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On Steady Solutions of a Generalized Whitham Equation

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Abstract

We study the steady solutions of a generalized Whitham equation $\eta_t + \frac{3c_o}{2}\eta\eta_x + L_w\eta_x = 0$, where L_w is the nonlocal Fourier multiplier operator given by the symbol $m_s(\xi) = (\tanh\xi/\xi)^s$ for $s \in (0,1)$, for which we investigate whether a similar local and global theory is available as for the Whitham equation, which is the case $s = \frac{1}{2}$. Using functional analysis, we prove that there is a curve of small amplitude sinusoidal waves bifurcating at wave speed $c = (\tanh(1))^s$, and these waves may be extended to large ones by global bifurcation. In our quest to understand the regularity of a possible highest wave for this generalized equation, we study the regularity of waves along the global bifurcation curve. We find that any highest wave of the generalized equation is α-Hölder continuous and has Hölder regularity C^α for $0 < \alpha < s < 1$, and $\alpha + s \leq 1$. In addition, we study the properties of the symbol $m_s(\xi)$, and the corresponding integral kernel. In view of the fact that some arguments were quite technical, we perform a brief background study of Banach algebras, Hölder and Schwartz spaces, Fréchet differentiability, completely monotone functions, the implicit function theorem on Banach spaces, and Fourier series.

Preface

This thesis marks the end of the two years master's degree programme in "Mathematical Sciences" at The Norwegian University of Science and Technology (NTNU) in Trondheim, within the specialization Analysis (Differential Equations). The thesis was performed throughout my 4^{th} semester of the master, spring 2017, at the Department of Mathematical Sciences under the Faculty of Information Technology and Electrical Engineering.

The thesis, which is an add up to the work of my supervisor M. Ehrnström, deals with the study of the generalized Whitham equation for which we investigate whether a similar local and global theory is available as for the Whitham equation, which is the case $s = \frac{1}{2}$ (see [1]). It is assumed from the reader only a basic knowledge of functional analysis, partial differential equations and Fourier analysis.

The thesis is structured as follows:

Section 1 introduces the Whitham's equation as a non-local model for a shallow water wave, for capturing the balance between linear dispersion and nonlinear effects. We next review some research on the Whitham equation and then present the contribution of this thesis.

Section 2 recalls some facts about Banach algebras, Hölder and Schwartz spaces, Fréchet differentiability, Completely monotone and Stieltjes functions, and The implicit function theorem which will appear frequently throughout the various sections.

Section 3 begins with the study of Fourier series of periodic functions and its convergence, differentiability, decay and convolution properties. It then gives a summary of the concept of Carleson-Hunt theorem and also briefly treats the Fourier transform on $\mathbb{R}$ and other spaces. The section ends with an introduction of Fourier multipliers on Hölder spaces.

Section 4 is devoted to the study of the generalized Whitham integral kernel $K_s(x) = \widehat{m}_s(x)$. It specifically gives the monotonicity and limiting properties of the generalized Whitham symbol. The section ends with some discussions about the convolution operator L_w.
Section 5 provides a prove of the existence of periodic traveling waves and then introduces the local bifurcation theory which will be extended to the global continuous curves of solutions in the next section. The approach in this section and the next section follows closely that of [2, 3].
Section 6 contains the main part of this thesis, where we investigate the global bifurcation for the generalized Whitham equation by an extension to the local bifurcation. It also gives some analysis of the uniform convergence of solution and also the characterization of blow-up.

A Ghanaian proverb reads "Knowledge is like a Baobab tree, one person's arms cannot encompass it". That is, it takes several arms held together to encompass it. In my quest for knowledge, I am privileged to have met many excellent persons who in diverse ways held my hand in the process. My primary debt of gratitude goes to God as my source of strength and spiritual guide. I am also grateful to the Norwegian government for granting me the opportunity and also providing funds for my studies.

I further wish to express my deepest and sincere gratitude to my adviser Professor Mats Ehrnström, whose expertise, understanding, generous guidance and support made it possible for me to work on such a topic. I am also grateful to my parents and family for their love and support during my studies, and lastly I thank the many excellent professors and students at NTNU whom I have learned much from during my time in Trondheim.

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1 Introduction

Waves on the surface of the ocean are a dramatic and beautiful phenomena that impact every aspect of life on the planet [4]. The behaviour of water waves and the propagation characteristics of light and sound are familiar from everyday experience. Wave motion is one of the broadest scientific subjects and unusual in that it can be studied at any technical level. One important area of study is the traveling water waves (the class of travelling waves which move progressively in one direction with fixed speed and shape).

In most times, the steady waves repeat themselves periodically, leading to periodic traveling waves. The bifurcation theory on the other hand is one of the methods used in proving the existence of such periodic traveling waves. Of particular interest is how the local bifurcation curves of solutions (that is the 2π-periodic, smooth, travelling-wave solutions) to the Whitham equation is extended to the global continuous curves of solutions.

1.1 Whitham's model equation

The water wave equations pose severe challenges for rigorous analysis, modeling, and numerical simulation, from a mathematical viewpoint. Although water waves have intrigued mankind for thousands of years, it was not until the middle of the nineteenth century that the modern theory appeared, principally in the work of Stokes. The nineteenth century also produced useful models for tidal waves, solitary waves, the Korteweg–de Vries (KdV) equation, the Boussinesq models for shallow water waves, the Kelvin–Helmholtz instability, Cauchy–Poisson circular waves, Gerstner's rotational waves, Stokes' model for the highest wave, and Kelvin's model for ship wakes [5].

The Korteweg-de Vries equation (KdV) was introduced in 1895 to model the behavior of long waves on shallow water in close agreement with the observations of J. S. Russell [6]. The KdV model admits solitary waves which present soliton interaction: two solitary waves keep their shape and size after interaction although the ultimate position of each wave has been affected by the nonlinear interaction [7]. KdV has a bi-Hamiltonian structure which permits to obtain very precise information about the structure of the equation by the inverse scattering method, the equation being integrable [8]. The main challenge of the KdV equation was that it could not describe the breaking of the wave.

In 1967, a British-born American mathematician, G.B. Whitham proposed in [9] a non-local shallow water wave model for capturing the balance between linear dispersion and nonlinear effects, so that one would have smooth periodic and solitary waves, but also the features of wave breaking and surface singularities. Whitham [7] emphasized that the breaking phenomena is one of the most intriguing long-standing problems of water wave theory, and since the KdV equation can not describe breaking, he suggested the model

$$\eta_t + \frac{3}{2}\frac{c_o}{h_o}\eta\eta_x + K_{h_o} * \eta_x = 0 \tag{1.1}$$

known as the Whitham equation. This equation combines a generic non-linear quadratic term with the exact linear dispersion relation for surface water waves on finite depth. Here, the kernel

$$K_{h_o} = \mathcal{F}^{-1}(c_{h_o}) \tag{1.2}$$

is the inverse Fourier transform of the phase speed

$$c_{h_o}(\xi) = \sqrt{\frac{g \tanh h_o\xi}{\xi}} \tag{1.3}$$

for the linearized water-wave problem; the constants g, h_o and $c_o =$

$\sqrt{gh_o}$ denote, respectively, the gravitational constant of acceleration, the undisturbed water depth, and the limiting long-wave speed. The function $\eta(t,x)$ describes the deflection of the fluid surface from the rest position at a point x at time t [7].

The Whitham equation (1.1) with the kernel (1.2) has some very interesting mathematical features. That is, it is generically non-local, making pointwise estimates difficult. Moreover, $c_{h_o}(\xi)$ has slow decay, and the kernel K_{h_o} is singular (it blows up at $x = 0$). This makes the Whitham equation in some important respects different from many other equations of the form (1.1) [3]. Whitham's actual motivation was to find a model that could feature the breaking of waves (wave breaking in this context describes a situation in which the spatial derivative of the function η becomes unbounded in finite time, while η itself remains bounded). Another interest was wave peaking which means that, a wave forms a sharp crest or peak, such as a stagnation point in the full water-wave problem [2, 10].

The Whitham equation captures the peaking phenomenon of the Stokes waves for the full water-wave problem. Interest in breaking, peaking and other phenomena connected with (1.1) has spawned a large amount of mathematical work. The monograph by Naumkin and Shishmarev [11] is devoted entirely to equations like (1.1).

1.2 A review of some research on the Whitham equation

A lot of research has being done on water wave models. Of particular interest is the Whitham equation as a model for water waves. Some highlights of the analytical and numerical research advancements of the Whitham equation are been introduce in this section.

Early years after Whitham [7, 9] introduced the Whitham equation, Gabov [12] and Zaitsev [13] made some studies on this equation. The monograph by Naumkin and Shishmarev [11] in the year 1994 is devoted entirely to the analysis of (1.1) for a mixture of kernels and also provided an affirmative answer to the question of wave breaking. In recent years, Hur [14] also dealt with the issue of wave breaking of bounded solutions with unbounded derivatives. Together with Tao [10], they show wave breaking for the Whitham equation in a range of fractional dispersion. Hur and Johnson [15] also show that periodic traveling waves with sufficiently small amplitudes of the Whitham equation are spectrally unstable to long-wavelengths perturbations if the wave number is greater than a critical value, bearing out the Benjamin-Feir instability of Stokes waves.

Borluk et al. [16] investigated the simulation properties of the Whitham equation as a model for waves at the surface of a body of fluid. They found out that the periodic traveling-waves solutions of the Whitham equation are good approximations to solutions of the full free-surface water wave problem. This was as a results of the comparison of numerical solutions of the Whitham equation to numerical approximations of solutions of the full Euler free-surface water-wave problem.

Ehrnström and Kalisch [2] in 2009 proved that there exist small-amplitude periodic traveling waves with sub-critical speeds and as the period of these traveling waves tends to infinity, their velocities approach the limiting long-wave speed c_o. They further shown that there can be no solitary waves with velocities much greater than c_o. Again after performing some numerical analysis, it was proven that there is a periodic wave of greatest height $\sim 0.642h_o$. In 2013, Ehrnström and Kalisch [3] proved the existence of a global bifurcation branch of 2π-periodic, smooth, traveling-wave solutions of the Whitham equation. Furthermore [3] showed that the solutions converge uniformly to a solution of Hölder regularity $\alpha \in (0, 1)$, except possibly at the highest crest point (where $\alpha \leq \frac{1}{2}$).

The kernel K_{h_o} of the Whitham equation has not thoroughly being understood. In 2009, [2] features the integrability of this kernel in certain L^p spaces and smoothness away from the origin. However, in a very recent time Ehrnström and Wahlén [1] provided an explicit representation formula for it and again shown that the integral kernel is completely monotone on the interval $(0, \infty)$ and also analytic with exponential decay away from the origin. They further proved the existence of a highest, cusped periodic traveling wave using the global bifurcation theory. Again, they found that the solution is P-periodic, even and strictly increasing on the interval $(-\frac{P}{2}, 0)$, satisfying $\varphi(0) = \frac{\mu}{2}$. The solution is furthermore smooth away from any crest, and obtains its optimal Hölder regularity $C^{\frac{1}{2}}(\mathbb{R})$ exactly at the crest, thereby resolving Whitham's conjecture.

The paper [17] identified a scaling regime in which the Whitham equation can be derived from the Hamiltonian theory of surface water waves. After integrating the Whitham equation numerically, they shown that the equation gives a close approximation of inviscid free surface dynamics as described by the Euler equations. They then concluded that in a wide parameter range of amplitudes and wavelengths, the Whitham equation performs on par with or better than the Korteweg-de Vries (KdV) equation, the Benjamin Bona Mahony (BBM) equation and the Padé model.

Sanford et al. [18] focused on the stability of solutions in view of [2]. The numerical results presented in [18] suggest that all large-amplitude solutions are unstable, while small-amplitude solutions with large enough wavelength L are stable. Additionally, [18] proved that the periodic solutions with wavelength smaller than a certain cut-off period always exhibit modulational instability. However, the cut-off wavelength is characterized by $kh_o = 1.145$, where $k = \frac{2\pi}{L}$ is the wave number and h_o is the mean fluid depth. The works by Benjamin and Hasselmann [19] also presented a detailed stability analysis for wave trains on water of arbitrary depth h_o, and calculated that small amplitude periodic traveling waves are unstable if the fundamental wave number k satisfies $kh_o > 1.363$.

The Periodic traveling waves to the KdV do not exhibit this property but are spectrally stable [20]. Bronski and Johnson [21] also investigated the spectral stability of a family of periodic standing wave solutions to the generalized KdV equation.

1.3 The work at hand

The existence of smooth, small-amplitude, periodic traveling-wave solutions and their properties was established and numerically investigated by Ehrnström and Kalisch [2]. In years later, they again in [3] worked on the steady solutions of the Whitham equation (that is traveling-wave solutions characterized by a constant speed and shape). They proved that the Whitham solutions are all smooth and subcritical, and that they converge uniformly to a wave of C^α-regularity, $\alpha < \frac{1}{2}$.

In this present work, we consider a general version of the Whitham equation defined in (1.1), (1.2) and (1.3). That is taking $g, h_o \sim 1$, we have the generalized Whitham equation to be

$$\eta_t + \frac{3c_o}{2}\eta\eta_x + K_s * \eta_x = 0. \tag{1.4}$$

We then define the generalized Whitham symbol as

$$m_s(\xi) = \widehat{K}_s(\xi) = \left(\frac{\tanh \xi}{\xi}\right)^s, \qquad 0 < s < 1, \tag{1.5}$$

whilst we have the generalized Whitham kernel defined by

$$K_s(x) = \mathcal{F}^{-1}\{m_s(\xi)\} = \frac{1}{2\pi}\int_{\mathbb{R}} m_s(\xi)e^{ix\xi}d\xi. \tag{1.6}$$

The aim of this thesis is to study the generalized Whitham equation (1.4) and to see if a similar local and global theory is available as for the Whitham equation with $s = \frac{1}{2}$ (see [1]). As one goal, we wanted to understand the regularity of a possible highest wave for the generalized equation (1.4). Although some steps in this direction have been achieved, the time frame of this master's thesis have not made a complete theory possible.

In our way towards this goal, however we have studied and investigated the symbol $m_s(\xi)$ and its Fourier transform using the theory of Stieltjes and completely monotone functions. It is also shown that any subset of solutions in the global branch contains a sequence which converges uniformly to some solution of Hölder class C^α for $\alpha \in (0, s)$. This required a study of Banach algebras, Hölder spaces, Fréchet differentiability, implicit function theorem in Banach spaces, and the bifurcation theory.

The bifurcation curve of the solution to the generalized Whitham equation is found to be a subcritical pitchfork bifurcation, which is of the same kind as the one described in [3]. The uniform convergence of the sequence of solutions is proved for the case where $\alpha \in (0, s)$ (s is defined in (1.5)) satisfy $\alpha + s \leq 1$. The case where $s = \frac{1}{2}$ and $\alpha < \frac{1}{2}$ is already included in [3, 1]. In the general case, when $\varphi \leq 2\mu$, it is found that the Whitham solution is α-Hölder continuous and has Hölder regularity C^α for $0 < \alpha < s < 1$ required that $\alpha + s \leq 1$. It is also proved that if $\varphi < 2\mu$ uniformly on $\mathbb{R}$, then the solution is smooth with all its derivatives bounded.

In addition, we deal with the existence of periodic traveling waves as proved by Ehrnström and Kalisch [2]. The local bifurcation theorem is studied and later extended to the global continuous curves of solutions in relation to the generalized Whitham equation in the very last section. Another object of study is the convolution operator L_w of the generalized Whitham equation which we find to be a symmetric bounded linear operator. In view of this, we introduce Fourier series and transform since some of the arguments were quite technical.

