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Hankel forms and Nehari's theorem

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ABSTRACT. The purpose of this thesis is to explore the relation between the classical Hardy space of analytic functions and the Hardy space of Dirichlet series. Two chapters are devoted to developing the basic properties of these spaces. In the remaining two chapters we study Nehari's theorem – both in the classical and multiplicative setting – as a concrete example of the usefulness of the interplay between the space of Dirichlet series and the space of analytic functions on the infinite-dimensional polydisc.

SAMMENDRAG. Formålet med denne oppgaven er å utforske sammenhengen mellom Hardy rommet av analytiske funksjonene i polydisken og Hardy rommet av Dirichlet rekker. To kapittler er satt av til å utforske egenskapene til disse rommene. I de to gjenværende kapittlene studeres Neharis teorem – både i den klassiske og multiplikative settingen – som et konkret eksempel på nytterdien av å utnytte samspillet mellom rommet av Dirichlet rekker og rommet av analytiske funksjoner på den uendelig-dimensjonale polydisken.

Preface

This thesis was written from January to June 2017 under supervision of Ole Fredrik Brevig, and marks the end of my time as a student at the Department of Mathematical Science and at the Teacher Education program at NTNU.

I thank Brevig for suggesting the fascinating topic at hand, and have found working with this thesis to be very educative, and it has allowed me to focus on the parts of mathematics that I have come to enjoy the most. I am also indebted to Brevig for taking his time to meet me twice a week, providing detailed feedback feedback on my drafts, and offering enlightening discussions when things looked dark. In addition, a big thanks to my friends here in Trondheim, who have made these years very memorable, and the studies that much easier. Finally, thank you to my family for always supporting me and building me up.

While there is nothing groundbreaking in this thesis, I feel like I have given the topic a coherent treatise from the classical to the cutting-edge results, while simplifying some proofs in the process. All in all I am fairly satisfied with the end product.

I hope you enjoy your reading.

Øistein Søvik
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Contents

Symbols and abbreviations

Symbol	**Description**	**Page**
$\mathbb{N}$	The natural numbers: $1, 2, 3, \ldots$.	5
$\mathbb{Z}$	The integers: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$.	5
$\mathbb{R}$	The real numbers.	5
$\mathbb{D}$	The unit disk.	5
$\mathbb{T}$	Boundary of the unit disk.	5
$o(f(n))$	Little O-notation: If $g \in o(f(x))$ then $g/f \to 0$ as $x \to \infty$.	35
$O(f(n))$	Big O-notation: $g \in O(f(x))$ if and only if there exists a positive real number C and a real number k such that $\lvert g(x)\rvert \leq Cf(x)$ for all $x \geq k$.	34
$\mathrm{d}m$	Normalized Lebesgue measure on $\mathbb{T}$ such that $m(\mathbb{T}) = 1$.	5
$\mathrm{d}\sigma$	Normalized Lebesgue measure on $\mathbb{D}$ such that $\sigma(\mathbb{D}) = 1$.	21
L^p	The space of Lebesgue integrable functions.	5
A^p	The Bergman space of analytic functions.	21
$B(z)$	The Blaschke product.	14
$\mathbb{C}$	The complex plane $\{\sigma + it : \sigma, t \in \mathbb{R}\}$.	8
$\mathbb{C}_\theta$	The complex half plane $\{\sigma + it : \sigma > \theta, t \in \mathbb{R}\}$.	44

Introduction

We begin by a short introduction to the topic at hand, through Hankel forms and Dirichlet series. After this a short overview of each chapter is given.

Hankel forms and Dirichlet series

A *Hankel form* in $\ell^2 \times \ell^2 \to \mathbb{C}$ is one of the form

$$\rho(a,b) := \sum_{m,n\geq 0} a_m b_m \rho_{m+n}\,,$$

and we say that the Hankel form is bounded if there exists a constant such that

$$\left|\sum_{m,n=0}^{\infty} a_m b_n \rho_{n+m}\right| \leq C\left(\sum_{m=0}^{\infty} |a_m|^2\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} |b_n|^2\right)^{\frac{1}{2}}.$$

Further we let $H^2(\mathbb{D})$ denote the Hilbert space of functions analytic in $\mathbb{D}$ with square-summable Taylor coefficients. Every function $\psi = \sum_j \rho_j z^j$ in $H^2(\mathbb{D})$ defines a Hankel form H_ψ by the relation

$$H_\psi(fg) = \langle fg, \varphi\rangle_{H^2}\,, \qquad f,g \in H^2\,.$$

The most important theorem for Hankel forms is the Nehari's theorem [33], which states that every bounded Hankel form is generated by a bounded symbol ψ on the torus $\mathbb{T}$. More precisely H_ψ extends to a bounded form on $H^2(\mathbb{T}) \times H^2(\mathbb{T})$ if and only if $\psi = P_+\varphi$ for a bounded function φ in $L^\infty(\mathbb{T})$. Where P_+ denotes the orthogonal projection $L^2(\mathbb{T}) \to H^2(\mathbb{T})$. An interesting question is whether the *multiplicative* Hankel forms

$$\varrho(a,b) := \sum_{m,n\geq 1} a_m b_m \varrho_{mn}\,,$$

exhibits the same properties as the (additive) Hankel forms. These can be viewed as the classical Hankel forms now on the infinite-dimensional polydisc. We let $\mathscr{H}^2$ denote the Hilbert space of Dirichlet series with square-summable coefficients in the half plane $\mathbb{C}_{1/2} = \{s \in \mathbb{C},\, \operatorname{Re} s > 1/2\}$. Every Dirichlet series $\psi = \sum_{n\geq 1} \rho_n n^{-s}$ in $\mathscr{H}^2$ defines a *multiplicative* Hankel form H_ψ by the relation

$$H_\psi(f,g) = \langle fg, \psi\rangle_{\mathscr{H}^2}\,, \qquad f,g \in \mathscr{H}^2\,,$$

The main purpose of this thesis is to explore to what extent Nehari's theorem holds for these multiplicative Hankel forms. This study is started by exploring the properties of the multiplicative analog of the Hilbert matrix whose analytic symbol

φ is the primitive of $\zeta(s+1/2)-1$ in $\mathscr{H}_0^2$. As shown in [10] this Hankel form is bounded with norm π. More explicitly written

$$\left|\sum_{n,m\geq 2}\frac{a_m b_n}{\sqrt{nm}\log(nm)}\right| \leq \pi\Big(\sum_{n\geq 2}|a_m|^2\Big)^{\frac{1}{2}}\Big(\sum_{n\geq 2}|b_n|^2\Big)^{\frac{1}{2}}$$

This raises the following question.

Question 1. Does the multiplicative Hilbert matrix have a bounded symbol?

To which the answer is still maybe. A key tool in the study of Dirichlet series and Hardy spaces is the Bohr lift. For any $n \in \mathbb{N}$, the fundamental theorem of arithmetic yields

$$n = \prod_{j\geq 1} p_j^{\kappa_j} ,$$

which associates the finite non-negative multi-index $\kappa(n) = (\kappa_1, \kappa_2, \kappa_3, \ldots)$ to n. The Bohr lift of the Dirichlet series is the power series

$$\mathscr{B}f(z) = \sum_{n\geq 1} a_n z^{\kappa(n)} ,$$

where $z = (z_1, z_2, z_3, \ldots)$. Under the Bohr lift, a formal computation shows that

$$\langle \mathscr{B}f\mathscr{B}g, \mathscr{B}\varphi\rangle_{L^2(\mathbb{T}^\infty)} = \langle fg, \varphi\rangle_{\mathscr{H}^2} ,$$

allowing us to compute the multiplicative Hankel form on the infinite polydisk $\mathbb{T}^\infty$. The study of Hankel forms on $\mathbb{T}^\infty$ was initiated by Helson [25, p. 52–54], who raised the following questions:

Question 2. Does every bounded multiplicative Hankel form have bounded symbol φ on the polytorus $\mathbb{T}^\infty$?

Question 3. Does every multiplicative Hankel form in the Hilbert Schmidt class have a bounded symbol?

We answer these questions in full detail chapter 4. By realizing Hankel forms as small operators on the polydisk and using ideas from Ortega-Cerdà and Seip [36], Bayart et al. [6], and Brevig and Perfekt [9] we answer Question 1 in the negative. By extending Carleman's inequality into the polydisk we obtain Helson's inequality, and using this inequality we prove that every multiplicative Hankel form in the Hilbert Schmidt class have a bounded symbol, thus confirming Question 3.

Overview of the thesis

Chapter 1. The first chapter is an introduction to the classical Hardy spaces on the disk. We prove the Riesz factorization theorem, and use the results to show that the space of polynomials are dense in H^p. We then extend the properties of the point estimate and Carleman's inequality to the polydisk; these results are respectively known as the Cole-Gamelin estimate and Helson's inequality.

Chapter 2. The second chapter introduces the Hankel forms and shows their relationship with functions in the Hardy space H^2. We study the bona fide example of a Hankel form, namely the Hilbert matrix. Then we use the weak-factorization of the Hardy space on the disk to prove Nehari's theorem for Hankel forms.

Chapter 3. In the third chapter, we study the Hardy space of Dirichlet series, and prove that this space behaves similarly to the Hardy spaces. In particular we prove Carlson's theorem, and use it to show that $\mathscr{H}^2$ is the closure of Dirichlet polynomials under the Besicovitch norm. Using an idea of Brevig and a bilinear form, a sharp estimate for an embedding inequality is obtained.

The Bohr correspondence is then introduced, and we use it to obtain the point-estimate for $\mathscr{H}^p$. Using the Bohr correspondence and idea of Saksman and Seip we offer an elementary proof that Hardy space $\mathscr{H}^p$ may be defined as the Banach space completion of Dirichlet polynomials in the Besicovitch norm, thus extending Carleson's theorem.

Chapter 4. In the last chapter we introduce the multiplicative Hankel forms, and study the multiplicative analogue to the Hilbert matrix. We prove that this Hankel form is bounded with same norm as the Hilbert matrix. The chapter ends by proving that Nehari's theorem does not hold in full generality, this is done by studying Hankel forms as small operators on the polydisc. We also show that Nehari's theorem holds under the restriction that the symbol is completely multiplicative or has square summable coefficients.

CHAPTER 1

Hardy spaces on the disc

This chapter begins with some preliminaries, before the classical definition of the Hardy space is presented together with some basic results on boundary behavior. This work is done in preparation for proving the Riesz factorization theorem, which has a number of interesting applications. In particular we show that every function $f \in H^1$ can be written as $f = gh$, where $g, h \in H^2$ and $\|f\|_1 = \|g\|_2 \|h\|_2$. At the end we show that the Hardy spaces may be defined as the closure of the polynomials in L^2, and extend some of our results to the polydisc.

1.1. Preliminaries

This section is devoted to introducing a series of necessary prerequisites. In particular we briefly introduce harmonic functions, the Poisson kernel and Möbius transformations. In particular we need some results about the radial limits of harmonic, and thus also analytic functions. For brevity the proofs are omitted, see Pavlović [38], Rudin [44, Chp. XI], or Duren [15, Chp. I] for reference.

Following the notation of standard literature we will denote the unit disk $\mathbb{D}$ as

$$\mathbb{D} = \{\, z \in \mathbb{C} \colon |z| < 1 \,\} .$$

Similarly $\mathbb{T}$, rather than $\partial\mathbb{D}$ will represent the boundary of the disk

$$\mathbb{T} = \{\, z \in \mathbb{C} \colon |z| = 1 \,\} = \{ e^{it} \colon t \in \mathbb{R}/2\pi\mathbb{Z} \} .$$

Functions defined on $\mathbb{T}$ will be identified with functions on $\mathbb{R}/2\pi\mathbb{Z}$, i.e. with functions on the real line, periodic of period 2π. Here $\mathbb{Z}$ denotes the set of integers $\{\dots, -1, 0, 1, \dots\}$, and similarly $\mathbb{N}$ represents the set positive integers $\{1, 2, \dots\}$.

Integrals on $\mathbb{T}$ will be with respect $\mathrm{d}m = \mathrm{d}\theta/2\pi$, the normalized Lebesgue measure such that $m(\mathbb{T}) = 1$. We will use the following notations to describe integrals on $\mathbb{T}$ and over the real numbers $\mathbb{R}$:

$$\int_{\mathbb{T}} f \,\mathrm{d}m := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \,\mathrm{d}\theta \quad \text{and} \quad \int_{\mathbb{R}} f \,\mathrm{d}x := \int_{-\infty}^{\infty} f(x) \,\mathrm{d}x ,$$

and the notation $\int_{\mathbb{T}} f(z) \,\mathrm{d}m(z)$ will be used whenever the need to specify which variable we are integrating over arises. Similarly, the notation

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n=-\infty}^{\infty} f(n) \quad \text{and} \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(n,m) = \sum_{m,n\geq 0} f(n,m) ,$$

will frequently be used, and the latter expression will naturally be extended to as many variables as needed. For $1 \leq p < \infty$, we will let $L^p(\mathbb{T})$ denote the Banach space consisting of all analytic functions satisfying

$$\|f\|_{L^p(\mathbb{T})} := \left(\int_{\mathbb{T}} |f|^p \, \mathrm{d}m\right)^{\frac{1}{p}} < \infty .$$

When $p = \infty$, we define L^∞ as the space of essentially bounded functions

$$\|f\|_{L^\infty(\mathbb{T})} := \sup_{0 \le \theta < 2\pi} \left|f(e^{i\theta})\right| .$$

For brevity we will write $L^p = L^p(\mathbb{T})$ when no confusion is possible.

1.1.1. Harmonic functions and the Poisson kernel

Definition (Harmonic functions). Let u be an analytic function in an open set Ω, such that $\partial u^2/\partial^2 x$ and $\partial u^2/\partial^2 y$ exists at every point of Ω. The *Laplacian* of u is defined as

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} .$$

If $u \in C^2(\Omega)$ is a twice continuously differentiable function in Ω and if $\Delta u = 0$,, at every point of Ω, then u is said to be *harmonic* in Ω.

Theorem 1.1. *A harmonic function u defined on a simply connected domain Ω can be represented in the form $u(z) = h(z) + \overline{g(z)}$, $z \in \Omega$, where h and g are analytic and uniquely determined up to an additive constant; conversely, if $u = h + \overline{g}$, where h and g are analytic, then f is harmonic.*

Using this theorem one can deduce various properties of harmonic functions from the corresponding properties of analytic functions and vice versa.

The Poisson integral and kernel. One of the most used and well known harmonic functions is the Poisson kernel, see [1, p. 166-168], [44, p. 110-112, Chp. XI] or Pavlović [38, Chp. III] for futher details.

Definition. For all $0 \le r < 1$ and $\theta \in [0, 2\pi)$, the *Poisson kernel* is defined as

$$P_r(\theta) := \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} = \frac{1 - r^2}{1 - 2r\cos\theta + r^2} . \tag{1.1}$$

Definition. The Poisson integral of a function $\phi \in L^p(\phi)$ is the harmonic function $P[\phi]$ defined by

$$P[\phi] := P_r * \phi := \int_{\mathbb{T}} P_r(t - \theta)\varphi(e^{i\theta}) \, \mathrm{d}m \qquad (re^{i\theta} \in \mathbb{D}) . \tag{1.2}$$

The notation $f * g$ is referred to as the convolution of f and g. Perhaps the most useful property of the Poisson integral is that it can be used to solve the Dirichlet problem for the disk:

Theorem 1.2. *If φ is a continuous function defined on $\mathbb{T}$, then φ has a unique continuous extension to $\mathbb{D}$ that is harmonic in $\mathbb{D}$; this extension equals $P[\phi]$.*

An immediate consequence is that set of all trigonometric polynomials is dense in each of the spaces $C(\mathbb{T})$, $L^p(\mathbb{T})$, this known Weierstrass approximation theorem.

1.1.2. The harmonic Hardy spaces

DEFINITION. Let $1 \leq p \leq \infty$. We denote by h^p the space of *harmonic* functions in $\mathbb{D}$ such that

$$h^p := \left\{ f \colon \|u\|_{h^p} < \infty \right\}. \tag{1.3}$$

Here $\|u\|_{h^p}$ is the norm of u, and defined as

$$\|u\|_{h^p} := \sup_{0 \leq r < 1} \left(\frac{1}{2\pi} \int_{\mathbb{T}} |f_r|^p \, \mathrm{d}m \right)^{1/p},$$

where the shorthand notation $f_r(e^{i\theta}) = f(re^{i\theta})$ was introduced. In the case $p = \infty$ the integral is to be interpreted as a supremum:

$$\|u\|_{h^\infty} := \sup_{z \in \mathbb{D}} |u(z)| .$$

That Theorem 1.2 extends to $1 < p \leq \infty$ is shown in the following theorem:

THEOREM 1.3. *The function u belongs to h^p $(1 < p \leq \infty)$ if and only if it is equal to the Poisson integral of some function $\phi \in L^p$. And if $f = P[\phi]$, then*

$$\|u\|_{h^p} = \|\phi\|_{L^p} .$$

THEOREM 1.4 (Fatou's Theorem [16], 1906). *Let $u \in h^p$ $(1 < p \leq \infty)$, then u has a radial limit at almost every point $e^{i\theta}$. In particular*

$$\lim_{r \to 1^-} f(re^{i\theta}) = \phi(e^{i\theta}) \qquad \text{for almost every } \theta \in [0, 2\pi) .$$

For a modern proof see Nikolski [34, p. 39]. The case L^1 is treated in Rudin [44, p. 244], and Duren [15, p. 5]. While Theorem 1.3 fails to hold for $p = 1$, the following is true:

COROLLARY 1.5. *Each function $u \in h^1$ has a radial limit almost everywhere.*

COROLLARY 1.6. *If u is the Poisson integral of a function $\varphi \in L^1$, then $u(re^{i\theta}) \to \varphi(\theta)$ almost everywhere.*

As an example let $\phi = \sum_{m=-\infty}^{\infty} a_m e^{im\theta}$ be a function such that $\phi \in L^1$. Then

$$\begin{aligned} P[\phi](re^{i\theta}) &= \int_{\mathbb{T}} \Big(\sum_{n=-\infty}^{\infty} r^{|n|} e^{I(\theta - t)n} \Big) \Big(\sum_{m=-\infty}^{\infty} a_m e^{imt} \Big) \, \mathrm{d}t \\ &= \sum_{m,n=-\infty}^{\infty} a_m e^{int} \int_{\mathbb{T}} e^{I(m-n)\theta} \, \mathrm{d}m = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{int} , \end{aligned}$$

So the operator $P \colon L^1 \to h^1$ is injective, as every function $\phi \in L^1$ may be seen as a boundary function of a function $u \in h^1$. However P is not onto as there exists functions in h^1, whose boundary function does not lie in L^1.

