [18].

During the photon era, the Universe is in a plasma phase, with free electrons and nuclei. Since photons interact strongly with the ionized particles the Universe was opaque. Eventually, the temperature becomes low enough for electrons and nuclei to form atoms. This period is called recombination, which is kind of a bad expression, as this suggests that they have previous been combined. The recombination happened quite rapidly, first with most ionized isotopes of beryllium, lithium, and helium,
and then most importantly with hydrogen at around 380,000 years after the Big Bang. Photons could now no longer interact with the neutral atoms, causing them to decouple from matter. Light would then travel unhindered and is what we today observe as the cosmic microwave background (CMB).

### 1.1.3 Structure formation

![Figure 1.2: The cosmic microwave background gives an imprint of the Universe at the time of photon decoupling at around 380,000 years after the Big Bang (redshift \( z = 1090 \) [19]). The fluctuations of around 1 part per 100,000 is the start of structure formations. Photo: ESA.](image)

**Dark ages (380,000 y → 400 million y).** After recombination and photon decoupling, matter is quite evenly distributed with only small fluctuations in density. These small fluctuations will eventually grow to form clusters and galaxies, and voids. There are few new sources of light, hence the name the dark ages. Future studies of the 21 cm hydrogen line might shed some more light on this relatively unknown era.

**Reionization (400 million y → 1 billion y).** As the matter clumps together they will form stars and quasars. These events will emit large amounts of radiation, which will reionize the Universe. The exact time when reionization starts is still a bit unclear, but the oldest galaxies we have observed dates back to roughly 400 million years after the Big Bang. Hopefully, future telescopes like the James Webb Space Telescope, scheduled to be launched in 2018 will give us more answers. The whole Universe is thought to be reionized at about 1 billion years after the Big Bang [20]. Because of the low density of the electrons and baryons, the interaction rate between them and the photons are so low that the opaqueness just barely increased and Universe remains transparent.
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Current era (1 billion y → 13.799 billion y). What happens after reionization is quite well understood. The Universe will continue to clump together and expand, leading to decreasing matter and radiation densities. The discovery of dark energy in 1998 [21, 22] gave us a new ingredient to include. Assuming this is a cosmological constant, its energy density is constant. This means that the Universe became dark energy dominated at around 9.8 billion years after the Big Bang [18].

1.2 Cosmological parameters

Today the ΛCDM (Lambda Cold Dark Matter) model is referred to as the Standard Model of Big Bang cosmology, as is the simplest model which reasonably describes the cosmos (e.g. the existence and structure of the CMB, the large-scale structure and distribution of galaxies, and the abundances of light elements through primordial nucleosynthesis). It can also be extended to include inflation. The ΛCDM is based on six independent (primary) parameters as shown in Table 1.1. Together with a few fixed parameters we also get some of the more famous calculated parameters shown in the same table.
## 1.2 Cosmological parameters

Table 1.1: Cosmological parameters according to the Planck 2015 results with the $TT, TE, EE + \text{lowP} + \text{lensing} + \text{ext}$ parameters with 68% confidence limits \cite{19}. $\Omega$ is the density compared to critical density, $h$ is the reduced Hubble constant, defined as $H_0/(100 \text{ km s}^{-1} \text{ Mpc}^{-1})$. What is listed as the sound horizon at last scattering is actually the Monte-Carlo calculated angular size of the sound horizon (of BAO) multiplied by a hundred. The reionization optical depth tells us about the opacity at the time of reionization. Perturbation amplitude tells us about the fluctuations in density in the early universe. The scalar spectral index tells us how the density fluctuations vary with scale.

<table>
<thead>
<tr>
<th>Description</th>
<th>Symbol:</th>
<th>Value:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Primary</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Physical baryon density</td>
<td>$\Omega_b h^2$</td>
<td>0.02230 ± 0.00014</td>
</tr>
<tr>
<td>Physical cold dark matter density</td>
<td>$\Omega_c h^2$</td>
<td>0.1188 ± 0.0010</td>
</tr>
<tr>
<td>Sound horizon at last scattering</td>
<td>$100\theta_{\text{MC}}$</td>
<td>1.04093 ± 0.00030</td>
</tr>
<tr>
<td>Reionization optical depth</td>
<td>$\tau$</td>
<td>0.066 ± 0.012</td>
</tr>
<tr>
<td>Perturbation amplitude</td>
<td>$\ln(10^{10} A_s)$</td>
<td>3.064 ± 0.023</td>
</tr>
<tr>
<td>Scalar spectral index</td>
<td>$n_s$</td>
<td>0.9667 ± 0.0040</td>
</tr>
<tr>
<td><strong>Calculated</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hubble constant</td>
<td>$H_0$</td>
<td>67.74 ± 0.46</td>
</tr>
<tr>
<td>Cosmological constant density</td>
<td>$\Omega_\Lambda$</td>
<td>0.6911 ± 0.0062</td>
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<tr>
<td>Matter density</td>
<td>$\Omega_m$</td>
<td>0.3089 ± 0.0062</td>
</tr>
<tr>
<td>Age of the Universe / Gyr</td>
<td>$t_0$</td>
<td>13.799 ± 0.021</td>
</tr>
<tr>
<td>Redshift at decoupling</td>
<td>$z_\text{a}$</td>
<td>1089.90 ± 0.23</td>
</tr>
</tbody>
</table>

1 $TT$: temperature power spectrum, $TE$: temperature-polarization cross spectrum, $EE$: polarization power spectrum, $\text{lowP}$: Planck polarization data in the low-$\ell$ likelihood, $\text{lensing}$: CMB lensing reconstruction, $\text{ext}$: External data from Baryon acoustic oscillations (BAO), Joint Light-curve Analysis (JLA), and the Hubble constant.
2 Particle Physics Summary

2.1 Standard Model of elementary particles

The Standard Model of particle physics is one of the most successful theories in physics and explains the existence and composition of all the known particles. Figure 2.1 shows the most familiar representation of the Standard Model of elementary particles, where the particles are divided into four categories: the quarks, the leptons, the force-carrying gauge bosons, and finally the Higgs boson. The spin of elementary particles comes in units of the reduced Planck constant, $\hbar$. Particles with half-integer spin ($\frac{1}{2}, \frac{3}{2}, \ldots$) are called fermions, while bosons have an integer spin number (0, 1, 2, ...). All matter particles (quarks and leptons) have spin-$\frac{1}{2}$, the gauge bosons have spin-$1$ and the Higgs boson has spin-$0$. Gravitation is not part of the Standard Model. However, most physicists believe that gravity is mediated by a massless particle called the graviton. This graviton should connect to what is called the stress-energy tensor. This is a second order tensor, i.e. a $4 \times 4$ matrix, and therefore the graviton must have spin-2. Fermions with the same quantum numbers cannot occupy the space and follow Fermi-Dirac statistics. Bosons, on the other hand, can occupy the same state. They follow Bose-Einstein statistics.

Quarks come in six flavors. We can further divide these into three generations, with each next generation being more massive, but otherwise possessing the same properties. Only the first generation particles are stable. The six quarks ($q$) are: up ($u$) and down ($d$), charm ($c$) and strange ($s$), and finally top ($t$) and bottom ($b$). The quarks also have their own antiparticle ($\bar{q}$) with opposite electric charge ($\bar{u}, \bar{d}, \bar{c}, \bar{s}, \bar{t}, \bar{b}$). Quarks have charge $\frac{2}{3}$ and $\frac{1}{3}$ (particles) and $-\frac{2}{3}$ and $+\frac{1}{3}$ (antiparticles). Quarks interact through all the four forces: the strong, electromagnetic, weak, and gravitational. Similar to electric charge, the strongly interacting quarks have color charge. There are three colors for particles and three colors for antiparticles, namely: red, green, and blue; and antired, antigreen and antiblue. Quarks are bounded by color confinement and can never be directly observed in isolation. They need to form color-neutral particles. These can be combinations of three colored quarks (rgb), or three anticolored quarks ($\bar{r}\bar{g}\bar{b}$). We call these particles baryons. The most common baryons are the proton and the neutron. A quark can also combine with an antiquark in a color-anticolor combination ($r\bar{r}$, $g\bar{g}$, $b\bar{b}$, or some superposition of these) to form mesons (e.g. pions). Current research also suggests the existence of more exotic quark compositions like tetraquarks [23, 24] and pentaquarks [25]. All particles made up of quarks are called hadrons.
Leptons are organized in much the same way as the quarks: they are fermions and come in three generations. We have the charged leptons: the electron ($e^-$), the muon ($\mu^-$), and the tau ($\tau^-$). Then we have their accompanying neutrinos, the electron neutrino ($\nu_e$), the muon neutrino ($\nu_\mu$), and the tau neutrino ($\nu_\tau$). The antiparticles of the charged leptons are normally expressed with a “+” superscript ($e^+$, $\mu^+$, $\tau^+$), while the antineutrinos use a bar-notation ($\bar{\nu}_e$, $\bar{\nu}_\mu$, $\bar{\nu}_\tau$). While it is clear that the charged leptons are Dirac fermions, that is, they are not their own antiparticles, this is unclear for the neutrinos. If they are their own antiparticles, they would be Majorana fermions.

The fundamental forces are carried by the so-called gauge bosons: The photon ($\gamma$), the eight gluons ($g$), and the $W^+$, $W^-$ and $Z^0$, are all mediators for the electromagnetic, strong and weak forces. All these bosons are spin 1 particles. In addition, there is the hypothetical graviton ($G$), which, as mentioned, should be a massless spin-2 particle mediating the gravitational force. The latest addition to the Standard Model is the Higgs boson ($H^0$), which is responsible for giving fundamental particles their mass.

The total number of elementary particles depend on how we count. Disregarding the graviton which is not part of the Standard Model (and might not even exist), the common practice is to categorize the elementary particles by 17 different entries, as is done in Figure 2.1. If we count antiparticles as separate particles the number increases to 30. Further differentiating between colors gives us 61. By including
### 2.2 Baryon-to-photon-ratio

Just after the Big Bang when the temperature was high, the Universe was filled with photons, particles, and antiparticles. Because of constant annihilations and pair-productions, all particles were more or less in equilibrium and as abundant as the other. As the temperature fell, and no pair production ceased, the observable Universe was left with an asymmetry between matter and antimatter. For some reason which neither the Standard Model of particle physics nor general relativity can give, there is more of the former than the latter. This is normally expressed through the baryon asymmetry problem, which is one of the big unanswered questions in physics. The asymmetry parameter is expressed as

\[ \eta = \frac{n_B - n_{\bar{B}}}{n_\gamma} \approx \frac{n_B}{n_\gamma}. \]  

(2.1)

Counting up the baryons and CMB-photons gives us \( \eta \) equal to roughly \( 6 \times 10^{-10} \). It is sometimes preferred to base the baryon asymmetry on entropy density instead of the photon density as entropy has, to a good approximation, stayed constant since at least the electroweak era\(^1\). In such case, the asymmetry parameter may be expressed as

\[ \eta_s = \frac{n_B - n_{\bar{B}}}{s}, \]  

(2.2)

\(^1\)A reasonable assumption is that entropy has stayed fairly constant since the end of inflation. In Standard Cosmology, entropy per comoving volume is roughly constant, only increasing by a small amount \([8]\).
where $s = 7.04n_\gamma$ at the present era, giving $\eta_s$ a value of roughly $10^{-10}$. This asymmetry led us to a Universe where we could have primordial nucleosynthesis (see appendix B).

### 2.3 Chemical potential

The chemical potential, $\mu$, is related to the willingness for a certain reaction to occur, and we can associate a chemical potential to every particle. First a few observations:

1. The sum of the chemical potentials $\mu_i$, is conserved in chemical equilibrium, and thus, also in thermodynamic equilibrium.
2. Since photons can be absorbed or emitted arbitrary in a reaction, the chemical potential for photons, $\mu_\gamma$, is zero [26].
3. A particle-antiparticle pair can annihilate into photons, the chemical potential of a particle and its associated antiparticle is equal to zero and opposite [26].

The chemical potential is derived from the fundamental thermodynamic relation

$$dU = TdS - PdV + \sum_{j=1}^{n} \mu_j dN_j,$$

and give the following definition

$$\mu_j = \left( \frac{\partial U}{\partial N_j}\right)_{S,V,N_i\neq j}.$$

The chemical potential is thus the change in internal energy by adding or removing one particle to or from the system, keeping the volume and entropy constant. We can introduce a chemical potential $\mu_j$ for each conserved charge $Q_j$. This is done by replacing the Hamiltonian $H$ of the system with $H = H - \mu_j N_{Q_j}$, where $N_{Q_j}$ is the number operator of particles with charge $Q_j$. In the Standard Model, there are five independent conserved charges. These are the electric charge, baryon number, electron-lepton number, muon-lepton number, and tau-lepton number. This means there are also five independent chemical potentials [26]. The chemical potentials are determined by the number densities. The electric charge density is very close to zero. The baryon density is estimated to be less than a billionth of the photon density [27, 28]. Lepton density is also thought to be very small, on the same order as the baryon number. According to Weinberg [26], for an early universe scenario, we can put all these numbers equal to zero to a good approximation. For a correct representation of the Universe, the chemical potentials cannot all cancel out—otherwise, there would be no matter present today.
3 Statistical Mechanics

Statistical mechanics looks at the behavior of quantum systems on larger scales, and is so a bridge from the microscopic quantum world to the macroscopic world. Using probability theory, statistical mechanics studies the average behavior of mechanical systems where the state of the system is uncertain. In this chapter I will give a brief introduction to the topic. Most importantly to the distribution function which is the basis for paper one: “On the Effective Degrees of Freedom in the Early Universe” [1].

3.1 Distribution function

There is a fundamental assumption in statistical mechanics, namely that “in thermal equilibrium every distinct state with the same total energy are equally probable” [29]. Statistical mechanics is all about counting these states. However, things behave very differently in the quantum world, so before we can start counting states we need to know a few things about how particles occupy these states. Namely if the particles are distinguishable? And if more than one particle occupy the same state? With these two properties in mind we can categorize particles into three groups. First we have classical particles, which are distinguishable and can occupy the same state. Particles following this behavior obey the classical Maxwell-Boltzmann probability distribution. Next we have bosons (e.g. photons), which as classical particles can occupy the same state, but are indistinguishable. Bosons follow the Bose-Einstein distribution. Finally we have fermions (e.g. electrons and protons). These particles are also indistinguishable, but only one fermion can occupy one state. We have summarized the properties in Table 3.1. On the fundamental level, particles can only be fermions or bosons.

<table>
<thead>
<tr>
<th>Particles:</th>
<th>Probability distribution:</th>
<th>Distinguishable?</th>
<th>Can occupy same state?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classical particles</td>
<td>Maxwell-Boltzmann</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Bosons</td>
<td>Bose-Einstein</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Fermions</td>
<td>Fermi-Dirac</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

The difference in these three probability distributions lies in how we count the microstates. A microstate describes the properties of the individual particles, like
position, velocity, spin, etc. A macrostate, on the other hand, describes the systems macroscopic properties (like pressure, volume, temperature, etc.). Interchanging two microstates do not change the macrostate of the system.

The occupation number $N_i$ tells us how many particles are in each state, $\psi_i$, but does not care about which particles are in which states. The collection of all occupation numbers is called a configuration. The configuration which can be achieved in most different ways is the most probable configuration. When we are working with very large numbers we get one remarkable feature — it turns out that the distribution of individual particle energies, when they are in equilibrium, is just the most probable configuration [29, 30] (a simple example for a two-state system showing these feature given in Figure 3.1). The probability that a state is occupied depends on the energy of the system and if we are dealing with fermion or bosons, and can be approximated to

$$N_i \simeq \begin{cases} 
g_i & 	ext{Maxwell-Boltzmann ,} \\
g_i & \text{Fermi-Dirac ,} \\
g_i & \text{Bose-Einstein ,}
\end{cases} \frac{e^{(\alpha+\beta E_i)}}{e^{(\alpha+\beta E_i)} + 1} \quad \frac{g_i}{e^{(\alpha+\beta E_i) + 1}} \quad \frac{g_i}{e^{(\alpha+\beta E_i) - 1}}$$

where $g_i$ is the degeneracy of state $i$ and $E_i$ is the energy of single particle in that

\textbf{Figure 3.1:} Example given a two-state classical system (particles are distinguishable and can occupy the same state). The probability that the particles will distribute themselves evenly between the two states increases with the number of particles in the system. For a large number of particles (e.g. Boltmann’s constant) this distribution is simply the most probable configuration.
3.2 Particle velocities in Maxwell-Boltzmann and Maxwell-Jüttner distributions

State. \( \alpha \) and \( \beta \) are Lagrange multipliers, which can be identified as [29]

\[
\beta = \frac{1}{k_B T}, \tag{3.2}
\]

\[
\alpha = -\frac{\mu(T)}{k_B T} = -\mu(T) \beta, \tag{3.3}
\]

\( \beta \) is called the thermodynamic beta — a system's reciprocal thermodynamic temperature, and \( \mu \) is the chemical potential. We can introduce \( \varepsilon_i \) as the energy associated with state \( i \), such that for a one-particle state, \( E_i = 0 \) when it is unoccupied, and \( E_i = \varepsilon_i \) when it is occupied by one particle. By dividing Eq. (3.1) by the degeneracy of its energy states, we find the most probable number of particles in a single “one-particle” state with energy \( \varepsilon \), that is number density per state \( n \):

\[
n(\varepsilon) = \begin{cases} 
\frac{1}{e^{(\varepsilon-\mu)/k_B T}} & \text{Maxwell-Boltzmann}, \\
\frac{1}{e^{(\varepsilon-\mu)/k_B T} + 1} & \text{Fermi-Dirac}, \\
\frac{1}{e^{(\varepsilon-\mu)/k_B T} - 1} & \text{Bose-Einstein}.
\end{cases} \tag{3.4}
\]

For fermions this number will always be between zero and one, while it can be any positive number for bosons. Using this we can set up the distribution functions as functions of momenta for classical particles, fermions, and bosons:

\[
f(p) = \frac{1}{e^{(E(p)-\mu)/(k_B T)}} \quad \text{(Classical)}, \tag{3.5a}
\]

\[
f(p) = \frac{1}{e^{(E(p)-\mu)/(k_B T)} + 1} \quad \text{(Fermions)}, \tag{3.5b}
\]

\[
f(p) = \frac{1}{e^{(E(p)-\mu)/(k_B T)} - 1} \quad \text{(Bosons)}. \tag{3.5c}
\]

3.2 Particle velocities in Maxwell-Boltzmann and Maxwell-Jüttner distributions

We can distinguish between ultra-relativistic, semi-relativistic, and non-relativistic velocities, according to the value of the Lorentz factor, \( \gamma \). Another way to look at it, is which energy-term is dominant — the kinetic term, the rest mass term, or both. The ratio of semi- and non-relativistic particles plays a central role in the production of viscous effects. We will here go through the distribution functions for particle velocities.
For a classical case the particle speed distribution, or probability density function (PDF), for particles with mass $m$ at a temperature $T$ is given by the Maxwell-Boltzmann distribution:

$$f(v) = \left( \frac{m}{2\pi kT} \right)^{\frac{3}{2}} 4\pi v^2 e^{-\frac{mv^2}{2kT}},$$

$$= \sqrt{\frac{2}{\pi}} \frac{v^2 e^{-v^2/(2a^2)}}{a^3},$$

with $a = \sqrt{(k_B T/m)}$ being the distribution parameter.

For a relativistic gas the equivalent is the Jüttner distribution function [31].

$$f(\gamma) = \frac{\beta \gamma^2 z}{K_2(z)e^{\gamma z}},$$

where $\beta = v/c$, $\gamma = (1 - (v/c)^2)^{-1/2}$ is the Lorentz factor, $z = (mc^2)/(k_B T)$ in the reciprocal dimensionless temperature, and $K_2(z)$ is the modified Bessel function of the second kind.