References for borrowed materials and proofs are provided throughout the text. Some results in Sections 2 and 3 are stated without proofs and specific references, since they are standard. The proofs in Sections 4, 5 and 6 are the author's own adaptions of the ones in [1, 2, 3], where the generalized Whitham equation, kernel and symbol have been taken into consideration.

2 Preliminaries

In this section we review some spaces and fundamental tools from real and functional analysis which are necessary in providing a firm base for the rest of the discussion. It must be noted that the various tools are not given in detailed but only a brief summary of what is actually needed for the discussion. We begin with Banach algebras, Hölder and Schwartz spaces. Next follows a general overview of Fréchet differentiability, completely monotone and Stieltjes functions. We then end the section with the introduction to the concept of the implicit function theorem.

The results in this section are mostly stated without proofs and specific references. The monograph by Marcoux [22] contains details on Banach algebra whilst we can find the remaining topics by the works of Buffoni and Toland [23], Miller and Samko [24], Shilling, Song and Vondracek [25] and Royster [26].

Throughout the various sections, the standard notation of mathematical analysis is used. For $1 \leq p < \infty$, the space $L^p(\Omega)$ is the set of measurable real-valued functions of a real variable whose p^{th} powers are Lebesgue integrable over a subset $\Omega \subseteq \mathbb{R}$. If $f \in L^p(\Omega)$, its norm is given by

$$\|f\|^p_{L^p(\Omega)} := \int_\Omega |f|^p dx. \tag{2.1}$$

The space $L^\infty(\Omega)$ consists of all measurable, essentially bounded functions with norm

$$\|f\|_{L^\infty(\Omega)} := \text{ess sup}_{x\in\Omega}|f(x)|. \tag{2.2}$$

2.1 Banach algebras

If we consider B as a Banach space over $\mathbb{C}$. We then say that B is a Banach algebra if there exists an operation from $B \times B$ to B,

$(x, y) \mapsto xy$, such that for all x, y and z in B and α in $\mathbb{C}$, we have

$$(xy)z = x(yz) \qquad \text{the operation is associative,}$$

$$\begin{aligned} (\alpha x + y)z &= \alpha xz + yz \\ z(\alpha x + y) &= \alpha zx + zy \qquad \text{the operation is bilinear,} \end{aligned}$$

$$\|xy\| \leq \|x\|\|y\| \qquad \text{the norm is sub-multiplicative.}$$

In [22], the set $(\mathrm{C(X)}, \|\cdot\|_\infty)$ of continuous functions on a compact Hausdorff space X, becomes a Banach algebra under pointwise multiplication of functions. That is, for $f, g \in (\mathbb{C}(\mathrm{X}), \|\cdot\|_\infty)$, we set $(fg)(x) = f(x)g(x)$ for all $x \in \mathrm{X}$.

Remark 2.1. *$C(X) = \{f : X \to \mathbb{C}; f$ is continuous$\}$.*

If Y is being considered as a Banach space, then according to [27] the set of continuous linear maps, $L(\mathrm{Y})$, from Y to itself is a non-commutative Banach algebra under composition.

2.2 Hölder and Schwartz spaces

Hölder spaces are basic in areas of functional analysis relevant to solving partial differential equations and in dynamical systems. The Hölder space with the Hölder norm is a Banach space [28].

Definition 2.1 (Hölder space)**.** *The space consisting of functions satisfying a Hölder condition (i.e for $c, \alpha > 0$ such that*

$$|f(x) - f(y)| \leq c|x - y|^\alpha \tag{2.3}$$

for all x and y in the domain of a real or complex-valued function f on d-dimensional Euclidean space) is called a Hölder space.

The Hölder space $C^{k,\alpha}(\Omega)$, where Ω is an open subset of some Euclidean space and $k \geq 0$ an integer, consist of functions on Ω having continuous derivatives up to order k and such that the k^{th} partial derivative are Hölder continuous with exponent α, where $0 < \alpha \leq 1$.

If Ω is open and bounded, then we can say that the Hölder space $C^{k,\alpha}(\bar{\Omega})$ consists of all functions, $u \in C^{k}(\bar{\Omega})$ for which the norm

$$\|u\|_{C^{k,\alpha}(\bar{\Omega})} = \sum_{|\gamma|\leq k} \|D^{\gamma}u\|_{C(\bar{\Omega})} + \sum_{|\gamma|=k} |D^{\gamma}u|_{C^{0,\alpha}(\bar{\Omega})} \tag{2.4}$$

is finite. We note that If $0 < \alpha < \beta$ and Ω is bounded, then the Hölder space $C^{\beta}(\Omega)$ is compactly embedded to $C^{\alpha}(\Omega)$.

Remark 2.2. *If Ω is open and bounded, then $C^{k,\alpha}(\bar{\Omega})$ is a Banach space with respect to the norm $\|\cdot\|_{C^{k,\alpha}}$.*

Definition 2.2 (Schwartz space)**.** *The Schwartz space $\mathscr{S}(\mathbb{R}^n)$ or space of rapidly decreasing functions on $\mathbb{R}^n$ is the topological vector space of functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $f \in C^{\infty}(\mathbb{R}^n)$ and $x^{\alpha}\partial^{\beta} f(x) \to 0$ as $|x| \to \infty$ for every pair of multi-indices $\alpha, \beta \in \mathbb{Z}^n_+$.*

If $\alpha, \beta \in \mathbb{Z}^n_+$ and $f \in \mathscr{S}(\mathbb{R}^n)$ then we have the family of semi-norms of f to be

$$\|f\|_{\alpha\beta} = \sup_{x\in\mathbb{R}^n} |x^{\alpha}\partial^{\beta} f(x)|. \tag{2.5}$$

The Schwartz space is a Fréchet space which have the property that the Fourier transform is a linear isomorphism, $\mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$, and if $f \in \mathscr{S}(\mathbb{R}^n)$ then f is uniformly continuous on $\mathbb{R}$. The Schwartz space also have the property that if $f, g \in \mathscr{S}(\mathbb{R}^n)$, then $fg \in \mathscr{S}(\mathbb{R}^n)$ and also if $1 \leq p \leq \infty$, then $\mathscr{S}(\mathbb{R}^n) \subset \mathrm{L}^p(\mathbb{R}^n)$ [29].

We next briefly discuss the Fréchet derivative and also refer the reader to [30, 31] for a detailed presentation.

2.3 Fréchet differentiability

Fréchet derivative is a derivative which is defined on the Banach Spaces. It extends the idea of the derivative from real-valued functions of one real variable to functions on Banach spaces. The Fréchet derivative has applications to nonlinear problems throughout mathematical analysis and physical sciences, particularly to the calculus of variations and much of nonlinear analysis and nonlinear functional analysis [31].

Definition 2.3 (Fréchet derivative)**.** *If we have a function f, which is defined to be an open subset of U of a Banach space X into the Banach space Y. We say f is Fréchet differentiable at $x \in U$ if there is a bounded and linear operator $T : X \mapsto Y$ such that*

$$\lim_{t \to 0} \frac{f(x+th) - f(x)}{t} = T_x(h) \tag{2.6}$$

is uniform for every $h \in S_X$. The operator T is called the Fréchet derivative of f at x.

Conversely, if we set $th = y$ and if $t \to 0$ then $y \to 0$. Therefore by this changes, we have $f : X \mapsto Y$ to be Fréchet differentiable at $x \in U$ if

$$\lim_{y \to 0} \frac{\|f(x+y) - f(x) - T(y)\|_Y}{\|y\|_X} = 0 \tag{2.7}$$

for all $y \in X$.

We have from [23] that a Fréchet derivative belongs to neither X nor Y, but rather is a bounded linear operator from X to Y (To say that $\cos x_0$ is the derivative at x_0 of the function $f : \mathbb{R} \mapsto \mathbb{R}$ given by $f(x) = \sin x$ means only that $df[x_0]x = x \cos x_0$ for all $x \in \mathbb{R}$). In practice, one can consider $\frac{d}{dt} f(x+ty)|_{t=0} = Df[x](y)$ for $x, y \in X$ and $t \in \mathbb{R}$, where the left hand side is defined as the Gâteaux derivative.

2.4 Completely monotone and Stieltjes functions

In this section, we present our exposition with a brief survey and analysis of completely monotone and Stieltjes functions. The reader is recommended to read [25, 32, 33] for a more detailed analysis.

Definition 2.4 (Completely monotonic function). *A function f is completely monotone on $[0,\infty)$ if it is continuous on $[0,\infty)$, infinitely differentiable on $(0,\infty)$ and also satisfies*

$$(-1)^k \frac{d^k}{dt^k} f(t) \geq 0, \quad \textit{for } t > 0 \textit{ and } k = 0, 1, 2, \ldots. \tag{2.8}$$

According to [24], if $f(t)$ and $g(t)$ are completely monotone, then $\alpha f(t) + \beta g(t)$, where α and β are non-negative constants, and $f(t)g(t)$ are also completely monotone. It is also proven that, if $h(t)$ is non-negative function with a completely monotonic derivative, then $f[h(t)]$ is also completely monotone.

There exist limits of $f^{(k)}$ as $t \to 0$ for any $k \geq 0$; if those limits are finite then f can be extended to $[0, +\infty)$ and (2.8) will also hold for $t = 0$ (with strict inequality for all k). Limits at zero need not be finite, as in $f(t) = \frac{1}{t}$, for example. It is clearly seen, that

$$\lim_{t \to +\infty} f^{(k)}(t) = 0 \tag{2.9}$$

for all $k \geq 1$. The limit of $f(t)$ at $+\infty$ must be finite and if it is non-zero, then it has to be positive (for example, $f(t) = 1 + e^{-t}$).

It is known (Bernstein's Theorem) that f is completely monotonic if and only if

$$f(t) = \int_0^\infty e^{-st} d\mu(s) \tag{2.10}$$

where μ is a non-negative measure on $[0,\infty)$ such that the integral converges for all $t > 0$. For a proof of this results, see Schilling and

Vondracek [25]. A consequence of Bernstein's theorem is that if f is completely monotone, then (2.8) holds with strict inequality for every t and every k, unless f is identically constant.

Remark 2.3. *Note that the measure μ in (2.10) is finite if and only if*

$$\lim_{t\to 0} f(t) < \infty.$$

Definition 2.5 (Stieltjes function)**.** *A function $f : (0,\infty) \to [0,\infty)$ is said to be a (non-negative) Stieltjes function if it admits a representation*

$$f(t) = \frac{\alpha}{t} + \beta + \int_{(0,\infty)} \frac{1}{t+s} d\mu(s) \qquad (t > 0), \tag{2.11}$$

where α and β are non-negative constants and μ is a positive measure on $[0,\infty)$ such that

$$\int_{(0,\infty)} (1+s)^{-1} d\mu(s) < \infty \tag{2.12}$$

Remark 2.4. *We note from [1] that, if f has a finite limit at the origin, then $\alpha = 0$ and $\int_{(0,\infty)} \frac{d\mu(s)}{t} < \infty$ by Fatou's lemma. Moreover, $\beta = \lim_{t\to\infty} f(t)$. The fact that Stieljes functions are completely monotone is proved in [25].*

The integral appearing in (2.11) is called the Stieltjes transform of the measure μ. It is apparent that by the dominated convergence theorem the Stieltjes function is completely monotone on $(0,\infty)$, thus it is a subclass of the completely monotonic function, but we must also note that not every completely monotone function is a Stieltjes function.

Theorem 2.5. *In [[25], Theorem 2.2], it is given that Stieltjes functions are completely monotone. A completely monotone function is a Stieltjes function if and only if the measure μ in (2.10) is absolutely* *continuous on* $(0,\infty)$ *and its Radon-Nikodym derivative is completely monotone.*

It turns out from [1], that any Stieltjes function has an analytic extension to the cut complex plane $\mathbb{C}\backslash(-\infty,0]$. This property gives a complete characterization of the class of Stieltjes functions. Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im m\, z > 0\}$ and $\mathbb{C}_- = \{z \in \mathbb{C} : \Im m\, z < 0\}$.

Theorem 2.6. *[[25], Corollary 7.4] Let* f *be a positive function on* $(0,\infty)$*. Then* f *is a Stieltjes function if and only if the limit* $\lim_{t\to 0} f(t)$ *exist in* $[0,\infty]$ *and* f *extends analytically to* $\mathbb{C}\backslash(-\infty,0]$ *such that* $\Im m\, z \cdot \Im m f(z) \leq 0$.

Remark 2.7. *From [1] we note that, positive constant functions are examples of Stieltjes functions. It follows easily by basic properties of analytic functions that a non constant Stieltjes function maps* $\mathbb{C}_+$ *to* $\mathbb{C}_-$*. We also note that if* f *is not identically 0, then* $1/f(z)$ *is a Nevanlinna function (A complex function which is an analytic function on the open upper half-plane and has non-negative imaginary part). The corresponding function* $1/f(t)$ *is then a complete Bernstein function by [25].*

Lemma 2.8. *[[1], Lemma 2.12] If* f *is a Stieltjes function, then so is* f^s *for any* $s \in (0,1]$.

We end the section by giving a brief explanation about the implicit function theorem and also refer the reader to [23, 26] for details.

2.5 The implicit function theorem

In mathematics, more specifically in multivariable calculus, the implicit function theorem is a tool that allows relations to be converted to functions of several real variables.

Theorem 2.9 (Implicit function theorem)**.** *Suppose X, Y and Z are Banach spaces and H is an open subset of $X\times Y$, such that the mapping $f : H \to Z$ is continuously Fréchet differentiable on H. If $(x_o, y_o) \in H$, $f(x_o, y_o) = 0$ and $Y \ni y \mapsto \partial f(x_o, y_o)(0, y)$ is a Banach space isomorphism from Y onto Z, then there exist an open subset $U \subset X$ and $V \subset Y$ such that $x_o \in U$ and $y_o \in V$ and a continuously Fréchet differentiable function $g : U \to V$ such that $f(x, g(x)) = 0$ and $f(x, y) = 0$ if and only if $y = g(x)$, for all $(x, y) \in U\times V$.*

Remark 2.10. *Note $H = \{(x, g(x)) : (x, y) \in U \times V\}$.*

In practice, if we consider a function $f : \mathbb{R}^3 \to \mathbb{R}$ (with continuous partial derivatives) given by $f(x, y, z) = x^2 + y^2 + z^2 - 1$. Suppose that (x_o, y_o, z_o) is a point satisfying $f(x_o, y_o, z_o) = 0$ and $\frac{\partial f}{\partial z}(x_o, y_o, z_o) \neq 0$ but $x_o \neq 1, -1$ and $y_o \neq 1, -1$. In this case there is an open disk $\mathrm{M} \subset \mathbb{R}^2$ containing (x_o, y_o) and an open interval $\mathrm{N} \subset \mathbb{R}$ containing z_o with the property that if $(x, y) \in \mathrm{M}$ then there is a unique element of N for which $f(x, y, g(x, y)) = 0$.

In other words, there is a function $g : \mathrm{M} \to \mathrm{N}$ so that $z = g(x, y)$ or, we solve for z in terms of the variables x and y. We say that equation $f(x, y, z) = 0$ has implicitly defined z as a function of x and y. In such a case, we are able to explicitly solve for z, for if $x > 0$ and $y > 0$, then $z = g(x, y) = \sqrt{1 - x^2 - y^2}$ (Note that the function g is differentiable).

On the other hand, if we were to have chosen $x_o = 1$ and $y_o = 1$, then we would not be able to find such a function g defined on an open interval containing 1, for some values of x and y would of necessity be sent to two different values of z.