1.1.3. Subharmonic functions

As usual a *domain* is an open connected set in the complex plane.

DEFINITION. A real-valued function $g(z)$ is said to be *subharmonic* if it has the following property: For each domain B with $B \subset D$, and for each function $U(z)$ harmonic in B, continuous in the closure $\overline{B}$, such that $g(z) \leq U(z)$ for ∂B, then

$$g(z) \leq U(z)$$

holds throughout B.

In particular if there is a function $U(z)$ harmonic in B with boundary values $g(z)$, then $g(z) \leq U(z)$ in B.

PROPOSITION 1.7. *If f is analytic in a domain D and $p > 0$, then $|f|^p$ is subharmonic in D.*

1.1.4. The Möbius group

A transformation of the form

$$T(z) = \frac{az+b}{cz+d}, \tag{1.4}$$

where $a, b, c, d \in \mathbb{C}$ and $ab - cd \neq 0$, is called a Möbius transformation[1]. Where $\mathbb{C} = \{\sigma + it \colon \sigma, t \in \mathbb{R}\}$ denotes the complex plane.

PROPOSITION 1.8. *The Möbius transformation is a conformal one-to-one mapping that sends circles and lines to circles or lines.*

Before moving on we would like to present two useful Möbius transformations. The shifted Cayley transformation

$$\mathscr{T}(z) = a + \frac{1+z}{1-z},$$

is a conformal one-to-one mapping of the open unit disk onto the open half plane $\mathbb{C}_a$. In particular if z lies on the boundary $\mathbb{T}$ we have

$$\mathscr{T}(e^{it}) = a + i\tan(t/2).$$

For any $\alpha \in \mathbb{D}$, define

$$\varphi_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}. \tag{1.5}$$

Fix $\alpha \in \mathbb{D}$. Then φ_α is a one-to-one mapping which carries $\mathbb{T}$ onto $\mathbb{T}$, $\mathbb{D}$ onto $\mathbb{D}$ and α to 0. We have

$$\phi_a'(z) = -\frac{1-|a|^2}{(1-\overline{a}z)^2}. \tag{1.6}$$

PROPOSITION 1.9 ([44, Thm. 12.6]). *Suppose T is an Möbius transformation (φ is one-to-one, $\varphi(\mathbb{D}) = \mathbb{D}$, $\alpha \in \mathbb{D}$, and $\varphi(\alpha) = 0$). Then there exists a constant $\theta \in [0, 2\pi)$, such that*

$$T(z) = e^{i\theta}\varphi_\alpha(z) \qquad z \in \mathbb{D}. \tag{1.7}$$

In other words, we obtain $T(z)$ by composing the mapping φ_α with a rotation.

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[1]This mapping is also referred to as a linear fractional transformation.

1.2. The Hardy space

In 1915 Godfrey Harold Hardy, published in the Proceedings of the London Mathematical Society a paper confirming a question posed by Landau [19]. In this paper, not only did Hardy generalize Hadamard's three-circle theorem, but he also put in place the first brick of a new branch of mathematics which bears his name: the theory of Hardy spaces H^p. For three decades afterwards mathematicians such as Hardy, Littlewood, Pólya, Riesz, Privalov, F. and V. Smirnov, and G. Szegö, expanded and developed the theory of the Hardy spaces. While most of this early work is concerned with properties of individual functions of class H^p, the development of functional analysis has stimulated a new interest in the H^p classes. For the interested reader an excellent exposition of the classical Hardy space is the monograph by Duren [15], other sources includes [29, 34] and the short treatise by Rudin [44, Chp. XVII].

In this section we shall look at properties of spaces which are represented by power series in $\mathbb{D}$, i.e functions of the form

$$f(z) = \sum_{n\geq 0} a_n z^n, \qquad z = re^{i\theta}\,. \tag{1.8}$$

When the power series in equation (1.8) converges we call f an *analytic* function. As before we will work in the unit disk $0 \leq r < 1$, and similar to how the $\|f\|_{L^p}$ norm was defined, we introduce

$$\|f\|_{H^p} := \left(\sup_{0\leq r<1} \int_{\mathbb{T}} \left|f_r\right|^p \mathrm{d}m \right)^{1/p} = \sup_{0\leq r<1} \|f_r\|_{L^p}\,. \tag{1.9}$$

and when $p = \infty$, we use let the norm be defined as the essential supremum of f:

$$\|f\|_{H^\infty} := \sup_{z\in\mathbb{D}} |f(z)|\,. \tag{1.10}$$

Definition. Let $1 \leq p \leq \infty$, the Hardy space $H^p(\mathbb{D})$ consists of those analytic functions in the unit disk $\mathbb{D}$ such that, $\|f\|_{H^p} < +\infty$.

As we will only work on the unit disk $\mathbb{D}$ will omit the domain and simply write H^p when no confusion is possible. We will first look at the particular case $p = 2$ and then extend the properties to $1 \leq p \leq \infty$.

1.2.1. The Hardy space H^2.

With the norm of H^2 defined as above, the definition of the inner-product follows naturally:

$$\langle f, g\rangle_{H^2}^2 := \lim_{r\to 1} \int_{\mathbb{T}} f_r \cdot \overline{g_r}\,\mathrm{d}m = \lim_{r\to 1} \langle f_r, g_r\rangle_{L^2}^2\,.$$

In addition, we introduce the notation $f^*(e^{i\theta}) := \lim_{r\to 1^-} f(re^{i\theta})$. The most essential properties of H^2 are encapsulated in the following theorem:

Theorem 1.10. *Let $f = \sum_{n\geq 0} a_n z^n$ and $g = \sum_{n\geq 0} \overline{b_n} z^n$ be analytic for $|z| < 1$, where $z = re^{i\theta}$. Then*

(1) $\langle f, g\rangle_{H^2}^2 = \sum_{n\geq 0} a_n b_n$.

(2) $\|f\|_{H^2} = \sum_{n\geq 0} |a_n|^2 = \|f^*\|_{L^2}$.
(3) $\|f_r\|_{L^2}$ *is a non-decreasing function of* r.
(4) H^2 *is a Hilbert space.*
(5) $|f(z)| \leq \|f\|_{H^2}/\sqrt{1-|z|^2}$.

It will be convenient to first prove the following lemma.

LEMMA 1.11. *Let* $z \in \mathbb{T}$*, then* $\{z^j\}_{j\geq 1}$ *forms an orthonormal set in* L^2*.*

PROOF. We start by introducing the Kronecker delta symbol δ_{jk}, defined as 1 if $j = k$, and 0 otherwise. Proving the lemma is the same as showing

$$\langle z^j, z^k \rangle_{L^2} = \int_{\mathbb{T}} z^j \cdot \overline{z}^k \,\mathrm{d}m = \delta_{jk} \,,$$

for every $j, k \in \mathbb{N}$. Since $z \in \mathbb{T}$ we can write $z = e^{i\theta}$, and our integral becomes

$$\int_{\mathbb{T}} z^j \cdot \overline{z}^k \,\mathrm{d}m = \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta(j-k)} \,\mathrm{d}\theta \,.$$

It is clear that the integral is 1 whenever $j = k$. Assume therefore that $j \neq k$,

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i\theta(j-k)} \,\mathrm{d}\theta = \frac{1}{2\pi i} \frac{e^{2\pi i(j-k)} - 1}{j-k}$$

which completes the proof since $e^{2\pi i(j-k)} = 1$ for every integer pair $j \neq k$. $\square$

PROOF OF THEOREM 1.10. We begin by applying Lemma 1.11 to the inner product of $f = \sum_{n\geq 0} a_n z^n$ and $g = \sum_{n\geq 0} \overline{b_n} z^n$:

$$\langle f_r, g_r \rangle_{L^2}^2 = \int_{\mathbb{T}} f_r \overline{g_r} \,\mathrm{d}m = \sum_{n,m\geq 0} a_n b_m r^{n+m} \int_{\mathbb{T}} z^n \cdot \overline{z}^m \,\mathrm{d}z = \sum_{n\geq 0} a_n b_n r^{2n} \,,$$

This proves that the inner product is increasing as a function of r, thus proving 3. Since $0 < r < 1$, we can apply the monotone convergence theorem on $\langle f_r, g_r \rangle_{L^2}^2$ to obtain item 1. The computation above also shows

$$\langle f_r, f_r \rangle_{L^2} = \|f_r\|_{L^2}^2 = \sum_{m\geq 0} |a_m|^2 r^{2n} \,, \tag{1.11}$$

and proves the first part of 2. Since $L^2(\mathbb{T})$ is a complete space, $f^* \in L^2(\mathbb{T})$ and we can compute the Fourier coefficients to be

$$\widehat{f^*}(n) = \int_0^{2\pi} f^*(e^{i\theta}) e^{-n\theta} \frac{\mathrm{d}\theta}{2\pi} = \lim_{r\to 1} \int_0^{2\pi} f_r(e^{i\theta}) e^{-in\theta} \frac{\mathrm{d}\theta}{2\pi} = \begin{cases} a_n & : \quad n \geq 0 \\ 0 & : \quad n < 0 \end{cases} ,$$

The second equality follows from the monotone convergence theorem since f^* is increasing. Combining this with Parseval's theorem shows

$$\|f^*\|_{L^2} = \sum_{n\geq 0} |a_n|^2 = \|f\|_{H^2} \,,$$

thus completing the proof of 2. To prove 4 we need to show that every sequence $f_r \to f$, as $r \to 1$ is Cauchy in $H^2(\mathbb{T})$.[2] Using Lemma 1.11 from above, and obvious modifications,

$$\|f_r - f_s\|_{L^2}^2 = \int_{\mathbb{T}} \big|\sum_{n\geq 1} a_n \big(r^n - s^n\big) z^n\big|^2 \,\mathrm{d}m = \sum_{n\geq 1} \big(r^n - s^n\big)|a_n|^2 \,.$$

But as $\sum_{n\geq 1} |a_n|^2 < \infty$ we get by the dominated convergence theorem that the last summand goes to zero when $r, s \to 1$. Thus, H^2 is a complete Hilbert space as $f^*(e^{i\theta}) = f(e^{i\theta})$ almost everywhere.

To prove that point-wise evaluation of functions in H^2 is a bounded functional we may apply the Cauchy–Schwarz inequality

$$|f(z)| \leq \sum_{n\geq 0} \big|a_n z^n\big| \leq \left(\sum_{n\geq 0} \big|z^{2n}\big|\right)^{\frac{1}{2}} \left(\sum_{n\geq 0} \big|a_n\big|^2\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{1-|z|^2}} \|f\|_{H^2} \,.$$

where the last equality followed from applying item 2 and the geometric series $\sum_{n\geq 0} r^n = 1/(1-r)$. This proves 5, and completes the proof of Theorem 1.10. $\square$

From the preceding discussion we see that the polynomials are dense in H^2, thus the mapping $f \mapsto f^*$ establishes an isometry between H^2 and the closure of the polynomials in L^2. Hence, H^2 may be defined as as:

(1) the set of analytic functions f in $\mathbb{D}$ such that $\lim_{r\to 1} \int_{\mathbb{T}} |f_r|^2 \,\mathrm{d}m < \infty$.
(2) the closure of the polynomials in $L^2(\mathbb{T})$.

That H^p can be seen as the closure of the polynomials in L^p and that Theorem 1.10 can be extended to $1 \leq p \leq \infty$ is true, but not entirely trivial. A key part in proving this will the the Riesz factorization theorem. A stepping stone in proving this is the following theorem.

The class H^p was introduced as the set of all functions $f(z)$ analytic in $|z| < 1$ for which the means $\|f_r\|_{L^p}$ are bounded. As seen from 1.10, $\|f_r\|_{L^2}$ is increasing as a function of r, and the case $p = \infty$ is trivial as $\|f_r\|_{L^\infty}$ increases with r from the maximum modulus principle. A natural question is therefore whether $\|f_r\|_{L^p}$ is always a non-decreasing function of r. This was proven by Hardy [19] and is considered the starting point of the theory of Hardy spaces.

THEOREM 1.12 (Hardy's convexity theorem). *For $|z| < 1$ let $f(z)$ be analytic, and let $1 \leq p \leq \infty$. Then $\|f_r\|_{L^p}$ is a non-decreasing function of r.*

PROOF. As pointed out in section 1.1.3 $|f|^p$ $(1 \leq p \leq \infty)$ is subharmonic if f is analytic. So it is enough to prove Theorem 1.12 for subharmonic functions. Let $g(z)$ be subharmonic in $|z| < 1$, and define

$$m(r) := \int_{\mathbb{T}} g_r \,\mathrm{d}m \,, \qquad 0 \leq r < 1 \,.$$

Choose $0 \leq r_1 < r_2 < 1$. Since $g(z)$ is subharmonic there exists a function U such that, $U(z)$ is harmonic in $|z| < r_2$, continuous in $|z| \leq r_2$, and equal to $g(z)$ for

[2] That we may associate $H^2(\mathbb{T})$ with a subspace of $L^2(\mathbb{T})$ follows from 2, and Fatou's Theorem 1.4

$|z| = r_2$. Hence, $g(z) \leq U(z)$ for $|z| \leq r_2$, so

$$m(r_1) \leq \int_{\mathbb{T}} U_{r_1} \, \mathrm{d}m = U(0) = \int_{\mathbb{T}} U_{r_2} \, \mathrm{d}m = m(r_2) \,,$$

by the mean-value property A.23. This proves that $m(r)$ is non-decreasing, and so $\|u\|_{L^p}$ is also non-decreasing. ☐

While not needed, it is also true that $\log \|f_r\|_{L^p}$ is a convex function of $\log r$, see Hardy [19] or Duren [15, p. 9].

Remark. Theorem 1.12 implies we may replace the sup in the definition of the H^p with a limit

$$\|f\|_{H^p} = \left(\lim_{r \to 1} \int_{\mathbb{T}} |f_r|^p \, \mathrm{d}m \right)^{\frac{1}{p}} ,$$

as the norm is increasing. The proof for $p = \infty$ follows again from the maximum modulus principle.

1.3. The zeroes of functions in H^p

Let $f \in L^p$, $(1 \leq p \leq \infty)$. We denote the *zero sequence* of f as $\mathscr{Z}(f)$ consisting of the elements

$$\left\{ z \in \mathbb{D} \colon f(z) = 0 \right\} , \tag{1.12}$$

in increasing order of magnitude. It is well known that for a analytic function in the unit disk, either $\mathscr{Z}(f) = \mathbb{D}$ or $\mathscr{Z}(f)$ has no limit points in $\mathbb{D}$. The first case bears little interest as by the maximum modulus principle it implies $f \equiv 0$. Thus, the zeroes of a non-zero analytic function $f \in L^p$ are isolated points in $\mathbb{T}$, and if the number of zeroes is infinite, the limit points have to lie outside $\mathbb{D}$ i.e. on the boundary $\mathbb{T}$. From the theorem of Weierstrass [44, Chapter 15] this is all we can say about the zeroes of analytic functions.

However, if we instead consider functions in H^p we can say much more about the distribution of zeroes in $\mathbb{D}$, namely that the zeroes have to converge with a certain rate toward the limit points on $\mathbb{T}$. The basis of deriving the rate of conversion of the zeroes of H^p is the following formula.

Theorem 1.13 (Jensen's Formula). *Let f be an analytic function in a region which contains the closed disk $\mathbb{D}_r$ of radius r and center 0. Denote $|\alpha_1| \leq |\alpha_2| \leq \ldots \leq |\alpha_n|$ the zeroes of f in the interior of $\mathbb{D}_r$ repeated according to multiplicity, and suppose that $f(0) \neq 0$. Then*

$$\log\bigl|f(0)\bigr| = \sum_{j=1}^{n} \log \frac{|\alpha_j|}{r} + \int_{\mathbb{T}} \log |f_r| \, \mathrm{d}m \ . \tag{1.13}$$

PROOF. If f is an analytic function, then $\log|f|$ is harmonic except at the zeroes of f.[3] If f is zero free in $|z| \le \rho$ and analytic, then

$$\log\big|f(0)\big| = \int_{\mathbb{T}} \log|f_\rho|\,\mathrm{d}m\,, \tag{1.14}$$

which is the mean-value property A.23 applied on the harmonic function $\log\big|f\big|$.