### 3.3 Ideal quantum gases in thermodynamic equilibrium

Calculating the macrostates for systems in thermodynamic equilibrium is quite straightforward. In this section I will derive the functions for number density ($n$), energy density ($\varepsilon$), pressure ($P$), and entropy density ($s$). This follows closely the theory described in the paper [1].
3.3 Ideal quantum gases in thermodynamic equilibrium

$$z = \frac{mc^2}{k_B T}$$

**Figure 3.3**: Maxwell-Jüttner distributions (MJD) of speeds in the $\gamma$-representation as functions of $z$, the dimensionless temperature. This is really just the ratio of mass over temperature. Log-log plot for large $\gamma$’s on the right.

### 3.3.1 Thermodynamic functions

When we are dealing with a large (ideally infinite) system we can approximate the conditions by using small unit cells — or cubic boxes with periodic boundary conditions. For boxes of volume $V = L^3$, solving the Schrödinger equation for a particle, we find the possible momentum eigenvalues

$$p = \frac{h}{L} (n_1 \bar{e}_x + n_2 \bar{e}_y + n_3 \bar{e}_z) ,$$

where $h$ is the Planck constant, $n_i = 0, \pm 1, \pm 2, \pm 3, ...$, and $\bar{e}_x, \bar{e}_y, \bar{e}_z$ are the standard unit vectors in three-dimensional Euclidean space. The number of states in momentum space is thus:

$$\frac{n_1 \bar{e}_x \Delta p_x \Delta p_y \Delta p_z}{L^3} = \frac{L^3}{h^3} ,$$

By adding the internal degrees of freedom ($g$) and dividing Eq. (3.9) with the volume ($L^3$) we find the density of states:

$$\text{dos} = \frac{g}{h^3} = \frac{g}{(2\pi \hbar)^3}$$

The energy of a particle with mass $m$ and momentum $p$ is $E(p) = \sqrt{m^2 c^4 + p^2 c^2}$.

In thermal equilibrium, the probability that a single-particle state with momentum $p$ and energy $E(p)$ is occupied is given by the Bose-Einstein or Fermi-Dirac distribution functions given in Eqs. (3.5b) and (3.5c). In order to find the total number of
3 Statistical Mechanics

particles occupying a state with energy $E$, we must find the density of states in phase space. We see from Eq. (3.8) that the number of possible states in momentum space is $L^3/\hbar^3$. By dividing by the volume, $L^3$, as well, we are left with the factor $(1/\hbar)^3$. If there is an additional degeneracy $g$ (for example, spin), we can write the density of states (dos) as

$$\text{dos} = \frac{g}{h^3} = \frac{g}{(2\pi)^3 \hbar^3}.$$  

The density of particles with momentum $p$ is then given by

$$n(p) = \frac{g}{(2\pi)^3 \hbar^3} \times f(p).$$  

The total density of particles, $n$, can then be written as an integral over three-momentum involving the distribution function as

$$n = \frac{g}{(2\pi)^3 \hbar^3} \int f(p) \, d^3p.$$  

By multiplying the distribution functions with the energy and integrating over three-momentum, we obtain the energy density $\epsilon$ of the system. The pressure, $P$, can be found in a similar manner by multiplying the distribution function with $|p|^2/(3E/c^2)$ (a nice derivation of this is shown by Baumann [32]). This yields the integrals

$$\epsilon = \frac{g}{(2\pi)^3 \hbar^3} \int E(p) f(p) \, d^3p,$$

$$P = \frac{g}{(2\pi)^3 \hbar^3} \int \frac{|p|^2}{3(E/c^2)} f(p) \, d^3p.$$  

The entropy density $s$ can be calculated from the thermodynamic relation

$$s = \frac{\epsilon + P - \mu_T}{T},$$  

where the index $\mu_T$ is the total chemical potential.

3.3.2 From momentum to energy integrals

It is sometimes more convenient to use energy, $E$, instead of the momentum, $p$, as the integration variable. By integrating over all angles, we can replace $d^3p$ by $4\pi p^2 \, dp$. Using the energy momentum relation, we find $p = \sqrt{E^2 - m^2c^2}/c$ and $\sqrt{p^2} = E \, dE$.

The integrals will then go from $mc^2$ to infinity instead of $p = 0$ to infinity. We can simplify these formulas further by introducing the dimensionless variables $u$, $z$, and $\hat{\mu}$.

$$u = \frac{E}{k_BT}, \quad z = \frac{mc^2}{k_BT}, \quad \hat{\mu} = \frac{\mu}{k_BT}.$$  

22
This yields the following expressions for the number density, energy density, and pressure for a species \( j \), and for all species (as this is simply the sum of all particle species).

\[
\begin{align*}
n_j(T) &= \frac{g_j}{2\pi^2\hbar^3} \int_{m_j c^2}^{\infty} \frac{E \sqrt{E^2 - m_j^2 c^4}}{e^{(E - \mu_j)/k_B T} \pm 1} \, dE \\
&= \frac{g_j}{2\pi^2} \left( \frac{k_B T}{\hbar c} \right)^3 \int_{z_j}^{\infty} \frac{u \sqrt{u^2 - z_j^2}}{e^{u - \mu_j} \pm 1} \, du \\
n(T) &= \sum_j n_j = \sum_j \frac{g_j}{2\pi^2} \left( \frac{k_B T}{\hbar c} \right)^3 \int_{z_j}^{\infty} \frac{u \sqrt{u^2 - z_j^2}}{e^{u - \mu_j} \pm 1} \, du \\
\epsilon_j(T) &= \frac{g_j}{2\pi^2 \hbar^3} \int_{m_j c^2}^{\infty} \frac{E^2 \sqrt{E^2 - m_j^2 c^4}}{e^{(E - \mu_j)/k_B T} \pm 1} \, dE \\
&= \frac{g_j}{2\pi^2} \left( \frac{k_B T}{\hbar c} \right)^4 \int_{z_j}^{\infty} \frac{u^2 \sqrt{u^2 - z_j^2}}{e^{u - \mu_j} \pm 1} \, du \\
\epsilon(T) &= \sum_j \epsilon_j = \sum_j \frac{g_j}{2\pi^2} \left( \frac{k_B T}{\hbar c} \right)^4 \int_{z_j}^{\infty} \frac{u^2 \sqrt{u^2 - z_j^2}}{e^{u - \mu_j} \pm 1} \, du \\
P_j(T) &= \frac{g_j}{6\pi^2 \hbar^3} \int_{m_j c^2}^{\infty} \frac{(E^2 - m_j^2 c^4)^{3/2}}{e^{(E - \mu_j)/k_B T} \pm 1} \, dE \\
&= \frac{g_j}{6\pi^2} \left( \frac{k_B T}{\hbar c} \right)^5 \int_{z_j}^{\infty} \frac{(u^2 - z_j^2)^{3/2}}{e^{u - \mu_j} \pm 1} \, du \\
P(T) &= \sum_j P_j = \sum_j \frac{g_j}{6\pi^2} \left( \frac{k_B T}{\hbar c} \right)^5 \int_{z_j}^{\infty} \frac{(u^2 - z_j^2)^{3/2}}{e^{u - \mu_j} \pm 1} \, du.
\end{align*}
\]

As shown in Eq. (3.16) we can find the entropy density for a single species \( j \) and the total entropy as:

\[
\begin{align*}
s_j(T) &= \frac{\epsilon_j + P_j - \mu_j n_j}{T} \\
s(T) &= \sum_j s_j = \sum_j \frac{\epsilon_j + P_j - \mu_j n_j}{T} = \frac{\epsilon + P - \sum_j \mu_j n_j}{T}.
\end{align*}
\]
3.3.3 Massless particle contributions

In Eqs. (3.18)–(3.20), we see how dimensionless units, \( u, z, \) and \( \hat{\mu} \), simplifies the integrals. In the ultrarelativistic limit, we can ignore the particle masses. Moreover, as we have set the chemical potentials to zero, we can easily solve the dimensionless integrals appearing in Eqs. (3.18b), (3.19b), and (3.20b) analytically. Since the integrals for energy density and pressure in the massless cases are the same, we find:

\[
\int_0^\infty \frac{u^2}{e^u \pm 1} \, du = \begin{cases} \frac{3}{2} \zeta(3) \approx 1.803 & \text{(Fermions)} \\ 2\zeta(3) \approx 2.404 & \text{(Bosons)} \end{cases}, \tag{3.22}
\]

\[
\int_0^\infty \frac{u^3}{e^u \pm 1} \, du = \begin{cases} \frac{7}{8} \pi^4 \approx 5.682 & \text{(Fermions)} \\ \pi^4 \approx 6.494 & \text{(Bosons)} \end{cases}, \tag{3.23}
\]

where \( \zeta(3) \) is the Riemann zeta function of argument 3. Using these results, we find the values for \( n, \epsilon, P, \) and indirectly \( s \) for massless bosons and fermions:

\[
n_b(T) = g \frac{\zeta(3)}{\pi^2} \frac{(k_B T)^3}{(hc)^3}, \quad n_f(T) = \frac{3}{4} g \frac{\zeta(3)}{\pi^2} \frac{(k_B T)^3}{(hc)^3}, \tag{3.24}
\]

\[
\epsilon_b(T) = g \frac{\pi^2}{30} \frac{(k_B T)^4}{(hc)^3}, \quad \epsilon_f(T) = \frac{7}{8} g \frac{\pi^2}{30} \frac{(k_B T)^4}{(hc)^3}, \tag{3.25}
\]

\[
P_b(T) = g \frac{\pi^2}{90} \frac{(k_B T)^4}{(hc)^3}, \quad P_f(T) = \frac{7}{8} g \frac{\pi^2}{90} \frac{(k_B T)^4}{(hc)^3}, \tag{3.26}
\]

\[
s_b(T) = g \frac{2\pi^2}{45} \frac{k_B^4 T^3}{(hc)^3}, \quad s_f(T) = \frac{7}{8} g \frac{2\pi^2}{45} \frac{k_B^4 T^3}{(hc)^3}. \tag{3.27}
\]

Here the subscript b is for bosons, and f is for fermions. We see that solving the integrals gives a difference between fermions and bosons, namely a factor \( \frac{3}{4} \) for the number density and \( \frac{7}{8} \) for energy density and pressure. We will call these two factors the “fermion prefactors”. We also see that the pressure is simply one third that of the energy density, while the entropy density can be found by multiplying the energy density by \( 4/(3T) \).

3.3.4 Effective degrees of freedom

In most cases we cannot ignore the particle masses. In these cases, we must solve the integrals in Eqs. (3.18b), (3.19b), and (3.20b) numerically. The integrals are decreasing functions of the temperature, and they vanish in the limit \( k_B T/mc^2 \to 0 \). We can normalize these by dividing their values by the case of the photon (but with \( g \) equal to one). As we recall, the photon has a bosonic nature with \( m = 0 \) and \( \mu = 0 \). This means that for massive particles at high temperature \( (k_B T \gg mc^2) \), one actual degree of freedom for bosons contributes as much as one degree of freedom for photons, and the fermions a little less. As the temperature drops, and less particles
are created, the effective contributions will be smaller. By including the intrinsic
degrees of freedom \((g)\), we find each particle species’ effective degree of freedom, \(g_{*j}\):

\[
g_{*n}^j(T) = \frac{g_j}{2\pi^2} \left(\frac{k_b T}{\hbar c}\right)^3 \int_{z_j}^{\infty} \frac{u \sqrt{u^2 - z_j^2}}{e^{u \pm 1}} \, du = \frac{g_j}{2\zeta(3)} \int_{z_j}^{\infty} \frac{u \sqrt{u^2 - z_j^2}}{e^u \pm 1} \, du , \tag{3.28}
\]

\[
g_{*e}^j(T) = \frac{g_j (k_b T)^4}{2\pi^2 (\hbar c)^4} \int_{z_j}^{\infty} \frac{u^2 \sqrt{u^2 - z_j^2}}{e^{u \pm 1}} \, du = \frac{15g_j}{\pi^4} \int_{z_j}^{\infty} \frac{u^2 \sqrt{u^2 - z_j^2}}{e^u \pm 1} \, du , \tag{3.29}
\]

\[
g_{*p}^j(T) = \frac{g_j (k_b T)^4}{2\pi^2 (\hbar c)^4} \int_{z_j}^{\infty} \frac{(u^2 - z_j^2)^{3/2}}{e^{u \pm 1}} \, du = \frac{15g_j}{\pi^4} \int_{z_j}^{\infty} \frac{(u^2 - z_j^2)^{3/2}}{e^u \pm 1} \, du , \tag{3.30}
\]

\[
g_{*s}^j(T) = \frac{3g_{*e}^j(T) + g_{*p}^j(T)}{4} . \tag{3.31}
\]

In Figure 3.4, we have plotted the effective degrees of freedom for massive bosons
(panel a) and fermions (panel b) with \(g = 1\) (and \(\mu = 0\)) as functions of the temperature. When the temperature is equal to the mass \((k_b T = mc^2)\), the effective degrees of freedom for energy density is approximately 0.9 for bosons and 0.8 for fermions,
compared to that of the photon. For number density, pressure, and entropy density,
they are a little lower.

The effective degrees of freedom are defined as functions of the corresponding variables and temperature. We find the total effective degrees of freedom for \(g_{*n}\), \(g_{*e}\),
and \(g_{*p}\) by summing Eqs. (3.28)–(3.31) over all particle species \(j\):

\[
g_{*n}(T) = \frac{\pi^2}{\zeta(3)} \frac{n(T)}{T^3} , \tag{3.32a}
\]

\[
= \sum_j \frac{g_j}{2\zeta(3)} \int_{z_j}^{\infty} \frac{u \sqrt{u^2 - z_j^2}}{e^{u \pm 1}} \, du , \tag{3.32b}
\]

\[
g_{*e}(T) = \frac{30c(T)}{\pi^2} \frac{n(T)}{T^4} , \tag{3.33a}
\]

\[
= \sum_j \frac{15g_j}{\pi^4} \int_{z_j}^{\infty} \frac{u^2 \sqrt{u^2 - z_j^2}}{e^u \pm 1} \, du , \tag{3.33b}
\]

\[
g_{*p}(T) = \frac{90P(T)}{\pi^2} \frac{n(T)}{T^4} , \tag{3.34a}
\]

\[
= \sum_j \frac{15g_j}{\pi^4} \int_{z_j}^{\infty} \frac{(u^2 - z_j^2)^{3/2}}{e^u \pm 1} \, du . \tag{3.34b}
\]
Figure 3.4: The effective degrees of freedom $g_n$, $g_\epsilon$, $g_p$, and $g_s$ for bosons (a) and fermions (b) per intrinsic degree of freedom. A more detailed look at each of the four $g_s$ is given in the lower four panels (c–f). Here the solid colored curves are for the bosons, and the dash-dotted colored curves are for the fermions. The grey dash-dotted curves represent the fermions’ contribution compared to its own relativistic value (such that it is 100% for $T \to \infty$). We have included the relative values at $k_B T = mc^2$ for the four cases (marked with “+” symbols). During particle annihilations, the energy density falls slower than the other quantities due to the impact of the rest mass energy. At temperatures close to the rest mass of some massive particle species, this rest mass is substantial to their total energy.
Finally, the effective degrees of freedom associated with entropy is then:

\[
g_{*s}(T) \equiv \frac{45}{2\pi^2} \frac{s(T)}{T^3} 
= \frac{3g_{*e}(T) + g_{*p}(T)}{4}.
\]
4 The Friedmann Equations and Expansion of the Universe

The Friedmann equations describe the geometric expansion or contraction of homogeneous and isotropic universes. They are solutions to Einstein’s field equations using the Friedmann-Lemaître-Robertson-Walker metric. They were derived by Alexander Friedmann in 1922 [33] and 1924 [34], and are valid for flat, positive, and negative spatial curvatures.

4.1 Newtonian gravity

In many cases, Newtonian gravity is a good approximation to describe gravity. From there, we remember that the force exerted on an object with mass \( m \) by an object of mass \( M \) is given by

\[
F = \frac{GMm}{r^2},
\]

(4.1)

where Newton’s gravitational constant, \( G \), is roughly \( 6.67 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2} \). The gravitational potential energy is

\[
V = -\frac{GMm}{r}.
\]

(4.2)

4.1.1 Deriving the Friedmann equations classically

One can partly derive Friedmann’s (first) equation from Newtonian gravity [7]. First, we start off with a proof given by Newton in his *Philosophiæ Naturalis Principia Mathematica* [35]. Consider a spherical distribution of matter, with a particle of mass \( m \) at a distance \( r \) from the center of mass. The particle only feel the gravitational attraction from the mass inside this \( r \)-radial sphere, and as if all of that were a single point at the center of mass. The gravitational attractions from outside the radius of the sphere cancel out (just as an object inside a hollow sphere will feel no gravity at all). The mass inside the sphere is given by

\[
M = \frac{4\pi}{3}r^3\rho,
\]

(4.3)
where $\rho$ is the mass density. The total energy for the particle is then the sum of kinetic and potential energy, respectively:

$$E_T = \frac{1}{2} m \dot{r}^2 - \frac{G M m}{r} = \frac{1}{2} m \dot{r}^2 - \frac{4 \pi r^3 \rho G m}{3}.$$  \hspace{1cm} (4.4)

Using the cosmological principle as a basis, that the Universe is homogeneous and isotropic, every location may be considered to be the center, so we may choose one arbitrarily. For a uniform expansion (or contraction) of space we can use comoving coordinates. These comoving distances ($x$) between objects carried along the expansion remain the same. The relation between the actual distance $r$ and $x$ is given by the scale factor, $a$, and is a function of time alone ($r = a(t)x$). The comoving distances do not change, such that $\dot{x} = 0$, this gives us:

$$E_T = \frac{m a^2 x^2}{2} - \frac{4 \pi G \rho a^2 x^2 m}{3}.$$ \hspace{1cm} (4.5)

Multiplying everything with $-2/(m x^2 a^2)$ and rearranging gives us

$$\left( \frac{\dot{a}}{a} \right)^2 + \frac{2 E_T}{m x^2 a^2} = \frac{8 \pi G}{3} \rho - \frac{k c^2}{a^2}.$$ \hspace{1cm} (4.6)

Setting $k = 2 E_T/(m c^2 x^2)$ we get

$$H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{8 \pi G}{3} \rho - \frac{k c^2}{a^2},$$ \hspace{1cm} (4.7)

which is the Friedmann equation without the cosmological constant. The unit of $k$ is inverse comoving length squared — the curvature of space.

### 4.1.2 The fluid equation

The fluid equation can be derived from the first law of thermodynamics for a reversible process (for which the heat transfer, $\delta Q$, is zero)

$$dU + P dV = 0,$$ \hspace{1cm} (4.8)

where $U = \rho c^2$ is the internal energy, $P$ is the pressure, and $V = 4 \pi a^3/3$ is the volume. If the change happens over a period of $dt$ we get

$$\frac{d}{dt} \left( \frac{4 \pi}{3} a^3 \rho c^2 \right) + P \frac{d}{dt} \left( \frac{4 \pi}{3} a^3 \right) = 0$$

$$3 \dot{a} \rho c^2 + \dot{a} P + 3 \dot{a} = 0$$

$$\dot{\rho} = -3 \frac{\dot{a}}{a} \left( \rho + \frac{P}{c^2} \right).$$ \hspace{1cm} (4.9)
From this we can see that the mass density decreases both due to an increased volume and because of pressure work done by the material on the Universe. The pressure is material dependent and can be both positive, zero, or negative, as we will get back to in Section 4.3.