Remark 2.11. *$z = g(x, y)$ is differentiable with the derivative given by*

$$\frac{\partial g}{\partial x} = -\frac{\partial f}{\partial x}/\frac{\partial f}{\partial z} \quad \text{and} \quad \frac{\partial g}{\partial y} = -\frac{\partial f}{\partial y}/\frac{\partial f}{\partial z}.$$

3 Fourier Series and Transform on $\mathbb{R}$

In the year 1807, the French mathematician and physicist, Fourier made an astonishing discovery through his deep analytical investigations into the partial differential equations modeling heat propagation in bodies. Fourier introduced the series for the purpose of solving the heat equation in metal plate and also investigated the decomposition of a periodic function f into a countable sum of sines and cosines [34], that is

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{p}\right) + b_n \sin\left(\frac{2\pi nx}{p}\right) \tag{3.1a}$$

or

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{\frac{2\pi inx}{p}} \tag{3.1b}$$

where p is the period of f and c_n is given by

$$c_n = \frac{1}{p}\int_{-p}^{p} f(x)e^{\frac{-2\pi inx}{p}}\, dx. \tag{3.2}$$

In using the orthogonality properties of sine and cosine, he found simple formulas for the coefficients a_o, a_n, b_n and c_n and then applied the techniques in the analysis of the heat equation with periodic boundary conditions. The infinite sum of the right hand side expressed in (3.1a) and (3.1b) are known as the Fourier series representation of f.

Fourier analysis is an essential component of much of modern applied (and pure) mathematics. It forms an exceptionally powerful analytic tool for solving a broad range of linear partial differential equations and it is also applicable in the field of physics, engineering, biology, finance, among others. Many modern technological advances, including television, music CDs and DVDs, cell phones, movies, computer graphics, image processing, and fingerprint analysis and storage, are, in one way or another, founded on the many ramifications of Fourier theory [35].

We begin our discussion by introduction Fourier series in Section 3.1. Section 3.2, introduces the periodic functions and extensions and in Section 3.3, we explore some fundamental properties of Fourier series related to convergence, differentiation, decay and convolution. Section 3.4 next gives the concept of Carleson-Hunt theorem on Fourier series. In Section 3.5, we discuss the Fourier transform on $L^1(\mathbb{R})$, $L^2(\mathbb{R})$ and the Schwartz space $\mathscr{S}(\mathbb{R})$ and finally we end with Section 3.6, which gives a brief summary about certain properties of the Fourier multiplier operators given by classical symbols.

The results in the various sections are mostly stated without proofs and specific references. The works by Cajori [34], Olver [35], Bogges and Narcowich [36], Zygmund [37], Jørsboe and Mejlbro [38], Strichartz [39], and Amann [40] covers all topics in this section.

3.1 Fourier series

The preceding section served to motivate the development of Fourier series as a tool for solving partial differential equations. Our immediate goal is to give a brief discussion about the Fourier series. A more detailed discussion can be found in [36, 35].

The coefficients of the full range Fourier series representation of f

on $(-p, p)$ in (3.1a) is defined by

$$a_o = \frac{1}{p}\int_{-p}^{p} f(x)dx, \tag{3.3a}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{2\pi nx}{p}\right)dx \qquad (n = 1, 2, 3, \ldots), \tag{3.3b}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{2\pi nx}{p}\right)dx \qquad (n = 1, 2, 3, \ldots). \tag{3.3c}$$

If f is $2p$-periodic then the series in (3.1a) is a representation of f. Fourier series is used in representing a given periodic function $f(x)$ in terms of cosine and sine functions. Calculation of a Fourier series boils down to computing the coefficients a_o, a_n and b_n and a firm graps of integration by parts is required to compute these calculations successfully.

In applications, it is found that most function are defined on a half-range interval $(0, p)$ and the $2p$-periodic extension of f can be defined to be an odd function or an even function. Fourier series could still be used to represent such functions defined on half-range intervals. The function f can be extended periodically with period p after which, the extended function can be represented by Fourier series which in general involves both sine and cosine terms.

Remark 3.1. *One of the draw backs in Fourier series is that in order for a function to have a Fourier series representation, the function must be periodic. A function f is odd if $f(-x) = -f(x)$ and even if $f(-x) = f(x)$ for all x.*

We next discuss the periodic functions and extensions of the Fourier series.

3.2 Periodic functions and extensions

Periodic functions are used throughout science to describe oscillations, waves and other phenomena that exhibit periodicity.

Definition 3.1 (Periodic functions)**.** *A function f is periodic with period $2p$ if*

$$f(x) = f(x + 2p) \tag{3.4}$$

for all x.

The most important examples are the trigonometric functions, which repeat over intervals of 2π radians.

Theorem 3.2. *If $f(x) = f(x + p)$ is periodic then $f(x) = f(x + 2p)$ is also periodic.*

Proof of Theorem 3.2. If we let $y = x + p$ then,

$$f(x + 2p) = f(y + p) = f(y) = f(x + p) = f(x).$$

Hence, for any integer n, $f(x + np) = f(x)$ for all x. □

The smallest positive number p for which (3.4) holds is called the fundamental period or simply the period of f.

Remark 3.3. *If f and g are periodic functions with period p then $\alpha f(x) + \beta g(x)$ and $f(x)g(x)$ are also periodic with period p, where α and β are constants. The function $f(x) = c$, where c is a constant is also a periodic function.*

All periodic functions are fully determined on $[0, p)$ or any half-open interval of length p. For example,

$$\int_t^{p+t} f(x)dx = \int_0^p f(x)dx \tag{3.5}$$

for any $t \in \mathbb{R}$.

Definition 3.2 (Periodic convolution)**.** *The p-periodic convolution $f *_p g$ between two p-periodic functions f and g is given by*

$$f *_p g = \int_0^p f(x-y)g(y)dy. \tag{3.6}$$

Theorem 3.4. *Let f be a function with a well-defined periodic summation f_s, where*

$$f_s(x) = \sum_{k=-\infty}^{\infty} f(x+kp). \tag{3.7}$$

If g is any other function for which the convolution $f_s *_p g$ exists, then the convolution $f_s *_p g$ is periodic.

Proof of Theorem 3.4.

$$\begin{aligned} f_s *_p g &= \int_{-\infty}^{\infty} f_s(x-y)g(y)dy \\ &= \sum_{k=-\infty}^{\infty} \int_{t+kp}^{t+(k+1)p} f_s(x-y)g(y)dy \end{aligned}$$

$y \mapsto y + kp$

$$\begin{aligned} &= \sum_{k=-\infty}^{\infty} \int_{t}^{t+p} f_s(x-y-kp)g(y+kp)dy \\ &= \int_{t}^{t+p} \left[f_s(x-y) \sum_{k=-\infty}^{\infty} g(y+kp) \right] \end{aligned}$$

$f_s(x-y-kp) = f_s(x-y)$ by periodicity and from (3.7), we can defined the function g by

$$g_s(y) = \sum_{k=-\infty}^{\infty} g(y+kp).$$

Hence, from (3.5) we conclude that

$$f_s *_p g = \int_0^p f_s(x-y)g_s(y).$$

□

Definition 3.3 (Periodic extensions). *If f is any function defined in the interval $(-p,p]$ or $[-p,p)$ then $2p$-periodic extension of f denoted $\tilde{f}$ is defined by*

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in (-p,p] \text{ or } x \in [-p,p) \\ \tilde{f}(x+2p) & \text{otherwise .} \end{cases}$$

Theorem 3.5. *[[35], Lemma 3.4] If $f(x)$ is any function defined for $-\pi < x \leq \pi$, then there is a unique 2π-periodic function $\tilde{f}$, known as the 2π-periodic extension of f, that satisfies $\tilde{f}(x) = f(x)$ for all $-\pi < x \leq \pi$.*

One can see [35] for a detailed prove. The construction of the periodic extension in Theorem 3.5, uses the value $f(\pi)$ at the right endpoint and requires $\tilde{f}(-\pi) = \tilde{f}(\pi) = f(\pi)$.

Alternatively. one could require $\tilde{f}(\pi) = \tilde{f}(-\pi) = f(-\pi)$, which, if $f(-\pi) \neq f(\pi)$, leads to a slightly different 2π-periodic extension of the function. There is no, a priori reason to prefer one over the other [35].

Remark 3.6. *A Fourier series can converge only to a 2π-periodic function.*

3.3 Convergence, differentiability, decay and convolution

The convergence, differentiation, decay and convolution of the Fourier series is briefly examine in this subsection and a more detailed discussion is presented in [36].

The convergence of Fourier series is somewhat important in the study of Fourier analysis. If we consider a 2π-periodic function which is integrable on the interval $[-\pi, \pi]$, then the Fourier coefficient defined in (3.2) can be redefined as

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx. \tag{3.8}$$

In a more careful investigation of convergence, the partial sums of Fourier series defined by

$$f_N(x) = \sum_{n=-N}^{N} c_n e^{inx} \tag{3.9}$$

is needed.

Definition 3.4 (Dirichlet kernel)**.** *The function*

$$D_N(x) = \sum_{n=-N}^{N} c_n e^{inx} = \frac{\sin(N + \frac{1}{2})x}{\sin \frac{x}{2}} \tag{3.10}$$

is called the Dirichlet kernel.

The Fourier partial sum of $f(x)$ can be expressed through the Dirichlet kernel:

$$\begin{aligned} f_N(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x-y) f(y) dy \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(y) f(x-y) dy. \end{aligned}$$

Theorem 3.7 (Riemann-Lebesgue Lemma)**.** *If $f \in L^1(-\pi, \pi)$ is a* *piecewise continuous function on the interval* $-\pi \leq x \leq \pi$*. Then*

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \;\rightarrow 0 \text{ as } n \rightarrow \pm\infty.$$

Proof of Theorem 3.7.

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\
-c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}e^{-i\pi}dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-in(x-\frac{\pi}{n})}dx
\end{aligned}$$

$y \mapsto x - \frac{\pi}{n}$

$$\begin{aligned}
-c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y+\frac{\pi}{n})e^{-iny}dy \\
-4\pi c_n &= \int_{-\pi}^{\pi}\left[f\left(y+\frac{\pi}{n}\right) - f(y)\right]e^{-iny}dy \\
4\pi|c_n| &\leq \int_{-\pi}^{\pi}\left|f\left(y+\frac{\pi}{n}\right) - f(y)\right|dy \\
&\rightarrow 0 \text{ as } n \rightarrow \pm\infty.
\end{aligned}$$

□

Theorem 3.8 (Uniform convergence)**.** *A sequence of the partial sums* $\{f_N(x)\}$ *is said to be uniformly convergent to the function* $f(x)$*, if the speed of convergence of the partial sums* $f_N(x)$ *does not depend on* x*.*

We say that the Fourier series of a function $f(x)$ converges uniformly to this function if

$$\lim_{N\rightarrow\infty}\left[\max_{x\in[-\pi,\pi]} |f(x) - f_N(x)|\right] = 0. \tag{3.11}$$

Lemma 3.9. *The Fourier series of a 2π-periodic continuous and piecewise smooth function converges uniformly.*

Theorem 3.10 (Convergence in L^2-norm)**.** *The space $L^2(-\pi, \pi)$ is formed by those functions for which*

$$\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty. \tag{3.12}$$

We will say that a function $f(x)$ is square-integrable if it belongs to the space L^2. If a function $f(x)$ is square-integrable, then

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - f_N(x)|^2 dx = 0. \tag{3.13}$$

That is the partial sums $f_N(x)$ converge to $f(x)$ in the norm L^2.

Remark 3.11. *The uniform convergence implies L^2-convergence. But the opposite is not true.*

Under appropriate hypotheses, if a series of functions converges, then one will be able to integrate or differentiate it term by term, and the resulting series should converge to the integral or derivative of the original sum [35].

Theorem 3.12 (Differentiation of Fourier series)**.** *If $f(x)$ defined in* (3.1a) *and* (3.1b) *has a piecewise C^2 and continuous 2π–periodic extension, then its Fourier series can be differentiated term by term, to produce the Fourier series for its derivative*

$$f'(x) \sim \sum_{n=1}^{\infty} [n\, b_n \cos(nx) - n\, a_n \sin(nx)] = \sum_{n=-\infty}^{\infty} inc_n e^{inx}. \tag{3.14}$$

Theorem 3.13 (Differentiation of Fourier transform)**.** *If we differen-*

tiate the basic inverse Fourier transform formula

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi \tag{3.15}$$

with respect to x, we obtain

$$f'(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\xi \hat{f}(\xi) e^{i\xi x} d\xi. \tag{3.16}$$

The resulting integral is itself in the form of an inverse Fourier transform, namely of i k $\hat{f}(\xi)$, which immediately implies the following key result.

Proposition 3.14. *The Fourier transform of the derivative $f'(x)$ of a function is obtained by multiplication of its Fourier transform by $i\xi$:*

$$\mathcal{F}[f'(x)] = i\xi \hat{f}(\xi). \tag{3.17}$$

Similarly, the Fourier transform of the product function $xf(x)$ is obtained by differentiating the Fourier transform of $f(x)$:

$$\mathcal{F}[x\ f(x)] = i\frac{d\hat{f}}{d\xi}. \tag{3.18}$$

Corollary 3.15. *The Fourier transform of $f^{(n)}(x)$ is $(i\xi)^n \hat{f}(\xi)$.*

The smoothness of the function $f(x)$ is manifested in the rate of decay of its Fourier transform $\hat{f}(\xi)$. The Fourier transform of a (nice) function must decay to zero at large frequencies: $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$ (This result can be viewed as the Fourier transform version of the Riemann– Lebesgue Lemma 3.7). If the n^{th} derivative $f^{(n)}(x)$ is also a reasonable function, then its Fourier transform $\widehat{f^{(n)}}(\xi) = (i\xi)^n \hat{f}(\xi)$ must go to zero as $|\xi| \to \infty$. This requires that $\hat{f}(\xi)$ go to zero more rapidly than $|\xi|^{-n}$. Thus, the smoother $f(x)$, the more rapid the decay of its Fourier transform. As a general rule of thumb, local features of $f(x)$, such as smoothness, are manifested by global features of $\hat{f}(\xi)$, such as the rate of decay for large $|\xi|$. The Symmetry Principle implies that the reverse is also true: global features of $f(x)$ correspond to local features of $\hat{f}(\xi)$. For instance, the degree of smoothness of $\hat{f}(\xi)$ governs the rate of decay of $f(x)$ as $x \to \pm\infty$ [35].

Uniform convergence of the Fourier series requires at the very least that the Fourier coefficients goes to zero : $c_n \to 0$ as $n \to \pm\infty$.

Theorem 3.16 (Decay)**.** *[[35], Theorem 3.31] Let $0 \leq k \in \mathbb{Z}$. If the Fourier coefficient of $f(x)$ satisfy*

$$\sum_{n=-\infty}^{\infty} |n|^k |c_n| < \infty, \tag{3.19}$$

then the Fourier series (3.1b) converges uniformly to a k-times continuously differentiable function $\tilde{f}(x) \in C^k$, which is the 2π-periodic extension of $f(x)$. Furthermore, for any $0 < l \leq k$, the l-times differentiated Fourier series converges uniformly to the corresponding derivative $\tilde{f}^{(l)}(x)$.

If the Fourier coefficients go to zero faster than any power of n, e.g., exponentially fast, then the function is infinitely differentiable. Analyticity is more delicate, and we refer the reader to [37] for details.