Order the zeros $\{\alpha_j\}_{n=1}^N$ of f in $\overline{\mathbb{D}}_r(0)$ according to their distance from origo i.e. such that $|\alpha_1| \le \cdots \le |\alpha_n| < r$ and $|\alpha_{n+1}| = \cdots = |\alpha_N| = r$. Define the function

$$g(z) = f(z)\prod_{j=1}^{n}\frac{r^2 - \overline{\alpha_j}z}{r(\alpha_j - z)}\prod_{j=n+1}^{N}\frac{\alpha_j}{\alpha_j - z}\,. \tag{1.15}$$

Inserting $z = 0$ into equation (1.15) and taking the logarithm gives

$$\log\big|g(0)\big| = \log\left(\big|f(0)\big|\prod_{j=1}^{n}\frac{r}{|\alpha_j|}\right) = \log\big|f(0)\big| + \sum_{j=1}^{n}\log\frac{r}{|\alpha_j|}\,. \tag{1.16}$$

On the other hand g has no zeroes in $\mathbb{D}$ and hence $\log|g|$ is harmonic, and so

$$\log|g(0)| = \int_{\mathbb{T}} \log\big|g_r\big|\,\mathrm{d}m\,, \tag{1.17}$$

again by the mean value property. Combining equations (1.16) and (1.17) gives

$$\int_{\mathbb{T}} \log\big|g_r\big|\,\mathrm{d}m = \log|f(0)| - \sum_{j=1}^{n}\log\frac{|\alpha_j|}{r}\,. \tag{1.18}$$

Let $|z| = r$, then the factors in (1.15) for $j \in [n+1, N]$ have absolute value 1. Since $\alpha_j = re^{i\theta_j}$ and $z = re^{i\theta}$ it follows that for every $n < j \le N$,

$$\frac{\alpha_j}{\alpha_j - z} = \frac{1}{1 - z/\alpha_j} = \frac{1}{1 - e^{i(\theta-\theta_j)}}\,. \tag{1.19}$$

Using this and that the first product in equation (1.15) equates to one for $z = re^{i\theta}$, we obtain the following expression for $\log\big|g(re^{i\theta})\big|$,

$$\log\big|g(re^{i\theta})\big| = \log\big|f(re^{i\theta})\big| - \sum_{j=n+1}^{N}\log\big|1 - e^{i(\theta-\theta_j)}\big|. \tag{1.20}$$

Integrating this expression over $\mathbb{T}$ gives

$$\int_{\mathbb{T}} \log\big|f_r\big|\,\mathrm{d}m = \int_{\mathbb{T}} \log\big|g_r\big|\,\mathrm{d}m - \sum_{j=n+1}^{N}\frac{1}{2\pi}\int_0^{2\pi}\log\big|1 - e^{i(\theta-\theta_n)}\big|\,\mathrm{d}\theta\,.$$

The last integral is evidently independent of θ_j and thus zero by Lemma A.11. Combining this with equation (1.18) completes the proof. ☐

[3]Recall that if D is a simply connected domain in $\mathbb{C}$ and h a non-vanishing holomorphic function on D then $h = e^g$ for some holomorphic function g. So, if D was simply connected we would know that $f = e^g$ for some holomorphic g, and then $\log|f| = \log|e^g| = \log(\exp(\mathrm{Re}(g)) = \mathrm{Re}(g)$ and since g is harmonic (g was holomorphic) we are done.

The next lemma proves a necessary condition on the zeros of a function f in order that $f \in H^p$ for some $1 \leq p \leq \infty$. We will later use it to prove that any function in H^p may be written as the product of a Blaschke product and a non-vanishing element of H^p.

LEMMA 1.14 (G. Szegö). *Let $f \in H^p$ $(1 \leq p \leq \infty)$ be an analytic function in $\mathbb{D}$ such that $f \not\equiv 0$ and $f(0) \not\equiv 0$. Further, let $\{\alpha_n\}_{n\geq 1}$ be the zeros of f, listed according to their multiplicities. Then these zeros satisfy the* Blaschke condition

$$\sum_{n\geq 1}\big(1-|\alpha_n|\big) < \infty\,. \tag{1.21}$$

PROOF. If f has a finite number of zeroes, then the sum is finite and the result follows. Therefore, we assume that f has an infinite number of zeroes, since $f \not\equiv 0$ they converge toward some points in the unit circle. Which is to say $\lim_{r\to 1}|z_n| = 1$.

Denote the number of zeroes of f in the closed disk $\overline{\mathbb{D}}_r$ by $N(r)$, where $r < 1$. Fix $K \in \mathbb{N}$, and choose $r < 1$ such that $N(r) > K$. By Jensen's fomula 1.13, for each $r \in (0,1)$, we have

$$\big|f(0)\big|\prod_{n=1}^{K}\frac{r}{|\alpha_n|} \leq \big|f(0)\big|\prod_{n=1}^{N(r)}\frac{r}{|\alpha_n|} = \exp\left(\int_{\mathbb{T}}\log\big|f_r\big|\,\mathrm{d}m\right) < \infty\,,$$

where the right hand side is bounded as $f \in H^p \subset H^1$. Hence, there exists some constant $C < \infty$ such that $\prod_{n=1}^{K}|\alpha_n| \geq r^K|f(0)|/C$. As the sum now is finite we can let $r \to 1$. Since the inequality holds for all K, we can let $K \to \infty$.

$$\prod_{n=1}^{\infty}\big|\alpha_n\big| \geq \frac{\big|f(0)\big|}{C} > 0\,.$$

Using $1 - x \leq e^{-x}$ now gives

$$\begin{aligned}0 < \prod_{n=1}^{\infty}\big|\alpha_n\big| &= \prod_{n=1}^{\infty}\big|1-\big(1-|\alpha_n|\big)\big| \\ &\leq \prod_{n=1}^{\infty}\exp\big[-\big(1-|\alpha_n|\big)\big] \leq \exp\left(-\sum_{n=1}^{\infty}\big(1-|\alpha_n|\big)\right).\end{aligned}$$

Since $e^{-x} \to 0$ as $x \to \infty$, the inequality above proves that $\sum_{n=1}^{\infty}\big(1-|\alpha_n|\big) < \infty$ as $\exp(-\sum_{n=1}^{\infty}\big(1-|\alpha_n|\big) > 0$. □

So the *Blashke condition* (1.21) is a necessary condition for the zeroes of an analytic function to belong to a Hardy space H^p. Surprisingly enough (1.21) is also sufficient condition for the existence of a function $f \in H^p$, which has zeros only at $\{\alpha_n\}_{n=1}^{\infty}$.

DEFINITION. A *Blaschke product* $B(z)$ is a product of Möbius transformations of the form

$$B(z) := z^k\prod_{n\geq 1}\frac{|\alpha_n|}{\alpha_n}\frac{\alpha_n - z}{1-\overline{\alpha_n}z}\,,$$

We define $B(z) = z^k$, when $\alpha = \{\alpha_j\}_{j\geq 0}$ is empty.

PROPOSITION 1.15 (Blaschke product). *Let $\{\alpha_n\}_{n=1}^{\infty}$ be sequence of complex numbers such that*

$$0 < |\alpha_1| \leq |\alpha_2| \leq \cdots < 1, \qquad \alpha_n \in \mathbb{D},$$

for all $n \in \mathbb{N}$, satisfying the Blaschke condition (1.21). Then the Blaschke product $B(z)$ has only zeroes only at the points α_n and a zero of order k at 0. In addition, $B(z)$ converges uniformly in each disk $|z| \leq R < 1$, we have $|B(e^{i\theta})| = 1$ almost everywhere and $|B(z)| < 1$ for all $z \in \mathbb{D}$.

PROOF. The function $B(z)$ is the product of the factors

$$b_n(z) := \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}. \tag{1.22}$$

Each factor b_n has a zero at $z = \alpha_n$ inside $\mathbb{D}$, and a pole at $z = \overline{\alpha}^{-1}$ outside the closed unit disk $\overline{\mathbb{D}}$. Thus, each factor b_n is analytic in $\mathbb{D}$ with precisely one zero at α_n. Assume that $|z| \leq R$ then,

$$\begin{aligned} \big|1 - b_n(z)\big| = \left|1 - \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}\right| &= \left| \frac{\big(a_n + |a_n| z\big)\big(1 - |a_n|\big)}{a_n\big(1 - \overline{a_n} z\big)} \right| \\ &\leq \left| \frac{1 + z|a_n|/a_n}{1 - z|a_n|/a_n} \right| \big|1 - |a_n|\big| \leq \frac{1+1}{1-R}\big(1 - |a_n|\big). \end{aligned}$$

Since $\sum_{n\geq 1}(1 - |a_n|) < \infty$ it follows that $B(z) = \prod_{n \geq 1} b_n(z)$ converges uniformly in each disk $|z| \leq R < 1$. That $|B(z)| < 1$ is clear since

$$|B(z)| = \left| \prod_{n\geq 1} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z} \right| \leq \prod_{n \geq 1} \left| \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z} \right| < 1,$$

as each partial product is less than 1 for $|z| < 1$. Hence, $|B(e^{i\theta})| \leq 1$ by the maximum modulus principle, and the radial limit $B(e^{i\theta})$ exists almost everywhere (1.4).

Let $f \in H^\infty \subseteq H^1$, from Theorem 1.12 $\|f_r\|_{L^1}$ is increasing and we have the bound

$$\|f_r\|_{L^1} \leq \|f\|_{L^1}. \tag{1.23}$$

We can apply the inequality above on the function $f = B/B_n$ where $B_n = \prod_{k=1}^n b_k$. Since $|B_n(e^{i\theta})| \equiv 1$ we get

$$\|(B)_r/(B_n)_r\|_{L^1} \leq \|B\|_{L^1}, \tag{1.24}$$

where the slightly convoluted notation $(B_n)_r = B_n(re^{i\theta})$ was introduced. As $B_n(z) \to B(z)$ uniformly on $|z| = r$ we have the inequality

$$1 \leq \|B\|_{L^1}. \tag{1.25}$$

Since $B(e^{i\theta}) \leq 1$ almost everywhere, this proves that $|B(e^{i\theta})| = 1$ almost everywhere. $\square$

1.3.1. The Riesz factorization theorem

Lemma 1.14 shows that the zeroes of any nonzero function in in an Hardy space forms a Blaschke product. Thus, we can try to divide out the zeros of f by dividing f by the corresponding Blaschke product B. Of course, the resulting quotient $g = f/B$ is again an analytic function in $\mathbb{D}$, and since B has absolute value 1 almost everywhere on the unit circle, we may expect that g have the same H^p-norm as the original f. That this reasoning is indeed correct was proven by F. Riesz in (1923) [41].

THEOREM 1.16 (F. Riesz). *Let $f \in H^p$, $(1 \le p \le \infty)$, $f \not\equiv 0$, and let B denote the Blaschke product formed with the zeroes of f in $\mathbb{D}$. If*

$$g := f/B\,,$$

then $g \in H^p$, g is free of zeroes in $\mathbb{D}$, and

$$\|g\|_{H^p} = \|f\|_{H^p}\,.$$

PROOF. From Lemma 1.14 it is clear that f and B has excactly the same zeroes. Clearly g is then analytic and free of zeroes on $\mathbb{D}$. Let $\{\alpha_n\}_{n\ge1}$ be the sequence if zeroes of f in $\mathbb{D}$, and let $b_n(z)$ denote the factor of the Blaschke product corresponding to the zero α_n as defined in equation (1.22). Further, let

$$B_N(z) = \prod_{n=1}^{N} b_n(z)\ , \qquad z \in \mathbb{D}\,,$$

be the partial Blaschke product formed with the first N zeroes of f, and define $g_N := f/B_N$. Proposition 1.15 shows that for every fixed N, we have $(B_N)_r = B_N(re^{i\theta}) \to 1$ uniformly as $r \to 1$. It follows that $(g_N)_r \to f$ and consequently that

$$\|g_N\|_{H^p} = \|f\|_{H^p}\,.$$

Since $|b_n(z)| < 1$ for all n and $z \in \mathbb{D}$, we have that

$$0 \le |g_1(z)| \le |g_2(z)| \le \cdots \le \infty \quad \text{and} \quad |g_n(z)| \to |g(z)|\,,$$

for every $z \in \mathbb{D}$. Fixing $0 < r < 1$ and applying Lebesgue monotone convergence theorem, one gets

$$\lim_{N\to\infty} \|(g_N)_r\|^p_{H^p} = \lim_{N\to\infty} \int_{\mathbb{T}} |(g_N)_r|^p \,\mathrm{d}m = \int_{\mathbb{T}} |g_r|^p \,\mathrm{d}m = \|g_r\|^p_{L^p}\,.$$

Since g_N is analytic in $\mathbb{D}$ and because $\|f_r\|_{L^p} \le \|f\|_{L^p}$ (see 1.12), the left-hand side is bounded from above by $\|f\|^p_{H^p}$ for every $0 < r < 1$. Letting $r \to 1$ we obtain $\|g\|_{H^p} \le \|f\|_{H^p}$. Moreover, since $B(z) \le 1$ for all $z \in \mathbb{D}$, we also have that $|g(z)| \ge |f(z)|$ for all $z \in \mathbb{D}$, this proves that we have equality, i.e that $\|g\|_{H^p} = \|f\|_{H^p}$. □

COROLLARY 1.17. *Suppose $1 \le p < \infty$, $f \in H^p$ and again let B be the Blaschke product formed by the zeroes of f. Then there exists a zero-free function $g \in H^2$ such that*

$$f = B \cdot g^{p/2}\,, \tag{1.26}$$

and

$$\|f\|_{H^p}^p = \|g\|_{H^2}^2 \,. \tag{1.27}$$

In particular, every $f \in H^1$ is a product

$$f = gh \,, \tag{1.28}$$

in which both factors are in H^2 and

$$\|f\|_{H_1} = \|g\|_{H_2} \cdot \|h\|_{H_2} \,. \tag{1.29}$$

PROOF. By Theorem 1.16 $f/B \in H^p$ and $\|f/B\|_{H^p} = \|f\|_{H^p}$ a.e. Since f/B has no zeroes in $\mathbb{D}$ there exists an analytic $\psi \in \mathbb{D}$ so that $e^{\psi} = f/B$. Let $g = e^{p\psi/2}$, then

$$|g|^2 = |f/B|^p, \tag{1.30}$$

and so it follows that $g \in H^2$ thus, proving equation (1.26). Equation (1.27) follows directly from integrating equation (1.30) over $\mathbb{T}$ and taking the supremum over r.

To prove equation (1.28) we can write (1.26) in the form $f = Bg = f_1 f_2$ with $f_1 = Bg^{1/2}$ and $f_2 = g^{1/2}$. Since $f_1, f_2 \in H^2$, we have

$$\|f_1\|_{H^2} = \|f_2\|_{H_2} = \|g\|_{H_2} = \|f\|_{H_1}^{1/2} \,.$$

Using the last equation twice proves (1.29), and we are done. □

1.3.2. Applications of the Riesz factorization theorem

PROPOSITION 1.18 (Mean convergence property). *If $f \in H^p$, ($1 \le p < \infty$) then*

$$\lim_{r\to 1} \|f_r\|_{L^p} = \|f\|_{L^p} \,, \tag{1.31}$$

and

$$\lim_{r\to 1} \left\|f_r - f\right\|_{L^p} = 0 \,. \tag{1.32}$$

PROOF. We have $\|g\|_{L^2}^2 = \|f\|_{L^p}^p$ from Corollary 1.17 so it is enough to prove equation (1.31) for H^2. However, this was shown in Theorem 1.10, and that we may replace the supremum by a limit follows from Theorem 1.12 as the norm is increasing as a function of r.

If $f(z) = \sum_{n\ge 1} a_n z^n$, then $|a_n|^2$ converges when $f \in H^2$. From Fatou's lemma A.26 ,

$$\|f_r - f\|_{L^2} \le \liminf_{\rho\to 1} \|f_r - f_\rho\| = \sum_{n=1} |a_n|^2 \big(1 - r^n\big)^2 \,. \tag{1.33}$$

Letting $r \to 1$ shows equation (1.32) for $p = 2$, since letting $r \to 1$ is no problem as the radial limit $\lim_{r\to 1} f(re^{i\theta})$ exists for almost every θ, 1.4.

We have proved equation (1.31) for all $1 \le p < \infty$, and equation (1.32) for $p = 2$. To deduce (1.32) from (1.31) we need the following lemma from measure theory. □

LEMMA 1.19 (Duren [15, p. 21]). *Let $\Omega \subset \mathbb{R}$ be a measurable subset, and let $\varphi_n \in L^p(\Omega)$, $1 \leq p < \infty$, and $n \in \mathbb{N}$. As $n \to \infty$ suppose that $\varphi_n(x) \to \varphi(x)$ for almost every $x \in \Omega$ and*

$$\int_\Omega |\varphi_n(x)|^p \,\mathrm{d}x \to \int_\Omega |\varphi(x)|^p < \infty \,.$$

Then,

$$\int_\Omega \big|\varphi_n(x) - \varphi(x)\big|^p \mathrm{d}x \to 0.$$

See Duren [15, p. 21] for proof. proposition 1.18 now follows from this lemma as $f(re^{i\theta}) \to f(e^{i\theta})$ almost everywhere from Fatou's Theorem 1.4 and we have already shown that $\lim_{r\to 1} \|f_r\|_{L^p} = \|f\|_{L^p}$.

LEMMA 1.20. *Let $1 \leq p \leq \infty$ and $0 \leq r < 1$. Then,*

$$\big|f(0)\big|^p \leq \|f_r\|^p_{H^p}. \tag{1.34}$$

PROOF. From the mean value theorem A.23 we have

$$f(0) = \int_{\mathbb{T}} f_r \,\mathrm{d}m \;. \tag{1.35}$$

Applying the triangle-inequality yields

$$|f(0)| \leq \int_{\mathbb{T}} |f_r| \,\mathrm{d}m \;.$$

Using Hölders inequality A.7 with $1/p + 1/q = 1$ the equation above can be written.

$$|f(0)| \leq \left(\int_{\mathbb{T}} |f_r|^p \,\mathrm{d}m\right)^{1/p} \left(\int_{\mathbb{T}} |1|^q \,\mathrm{d}\theta\right)^{1/q}. \tag{1.36}$$

Raising both sides of the inequality to the power p completes the proof. $\square$

With the help of the mean convergence property we are now ready to generalize some properties from Theorem 1.10 to H^p $(1 \leq p \leq \infty)$.