4.1.3 The acceleration equation

The rate of change of the expansion is given by the acceleration equation. It can be derived by first differentiating Eq. (4.7) with respect to time to get

$$2\frac{\ddot{a}}{a} \frac{\dot{a}}{a^2} - 2\frac{k c^2 \dot{a}}{a^2} a = -\frac{8\pi G}{3} \rho ,$$  \hspace{1cm} (4.10)

Substituting $\dot{\rho}$ with Eq. (4.9) we get

$$2\frac{\ddot{a}}{a} \frac{\dot{a}}{a^2} - 2\frac{k c^2 \dot{a}}{a^2} a = -8\pi G \frac{\dot{a}}{a} \left( \rho + \frac{P}{c^2} \right) ,$$  \hspace{1cm} (4.11)

and dividing everything with $2\dot{a}/a$

$$\frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} + \frac{k c^2}{a^2} \right) = -4\pi G \left( \rho + \frac{P}{c^2} \right) .$$  \hspace{1cm} (4.12)

Putting Eq. (4.7) in we get

$$\frac{\ddot{a}}{a} - \frac{8\pi G}{3} \rho = -\frac{12\pi G}{3} \left( \rho + \frac{P}{c^2} \right)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left( \rho + \frac{3P}{c^2} \right) ,$$  \hspace{1cm} (4.13)

which is the acceleration equation.

4.2 Friedmann equations from general relativity

The three equations, Eqs. (4.7), (4.9), (4.13), which we just derived classically should be derived from Einstein’s field equations if we want the full picture. Friedmann did this back in the early twenties, using what is now called the Friedmann-Lemaître-Robertson-Walker metric [33, 34]. This gives the Friedmann equation and the acceleration equation in the general form which also include the cosmological constant (the fluid equation remains unchanged). In many cases it is more convenient to express the Friedmann equations using energy density, $\varepsilon = \rho c^2$. We then end up with

Friedmann Eq. \hspace{1cm} \left( \frac{\dot{a}}{a} \right)^2 + \frac{k c^2}{a^2} = \frac{8\pi G}{3c^2} \varepsilon + \frac{\Lambda c^2}{3} , \hspace{1cm} (4.14)

Acceleration Eq. \hspace{1cm} \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} (\varepsilon + 3P) + \frac{\Lambda c^2}{3} , \hspace{1cm} (4.15)

Fluid Eq. \hspace{1cm} \dot{\varepsilon} = -3 \frac{\dot{a}}{a} (\varepsilon + 3P) . \hspace{1cm} (4.16)
If we are using $c = 1$, which is common, the Friedmann equations look the same using $\rho$ or $\varepsilon$.

### 4.3 Equations of state and the evolution of our Universe

In cosmology, we have four main states in the sense of how its energy density evolves when space expands. These are the two trivial states of radiation (relativistic particles) and cold matter (non-relativistic particles), and the two less trivial: curvature and vacuum energy (cosmological constant). How these states evolve depends on how their pressures are related to their energy densities. This relation is expressed by the equation of state by the dimensionless number $w_i$, which is the ratio of pressure over energy density given by

$$w_i = \frac{P_i}{\rho_i c^2} = \frac{P_i}{\varepsilon_i},$$

where $w_i$ has the following values:

- **Cold matter**
  - $P = 0$\quad $\rightarrow$\quad $w = 0$. (4.18a)
- **Radiation**
  - $P = \frac{1}{3} \varepsilon$\quad $\rightarrow$\quad $w = \frac{1}{3}$. (4.18b)
- **Curvature**
  - $P = -\frac{1}{3} \varepsilon$\quad $\rightarrow$\quad $w = -\frac{1}{3}$. (4.18c)
- **Vacuum energy**
  - $P = -\varepsilon$\quad $\rightarrow$\quad $w = -1$. (4.18d)

We can use this to get the equations of state for our cases. From the thermodynamic relation we have $dU = T \, dS - P \, dV$, with no increase in the entropy, such that

$$\varepsilon \, da^3 + a^3 \, d\varepsilon + P \, da^3 = 0$$

$$\varepsilon \, da^3 + a^3 \, d\varepsilon + w \varepsilon a^3 = 0$$

$$\varepsilon (1 + w) \, da^3 + a^3 \, d\varepsilon = 0$$

$$\frac{d\varepsilon}{\varepsilon} = -(1 + w) \frac{da^3}{a^3}$$

$$\ln(\varepsilon) = -(1 + w) \ln(a^3) + C$$

$$\ln(\varepsilon) = \ln \left( a^{-3(1+w)} \right) + C$$

$$\varepsilon = C a^{-3(1+w)},$$

where $C$ is a constant to be defined shortly.

The curvature of our Universe depends on the Hubble parameter and the energy density. A high density leads to a positive curvature and a low density to a negative curvature. The critical density is the density which gives the Universe a flat curvature, that is $k$ equal to zero. We can find the conditions for this by rearranging Eq. (4.14):

$$\frac{k}{a^2} = \frac{8 \pi G \varepsilon}{3 c^2} - H^2 = 0.$$
Solving this for $\varepsilon$ gives us the critical density $\varepsilon_c$:

$$\varepsilon_c = \frac{3H^2c^2}{8\pi G}.$$  \hspace{1cm} (4.21)

The critical density is crucial for the evolution of the Universe, and the actual densities can be expressed as a ratio in comparison to this density. We call the ratio the density parameter, and define it as:

$$\Omega_i = \frac{\varepsilon_i}{\varepsilon_c} = \frac{8\pi G}{3H^2c^2} \varepsilon_i,$$  \hspace{1cm} (4.22)

where the index $i$ can be the one of the states from Eq. (4.18). We can then express the energy densities as

$$\varepsilon_i = \varepsilon_c \Omega_i.$$  \hspace{1cm} (4.23)

We now have two different equations for $\varepsilon$ in Eqs. (4.19, 4.23). By defining the scale factor at a given time, $t_0$ (e.g. present time) to be $a = 1$, we get a value for the constant $C$ in Eq. (4.19) for that time to be $\varepsilon_c\Omega_0$. Or as expressed by its components:

$$C_i = \varepsilon_c\Omega_{i,0}.$$  \hspace{1cm} (4.24)

The time-varying energy density will thus be

$$\varepsilon_i = \varepsilon_c\Omega_{i,0}a^{-3(1+w_i)}.$$  \hspace{1cm} (4.25)

The current value of $\Omega$, that is the sum over all $\Omega_i$ looks like it is very close to one. This is a peculiar thing, since if $\Omega$ were ever slightly different from one, it would rapidly move away from this ratio. This is known as the flatness problem. One of the most popular theories which solves this conundrum is inflation theory. In this scenario the observable Universe is just a very small part of something much larger, so even though the whole Universe might be curved, it will appear flat to us. Assuming a flat non-curved Universe (or alternately put the curvature as part of the energy density) we can set up the first Friedmann equation (Eq. (4.14)) to account for the different states of matter.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \varepsilon_c \sum_i \Omega_{i,0}a^{-3(1+w_i)}$$

$$= H_0^2 \sum_i \Omega_{i,0}a^{-3(1+w_i)}.\hspace{1cm} (4.26)$$

We can rewrite this as

$$\dot{a}^2 = H_0^2 \sum_i \Omega_{i,0}a^{-3(1+w_i)}.$$  \hspace{1cm} (4.27)
Taking the time-derivative on the left hand side in two different ways gives us
\[
\frac{d}{dt} \dot{a}^2 = 2 \ddot{a}
\]
\[
= \frac{d \dot{a}^2}{da} \frac{da}{dt} = \dot{a} \frac{d \dot{a}^2}{da}
\]
\[
\rightarrow \ddot{a} = \frac{1}{2} \frac{d \dot{a}^2}{da} .
\]
(4.28)

Taking the \(d/da\)-derivative on the right hand side of Eq. (4.27) gives us
\[
H_0^2 \sum_i (1 + 3w_i) \Omega_{i,0} a^{-(2+3w_i)} .
\]
(4.29)

By further combining Eq. (4.28) and Eq. (4.29) we get the following equation for the acceleration of the Universe, telling us how the Universe evolved in the past, and will evolve in the future:
\[
\ddot{a} = -\frac{H_0^2}{2} \sum_i \frac{1 + 3w_i}{a^{2+3w_i}} \Omega_{i,0} .
\]
(4.30)

By using the latest estimates for the Hubble variable and the density parameters we can calculate that the radiation dominated era ended roughly 50,000 years after the Big Bang, and the following matter dominated era ended about 10 billion years later. These transitions happens gradually as is shown in Figure 4.1. The evolution of the scale factor is shown in Figure 4.1 together with scenarios using different energy densities.

### 4.4 Adding viscosity to the Friedmann equations

If the cosmic fluid has a bulk viscosity, the pressure will be reduced accordingly such that we get an effective pressure, \(P_{\text{eff.}} = P - P_{\text{visc.}}\), where \(P\) is the thermodynamic equilibrium pressure and \(P_{\text{visc.}}\) is the viscous pressure given as \(3 \eta_v H\), where \(\eta_v\) is the bulk viscosity [36]. We can then express the effective pressure as a function of the viscosity:
\[
P_{\text{eff.}} = P - 3 \eta_v H .
\]
(4.31)

By simulating the early universe, where we can put both \(k\) and \(\Lambda\) equal to zero, we can incorporate the viscous pressure into the accelerating and fluid equation (the first Friedmann equation remains unchanged as it is independent of the pressure):

**Visc. acceleration Eq.**
\[
\dot{a} \frac{\ddot{a}}{a} = -\frac{H^2}{2} - \frac{4 \pi G}{c^2} (P - 3 \eta_v H) .
\]
(4.32)

**Visc. fluid Eq.**
\[
\dot{\varepsilon} = -3H (\varepsilon + P) + 9 \eta_v H^2 .
\]
(4.33)
4.4 Adding viscosity to the Friedmann equations

Figure 4.1: Energy contribution in the Universe by radiation, matter and a cosmological constant ($\Lambda$). The energy density for these three cases evolves differently during expansion ($\varepsilon_r \propto a^{-4}, \varepsilon_m \propto a^{-3}, \varepsilon_\Lambda \propto 1$). This leads to the Universe going through different eras. Radiation dominated for the first 44 700 years after the Big Bang, followed by matter. We are currently living in an era where dark energy (e.g. a cosmological constant) is the dominant energy contributor.
4 The Friedmann Equations and Expansion of the Universe

Figure 4.2: Mapping the evolution of the Universe by $\Omega$ contributions. In (hypothetical) scenarios with no dark energy, the curvature (and overall geometry) is determined by the matter contribution. $\Omega_m$ below, equal to, or above the critical density (here in magenta, blue and red colors) will give negative, flat and positive curvatures. Using the current $\Lambda$CDM parameters as of 2015 [19] gives us the green plot. Baryonic matter (5%) and dark matter (26%) make up the 31% matter contribution, while dark energy (here in the form of the cosmological constant) make up 69%.

4.5 Viscosity by numbers and illustrations

The concept of shear and bulk viscosity is discussed in my second paper “Viscosity in a lepton-photon universe” [2], and shown briefly here in Figure 4.3 and 4.4. The first numerical calculation of shear viscosity $\eta_s$ was done by Misner in 1967, where he studied a mixture of massive particles interacting with massless photons and neutrinos. He got a result of [37]:

$$\eta_s = \frac{4}{15} \varepsilon \tau \approx 1.23 \times 10^{38} T^{-1}, \quad (4.34)$$

where $\tau$ is the mean free time of flight. In 1978, Caderni, Fabbri, Van den Horn, and Sisskens came of with a much smaller value of the $\eta_s$ during the lepton era [38]:

$$\eta_s = 2.20 \times 10^{35} T^{-1}. \quad (4.35)$$

In the early eighties, a lot of progress were done on the topic of relativistic kinetic theory by, among others, Hoogeveen, et al. This group estimated the shear viscosity to [39]:

$$\eta_s = 9.81 \times 10^{35} T^{-1}. \quad (4.36)$$

The two latter papers also gave estimates to the bulk viscosity during the lepton era. These vary much more with temperature but for a temperature of $10^{12}$ K, Caderni’s term
4.6 The deceleration parameter

The deceleration parameter is used to make corrections when calculating the luminosity distance. It is actually just the acceleration equation divided by the Friedmann equation, and is defined as

\[
q = \frac{\ddot{a}a}{a^2} = \frac{1}{2} \frac{\sum(1 + 3w_i)\Omega_i}{\sum \Omega_i} .
\]  

(4.39)

The sum of the density parameters is very close to one (in many models it is defined as one), so we can remove the denominator (otherwise \( \sum \Omega_i \) varies over time). By inserting the values for \( w \) we can express Eq. (4.39) as

\[
q = \frac{1}{2} \Omega_M + \Omega_r - \Omega_{\Lambda} .
\]  

(4.40)

It is worth noting that the deceleration parameter is independent of the curvature term (because \( w_k = -1/3 \) and hence the nominator becomes zero). We can easily

Figure 4.3: Concept drawing of shear viscosity for a flow. The shear stress increases as the particles travel farther in the perpendicular direction of the flow. Weakly interacting particles is thus the main contributor to shear viscosity.

group found

\[
\eta_v = 5.85 \times 10^{12} \text{g cm}^2 \text{s}^{-1} ,
\]

while Hoogeveen’s group came to a result of

\[
\eta_v = 2.30 \times 10^{13} \text{g cm}^2 \text{s}^{-1} ,
\]

which is roughly 5 times larger.

4.6 The deceleration parameter

The deceleration parameter is used to make corrections when calculating the luminosity distance. It is actually just the acceleration equation divided by the Friedmann equation, and is defined as

\[
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\]  

(4.39)

The sum of the density parameters is very close to one (in many models it is defined as one), so we can remove the denominator (otherwise \( \sum \Omega_i \) varies over time). By inserting the values for \( w \) we can express Eq. (4.39) as

\[
q = \frac{1}{2} \Omega_M + \Omega_r - \Omega_{\Lambda} .
\]  

(4.40)

It is worth noting that the deceleration parameter is independent of the curvature term (because \( w_k = -1/3 \) and hence the nominator becomes zero). We can easily
Concept drawing of bulk viscosity. If we have two types of particles, which wants to cool down at different rates we get a heat transfer from the hotter particles to the colder particles. In my papers [2, 3], a model universe consisting only of the known leptons and the photons have been used. In this scenario, the first particles which stops behaving purely relativistic are the tau-particles. As the rest mass becomes a significant contribution to the tau’s total energy, it will start to cool down slightly faster, and then more so as the temperature drops further. We can divide the heat transfer into two events. First, we have the very-frequent-low-momentum-exchange heat transfer between the taus and the other electromagnetically charged particles. All these particles are thus cooled down slightly faster than before. The second event is the heat exchange between all the charged leptons and the neutrinos. The neutrinos, which only interact weakly and thus have kept their temperature for a longer time, will give rise to a much larger momentum transfer.
calculate today’s value of $q_0$ using the latest data. The radiation contribution today is negligible, and we get:

$$q_0 = -0.54$$

which we see is a negative value, meaning that the expansion of the Universe is accelerating.
5 Kinetic Theory

5.1 Four-vectors, velocities and momenta

Four-vectors are important in relativistic kinetic theory, and a short primer is given here. The familiar spatial vectors from Euclidean geometry has to be replaced with new ones containing the time coordinate as well. We will use a metric signature of \((+ \, - \, - \, -)\), which in Minkowski space gives the following matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}.
\]

We will use the following notation for variables for positions, velocities, and momenta

\[ x^\mu = (ct, x, y, z) = (ct, \mathbf{x}) , \quad (5.1) \]
\[ u^\mu = (c, v_x, v_y, v_z) = \gamma (c, \mathbf{v}) , \quad (5.2) \]
\[ p^\mu = (E/c, p_x, p_y, p_z) = (E/c, \mathbf{p}) , \quad (5.3) \]

where \(\gamma = \left[1 - (v/c)^2\right]^{-1/2}\) is the Lorentz factor. The momentum four-vector is defined as \(p^\mu = mu^\mu\). We have used bold parameters to represent the three spatial variables. Thus for a particle with mass \(m\) and momenta \(\mathbf{p}\), the energy is given as

\[ E = cp^0 = c\sqrt{p^2 + m^2c^2} . \quad (5.4) \]

The magnitudes of the four-velocity and four-momentum is given as

\[ u \cdot u = u^\mu u^\nu = g_{\mu\nu} u^\mu u^\nu = c^2 \quad (5.5) \]
\[ p \cdot p = p^\mu p^\nu = g_{\mu\nu} p^\mu p^\nu = (E/c)^2 - \mathbf{p}^2 = mc^2 \quad (5.6) \]

where \(g_{\mu\nu}\) is the metric. Finally we may write down the four-flow which is the relativistic equivalence of particle flow, \(j(x, t)\):

\[ N^\mu(x) = (cn(x, t), j(x, t)) \quad (5.7) \]

5.2 Cross sections and mean free paths

Charged leptons interact via the weak and electromagnetic forces, neutrinos only through the weak force, and photons only through the electromagnetic force. If we limit ourselves to the lepton era we can distinguish between four different collisions:
1. $\nu\nu$ and $\nu e$ collisions → governed by the weak force ($\text{wk}$),

2. $ee$ and $e\gamma$ collisions → governed by the electromagnetic force ($\text{em}$),

3. $\gamma\gamma$ collisions,

4. $\nu\gamma$ collisions.

As functions of the mean kinetic energy, $k_B T$, we have approximate expressions for these four cross sections \([39, 40]\):

\[
\sigma_{\text{wk}} \approx \left( \frac{G k_B T}{2\pi \hbar^2 c^2} \right)^2, \quad (5.8)
\]

\[
\sigma_{\text{em}} \approx \left( \frac{\alpha \hbar c}{k_B T} \right)^2, \quad (5.9)
\]

\[
\sigma_{\gamma\gamma} \approx \left( \frac{\alpha^2 \hbar c}{k_B T} \right)^2, \quad (5.10)
\]

\[
\sigma_{\nu\gamma} \approx 0. \quad (5.11)
\]

The mean free path is the inverse of the number density and cross section:

\[
l = \frac{1}{n\sigma}. \quad (5.12)
\]

As we see from Eqs. (5.8)–(5.10), the weak and electromagnetic cross sections behave quite differently as functions of temperature. One gets smaller as the temperature drops, and the other one increases. The electromagnetic force and the weak force merge together to what we call the electroweak force at around 100 GeV (around the masses of the intermediate vector bosons — the W’s and Z). These simple approximated cross sections and mean free paths are shown in Fig. 5.1. For a more rigorous estimate we need to take the particle flavors into account.

### 5.3 Weinberg-Salam model for weak interactions

We can make a more precise estimate of the weak cross sections by using the Weinberg-Salam model for electroweak interactions. For the lepton era (at energies below the mass-equivalent of the intermediate vector bosons: $W^+$, $W^-$, and $Z^0$) we can use a current-current coupling approximation \([39]\). The charged current is the exchange of either $W^+$ or $W^-$ bosons, and the neutral current is the exchange of the $Z^0$ boson. This is equivalent to the photon exchange in the electromagnetic interaction. For elastic collisions we have:

\[
k + l \rightarrow k + l. \quad (5.13)
\]
5.3 Weinberg-Salam model for weak interactions

Figure 5.1: Simplified plots of cross sections (left plot) and mean free paths (right plot) for electromagnetically interacting and weakly interacting particles. At around 100 GeV the two forces merge together to the electroweak force.

The charged leptons will predominantly interact with each other (and the photon) through the electromagnetic force, while the interactions involving neutrinos can only happen through the weak force. For the interactions involving neutrinos we have:

\[ k = \nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau, \]
\[ l = \nu_e, \bar{\nu}_e, \nu_\mu, \bar{\nu}_\mu, \nu_\tau, \bar{\nu}_\tau, e, \bar{e}, \mu, \bar{\mu}, \tau, \bar{\tau}. \]  

Before we go any further we should introduce what are called the Mandelstam variables.