Theorem 3.17 (Convolution theorem)**.** *If $f, g \in L^1(\mathbb{R})$, then*

$$\widehat{f * g} = \sqrt{2\pi} \hat{f} \cdot \hat{g}. \tag{3.20}$$

If additionally $\hat{f}, \hat{g} \in L^1(\mathbb{R})$, then

$$\widehat{f \cdot g} = \sqrt{2\pi} \hat{f} * \hat{g}. \tag{3.21}$$

Thus (3.20) and (3.21) hold for all $f, g \in S(\mathbb{R})$. Moreover, if $f, g \in$

$L^2(\mathbb{R})$, *then*

$$f * g = \sqrt{2\pi}\mathcal{F}^{-1}(\hat{f}\cdot\hat{g}) \qquad \text{and} \qquad \widehat{f\cdot g} = \sqrt{2\pi}\hat{f} * \hat{g}.$$

3.4 The Carleson-Hunt theorem on Fourier series

Carleson-Hunt theorem is a fundamental result in mathematical analysis establishing the pointwise (Lebesgue) almost everywhere convergence of Fourier series of L^p functions for $p \in (1, \infty)$ [38]. If we consider the Fourier coefficients on $2p$-periodic functions on $\mathbb{R}$ defined by

$$\hat{f}_n := \int_{-p}^{p} f(x) e^{-\frac{inx\pi}{p}} dx. \tag{3.22}$$

We write

$$f(x) \sim \frac{1}{2p} \sum_{n\in\mathbb{Z}} \hat{f}_n e^{\frac{inx\pi}{p}} \tag{3.23}$$

to indicate that, under certain conditions on f, this infinite trigonometric series converges to f pointwise, uniformly, or in norm. For example [2], if $f \in L^p(-p, p)$, $p > 1$, then the Carleson-Hunt theorem [38] guarantees that the series converges to $f(x)$ almost everywhere. If, in addition, $f(x)$ is an even function, the series can be written as

$$\begin{aligned} f(x) &\sim \frac{1}{2p}\hat{f}_o + \frac{1}{p}\sum_{n=1}^{\infty} \hat{f}_n \cos\left(\frac{nx\pi}{p}\right) \\ &= \frac{1}{p}\sum_{n=0}^{\infty}{}' \hat{f}_n \cos\left(\frac{nx\pi}{p}\right), \end{aligned}$$

where the prime indicates that the first term of the sum is multiplied by 1/2.

We next examine the the Fourier transform on the spaces $L^1(\mathbb{R})$, $L^2(\mathbb{R})$ and $\mathscr{S}(\mathbb{R})$.

3.5 The Fourier transform on $L^1(\mathbb{R})$, $L^2(\mathbb{R})$ and the Schwartz space $\mathscr{S}(\mathbb{R})$

The extension of the Fourier calculus to the entire real line leads naturally to the Fourier transform, a powerful mathematical tool for the analysis of aperiodic functions.

Theorem 3.18 (Fourier transform formula)**.** *The Fourier transform $\mathcal{F}(f) = \hat{f}$ of an aperiodic function f is defined by*

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} dx. \tag{3.24}$$

Theorem 3.19 (Fourier inversion formula)**.** *If both $f, \hat{f} \in L^1(\mathbb{R})$, then*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(\xi) e^{i\xi x} d\xi \tag{3.25}$$

for almost everywhere $x \in \mathbb{R}$.

Remark 3.20. *It is not always the case that $\hat{f}$ is integrable whenever f is. But if $f \in L^2(\mathbb{R})$, with f, f' and f'' in $L^1(\mathbb{R})$, we do have $\hat{f} \in L^1(\mathbb{R})$.*

Lemma 3.21. *If $f \in L^1(\mathbb{R})$, then $|\hat{f}(\xi)| \leq \frac{1}{\sqrt{2\pi}} \|f(x)\|_{L^1}$.*

Proof of lemma 3.21.

$$\begin{aligned}
|\hat{f}(\xi)| &= \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx \right| \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| |e^{-i\xi x}| dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| dx \\
&= \frac{1}{\sqrt{2\pi}} \|f(x)\|_{\mathrm{L}^1}.
\end{aligned}$$

$\square$

Lemma 3.22. *If $f_n \to f$ in L^1, then $\hat{f}_n \to \hat{f}$ in L^∞.*

Proof of Lemma 3.22.

$$\begin{aligned}
|\hat{f}(\xi) - \hat{f}(\xi)| &= \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f_n(x) - f(x)) e^{-i\xi x} dx \right| \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f_n(x) - f(x)||e^{-i\xi x}| dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f_n(x) - f(x)| dx \\
&= \frac{1}{\sqrt{2\pi}} \|f_n(x) - f(x)\|_{\mathrm{L}^1}
\end{aligned}$$

$\to 0$ as $n \to \infty$ by assumption. □

Lemma 3.23. *$\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$ by the Riemann-Lebesgue Lemma 3.7.*

Remark 3.24. *$\hat{f}(\xi)$ is uniformly continuous in $\mathbb{R}$ [36].*

Theorem 3.25 (Parseval's theorem)**.** *If f belongs to $L^2[-\pi, \pi]$, then*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx. \tag{3.26}$$

Theorem 3.26 (Plancherel's theorem)**.** *The Fourier transform extends uniquely to a unitary operator $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$. That is*

$$\langle \hat{f}, \hat{g} \rangle_{L^2(\mathbb{R})} = \langle f, g \rangle_{L^2(\mathbb{R})} \tag{3.27}$$

for all $f, g \in L^2(\mathbb{R})$.

Proposition 3.27. *If $f \in L^1(\mathbb{R})$, $f \in L^2(\mathbb{R})$ and also if $\hat{f}$ is as defind in (3.24), then $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$.*

Remark 3.28. *Fourier transforms on $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ coincide on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.*

The Fourier transform is a linear isomorphism $\mathcal{F} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$, and if $f \in \mathscr{S}(\mathbb{R})$ then f is uniformly continuous on $\mathbb{R}$. If f and g belongs to the class $\mathscr{S}(\mathbb{R})$ of rapidly decreasing functions, then $\widehat{f * g}$, $\hat{f}$ and $\hat{g}$ all exist in $\mathscr{S}'(\mathbb{R})$ (the tempered distributions). The space of tempered distributions $\mathscr{S}'(\mathbb{R})$ is defined as the (continuous) dual of the Schwartz space$\mathscr{S}(\mathbb{R})$. We refer the reader to [39] for a precise details on the Fourier transform on $\mathscr{S}$ and also the Fourier transform of tempered distribution.

Finally, we end the section with a brief discussion of Fourier multipliers on Hölder spaces.

3.6 Fourier multipliers on Hölder spaces

We introduce a brief summary of certain properties of the Fourier multiplier operators, given by classical symbols for the purpose of our analysis. We refer the reader to [40, 39] for a more detailed argument.

A smooth, real-valued function g on $\mathbb{R}$ is said to be in the symbol class $\mathcal{S}^m$ if for some constant $c > 0$ and any non-negative integer k, the estimate

$$|\partial_\xi^k g(\xi)| \leq c(1 + |\xi|)^{m-k} \tag{3.28}$$

holds. If $\alpha \geq 0$ is real, we may consider those functions in L^2 such that

$$\int (1 + |\xi|^2)^\alpha |\hat{g}(\xi)|^2 d\xi \tag{3.29}$$

is finite to define the Sobolev space H^2_α.

Remark 3.29. *Notice that since $1 \leq (1+|\xi|^2)^\alpha$ the finiteness of this integral implies $\int |\hat{f}(\xi)|^2 d\xi < \infty$ which implies $f \in L^2$ by the Plancherel theorem.*

4 The Generalized Whitham Kernel

In this section, we discuss the generalized Whitham kernel and its properties. We will first review the monotonicity property of the generalized Whitham kernel and next discuss the limit property of the generalized Whitham symbol. We then finally end the section with some discussion on the convolution operator L_w of the generalized Whitham kernel. One should note that not all theorems are proved, hence we refer the reader to the necessary reference for a detailed proof.

Whitham [9] introduced the Whitham equation (1.1) after recognizing the problems of the Korteweg-de Vries (KdV) equation (a model equation for water waves). The equation was introduced with the kernel defined in (1.2). A more precise details about the Whitham kernel (1.2) is presented in [1, 2].

In our discussion we will consider $g, h_o \sim 1$ in (1.3) and examine the generalized Whitham kernel defined by

$$K_s(x) = \mathcal{F}^{-1}\{m_s(\xi)\} = \frac{1}{2\pi}\int_{\mathbb{R}} m_s(\xi)e^{ix\xi}d\xi \tag{4.1}$$

where $m_s(\xi)$ is the generalized Whitham symbol for which we will define as

$$m_s(\xi) = \widehat{K}_s(\xi) = \left(\frac{\tanh\xi}{\xi}\right)^s, \qquad 0 < s < 1. \tag{4.2}$$

4.1 Monotonicity property of the generalized Whitham kernel

Our aim is to show that the generalized Whitham symbol (4.2) belongs to the class of completely monotone functions. A more general theory can be found in the monograph [25], although we only skew the discussion to the generalized Whitham symbol.

The generalized Whitham symbol can be represented as $m_s(\xi) = f(\xi^2)$, where

$$f(\lambda) = \left(\frac{\tanh\sqrt{\lambda}}{\sqrt{\lambda}}\right)^s, \qquad \lambda \geq 0 \text{ and } 0 < s < 1. \tag{4.3}$$

It is clearly seen that $f(\lambda)$ is positive on the interval $(0, \infty)$ and also has a finite limit as $\lambda \to 0$. That is

$$\begin{aligned}
\lim_{\lambda\to 0} f(\lambda) &= \lim_{\lambda\to 0}\left(\frac{\tanh\sqrt{\lambda}}{\sqrt{\lambda}}\right)^s \\
&= \lim_{\lambda\to 0}\left(\frac{\sinh\sqrt{\lambda}}{\sqrt{\lambda}}\cdot\frac{1}{\cosh\sqrt{\lambda}}\right)^s \\
&= \left(\lim_{\lambda\to 0}\frac{\sinh\sqrt{\lambda}}{\sqrt{\lambda}}\cdot\lim_{\lambda\to 0}\frac{1}{\cosh\sqrt{\lambda}}\right)^s \\
&= 1 < \infty.
\end{aligned}$$

Theorem 4.1. *[[1], Proposition 2.20] Let g and f be two functions satisfying $g(\xi) = f(\xi^2)$. Then g is the Fourier transform of an even, integrable and completely monotone function if and only if f is Stieltjes with $\lim_{\lambda\to 0} f(\lambda) < \infty$ and $\lim_{\lambda\to\infty} f(\lambda) = 0$.*

Proof of Theorem 4.1. See [1], Proposition 2.20 for proof. □

Proposition 4.2. *$(h(\lambda))^s$ is a Stieljes function for any $s \in (0, 1)$.*

Proof of Proposition 4.2. We can observe that the function f in (4.3) has a limit 0 as $\lambda \to \infty$ and 1 as $\lambda \to 0$ (see Section 4.2). It is then left to show that f is a Stieltjes function and to proof this we consider a function h which is defined by

$$h(\lambda) = \left(\frac{\tanh\sqrt{\lambda}}{\sqrt{\lambda}}\right), \qquad \lambda \geq 0. \tag{4.4}$$

and $f(\lambda) = (h(\lambda))^s$. It is noted that the reciprocal of $h(\lambda)$

$$\lambda \mapsto \frac{\sqrt{\lambda}}{\tanh\sqrt{\lambda}}$$

is positive on $(0,\infty)$ with the finite limit 1 as $\lambda \to 0$, and extends to an analytic function on $\mathbb{C}\backslash(-\infty, 0]$ if we let $\sqrt{\lambda}$ denote the principal branch of the square root. It also maps $\mathbb{C}_+$ to $\mathbb{C}_+$. By a straightforward calculation it can be shown that

$$\begin{aligned}
\Im m\left(\frac{z}{\tanh z}\right) &= \Im m\left(\frac{z\cosh z}{\sinh z}\right) \\
&= \Im m\left(\frac{z(e^z + e^{-z})}{(e^z - e^{-z})} \cdot \frac{\overline{(e^z - e^{-z})}}{\overline{(e^z - e^{-z})}}\right) \\
&= \frac{\Im m\, z(2\sinh(2\Re e\, z) + 2i\sin(2\Im m\, z))}{|e^z - e^{-z}|^2} \\
&= \frac{2}{|e^z - e^{-z}|^2}(\Im m\, z\sinh(2\Re e\, z) - \Re e\, z\sin(2\Im m\, z)) \\
&> \frac{4}{|e^z - e^{-z}|^2}(\Im m\, z\,\Re e\, z - \Re e\, z\,\Im m\, z) \\
&= 0
\end{aligned}$$

when $\Re e\, z$, $\Im m\, z > 0$ from which it follows that $\Im m(\sqrt{\lambda}/\tanh\sqrt{\lambda}) > 0$ when $\Im m\,\lambda > 0$. This implies that $\lambda \mapsto \tanh\sqrt{\lambda}/\sqrt{\lambda}$ satisfies the conditions of Theorem 2.6 and Remark 2.7, hence the function h is a Stieltjes function. In agreement with Lemma 2.8, we can then say that $(h(\lambda))^s = f(\lambda)$ is a Stieltjes function. □

Remark 4.3. *It must be noted that* $\sinh(z) = -i\sin(iz)$, $\sinh z \geq z$ *and* $\sin z \leq z$, *for* $z \geq 0$.

The generalized Whitham kernel $K_s(x)$ in (4.1) is completely monotone on $(0,\infty)$. In particular, it is positive, strictly decreasing and strictly convex for $x > 0$ as proved by [1].

Ehrnström and Wahlén remarked in [1] that an alternative approach to obtaining the positivity and monotonicity properties of the Whitham kernel is to study the functions $-xD_xK(x)$ and $x^2D_x^2K(x)$. And that these functions are regular at the origin and one can show that their Fourier transforms $D_\xi(\xi m_s(\xi))$ and $D_\xi^2(\xi^2 m_s(\xi))$, respectively, are positive definite.

4.2 Limit property of the generalized Whitham symbol

Limits are essential to mathematical analysis in general and are used to define continuity, derivatives and integrals. We will in this section examine the limit properties of the generalized Whitham symbol and kernel.