LEMMA 1.21 (Point-estimate). *Suppose $1 \leq p < \infty$ and $f \in H^p$, then*

$$|f(z)| \leq \frac{\|f\|_{H^p}}{(1 - |z|^2)^{1/p}} \qquad \text{for all } z \in \mathbb{D}\,.$$

PROOF. Following the lines of [49] we consider

$$F_r(w) = f\big(r\frac{z - w}{1 - \overline{z}w}\big)\frac{\big(1 - |z|^2\big)^{1/p}}{\big(1 - \overline{z}w\big)^{2/p}} = f\big(r\varphi_z(w)\big)\big[-\varphi_z'(w)\big]^{1/p}, \tag{1.37}$$

for $0 < r < 1$. The idea is now to integrate $|F_r(e^{i\theta})|^p = \big|f\big(r\varphi_z(e^{i\theta})\big)\big|^p \big|\varphi_z'(e^{i\theta})\big|$ with the substitution $\varphi_z(e^{i\theta}) \mapsto e^{i\theta}$ such that $\mathrm{d}\theta \mapsto \varphi_z'(e^{i\theta})\,\mathrm{d}\theta$. So

$$\int_{\mathbb{T}} \big|F_r(e^{i\theta})\big|^p \,\mathrm{d}\theta = \int_{\mathbb{T}} \big|f(re^{i\theta})\big|^p \,\mathrm{d}\theta = \|f_r\|^p_{L^p} \leq \|f\|^p_{H^p}. \tag{1.38}$$

From Lemma 1.20 we get the following inequality for the integral

$$\int_{\mathbb{T}} \left|F_r(e^{i\theta})\right|^p \mathrm{d}\theta \geq \left|F_r(0)\right|^p = \left|f(r\varphi_z(0))\right|^p \varphi_z'(0) = \left|f(z)\right|^p \big(1 - |z|^2\big). \qquad (1.39)$$

Comparing equation (1.38) and (1.39) completes the proof. $\square$

1.4. Boundary functions

From Theorem 1.4 we have seen that every function $f \in H^p$ has a nontangential limit $f(e^{i\theta})$ at almost every boundary point. Let $H^p(\mathbb{T})$ denote the set of boundary functions $f(e^{i\theta})$. We know from Cauchy's integral formula that a holomorphic function is uniquely determined by its boundary value Proposition A.23, so a Hardy space can be identified with a subspace of the $L^p(\mathbb{T})$.

For the study of Dirichlet series it will be of interest to characterize H^p in terms of these boundary functions. Let $1 \leq p \leq \infty$ from Weierstrass approximation theorem the set of trigonometric polynomials are dense in $L^p(\mathbb{T})$. Thus, a function f in $L^p(\mathbb{T})$ may be written as

$$f(e^{i\theta}) = \sum_{k\in\mathbb{Z}} c_k e^{ik\theta}\,, \qquad (1.40)$$

where c_k are the Fourier coefficients. Similarly, H^p contains functions on the form

$$P(e^{i\theta}) = \sum_{k=0}^{n} a_k e^{ik\theta}\,, \qquad (1.41)$$

where a_k are complex constants and these functions will be called polynomials in $\mathbb{T}$. The main result is that the polynomials (1.41) are dense in $H^p(\mathbb{D})$.

THEOREM 1.22. *For every $1 \leq p < \infty$, $H^p(\mathbb{T})$ is the closure of the set of polynomials in $e^{i\theta}$.*

PROOF. We begin by considering the analytic function $f(z) = \sum_{n\geq 1} a_n z^n$, $f \in H^p(\mathbb{D})$ and let $S_N f(z) = \sum_{n=1}^{N} a_n z^n$ denote the n'th partial sum of the Taylor series of f at the origin. Proving Theorem 1.22 is the same as proving that for every $\varepsilon > 0$, there exists a $k \in \mathbb{N}$, such that

$$\|S_N f - f\|_{H^p} \leq \varepsilon\,,$$

for every $N \geq k$. The idea is to go a small distance λ into the disk, and prove that the result holds for every $0 < \lambda < 1$. In other words

$$\|S_{N_\lambda} f - f\|_{H^p} \leq \varepsilon\,.$$

As before we write, $f_\lambda(z) = f(\lambda z)$. Since $f \in H^p$ has a bounded norm on the boundary it follows from Proposition 1.18 that we can choose an ε such that

$$\|f_\lambda - f\|_{H^p} < \frac{\varepsilon}{2}\,.$$

Similarly, since $S_N f \to f(z)$ uniformly on the circle $|z| = \lambda$, we have by the Taylor approximation that for every $f(\lambda z)$, $0 < \lambda < 1$ then

$$\|S_{N_\lambda} f - f_\lambda\|_p \leq \frac{\varepsilon}{2}\,,$$

for sufficiently large enough N_λ. Applying Minkowski's inequality A.9 we obtain

$$\|S_{N_\lambda} f - f\|_{H^p} \le \|S_{N_\lambda} f - f_\lambda\|_{H^p} + \|f_\lambda - f\|_{H^p} \le \varepsilon\,. \tag{1.42}$$

Thus, proving that the boundary function $f(e^{i\theta})$ belongs to the L^p closure of the polynomials in $e^{i\theta}$.

To complete the proof we will show that for if $P_N(e^{i\theta}) = \sum_{k=0}^{n} a_k e^{ik\theta}$ is some polynomial such that P_N converges to $f \in L^p$ then $f \in H^p$. The strategy will be to show that all the negative Fourier-coefficients to f is zero.

$$a_{-k} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{ik\theta}\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \big(f(e^{i\theta}) - P_N(e^{i\theta})\big)e^{ik\theta}\,\mathrm{d}\theta\,,$$

the last equality follows since the negative Fourier-coefficients of P_N are zero. By taking the absolute value and using the triangle inequality we obtain

$$|a_{-k}| \le \frac{1}{2\pi}\int_0^{2\pi} \big|f(e^{i\theta}) - P_N(e^{i\theta})\big|\,\mathrm{d}\theta = \|f - P_n\|_{L^1} \le \|f - P_n\|_{L^p}\,.$$

Since $P_N \to f$ the norm above can be made arbitrary small, thus proving that $a_{-k} = 0$ for all $k \in \mathbb{N}$. Hence, $f \in H^p$ and we are done. □

To show that Theorem 1.22 does not extend to $p = \infty$ consider the function

$$g(z) = \exp\left(\frac{z-1}{z+1}\right), \qquad z \in \mathbb{D} \cup \mathbb{T}\,.$$

It is clear that $g(z)$ is analytic for $z \in \mathbb{D}$ because it is the composition of analytic functions, and the only singular point $z = -1$ lies on the boundary $\mathbb{T}$. The function in the exponent $(z-1)/(z+1)$ is a Möbius transformation and maps $\mathbb{D}$ onto the left half plane $\mathbb{C}_- = \{z \in \mathbb{C} \colon \operatorname{Re}(z) < 0\}$. Since the exponential function is bounded on $\mathbb{C}_-$: $|e^{\sigma+it}| = e^{\sigma} \le 1$, this shows that $g(z)$ is also bounded.

In fact, on the boundary we have $|g(e^{i\theta})| = 1$, for almost every θ. Hence, g is Lebesgue integrable and $g \in L^\infty$. Does this imply that $g \in H^\infty$? Hardly. Look at $g(r)$ as $r \to -1$. Thus, there exists a function in L^∞ which does not extend analytically into $H^\infty(\mathbb{D})$.

COROLLARY 1.23. *If $1 \le p \le \infty$, H^p is a Banach space.*

Consequently, H^p could be defined as the subspace of those L^p functions which all negative Fourier coefficients are equal to zero:

DEFINITION. The Hardy space H^p for $1 \le p \le \infty$ is the subspace of $L^p(\mathbb{T})$ consisting of functions f such that $\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\,\mathrm{d}t = 0$ for all $n < 0$.

$$H^p = \big\{f \in L^p : \hat{f}(n) = 0 \;\forall\, n < 0\big\}\,. \tag{1.43}$$

Since the polynomials are dense in H^p for $1 \le p < \infty$, we will henceforth make no distinction between the spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$. Thus, formally defining H^p as the closure of all polynomials with respect to the norm on the boundary

$$\|f\|_{H^p} = \left(\int_{\mathbb{T}} |f|^p\,\mathrm{d}m\right)^{\frac{1}{p}}\,.$$

We take the expression above as a radial limit when necessary, that is when $p = \infty$.

1.5. Carleman's inequality

The purpose of this section is to prove Carlemans's inequality, and we offer some historical context for the inequality. The last part is devoted to viewing some generalizations of this inequality.

The circle is uniquely characterized by the property that among all simple closed plane curves of given length L, the circle of circumference L encloses maximum area. This property is most succinctly expressed in the *isoperimetric inequality*

$$A \leq L^2/4\pi \,. \tag{1.44}$$

Here A is the area enclosed by a curve C of length L, and where equality holds if and only if C is a circle. There are many known proofs of this fact. More than one idea can be found in the expository paper by Osserman [37], along with a brief histor of the problem. It was Carleman [12] who in 1921 gave the first proof based on complex analysis, in the special case of a Jordan domain bounded by a smooth curve. In this section we will see that the theory of the Hardy spaces gives an elementary proof of the inequality. In modern notation equation (1.44) may be rewritten as

$$\int_{\mathbb{D}} |\tau|^2 \,\mathrm{d}\sigma \leq \left(\frac{1}{2\pi} \int_0^{2\pi} |\tau(e^{i\theta})| \,\mathrm{d}\theta \right)^2 \tag{1.45}$$

where τ is a conformal mapping of $\mathbb{D}$ onto A, and is known as *Carleman's inequality*. Here $\mathrm{d}\sigma$ denotes the Lebesgue measure on $\mathbb{D}$ normalized so that the measure of $\mathbb{D}$ is 1. In terms of real (rectangular and polar) coordinates, we have

$$\mathrm{d}\sigma = \frac{1}{\pi}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{\pi} r \,\mathrm{d}r\,\mathrm{d}\theta, \qquad z = x + iy = re^{i\theta} \,.$$

Note that in light of Theorem 1.22 the right-handside of equation (1.44) is nothing more than $\|\tau\|_{H^1}^2$. Similarly, we define

$$\|f\|_{A^p} := \left(\int_{\mathbb{D}} |f|^p \,\mathrm{d}\sigma \right)^{\frac{1}{p}} . \tag{1.46}$$

In the passing we mention that the space that contains all analytic functions such that $\|f\|_{A^p} < \infty$ is called the *Bergman space* A^p, and it has a theory nearly as rich as the Hardy spaces, see Duren [15, p. 250] for a brief overview. We will only make use of the Bergman spaces for its convenient notation though. Thus, Carleman's inequality (1.45) may be restated as

$$\|f\|_{A^2} \leq \|f\|_{H^1} \,. \tag{1.47}$$

LEMMA 1.24. *Let $f(z) = \sum_{n \geq 0} a_n z^n$ be analytic in A^2, then*

$$\|f\|_{A^2} = \left(\sum_{j \geq 0} \frac{|a_j|^2}{1 + j} \right)^{\frac{1}{2}} . \tag{1.48}$$

PROOF. Recall from Lemma 1.11 that $\int_{\mathbb{T}} z^j \overline{z}^k \,\mathrm{d}m = \delta_{jk}$ by the orthogonality of the trigonometric system. Thus,

$$\int_{\mathbb{D}} z^j \overline{z}^k \,\mathrm{d}\sigma = 2\int_0^1 r^{i+j} \int_{\mathbb{T}} z^j \overline{z}^k \,\mathrm{d}m \,\mathrm{d}r = \frac{2\delta_{jk}}{2+j+k} = \frac{\delta_{jk}}{1+j} \,.$$

Applying this to equation (1.46) the norm of the A^2 space becomes

$$\|f\|_{A^2} = \left(\int_{\mathbb{D}} |f|^2 \,\mathrm{d}\sigma\right)^{\frac{1}{2}} = \left(\sum_{j,k\geq 0} a_j \overline{a_k} \int_{\mathbb{D}} z^j \overline{z}^k \,\mathrm{d}\sigma\right)^{\frac{1}{2}} = \left(\sum_{k\geq 0} \frac{|a_j|^2}{1+j}\right)^{\frac{1}{2}} . \qquad \square$$

By using the Riesz factorization theorem among other results Vukotić presented in [48] a modern and natural way of generalizing (1.47).

PROPOSITION 1.25. *For $1 \leq p < \infty$, every $f \in H^p$ belongs to A^{2p}, and*

$$\|f\|_{A^{2p}} \leq \|f\|_{H^p} \,.$$

PROOF. We begin by considering the case $p = 2$ first. Since f is analytic, we can write

$$f(z) = \sum_{n\geq 0} a_n z^n \,,$$

which converges for $z \in \mathbb{D}$. Squaring this we obtain

$$f^2 = \sum_{n\geq 0} A_n z^n \quad \text{where} \quad A_n = \sum_{k=0}^{n} a_k a_{n-k} \,. \tag{1.49}$$

Using Lemma 1.24 on f^2, now gives

$$\|f\|_{A^4}^4 = \|f^2\|_{A^2}^2 = \sum_{n\geq 0} \frac{|A_n|^2}{1+n} = \sum_{n\geq 0} \frac{1}{1+n} \left|\sum_{k\geq 0} a_k a_{n-k}\right|^2 .$$

The last equation can be turned into an inequality by applying the Cauchy-Schwarz inequality $n+1$ times

$$\|f\|_{A^4}^4 \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_k a_{n-k}|^2 = \left(\sum_{n\geq 0} |a_n|^2\right)^2 = \|f\|_{H^2}^4 \,.$$

This proves the case $p = 2$. Assume that $p \geq 1$, if $f \equiv 0$ we are done. If $f \neq 0$ then f has a finite number of zeroes, and in particular from Riesz factorization theorem 1.16 we can write $f(z) = g(z)B(z)$, where B is a Blaschke product and $g \in H^2$ is zero free in $\mathbb{D}$. Furthermore, from Corollary 1.17 it is known that $g^{p/2}$ is in H^1, since g does not vanish in $\mathbb{D}$, and $\|f\|_{H^p} = \|g\|_{H^p}$. Thus,

$$\|f\|_{A^{2p}} \leq \|g\|_{A^2 p} = \|g^{p/2}\|_{A^4}^{p/2} \leq \|g^{p/2}\|_{H^2}^{p/2} = \|g\|_{H^p} = \|f\|_{H^p} \,,$$

where $|B(z)| < 1$ in $\mathbb{D}$ from 1.15 was used to prove the first inequality and the second inequality follows from $p = 2$. Which completes the proof. $\square$

The classical isoperimetric inequality now follows directly.

Corollary 1.26. *Let G be a Jordan domain with rectifiable boundary of length $L(\partial G)$, and area $A(G)$ Then there holds the inequality*

$$A(G) \leq L(\partial G)^2/4\pi\,.$$

Proof. Appealing to the Riemann mapping theorem, we can choose a conformal mapping F of $\mathbb{D}$ onto Ω. Then,

$$L(\partial G) = \lim_{r\to 1^-} L(F(\{|z|=r\})) = \lim_{r\to 1^-}\int_0^{2\pi} |F'(re^{i\theta})|\,\mathrm{d}\theta = 2\pi\|F'\|_{H^1}\,.$$

From Duren [15, p. 44] we have that if $f(z)$ maps $|z|<1$ conformally onto the interior of a Jordan curve C, then C is rectifiable if and only if $\tau \in H^1$.

Also $A(\Omega) = \|\tau\|_{A^2}^2$, so the isoperimetric inequality (1.44) follows from $\|f\|_{A^2}^2 \leq \|f\|_{H^1}^2$ applied to $f=\tau$. $\square$

Generalizations for the weighted Bergman space. While not needed for this thesis, we offer a small digression as to how Carleman's inequality extends to the weighted Bergman space. See the monograph Hedenmalm, Korenblum, and Zhu [23] for further references on the Bergman spaces. Let $\alpha > 1$ and $1 \leq p < \infty$, and define the (weighted) Bergman space $A^p_\alpha(\mathbb{D})$ as the space of analytic functions in $\mathbb{D}$ that are finite with respect to the norm

$$\|f\|_{A^p_\alpha} := \left(\int_{\mathbb{D}} |f(w)|^p(\alpha-1)(1-|w|^2)^{\alpha-2}\,\mathrm{d}\sigma(w)\right)^{1/p}$$

Here $\mathrm{d}\sigma$ denotes the Lebesgue area measure, normalized so that $\mathrm{d}\sigma(\mathbb{D})=1$. It will be convenient to let $\mathrm{d}\sigma_\alpha(w) = (\alpha-1)(1-|w|)^{\alpha-2}\,\mathrm{d}\sigma$ and to let $\mathrm{d}\sigma_1 = \mathrm{d}m$ denote the normalized Lebesgue measure on the torus $\mathbb{T}$.

The following inequality is due to Burbea [11, Cor. 3.4] who generalized Carleman's inequality.