5.3.1 Mandelstam variables

For scattering processes with two particles \((k\text{ and }l)\) going in and two particles going out, we can use the center-of-momentum frame as is done in Figure 5.2. The four-momentum of the ingoing particles are thus \(p_k\) and \(p_l\), and the two particles going out have four-momentum \(p'_k\) and \(p'_l\). This gives us three possible scenarios, or channels, which represent the different outcomes, namely the \(s\), \(t\), and \(u\) channel (shown in Figure 5.3). The associated variables are called the Mandelstam variables and are related to energy, momentum, and angles in the scattering process. They are defined as:

\[ s = (p_k + p_l)^2 = (p'_k + p'_l)^2, \]
\[ t = (p_k - p'_k)^2 = (p_l - p'_l)^2. \]  

43
Here $s$ is the square of the center-of-mass energy (invariant mass) for the two incoming particles $k$ and $l$. $t$ is the square of the four-momentum transfer, and is related to the scattering angle in the center of momentum system, which is defined later in Eq. (5.23). $p'_k$ is the four-momentum of particle $k$ after the collision.

\[ u = (p_k - p'_k)^2 = (p_l - p'_l)^2. \]  

Figure 5.3: Feynman diagrams for $s$ (space), $t$ (time), and $u$ channel. They represent the different possible scatterings with two incoming and two outgoing particles. The $s$-channel represents the two incoming particles, $p_k$ and $p_l$, merging together to form an intermediate particle, which then splits into particles $p'_k$ and $p'_l$. In the $t$-channel particle $p$ emits a particle, becoming particle $p'_k$, and particle $p_l$ absorbs the intermediate particle and becoming particle $p'_l$. For the $u$-channel the roles of the two outgoing particles are changed, such that $p_k$ becomes $p'_l$ and $p_l$ becomes $p'_k$.

### 5.3.2 Weak currents

The effective interaction Lagrangian density for the currents are [39]:

\[ \mathcal{L}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \left( J^{\lambda^\dagger} J_\lambda + 2 J^{n\lambda} J_n^\dagger \right), \]  

where $J^{\lambda}$ and $J^{n\lambda}$ are the charged and neutral currents, and $\lambda$ is the four-vector index (used instead of $\mu$ to avoid confusion with the symbol for the muon particle),
and the $\dagger$ symbol is for the Hermitian conjugate. The three (six if we count anti particles) charged lepton contributions to these are:

\begin{align*}
J_c^{\lambda e} & = \bar{\nu}_e \gamma^\lambda (1 - \gamma^5) e, \\
J_c^{\lambda \mu} & = \bar{\nu}_\mu \gamma^\lambda (1 - \gamma^5) \mu, \\
J_c^{\lambda \tau} & = \bar{\nu}_\tau \gamma^\lambda (1 - \gamma^5) \tau, \\
J_n^{\lambda e} & = \frac{1}{2} \bar{\nu}_e \gamma^\lambda (1 - \gamma^5) \nu_e - \frac{1}{2} \bar{\nu}_e \gamma^\lambda \left(1 - 4 \sin^2 \theta_W - \gamma^5\right) e, \\
J_n^{\lambda \mu} & = \frac{1}{2} \bar{\nu}_\mu \gamma^\lambda (1 - \gamma^5) \nu_\mu - \frac{1}{2} \bar{\nu}_\mu \gamma^\lambda \left(1 - 4 \sin^2 \theta_W - \gamma^5\right) \mu, \\
J_n^{\lambda \tau} & = \frac{1}{2} \bar{\nu}_\tau \gamma^\lambda (1 - \gamma^5) \nu_\tau - \frac{1}{2} \bar{\nu}_\tau \gamma^\lambda \left(1 - 4 \sin^2 \theta_W - \gamma^5\right) \tau.
\end{align*}

The $\gamma^\lambda$'s here are the four gamma matrices (also known as $4 \times 4$ Dirac matrices), and $\gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$. They are given in Appendix C.

The amplitude for any $k + l \rightarrow k + l$ reaction is the sum of the aforementioned $s$-channel, $t$-channel, and $u$-channel contribution. According to Hoogeveen [39] we can write all these using the $s$ and $t$ channel only with the addition of two matrices: $v$ and $a$ (as given in Tables 5.1 and 5.2). This leads to the following equations for the weak cross sections:

\begin{equation}
\frac{d\sigma_{kl}}{d\Omega_{\text{CM}}} = \frac{G_F^2 s}{32\pi^2 \hbar^4 c^2} S f(s, t),
\end{equation}

where $G_F$ is the Fermi coupling constant. $s$ is the square of center-of-mass collision energy, and was defined in Eq. (5.16a). $S$ is a prefactor which compensates for the number of identical particles in the final state ($n_f!$), and contains a factor 2 for each neutrino in the initial state (incoming particles). This latter factor comes from the fact that neutrinos only have one polarization state, while the other particles have 2 (spin up and spin down), which gives us the following value:

\begin{equation}
S = \frac{2^{n_\nu}}{n_\ell!}.
\end{equation}

The final variable in Eq. (5.19) is $f(s, t)$, which is a function of the Mandelstam variables, $s$ and $t$ and two prefactors, $a$ and $v$. $a$ and $v$ relate the current couplings to flavors of the $k$ and $l$ particle. This gives us [39]:

\begin{equation}
f(s, t) = (v + a)^2 \left(\frac{s - m^2 c^2}{s}\right)^2 + (v - a)^2 \left(\frac{s + t - m^2 c^2}{s}\right)^2 + (v^2 - a^2) \frac{2m^2 c^2 t}{s^2},
\end{equation}

where $m$ is the mass of the charged lepton. $T$ $a$ and $v$ matrices are given in Tables 5.1 and 5.2.
5 Kinetic Theory

Table 5.1: a matrix.

<table>
<thead>
<tr>
<th>$\nu_e$</th>
<th>$\bar{\nu}_e$</th>
<th>$\nu_\mu$</th>
<th>$\bar{\nu}_\mu$</th>
<th>$\nu_\tau$</th>
<th>$\bar{\nu}_\tau$</th>
<th>$e^-$</th>
<th>$e^+$</th>
<th>$\mu^-$</th>
<th>$\mu^+$</th>
<th>$\tau^-$</th>
<th>$\tau^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1</td>
<td>$1/2$</td>
<td>$-1/2$</td>
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Table 5.2: e matrix. $w^+$ and $w^-$ are abbreviations for $\sin^2 \theta_W \pm 1/2$, which are equal to 0.7223 and $-0.2777$, respectively.

<table>
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<th>$\nu_e$</th>
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We may rewrite Eq. (5.16a) as

$$P_{kl}^2 = s = (p_k + p_l)^2,$$

and then Eq. (5.16b) as

$$t = m_k^2 c^2 + m_l^2 c^2 - P_{kl}^2 + \frac{(P_{kl}^2 + m_k^2 c^2 - m_l^2 c^2)(P_{kl}^2 - m_k^2 c^2 + m_l^2 c^2)}{2P_{kl}^2} + \frac{[P_{kl}^2 - (m_k c + m_l c)^2][P_{kl}^2 - (m_k c - m_l c)^2]}{2P_{kl}^2} \cos \theta_{kl},$$

where $\theta_{kl}$ is the scattering angle in the center-of-momentum of the system, as is shown if Figure 5.2. We can define the angle $\theta$ as [40]

$$\cos \theta = \left[ \frac{\mathbf{p}_k \cdot \mathbf{p}'_l}{|\mathbf{p}_k| |\mathbf{p}'_l|} \right]_{\text{CM}} = \frac{(p'_{kl}^2 + p''_{kl}^2)(p'_{kl} + p''_{kl})}{(p_k - p_l)^2},$$

and relate it to the parameter, $t$, through the relation

$$\cos \theta - 1 = \frac{2t}{s - 4m^2 c^2},$$

We don’t include interactions involving two charged leptons, as they interact mainly through the electromagnetic force.

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5.4 Relativistic kinetic equation

For pure neutrino-neutrino interactions (here we treat them as massless, which is very close to reality), Eq. (5.23) and Eq. (5.21) simplifies significantly to

\[ t_{m=0} = \frac{P_{kl}}{2} (\cos \theta - 1), \]  

(5.26)

\[ f(s, t)_{m=0} = (v + a)^2 + \frac{(v - a)^2}{4} (1 + \cos \theta)^2. \]  

(5.27)

For the interaction between neutrinos and the massive leptons, we need to do the full calculation.

5.4 Relativistic kinetic equation

The classical Boltzmann transport equation, or kinetic equation, was derived by Ludwig Boltzmann in 1872. It describes macroscopic quantities in non-equilibrium systems, such as energy and number of particles, in terms of statistical behavior. The relativistic version of the kinetic equation was developed gradually during the last century. We will here give a short version following the theory developed by de Groot, van Leeuwen, and van Weert in the book “Relativistic kinetic theory” [40]. To quote them: “The kinetic equation is a closed equation (meaning it can be calculated in a finite number of operations, and usually do not contain any limits) for the space-time behavior of the distribution function”. In relativistic kinetic theory, all the macroscopic quantities are defined with help from the distribution functions. In Section 3.1, Eqs. (3.5), we defined those in the cases of thermodynamic equilibrium. For non-equilibrium states, we need to include the positions as well such that we get \( f(x, p) \). The relativistic kinetic equation requires, first of all, that the number of particles in the system is large enough that a statistical approach is justified. A second requirement is that the equations are covariant\(^2\). The kinetic equation also assumes three properties (both in the classical and relativistic case):

1. Only two particle interactions are considered.

2. The hypothesis of molecular chaos. By that, we assume that the velocities of the colliding particles are uncorrelated and independent of position. The number of (binary) collisions is proportional to the product of the distributions functions of the two colliding particles. It is also proportional to a transition rate, which is a measure of the probability of the collision process.

3. The change in the distribution function is negligible in spacetime. That is: the characteristic lengths and times are negligibly small.

\(^2\)An equation is Lorentz covariant if all its key properties are valid in all inertial reference frames.
5.4.1 Kinetic equation without collisions

Consider a small volume $\Delta^3\sigma$ located at position $x$. The changes in its three-surface element changes is given by the four vector $d^3\sigma_\mu$. In the Lorentz frame this vector is purely time-like ($d^3x, 0, 0, 0$). The number of particle world lines crossing the segment $\Delta^3\sigma$ with momentum in the range $\Delta p$ around $\vec{p}$ is given by:

$$\Delta N(x, p) = \int_{\Delta^3\sigma} \int_{\Delta^3p} d^3\sigma_\mu \frac{d^3p}{p^0} p^\mu f(x, p) \, .$$  \hspace{1cm} (5.28)

In the Minkowski space element $\Delta^4x$ the net flow through the surface $\Delta^3\sigma$ vanishes such that Eq. (5.28) becomes zero. By using the divergence theorem we can rewrite this as

$$\int_{\Delta^4x} \int_{\Delta^3p} d^4x \frac{d^3p}{p^0} p^\mu \hat{c}_\mu f(x, p) \, ,$$  \hspace{1cm} (5.29)

where $\hat{c}_\mu = \hat{c}/\hat{c}x^\mu = (c^{-1} \hat{c}_t, \nabla)$ is the differential with respect to the space-time coordinates. However, since the intervals $\Delta^4x$ and $\Delta^3p$ are arbitrary, we get:

$$p^\mu \hat{c}_\mu f(x, p) = 0 \hspace{0.5cm} 4\text{-vector notation}$$  \hspace{1cm} (5.30a)

$$(\hat{c}_\mu + \vec{u} \cdot \nabla) f(x, p) = 0 \hspace{0.5cm} 3\text{-vector notation}$$  \hspace{1cm} (5.30b)

which is the collisionless relativistic transport equation.

5.4.2 Kinetic equation with collisions

If we have collisions between particles within the range of $\Delta^4x$ and a given momentum, changes. For a particle $k$, with momenta in the range $\Delta^3p_k$ the change is equal to

$$\Delta^4x \frac{\Delta^3p}{p^0} C(x, p) \, ,$$  \hspace{1cm} (5.31)

where $C(x, p)$ is an invariant function. Using the theory of molecular chaos, the average number of collisions in a volume element $\Delta^3x$ during a time interval $\Delta t$ is proportional to

1. The average number of particles per unit volume with three momentum in the range $(p_k, p_k + \Delta p_k) = \Delta^3p_k f(x, p_k)$.
2. The average number of particles per unit volume with three momentum in the range $(p_l, p_l + \Delta p_l) = \Delta^3p_l f(x, p_l)$.
3. The intervals $\Delta^3p'_k$, $\Delta^3p'_l$, and $\Delta^4x$.

The proportionality factor is denoted as:

$$W(p_k, p_l, p'_k, p'_l) \frac{\Delta^3p}{p^0_k p^0_l p^0_k' p^0_l'} \, ,$$  \hspace{1cm} (5.32)
where the nominator \( W(p_k, p_l; p'_k, p'_l) \) is called the transition rate. It is Lorentz scalar (a scalar which is invariant under a Lorentz transformation), and depends only on the four-momenta before and after collision. We can rewrite Eq. (5.32) using three vectors as:

\[
w(p_k, p_l; p'_k, p'_l) = \frac{cW(p_k, p_l; p'_k, p'_l)}{p_k \cdot p_l = p'_k \cdot p'_l}.
\] (5.33)

The quantity \( w(p_k, p_l; p'_k, p'_l) \Delta^3 p_k \Delta^3 p_l \) is the transition probability per unit volume and per unit time that two particles having momenta \( p_k \) and \( p_l \) are scattered with an outgoing momenta of \( (p_k \pm \Delta p_k) \) and \( (p_l \pm \Delta p_l) \).

According to the theory of molecular chaos the total number of particles lost to collisions in the range \( \Delta^4 x \) and \( (p + \Delta p) \) is found by integrating the proportionality factor over all values of \( p_k, p'_k, \) and \( p'_l \). We also gain particles due to collisions, that is particles with initial momenta \( (p'_k, p'_l) \) ending up with momenta \( (p_k, p_l) \). We also have to include a factor \( 1/2 \), since the two momenta states \( (p'_k, p'_l) \) and \( (p_l, p_k) \) can not be distinguished. The resulting net change of particles in the interval \( \Delta^4 x \) and \( \Delta^3 p \) is the collision function \( C(x, p_k) \)

\[
C(x, p_k) = \frac{1}{2} \int \frac{d^3 p_l}{p_l^0} \frac{d^3 p'_k}{p'_k^0} \frac{d^3 p'_l}{p'_l^0} \left[ f(x, p'_k) f(x, p'_l) W(p'_k, p'_l; p_k, p_l) - f(x, p_k) f(x, p_l) W(p_k, p_l; p'_k, p'_l) \right].
\] (5.34)

We then get the following explicit equation for the relativistic transport equation with elastic collision:

\[
p^\mu_k \partial_\mu f(x, p_k) = C(x, p_k)
\]

\[
= \frac{1}{2} \int \frac{d^3 p_l}{p_l^0} \frac{d^3 p'_k}{p'_k^0} \frac{d^3 p'_l}{p'_l^0} \left[ f(x, p'_k) f(x, p'_l) W(p'_k, p'_l; p_k, p_l) - f(x, p_k) f(x, p_l) W(p_k, p_l; p'_k, p'_l) \right].
\] (5.35)

In non-relativistic theory, \( p^\mu \partial_\mu f(x, p_k) \) is called the streaming term, while \( C(x, p_k) \) is also called the collision term.

If \( k \) and \( l \) are different particles (mixture) we need to sum over all particle species

\[
p^\mu_k \partial_\mu f(x, p_k) = \sum_{i=1}^{N} C_{kl}(x, p_k),
\] (5.36)

with the collision matrix given by

\[
C_{kl} = \gamma_{kl} \int \frac{d^3 p_l}{p_l^0} \frac{d^3 p'_k}{p'_k^0} \frac{d^3 p'_l}{p'_l^0} \left[ f(x, p'_k) f(x, p'_l) W_{kl}(p'_k, p'_l; p_k, p_l) - f(x, p_k) f(x, p_l) W_{kl}(p_k, p_l; p'_k, p'_l) \right]
\]

\[
k, l = 1, 2, 3, \ldots, N,
\] (5.37)
where $\gamma_{kl} = 1 - \frac{1}{2}\delta_{kl}$, $\delta_{kl}$ being the Kronecker delta. Finally, if we have inelastic scattering, we get $k + l \rightarrow i + j$, and a transition rate given as

$$W_{kl|ij} = W_{kl|ij}(p_k, p_l, p_i, p_j)$$

(5.39)

and a collision matrix

$$C_{kl} = \frac{1}{2} \sum_{i,j=1}^{N} \int \frac{d^3p_l}{p_l^0} \frac{d^3p_i}{p_i^0} \frac{d^3p_k}{p_k^0} \left[ f(x, p_i) f(x, p_j) W_{ij|kl} - f(x, p_k) f(x, p_l) W_{kl|ij} \right]$$

(5.40)

$$k, l = 1, 2, 3, \ldots, N.$$  

(5.41)

For reactions $k + l \rightarrow i + j$ for the $i$ and $j$ particles are heavier, the transition rate $W_{kl|ij}$ should go zero at a certain threshold.
Appendices
We will start with a short history of modern cosmology. This section is based upon several sources. Among them are the books by Guth, Lederman, Randall and Panek [41, 42, 43, 44]. These are highly recommended introductory texts meant for the layman.

One could say that modern cosmology started with Albert Einstein. Not long after publishing his theory on special relativity in 1905 (which concerns space and time in inertial frames [45]), Einstein began thinking about incorporating gravity into his theory. The first step was his principle of equivalence, which roughly says that there is no difference between inertial mass and gravitational mass [46]. It would, however, take almost another decade before all the pieces fell into place. In 1915 he published his famous field equations which describe gravity as spacetime being curved by matter and energy [47]. The year after he published his final version of the general theory of relativity [48] Einstein quickly understood the implications his theory would have for cosmology, namely that the Universe could not be static. In order to fix this “problem”, he introduced the famous cosmological constant in his 1917 paper “Cosmological Considerations in the General Theory of Relativity” [49], an act which he supposedly would call the biggest blunder of his life [50, 51]. Also, his fix was not a stable solution.

Karl Schwarzschild was the first to find an exact solution to Einstein’s field equations, for the limited case of a spherical non-rotating mass [52]. He also found the equation for the event horizon [53]. In 1917 Willem de Sitter found another solution — one for a massless universe with a positive cosmological constant [54]. Arguably the most important solution was first found by Alexander Friedmann in 1922 [33] and 1924 [34] and describes a homogeneous and isotropic expanding or contracting universe.

In 1915 V.M. Slipher was the first to measure the radial velocity of galaxies [58]. In 1927 the Belgium priest and physicist, Georges Lemaître proposed a theory of an expanding Universe [59]. Combining his own measurements with Slipher’s, Edwin Hubble published his paper about the relation between distance and radial velocity of galaxies in 1929 [56] (see Figure A.1).

After the discovery of the cosmic expansion, two competing models arose: the Big Bang theory and the Steady State theory. The Big Bang theory is often credited to Lemaître who in 1931 wrote a short article [60] suggesting that that the Universe began as a dense “primeval atom”. The most important implication of this theory is that as we go back in time, the Universe gets denser and hotter — and thus
A Short history of Modern Cosmology

Figure A.1: Two Hubble diagrams, showing radial velocity vs. distance (as replotted in [55]). Hubble’s original data plotted from 1929 on the left [56]. By looking at much larger scales (and using newer data [57]) the straight line representing the Hubble constant fits much better.

both have a beginning and a finite size. The Steady State theory was proposed by Fred Hoyle, among others, in 1948 [61]. This theory suggests that the Universe continuously creates new matter as it expands. This would happen at a pace which would perfectly balance the dilution caused by expansion. In the Steady State model, the Universe on a large scale will always have the same density and always look the same. This theory is truthful to the Perfect Cosmological Principle, which says that the Universe is homogeneous and isotropic both in space and time. A steady state expanding universe does not run into Olbers’ paradox (If the Universe is infinite, how can the night be dark?) as the light gets redshifted. The total energy flux from the sky thus remains finite.

The same year as Hoyle’s Steady State paper; Alpher, Bethe, and Gamow published their famous paper on Big Bang nucleosynthesis [62], predicting the proportion of heavier elements in the Universe. Bethe was actually only added by his friend Gamow to make a wordplay on $\alpha, \beta, \gamma$. This undoubtedly brought a lot of extra attention to the paper as describes in Figure A.2. The same Alpher, together with Herman predicted that a Big Bang model would cause a cosmic microwave background [63].