It is clearly seen that the function $m_s(\xi)$ in (4.2) is real analytic, even and strictly decreasing on $(0, \infty)$. The generalized Whitham symbol takes the following limits:

$$\begin{aligned}
\lim_{\xi \to 0} m_s(\xi) &= \lim_{\xi \to 0} \left(\frac{\tanh \xi}{\xi} \right)^s \\
&= \lim_{\xi \to 0} \left(\frac{\sinh \xi}{\xi} \cdot \frac{1}{\cosh \xi} \right)^s \\
&= \left(\lim_{\xi \to 0} \frac{\sinh \xi}{\xi} \cdot \lim_{\xi \to 0} \frac{1}{\cosh \xi} \right)^s \\
&= 1 < \infty
\end{aligned}$$

$$\begin{aligned}
\lim_{\xi\to\infty} m_s(\xi) &= \lim_{\xi\to\infty}\left(\frac{\tanh\xi}{\xi}\right)^s \\
&= \lim_{\xi\to\infty}\left(\frac{\sinh\xi}{\xi}\cdot\frac{1}{\cosh\xi}\right)^s \\
&= \left(\lim_{\xi\to\infty}\frac{\sinh\xi}{\xi}\cdot\lim_{\xi\to\infty}\frac{1}{\cosh\xi}\right)^s \\
&= 0
\end{aligned}$$

since $\lim_{\xi\to\infty} 1\backslash\cosh\xi$ rapidly turns to 0. This also holds by Lemma 3.23. Consequently,

$$\int_{-\infty}^{\infty} K_s(x)dx = 1. \tag{4.5}$$

Proof of (4.5). If $f \in \mathrm{L}^1(\mathbb{R})$, then

$$\int_{\mathbb{R}} f(x)dx = \hat{f}(0) = \int_{\mathbb{R}} f(x)e^{ix\xi}\Big|_{\xi=0} dx$$

$$\int_{\mathbb{R}} K_s(x)dx = \hat{K}_s(0) = \left(\frac{\tanh\xi}{\xi}\right)^s\Big|_{\xi=0} = 1.$$

□

We can therefore deduce from the proof of (4.5) that

$$\|K_s\|_{L^1(\mathbb{R})} = \left\|\mathcal{F}^{-1}\left\{\left(\frac{\tanh\xi}{\xi}\right)^s\right\}\right\|_{L^1(\mathbb{R})} = 1.$$

Thus, it can be shown that $K_s \in \mathrm{L}^1(\mathbb{R})$ in the following way. Since the function $m_s(\xi)$ is analytic, the inverse Fourier transform has rapid decay. Thus, splitting the integral according to

$$\begin{aligned}\|K_s\|_{L^1(\mathbb{R})} &= \int_{\mathbb{R}} |K_s(x)|dx \\ &= \int_{|x|\leq 1} |K_s(x)|dx + \int_{|x|\geq 1} |K_s(x)|dx\end{aligned}$$

it is plain that K_s has finite $L^1(\mathbb{R})$-norm. In fact, this argument establishes more generally that $K_s \in L^p(\mathbb{R})$ for $1 \leq p < 2$, [2].

Remark 4.4. *We note that the smooth and even function $m_s(\xi)$ is increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$, reaching its global maximum of unit size at $\xi = 0$. As $|\xi| \to \infty$, it vanishes with the rate $|\xi|^{-s}$.*

We finally in the next section briefly discuss some properties of the convolution operator and also examine how it acts on periodic functions.

4.3 The convolution operator L_w

The convolution operator from the Whitham map is much needed in our bifurcation analysis and it is necessary that we know it properties. We refer the reader to [2, 12, 13] for more details. We define the convolution operator by

$$L_w := K_s * . \tag{4.6}$$

Theorem 4.5 (Bounded linear operator). *L_w is a bounded linear operator on $L^2(\mathbb{R})$, if for $f \in L^2(\mathbb{R})$ then $\|L_w f\|_{L^2(\mathbb{R})} \leq \|f\|_{L^2(\mathbb{R})}$.*

Proof of Theorem 4.5. In applying Theorems 3.26 and 3.17, we have that

$$\begin{aligned}\|L_w f\|_{L^2(\mathbb{R})} &= \|\mathcal{F}(L_w f)\|_{L^2(\mathbb{R})} \\ &= \left\| \left(\frac{\tanh \xi}{\xi}\right)^s \hat{f}(\xi) \right\|_{L^2(\mathbb{R})} \\ &\leq \|\hat{f}\|_{L^2(\mathbb{R})} \\ &= \|f\|_{L^2(\mathbb{R})}.\end{aligned}$$

□

Theorem 4.6 (Symmetric bounded linear operator)**.** *The operator L_w is symmetric on $L^2(\mathbb{R})$; if $f, g \in L^2(\mathbb{R})$, then $(L_w f, g)_{L^2(\mathbb{R})} = (f, L_w g)_{L^2(\mathbb{R})}$.*

Proof of Theorem 4.6. We apply Theorem 3.26 and also suppose $f, g \in L^2(\mathbb{R})$, then we have that

$$\begin{aligned}(L_w f, g)_{L^2(\mathbb{R})} &= (\mathcal{F}(L_w f), \mathcal{F}(g))_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} \mathcal{F}(L_w f)\overline{\mathcal{F}(g)} d\xi \\ &= \int_{\mathbb{R}} \left(\frac{\tanh \xi}{\xi}\right)^s \hat{f}(\xi)\overline{\hat{g}(\xi)} d\xi \\ &= (f, L_w g)_{L^2(\mathbb{R})}.\end{aligned}$$

It follows that L_w is a symmetric bounded linear operator on the space $L^2(\mathbb{R})$. □

We next discuss how the convolution operator acts on periodic functions. If $f \in L^\infty(\mathbb{R})$ is periodic and even and that since K_s is in $L^1(\mathbb{R})$, then by Theorem 3.4 we can write the integral

$$\begin{aligned}\int_{-\infty}^{\infty} K_s(x-y)f(y)dy &= \sum_{n=-\infty}^{\infty}\int_{-p}^{p} K_s(x-y+2np)f(y)dy\\ &= \int_{-p}^{p}\left(\sum_{n=-\infty}^{\infty} K_s(x-y+2np)\right)f(y)dy\\ &= \int_{-p}^{p} T(x-y)f(y)dy.\end{aligned}$$

The definition $T(x)$ shows that it is $2p$-periodic, even and continuous on $[-p,p]\backslash\{0\}$ by (3.7). It is proved by Ehrnström and Kalisch in [2] that $T(x)$ belongs to $L^p(-p,p)$, for $1\leq p<2$ using Minkowski's inequality. Therefore, according to Carleson-Hunt theorem [38], $T(x)$ can be approximated pointwise by its Fourier series. Thus from Section 3.4, we have

$$T(x)=\frac{1}{p}{\sum_{n=0}^{\infty}}'\widehat{T}_n\cos\left(\frac{n\pi x}{p}\right) \qquad a.e., \tag{4.7}$$

where the Fourier coefficients of T are given by

$$\widehat{T}_n=\int_{-p}^{p}\sum_{k=-\infty}^{\infty}K_s(x+2kp)e^{-\frac{ixn\pi}{p}}dx$$

$x\mapsto x+2kp$

$$\begin{aligned}&=\sum_{k=-\infty}^{\infty}\int_{-p}^{p}K_s(x+2kp)e^{-\frac{i(x+2kp)n\pi}{p}}dx\\ &=\int_{-\infty}^{\infty}K_s(x)e^{-\frac{ixn\pi}{p}}dx\\ &=\widehat{K}_s\left(\frac{n\pi}{p}\right).\end{aligned}$$

One can observe that the periodic problem is given by the same multiplier as the problem at hand, hence we have the representation

$$\begin{aligned} L_w f &= K_s * f(x) \\ &= \frac{1}{p} \sum_{n=0}^{\infty}{}' \hat{f}_n \widehat{T}_n \cos\left(\frac{n\pi x}{p}\right) \\ &= \frac{1}{p} \sum_{n=0}^{\infty}{}' \hat{f}_n \widehat{K}_s\left(\frac{n\pi}{p}\right) \cos\left(\frac{n\pi x}{p}\right). \end{aligned}$$

We will now in the next section discuss the local bifurcation for the Whitham equation which will later be extended to the global continuous curves of solutions in Section 6.

5 Local Bifurcation for the Whitham Equation

In this section, we will discuss the local bifurcation in relation to the Whitham equation by first investigating the existence of traveling waves. The section is then ended with a discussion of the local bifurcation theory (the reader is referred to [3, 2] for a more detailed work).

The solution of the Whitham equation is defind on the space C^{α}_{even}, $\alpha \in (0,1)$, that is the space of even and α-Hölder continuous real-valued functions on the unit circle $\mathbb{S}$. We also take into consideration that the convolution operator (4.6) is a bounded linear operator (Theorem 4.5) on $C^{\alpha}_{\text{even}}(\mathbb{S}) \to C^{\alpha+s}_{\text{even}}(\mathbb{S})$ for $\alpha + s \notin \mathbb{Z}$. Bifurcation theory is the mathematical study of changes in the qualitative or topological structure of a given family, such as the integral curves of a family of vector fields, and the solutions of a family of differential equations [41, 42].

Most commonly applied to the mathematical study of dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behaviour. Bifurcations occur in both continuous systems (described by ODE's or PDE's) and discrete systems (described by maps). Two main principal classes are known as the local and global bifurcation [43].

5.1 Existence of periodic traveling waves

In considering steady solutions with the propagation speed $c > 0$ of a right-going traveling wave, we make the usual ansatz $\eta(x,t) = \varphi(x -$

ct). Using this form, the equation (1.4) transforms into

$$-c\varphi' + \frac{3c_o}{2}\varphi\varphi' + K_s * \varphi' = 0 \tag{5.1}$$

which may be integrated to

$$-c\varphi + \frac{3c_o}{4}\varphi^2 + K_s * \varphi = \beta \tag{5.2}$$

for some real constant β. For solutions $\varphi \in \mathrm{L}^2(\mathbb{R})$, it appears that the convolution $K_s * \varphi$ is in $\mathrm{L}^2(\mathbb{R})$ since K_s is in $\mathrm{L}^1(\mathbb{R})$ as shown in Section 4.3. Therefore, the left-hand side must vanish as $|x| \to \infty$, and we shall consider the only case for which $\beta = 0$ [2]. The scalings $\frac{3}{4}\varphi \mapsto \varphi$ and $\frac{1}{c_o}K_s \mapsto K_s$ then yield the normalised equation

$$-\mu\varphi + \varphi^2 + K_s * \varphi = 0 \tag{5.3}$$

where $\mu := c\backslash c_o$ is the non-dimensional wave speed.

Alternatively, we can also consider the scalings $\frac{3c_o}{4c}\varphi \mapsto \varphi$ which yields the normalised equation

$$-\varphi + \varphi^2 + \frac{1}{c}K_s * \varphi = 0. \tag{5.4}$$

We refer reader to the Crandall-Rabinowitz bifurcation theorem [2, 44] for details on the following theorem and lemma. The proof of the theorem is an adaption of the one in [2], but for general $s \in (0, 1)$.

Theorem 5.1. *For a given $p > 0$ and a given depth $h_o \sim 1$, there exists a local bifurcation curve of, $2p$-periodic, even and continuous solutions of the weak Whitham equation (5.4). Those solutions are perturbations in the direction of $\cos(\pi x/p)$, and their wave speed at the bifurcation* *point is determined by the full dispersion relation*

$$c^* = \left(\frac{p\,\tanh(\pi/p)}{\pi}\right)^s \tag{5.5}$$

In particular, as $p \to \infty$ *we have* $c^* \to 1$*.*

Here and elsewhere, D_c is the Frechet derivative with respect to c.

Lemma 5.2. *Let* W *be a Banach algebra,* $c \in I := (0,1)$ *a parameter, and let* $\mathcal{L} : W \to W$ *be the Fréchet derivative at* 0 *with respect to* u *of the Whitham map*

$$u \mapsto u - \frac{1}{c} K_s * u - u^2 = F(u,c). \tag{5.6}$$

Suppose that $\mathcal{L}$ *and* $D_c\mathcal{L}$ *exist and are continuous from* $W \to W$*, and that for some* $c^* \in I$ *the following conditions hold:*

(i) $\mathit{dim}\ \mathit{ker}(\mathcal{L}) = 1$*;*

(ii) $W = \mathit{ker}(\mathcal{L}) \oplus \mathit{ran}(\mathcal{L})$*;*

(iii) $(D_c\mathcal{L})\mathit{ker}(\mathcal{L}) \cap \mathit{ran}(\mathcal{L}) = 0$*.*

Then there exist $\varepsilon > 0$ *and a continuous bifurcation curve* $\{(c_\kappa, \varphi_\kappa) : |\kappa| < \varepsilon\}$ *with* $c_\kappa|_{\kappa=0} = c^*$*, such that* φ_0 *is the vanishing solution of* (5.4)*, and* $\{\varphi_\kappa\}_\kappa$ *is a family of nontrivial solutions with corresponding wave speeds* $\{c_\kappa\}_\kappa$*. Moreover, we have*

$$\mathit{dist}(\varphi_\kappa, \mathit{ker}(\mathcal{L})) = o(\kappa) \quad \mathit{in}\ W. \tag{5.7}$$

Proof of Theorem 5.1. We begin the proof by first defining the Fréchet derivatives $\mathcal{L}$ and $D_c\mathcal{L}$:

$$\mathcal{L} = D_u F(0,c) = 1 - \frac{1}{c} K_s *$$
$$D_c\mathcal{L} = D_c D_u F(0,c) = \frac{1}{c^2} K_s * .$$

In search for a traveling solution we consider first the linearized equation

$$\mathcal{L}\psi = \psi - \frac{1}{c} K_s * \psi = 0. \tag{5.8}$$

For $\psi \in L^\infty(\mathbb{R})$, the Fourier transform of $\mathcal{L}\psi$ is given by

$$\begin{aligned}
\widehat{\mathcal{L}\psi} &= \hat{\psi} - \frac{1}{c}\widehat{K_s * \psi} \\
&= \hat{\psi} - \frac{1}{c}\hat{K}_s\hat{\psi} \\
&= \hat{\psi}\left(1 - \frac{1}{c}\left(\frac{\tanh\xi}{\xi}\right)^s\right) \\
&= \hat{\psi}\left(c - \left(\frac{\tanh\xi}{\xi}\right)^s\right) = 0.
\end{aligned}$$

This makes sense in the settings of distributions (see [39]).

If $c > 1$, we have that $\hat{\psi}(\xi) = 0$ for all ξ; If $c = 1$ implies $\hat{\psi}(\xi) = 0$ for all $\xi \neq 0$. The support of $\hat{\psi}$ is then define by $\operatorname{supp}(\hat{\psi}) \subseteq \{0\}$ which also implies that $\hat{\psi}$ is a linear combination of $\delta_0, \delta_0', \delta_0'', \ldots$ and that $\mathcal{F}\delta_0 = 1$, hence $\psi =$ constant; and if $c < 1$, then there exist $\pm\xi_o$ such that $c = (\tanh\xi_o/\xi_o)^s$ and $\hat{\psi}(\xi) = 0$ for all $\xi \neq \pm\xi_o$. The support of $\hat{\psi}$ is then define by $\operatorname{supp}(\hat{\psi}) \subseteq \{\pm\xi_o\}$ which also implies that $\psi = \alpha\cos(\xi_o x)$ for $\alpha \in \mathbb{R}$.

In summary, the nontrivial even solutions of the linear problem are thus given by

$$\begin{cases} \psi(x) = \alpha & \text{if } \ c = 1, \\ \psi(x) = \alpha\cos(\xi_o x) & \text{if } c < 1. \end{cases} \tag{5.9}$$

We note that the constant solutions different from zero are nonphysical, hence we get rid of it in this analysis. The speed $c > 0$ will be our bifurcation parameter as we fix the depth $h_o \sim 1$ and half wavelength $p > 0$ for the purpose of finding even periodic small amplitude solutions by bifurcating from a curve of trivial flows. It is then clear from (5.9) that, in any real linear space of $2p$-periodic and even functions,

$$\dim \ker(\mathcal{L}) = 1$$

if and only if $\xi = n\pi/p,\ n \in \mathbb{Z}^+$ [2]. We have a unique c as in (5.5), if we settle for the lowest mode by taking $n = 1$.