Proposition 1.27 (Burbea). *Suppose that $f \in H^2$, then for every integer $k \geq 2$*

$$\|f\|_{A^{2k}_k} = \left(\int_{\mathbb{D}} |f(z)|^{2k}\,\mathrm{d}\sigma_k(z)\right)^{\frac{1}{2k}} \leq \|f\|_{H^2}. \tag{1.50}$$

Let $C_\alpha(j)$ denote the coefficients of the binomial series

$$\frac{1}{(1-z)^\alpha} = \sum_{j=0} C_\alpha(j)z^j\,, \qquad C_\alpha(j) = \binom{j+\alpha-1}{j}. \tag{1.51}$$

Notice that $C_1(j) = 1$ for every j. Identifying $C_\alpha(j)$ as the coefficients of the binomial series $(1-z)^{-\alpha}$, we find that

$$C_{\alpha k}(j) = \sum_{j_1+j_2+\cdots+j_k=j} C_\alpha(j_1)C_\alpha(j_2)\cdots C_\alpha(j_k)\,. \tag{1.52}$$

In particular if α is an integer, then $C_\alpha(j)$ denotes the number of ways to write j as a sum of α non-negative integers. Hence,

$$\sum_{j+k=l} C_\alpha(j)C_\beta(k) = C_{\alpha+\beta}(l)\,.$$

To prove equation (1.50) we will need to compute the norm of the weighted Bergman space

LEMMA 1.28. *Let $f = \sum_{n\geq 0} a_n z^k$ be in A^2_α, then*

$$\|f\|^2_{A^2_\alpha} = \sum_{n\geq 0} \frac{|a_n|^2}{C_\alpha(n)} .$$

PROOF. Since $f \in A^2_\alpha$, we may interchange the integral and summation as needed.

$$\begin{aligned}
\|f\|^2_{A^2_\alpha} &= \int_{\mathbb{D}} |f(z)|^2 \, \mathrm{d}\sigma_\alpha(z) \\
&= \int_{\mathbb{D}} \sum_{n,m\geq 0} a_n \overline{a_m} r^{n+m} e^{i(n-m)\theta} (\alpha-1)(1-r^2)^{\alpha-2} \, \mathrm{d}\sigma(w) \\
&= 2(\alpha-1) \int_0^1 \sum_{n\geq 0} |a_n|^2 r^{2n} (1-r^2)^{\alpha-2} r \, \mathrm{d}r \\
&= \sum_{n\geq 0} |a_n|^2 (\alpha-1) \int_0^1 t^n (1-t)^{\alpha-2} \, \mathrm{d}t
\end{aligned}$$

Where the substitution $r^2 \mapsto t$ was used. Using Corollary A.20 the integral becomes

$$\int_0^1 t^n (1-t)^{\alpha-2} \, \mathrm{d}t = B(n+1, \alpha-1) = \frac{1}{\alpha-1} \binom{n+\alpha-1}{n}^{-1} = \frac{1}{(\alpha-1)C_\alpha(n)} ,$$

and we are done. □

PROOF OF PROPOSITION 1.27. Again let $f(z) = \sum_{j\geq 0} a_j z^j$. The idea is to use $|f|^{2k} = |f^k|^2$ and use Lemma 1.28.

$$\begin{aligned}
\|f\|^{2k}_{A^{2k}_k} &= \sum_{j\geq 0} \frac{1}{C_k(j)} \left| \sum_{j_1+j_2+\cdots+j_k=j} a_{j_1} \cdots a_{j_k} \right|^2 \\
&\leq \sum_{j\geq 0} \left(\sum_{j_1+j_2+\cdots+j_k=j} |a_{j_1}|^2 \cdots |a_{j_k}|^2 \right) = \left(\sum_{j\geq 0} |a_j|^2 \right)^{2k} = \|f\|^{2k}_{H^2} ,
\end{aligned}$$

where equation (1.52) was used in the second to last inequality. □

Even though the above proof was relative easy, it is not known whether Proposition 1.27 holds for non-integer k.

1.6. Hardy spaces on the polydisc

In this section we give a brief introduction to the Hardy spaces of the countably infinite polydisk, $H^p(\mathbb{D}^\infty)$, which in recent years have recieved considerable interest and study, emerging from the fundamental papers [24, 14]. Much of the renewed interest is due to a simple observation of Bohr [7], which facilitates a link between Dirichlet series and function theory in polydiscs.

The standard reference for the Hardy space on the polydisc is the classical monograph [44] by Rudin. We will frequently use polynomials in several complex variables, and for bookkeeping the following multi-index notation is introduced.

Definition. An m'th order *multi-index* on $\mathbb{C}^n$ is the following vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha)$ where $\alpha_i \in \{0, 1, \ldots, m\}$. Furthermore $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n = m$. For $z \in \mathbb{C}^n$ we take

$$z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}.$$

Any m'th degree polynomial on $\mathbb{T}^n$ can thus be represented as

$$P(z) = \sum_{|\alpha| \le m} a_\alpha z^\alpha \,,$$

where we assume that there exists some $a_\alpha \neq 0$ and $|\alpha| = m$. Similarly we denote an analytic function on $\mathbb{T}^n$ as

$$f(z) = \sum_{\alpha \ge 0} a_\alpha z^\alpha \,,$$

with $\alpha = (\alpha_1, \alpha_2, \cdots)$.

Definition. Let U be an open subset of $\mathbb{C}^n$. A function $F \colon U \to \mathbb{C}$ is called *analytic* if it is continuous and analytic in each variable.

In one dimension we have studied the unit disk and the unit torus:

$$\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\} \,,$$
$$\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\} \,,$$

where it was clear that $\partial\mathbb{D} = \mathbb{T}$, and we made no distinction between the spaces. It is natural to consider:

$$\mathbb{D}^n := \{z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \colon z_i \in \mathbb{D}\} \,,$$
$$\mathbb{T}^n := \{z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \colon z_i \in \mathbb{T}\} \,.$$

However, if we let $z = (1, 0, \ldots, 0)$ then z is on $\partial\mathbb{D}^n$ but not on $\mathbb{T}$, so for $n \ge 1$, we see that $T^n \nsubseteq \partial\mathbb{D}^n$. Thus, some extra care is needed to define $H^p(\mathbb{D}^\infty)$, since functions in $H^p(\mathbb{D}^\infty)$ will generally not be well defined in the whole set $\mathbb{D}^\infty$.

However, similar to the one-dimensional case the radial boundary limit

$$f^*(z) = \lim_{r \to 1^-} f(rz) \,,$$

exists for almost every $z \in \mathbb{T}^d$, and we can write

$$\|f\|_{H^p(\mathbb{D}^d)}^p = \int_{\mathbb{T}^d} |f^*|^p \, \mathrm{d}m_d \;. \tag{1.53}$$

This means that $H^p(\mathbb{D}^d)$ is a subspace of $L^p(\mathbb{T}^d, m_d)$. Moreover, again as in the one-dimensional case, for every $f \in H^p(\mathbb{D}^d)$ we have

$$\lim_{r \to 1^{-1}} \|f - f_r\|_{H^p(\mathbb{D}^d)} = 0 . \tag{1.54}$$

Which implies that the polynomials are dense in $H^p(\mathbb{D}^d)$. Thus, it will be convenient to define the space $H^p(\mathbb{D}^d)$ as the Banach space completion of the polynomials $F(z) = \sum_{n=0}^{N} a_n z^{\kappa(n)}$ in the norm

$$\|f\|_{H^p(\mathbb{T}^d)} := \left(\int_{\mathbb{T}^d} |f|^p \, \mathrm{d}m_d \right)^{\frac{1}{2}} .$$

As before we make no distinction between $H^p(\mathbb{T}^d)$ and $H^p(\mathbb{D}^d)$. A convenient method to obtain equations (1.53) and (1.54) is to apply the L^p-boundedness of the radial maximal function on $H^p(\mathbb{D}^d)$ for all $p > 0$. By Fubini's theorem, the boundedness of the maximal function then reduces to the classical one-dimensional estimate, see [42] for details.

To define $\mathbb{D}^\infty$, it will be convenient to introduce the set $\mathbb{D}^\infty_{\mathrm{fin}}$ which consists of elements $z = \{z_j\}_{j \geq 1} \in \mathbb{D}^\infty$ such that $z_j \neq 0$ only for finitely many j. It is clear that the function f can be written as a convergent Taylor series

$$f(z) = \sum_{\alpha \in \mathbb{N}^\infty_{\mathrm{fin}}} a_\alpha z^\alpha \, , \quad z \in \mathbb{D}^\infty_{\mathrm{fin}} \, ,$$

and the coefficients c_k determine f uniquely.

Definition. Let $p \geq 1$. The space $H^p(\mathbb{D}^\infty)$ is the space of analytic functions on $\mathbb{D}^\infty_{\mathrm{fin}}$ obtained by taking the closure of all polynomials in the norm

$$\|f\|_{H^p(\mathbb{T}^\infty)} := \left(\int_{\mathbb{T}^\infty} |f|^p \, \mathrm{d}m_\infty \right)^{\frac{1}{p}} .$$

Here $\mathrm{d}m_\infty$ denotes the Haar measure, we refer to [24] for the details, mentioning only that the Haar measure of $\mathbb{T}^\infty$ is simply the product of the normalized Lebesgue measures in each variable.

Lemma 1.29. *For any multi-indices α and β on $\mathbb{C}^d$ we have*

$$\int_{\mathbb{T}^d} z^\alpha \cdot \overline{z}^\beta \, \mathrm{d}m_d(z) = \delta_{\alpha\beta} \, . \tag{1.55}$$

Proof. Recall from Lemma 1.11 that in the one dimensional case

$$\int_{\mathbb{T}} z^k \cdot \overline{z}^j \, \mathrm{d}m(z) = \delta_{ij}$$

holds for all non-negative integers k and j, by the orthogonality of the trigonometric system. Applying this for each of the d variables completes the proof. $\square$

1.6.1. The Cole-Gamelin estimate

We now wish to show that point evaluations at a point z in

$$\mathbb{D}^\infty \cap \ell^2 = \{z \in \mathbb{D}^\infty \ : \ \sum_{j\geq 1} |z_j|^2 < +\infty\},$$

extends continuously to $H^p(\mathbb{T}^\infty)$. This was first shown by Cole and Gamelin in [14].

PROPOSITION 1.30 (Cole-Gamelin). *Let $f \in H^p(\mathbb{T}^\infty)$, where $p \geq 1$ then*

$$|f(z)| \leq \left(\prod_{j\geq 1} \frac{1}{1-|z_j|^2}\right)^{\frac{1}{p}} \|f\|_{H^p(\mathbb{D}^\infty)},$$

and the inequality is sharp.

PROOF. Let $P(z)$ be a polynomial with $z \in \mathbb{D}^d$,

$$|P(z_1, z_2, \cdots, z_d)|^p .$$

By applying the standard point-estimate 1.21 to z_1 we obtain

$$|P(z_1, z_2, \cdots, z_d)|^p \leq \frac{1}{1-|z_1|^2} \int_{\mathbb{T}} |P(w_1, z_2, \cdots, z_d)|^p \, \mathrm{d}m(w_1). \tag{1.56}$$

Applying 1.21 to z_2 in equation (1.56) gives

$$|P(z_1, z_2, \cdots, z_d)|^p \leq \prod_{j=1}^{2} \frac{1}{1-|z_j|^2} \int_{\mathbb{T}^2} |P(w_1, w_2, \cdots, z_d)|^p \, \mathrm{d}m_2(w_1, w_2),$$

by repeating this process and applying the point estimate to each variable we obtain

$$|P(z_1, z_2, \cdots, z_d)|^q \leq \prod_{m=1}^{d} \frac{1}{1-|z_m|^2} \int_{\mathbb{T}^d} |P|^q \, \mathrm{d}m_d .$$

Letting $d \to \infty$ completes the proof. That this inequality is sharp follows since the point-estimate in one variable is sharp. □

1.7. Helson's inequality

The purpose of this section will be to generalize Carleman's inequality (1.47)

$$\left(\sum_{k\geq 0} \frac{|a_k|^2}{1+k}\right)^{\frac{1}{2}} \leq \int_{\mathbb{T}} |f| \, \mathrm{d}m, \tag{1.57}$$

to $\mathbb{T}^\infty$. In other words we wish to prove that

THEOREM 1.31 (Helson's inequality). *Given $f \in H^1(\mathbb{D}^\infty)$ then*

$$\left(\sum_{\alpha\geq 0} \frac{|a_\alpha|^2}{(1+\alpha_1)(1+\alpha_2)\cdots}\right)^{\frac{1}{2}} \leq \|f\|_{H^1(\mathbb{D}^\infty)}, \tag{1.58}$$

where $\alpha \geq 0$ means the unbounded multi-index $\alpha = (\alpha_1, \alpha_2, \cdots)$.

PROOF. As before it suffices to prove Theorem 1.31 in d variables, and then take closure of the analytic polynomials. We let $f(z)$ be a polynomial of $j = 1, \dots, d$ variables, and following Helson [26] define the operator T_j as

$$T_j \sum_{\alpha \geq 0} a_\alpha z^\alpha = \sum_{\alpha \geq 0} \frac{a_\alpha}{\sqrt{1+\alpha_j}} z^\alpha \,.$$

Where again the multi-index notation defined in section 1.6 was used. Helson's inequality (1.58) can now be written as

$$\|T_1 \cdots T_d f\|_{H^2(\mathbb{T}^d)} \leq \|f\|_{H^1(\mathbb{T}^d)} \,. \tag{1.59}$$

The idea is now to apply Carleman's inequality (1.57) to the first variable of $f(z_1, \cdots, z_d)$ in the left hand-side of equation (1.59)

$$\|T_1 \cdots T_d f\|_{H^2(\mathbb{T}^d)} \leq \left(\int_{\mathbb{T}} \left(\int_{\mathbb{T}^{d-1}} |T_2 \cdots T_d f| \,\mathrm{d}m_{d-1} \right)^2 \mathrm{d}m_1 \right)^{\frac{1}{2}} . \tag{1.60}$$

The next step is to use Minkowski's continuous inequality A.10;

$$\left[\int_X \left(\int_Y |f(x,y)| \mathrm{d}\nu(y) \right)^2 \mathrm{d}\mu(x) \right]^{\frac{1}{2}} \leq \int_Y \left(\int_X |f(x,y)|^2 \mathrm{d}\mu(x) \right)^{\frac{1}{2}} \mathrm{d}\nu(y) \,,$$

to the right-hand side of (1.60), thus reversing the order of integration

$$\|T_1 \cdots T_d f\|_{H^2(\mathbb{T}^d)} \leq \int_{\mathbb{T}} \left(\int_{\mathbb{T}^{d-1}} |T_2 \cdots T_d f|^2 \,\mathrm{d}m_1 \right)^{\frac{1}{2}} \mathrm{d}m_{d-1} \,.$$

We now have one fewer Ts, and one variable removed from the inner integral. Repeating this process of alternating between Carleman's and Minkowski's inequality $d-1$ times to the right hand-side of equation (1.59) we obtain

$$\|T_1 \cdots T_d f\|_{H^2(\mathbb{T}^d)} \leq \int_{\mathbb{T}^{d-1}} \left(\int_{\mathbb{T}} |T_d f|^2 \,\mathrm{d}m_{d-1} \right)^{\frac{1}{2}} \mathrm{d}m_1 \,.$$

A final application of Carleman's inequality and Minkowski's inequality shows

$$\|T_1 \cdots T_d f\|_{H^2(\mathbb{T}^d)} \leq \left(\int_{\mathbb{T}^d} |f| \,\mathrm{d}m_d \right)^{\frac{1}{1}} = \|f\|_{H^1(\mathbb{T}^d)} \,.$$

Taking the closure of the analytic polynomials now completes the proof. □

CHAPTER 2

Hankel forms

The purpose of this section is to introduce the bilinear forms, the Hilbert matrix, and prove Nehari's theorem in its original form.

It was Hankel [18] who in 1861 began the study of finite matrices whose entries depend only on the sum of the coordinates, and therefore such objects are called *Hankel matrices*. In particular, Hankel forms was first represented by matrices $(a_{n+k})_{n,k\geq 0}$ where $(a_n)_{n\geq 0}$ is a sequence of complex numbers and was originaly used to study moment problems.

The theory on Hankel forms had a latent development, but after the work of Nehari [33] [1957] and Hartman [22] [1958], the theory rapidly evolved. The classical framework for the theory of Hankel operators is the sequence space

$$\ell^2 = \left\{ x = (x_k)_{k\geq 0} \colon \|x\|_{\ell^2}^2 = \sum_{k\geq 0} |x_k|^2 < \infty \right\},.$$

We will identify ℓ^2 with the Hardy space H^2 of analytic functions in $\mathbb{D}$

$$H^2 = \left\{ f(z) = \sum_{n\geq 0} a_n z^n \colon \|f\|_{H^2}^2 = \sum_{n\geq 0} |a_n|^2 < \infty \right\}.$$

We see from the expression above that the vector $f = (a_0, a_1, \cdots) \in \ell^2$ is identified with the analytic function $f(z) = \sum_{n\geq 0} a_n z^n \in H^2$ and vice versa. This identification was explicitly shown in Theorem 1.10 and will be used frequently.

2.1. Bilinear forms

DEFINITION. Let $a = \{a_n\}_{n\geq 1}$, $b = \{b_n\}_{n\geq 0}$ be two sequences in ℓ^2. Then the map $A \colon \ell^2 \times \ell^2 \to \mathbb{C}$ defined by

$$A(a,b) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{m,n} a_m b_n \,,$$

is a bilinear form.