The death of the Steady State theory came in 1964 when the two astronomers Arno Penzias and Robert Wilson at Bell Labs discovered the 2.7 K microwave background more or less by accident [65]. The noise was interpreted as black body radiation from the Big Bang by Robert Dicke, James Peebles, Peter Roll, and David T. Wilkinson [66].

A few years later, in 1967, Andrei Sakharov wrote a famous paper about the requirements for a baryon-antibaryon asymmetry [67], which is one of the main unanswered questions in physics: why is there more matter than antimatter?
In the late sixties, the Big Bang theory was presented with several problems. Charles Misner and Robert Dicke formally presented the horizon problem [68] and flatness problem [69]. The first being that distant regions in the Universe, according to the standard Big Bang theory, have never been in contact with each other, and hence it is peculiar why they would have the same temperature. The flatness problem is about the overall curvature of the observable universe and why it appears to be so close to flat. The curvature is related to the overall energy-density in the Universe, and it can be shown that if this is slightly off the critical density, it will quickly diverge. This means that if the Universe is flat now, it must have been incredibly flat in its very early stages. It can be shown that if the observable universe had just one single gram of extra matter it would tip off balance [70].

In 1970 Vera Rubin and Kent Ford found evidence for dark matter by measuring galaxy rotation curves at large radii [71], although it was already postulated by both Jan Oort and Fritz Zwicky in 1932 and 1933 [72].

In 1980 Alan Guth, among others, came up with the inflationary theory [11], proposing that the Universe went through a period of extremely rapid exponential expansion from when it was around $10^{-36}$ to $10^{-32}$ seconds old. This theory solves the magnetic monopole problem, which was one of the original motivations for his theory, as well as the flatness and horizon problem.

The last 25 years have been a golden age for experimental cosmology and astronomy. There has been a boost in new telescopes and technologies. The COBE (COsmic Background Explorer) satellite was launched in 1989 to measure the cosmic microwave background radiation (CMB). When adjusted to the cosmic frame the results matched that of a blackbody spectrum to a precision of 1 to 100,000. It also gave the first (and very famous) picture of the full-sky anisotropy of the CMB. A year later, the Hubble Space Telescope was launched, giving us some of the most spectacular pictures of our universe. As of 2015, on its 25th year, the Hubble telescope is still operational. By observing distant Type Ia supernovae in the late 90s, two teams (the High-Z Supernova Search Team and Supernova Cosmology Project) found evidence that the Universe is expanding at an accelerating rate [21, 22]. The most accepted
A Short history of Modern Cosmology

hypothesis to explain these observations is dark energy, an unknown form of energy which permeates all space. If this energy is constant in space and time it will act as the cosmological constant, $\Lambda$, which Einstein introduced in 1917. There can, however, be different solutions, such as quintessence which is a dynamic quantity which can vary in space and/or time.

The COBE mission had two successors: WMAP (Wilkinson Microwave Anisotropy Probe), launched in 2001 and then the Planck satellite which was launched in 2009. Data from these two missions [73, 27] have been the main contributions for estimating several important cosmological parameters, like the age of the universe, the energy constituents, and the Hubble parameter. The $\Lambda$CDM (cosmological constant with cold dark matter) model is considered the Standard Model of cosmology. The 2015 result from the Planck mission suggests a universe consisting of roughly 5% baryonic matter, 26% dark matter, and 69% dark energy [19]. The results from the Planck mission are discussed a bit further in Section 1.2.

It seems almost poetic that this section which started with Einstein will also end with him. On February 11, 2016, one of the biggest scientific discoveries was announced — namely the first direct detection of gravitational waves [74]. These waves were predicted by Einstein in 1916 [75, 76] and was indirectly discovered in 1974 by measuring the orbits of the Hulse-Taylor binary system [77]. The measured changes in the orbit of this system and others due to the radiation of gravitational waves are in very good agreement with the theory. The first direct detection of gravitational waves, however, came almost a hundred years after Einstein’s prediction. This happened on September 14, 2015, when the gravitational waves from two merging black holes were detected here on Earth. Gravitational waves open up a completely new field of astronomy as we will be able to observe the Universe not only through the electromagnetic spectrum but through the ripples of spacetime itself.
At temperatures above a few MeV, the neutrons and protons were essentially in the same abundance, being kept in equilibrium by the reactions

\[ n + \nu_e \rightarrow p + e^- , \]  

\[ n + e^+ \rightarrow p + \nu_e . \]

The neutron is slightly heavier than the proton, and this will significantly affect the ratio of neutrons to protons when the temperature \((k_B T)\) is in the same order as the mass difference between the two particles. This neutron-to-proton-ratio given as \[ \frac{n_n}{n_p} = \exp \left( -\frac{m_n c^2 - m_p c^2}{k_B T} \right) = \exp \left( -\frac{1.29 \text{ MeV}}{k_B T} \right) . \]

Luckily for us, at \(k_B T \sim 1\) MeV, the cross-sections for the reactions given in Eqs. (B.1, B.2) are very small. The reaction involve weak force, which effectively the reactions stop and the neutrons freeze out at around 0.8 MeV [18]. This happens at \(t = 1\) s, and the neutron-to-proton-ratio is down to \(1/6\). As we know, free neutrons have a mean life of about 15 minutes, resulting in a slighter lower number of neutron-to-proton-ratio after nucleosynthesis is complete. Around 10 seconds after the Big Bang, the production of nuclei with more that one nucleon start, what we call primordial nuclear synthesis. The majority of nucleosynthesis happens between three and twenty minutes, as is shown in the left panel of Figure B.1. Only light elements are created during this period, the most important one being helium-4, and a smaller amount of deuterium (hydrogen-2), helium-3, lithium-7, and the two radioactive elements tritium (hydrogen-3) and beryllium-7. The latter two would decay into helium-3 and lithium-7. A reaction chain is given in the right panel in Figure B.1. All other elements are believed to have been created much later. The main source being evolving and exploding stars (stellar and supernova nucleosynthesis). Some of the lighter elements are also created by cosmic rays spallation. After the primordial nuclear synthesis ended the Universe consisted (by mass) of roughly 75% of hydrogen-1, about 25% helium-4, a few times \(10^{-5}\) of deuterium and helium-3, and just trace amounts (in the order of \(10^{-10}\)) of lithium [81].
Figure B.1: Model for elements created during Big Bang nucleosynthesis (BBN). Timeline on right and main reactions on left. Timeline from [78]. Recent studies suggest that the BBN is not the source of Lithium-6 [79, 80].
C Gamma matrices

The $\gamma$-matrices, also known as the $4 \times 4$ Dirac matrices being used in Section 5.3 are defined using contravariant representation and metric signature of $(+ - - -)$ as:

\[
\begin{align*}
\gamma^0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \\
\gamma^1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\
\gamma^2 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \\
\gamma^3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

The product of the four gives us $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ which we can represent as

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]
Bibliography


Bibliography


Bibliography


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Part II

Papers
Paper I

On Effective Degrees of Freedom in the Early Universe
Galaxies 4 (2016) no. 78
1. Introduction

The early Universe was filled with different particles. A tiny fraction of a second after the Big Bang, when the temperature was $10^{16}$ K $\approx 1$ TeV, all the particles in the Standard Model were present, and roughly in the same abundance. Moreover, the early Universe was in thermal equilibrium. At this time, essentially all the particles moved at velocities close to the speed of light. The average distance travelled and lifetime of these ultra-relativistic particles were very short. The frequent interactions led to the constant production and annihilation of particles, and as long as the creation rate equalled that of the annihilation rate for a particle species, their abundance remained the same. The production of massive particles requires high energies, so when the Universe expanded and the temperature dropped, the production rate of massive particles could not keep up with their annihilation rate. The heaviest particle we know about, the top quark and its antiparticle, started to disappear just one picosecond ($10^{-12}$ s) after the Big Bang. During the next minutes, essentially all the particle species except for photons and neutrinos vanished one by one. Only a very tiny fraction of protons, neutrons, and electrons, what makes up all the matter in the Universe today, survived due to baryon asymmetry (the imbalance between matter and antimatter in the Universe). The fraction of matter compared to photons and neutrinos is less than one in a billion, small enough to be disregarded in the grand scheme for the first stages of the Universe.

We know that the early Universe was close to thermal equilibrium from studying the Cosmic Microwave Background (CMB) radiation. Since its discovery in 1964 [1], the CMB has been thoroughly measured, most recently by the Planck satellite [2]. After compensating for foreground effects, the CMB almost perfectly fits that of a black body spectrum, deviating by about one part in a hundred thousand [3]. It remained so until the neutrinos decoupled. For a system in thermal equilibrium, we can use statistical mechanics to calculate quantities such as energy density, pressure, and entropy density. These quantities all depend on the number density of particles present at any given time. How the different particles contribute to these quantities depends of their nature—most important
being their mass and degeneracy. The complete contribution from all particles is a result of the sum of all the particle species’ effective degrees of freedom. We call these temperature-dependent functions \( g_\star \), and we have one for each quantity, such as \( g_{\star n} \) related to number density, and \( g_{\star \epsilon}, g_{\star p}, \text{and } g_{\star s} \) related to energy density, pressure, and entropy density, respectively.

In this paper, we will show how to calculate these four quantities \((n, \epsilon, P, s)\), as well their associated effective degrees of freedom \((g_{\star n}, g_{\star \epsilon}, g_{\star p}, g_{\star s})\). These latter functions describe how the number of different particles evolve, and we have plotted these values in Figure 1. Throughout this paper, we will look more closely at five topics. After first having a quick look at the elementary particles of the Standard Model and their degeneracy (Section 3), we address the standard approach when everything is in thermal equilibrium in Section 4. Next, we take a closer look at the behavior during the QCD phase transition; i.e., the transition from a quark-gluon plasma (QGP) to a hot hadron gas (HG) in Section 5. We then look at the behavior during neutrino decoupling (Section 6). For the fifth topic, we study how the temperature decreases as function of time (Section 8). In Appendix A, we have also included a table with the values for all four \( g_{\star s} \), as well as time, from temperatures of 10 TeV to 10 keV. This article was inspired by the lecture notes by Baumann [4] and Kurki-Suonio [5]. Other important books on the subject are written by Weinberg [6,7], Kolb and Turner [8], Dodelson [9], Ryden [10], and Lesgourgues, Mangano, Miele, and Pastor [11].

![Figure 1. The evolution of the number density \((g_{\star n})\), energy density \((g_{\star \epsilon})\), pressure \((g_{\star p})\), and entropy density \((g_{\star s})\) as functions of temperature.](image_url)

2. Notations and Conventions

The effective degrees of freedom of a particle species is defined relative to the photon. This is not just an arbitrary choice, but chosen since the photon is massless, and whose density history is best known. The most important source of information about the early Universe comes from the CMB photons. Even though the photon is the natural choice as a reference particle, technically any particle could be used. Additionally, when talking about effective degrees of freedom, we most often do so in the context of energy density \( g_{\epsilon} \), which in most textbooks is just called “\( g \).” Here, we use the notation \( g_{\epsilon} \) for that matter, and \( g_\star \) as a collective term for all four quantities.

The term “particle annihilations” frequently appears in this paper. Strictly speaking, we have particle creations and annihilations all the time, but in this context, “particle annihilations” refers to periods where the annihilation rate is (noticeable) faster than the production rate for a particle species.

In many textbooks, the value of the speed of light \( c \), the Boltzmann constant \( k_B \), and the Planck reduced constant \( \hbar \) are set to unity. We have chosen to keep these units in our equations to avoid any problems with dimensional analysis during actual calculations. One of the advantages of using
$\hbar = c = k_B = 1$ is that we can use temperature, energy, and mass interchangeably. For our equations, we use $k_B T$ and $mc^2$ when we want to express temperature and mass in units of MeV, but in the main text, when we talk about temperature and mass, it is implied that these are $k_B T$ and $mc^2$.

Simplifications are important when we first want to approach a new subject. One of our assumptions in this paper is that the early Universe was in total thermal equilibrium. There were, however, periods where this was not so. In those cases, viscous effects drove the system (the Universe) towards equilibrium. This increased the entropy. For our purposes, all viscous effects have been neglected. Some relevant papers address this issue [12–14].

3. The Standard Model Particles and Their Degeneracy

Let us start by looking at the degeneracy of the different particle species—their intrinsic degrees of freedom, $g$. The Standard Model of elementary particles are often displayed as in Figure 2. The quarks, leptons, and neutrinos are grouped into three families, shown as the first three columns. These are all fermions. The two last columns are the bosons. The fourth column consists of force mediator particles, also called gauge bosons. These are the eight gluons, the photon, and the three massive gauge bosons. The Higgs boson that was discovered at CERN in 2012 [15,16] comprises the fifth and last column.

A particle’s degeneracy depends on its nature and which properties it possesses. We have listed these as four different columns in Table 1. They are: (1) Number of different flavors. These are different types of particles with similar properties, but different masses. These are listed as separate entries in Figure 2; (2) Existence of anti-particles. Antiparticles have different charge, chirality, and color than their particle companion. Not all particles have anti-partners (e.g., the photon); (3) Number of color states. Strongly interacting particles have color charge. For quarks and their anti-partners, there are three possibilities (red, green, blue, or antired, antigreen, antiblue). Gluons have eight possible color states. These are superpositions of combined states of the three plus three colors; (4) Number of possible spin states. We remember from quantum mechanics that all bosons have integer spins, while fermions have half integer spins, both in units of $\hbar$. The spin alignment of a particle in some direction is called its polarization. Quarks and the charged leptons have two possible polarizations: $+\frac{1}{2}$ or $-\frac{1}{2}$. Another way of saying this is that they can be either left-handed or right-handed. Neutrinos, on the other hand, can only be left-handed (and antineutrinos only right-handed), so they only have one spin state. Actually,
whether neutrinos are Dirac or Majorana fermions is still an open question. Majorana fermions are their own antiparticles, while Dirac fermions have distinct particles and antiparticles. In the latter case, we expect there to be additional right-handed neutrinos and left-handed antineutrinos, whose weak interaction is suppressed. These “new” neutrinos are expected to have negligible density compared to the left-handed neutrinos and right-handed antineutrinos [17]. The book by Lesgourgues, Mangano, Miele, and Pastor [11] also discusses this topic in detail. The massive spin-1 bosons ($W^\pm$ and $Z^0$) have three possible polarizations ($-1, 0, 1$): one longitudinal and two transverse. The massless spin-1 bosons (photons and gluons) have only two possible polarizations, namely the transverse ones. The Higgs particle is a scalar particle and has spin-0. Finally, we should say that hadrons can have multiple possible spin states, depending on their composition.

### Table 1. The Standard Model of elementary particles and their degeneracies.

<table>
<thead>
<tr>
<th>Particles</th>
<th>Flavors</th>
<th>Particle + Antiparticle</th>
<th>Colors</th>
<th>Spins</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quarks (u, d, c, s, t, b)</td>
<td>6</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>72</td>
</tr>
<tr>
<td>Charged leptons (e, $\mu$, $\tau$)</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>Neutrinos ($\nu_e$, $\nu_\mu$, $\nu_\tau$)</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>Gluons (g)</td>
<td>1</td>
<td>1</td>
<td>8</td>
<td>2</td>
<td>16</td>
</tr>
<tr>
<td>Photon ($\gamma$)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Massive gauge bosons ($W^\pm$, $Z^0$)</td>
<td>2</td>
<td>2, 1</td>
<td>1</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>Higgs bosons ($H^0$)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>All elementary particles</td>
<td>17</td>
<td>11</td>
<td>18</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

At high temperatures where all the particles of the Standard Model are present, we have 28 bosonic and 90 fermionic degrees of freedom. It turns out that fermions do not contribute as much as bosons, since they cannot occupy the same state. We will get back to this in the next section, and just say that fermions have $28 + \frac{7}{8} \times 90 = 106.75$ effective degrees of freedom for energy density, pressure, and entropy density. For the number density, the effective degrees of freedom is $28 + \frac{3}{4} \times 90 = 95.5$.

4. Statistical Mechanics of Ideal Quantum Gases in Thermodynamic Equilibrium

In this section, we briefly review the statistical mechanics of ideal quantum gases in thermal equilibrium. We also introduce the concept of effective number of degrees of freedom for a particle species, and how to count these as functions of the temperature.

4.1. Thermodynamic Functions

In order to calculate the thermodynamic functions, we need to know the single-particle energies of the system. We consider a cubic box with periodic boundary conditions, and with sides of length $L$ and volume $V = L^3$. Solving the Schrödinger equation for a particle, we find the possible momentum eigenvalues

$$\hat{\vec{p}} = \frac{\hbar}{L} (n_1 \vec{e}_x + n_2 \vec{e}_y + n_3 \vec{e}_z),$$

where $\hbar$ is the Planck constant, $n_i = 0, \pm 1, \pm 2, \pm 3, \ldots$, and $\vec{e}_x, \vec{e}_y, \vec{e}_z$ are the standard units vectors in three-dimensional Euclidean space. The energy of a particle with mass $m$ and momentum $\vec{p}$ is $E(\vec{p}) = \sqrt{m^2 c^4 + \vec{p}^2 c^2}$.

In thermal equilibrium, the probability that a single-particle state with momentum $\vec{p}$ and energy $E(\vec{p})$ is occupied is given by the Bose–Einstein or Fermi–Dirac distribution functions

$$f(\vec{p}) = \frac{1}{e^{(E(\vec{p}) - \mu)/k_B T} + 1},$$

where the upper sign is for fermions and the lower sign for bosons. Moreover, $k_B$ is the Boltzmann constant and $\mu$ is the chemical potential. In order to find the total number of particles occupying a state
with energy $E$, we must find the density of states in phase space. We see from Equation (1) that the number of possible states in momentum space is $L^3/h^3$. By dividing by the volume, $L^3$, as well, we are left with the factor $(1/h)^3$. If there is an additional degeneracy $g$ (for example, spin), we can write the density of states (dos) as

$$\text{dos} = \frac{g}{L^3} = \frac{g}{(2\pi)^3 h^3}. \quad (3)$$

The density of particles with momentum $\hat{p}$ is then given by

$$n(\hat{p}) = \frac{g}{(2\pi)^3 h^3} \times f(\hat{p}). \quad (4)$$

The total density of particles, $n$, can then be written as an integral over three-momentum involving the distribution function as

$$n = \frac{g}{(2\pi)^3 h^3} \int f(\hat{p}) d^3\hat{p}. \quad (5)$$

By multiplying the distribution function (2) with the energy and integrating over three-momentum, we obtain the energy density $\epsilon$ of the system. The pressure, $P$, can be found in a similar manner by multiplying the distribution function with $|\vec{p}|^2/(3E/c^2)$ (a nice derivation of this is shown by Baumann [4]). This yields the integrals

$$\epsilon = \frac{g}{(2\pi)^3 h^3} \int E(\hat{p}) f(\hat{p}) d^3\hat{p}, \quad (6)$$

$$P = \frac{g}{(2\pi)^3 h^3} \int \frac{|\vec{p}|^2}{3E/c^2} f(\hat{p}) d^3\hat{p}. \quad (7)$$

Finally, let us mention the entropy density $s$. It can be calculated from the thermodynamic relation

$$s = \frac{\epsilon + P - \mu_T}{T}, \quad (8)$$

where the index $\mu_T$ is the total chemical potential. We will get back to chemical potentials in Section 4.3.