We now introduce the commuting Banach algebra

$$W := \left\{ u(x) = \frac{1}{p} \sum_{n=0}^{\infty}{}' \hat{u}_n \cos\left(\frac{n\pi x}{p}\right) :\ \|u\| := \frac{1}{p} \sum_{n=0}^{\infty}{}' |\hat{u}_n| < \infty \right\}. \tag{5.10}$$

as we look for even, continuous and periodic solutions [see Section 3.4]. One much note that each member of W is uniformly continuous on all of $\mathbb{R}$. We then consider the Whitham equation as a continuous map (5.6) from W to itself. We have at the very end of Section 4.3 that the periodic problem is given by the same multiplier as the problem at hand. As such,

$$\begin{aligned}
\mathcal{L}u &= \left[1 - \frac{1}{c} K_s *\right] u \\
&= u - \frac{1}{c} K_s * u \\
&= \frac{1}{p} \sum_{n=0}^{\infty}{}' \hat{u}_n \cos\left(\frac{n\pi x}{p}\right) - \frac{1}{c}\frac{1}{p} \sum_{n=0}^{\infty}{}' \hat{u}_n \hat{T}_n \cos\left(\frac{n\pi x}{p}\right) \\
&= \frac{1}{p} \sum_{n=0}^{\infty}{}' \hat{u}_n \left(1 - \frac{1}{c}\hat{T}_n\right) \cos\left(\frac{n\pi x}{p}\right).
\end{aligned}$$

Hence

$$\mathcal{L}u \sim \frac{1}{p}\sum_{n=0}^{\infty}{}' \hat{u}_n \left(1 - \frac{1}{c}\hat{T}_n\right) \cos\left(\frac{n\pi x}{p}\right) \tag{5.11}$$

holds almost everywhere on $[-p, p]$. By Theorem 3.7 (Riemann-Lebesgue lemma), $\hat{T}_n \to 0$ as $n \to \infty$, which implies that the right-hand side is in W, hence continuous, and by the definition of the norm in (5.10) we have that

$$\begin{aligned}
\|\mathcal{L}u\| &= \left\| \frac{1}{p}\sum_{n=0}^{\infty}{}' \hat{u}_n \left(1 - \frac{1}{c}\hat{T}_n\right) \cos\left(\frac{n\pi x}{p}\right) \right\| \\
&= \frac{1}{p}\sum_{n=0}^{\infty}{}' \left| \hat{u}_n \left(1 - \frac{1}{c}\hat{T}_n\right) \right| \\
&= \frac{1}{p}\sum_{n=0}^{\infty}{}' |\hat{u}_n| \left| 1 - \frac{1}{c}\hat{T}_n \right| \\
&\le \left(1 + \frac{1}{c}\max_n\{\hat{T}_n\}\right) \frac{1}{p}\sum_{n=0}^{\infty}{}' |\hat{u}_n| \\
&= \left(1 + \frac{1}{c}\max_n\{\hat{T}_n\}\right) \|u\|
\end{aligned}$$

so that $\mathcal{L} : W \to W$ is continuous. Since also the left-hand side is continuous, (5.11) is an equality, which in its turn implies that the full nonlinear Whitham map

$$u \mapsto u - \frac{1}{c}K_s * u - u^2 = \mathcal{L}u - u^2 \tag{5.12}$$

is a continuous endomorphism on W, since this is an algebra. The fact that $\ker(\mathcal{L}) = \mathrm{span}_{\mathbb{R}}(\cos(\pi x/p))$ corresponds to

$$\hat{T}_{n=1} = \hat{T}(1) = \hat{K}_s\left(\frac{\pi}{p}\right) = \left(\frac{p\ \tanh(\pi \backslash p)}{\pi}\right)^s = c \tag{5.13a}$$

and

$$\hat{T}(n) \neq c \quad \text{for} \;\; n \neq 1. \tag{5.13b}$$

To show that codim ran$(\mathcal{L})$ is one dimensional, we consider a given $u \in W$. Take $u^* \in W$ with $\widehat{u^*}(1) = 0$, and $\widehat{u^*}(n) = \hat{u}(n)$ for $n \neq 1$. Then the function

$$v(x) = \frac{1}{p} \sum_{n=0}^{\infty}{}' \frac{\widehat{u^*}(n)}{1 - \frac{1}{c}\hat{T}(n)} \cos\left(\frac{n\pi x}{p}\right) \tag{5.14}$$

is well defined and belongs to W (this can be seen from Section 4.3, but it also follows the Riemann-Lebesgue lemma in combination with (5.13)). From (5.11), we can rewrite $v(x)$ by

$$v(x) = \mathcal{L}^{-1} u^*(x). \tag{5.15}$$

Consequently,

$$\begin{aligned}
u(x) &= \frac{1}{p} \sum_{n=0}^{\infty}{}' \hat{u}(n) \cos\left(\frac{n\pi x}{p}\right) \\
&= \frac{1}{p} \sum_{n=0}^{\infty}{}' \hat{u}(n) \frac{1 - \frac{1}{c}\hat{T}(n)}{1 - \frac{1}{c}\hat{T}(n)} \cos\left(\frac{n\pi x}{p}\right) + \frac{\hat{u}(1)}{p} \cos\left(\frac{\pi x}{p}\right) \\
&= \frac{1}{p} \sum_{n=0}^{\infty}{}' \widehat{u^*}(n) \frac{1 - \frac{1}{c}\hat{T}(n)}{1 - \frac{1}{c}\hat{T}(n)} \cos\left(\frac{n\pi x}{p}\right) + \frac{\hat{u}(1)}{p} \cos\left(\frac{\pi x}{p}\right) \\
&= \mathcal{L} v(x) + \frac{\hat{u}(1)}{p} \cos\left(\frac{\pi x}{p}\right)
\end{aligned}$$

so that $W = \ker(\mathcal{L}) \oplus \operatorname{ran}(\mathcal{L})$. The derivative with respect to the bifurcation parameter c is

$$(D_c \mathcal{L}) u = \frac{1}{c^2} K_s * u. \tag{5.16}$$

Hence, by the same arguments as above, we have that

$$(D_c\mathcal{L})u = \frac{1}{pc^2}\sum_{n=0}^{\infty}{}'\hat{u}(n)\hat{T}(n)\cos\left(\frac{n\pi x}{p}\right) \tag{5.17}$$

is bounded as a map on W, that is $\|(D_c\mathcal{L})u\| \leq (\frac{1}{c^2}\max_n\{\hat{T}(n)\})\|u\|$. In particular,

$$(D_c\mathcal{L})\ker(\mathcal{L}) \cap \operatorname{ran}(\mathcal{L}) = \ker(\mathcal{L}) \cap \operatorname{ran}(\mathcal{L}) = 0. \tag{5.18}$$

$\square$

5.2 Local bifurcation theory

Local bifurcation occurs when a parameter change causes the stability of an equilibrium (or fixed point) to change [41]. Let $C^{\alpha}_{even}(\mathbb{S}), \alpha \in (0,1)$ denote the space of even and α-Hölder continuous real-valued functions on the unit circle $\mathbb{S}$. The Whitham symbol in (4.2) is considered as a generic non-local smoothing operator in the form of a Fourier multiplier, that is

$$m(\xi) = \left(\frac{\tanh\xi}{\xi}\right)^s \lesssim \frac{1}{|\xi|^s} \approx \frac{1}{(1+|\xi|)^s}, \quad |\xi| \gg 1. \tag{5.19}$$

We can then say that m belongs to the symbol class $\mathcal{S}^{-s}(\mathbb{R})$ and therefore its estimate is given by

$$|\partial^k_\xi m(\xi)| \leq \frac{1}{|\xi|^{s+k}} \approx \frac{1}{(1+|\xi|)^{s+k}}. \tag{5.20}$$

Remark 5.3. *We must also note that*

$$\frac{\tanh\xi}{\xi} \lesssim \frac{1}{1+|\xi|} \approx \frac{1}{(1+|\xi|^2)^{\frac{1}{2}}}. \tag{5.21}$$

To illustrate how the analysis used for the Whitham equation can be applied to a larger class of equation, a local bifurcation is performed for the Whitham equation.

We shall make use of ([3], Theorem 3.1), which we state in a form suitable for our purposes. The proof of Theorem 5.4 is almost the same as the one in [3], but we discard the KdV equation and only consider the generalized Whitham equation (1.4).

Theorem 5.4 (Functional-analytic formulation)**.** *For a fix α and $\mu > 0$, the solutions in $C^{\alpha}_{even}(\mathbb{S})$ of the Whitham equation (5.3), coincide with the kernel of an analytic operator $F : C^{\alpha}_{even}(\mathbb{S}) \times \mathbb{R}_{>0} \to C^{\alpha}_{even}(\mathbb{S})$ given by*

$$F(\varphi, \mu) = \mu\varphi - L_w\varphi + N(\varphi) \tag{5.22}$$

where L_w is bounded linear and compact and the non-linear operator $N(\varphi)$ has zero linear part, meaning that $D_\varphi N[0, \mu] = 0$. Thus $D_\varphi N[0, \mu]$ is Fredholm of index 0.

Remark 5.5. *The operators L and N are independent of μ.*

Proof of Theorem 5.4. We first consider the Whitham equation (5.3) and defined L_w as in (4.6). $C^{\alpha}_{even}(\mathbb{S})$ is a sub-algebra of the Wiener algebra of 2π-periodic functions with absolutely converging Fourier series [45]. Hence for $f \in C^{\alpha}_{even}(\mathbb{S})$ and from (3.1a) we have that

$$f(x) = \sum_{k=0}^{\infty} a_k \cos(kx) \qquad \text{and} \qquad \sum_{k=0}^{\infty} |a_k| < \infty.$$

Now, from (4.6) we have

$$\begin{aligned} L_w f(x) &= K_s * f(x) && (5.23)\\ &= \sum_{k=0}^{\infty} a_k \left(\frac{\tanh k}{k}\right)^s \cos(kx). && (5.24) \end{aligned}$$

The Fourier multiplier symbol in the above expression belongs to the symbol class $\mathcal{S}^{-s}(\mathbb{R})$ as shown in (5.19) and $L_w f$ is bounded linear operator on $C^{\alpha}_{even}(\mathbb{S}) \to C^{\alpha+s}_{even}(\mathbb{S})$ for $\alpha + s \notin \mathbb{Z}$. That is from (3.29), (5.19) and (5.21) we have that

$$\begin{aligned}
|L_w f|^2_{H^\alpha(\mathbb{R})} &= \int |(\widehat{L_w f})(\xi)|^2 (1+|\xi|^2)^\alpha d\xi \\
&= \int \left| \left(\frac{\tanh \xi}{\xi} \right)^s \hat{f}(\xi) \right|^2 (1+|\xi|)^\alpha d\xi \\
&\lesssim \int \left| \frac{\hat{f}(\xi)}{(1+|\xi|)^s} \right|^2 (1+|\xi|)^\alpha d\xi \\
&\approx \int \left| \frac{\hat{f}(\xi)}{(1+|\xi|^2)^{\frac{s}{2}}} \right|^2 (1+|\xi|)^\alpha d\xi \\
&= \int |\hat{f}(\xi)|^2 (1+|\xi|^2)^{\alpha - s} d\xi \\
&= |f|^2_{H^{\alpha-s}(\mathbb{R})}.
\end{aligned}$$

Since $L_w : H^{\alpha-s} \to H^\alpha$ is continuous implies that $L_w : H^\alpha \to H^{\alpha+s}$ is also continuous, hence L_w is invertible with bounded linear inverse $L_w^{-1} : C^{\alpha+s}_{even}(\mathbb{S}) \to C^{\alpha}_{even}(\mathbb{S})$. Due to the compactness of the embedding $C^{\beta}_{even}(\mathbb{S}) \hookrightarrow C^{\alpha}_{even}(\mathbb{S})$, $\beta > \alpha$, the operator is compact on $C^{\alpha}_{even}(\mathbb{S})$. We then define the mapping $L_w : C^{\alpha}_{even}(\mathbb{S}) \times \mathbb{R}_{>0} \to C^{\alpha}_{even}(\mathbb{S})$ by

$$F_w(\varphi, \mu) := \mu\varphi - L_w\varphi - \varphi^2, \tag{5.25}$$

where F_w is analytic. We also have $F_w(0, \mu) = 0$, and the linearization $D_\varphi F_w[0, \mu] = \mu - L_w$ is Fredholm of index 0. □

We restate ([3], Corollary 3.2) as we consider the general $s \in (0, 1)$. The proof is almost the same as the one in [3], hence we refer the reader to the exact reference for proof.

Proposition 5.6. *For each integer $k \geq 1$, there exist $\mu_k := (\tanh(k)/k)^s$*

and a local, analytic curve

$$\varepsilon \mapsto (\varphi(\varepsilon), \mu(\varepsilon)) \in C^{\alpha}_{even}(\mathbb{S}) \times (0,1)$$

of nontrivial $2\pi/k$-periodic Whitham solutions with $D_{\varepsilon}\varphi(0) = \cos(kx)$ that bifurcates from the trivial solution curve $\mu \mapsto (0,1)$ at $(\varphi(0), \mu(0)) = (0, \mu_k)$. In a neighborhood of the bifurcation point $(0, \mu_k)$ these are all nontrivial solutions of $F_w(\varphi, \mu) = 0$ in $C^{\alpha}_{even}(\mathbb{S}) \times (0,1)$, and there are no other bifurcation points $\mu > 0, \mu \neq 1$, for solutions in $C^{\alpha}_{even}(\mathbb{S})$. At $\mu = 1$ the trivial solution curve $\mu \mapsto (0, \mu)$ intersects the curve $\mu \mapsto (\mu - 1, \mu)$ of constant solutions $\varphi = \mu - 1$; together these constitute all solutions in $C^{\alpha}_{even}(\mathbb{S})$ in a neighborhood of $(\varphi, \mu) = (0, 1)$.

Proof of Proposition 5.6. See Corollary 3.2 [3] for proof. □

We finally in the next section discuss the global bifurcation for the Whitham equation.

6 Global Bifurcation for the Whitham Equation

In this final section we study the global bifurcation for the Whitham equation (1.1) and some properties along the bifurcation branch (Uniform convergence and the characterization of blow-up).

The sections are being structured as follows. In Section 6.1 we discuss the boundedness and smoothness of the Whitham solution, while Section 6.2 discusses the global bifurcation theory. Section 6.3 next introduces the Lyapunov-Schmidt reduction theory in relation to the global bifurcation of the Whitham equation. Section 6.4 proceeds by establishing the bifurcation formulas and finally in Section 6.5, we discuss the properties along the bifurcation Branch.

The discussions in this chapter follows a similar pattern as presented by Ehrnström and Kalisch in [3]. In this present discussion we consider the generalized Whitham symbol described in (4.2) instead of

$$\hat{K}(\xi) = \sqrt{\frac{\tanh \xi}{\xi}} \tag{6.1}$$

as defined in [3]. The theorems in this section are true for the convolution operator L_w, which maps C^α into $C^{\alpha+s}$ for $\alpha + s \notin \mathbb{Z}$, but for simplicity we will assume $\alpha + s \leq 1$ where $\alpha < s$.

6.1 Boundedness and smoothness of the Whitham solution

Global bifurcations occur when 'larger' invariant sets, such as periodic orbits, collide with equilibria. This causes changes in the topology of the trajectories in the phase space which cannot be confined to a small neighbourhood, as is the case with local bifurcations. In fact, the changes in topology extend out to an arbitrarily large distance (hence 'global') [42].

Let F_w be the Whitham operator from Theorem 5.4, defined by (5.24) and (5.25). With

$$U := \left\{ (\varphi, \mu) \in C^{\alpha}_{even}(\mathbb{S}) \times (0,1) : \varphi < \frac{\mu}{2} \right\}, \tag{6.2}$$

we let

$$S := \{(\varphi, \mu) \in U : F_w(\varphi, \mu) = 0\} \tag{6.3}$$

be our set of solutions (we refer reader to [1] for a detailed proof of the choice of U and S).

The Lemmas 6.1, 6.2 and 6.3 follows strictly with few details added to the ones found in ([3], Section 4). We also assume that these lemmas are true for the generalized Whitham equation.