Where the double sequence $\{A_{m,n}\}_{m,n\geq 0}$ can be viewed as a matrix with indices $A_{m,n}$. As a reminder we will from here on out use the notation $\sum_{m,n\geq 0}$ to denote double series, when no confusion is possible. A bilinear form is said to be bounded if there exists a positive constant K such that

$$\left|\sum_{m,n\geq 0}^{\infty} A_{m,n} a_m b_n\right| \leq K\left(\sum_{n\geq 0}|a_n|^2\right)^{\frac{1}{2}}\left(\sum_{n\geq 0}|b_n|^2\right)^{\frac{1}{2}}. \tag{2.1}$$

The smallest number K for which the inequality holds is referred to as the *norm*, and we write $\|A\| = K$. This is attained when $|A(a,b)|/\|a\|\|b\|$ is maximized, thus we define the operator norm of A as

$$\|A\| := \sup_{\substack{a,b\in\ell^2\\ a,b\neq 0}} \frac{|A(a,b)|}{\|a\|\|b\|} = \sup_{\substack{a,b\in\ell^2\\ \|a\|=\|b\|=1}} |A(a,b)|\,, \tag{2.2}$$

where the norms of a and b is the ℓ^2 norm.

Among the numerous bilinear forms which have been studied [20, Chp. VIII, IX], there are some whose coefficients $A_{m,n}$ of the special types $\alpha(n+m)$, where the function $\alpha(n)$ is defined for integral values of n. We will denote these bilinear forms as Hankel forms:

Definition. For a sequence $\rho = (\rho_1, \rho_2, \rho_3, \ldots) \in \ell^2$ its corresponding Hankel form on $\ell^2 \times \ell^2$ is given by

$$\rho(a,b) := \sum_{m,n\geq 0} a_m b_m \rho_{m+n}\,, \tag{2.3}$$

which initially is defined for $a, b \in \ell^2$.

Proposition 2.1. *Let f,g be in H^2. Then,*

$$H_\varphi(fg) := \langle fg, \varphi\rangle_{H^2} = \rho(a,b)\,, \tag{2.4}$$

induces a Hankel form on $\ell^2 \times \ell^2$ where

$$\varphi(z) = \sum_{n\geq 0} \overline{\rho_n} z^n.$$

The function φ is called the *symbol* of H.

Proof. Since $f, g \in H^2$ this implies that f,g are analytic functions on the form

$$f(z) = \sum_{m\geq 0} a_m z^m \quad \text{and} \quad g(z) = \sum_{n\geq 0} \overline{b_n} z^n.$$

Using equation (2.4), a computation at the level of coefficients shows

$$\begin{aligned} H_\varphi(fg) &= \int_{\mathbb{T}} fg\cdot\overline{\varphi}\,\mathrm{d}m \\ &= \sum_{m,n,k\geq 0} a_m b_n \rho_k \int_{\mathbb{T}} z^{m+n}\cdot\overline{z}^k\,\mathrm{d}m = \sum_{m,n\geq 0} a_m b_n \rho_{n+m} = \rho(a,b). \end{aligned}$$

The integral was evaluated using Lemma 1.11, since z^{m+n} and $\overline{z}^k$ are orthogonal on L^2. ∎

Similar to equation (2.1), we define H_φ to be bounded *bounded* if there exists a real number k such that for all $f, g \in H^2$

$$\big|H(fg)\big| \leq k\|f\|_{H^2}\|g\|_{H^2}$$

The smallest possible k is obtained by maximizing $\big|H(fg)\big|/\|f\|\|g\|$. Thus we define the norm of H_φ as

$$\|H_\varphi\| = \sup_{\substack{f,g\in H^2\\ f,g\neq 0}} \frac{|\langle fg, \varphi\rangle_{H^2}|}{\|f\|\|g\|} = \sup_{\substack{f,g\in H^2\\ \|f\|=\|g\|=1}} |\langle fg, \varphi\rangle_{H^2}|\,, \tag{2.5}$$

where the norm of f and g is the H^2 norm. If φ is in H^∞, we obtain a very simple bound for $\|H_\varphi\|$.

PROPOSITION 2.2. *Let H_φ be a Hankel form, and let φ in H^∞. Then*

$$\|H_\varphi\| \leq \|\varphi\|_{H^\infty}\,. \tag{2.6}$$

PROOF. A direct computation of the inner product yield,

$$\begin{aligned} |H_\varphi(fg)| = \left|\int_{\mathbb{T}} f\cdot\overline{g}\cdot\varphi\,\mathrm{d}m\right| &\leq \sup_{z\in\mathbb{T}}|\varphi(z)|\left|\int_{\mathbb{T}} f\cdot\overline{g}\,\mathrm{d}m\right| \\ &\leq \sup_{z\in\mathbb{T}}|\varphi(z)|\left(\int_{\mathbb{T}}|f|^2\,\mathrm{d}m\right)^{\frac{1}{2}}\left(|g|^2\,\mathrm{d}m\right)^{\frac{1}{2}}, \end{aligned}$$

where the first inequality follows by taking out by taking the supremum of φ, and the latter from Cauchy–Schwarz. As the last expression is $\|\varphi\|_{L^\infty}\|f\|_{H^2}\|g\|_{H^2}$, the proposition follows directly from equation (2.5). □

2.2. The Hilbert matrix

As seen in Proposition 2.1 every Hankel form can be viewed as the inner product of two functions in H^2. Another simple integral that produces Hankel forms is the following,

$$H(fg) = \int_0^1 f(z)g(z)\,\mathrm{d}z\,, \qquad f,g\in H^2. \tag{2.7}$$

PROPOSITION 2.3. *The integral (2.7) is a Hankel form with symbol*

$$\varphi(z) = \sum_{k\geq 0}\frac{1}{k+1}z^k\,.$$

PROOF. To see that φ is the symbol, we compute $H(fg)$ at the level of coefficients:

$$\int_0^1 f(z)g(z)\,\mathrm{d}z = \sum_{m,n\geq 0} a_m b_n \int_0^1 z^{m+n}\,\mathrm{d}z = \sum_{m,n\geq 0}\frac{a_m b_n}{m+n+1}\,. \tag{2.8}$$

Where the interchange of the sum and integral follows since $f, g \in H^2$. Comparing this with equation (2.3) we see that

$$\rho_{m+n} = \frac{1}{m+n+1}\,. \tag{2.9}$$

Thus the symbol can be written

$$\varphi(z) = \sum_{k\geq 0} \frac{1}{1+k} z^k = \int_0^1 \left(\sum_{k\geq 0}^{\infty} (zw)^k \right) \mathrm{d}w \ . \qquad \square$$

We can view ρ_{n+m} of $H(fg)$ as the coefficients of the following matrix

DEFINITION. Let $M \colon \ell^2 \to \ell^2$ be the following matrix

$$M := \left(\frac{1}{n+m+1} \right)_{m,n\geq 0} = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

We define M as the *Hilbert matrix*.

Since φ is not bounded, we obtain from Proposition 2.2 the un-interesting bound $\|H_\varphi\| \leq \infty$. The purpose of the remaining part of this section is to prove that $\|H_\varphi\|$ in fact is bounded.

THEOREM 2.4. *The Hankel form H_φ is a strictly positive and bounded on H^2 and $\|H\| = \pi$.*

This is the same as proving

$$\sum_{n,m\geq 0} \frac{a_n b_n}{n+m+1} \leq \pi \left(\sum_{n\geq 0} |a_n|^2 \right)^{\frac{1}{2}} \left(\sum_{n\geq 0} |b_n|^2 \right)^{\frac{1}{2}} , \tag{2.10}$$

where the constant π can not be improved, in other words showing that $\|M\| = \pi$. Equation (2.10) is part of a family of inequalities all known as *Hilbert's inequality.*

The history of the Hilbert's inequality is briefly explained in Hardy, Littlewood, and Pólya [20, Chp. IX]. According to this Hilbert first proved his double series theorem in his lectures on integral equations. The theorem states that there exists some positive constant C, such that for any real square summable sequence $\{a_n\}$ one has

$$\sum_{m,n\geq 1} \frac{a_m a_n}{m+n} \leq C \sum_{m\geq 1} a_m^2 \, . \tag{2.11}$$

Hilbert proved this equation with the constant $C = 2\pi$, and later Shur improved this bound, proving that the optimal constant was $C = \pi$. We will not prove equation (2.10) here, but instead prove the weaker version (2.11). The reason for this is twofold.

Firstly it was shown in Hardy, Littlewood, and Pólya [20, p. 233] that equation (2.11) may be sharpened into (2.10) by using a discretization of the continuous version of (2.11) and the Hermite–Hadamard inequality. The details are omitted as we will later prove the strongest version of Hilbert's inequality using Nehari's theorem.

Secondly, the proof for the weaker version (2.11) has a natural extension to Hankel forms for Dirichlet series. The same does not hold for equation (2.10), see Brevig and Perfekt [9] for further details. To simplify the writing, Hilbert assumed in (2.11) that $a_n = b_n$. While it is a well-known fact that this restriction does not change the bound of the form, we will instead follow Steele [47] and prove

$$\Big|\sum_{m,n\geq 0} \frac{a_m b_n}{n+m}\Big| \leq \pi\Big(\sum_{m\geq 0} a_m^2\Big)^{\frac{1}{2}}\Big(\sum_{n\geq 0} a_n^2\Big)^{\frac{1}{2}} . \tag{2.12}$$

A naive first attempt to prove equation (2.12) would be to use Cauchy-Schwarz

$$\Big(\sum_{m,n\geq 1} \alpha_s \beta_s\Big)^2 \leq \sum_{m,n\geq 1} \alpha_s^2 \sum_{m,n\geq 1} \beta_s^2 \,, \tag{2.13}$$

directly with

$$\alpha_s = \frac{a_m}{\sqrt{n+m}} \,, \qquad \beta_s = \frac{b_n}{\sqrt{n+m}} \,, \qquad s = (n,m) \,.$$

By design, the products $\alpha_s\beta_s$ recapture the terms one finds on the left-hand side of Hilbert's inequality, but the bound one obtains from Cauchy's inequality (2.13) turns out to be disappointing. Specifically with the choices above we have

$$\Big(\sum_{m,n\geq 1} \frac{a_n b_n}{n+m}\Big)^2 \leq \sum_{m,n\geq 1} \frac{a_m^2}{n+m} \sum_{m,n\geq 1} \frac{b_n^2}{n+m} \,, \tag{2.14}$$

where unfortunately the right-hand side diverges. The first factor diverges like an harmonic series when we sum over n, and similarly β_s^2 diverges when we sum over m. Thus, we will instead look at the parametric family

$$\alpha_s = \frac{a_m}{n+m}\left(\frac{m}{n}\right)^{\lambda} \,, \qquad \beta_s = \frac{b_n}{n+m}\left(\frac{n}{m}\right)^{\lambda} \,, \qquad s = (n,m) \,,$$

where $0 < \lambda < 1$ will be chosen later. The reason for the choice above is simple, For large n, $\alpha_s \sim a_m/n^{\lambda+1}$. So α_s behaves like a real Dirichlet series for n, and hence converges for $\lambda > 0$. Applying Cauchy–Schwarz (2.13) on $\sum \alpha_s\beta_s$ yield

$$\left(\sum_{m,n\geq 1} \frac{a_m b_n}{n+m}\right)^2 \leq \sum_{m,n\geq 1} \frac{a_m^2}{n+m}\left(\frac{m}{n}\right)^{2\lambda} \sum_{m,n\geq 1} \frac{b_n^2}{n+m}\left(\frac{n}{m}\right)^{2\lambda} . \tag{2.15}$$

We will now bound the right-hand side by an integral estimate, and by symmetry, we only need to consider one of the factors. For any non-negative decreasing function $f\colon [0,\infty) \to \mathbb{R}$, we have

$$\sum_{n\geq 1} f(n) \leq \int_0^\infty f(x)\,\mathrm{d}x \,. \tag{2.16}$$

Specifically for the function $f(x) = m^{2\lambda}x^{2\lambda}(m+x)^{-1}$,

$$\sum_{n\geq 1} \frac{(m/n)^{2\lambda}}{n+m} \leq \int_0^\infty \frac{(m/x)^{2\lambda}}{m+x}\,\mathrm{d}x \overset{x\mapsto my}{=} \int_0^\infty \frac{y^{-2\lambda}}{1+y}\,\mathrm{d}y = \frac{\pi}{\sin 2\pi\lambda} \,, \tag{2.17}$$

where the last equality follows from Lemma A.18. As $\sin 2\pi\lambda$ is maximal when $\lambda = 1/4$, we obtain π as the constant. Even though $\lambda = 1/4$ gives the best constant, it does not prove that there does not exists a smaller constant $C < \pi$ that satisfies (2.11).

PROPOSITION 2.5 (Hilbert's inequality). *Let a_n, b_n be real square summable sequences, then*

$$\left|\sum_{m,n\geq 1}\frac{a_m b_n}{n+m}\right| \leq \pi\Big(\sum_{m\geq 1} a_m^2\Big)^{\frac{1}{2}}\Big(\sum_{n\geq 1} a_n^2\Big)^{\frac{1}{2}}, \tag{2.18}$$

and the constant π can not be improved.

PROOF. Using the Cauchy–Schwarz inequality on the sequences

$$\alpha_{m,n} = \frac{a_m}{n+m}\left(\frac{m}{n}\right)^{1/4}, \qquad \beta_{m,n} = \frac{b_n}{n+m}\left(\frac{n}{m}\right)^{1/4},$$

we obtain the following bound

$$\left|\sum_{m,n\geq 1}\frac{a_m b_n}{n+m}\right|^2 \leq \Big(\sum_{m\geq 1} a_m^2 \sum_{n\geq 1}\frac{1}{n+m}\sqrt{\frac{n}{m}}\Big)\Big(\sum_{n\geq 1} b_n^2 \sum_{m\geq 1}\frac{1}{n+1}\sqrt{\frac{m}{n}}\Big).$$

By taking the square root and applying the integral estimate equation (2.17), we obtain (2.18). To prove that the inequality in (2.18) is sharp, we note that both sides of converges for

$$a_n(\varepsilon) := b_n(\varepsilon) := n^{-(1+\varepsilon)/2}, \qquad \varepsilon > 0,$$

but fails for $\varepsilon = 0$. The idea is to 'stress' the inequality by seeing what happens as $\varepsilon \to 0$. By the standard integral estimate (2.16)

$$\sum_{m\geq 1} m^{-1-\varepsilon} \leq \int_1^\infty x^{-1-\varepsilon}\,\mathrm{d}x = \frac{1}{\varepsilon}.$$

Thus, we obtain the following estimates for the sums

$$\sum_{m\geq 1} a_m^2 = \sum_{m\geq 1} b_m^2 = \frac{1}{\varepsilon} + O(1), \tag{2.19}$$

where $O(1)$ is some function that is bounded as $\varepsilon \to 0$. Similarly,

$$\sum_{n,m\geq 1}\frac{a_m b_n}{m+n} \geq \int_1^\infty\int_1^\infty (xy)^{-(1+\varepsilon)/2}\frac{\mathrm{d}x\,\mathrm{d}y}{x+y} = \int_1^\infty x^{-1-\varepsilon}\int_{1/x}^\infty u^{-(1+\varepsilon)/2}\frac{\mathrm{d}u\,\mathrm{d}x}{1+u}.$$

We now need an estimate for the last integral. A standard calculation shows

$$\int_0^{1/x}\frac{u^{-(1-\varepsilon)/2}}{1+u}\,\mathrm{d}u \leq \int_0^{1/x} u^{-(1+\varepsilon)/2}\,\mathrm{d}u = \frac{2}{1-\varepsilon}x^{(\varepsilon-1)/2} < \frac{x^{-1/2}}{1/2}.$$

Hence the error in replacing the lower limit in the inner integral by 0 is less than $x^{-\alpha}/\alpha$, where α is positive and independent of ε. Integration gives

$$\int_1^\infty x^{-1-\varepsilon}\cdot x^{-\alpha}/\alpha\,\mathrm{d}x = \frac{1}{\alpha(\alpha+\varepsilon)} < \frac{1}{\alpha^2}.$$

Using this and Lemma A.18 we obtain

$$\begin{aligned}\sum_{n,m\geq 1}\frac{a_m b_n}{m+n} &= \int_1^\infty x^{x-1-\varepsilon}\left(\int_0^\infty u^{-(1+\varepsilon)/2}\frac{\mathrm{d}u}{1+u}+O(x^{-\alpha}/\alpha)\right)\\ &= \frac{1}{\varepsilon}\frac{\pi}{\sin\pi/2}+O(1/\alpha^2) = \frac{1}{\varepsilon}\left\{\frac{\pi}{\sin\pi/2}+o(1)\right\},\end{aligned} \tag{2.20}$$

for sufficiently small ε. Here $o(1)$ is some constant that tends to 0 as $\varepsilon\to 0$, to be precise $o(1)=\varepsilon\cdot O(1/\alpha^2)$. Combining equations (2.19) and (2.20) we obtain

$$\sum_{n,m\geq 1}\frac{a_m b_n}{m+n}\geq \pi\left(\sum_{n\geq 1}a_m^2\right)^{\frac{1}{2}}\left(\sum_{n\geq 1}b_n^2\right)^{\frac{1}{2}},$$

for ε sufficiently small. This proves that π is the best possible constant for (2.18). □

Corollary 2.6 (Hilbert's inequality). *Let a_n, b_n be real square summable sequences and $1<p<\infty$. If $q\in\mathbb{R}$ satisfies $1/p+1/q=1$. Then,*

$$\sum_{m,n\geq 1}\frac{a_m b_n}{n+m}\leq\frac{\pi}{\sin\pi/p}\Big(\sum_{n\geq 1}a_n^p\Big)^{\frac{1}{p}}\Big(\sum_{n\geq 1}b_n^q\Big)^{\frac{1}{q}}, \tag{2.21}$$

and $\pi/\sin(\pi/p)$ is the best possible constant.