4.2. From Momentum to Energy Integrals

It is sometimes more convenient to use energy, $E$, instead of the momentum, $\vec{p}$, as the integration variable. By integrating over all angles, we can replace $d^3\hat{p}$ by $4\pi |\vec{p}|^2 dE$. Using the energy momentum relation, we find $|\vec{p}| = \sqrt{E^2 - m^2 c^2}$ and $c \vec{p} \cdot d\vec{p} = E dE$. We can simplify these formulas further by introducing the dimensionless variables $u$, $z$, and $\hat{\mu}$.

$$u = \frac{E}{k_B T}, \quad z = \frac{mc^2}{k_B T}, \quad \hat{\mu} = \frac{\mu}{k_B T}. \quad (9)$$

This yields the following expressions for the number density, energy density, and pressure for a species $j$, and for all species (as this is simply the sum of all particle species).

$$n_j(T) = \sum \frac{g_j}{2\pi^2 h^3} \int_{u_j, z_j}^\infty \frac{E}{e^{E-n_j/k_BT} + 1} dE \quad (10a)$$

$$n_j(T) = \sum \frac{g_j}{2\pi^2 h^3} \left( \frac{k_B T}{\hbar c} \right)^3 \int_{u_j}^\infty \frac{u}{e^{u-z_j} + 1} du \quad (10b)$$

$$n(T) = \sum_j n_j = \sum_j \frac{g_j}{2\pi^2 h^3} \left( \frac{k_B T}{\hbar c} \right)^3 \int_{u_j}^\infty \frac{u}{e^{u-z_j} + 1} du \quad (10c)$$
As shown in Equation (8) we can find the entropy density for a single species \( j \) and the total entropy as:

\[
\begin{align*}
\epsilon_j(T) &= \frac{S_j}{2\pi^2\hbar^3} \int_{m\epsilon^2}^{\infty} \frac{E^2 - m_j^2 c^4}{e^{\frac{(E - m_j)c^2}{\hbar k_B T}} + 1} \, dE, \\
\epsilon(T) &= \sum_j \epsilon_j = \sum_j \frac{S_j}{2\pi^2\hbar^3} \left(\frac{k_B T}{\hbar c}\right)^4 \int_{z_j}^{\infty} \frac{u^2 \sqrt{u^2 - z_j^2}}{e^{\frac{u^2 - \mu_j}{\hbar c k_B T}} + 1} \, du,
\end{align*}
\]

\[
\begin{align*}
P_j(T) &= \frac{S_j}{6\pi^2\hbar^3} \int_{m\epsilon^2}^{\infty} \frac{(E^2 - m_j^2 c^4)^{3/2}}{e^{\frac{(E - m_j)c^2}{\hbar k_B T}} + 1} \, dE, \\
P(T) &= \sum_j P_j = \sum_j \frac{S_j}{6\pi^2\hbar^3} \left(\frac{k_B T}{\hbar c}\right)^4 \int_{z_j}^{\infty} \frac{(u^2 - z_j^2)^{3/2}}{e^{\frac{u^2 - \mu_j}{\hbar c k_B T}} + 1} \, du.
\end{align*}
\]

4.3. Chemical Potentials

Before we proceed, we briefly discuss the chemical potentials. We recall from statistical mechanics that we can introduce a chemical potential \( \mu_j \) for each conserved charge \( Q_j \). This is done by replacing the Hamiltonian \( H \) of the system with \( H - \mu_j N_{Q_j} \), where \( N_{Q_j} \) is the number operator of particles with charge \( Q_j \).

In the Standard Model, there are five independent conserved charges. These are electric charge, baryon number, electron-lepton number, muon-lepton number, and tau-lepton number. This means there are also five independent chemical potentials [7]. The chemical potentials are determined by the number densities. The electric charge density is very close to zero. The baryon density is estimated to be less than a billionth of the photon density [18,19]. Lepton density is also thought to be very small, on the same order as the baryon number. According to Weinberg [7], for an early Universe scenario, we can put all these numbers equal to zero to a good approximation. For a correct representation of the Universe, the chemical potentials cannot all cancel out—otherwise, there would be no matter present today. For more general calculations including chemical potentials, the book by Weinberg is recommended [6]. The implications of a large neutrino chemical potential is discussed by Pastor and Lesgourgues [20]. Mangano, Miele, Pastor, Pisanti, and Sarikasa discuss the chemical potentials and their influence on the effective number of neutrino species [21] (we will briefly mention effective neutrino species in Section 6.2).
4.4. Massless Particle Contributions

In Equations (10)–(12), we see how dimensionless units, $u$, $z$, and $\mu$, simplifies the integrals. In the ultrarelativistic limit, we can ignore the particle masses. Moreover, as we have set the chemical potentials to zero, we can easily solve the dimensionless integrals appearing in Equations (10b), (11b), and (12b) analytically. Since the integrals for energy density and pressure in the massless cases are the same, we find:

$$\int_0^\infty \frac{u^5}{e^{u^2}+1} \, du = \begin{cases} 2\zeta(3) \approx 1.803 & \text{(Fermions)}, \\ 2\zeta(3) \approx 2.404 & \text{(Bosons)}, \end{cases}$$

(14)

and

$$\int_0^\infty \frac{u^5}{e^{u^2}+1} \, du \geq \begin{cases} \frac{\sqrt{3} \pi}{12} \approx 5.682 & \text{(Fermions)}, \\ \frac{\sqrt{3} \pi}{12} \approx 6.494 & \text{(Bosons)}, \end{cases}$$

(15)

where $\zeta(3)$ is the Riemann zeta function of argument 3. Using these results, we find the values for $n$, $\epsilon$, $P$, and indirectly $s$ for massless bosons and fermions:

$$n_b(T) = \frac{\zeta(3)}{\pi^2} \left(\frac{k_b T^3}{\hbar c}\right)^2, \quad n_f(T) = \frac{3}{4} \frac{\zeta(3)}{\pi^2} \left(\frac{k_b T^3}{\hbar c}\right)^2,$$

(16)

$$\epsilon_b(T) = \frac{\pi^2}{30} \left(\frac{k_b T}{\hbar}\right)^4, \quad \epsilon_f(T) = \frac{7}{8} \frac{\pi^2}{30} \left(\frac{k_b T}{\hbar}\right)^4,$$

(17)

$$P_b(T) = \frac{\pi^2}{90} \left(\frac{k_b T}{\hbar}\right)^4, \quad P_f(T) = \frac{7}{8} \frac{\pi^2}{90} \left(\frac{k_b T}{\hbar}\right)^4,$$

(18)

$$s_b(T) = \frac{2\pi^2}{45} \left(\frac{k_b T}{\hbar}\right)^4, \quad s_f(T) = \frac{7}{8} \frac{2\pi^2}{45} \left(\frac{k_b T}{\hbar}\right)^4.$$  

(19)

Here the subscript $b$ is for bosons, and $f$ is for fermions. We see that solving the integrals gives a difference between fermions and bosons, namely a factor $\frac{3}{4}$ for the number density and $\frac{7}{8}$ for energy density and pressure. We will call these two factors the “fermion prefactors”. We also see that the pressure is simply one third that of the energy density, while the entropy density can be found by multiplying the energy density by $4/(3T)$.

4.5. Effective Degrees of Freedom

In most cases we cannot ignore the particle masses. In these cases, we must solve the integrals in Equations (10b), (11b), and (12b) numerically. The integrals are decreasing functions of the temperature, and they vanish in the limit $k_b T/m_c^2 \to 0$. We can normalize these by dividing their values by the case of the photon (but with $g$ equal to one). As we recall, the photon has a bosonic nature with $m = 0$ and $\mu = 0$. This means that for massive particles at high temperature ($k_b T \gg m_c^2$), one actual degree of freedom for bosons contributes as much as one degree of freedom for photons, and the fermions a little less. As the temperature drops, and less particles are created, the effective contributions will be smaller. By including the intrinsic degrees of freedom ($g$), we find each particle species’ effective degree of freedom, $g_\epsilon$:

$$g_{\epsilon n_i}(T) = \frac{g_i \left(\frac{k_b T}{\hbar}\right)^3}{\frac{1}{\pi^2} \left(\frac{k_b T}{\hbar}\right)^3} \int_0^\infty \frac{u^5}{e^{u^2}+1} \, du = \frac{g_i}{\zeta(3)} \int_0^\infty \frac{u^5}{e^{u^2}+1} \, du,$$

(20)
Equations (20)–(23) over all particle species to that of the photon. For number density, pressure, and entropy density, they are a little lower.

The effective degrees of freedom are defined as functions of the corresponding variables. A more detailed look at each of the four vertical panels (c–f) is given in the lower four panels (c–f). Here the solid colored curves are for the bosons, and the dash-dotted colored curves are for the fermions. The grey dash-dotted curves represent the fermions’ contribution compared to that of the photon. For number density, pressure, and entropy density, they are a little lower.

Figure 3. The effective degrees of freedom \( g_{*n}, g_{*s}, g_{*p}, \) and \( g_{*s} \) for bosons (a) and fermions (panel b) with \( g = 1 \) (and \( \mu = 0 \)) as functions of the temperature. We have also listed the results in Table B1 in Appendix B. When the temperature is equal to the mass (\( k_B T = mc^2 \)), the effective degrees of freedom for energy density is approximately 0.9 for bosons and 0.8 for fermions, compared to that of the photon. For number density, pressure, and entropy density, they are a little lower.

In Figure 3, we have plotted the effective degrees of freedom for massive bosons (panel a) and fermions (panel b) with \( g = 1 \) (and \( \mu = 0 \)) as functions of the temperature. We have also listed the results in Table B1 in Appendix B. When the temperature is equal to the mass (\( k_B T = mc^2 \)), the effective degrees of freedom for energy density is approximately 0.9 for bosons and 0.8 for fermions, compared to that of the photon. For number density, pressure, and entropy density, they are a little lower.

The effective degrees of freedom are defined as functions of the corresponding variables and temperature. We find the total effective degrees of freedom for \( g_{*n}, g_{*s}, \) and \( g_{*p} \) by summing Equations (20)–(23) over all particle species \( j \):

\[
g_{*e}(T) = \frac{2}{\pi^2} \int_0^\infty \frac{u^2 \sqrt{u^2 - z_j^2}}{e^{2u} + 1} \, du,
\]

\[
g_{*p}(T) = \frac{2}{\pi^2} \int_0^\infty \frac{(u^2 - z_j^2)^{3/2}}{e^{2u} + 1} \, du,
\]

\[
g_{*s}(T) = \frac{3g_{*e}(T) + g_{*p}(T)}{4}.
\]
Finally, the effective degrees of freedom associated with entropy is then:

\[ g \star s (T) \equiv \frac{45}{2\pi^2} s(T) T^3 \]

\[ = \sum_j \frac{15g_j}{\pi^2} \int_{z_j}^{\infty} \frac{u^3}{u^2 - z_j^2} \frac{du}{e^{u \pm 1}} \]  \quad (27a)

\[ \frac{3g \star \epsilon(T) + g \star p(T)}{4} \]  \quad (27b)

5. Particle Evolution During the Cooling of the Universe

Our analysis starts with all the particles of the Standard Model present. As the Universe expands and cools, the annihilation rate of the more massive particles will become smaller and smaller compared to their creation rate. As the heavier particles disappear, this again will lead to a relatively larger creation rate for all the remaining lighter particle species. The overall number of particles in a comoving volume will thus remain (almost) constant. A few minutes after the Big Bang, when the temperature was down to 10 keV (corresponding to 100 million Kelvin), the Universe was mainly filled with photons and neutrinos. As we mentioned in Sections 1 and 4.3, a small—and at this stage, negligible—portion of matter survived due to the baryon asymmetry. Without the presence of antiparticles, the matter particles (i.e., nucleons and electrons) thus survived and “froze out” when their reaction rate (i.e., annihilation and creation rate) became slower than the expansion rate of the Universe (or equivalently, when the time scale of the weak interaction became longer than the age of the Universe) \[8\]. This process has some similarities with the decoupling of the neutrinos (which we will discuss in more detail in Section 6). These relic matter particles still interact with the photons and remain in thermal equilibrium until after the photon decoupling at around 380,000 years after the Big Bang [2]. Although negligible in the early stages of the Universe, matter eventually became the dominant energy contributor around 47,000 years after the Big Bang [10]. This is because non-relativistic (cold) matter receive their energy mainly from their rest mass. The energy density for cold matter goes as \(T^{-3}\). This is solely due to the dilution of the particles. The kinetic contribution to the energy is negligible. Radiation (massless particles) goes as \(T^{-4}\), because it is also subject to redshift as the Universe expands. A simple overview of events which affects the four \(g_s\) is given in Table 2.
Table 2. List of events which impacts $g_{\star n}, g_{\star \epsilon}, g_{\star p},$ and $g_{\star s}$. For the particle annihilation events, we have here used the particle masses as a reference. By combining this Table with Table B1 in Appendix B, we get a more precise picture.

<table>
<thead>
<tr>
<th>Event</th>
<th>Temperature</th>
<th>$g_{\star n}$</th>
<th>$g_{\star \epsilon}$</th>
<th>$g_{\star p}$</th>
<th>$g_{\star s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annihilation of $t\bar{t}$ quarks</td>
<td>$&lt;173.3$ GeV</td>
<td>95.5</td>
<td>106.75</td>
<td>106.75</td>
<td>106.75</td>
</tr>
<tr>
<td>Annihilation of Higgs boson</td>
<td>$&lt;125.6$ GeV</td>
<td>86.5</td>
<td>96.25</td>
<td>96.25</td>
<td>96.25</td>
</tr>
<tr>
<td>Annihilation of $Z^0$ boson</td>
<td>$&lt;91.2$ GeV</td>
<td>82.5</td>
<td>92.25</td>
<td>92.25</td>
<td>92.25</td>
</tr>
<tr>
<td>Annihilation of $W^+W^-$ bosons</td>
<td>$&lt;80.4$ GeV</td>
<td>76.5</td>
<td>86.25</td>
<td>86.25</td>
<td>86.25</td>
</tr>
<tr>
<td>Annihilation of $b\bar{b}$ quarks</td>
<td>$&lt;4190$ MeV</td>
<td>67.5</td>
<td>75.75</td>
<td>75.75</td>
<td>75.75</td>
</tr>
<tr>
<td>Annihilation of $\tau^+\tau^-$ leptons</td>
<td>$&lt;1777$ MeV</td>
<td>64.5</td>
<td>72.25</td>
<td>72.25</td>
<td>72.25</td>
</tr>
<tr>
<td>Annihilation of $\ell^+\ell^-$ quarks</td>
<td>$&lt;1290$ MeV</td>
<td>55.5</td>
<td>61.75</td>
<td>61.75</td>
<td>61.75</td>
</tr>
<tr>
<td>QCD transition $^\dagger$</td>
<td>$150–214$ MeV</td>
<td>15.5</td>
<td>17.25</td>
<td>17.25</td>
<td>17.25</td>
</tr>
<tr>
<td>Annihilation of $\pi^+\pi^-$ mesons</td>
<td>$&lt;139.6$ MeV</td>
<td>13.5</td>
<td>15.25</td>
<td>15.25</td>
<td>15.25</td>
</tr>
<tr>
<td>Annihilation of $\pi^0$ mesons</td>
<td>$&lt;135.0$ MeV</td>
<td>13.5</td>
<td>14.25</td>
<td>14.25</td>
<td>14.25</td>
</tr>
<tr>
<td>Annihilation of $\mu^+\mu^-$ leptons</td>
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<td>9.5</td>
<td>10.75</td>
<td>10.75</td>
<td>10.75</td>
</tr>
<tr>
<td>Neutrino decoupling</td>
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<td>6.863</td>
<td>6.863</td>
<td>7.409</td>
</tr>
<tr>
<td>Annihilation of $e^+e^-$ leptons</td>
<td>$&lt;511.0$ keV</td>
<td>3.636</td>
<td>3.363</td>
<td>3.363</td>
<td>3.909</td>
</tr>
</tbody>
</table>

$^\dagger$ Using lattice QCD, this transition is normally calculated to $150–170$ MeV.

5.1. Quark-Gluon Plasma vs. Hadron Gas

In the early Universe, quarks and gluons moved freely around. A gas consisting of quarks and gluons at high temperature is referred to as a quark-gluon plasma, in analogy with an ordinary electromagnetic plasma. This is in contrast to today, where we do not observe free quarks, but only hadrons (e.g., pions and nucleons) that are bound states of either three quarks, three antiquarks, or a quark-antiquark pair. These different combinations are called baryons and mesons, and are both bound together by the gluons. While quarks and gluons carry color charge, the hadrons we observe are color neutral. At some critical temperature of the Universe $T_c$, a phase transition from a quark-gluon plasma to a hadronic phase took place. We call the gas formed immediately after the phase transition a hadron gas. This is similar to the formation of atoms, where the nucleus and electrons are bound together by electric forces. The aforementioned phase transition took place when the temperature of the Universe was approximately $150–170$ MeV [22,23]. The transition temperature can be calculated by so-called lattice Monte Carlo simulations. Although the study of the phase transition from a quark-gluon plasma to a hadron gas is rather difficult, we can get an estimate of the critical temperature by evaluating the effective degrees of freedom for the energy density. This estimate could be thought of as an upper limit bound, as we cannot have an increase in $g_{\epsilon \epsilon}$ (i.e., the energy density) as the universe expands.

5.2. Effective Degrees of Freedom in the QGP and HG Phases

Let us start an analysis at very high temperature, where all the elementary particles are present and effectively massless. $g_{\epsilon \epsilon}$ is therefore at a maximum. As the temperature decreases, the various
particles annihilate, and $g_\star$ falls accordingly. We trace the number of effective degrees of freedom as a function of the temperature in Figure 4a. Here, the yellow dotted curve shows the effective degrees of freedom in the quark-gluon plasma phase (if it would exist for all temperatures). Without a phase transition, the quarks disappear only when the temperatures drop below their rest mass value. At the far right (colder) part of the scale, the gluons are still present together with the photons and neutrinos. In a similar manner, we can trace the effective degrees of freedom in a hadronic phase (if it would exist for all temperatures), as shown in the purple dash-dotted curve. As in the real world, relatively speaking we only have photons and neutrinos present at low temperatures. As we go left to higher temperatures, the first increase in $g_\star$ is caused by the presence of electron–positron pairs. The muons and the lightest mesons (namely, the pions), are the next particles to appear. We then get a very steep increase in $g_\star$, starting at around 100 MeV. This is due to the appearance of many heavier hadrons, whose numbers grow almost exponentially as the temperature increases.

\[
\begin{align*}
\text{Figure 4.} & \quad \text{Panel (a) shows the effective degrees of freedom (dof) for } g_\star \text{ in the quark-gluon and hadronic phases as functions of temperature. The yellow dotted curve represents the quark-gluon degrees of freedom and the purple dash-dotted curve is for the hadronic equivalent. In panel (b), we have zoomed in around the phase transition and plotted } g_\star \text{ for three different transition temperatures: } k_B T_c = 214 \text{ MeV in solid green, } k_B T_c = 170 \text{ MeV in dash-dotted red, and } k_B T_c = 150 \text{ MeV in dashed blue.}
\end{align*}
\]

We can now define a “cross-over” temperature $T_\star$, which is the temperature at which the two curves intersect. Hence, the phase with the lower number of effective degrees of freedom for energy density wins (in QCD theory, one normally compares the pressure of the two phases, and the phase with the higher pressure wins). Using the particles listed by the Particle Data Group [19] (and listed in Appendices C and D), this yields $k_B T_\star = 214$ MeV. However, if there are more possible baryonic states (which there most likely are), this temperature will be lower. This cross-over temperature could be thought of as the QCD transition temperature. To get a more accurate estimate for the transition temperature, one can use the numerical method called lattice simulations. Using this latter method, one obtains a transition temperature $k_B T_c$ in the 150–170 MeV range. The value depends on the number of quarks and their mass used for the calculation. Thus, our simple estimate gives us the correct order of magnitude, but a bit too high. Speculatively, however, it is possible that it can be thought of as an upper bound.

In Figure 4b, we zoom in around the transition temperature. We recognize the partly covered yellow and purple curves from panel-a, representing the QGP and HG scenarios. The green curve represents a transition temperature of the aforementioned 214 MeV. If we insist on a critical temperature of 170 MeV, we follow the yellow curve for the QGP to the right, and as we hit this temperature, we jump down to the HG curve. This discontinuous curve for $g_\star$ is shown in dash-dotted red color.
We will later see (Section 8) that this can be interpreted as the temperature remaining constant over a time while the degrees of freedom are reduced. The same remarks apply to the blue curve, which represents a 150 MeV transition.