Lemma 6.1 (L^∞-bound)**.** *Let $\mu > 0$. Any bounded Whitham solution satisfies*

$$\|\varphi\|_\infty \leq \mu + \|L_w\|_{\mathcal{L}(L^\infty(\mathbb{S}))} \tag{6.4}$$

where $\mathcal{L}(X)$ denotes the Banach algebra of bounded linear operators on a Banach space X.

Proof of Lemma 6.1. From $\mu\varphi - L_w\varphi - \varphi^2 = 0$ we have that

$$\begin{aligned} |\varphi|^2 &= |\mu\varphi - L_w\varphi| \\ &\leq \mu|\varphi| + |L_w\varphi| \\ &\leq \mu|\varphi| + \|L_w\|_{\mathcal{L}(L^\infty(\mathbb{S}))}\|\varphi\|_\infty. \end{aligned}$$

We take the supremum and divide by $\|\varphi\|_\infty$, then we have (6.4). □

Lemma 6.2 (Fredholm)**.** *The Fréchet derivative $D_\varphi F_w[\varphi, \mu]$ is a Fredholm operator of index 0 for all $(\varphi, \mu) \in U$.*

Proof of Lemma 6.2. We have

$$D_\varphi F_w[\varphi,\mu] = (\mu - 2\varphi)\mathrm{id} - L_w,$$

and, for any given $(\varphi,\mu) \in U$, that $(\mu - 2\varphi)\mathrm{id} \in \mathcal{L}_{\mathrm{is}}(C^\alpha_{\mathrm{even}}(\mathbb{S}))$. In view of that L_w is compact on $C^\alpha(\mathbb{S})$, the operator $D_\varphi F_w[\varphi,\mu]$ is Fredholm. The linearization $D_\varphi F_w[0,\mu]$ has Fredholm index zero along the trivial solution curve; we have

$$\tau \mapsto (\mu - 2\tau\varphi)\mathrm{id} - L_w \in C([0,1], \mathcal{L}(C^\alpha(\mathbb{S})),$$

and since the index is continuous in the operator-norm topology, it follows that it is zero also at (φ,μ). □

Lemma 6.3. *Whenever $(\varphi,\mu) \in S$ the function φ is smooth, and bounded and closed sets of S are compact in $C^\alpha_{even}(\mathbb{S}) \times (0,1)$.*

Proof of Lemma 6.3. We write the Whitham equation (5.3) in the form

$$\left(\varphi - \frac{\mu}{2}\right)^2 = \frac{\mu^2}{4} - L_w\varphi,$$

where we have φ to be

$$\varphi = \tilde{F}(\varphi,\mu) := \frac{\mu}{2} - \left(\frac{\mu^2}{4} - L_w\varphi\right)^{\frac{1}{2}}. \tag{6.5}$$

The mapping L_w is bounded and linear $C^\alpha(\mathbb{S}) \to C^{\alpha+s}(\mathbb{S})$ (proof of Theorem 5.4), and $x \mapsto \sqrt{x}$ is real analytic for $x > 0$. Consequently, if we let

$$V := \left\{(\varphi,\mu) \in C^\alpha(\mathbb{S}) \times (0,1) : \frac{\mu^2}{4} > L_w\varphi\right\},$$

then $\tilde{F}$ is real analytic $V \to C^{\alpha+s}(\mathbb{S})$. The space $C^{\alpha+s}(\mathbb{S})$ is relatively compact in $C^\alpha(\mathbb{S})$, whence $\tilde{F}$ maps bounded subsets of V into pre-compact sets. We may then prove:

(i) (Smoothness). For any $\varphi \in S$ there exists a constant R_1 such that sup$\varphi \leq R_1 < \mu/2$. Since φ is a fixed point of $\tilde{F}(,\mu)$ we have $(\varphi, \mu) \in V$. A straightforward induction argument reveals that $\varphi \in C^\infty(\mathbb{S})$.

(ii) (Compactness). Let $K \subset S$ be bounded and closed in the $C^\alpha(\mathbb{S}) \times \mathbb{R}$-topology. Then $K \subset V$, and $\{\varphi : (\varphi, \mu) \in K\} = \tilde{F}K$ is pre-compact in $C^\alpha(\mathbb{S})$. Any sequence $\{(\varphi_j, \mu_j)\}_{j\geq 1} \subset K$ thus converges to a pair (φ_0, μ_0) in the $C^\alpha(\mathbb{S}) \times \mathbb{R}$-topology. The fact that K is closed implies that $(\varphi_0, \mu_0) \in K$, whence K is compact.

□

We next introduce the concept of global bifurcation in relation to the Whitham equation.

6.2 Global bifurcation theory

We shall make use of the global one-dimensional branches theorem ([23], Theorem 9.1.1), which we state in the form suitable for our purposes.

Theorem 6.4 (Global bifurcation). *Suppose* $(0, \mu) \in U$ *and* $F_w(0, \mu) = 0$ *for all* $\mu \in \mathbb{R}$. *If also the Lemmas 6.2 and 6.3 hold then, the local bifurcation curves* $\varepsilon \mapsto (\varphi(\varepsilon), \mu(\varepsilon))$ *of solutions to the Whitham equation from Proposition 5.6 extend to global continuous curves of solutions* $\mathbb{R}_{\geq 0} \to S$, *with* S *as in* (6.3). *One of the following alternatives holds:*

(i) $\|(\varphi(\varepsilon)\|_{C^\alpha(\mathbb{S})} \to \infty$ *as* $\varepsilon \to \infty$.

(ii) $(\varphi(\varepsilon), \mu(\varepsilon))$ *approaches the boundary of* S *as* ε *tend to* ∞.

(iii) The function $\varepsilon \mapsto (\varphi(\varepsilon), \mu(\varepsilon))$ *is* T*-periodic, for some* $T \in (0, \infty)$.

We can rely on Lemma 6.2 and 6.3 and show that, for $\varepsilon > 0$ taken to be sufficiently small, $\mu(\varepsilon)$ is not identically equal to a constant; that is if any of the derivatives $\mu^{(k)}(0) \neq 0$. In our case it turns out that $\dot{\mu}(0) = 0$, however one can show that $\ddot{\mu}(0) \neq 0$.

We apply the Lyapunov-Schmidt reduction in other to establish this. In other to discuss the Lyapunov-Schmidt reduction and the bifurcation formulas in the next two sections we first defind some equations that are suitable for our purposes. Let $\mu^* := \mu_1$ be the bifurcation point from Proposition 5.6 and let

$$\varphi^*(x) := \cos(x). \tag{6.6}$$

Let furthermore

$$M := \left\{ \sum_{k \neq 1} a_k \cos(kx) \in C^\alpha(\mathbb{S}) \right\}, \tag{6.7}$$

and

$$N := \ker(D_\varphi F_w[0, \mu^*]) = \operatorname{span}(\varphi^*). \tag{6.8}$$

Then $C^\alpha_{\text{even}}(\mathbb{S}) = M \oplus N$ and we can use the canonical embedding $C^\alpha(\mathbb{S}) \hookrightarrow L^2(\mathbb{S})$ to define a continuous projection

$$\Pi\, \varphi := \langle \varphi, \varphi^* \rangle_{L^2(\mathbb{S})} \varphi^*, \tag{6.9}$$

with

$$\langle u, v \rangle_{L^2(\mathbb{S})} := \frac{1}{\pi} \int_{\mathbb{S}} uv\, dx. \tag{6.10}$$

6.3 Lyapunov-Schmidt reduction

The Lyapunov-Schmidt procedure is a method for reducing the question of existence of solutions to an infinite-dimensional equation, locally in a neighbourhood of a known solution, to an equivalent one involving an equation in finite dimensions, quite commonly (though not always) in just two dimensions [23].

Theorem 6.5 (Lyapunov-Schmidt Reduction [44]). *There exist a neighborhood $\mathcal{O} \times Y \subset U$ around $(0, \mu^*)$ in which the problem*

$$F_w(\varphi, \mu) = 0 \tag{6.11}$$

is equivalent to that

$$\Phi(\varepsilon\varphi^*, \mu) := \Pi F_w(\varepsilon\varphi^* + \psi(\varepsilon\varphi^*, \mu), \mu) = 0 \tag{6.12}$$

for functions $\psi \in C^\infty(\mathcal{O}_N \times Y, M)$, $\Phi \in C^\infty(\mathcal{O}_N \times Y, N)$, and $\mathcal{O}_N \subset N$ an open neighborhood of the zero function in N. One has

$$\begin{aligned} \Phi(0, \mu^*) &= 0, \\ \psi(0, \mu^*) &= 0, \\ D_\varphi\psi(0, \mu^*) &= 0, \end{aligned}$$

and solving the finite-dimensional problem (6.12) provides a solution

$$\varphi = \varepsilon\varphi^* + \psi(\varepsilon\varphi^*, \mu) \tag{6.13}$$

of the infinite-dimensional problem (6.11).

We next discuss the concept of bifurcation formulas in relation to the solution curve (bifurcation curve) of the Whitham equation.

6.4 Bifurcation formulas

The shape of the bifurcation curve follows from the bifurcation formulas. If $D^2_{\varphi\varphi}F_w[0, \mu^*](\varphi^*, \varphi^*) \notin R(D_\varphi F_w[0, \mu^*])$, the number $\dot{\mu}(0)$ is nonzero, and the bifurcation is called *transcritical* (see Figure 1).

However, if $D^2_{\varphi\varphi}F_w[0,\mu^*](\varphi^*,\varphi^*) \in R(D_\varphi F_w[0,\mu^*])$ then $\dot{\mu}(0)$ and the local shape of the curve is determined by $\ddot{\mu}(0)$. Now, if $\dot{\mu}(0) < 0$, the bifurcation is *subcritical*, and if $\dot{\mu}(0) > 0$, it is *supercritical*. In both cases the diagram is referred to as a *pitchfork bifurcation* (see Figure 1).

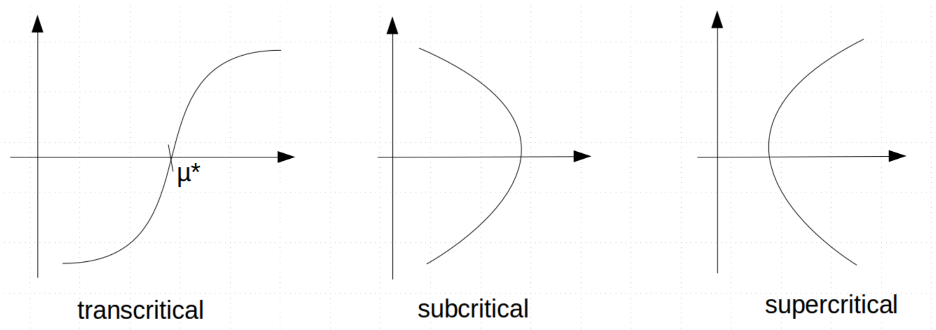

Figure 1: An illustration of the pitchfork bifurcation.

The bifurcation formulas in ([3], Theorem 4.6) is modified with general $s \in (0,1)$. The proof is also an adaption of the one in [3].

Theorem 6.6 (Bifurcation Formulas)**.** *Let*

$$\mu^* = (\tanh(1))^s,$$
$$C_1 = \frac{1}{\mu^* - 1},$$

and

$$C_2 = \frac{1}{2\left(\mu^* - \left(\frac{\tanh(2)}{2}\right)^s\right)}.$$

The main bifurcation curve ($k = 1$) for the Whitham equation found *in Proposition 5.6 satisfies*

$$\varphi(\varepsilon) = \varepsilon \cos(x) + \varepsilon^2 \left(\frac{1}{2} C_1 + C_2 \cos(2x) \right) + \mathcal{O}(\varepsilon^3), \tag{6.14}$$

and

$$\mu(\varepsilon) = \mu^* + \varepsilon^3 (C_1 + C_2) + \mathcal{O}(\varepsilon^3), \tag{6.15}$$

in $C^\alpha_{even}(\mathbb{S}) \times (0,1)$ *as* $\varepsilon \to 0$*. In particular,* $\ddot{\mu}(0) < 0$ *and Proposition 5.6 describes a subcritical pitchfork bifurcation.*

Proof of Theorem 6.6. The analysis for μ is perform first, followed by that of φ. It is known that $\varepsilon \mapsto \mu(\varepsilon)$ is analytic at $\varepsilon = 0$ and that $\mu(0) = \mu^*$, however it remains to show that $\dot{\mu}(0) = 0$ and also to determine $\ddot{\mu}(0)$. We refer to [[44], Section I.6] for the bifurcation formulas used in this proof. We have that

$$D^2_{\varphi\varphi} F_w[0, \mu^*](\varphi^*, \varphi^*) = -2\varphi^{*2},$$
$$D^2_{\varphi\mu} F_w[0, \mu^*]\varphi^* = \varphi^*,$$

and the value of $\dot{\mu}(0)$ may be explicitly calculated as

$$\dot{\mu}(0) = -\frac{1}{2} \frac{\langle D^2_{\varphi\varphi} F_w[0, \mu^*](\varphi^*, \varphi^*), \varphi^* \rangle_{L^2(\mathbb{S})}}{\langle D^2_{\varphi\mu} F_w[0, \mu^*]\varphi^* \rangle_{L^2(\mathbb{S})}} = 0,$$

since

$$\int_{\mathbb{S}} \cos^3(x)\, dx = 0.$$

Moreover, when $\dot{\mu}(0) = 0$ one has that

$$\ddot{\mu}(0) = -\frac{1}{3} \frac{\langle D^3_{\varphi\varphi\varphi} \Phi[0, \mu^*](\varphi^*, \varphi^*, \varphi^*), \varphi^* \rangle_{L^2(\mathbb{S})}}{\langle D^2_{\varphi\mu} F_w[0, \mu^*]\varphi^* \rangle_{L^2(\mathbb{S})}}. \tag{6.16}$$

Since $D^2_{\varphi\mu} F_w[0, \mu^*] = \mathrm{id}$ we find that the denominator is of unit size.