Proof. To prove the more general case we may apply Hölders inequality for sums

$$\begin{aligned}\sum_{m,n\geq 1}\frac{a_m b_n}{n+m} &\leq \left(\sum_{m\geq 1}a_m^p\sum_{n\geq 1}\frac{(m/n)^{1/q}}{n+m}\right)^{\frac{1}{p}}\left(\sum_{n\geq 1}b_n^q\sum_{m\geq 1}\frac{(n/m)^{1/p}}{n+m}\right)^{\frac{1}{q}}\\ &\leq\left(\frac{\pi}{\sin\pi/q}\right)^{\frac{1}{p}}\left(\frac{\pi}{\sin\pi/p}\right)^{\frac{1}{q}}\left(\sum_{m\geq 1}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n\geq 1}b_n^q\right)^{\frac{1}{q}}\\ &\leq\frac{\pi}{\sin\pi/p}\left(\sum_{m\geq 1}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n\geq 1}b_m^q\right)^{\frac{1}{q}}.\end{aligned}$$

The first equality follows from the integral estimate and Lemma A.18, the second from equation (A.21). Proving that $\pi/\sin(\pi/p)$ is the best constant, can be done similarly as in the case $p=2$, now with the sequences

$$a_n = m^{-(1+\varepsilon)/p}\,,\quad b_n = n^{-(1+\varepsilon)/q}\,,\qquad \varepsilon>0\,. \tag{2.22}$$

See Hardy, Littlewood, and Pólya [20, p. 232] for the details. □

2.3. Nehari's theorem and weak product spaces

In this section we aim to introduce and prove Nehari's theorem, and use it to study the strongest form of Hilbert's' inequality. From Proposition 2.2 we have that the Hankel form

$$\rho(a,b) = \sum_{m,n\geq 0} a_m b_n \rho_{m+n}\,,$$

is bounded if the symbol

$$\varphi(z) = \sum_{n\geq 0} \overline{\rho_n} z^n\,,$$

is bounded in the essential supremum norm. Nehari's theorem gives the reverse implication, namely that every bounded Hankel form has a bounded symbol, and the smallest such symbol coincides with the norm. As we will see this is equivalent to that $H^1(\mathbb{T})$ admits weak factorization. Before we can give the formal definition of Nehari's theorem, we need to introduce the following projection.

Definition (Riesz projection). We define, $P_+\colon L^2(\mathbb{T}) \to H^2(\mathbb{T})$, by

$$P_+f(z) = \sum_{k\geq 0} c_n z^k\,, \qquad z\in\mathbb{T}\,,$$

as the *Riesz projection.* Here $f(z) = \sum_{k\in\mathbb{Z}} c_k z^k$ is a complex-valued function on $\mathbb{T}$, and $\{c_k\}_{k\in\mathbb{Z}}$ are the Fourier coefficients of $f\in L^p(\mathbb{T})$.

Lemma 2.7. *The Riesz projection P_+, is a nonzero orthogonal projection. In other words $\|P\| = 1$.*

Proof. It is clear from the definition of the Riesz projection that $P_+^2 = P_+$ and $\langle P_+f, g\rangle_{L^2} = \langle f, P_+y\rangle_{L^2}$ holds for all $f, g \in L^2$. Thus showing that P_+ is an orthogonal projection from L^2 to H^2. For the second statement let $f\in H^2$ and $P_+f\neq 0$, then the use of the Cauchy-Schwarz inequality implies that

$$\|P_+f\| = \frac{\langle P_+f, P_+f\rangle}{\|P_+\|} = \frac{\langle f, P_+^2 f\rangle}{\|P_+f\|} = \frac{\langle f, P_+f\rangle}{\|P_+f\|} \leq \|f\|_{L^2}\,.$$

Therefore $\|P_+\|\leq 1$. However if $P_+\neq 0$, then there exists an $f\in L^2$ with $P_+f\neq 0$ and $\|P_+(P_+f)\| = \|P_+\| \geq 1$. ☐

Theorem 2.8 (Nehari's Theorem). *The Hankel form H_φ is bounded on H^2 if and only if there exists a function $\psi\in L^\infty$ such that $P_+\psi = \varphi$ and*

$$\|H_\varphi\| = \inf_{P_+\psi=\varphi} \|\psi\|_{L^\infty}, \tag{2.23}$$

where $P_+\colon L^2\to H^2$ is the Riesz projection.

2.3.1. Weak product spaces

Weak products on Hardy spaces have their origin in the work of Coifman, Rochberg, and Weiss [13]. We define the weak product $H^2 \odot H^2$ as the Banach space completion of the finite sums $f = \sum_k g_k h_k$ where $f_k, g_k \in H^2$, under the norm

$$\|f\|_{H^2 \odot H^2} := \inf \sum_k \|g_k\|_{H^2} \|h_k\|_{H^2} \,, \tag{2.24}$$

where the infimum is taken over all finite representations of f as a sum of products. In other words $H^2 \odot H^2$ is the closure of all finite sums $f = \sum_j g_j h_j$, for $g_j, h_j \in H^2$ under the norm (2.24).

Proposition 2.9. *Suppose that $g \in H^p$ and $h \in H^q$ with $1 < p \leq q < \infty$ and $1/p + 1/q = 1/s \leq 1$. Then $\|gh\|_{H^s} \leq \|g\|_{H^p} \|h\|_{H^q}$, and $gh \in H^s$.*

Proof. This follows directly from Hölders inequality

$$\|gh\|_{H^s} \leq \|g\|_{H^p} \cdot \|h\|_{H^q}, .$$

Since $g \in H^p$ and $h \in H^q$, then $\|g\|_{H^p} \|h\|_{H^q} < \infty$ and so $gh \in H^s$. □

Proposition 2.9 allows us now to prove the following tautology.

Proposition 2.10. *Let $H^2 \odot H^2$ be defined as above. Then $H^1 = H^2 \odot H^2$, meaning every function in H^1 lies in $H^2 \odot H^2$ and vice versa. In addition the norms are equal:*

$$\|f\|_{H^2 \odot H^2} = \|f\|_{H^1} \,.$$

Proof. Proposition 2.9 proves the inclusion $H^2 \odot H^2 \subset H^1$. However from Corollary 1.17, every function f in H^1 can be written as a product gh, where g and h are functions in H^2. This proves the inclusion $H^1 \subset H^2 \odot H^2$, and thus we have $H^1 = H^2 \odot H^2$. Corollary 1.17 says that every function $f \in H^1$ we have $\|f\|_{H^1} = \|g\|_{H^2} \|h\|_{H^2}$. Thus,

$$\|f\|_{H^1} = \|g\|_{H^2} \|h\|_{H^2} \geq \inf_j \sum_j \|g_j\|_{H^2} \|h_j\|_{H^2} = \|f\|_{H^2 \odot H^2} \,.$$

From the definition we have $f = \sum_j g_j h_j$, and the triangle inequality

$$\|f\|_{H^1} = \Big\|\sum_j g_j h_j\Big\|_{H^1} \leq \sum_j \|g_j\|_{H^2} \|h_j\|_{H^2} \,.$$

Taking the infinum with respect to all representations g_j, h_j proves that $\|f\|_{H^1} \leq \|f\|_{H^2 \odot H^2}$. □

2.3.2. Nehari's Theorem

LEMMA 2.11. *Suppose that φ generates a Hankel form on $H^2 \times H^2$. Then,*

$$\|H_\varphi\| = \|\varphi\|_{(H^2 \odot H^2)^*} . \tag{2.25}$$

PROOF. By explicitly writing out both sides of equation (2.25), we have

$$\|H_\varphi\| = \sup_{f,g \in H^2} \frac{\left|\langle fg, \varphi\rangle_{H^2}\right|}{\|f\|_{H^2}\|g\|_{H^2}} , \tag{2.26}$$

$$\|\varphi\|_{(H^2 \odot H^2)^*} = \sup_{f \in H^2 \odot H^2} \frac{\left|\langle f, \varphi\rangle\right|}{\|f\|_{(H^2 \odot H^2)^*}} . \tag{2.27}$$

To prove 2.11 we will first prove that (2.26) is greater or equal to equation (2.27), and then prove the reverse inequality. Let $f \in H^2 \odot H^2$, since $f = \sum_j g_j h_j$, we get

$$|\langle f, \varphi\rangle_{H^2}| \leq \sum_j \left|\langle g_i h_i, \varphi\rangle_{H^2}\right| \leq \|H_\varphi\| \sum_j \|g_i\|_{H^2}\|h_i\|_{H^2} .$$

Where the first inequality follows from the definition of f and the triangle-inequality, and the second from (2.26). By taking the infinum over all finite representations we obtain

$$|\langle f, \varphi\rangle_{H^2}| \leq \|H_\varphi\|\|f\|_{H^2 \odot H^2} ,$$

by the definition of the norm of $H^2 \odot H^2$. This proves the inequality $\|\varphi\|_{(H^2 \odot H^2)^*} \leq \|H_\varphi\|$. Similarly,

$$\begin{aligned}\left|\langle gh, \varphi\rangle_{H^2}\right| &\leq \|gh\|_{H^2 \odot H^2}\|\varphi\|_{(H^2 \odot H^2)^*} \\ &\leq \|g\|_{H^2}\|h\|_{H^2}\|\varphi\|_{(H^2 \odot H^2)^*} .\end{aligned}$$

First inequality follows from equation (2.27), and the second from the definition of $H^2 \odot H^2$. Proposition 2.10. Since $\inf \sum_j \|g_j\|\|h_j\| \leq \|g\|_{H^2}\|f\|_{H^2}$. This proves that $\|H_\varphi\| \leq \|\varphi\|_{(H^2 \odot H^2)^*}$ and thus our claim is proven. ☐

Before we can prove Nehari's theorem we need to recall a few general concepts about Banach space from functional analysis. Let X be a Banach space, and let S be a closed subspace. A *coset of X modulo S* is a subset $\xi = x + S$ consisting of all $x + y$, where x is some fixed member of X and $y \in S$. Two cosets are either identical or disjoint. The *quotient space* X/S has as ts elements all distinct cosets of X modulo S. Finally, the norm of a coset $\xi = x + S$ is defined by

$$\|\xi\| = \inf_{y \in S} \|x + u\| . \tag{2.28}$$

Under this given norm, X/S is complete, and therefore itself a Banach space.

The *annhilator* of the subspace S is the set $S^\perp$ of all linear functionals $\varphi \in X^*$ such that $\varphi(x) = 0$ for all $x \in S$. It can be verified that $S^\perp$ is a subspace of X^*.

PROPOSITION 2.12 (Duren [15, p. 113]). *The space $(X/S)^*$ is isometrically isomorphic to $S^\perp$. Furthermore, for each fixed $x \in X$,*

$$\max_{\varphi \in S^\perp, \|\varphi\| \leq 1} \left|\varphi(x)\right| . = \inf_{y \in S} \|x + y\| ,$$

Where max indicates that the supremum is attained.

PROOF. For each fixed $\varphi \in S^*$, the class of all extensions $\psi \in X^*$ is a coset in $X^*/S^\perp$. It is clear that this correspondence between S^* and $X^*/S^\perp$ is an isomorphism. In fact, $\|\varphi\| \leq \|\psi\|$ for every extension ψ; and, by the Hahn-Banach theorem, there is at least one extension for which $\|\varphi\| = \|\psi\|$. Thus, for the coset of extensions of φ, the infimum defining the norm is attained, and is equal to $\|\varphi\|$. $\square$

As we saw in Chapter 1, the polynomials are dense in H^p, if $1 \leq p < \infty$ and H^p is a Banach space. If $1 \leq p \leq \infty$, the set of boundary functions of H^p is the subspace of L^p for which

$$c_n = \int_0^{2\pi} e^{in\theta} f(e^{i\theta})\,\mathrm{d}\theta = 0\,, \quad n = 1, 2, \dots\,,$$

the negative Fourier-coefficients vanish. In particular if each $f \in H^p$ is identified with its boundary function, H^p can be regarded as a subspace of L^p. According to the Riesz representation theorem, every bounded linear functional ψ on L^p $(1 \leq p < \infty)$ has a unique representation

$$\psi(f) = \int_{\mathbb{T}} f g\,\mathrm{d}m\,, \qquad g \in L^q\,, \tag{2.29}$$

where $1/p + 1/q = 1$. In fact, $\|\psi\| = \|g\|_q$, and $(L^p)^*$ is isometrically isomorphic to L^q. Since H^p is a subspace of L^p, then Proposition 2.12 can be used to describe $(H^p)^*$ if the annhilator of H^p in $(L^p)^*$ can be determined. But if $g \in L^q$ annhilates every H^p function, then surely

$$\int_0^{2\pi} e^{in\theta} g(e^{i\theta})\,\mathrm{d}\theta = 0 \quad n = 1, 2, \dots\,.$$

Therefore $g(e^{i\theta})$ is the boundary function of some $g(z) \in H^q$, and $g(0) = 0$. We will denote this class of functions as H_0^q. Conversely, if $g \in H_0^q$, it follows that

$$\int_{\mathbb{T}} f g\,\mathrm{d}m = 0\,,$$

for every $f \in H^p$. Hence H_0^q is the annhiliator of H^p, and it follows form Proposition 2.12, that $(H^p)^*$ is isometrically isomorphic to L^q/H_0^q. Actually, we can do a little better and replace L^q/H_0^q by L^q/H^q, since the correspondence $\xi \leftrightarrow e^{i\theta}\xi$ between cosets of the two spaces itself is an isometric isomorphism. In summary:

LEMMA 2.13. *For $1 \leq p < \infty$, the space $(H^p)^*$ is isometrically isomorphic to L^q/H^q, where $1/p + 1/q = 1$.*

PROOF OF NEHARI'S THEOREM 2.8. Combining Propositions 2.10 and 2.10 we see that

$$\|H_\varphi\| = \|\varphi\|_{(H^1)^*}\,.$$

From Lemma 2.13 we have shown that the dual space of H^1 is isometrically isomorphic to L^∞/H^∞, Since this is a subspace of L^∞ the Hahn-Banach theorem A.21 states that φ extends to a bounded linear functional on L^∞. Thus proving that

$$\|H_\varphi\| = \inf_{P_+\psi = \varphi} \|\psi\|_{L^\infty}\,, \tag{2.30}$$

where $P_+ \colon L^2 \to H^2$ is the Riesz projection. This completes the proof. $\square$

2.3.3. Bounded symbol of the Hilbert matrix

Let us indicate an alternative proof (in fact, the original approach of Hilbert) of the fact that the usual Hilbert matrix has norm π. The strongest version of Hilbert's inequality is

$$\left|\sum_{\substack{m+n>0\\ m\neq n}} \frac{a_m b_n}{m+n}\right| \leq C\left(\sum_{m\geq 0}|a_m|^2\right)^{\frac{1}{2}}\left(\sum_{n\geq 0}|b_n|^2\right)^{\frac{1}{2}},, \tag{2.31}$$

which also can be stated for two-tailed sequences $\{a_m\}_{m\in\mathbb{Z}}$ and $\{b_n\}_{n\in\mathbb{Z}}$. Similar to the classical Hilbert inequality the symbol to equation (2.31) is

$$\varphi(\theta) = \sum_{n\geq 1}\frac{1}{n}e^{in\theta} = \sum_{n\geq 1}\hat{\varphi}(n)e^{in\theta} = \log\bigl(1-e^{i\theta}\bigr)\,.$$

Note however that gives the bound $C \leq \|\varphi\|_{L^\infty} = \infty$. To obtain a better bound we note that since $\varphi \in H^2$ we have $\hat{\varphi}(n) = 0$ for all $n = -1, -2, \ldots$. Thus we may add as many negative Fourier coefficients to φ and still have a symbol for (2.31). By adding every negative Fourier coefficient we obtain

$$\psi(\theta) = \sum_{n=-\infty}^{-1}\frac{1}{n}e^{in\theta} + \sum_{n=1}^{\infty}\frac{1}{n}e^{in\theta} = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{1}{n}e^{in\theta}\,. \tag{2.32}$$

Thus, on one hand we have

$$\varphi(\theta) = P_+\psi(\theta)\,, \tag{2.33}$$

while on the other

$$\psi(\theta) = 2i\sum_{n\geq 1}\frac{\sin(n\theta)}{n} = i(\pi-\theta)\,.$$

Here the first equality follows from splitting and regrouping the series, while the second follows from Proposition A.15. We thus obtain the following bound

$$\|H\| \leq \|\psi\|_{L^\infty} = \pi\,.$$

Similar what was done in section 2.2 we can prove that the constant π in equation (2.31) is sharp, by stressing the inequality. Since the inequality is sharp, Nehari's theorem tells us that we have found a function $\psi \in L^2$ such that $\varphi = P_+\psi$ and $\|H_\varphi\| = \|\psi\|_{L^\infty} = \pi$.

2.3.4. Nehari's theorem on the polydisc

We briefly mention that the proof above can be extended into the polydisc $d > 1$, and as in the one dimensional case the following two statements are equivalent:

- H_ψ is a bounded Hankel form if and only if the symbol φ in $L^\infty(\mathbb{T}^d)$ is bounded.
- $H^1(\mathbb{T}^d)$ admits weak factorization.

Factorization on the polydisc $\mathbb{D}^d$ is however a much subtler matter. That $H^1(\mathbb{D}^d)$ admits a weak factorization is a a highly nontrivial result that was first proven for $d = 2$ by Ferguson and Lacey [17], and extended to $d > 2$ by Lacey and Terwilleger [30].

THEOREM 2.14 (Ferguson–Lacey, Lacey–Terwilleger). *$H^1(\mathbb{D}^d)$ admits weak factorization, in other words*

$$H^1(\mathbb{D}^d) = H^2(\mathbb{D}^d) \odot H^2(\mathbb{D}^d)\,, \qquad 1 < d < \infty\,.$$

Our purpose is to explore to explore whether the two statements above are equivalent in the infinite dimensional polydisc $\mathbb{T}^\infty$.