5.3. A Closer Look at Each Particle Group

Let us have a closer look at how each group of particle species contributes to \( g_\star \). Figure 5a shows how the different particle groups contribute to the energy density as the temperature of the Universe drops. Let us look at the simplest case first—the photon (shown as the black dashed line). It always has two degrees of freedom, and thus a constant contribution, \( g_\star \nu_\gamma \), equal to two. The charged leptons (\( l \)) consist of the taus, muons, electrons, and their antiparticles. They are fermions, with two possible spin states. Each generation has a degeneracy of 3.5, which adds up to 10.5 at high temperatures. The magenta dash-dotted curve in Figure 5a shows how the charged lepton contribution drops around the time when the temperature \( (k_B T) \) goes below that of the particle masses \( (mc^2) \). The tau and antitau have a mass of 1777 MeV, so when the temperature drops below this value, their abundance will drop, and at a few hundred MeV they are all but gone, and \( g_\star l \) will have dropped to about 7. The same process happens for the muons and electrons from \( k_B T \sim 100 \text{ MeV} \) and \( k_B T \sim 0.5 \text{ MeV} \), when the value of \( g_\star l \) drops to 3.5, and finally zero. The case is more or less the same for the massive bosons (\( W^\pm, Z^0, H^0 \)). They have a total degeneracy of 10, and all have masses of around 100 GeV, which means that their annihilations will overlap as seen in the red dotted curve. For neutrinos (solid blue curve), we see a fall in \( g_\star \nu \) after they have decoupled, and the electron–positrons start to annihilate. We look closer at this in Section 6.1.

![Figure 5](image-url)

**Figure 5.** Panel (a) shows the contribution to the effective degrees of freedom (dof) for energy density from all particle groups. The drop in each group's \( g_\star \) value corresponds to ongoing annihilations of particles at that temperature. Panel (b) shows the total hadron contribution (green solid curve) to \( g_\star \) around the QCD phase transition temperature. We have further divided this into a baryon part (blue dotted curve) and a meson part (red dash-dotted curve). We have also plotted the pions specifically (black dashed curve), as they are the main hadronic contributor to \( g_\star \) at low temperatures. The two plots clearly show how fast the hadronic contribution increases at temperatures beyond 100 MeV. In both panels, we have marked the contribution to \( g_\star \) from hadrons, baryons, and mesons, at the three transition values of 214 MeV (\( \circ \) symbols), 170 MeV (\( \triangle \) symbols), and 150 MeV (\( \vartriangle \) symbols), respectively.

For the color-charged particles (gluons and quarks), things are a bit more complicated due to the differences before and after the QCD phase transition. In Figure 5a, we have plotted both the quark-gluon plasma and hadron gas without any transition. Instead, we have marked their value at three different transition values: \( k_B T_c = 214 \text{ MeV} \) (marked with \( \circ \)), \( k_B T_c = 170 \text{ MeV} \) (marked with \( \triangle \)), and \( k_B T_c = 150 \text{ MeV} \) (marked with \( \vartriangle \)).
and $k_B T_c = 150$ MeV (marked with $\nabla$). The case for the gluons is straightforward—they have 16 degrees of freedom for $T > T_c$, and zero after. Quarks—being massive—begin with 63 effective degrees of freedom, which will gradually decrease as the top, bottom, and charm particles disappear. At the time of the phase transition, this value is down to about $\sim 32$, depending on $T_c$.

After the phase transition, we need to count the hadronic degrees of freedom. We can distinguish these by baryons and mesons, as is done in Figure 5b. The only hadrons with masses less than $k_B T_c$ are the three pions, which for $T = T_c$ have roughly three degrees of freedom. There are, however, many heavier hadrons, which single-handedly do not contribute much at low temperatures, but the sheer number of different hadronic states results in a collective significant contribution. Going from low to high temperatures in Figure 5b, the effective degrees of freedom from mesons (red dash-dotted curve) and baryons (blue dotted curve) increase almost exponentially. This value is quite different at different $T_c$. Following the hadrons (green curve) from right to left, we see that at $k_B T = 150$ MeV, the hadrons make up roughly 12 effective degrees of freedom. At $k_B T = 170$ MeV, this number is approximately 19, and at $k_B T = 214$ MeV, we have roughly 48—which is the same as the $16 + 32$ degrees of freedom from the free quarks and gluons.

6. Decoupling

As we mentioned in Section 1, particles are kept in thermal equilibrium by constantly colliding (interacting) with each other. The collision rate depends on two factors—the cross section $\sigma$ and the particle density $n$. The cross section depends on several factors, but the most important one is by which forces the particles interact. Those which feel the strong and electromagnetic force interact strongly, while those which only feel the weak force interact much weaker. The cross sections related to the different forces depend on the temperature, or more correctly on the energy involved in the reaction. How these interaction strengths change are different for the four forces. In general, they become closer in strength for higher temperatures.

When the Universe expands, dilutes, and cools, particles travel farther and farther before interacting. That is, their mean free path and lifetime increases. As mentioned in Section 5, at some time the interaction rate for some particles can become slower than the expansion rate of the Universe, and (on average) those particles will never interact again. The time at which this happens is defined as the time of decoupling. For neutrinos, this happened about one second after the Big Bang (and we will get back to this in the next section), and for photons this happened about 380,000 years later (due to recombination and forming of neutral atoms). Let us look at the general case. First we need to introduce the concept of comoving coordinates and volumes. Comoving coordinates move with the rest frame of the Universe; i.e., they do not change as the Universe expands. An analogy of this would be to draw dots on a balloon. The actual distance between the dots increases as the balloon is inflated, but their comoving distance remains the same. For a comoving volume with constant entropy $S$, we can write

$$S = s(T)a^3 = g_{\ast}(T)\frac{2\pi^2}{45}T^3a^3 = \text{constant}$$

$$\rightarrow g_{\ast}(T)T^3a^3 = \text{constant} . \quad (28)$$

One of the consequences of this is that the temperature will fall slower during particle annihilations (i.e., when the effective degrees of freedom decreases). To understand this, we need to look at what is going on during particle creations and annihilations, as well as rest mass energy vs. kinetic energy.

During reactions where we go from two massive particles to two lighter particles, the excess rest mass energy will be converted to kinetic energy. Thus, the lighter particles will on average have a higher kinetic energy than the other particles in the thermal “soup”. Normally, this is countered by the reversed reaction—namely, reactions where two lighter particles create two more massive ones with less kinetic energy. Throughout periods where we have particle annihilations, there will be a net
flow of massive particles to lighter particles plus kinetic energy. Hence, the temperature will fall slower in these periods.

In order to maintain thermal equilibrium, particles need to constantly interact. That is, there needs to be some coupling between them (directly or indirectly). If some particles decouple, it means that they on average will never interact again, so if a particle species has decoupled before an annihilation process starts, their temperature will decrease independently of those which are still coupled together. As a result, there will be two different temperatures: the photon-coupled temperature ($T_{\gamma}$) (those particles that directly or indirectly interact with the photons), and the decoupled-particle temperature ($T_{dc}$). Solving Equation (28) before and after an annihilation process (indicated by subscripts “1” and “2”) for the photon-coupled ($\gamma_c$) and decoupled (dc) particles gives us

$$g_{\gamma c1} T_{\gamma c1}^3 = g_{\gamma c2} T_{\gamma c2}^3,$$

$$g_{dc1} T_{dc1}^3 = g_{dc2} T_{dc2}^3. \tag{30}$$

After decoupling, but before an annihilation process, the two temperatures are the same. Well, close enough, as we will briefly discuss in Section 8. Once a photon-coupled particle species start to annihilate, the degrees of freedom for (all) the coupled particles will reduce, while it will remain the same for the decoupled ones. Solving for the decoupled temperature after annihilation gives us:

$$T_{dc2}^3 = \frac{g_{\gamma c2} T_{\gamma c2}^3}{g_{\gamma c1}} \tag{31}$$

which we normally write as

$$T_{dc} = \sqrt[3]{\frac{g_{\gamma c2}}{g_{\gamma c1}}} T. \tag{32}$$

In principle, we can do this for more than one decoupled particle species, and get two or more different temperatures for the decoupled particles.

6.1. Neutrino Decoupling

Before they are decoupled, neutrinos are kept in thermal equilibrium with the photon-coupled particles mainly via weak interactions with electrons and positrons. Around one second after the Big Bang, the rate of the neutrino–electron interactions becomes slower than the rate of expansion of the Universe, $H$. The collision rate between neutrinos and electrons (and its antiparticle), $\Gamma_v$, is given by $[6,17]$:

$$\Gamma_v = n_e \sigma_{\nu e} = \left(\frac{k_B T}{\hbar c}\right)^3 \left(\frac{\hbar c G_{\nu e} k_B T}{k_B T}\right)^2 \approx \frac{G^2_{\nu e} (k_B T)^5}{\hbar c}, \tag{33}$$

where $n_e$ is the number density of electrons and $\sigma_{\nu e}$ is the neutrino–electron scattering cross section. $G_{\nu e} = G_F/\hbar c = 1.166 \times 10^{-8}$ GeV$^{-2}$ is the weak coupling constant $[24,25]$. By using the equation for energy density, either from Equation (11c), or better, by fast-forwarding to Equation (48), the expansion rate at the same time is given by the first Friedmann equation:

$$H = \sqrt{\frac{8 \pi G_0}{3 c^2} \epsilon} = \sqrt{\frac{8 \pi G}{3 c^2} \rho_{\gamma}(T) \frac{T^4}{\hbar c}} \approx \sqrt{\frac{5G (k_B T)^4}{(hc)^3}}. \tag{34}$$
The prefactors in $\Gamma$ and $H$ roughly cancel each other, such that we end up with

$$\frac{\Gamma}{H} = G_{\text{vac}}^2 \sqrt{\frac{\hbar c}{8G}} (k_B T)^3 \approx \left(\frac{T}{10^{10} \text{ K}}\right)^3. \quad (35)$$

This is a rough estimate, but one that is most commonly used (e.g., by Weinberg [6]). Being relativistic, the neutrino temperature $T_\nu$ scales as $a^{-1}$, while the energy density and number density scale as $a^{-4}$ and $a^{-3}$, respectively.

### 6.2. Neutrino Temperature and Entropic Degrees of Freedom

Let us look more closely at the effective degrees of freedom at the time just after the neutrinos decouple. For the entropy density before the electrons and positrons annihilate, they have 10.75 degrees of freedom, divided as 5.25 for the neutrinos and $2 + \frac{3}{8} = 2.125$ for the photon plus the electron and positron. The latter one is reduced to just 2 once all the electrons and positrons have annihilated (i.e., $g_{e\gamma}/g_{e\nu} = 2/5.5 = 4/11$). We now have a higher photon temperature and a lower neutrino temperature. Using Equation (32), we find the neutrino temperature after all electrons and positrons have annihilated to be

$$T_\nu = \sqrt[3]{\frac{4}{11}} T \approx 0.71T. \quad (36)$$

Hence, after the electron–positron annihilation, the neutrino temperature is 71% that of the photon temperature. Measurements of the Cosmic Microwave Background (CMB) radiation is found to be 2.73 K. This means that the neutrino background temperature should be 1.95 K (it should be mentioned that no measurement of the cosmic neutrino background have been made, or is likely to be made in the near future that would confirm this prediction).

The colder neutrinos do not contribute as much as the hotter particles to the four different $g_s$, and this has to be taken into account when we calculate the different effective degrees of freedom. In general, after a particle species decouples, we need to introduce a species-dependent temperature ratio into our equations; that is, $T \to T(T_j/T)$. Here $T_j$ is the temperature of the decoupled particle species, while $T$ is the photon-coupled (reference) temperature. We thus get the following $g_\ast n$, $g_\ast e$, $g_\ast p$, and $g_\ast s$ after electron–positron annihilation is completed

$$g_{\ast n} = 2 + 6 \times \frac{3}{4} \left(\frac{T_\nu}{T}\right)^3 = 2 + 6 \times \frac{3}{4} \frac{4}{11} - \frac{40}{11} = 3.636. \quad (37)$$

$$g_{\ast e} = g_{\ast p} = 2 + 6 \times \frac{7}{8} \left(\frac{T_\nu}{T}\right)^4 = 2 + 6 \times \frac{7}{8} \left(\frac{4}{11}\right)^{4/3} \approx 3.363, \quad (38)$$

$$g_{\ast s} = 2 + 6 \times \frac{7}{8} \left(\frac{T_\nu}{T}\right)^3 = 2 + 6 \times \frac{7}{8} \frac{4}{11} = \frac{43}{11} \approx 3.909. \quad (39)$$

The neutrino contribution during electron–positron annihilation is found by subtracting the electron–positron contribution in the following way:

$$S_{\ast \nu \nu} = \frac{6 \times 3}{4} \left\{ \frac{4}{11} + \left(1 - \frac{4}{11}\right) \frac{4}{3} \frac{4}{3} S_{\ast e\nu}\right\}, \quad (40)$$

$$S_{\ast \nu e} = \frac{6 \times 7}{8} \left\{ \left(\frac{4}{11}\right)^{4/3} + \left(1 - \frac{4}{11}\right)^{4/3} \right\} \frac{8}{4 \times 7 S_{\ast e\nu}}, \quad (41)$$

$$S_{\ast \nu p} = \frac{6 \times 7}{8} \left\{ \left(\frac{4}{11}\right)^{4/3} + \left(1 - \frac{4}{11}\right)^{4/3} \right\} \frac{8}{4 \times 7 S_{\ast e\nu}}, \quad (42)$$
where the four \( g_{\ast \nu} \) are the effective electron–positron contributions.

In reality, as can be seen in Figure 3 and Table B1, the first electron–positron annihilations began slightly before the neutrino decoupling was complete. Hence, some of the energy from the decaying electron–positron pairs heated up the neutrinos. This caused a small deviation from the above-mentioned values, which resulted in effective numbers of neutrino species slightly larger than three. This number is given to be 3.046 by Mangano [26] and 3.045 by de Salas and Pastor [27]. By using Mangano’s result, a compensated result will be

\[
g_{\ast n} = 2 + 2 \times 3.046 \times \frac{3}{4} = 3.661, \tag{44}
\]

\[
g_{\ast e} = g_{\ast p} = 2 + 2 \times 3.046 \times \frac{7}{8} \left( \frac{4}{11} \right)^{4/3} \approx 3.384, \tag{45}
\]

\[
g_{\ast s} = 2 + 2 \times 3.046 \times \frac{7}{8} \left( \frac{4}{11} \right) \approx 3.938. \tag{46}
\]

7. Functions for \( n, \epsilon, P, \) and \( S, \) and Their Implications

We can now express the complete number density, energy density, pressure, and entropy density in terms of their effective degrees of freedom:

Number density:

\[
n(T) = \frac{\zeta(3)}{\pi^2} g_{\ast n}(T) \left( \frac{k_B T}{\hbar c} \right)^3, \tag{47}
\]

Energy density:

\[
\epsilon(T) = \frac{\pi^2}{30} g_{\ast e}(T) \left( \frac{k_B T}{\hbar c} \right)^4, \tag{48}
\]

Pressure:

\[
P(T) = \frac{\pi^2}{90} g_{\ast p}(T) \left( \frac{k_B T}{\hbar c} \right)^4, \tag{49}
\]

Entropy density:

\[
s(T) = \frac{2\pi^2}{45} g_{\ast s}(T) \left( \frac{k_B T}{\hbar c} \right)^3. \tag{50}
\]

We have plotted these functions as well as the \( g_\ast \) values in Figure 6. The energy density and pressure have the same dimension, while the dimensions of entropy density and number density differ by the Boltzmann constant (unit: J K\(^{-1}\)).

When the prefactors are accounted for, the difference in \( s \) and \( n, \) and \( P \) and \( \epsilon, \) lies in the deviations between \( g_{\ast s}, \) \( g_{\ast n}, \) \( g_{\ast p}, \) and \( g_{\ast e}. \) So, let us discuss a bit more about what is actually happening. Both the increase in entropy per particle and the decrease in pressure (which we see as bumps and dips in panels (g) and (h) in Figure 6) are due to the presence of particles at semi- and non-relativistic temperatures. Before we go any farther, we should address the QCD phase transition. As not all four \( g_\ast \) can be continuous (as we see in panels (b)–(d) in Figure 6), we get inconsistencies and some unphysical results. For most of our plots, we use \( T_c = 214 \) MeV, keeping \( g_{\ast e} \) continuous, leaving \( g_{\ast n}, \) \( g_{\ast p}, \) and \( g_{\ast s} \) discontinuous at this point.

We see from panel (a) in Figure 6 that both number density and entropy density decrease more rapidly during annihilation periods. However, this is a bit deceiving, since we are looking at their values as functions of temperature. In fact, the total entropy stays constant (it actually increases ever so slightly if we do not have perfect thermal equilibrium). Both \( s \) and \( n \) fall a bit as we cross the transition temperature. We thus get a jump in the entropy per particle at this time, as can be seen in panel (g) in Figure 6. The other bumps in entropy per particle are continuous. The entropy per particle will start to rise when the rest mass of some massive particles becomes more significant. \( s \) then flattens out and drops again as these particles gradually become less numerous. After all these massive particles have annihilated and disappeared, the value of \( s \) returns to its original value (before the annihilations...
We have used a naive definition for our QCD phase transition—namely, that of the lowest energy pressure are just after the QCD phase transition and in the middle of the electron–positron annihilations. The pressure is thus at its lowest at times where the ratio of particles after they annihilate. It is important to emphasize that it is not the total entropy that changes, but rather the (total) particle number that falls and rises again. The change in entropy density after the neutrino decoupling, as we can see at the lowest temperature in panel (b) in Figure 6, is due to the fact that the neutrinos have a lower temperature and thus contribute less.

The same explanation goes for the fall in pressure, as seen in panel (h) in Figure 6. Non-relativistic particles exert (relatively) zero pressure. The pressure is thus at its lowest at times where the ratio of semi- and non-relativistic particles are at their highest. We see that the two most significant drops in pressure are just after the QCD phase transition and in the middle of the electron–positron annihilations. We have used a naive definition for our QCD phase transition—namely, that of the lowest energy density. In reality, this transition is quite complex, and we should interpret our result with a grain of
salt. With that in mind, we go from an almost pure relativistic gas (QGP) to a case where the majority of the particles are semi- or non-relativistic (HG)—which is the reason for the jump down in pressure at $T = 214$ MeV.

As we will get back to in the next section, the Universe expands faster when it is matter-dominated as compared to when it is radiation-dominated. So even though the early Universe was the latter, we know from our study that we have periods with a significant fraction of semi-relativistic particles. One can thus argue that $a$ should grow slightly faster at these times.

8. Time–Temperature Relation

As mentioned in Section 1, the measurements of the CMB thermal spectrum is very close to that of a perfect black body [28]. The early Universe should be very homogeneous, with the same features everywhere. How fast the early Universe expands depends on which energy contributor is dominating—the relativistic particles (radiation), or non-relativistic particles (cold matter). By solving the Friedmann equations for a flat adiabatic Universe with no cosmological constant, we find the relation between the scale factor ($a$) and time ($t$) to be $a = t^{1/2}$ for a pure radiation case, and $a = t^{2/3}$ for a pure cold matter case. Simple derivations for this are given by Ryden [10] and Liddle [29]. Similarly, a relation between the scale factor and temperature for the two extreme cases is given as $T = a^{-1}$ and $T = a^{-2}$ for the two cases [8,30]. This gives us the following relation between the three quantities:

\[
\begin{align*}
\text{Just radiation:} & \quad T \propto t^{-1/2} \propto a^{-1}, \\
\text{Just cold matter:} & \quad T \propto t^{-2/3} \propto a^{-2}.
\end{align*}
\]

(51) (52)

For a mixture of both types of particles, we should have something in between the two single-component cases. So, if radiation is the more dominant energy contributor, $a$ grows almost proportional to $t^{1/2}$, or more proportional $t^{2/3}$ for the matter case. Regarding temperature, for a radiation-dominated scenario, the relation between temperature and time (after the Big Bang) can be calculated as a function of $g_{*c}$, as follows [19]:

\[
t = \sqrt[3]{\frac{32\pi^2G g_{*c}(T)}{90h^3c^5}} (k_B T)^{-2} = \frac{2.4}{\sqrt{g_{*c}(T)}} T_{\text{MeV}}^{2}.
\]

(53)

The Universe becomes matter-dominated at roughly $10^5$ years after the Big Bang, long after the scope of this article. However, it should be noted that the temperature of both photons and matter drop as the inverse of the scale factor, even long after this radiation–matter equality. This is because temperature is determined by the kinetic energy of the particles, while the definition of a radiation- or matter-dominated Universe is that of the total energy (where rest mass is included). Being outnumbered more than a billion to one, matter is unable to cool down the photons, and the temperature of the Universe drops as the inverse of the scale factor until matter decouples from the photons.