One calculates

$$D_\varphi \Phi[\varphi,\mu]\varphi^* = \Pi D_\varphi F_w[\varphi+\psi(\varphi,\mu),\mu](\varphi^* + D_\varphi\psi(\varphi,\mu)\varphi^*),$$

$$\begin{aligned} &D^2_{\varphi\varphi}\Phi[\varphi,\mu](\varphi^*,\varphi^*)\\ &= \Pi\, D^2_{\varphi\varphi}F_w[\varphi+\psi(\varphi,\mu),\mu](\varphi^*+D_\varphi\psi(\varphi,\mu)\varphi^*,\varphi^*+D_\varphi\psi(\varphi,\mu)\varphi^*)\\ &\quad + \Pi D_\varphi F_w[\varphi+\psi(\varphi,\mu),\mu]D^2_{\varphi\varphi}\psi[\varphi,\mu](\varphi^*,\varphi^*), \end{aligned}$$

and, in view of that F_w is quadratic in φ,

$$\begin{aligned} &D^3_{\varphi\varphi\varphi}\Phi[\varphi,\mu](\varphi^*,\varphi^*,\varphi^*)\\ =&3\,\Pi\, D^2_{\varphi\varphi}F_w[\varphi+\psi(\varphi,\mu),\mu](\varphi^*+D_\varphi\psi(\varphi,\mu)\varphi^*,D^2_{\varphi\varphi}\psi[\varphi,\mu](\varphi^*,\varphi^*))\\ &+ \Pi D_\varphi F_w[\varphi+\psi(\varphi,\mu),\mu]D^3_{\varphi\varphi\varphi}\psi[\varphi,\mu](\varphi^*,\varphi^*,\varphi^*). \end{aligned}$$

Applying the form of $D_\varphi F_w$ together with that

$$\psi(0,\mu^*) = D_\varphi\psi[0,\mu^*]\varphi^* = 0$$

one finds that

$$\begin{aligned} D^3_{\varphi\varphi\varphi}\Phi[0,\mu^*](\varphi^*,\varphi^*,\varphi^*) = &\,\Pi\,(\mu^*\mathrm{id}-L_w)D^3_{\varphi\varphi\varphi}\psi[0,\mu^*](\varphi^*,\varphi^*,\varphi^*)\\ &- 6\,\Pi\,\varphi^* D^2_{\varphi\varphi}\psi[0,\mu^*](\varphi^*,\varphi^*). \end{aligned} \tag{6.17}$$

We have $\mathrm{ran}(\mu^*\mathrm{id}-L_w) = M$, so that $\Pi(\mu^*\mathrm{id}-L_w) = 0$. We thus need to determine $\varphi^* D^2_{\varphi\varphi}\psi[0,\mu^*](\varphi^*,\varphi^*)$. Since $D_\varphi F_w[0,\mu^*] = \mu^*\mathrm{id} - L_w$ is an isomorphism on M, it is possible (see again [[44] Section I.6]) to rewrite $D^2_{\varphi\varphi}\psi[0,\mu^*](\varphi^*,\varphi^*)$ as

$$
\begin{aligned}
D^2_{\varphi\varphi}\psi[0,\mu^*](\varphi^*,\varphi^*) &= -(D_\varphi F_w[0,\mu^*])^{-1}(\mathrm{id}-\Pi)D_{\varphi\varphi}F_w[0,\mu^*](\varphi^*,\varphi^*) \\
&= -(D_\varphi F_w[0,\mu^*])^{-1}(\mathrm{id}-\Pi)(-2\varphi^{*2}) \\
&= (D_\varphi F_w[0,\mu^*])^{-1}(2\cos^2(x)) \\
&= (D_\varphi F_w[0,\mu^*])^{-1}(1+\cos(2x)) \\
&= \frac{1}{\mu^*-1} + \frac{\cos(2x)}{\mu^* - \left(\frac{\tanh(2)}{2}\right)^s}.
\end{aligned}
\tag{6.18}
$$

After multiplication with $\cos(x)$ this equals

$$
\frac{\cos(x)}{\mu^*-1} + \frac{\cos(x)}{2\left(\mu^* - \left(\frac{\tanh(2)}{2}\right)^s\right)} + \frac{\cos(3x)}{2\left(\mu^* - \left(\frac{\tanh(2)}{2}\right)^s\right)}.
$$

In view of (6.16) and (6.17) the coefficient in front of $\cos(x)$ equals $\frac{1}{2}\ddot{\mu}(0)$. All taken into consideration, we obtain (6.15) via a Maclaurin series, and one easily checks that $\ddot{\mu}(0) < 0$.

To prove (6.14), we make use of the formula

$$
\varphi(\varepsilon) = \varepsilon\varphi^* + \psi(\varepsilon\varphi^*, \mu(\varepsilon)) \tag{6.19}
$$

from the Lyapunov-Schmidt reduction (cf. Theorem 6.5). We already know that $\varphi(0) = 0$ and $\dot{\varphi}(0) = \cos(x)$, so it remains to calculate $\ddot{\varphi}(0)$. It follows from (6.19) that

$$
\begin{aligned}
\ddot{\varphi}(\varepsilon) =& D^2_{\varphi\varphi}\psi[0,\mu^*](\varphi^*,\varphi^*) + 2D^2_{\varphi\mu}\psi[0,\mu^*](\varphi^*,\dot{\mu}(0)) \\
&+ D^2_{\mu\mu}\psi[0,\mu^*](\dot{\mu}(0),\dot{\mu}(0)) + D_\mu\psi[0,\mu^*]\dot{\mu}(0).
\end{aligned}
$$

Since $\psi(0,\mu) \equiv 0$ where ψ exists, we have $D_\mu\psi(0,\mu^*) = 0$. Combining this with $\dot{\mu}(0) = 0$ one finds that

$$\ddot{\varphi}(0) = D^2_{\varphi\varphi}\psi[0, \mu^*](\cos(x), \cos(x)),$$

so that the proposition now follows from (6.18). □

Remark 6.7. *We note that*

$$(\mu^* - L_w)^{-1} \sum_{k=0} a_k \cos(kx) = \sum_{k=0} \frac{a_k}{\mu^* - \left(\frac{\tanh(k)}{k}\right)^s} \cos(kx).$$

We next discuss some properties along the bifurcation branch of the Whitham equation.

6.5 Properties along the bifurcation branch

In considering a sequence of Whitham solutions $(\varphi_n, \mu_n) \in S$ where $\mu_n \in (0, 1)$, then Lemma 6.1 implies that φ_n is uniformly bounded in $C(\mathbb{S})$. That is

$$\|\varphi\|^2_\infty \leq \|\mu\varphi\|_\infty + \|L_w\|_{L^\infty(\mathbb{R})}\|\varphi\|_\infty = (|\mu| + 1)\|\varphi\|_\infty,$$

so that $(\varphi_n)_n$ is bounded whenever $(\mu_n)_n$ is bounded. We know that the kernel K_s of the Whitham equation is integrable and continuous almost everywhere, hence we can claim that any uniformly bounded sequence of Whitham solutions (i.e in $L_\infty(\mathbb{R})$) is equicontinuous (Proof of Theorem 4.1, [2]). The Arzela-Ascoli Lemma can be applied to conclude that a subsequence of φ_n converges in $C(\mathbb{S})$, when dealing with periodic solutions.

Theorem 6.8 (Uniform Convergence)**.** *Any sequence of Whitham solutions $(\varphi_n, \mu_n) \in S$ has a subsequence which converges uniformly to a solution φ. If $\varphi < 2\mu$ uniformly on $\mathbb{R}$, then the solution is smooth with* *all its derivatives bounded. If φ attains the value $\frac{\mu}{2}$, then the solution is α-Hölder continuous for $0 < \alpha < s$ with $\alpha + s \leq 1$.*

Proof of Theorem 6.8. We know from the proof of Theorem 5.4 that L_w maps $C^\alpha(\mathbb{S})$ into $C^{\alpha+s}(\mathbb{S})$ for $\alpha + s \notin \mathbb{Z}$ and $\alpha, s \in (0,1)$, we see from (6.5) that $\varphi \in C^\alpha(\mathbb{S})$ wherever $2\varphi \neq \mu$. On the other hand, when $\varphi(x) = \frac{\mu}{2}$ we have

$$\begin{aligned}
|\varphi(x) - \varphi(y)| &= \frac{\mu}{2} - \varphi(y) \\
&= \Big(\frac{\mu^2}{4} - L_w\varphi(y)\Big)^{\frac{1}{2}} \\
&= (L_w\varphi(x) - L_w\varphi(y))^{\frac{1}{2}} \\
&\leq C|x-y|^{\frac{\alpha+s}{2}}.
\end{aligned}$$

This means that if L_w is α-Hölder continuous at x, then φ is α-Hölder continuous at the same point. If $\varphi \in C^\alpha(\mathbb{S})$ then $L_w\varphi \in C^{\alpha+s}$ and φ has Hölder regularity $\frac{1}{2}(\alpha+s)$ at x. In view of that $\frac{1}{2}(\alpha+s) > \alpha$ for $\alpha < s$, this shows that for any such α, the function φ has the corresponding Hölder regularity at $x = 0$. In particular if we choose $\alpha = 0$ and $s = \frac{1}{2}$ then by repeating the argument one can show that $\varphi \in C^{\frac{1}{2}}$ wherever $2\varphi \neq \mu$, and $\varphi \in C^{\frac{1}{4}}$. Now the estimate

$$|L_w\varphi(x) - L_w\varphi(y)| \leq |x-y|^{\alpha+s} \tag{6.20}$$

if $\alpha + s < 1$ and $s \leq \frac{1}{2}$ implies $\alpha < \frac{1}{2}$. Thus $\varphi \in C^\alpha(\mathbb{S})$ for all $\alpha < \frac{1}{2}$.

We next establish an argument for $\alpha + s = 1$ for $s \in (\frac{1}{2}, 1)$. Assuming $s > \frac{1}{2}$ and also considering $\varphi \in C^\alpha$, we have (6.20) when the same argument for $s \leq \frac{1}{2}$ is used. From the proof of Theorem 5.4 the solution φ is even and it maximum is attained at 0 (Proposition 5.6). This implies that $L_w\varphi$ is also even and it maximum is attained at 0. That is if $L_w\varphi \in C^1$ then $(L_w\varphi)'(0) = 0$. Now $L_w\varphi \in C^{1+(\alpha+s-1)}$ implies that $(L_w\varphi)' \in C^{\alpha+s-1}$ and $(L_w\varphi) \in C^1$, in applying the mean value theorem (Remark 6.9) at $x = 0$, we have that

$$|L_w\varphi(0) - L_w\varphi(y)|^{\frac{1}{2}} \leq |y|^{(\alpha+s)\frac{1}{2}}.$$

Hence for $\varphi \in C^\alpha$ implies $|\varphi(0) - \varphi(y)| \leq |y|^\alpha$, for all $\alpha < s$ since we considered $\alpha + s = 1$ for $s \in (\frac{1}{2}, 1)$.

We will now prove that if $\varphi < \frac{\mu}{2}$ uniformly on $\mathbb{R}$, then $\varphi \in C^\infty(\mathbb{R})$ and all of it derivatives are uniformly bounded on $\mathbb{R}$. Assuming that $\varphi < \frac{\mu}{2}$ uniformly on $\mathbb{R}$ and we have the operator L_w which maps $C^\alpha(\mathbb{R})$ into $C^{\alpha+s}(\mathbb{R})$ and $L^\infty(\mathbb{R}) \subset C^0(\mathbb{R})$ into $C^s(\mathbb{R}) \subset L^\infty(\mathbb{R})$. We note from the proof of Lemma 6.3 that

$$\varphi = \tilde{F}(\varphi, \mu) := \frac{\mu}{2} - \left(\frac{\mu^2}{4} - L_w\varphi\right)^{\frac{1}{2}}.$$

That is the Nemytskii operator

$$v \mapsto \frac{\mu}{2} - \left(\frac{\mu^2}{4} - L_w\varphi\right)^{\frac{1}{2}}$$

maps $C^\alpha(\mathbb{R}) \cap L^\infty(\mathbb{R})$ into itself for $v < \frac{\mu^2}{4}$ and $\alpha > 0$ (see Theorem 2.87, [46] and note that $C^\alpha(\mathbb{R}) = B^\alpha_{p,q}$, where we take $p = q = \infty$ in our analysis). Now all the three mappings are continuous and since $\varphi < \frac{\mu}{2}$, it follows that $L_w\varphi < \frac{\mu^2}{4}$, and therefore

$$\left[L_w \mapsto \frac{\mu}{2} - \left(\frac{\mu^2}{4} - L_w\varphi\right)^{\frac{1}{2}}\right] \circ [\varphi \mapsto L_w\varphi] : C^\alpha(\mathbb{R}) \cap L^\infty(\mathbb{R}) \hookrightarrow C^{\alpha+s}(\mathbb{R})$$

for all $\alpha \leq 0$. Hence, the equality

$$\varphi = \frac{\mu^2}{4} - \left(\frac{\mu^2}{4} - L_w\varphi\right)^{\frac{1}{2}}$$

guarantees that $\varphi \in C^\infty(\mathbb{R})$ with uniformly bounded derivatives as long as $\varphi \in L^\infty(\mathbb{R})$. □

Remark 6.9. *The mean value theorem holds, if a function f is continuous on the closed interval $[x, y]$ and differentiable on the open interval (x, y), then there exist a point ξ in (x, y) such that*

$$|f(x) - f(y)| = |x - y||f'(0) - f'(\xi)|, \quad \text{for } f'(0) = 0.$$

If $f \in C^{1+\alpha}$ implies $f' \in C^\alpha$ and $f \in C^1$, and also if $|y| > |x|$ then

$$\begin{aligned} |f(x) - f(y)| &= |x - y||f'(0) - f'(\xi)| \\ &\leq |x - y||\xi|^\alpha \\ &\leq |x - y||y|^\alpha. \end{aligned}$$

Now at the point $x = 0$, we have that

$$|f(0) - f(y)| \leq |y|^{1+\alpha}.$$

The proof of Theorem 6.10 is an adaption of the one in [3], but with general $s \in (0, 1)$.

Theorem 6.10 (Characterization of Blow-up)**.** *Alternative (i) in Theorem 6.4 can happen only if*

$$\liminf_{\varepsilon \to \infty} \inf_{x \in \mathbb{R}} \left(\frac{\mu(\varepsilon)}{2} - \varphi(x; \varepsilon) \right) = 0. \tag{6.21}$$

In particular, alternative (i) implies alternative (ii).

Proof of Theorem 6.10. Assume that

$$\liminf_{\varepsilon \to \infty} \inf_{x \in \mathbb{R}} \left(\frac{\mu(\varepsilon)}{2} - \varphi(x; \varepsilon) \right) \geq \delta,$$

for some $\delta > 0$. Any such solution of the Whitham equation satisfies

$$\mu(\varphi(x) - \varphi(y)) + (\varphi(x))^2 - (\varphi(y))^2 + L_w\varphi(y) - L_w\varphi(x)$$

$$\begin{aligned} |\varphi(x) - \varphi(y)| &= \frac{|L_w\varphi(x) - L_w\varphi(y)|}{\mu - \varphi(x) - \varphi(y)} \\ &\leq \frac{|L_w\varphi(x) - L_w\varphi(y)|}{2\delta}. \end{aligned}$$

Since L_w is continuous $C^\alpha(\mathbb{S}) \to C^{\alpha+s}(\mathbb{S})$ and the family $\{\varphi(\varepsilon)\}_\varepsilon$ is uniformly bounded in $C^\alpha(\mathbb{S})$ (cf. Lemma 6.1), it follows that $\{\varphi(\varepsilon)\}_\varepsilon$ is uniformly bounded in $C^{\alpha+s}(\mathbb{S})$ too. Now if we take $\alpha = 0$, then $L_w : C^0(\mathbb{S}) \to C^s(\mathbb{S})$ is also continuous and hence $\varphi \in C^s$ which also implies that $\varphi \in C^{ks}$ for $k \in \mathbb{Z}$ and $ks < 1$. Therefore we have that

$$\|\varphi(\varepsilon)\|_{C^\alpha(\mathbb{S})} \leq C\delta^{-k}, \quad \alpha \in (0,1),$$

for some constant C depending only on L_w. It must be noted that μ is bounded and that $\|\varphi(\varepsilon)\|_{C^\alpha(\mathbb{S})} \to \infty$ is possible only if (6.21) holds. □

$\mu(\varepsilon)$ is bounded and hence according to Theorem 6.8, there is a subsequence $(\varphi_{n_k})_k$ which converges uniformly to a solution φ_o as $k \to \infty$. If we consider μ_o as the wave speed associated to φ_o, then by the nodal properties of φ_{n_k}, it follows that $\varphi_o(0) = \frac{\mu_o}{2}$ and $\varphi \in C^\alpha(\mathbb{S})$. The solution of the Whitham equation (5.3) is even, strictly increasing on $(-\pi, 0)$, smooth on $\mathbb{S}$, and has Hölder regularity $\frac{1}{2}(\alpha + s)$ at x.

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