On the other hand, when massive particles die out, their annihilation energy is transferred to the remaining particles in the thermal bath. This should make the temperature drop slower as a function of time. If we assume that this latter argument is dominant, we will get a Universe which drops in temperature more slowly when $g_{*c}$ is decreasing. Figure 7 shows the temperature as a function of time assuming a pure radiation-dominated Universe, as given by Equation (53). During particle annihilations, we have a smooth continuous function, but this is not the case for the QGP-to-HG transition using our models. Here we have to consider the three different transition temperatures separately. Using $k_B T_c = 214$ MeV, we have a scenario where the temperature will drop more slowly
right after $T_c$. Using $k_B T_c = 170$ MeV and $k_B T_c = 150$ MeV, the degrees of freedom ($g_{\star\epsilon}$) will jump down at $T_c$. For $k_B T_c = 170$ MeV, the value of $g_{\star\epsilon}$ falls from around 62 to 33, while for $k_B T_c = 150$ MeV, this value falls from around 61 to 26. This would, however, take some time. Using our simple model, the temperature and energy density of the Universe would stay constant as the Universe expands until it reaches its pure hadron-gas state.

![Figure 7](https://example.com/figure7.png)  
**Figure 7.** Energy and temperature as functions of time using the three different transition temperatures. For the $k_B T_c = 214$ MeV transition, the temperature will drop slower after $T_c$, but nonetheless always decrease over time. For $k_B T_c = 170$ MeV and $k_B T_c = 150$ MeV, there will be a period with constant temperature and energy density while $g_{\star\epsilon}$ decreases from its quark-gluon value to its hadron gas value.

The relation made here between time and temperature during the QCD transition is naive and simple, and the numerical values thereafter. Small deviations from our plot during the QCD transition (and for that matter, during regular particle annihilations) only affect the period on hand, and become negligible as time go on.

9. On the QCD Phase Transition and Cross-Over Temperature

Our method of using $k_B T_c = 214$ MeV is based on a calculation where we add all the $g_{\star\epsilon}$ from the quark-gluon state on one hand (easy), and all the $g_{\star\epsilon}$ from the hadrons on the other hand (not so easy). We have used the hadronic particles as listed in Appendices C and D. These are the particles listed by the Particle Data Group [19]. There are additional candidates to these lists—some good candidates, and some more speculative. There could also be more hadronic states which are hard to detect. For every new hadron we add to our model, the cross-over temperature decreases. Not so much for the most massive candidates, but more so for the lighter ones.

The phase transition we have used is of first order. Only $g_{\star\epsilon}$ is continuous at this point, and $g_{\star n}$, $g_{\star p}$, and $g_{\star s}$ are not. This will lead to some unphysical consequences—such as an instantaneous increase of the scale factor by roughly 4% if we assume constant entropy. A proper theory about the QCD phase transition is required to address this issue, which is beyond the scope of this article. The theory we have presented here should be valid to a good approximation, before and after the QCD transition.

10. Conclusions

Our knowledge of the very first stages of Universe is limited. In order to know for sure what is happening at these extreme energies and temperatures, we want to recreate the conditions using particle accelerators. The Large Hadron Collider at CERN can collide protons together at energies of 13 TeV, and with their discovery of the Higgs boson, all the elementary particles predicted by the Standard Model of particle physics have been found. This is, however, most likely not the complete
story. Dark matter particles are the hottest candidates to be added to our list of particles, and there is almost sure to be more particles at even higher temperatures, such as at the Grand Unified Theory (GUT) scale of $T \sim 10^{16}$ GeV.

We have here used the statistical physics approach to counting the effective degrees of freedom in the early Universe at temperatures below 10 TeV. Some simplifications have been used, such as setting the chemical potential equal to zero for all particles. The aim of this article was to give a good qualitative introduction to the subject, as well as providing some quantitative data in the form of plots and tables.

The early Universe is often thought of as being pure radiation (just relativistic particles). However, when the temperature drops to approximately that of the rest mass of some massive particles, we get interesting results, where we have a mix of relativistic particles and semi- and non-relativistic ones. This mix is most prominent during the electron–positron annihilations, and just after the phase transition from a quark-gluon plasma to a hadron gas. Approaching this, using our “no-chemical potential” distribution functions shows us how the entropy per particle increases when the ratio of semi- and non-relativistic particles becomes significant.

The number of effective degrees of freedom for hadrons changes very quickly around the QCD transition temperature. We found a cross-over temperature of 214 MeV using the known baryons and mesons. As there could be many more possible hadronic states than we have accounted for, this cross-over temperature could be lower. This is not meant as a claim of a new QCD transition temperature, but rather as an interesting fact. Our first-order approach based purely on the distribution functions has inconsistencies at the cross-over temperature.

We have listed the effective degrees of freedom for number density ($g_{\text{n}}$), entropy density ($s\star$), pressure ($p\star$), entropy density ($s\star$), and time ($t$) as function of temperature ($T$) in Table A1 in Appendix A. Table B1 in Appendix B lists the different effective contributions to a single intrinsic degree of freedom, corresponding to our plot in Figure 3.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Table for Time, $g_{\text{n}}$, $s\star$, $p\star$, and $g_{s\star}$

Table A1. Values for time, $g_{\text{n}}$, $s\star$, $p\star$, and $g_{s\star}$ from $T = 10$ TeV to 10 keV. In the region between 150–214 MeV, the values for all five quantities depend on the model’s transition temperature. We should emphasize that the values for this period are based on our simple no-chemical potential model.

<table>
<thead>
<tr>
<th>$b \bar{b}$ ($T$ (eV))</th>
<th>$T$ (K)</th>
<th>Time (s)</th>
<th>$g_{\text{n}}$</th>
<th>$g_{s\star}$</th>
<th>$s\star$</th>
<th>$p\star$</th>
<th>$g_{s\star}$</th>
<th>Tr. Temp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 TeV</td>
<td>1.16 $\times 10^{10}$</td>
<td>2.32 $\times 10^{-35}$</td>
<td>95.50</td>
<td>106.75</td>
<td>106.75</td>
<td>106.75</td>
<td>All</td>
<td></td>
</tr>
<tr>
<td>5 TeV</td>
<td>5.80 $\times 10^{10}$</td>
<td>9.29 $\times 10^{-15}$</td>
<td>95.49</td>
<td>106.75</td>
<td>106.75</td>
<td>106.75</td>
<td>All</td>
<td></td>
</tr>
<tr>
<td>2 TeV</td>
<td>2.32 $\times 10^{10}$</td>
<td>5.81 $\times 10^{-14}$</td>
<td>95.47</td>
<td>106.74</td>
<td>106.73</td>
<td>106.74</td>
<td>All</td>
<td></td>
</tr>
<tr>
<td>1 TeV</td>
<td>1.16 $\times 10^{10}$</td>
<td>2.32 $\times 10^{-13}$</td>
<td>95.39</td>
<td>106.72</td>
<td>106.65</td>
<td>106.70</td>
<td>All</td>
<td></td>
</tr>
<tr>
<td>500 GeV</td>
<td>5.80 $\times 10^{10}$</td>
<td>9.30 $\times 10^{-13}$</td>
<td>95.11</td>
<td>106.61</td>
<td>106.38</td>
<td>106.56</td>
<td>All</td>
<td></td>
</tr>
<tr>
<td>200 GeV</td>
<td>2.32 $\times 10^{10}$</td>
<td>5.83 $\times 10^{-12}$</td>
<td>93.55</td>
<td>105.90</td>
<td>104.75</td>
<td>105.61</td>
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<td></td>
</tr>
<tr>
<td>100 GeV</td>
<td>1.16 $\times 10^{10}$</td>
<td>2.36 $\times 10^{-11}$</td>
<td>89.89</td>
<td>103.53</td>
<td>100.80</td>
<td>102.85</td>
<td>All</td>
<td></td>
</tr>
<tr>
<td>50 GeV</td>
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<td>9.73 $\times 10^{-11}$</td>
<td>83.53</td>
<td>97.40</td>
<td>93.94</td>
<td>96.53</td>
<td>All</td>
<td></td>
</tr>
<tr>
<td>20 GeV</td>
<td>2.32 $\times 10^{10}$</td>
<td>6.40 $\times 10^{-10}$</td>
<td>77.39</td>
<td>88.45</td>
<td>87.22</td>
<td>88.14</td>
<td>All</td>
<td></td>
</tr>
<tr>
<td>10 GeV</td>
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<td>2.58 $\times 10^{-9}$</td>
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<td>86.22</td>
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</tr>
<tr>
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<td>1.04 $\times 10^{-8}$</td>
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<td>85.60</td>
<td>84.68</td>
<td>85.37</td>
<td>All</td>
<td></td>
</tr>
<tr>
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<td>6.61 $\times 10^{-8}$</td>
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<td>82.50</td>
<td>79.69</td>
<td>81.80</td>
<td>All</td>
<td></td>
</tr>
<tr>
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<td>76.34</td>
<td>72.97</td>
<td>75.50</td>
<td>All</td>
<td></td>
</tr>
<tr>
<td>500 MeV</td>
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<td>1.15 $\times 10^{-6}$</td>
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<td>69.26</td>
<td>66.43</td>
<td>68.55</td>
<td>All</td>
<td></td>
</tr>
<tr>
<td>214 MeV</td>
<td>2.48 $\times 10^{10}$</td>
<td>6.63 $\times 10^{-6}$</td>
<td>55.37</td>
<td>62.69</td>
<td>61.52</td>
<td>62.25</td>
<td>All</td>
<td></td>
</tr>
<tr>
<td>214 MeV</td>
<td>2.48 $\times 10^{10}$</td>
<td>6.63 $\times 10^{-6}$</td>
<td>55.37</td>
<td>62.69</td>
<td>61.52</td>
<td>62.25</td>
<td>All</td>
<td></td>
</tr>
<tr>
<td>200 MeV</td>
<td>2.32 $\times 10^{10}$</td>
<td>8.42 $\times 10^{-6}$</td>
<td>55.37</td>
<td>62.69</td>
<td>61.52</td>
<td>62.25</td>
<td>170 + 150 MeV</td>
<td>All</td>
</tr>
<tr>
<td>214 MeV</td>
<td>2.48 $\times 10^{10}$</td>
<td>6.63 $\times 10^{-6}$</td>
<td>62.69</td>
<td>70.75</td>
<td>69.26</td>
<td>70.47</td>
<td>214 MeV</td>
<td>All</td>
</tr>
</tbody>
</table>
Appendix B. Effective Contribution to One Single Intrinsic Degree of Freedom

Table A1. Continued.

<table>
<thead>
<tr>
<th>$h_B T$ (eV)</th>
<th>$T$ (K)</th>
<th>Time (s)</th>
<th>$g_{fs}$</th>
<th>$g_{ff}$</th>
<th>$g_{sf}$</th>
<th>$g_{sf}$</th>
<th>Tr. Temp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>190 MeV</td>
<td>2.20×10^{11}</td>
<td>7.61×10^{-6}</td>
<td>55.21</td>
<td>62.21</td>
<td>61.34</td>
<td>61.99</td>
<td>170 + 150 MeV</td>
</tr>
<tr>
<td>180 MeV</td>
<td>2.09×10^{11}</td>
<td>1.20×10^{-5}</td>
<td>55.10</td>
<td>62.03</td>
<td>61.21</td>
<td>61.83</td>
<td>170 + 150 MeV</td>
</tr>
<tr>
<td>170 MeV</td>
<td>1.97×10^{11}</td>
<td>9.42×10^{-6}</td>
<td>54.99</td>
<td>61.87</td>
<td>61.07</td>
<td>61.67</td>
<td>170 + 150 MeV</td>
</tr>
<tr>
<td>160 MeV</td>
<td>1.86×10^{11}</td>
<td>1.44×10^{-5}</td>
<td>54.88</td>
<td>61.72</td>
<td>61.54</td>
<td>61.52</td>
<td>170 + 150 MeV</td>
</tr>
<tr>
<td>150 MeV</td>
<td>1.74×10^{11}</td>
<td>1.14×10^{-5}</td>
<td>54.88</td>
<td>61.72</td>
<td>61.54</td>
<td>61.52</td>
<td>150 MeV</td>
</tr>
<tr>
<td>140 MeV</td>
<td>1.62×10^{11}</td>
<td>2.08×10^{-5}</td>
<td>17.93</td>
<td>26.31</td>
<td>20.13</td>
<td>24.77</td>
<td>214 + 170 MeV</td>
</tr>
<tr>
<td>130 MeV</td>
<td>1.51×10^{11}</td>
<td>1.92×10^{-5}</td>
<td>6.54</td>
<td>41.45</td>
<td>60.65</td>
<td>61.25</td>
<td>170 MeV</td>
</tr>
<tr>
<td>120 MeV</td>
<td>1.40×10^{11}</td>
<td>1.76×10^{-5}</td>
<td>6.54</td>
<td>41.45</td>
<td>60.65</td>
<td>61.25</td>
<td>150 MeV</td>
</tr>
<tr>
<td>110 MeV</td>
<td>1.29×10^{11}</td>
<td>1.60×10^{-5}</td>
<td>6.54</td>
<td>41.45</td>
<td>60.65</td>
<td>61.25</td>
<td>130 MeV</td>
</tr>
<tr>
<td>100 MeV</td>
<td>1.13×10^{11}</td>
<td>1.44×10^{-5}</td>
<td>6.54</td>
<td>41.45</td>
<td>60.65</td>
<td>61.25</td>
<td>110 MeV</td>
</tr>
</tbody>
</table>

Table B1. The effective contribution to a single degree of freedom as function of temperature ($k_B T$) over mass (m_{c}).
Appendix C. List of Mesons and Their Degeneracy

### Table C1. Pseudoscalar mesons.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Flavours</th>
<th>Spin States</th>
<th>Color States</th>
<th>Bose or Fermi</th>
<th>Mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi^0 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>134.9766</td>
</tr>
<tr>
<td>( \pi^\pm )</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>139.5701</td>
</tr>
<tr>
<td>( K^\pm )</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>493.677</td>
</tr>
<tr>
<td>( K^1, K^{0^0} )</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>497.614</td>
</tr>
<tr>
<td>( \eta )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>547.862</td>
</tr>
<tr>
<td>( \eta' )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>957.78</td>
</tr>
<tr>
<td>( D^0, D^0 )</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1864.84</td>
</tr>
<tr>
<td>( D^\star, D^\star )</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1869.61</td>
</tr>
<tr>
<td>( \eta_c )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2983.6</td>
</tr>
<tr>
<td>( B^\pm )</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>5279.26</td>
</tr>
<tr>
<td>( B^\pm, B^\pm )</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>5279.58</td>
</tr>
<tr>
<td>( B^\pm, B^\pm )</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>5366.77</td>
</tr>
<tr>
<td>( \eta_b )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>9398.0</td>
</tr>
</tbody>
</table>

### Pseudoscalar Mesons

\[ g = 27 \]

### Table C2. Vector mesons.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Flavours</th>
<th>Spin States</th>
<th>Color States</th>
<th>Bose or Fermi</th>
<th>Mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho^\pm )</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>775.11</td>
</tr>
<tr>
<td>( \rho^0 )</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>775.26</td>
</tr>
<tr>
<td>( \omega )</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>782.65</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>891.66</td>
</tr>
<tr>
<td>( \kappa^*, \kappa^{*0} )</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>895.81</td>
</tr>
<tr>
<td>( \phi )</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1019.461</td>
</tr>
<tr>
<td>( D^{*0}, D^{*0} )</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>2006.96</td>
</tr>
<tr>
<td>( D^{<em>+}, D^{</em>+} )</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>2010.26</td>
</tr>
<tr>
<td>( D_{1+}, D_{1+} )</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>2112.1</td>
</tr>
<tr>
<td>( f/\phi )</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
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</tr>
<tr>
<td>( B^{*+} )</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>5325.2</td>
</tr>
<tr>
<td>( B^{*0}, B^{*0} )</td>
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<td>3</td>
<td>1</td>
<td>1</td>
<td>5325.4</td>
</tr>
<tr>
<td>( B_{1+}, B_{1+} )</td>
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<td>3</td>
<td>1</td>
<td>1</td>
<td>5415.4</td>
</tr>
<tr>
<td>( Y (1S) )</td>
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<td>3</td>
<td>1</td>
<td>1</td>
<td>9460.30</td>
</tr>
</tbody>
</table>

### Vector Mesons

\[ g = 69 \]

* These are listed without 0 superscript by PDG [19].

### Table C3. Excited unflavoured mesons.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Flavours</th>
<th>Spin States</th>
<th>Color States</th>
<th>Bose or Fermi</th>
<th>Mass</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>500</td>
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<tr>
<td>( f_0(980) )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>980</td>
</tr>
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<td>( a_0(980) )</td>
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<td>1</td>
<td>980</td>
</tr>
<tr>
<td>( b_1(1170) )</td>
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<td>1</td>
<td>1</td>
<td>1170</td>
</tr>
<tr>
<td>( b_1(1235) )</td>
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<td>3</td>
<td>1</td>
<td>1</td>
<td>1235</td>
</tr>
<tr>
<td>( a_0(1260) )</td>
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<td>3</td>
<td>1</td>
<td>1</td>
<td>1260</td>
</tr>
<tr>
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<td>1</td>
<td>1270</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
<td>1285</td>
</tr>
<tr>
<td>Symbol</td>
<td>Flavours</td>
<td>Spin States</td>
<td>Color States</td>
<td>Bose or Fermi</td>
<td>Mass</td>
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<td>--------------</td>
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<td>------</td>
</tr>
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<tr>
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<tr>
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<td>5</td>
<td>1</td>
<td>1</td>
<td>1320</td>
</tr>
<tr>
<td>h(1370)</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1370</td>
</tr>
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Excited Unflavoured Mesons

\( g = 499 \)

\(^2\) We have used the \((JPC) = (4^{++})\) for \(f_1\).
**Table C4.** Excited strange mesons.

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<th>Color States</th>
<th>Bose or Fermi</th>
<th>Mass</th>
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Excited Strange Mesons $g = 194$

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**Appendix D. List of Baryons and Their Degeneracy**

**Table D1.** Spin $\frac{1}{2}$ baryons.

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<th>Color States</th>
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<th>Mass</th>
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Spin $\frac{1}{2}$ Baryons $g = 84$
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<th>Mass</th>
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Spin $\frac{3}{2}$ Baryons

$g = 133$

### Table D3. Excited N baryons.

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<th>Mass</th>
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Excited N Baryons

$g = 248.5$
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<td>1750</td>
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Excited Δ Baryons

$g = 238$

Table D5. Excited Λ baryons.

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Excited Λ Baryons

$g = 133$
Table D6. Excited Σ baryons.

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Excited Σ Baryons $g = 133$

Table D7. Excited Ξ baryons.

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Excited Ξ Baryons $g = 17.5$

References


Paper II

Viscosity in a Lepton-Photon Universe
Astrophysics and Space Science 361 (2016) no. 8
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Paper III

Entropy in a Lepton-Photon Universe
Astrophysics and Space Science 362 (2017) no. 2
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