CHAPTER 3

The Hardy space of Dirichlet series

In this chapter we will study Dirichlet series of the form

$$f(s) = \sum_{n\geq 1} a_n n^{-s}\,, \tag{3.1}$$

where $s = \sigma + it$ is a complex variable. Such series has a long history beginning in the nineteenth century, and the interest was due mainly to the central role that Dirichlet series play in analytic number theory. The general theory of Dirichlet series was developed by Hadamard, Landau, Hardy, Riesz, Schnee, and Bohr, to name a few. However, this research took place before the modern interplay between function theory and functional analysis, as well as the advent of the field of several complex variables, and thus the field was in many ways dormant until the late 1990s [46].

Much renewed interest in Dirichlet series is due to the 1997 paper of Hedenmalm, Lindqvist, and Seip [24] which introduced $\mathscr{H}^2$, the Hilbert space of Dirichlet series with square summable coefficients[1]. This chapter starts with the study this classical space. Then he Bohr correspondence is introduced, which we will use to create an analouge space to H^p for Dirichlet series.

3.1. Preliminaries

Similar to Chapter 1, we will briefly recall some classical facts about the Dirichlet series. Our main reference is Apostol [2, Chp. XI].

Definition. An *arithmetical function* is a function $f : \mathbb{N} \to \mathbb{C}$.

Definition. An arithmetical function f is called multiplicative if f is not identically zero and if

$$f(mn) = f(m)f(n), \qquad \gcd(m,n) = 1\,. \tag{3.2}$$

A multiplicative function f is called *completely multiplicative* if we also have

$$f(mn) = f(n)f(m), \qquad m,n \in \mathbb{N}.$$

Remark. We note that the function $f(n) = n^{-s}$, where s is a fixed real or complex number is completely multiplicative. Fixing $n = 1$, we see that (3.2) implies that $f(1) = 1$.

[1]Note that Hedenmalm, Lindqvist, and Seip used the notation for $\mathscr{H}$ for this space. After the work of Bayart [4] the notation changed to $\mathscr{H}^2$.

The study of half-planes will be important in this chapter, and for that reason we introduce the notation

$$\mathbb{C}_\theta := \{s = \sigma + it\colon \sigma > \theta\}, \tag{3.3}$$

where θ can be any real number. In contrast to the power series the regions of convergence for Dirichlet series differ when we consider pointwise convergence, uniform convergence or absolute convergence. Given $f(s)$ we can define at least three abcissas σ_c , σ_u, σ_a of convergence.

(1) σ_c is the smallest σ such that $f(s)$ is convergent in $\mathbb{C}_{\sigma_c}$.
(2) σ_u is the smallest σ such that $f(s)$ converges *uniformly* in $\mathbb{C}_{\sigma_u+\varepsilon}$ for any $\varepsilon > 0$.
(3) σ_a is the smallest σ such that $f(s)$ converges *absolutely* in $\mathbb{C}_{\sigma_a}$.

THEOREM 3.1. *If the series $\sum_{n\geq 1} a_n n^{-s}$ does not converge everywhere or diverge everywhere, then there exists a real number σ_c, called the abscissa of convergence, such that the series converges for all s in the half-plane $\sigma > \sigma_c$ and diverges for all s in the half-plane $\sigma < \sigma_c$.*

THEOREM 3.2. *Suppose the series $\sum |a_m n^{-s}|$ does not converge for all s or diverge for all s. Then there exists a real number σ_a called the abscissa of absolute convergence, such that the series $\sum a_m n^{-s}$ converges absolutely if $\sigma > \sigma_a$, but does not converge absolutely if $\sigma < \sigma_a$.*

Since absolute convergence implies convergence we have trivially $-\infty \leq \sigma_c \leq \sigma_u \leq \sigma_a \leq \infty$. See figure 1 for a comparison of the different abscissas.

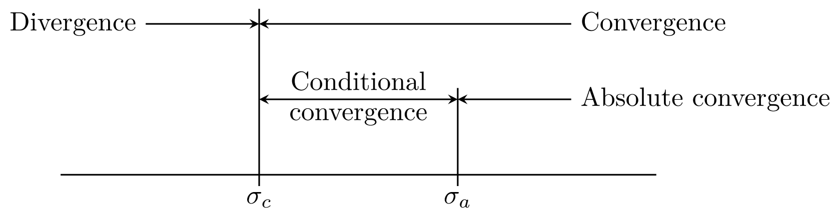

FIGURE 1.

REMARK. The function $f(s)$ may continue analytically in a region bigger than $\mathbb{C}_{\sigma_c}$. Let $g(s) = \sum_{n\geq 1}(-1)^n n^{-s} = 2\zeta(s)(2^{-s} - 1/2)$, then $\sigma_c = 0$, but $\sigma_a = \sigma_u = 1$.

We want to clearify what the relation between σ_u and σ_a is. Of relevance is the *abscissa of regularity and boundedness* which was studied by Bohr:

(4) σ_b is the smallest σ such that $f(s)$ converges at some point s and it is bounded in $\mathbb{C}_{\sigma_b+\varepsilon}$ for any $\varepsilon > 0$.

Bohr [7] proved further that

LEMMA 3.3. *For all Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$, we have $\sigma_u = \sigma_b$.*

In particular this means that whenever we have an analytic function f that coincides with a Dirichlet series in a half plane $\mathbb{C}_a$ and it is holomorphic and bounded up to $\mathbb{C}_b$ with $b < a$, then the Dirichlet series converges uniformly to f up to $\mathbb{C}_b$.

3.2. The Hardy-Hilbert space $\mathscr{H}^2$

As done by Hedenmalm, Lindqvist, and Seip [24], we introduce the space

$$\mathscr{H}^2 := \left\{ \sum_{n\geq 1} a_n n^{-s} : \sum_{n\geq 1} |a_n|^2 < +\infty \right\}.$$

In other words $\mathscr{H}^2$ is the Hilbert space of Dirichlet series with square summable coefficients, and is a natural analogue of $H^2(\mathbb{T})$ for Dirichlet series.

DEFINITION. Given two Dirichlet series $f(s) = \sum a_n n^{-s}$ and $g(s) = \sum \overline{b_m} m^{-s}$ in $\mathscr{H}^2$. We define

$$\langle f, g\rangle_{\mathscr{H}^2} = \sum_{m,n\geq 1} a_n b_m \tag{3.4}$$

as the inner product on $\mathscr{H}^2$. This induces the following norm on $\mathscr{H}^2$

$$\|f\|_{\mathscr{H}^2} := \langle f, f\rangle_{\mathscr{H}^2} = \sum_{n\geq 1} |a_n|^2. \tag{3.5}$$

In Theorem 1.10 we showed the basic properties of the Hardy space. The purpose of this section is to develop similar results for $\mathscr{H}^2$

THEOREM 3.4. *Let $f(s) = \sum_{m\geq 1} a_m n^{-s}$, $g(s) = \sum_{n\geq 1} a_n n^{-s}$ be in $\mathscr{H}^2$. Then*

(1) The largest half-plane of convergence for a Dirichlet series $f(s)$ in $\mathscr{H}^2$ is $\mathbb{C}_{1/2}$. Meaning there exists Dirichlet series in $\mathscr{H}^2$ that does not converge in the any half-plane bigger than $\mathbb{C}_{1/2}$.

(2) We have the point estimate

$$|f(\sigma + it)| \leq \zeta(2\sigma)^{1/2} \|f\|_{\mathscr{H}^2}.$$

(3) The space $\mathscr{H}^2$ is the closure of Dirichlet polynomials $P(s) = \sum_{n=1}^{N} a_n n^{-s}$ under the norm

$$\|P\|_{\mathscr{H}^2} := \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |P(it)|^2 \,\mathrm{d}t \right)^{\frac{1}{2}} \tag{3.6}$$

(4) For $\sigma \geq 1/2$ we have the following embedding inequalities

$$\frac{1}{\pi}\int_{\mathbb{R}} |f(1/2+it)|^2 \frac{\mathrm{d}t}{1+t^2} \leq \widetilde{C}\|f\|^2_{\mathscr{H}^2},$$

$$\sup_{\tau\in\mathbb{R}} \int_{\tau}^{\tau+1} |f(\sigma+it)|^2 \,\mathrm{d}t \leq C^2 \|f\|^2_{\mathscr{H}^2}.$$

By applying Cauchy–Schwarz on (3.1) we obtain two important properties

$$|f(s)| \leq \sum_{n\geq 1} \left|a_n n^{-s}\right| \leq \left(\sum_{n\geq 1} |a_n|^2\right)^{\frac{1}{2}} \left(\sum_{n\geq 1} n^{-2\sigma}\right)^{\frac{1}{2}}. \tag{3.7}$$

Since $n^s = e^{s\log n} = e^{(\sigma+it)\log n} = n^\sigma e^{it\log n}$, implies $|n^s| = |n^\sigma|$. So the abscissa of absolute convergence is at most $1/2$ for Dirichlet series in $\mathscr{H}^2$. In addition this shows the point-estimate

$$|f(s)| \leq |\zeta(2\sigma)|^{1/2}\|f\|_{\mathscr{H}^2}\,,$$

where again $s = \sigma + it$ is a complex variable. To prove that $\mathbb{C}_{1/2}$ is the largest half-plane of convergence for $\mathscr{H}^2$ consider $f(s) = \zeta(1/2+\varepsilon+s) \in \mathscr{H}^2$, where $\varepsilon > 0$. Then f converges in the half-plane $\mathbb{C}_{1/2+\varepsilon}$ and diverges otherwise. Before we can define the inner product on $\mathscr{H}^2$ it will be useful to show the following:

THEOREM 3.5. *Given two Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$ and $g(s) = \sum_{m\geq 1} \overline{b_m} n^{-s}$ with abscissae of absolute convergence σ_1 and σ_2, respectively. Then for all $\alpha > \sigma_1$ and $\beta > \sigma_2$ we have*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(\alpha+it)\overline{g(\beta+it)} = \sum_{n\geq 1} \frac{a_n b_n}{n^{\alpha+\beta}} \tag{3.8}$$

PROOF. Expanding we have

$$\begin{aligned} f(a+it)\overline{g(b+it)} &= \left(\sum_{n\geq 1} \frac{a_m}{m^{\alpha+it}}\right)\left(\sum_{n\geq 1} \frac{b_n}{n^{\beta-it}}\right) = \sum_{n,m\geq 1} \frac{a_m b_n}{m^\alpha n^\beta}\left(\frac{n}{m}\right)^{it} \\ &= \sum_{n\geq 1} \frac{a_n b_n}{n^{\alpha+\beta}} + \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{a_m b_n}{m^\alpha n^\beta}\left(\frac{n}{m}\right)^{it} \end{aligned}$$

Now by the triangle-inequality

$$\sum_{m,n\geq 1} \left|\frac{a_m b_n}{m^a n^b}\right| \leq \sum_{m\geq 1} \frac{|a_m|}{m^\alpha} \sum_{n\geq 1} \frac{|b_n|}{n^\beta}\,,$$

so the series is absolute convergent, and this convergence is also uniform for all t. Hence we may integrate term by term and divide by $2T$ to obtain

$$\int_{-T}^{T} f(\alpha+it)g(\beta-it)\frac{\mathrm{d}t}{2T} = \sum_{n\geq 1} \frac{a_n b_n}{n^{\alpha+\beta}} + \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{a_m \overline{b_n}}{m^a n^a} \int_{-T}^{T} e^{it\log(n/m)}\frac{\mathrm{d}t}{2T} \tag{3.9}$$

For $m \neq n$ we can write the last integral as

$$\frac{1}{2T}\int_{-T}^{T} e^{it\log(n/m)}\,\mathrm{d}t = \frac{\sin\big[T\log(n/m)\big]}{T\log(n/m)} = O\left(\frac{1}{T}\right)$$

Again the double series converges uniformly with respect to T since $(\sin x)/x$ is bounded for every x. Hence, we can let $T \to \infty$ in equation (3.9) to obtain the statement of the theorem. □

COROLLARY 3.6 (Carlson's theorem). *Let $f(s) = \sum a_n n^{-s}$ be analytic in $\mathbb{C}_0$ and bounded in every half-plane* $\mathrm{Re}(s) > \delta$ *with $\delta > 0$. Then, for each $\delta > 0$,*

$$\sum_{n\geq 1} |a_n|^2 n^{-2\sigma} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(\sigma+it)|^2\,\mathrm{d}t\,. \tag{3.10}$$

Letting $\sigma \to 0$ in (3.10), and comparing with (3.5) immediately gives the corollary.

COROLLARY 3.7. *If* $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ *is convergent and bounded in* $\mathbb{C}_0$*, then* $f \in \mathscr{H}^2$ *and*

$$\|f\|_{\mathscr{H}^2}^2 = \lim_{\sigma\to 0}\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(\sigma+it)\overline{f(\sigma+it)}\,\mathrm{d}t\right)^{1/2}. \tag{3.11}$$

Note that this together with Lemma 3.3, implies Bohr's inequality $|\sigma_a - \sigma_u| \leq 1/2$. However as the largest halfplance of convergence for Dirichlet series in $\mathscr{H}^2$ is $\mathbb{C}_{1/2}$, we introduce $\mathscr{P}$ the set of all *Dirichlet polynomials*,

$$P(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \qquad s \in \mathbb{C}\,.$$

As the Dirichlet polynomials converges in $\mathbb{C}_+$, and is bounded we obtain from Corollary 3.7 the following equivalent definition of $\mathscr{H}^2$

DEFINITION. The space $\mathscr{H}^2$ is the closure of Dirichlet polynomials $P(s) = \sum_{n=1}^{N} a_n n^{-s}$ under the norm

$$\|P\|_{\mathscr{H}^2} := \left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |P(it)|^2\,\mathrm{d}t\right)^{\frac{1}{2}} \tag{3.12}$$

3.2.1. The embedding constant

As mentioned in the introduction, functions in $\mathscr{H}^2$ are analytic in the half-plane $\mathbb{C}_{1/2}$. It is therefore interesting to investigate how they behave along the along the abscissa $\sigma = 1/2$. In this context, the most important question is the embedding problem, first considered implicitly by Montgomery and Vaughan, and addressed again by Hedenmalm, Lindqvist, and Seip. We will see in this section a practical application of a bilinear form to obtain a sharp estimate for one such embedding. The Embedding inequality can be formulated as follows:

THEOREM 3.8. *The embedding inequality*

$$\sup_{\tau\in\mathbb{R}}\int_{\tau}^{\tau+1} |f(\sigma+it)|^2\,\mathrm{d}t \leq C\|f\|_{\mathscr{H}^2}^2\,, \tag{3.13}$$

holds for every f *in* $\mathscr{H}^2$, $\sigma > 1/2$ *and* C *is a constant independent of* σ.

An important consequence of this theorem is that it shows that the Dirichlet series in $\mathscr{H}^2$ are locally L^2-integrable on the line $\mathrm{Re}(s) = 1/2$. It suffices to obtain (3.13) for finite Dirichlet series f, since on compact subsets of $\mathbb{C}_{1/2}$, elements of $\mathscr{H}^2$ are uniformly approximated by them. Moreover, by the Poisson integral formula, we see that it suffices to consider the limit case $\sigma = 1/2$. Thus the embedding inequality may be restated as:

$$\left(\int_{\tau}^{\tau+1} |f(1/2+it)|^2\right)^{1/2} \mathrm{d}t \leq C\|f\|_{\mathscr{H}^2} \tag{3.14}$$

There are several proof of equation (3.14) in standard litterature, Hedenmalm, Lindqvist, and Seip used a version of the classical Plancherel–Polya inequality [24, Thm. 4.11], while Olsen and Saksman prefered methods from Fourier analysis [35, pp. 36–37]. Lastly one may also prove the inequality using a general Hilbert–type inequality due to Montgomery and Vaughan [31]. It should be pointed out that these proofs do not give a precise value for the constant.

However, by employing the embedding of $\mathscr{H}^2$ into the conformally invariant Hardy space of $\mathbb{C}_{1/2}$, Brevig [8] was able to obtain an optimal value for the equivalent embedding:

THEOREM 3.9 (The Embedding Inequality). *Suppose that* $f(s) = \sum_{n\geq 1}^{\infty} a_n n^{-s}$ *is in* $\mathscr{H}^2$. *Then*

$$\left(\frac{1}{\pi}\int_{\mathbb{R}} |f(1/2+ix)|^2 \frac{\mathrm{d}x}{1+x^2}\right)^{\frac{1}{2}} \leq \widetilde{C}\|f\|_{\mathscr{H}^2}\,, \tag{3.15}$$

and the constant $\widetilde{C} = \sqrt{2}$ *is optimal.*

We will first prove Theorem 3.9 and then show that equation (3.14) holds if and only if Theorem 3.9 is true.

The left-hand side of equation (3.15) is the norm of the conformally invariant Hardy space in the half plane $\mathbb{C}_{1/2}$, which we denote H_i^2. It consists of those functions f such that $f \circ \mathscr{T} \in H^2(\mathbb{T})$, where $\mathscr{T}$ is the following mapping from $\mathbb{D}$ to $\mathbb{C}_{1/2}$,

$$\mathscr{T}(z) := \frac{1}{2} + \frac{1-z}{1+z}$$

The shifted Cayley transform $\mathscr{T}$ appeared in the transference principle of Queffélec and Seip [40]. Now, the norm of H_i^2 can be evaluated as follows:

$$\begin{aligned}\|f\|_{H_i^2} &:= \|f\circ T\|_{H^2}\\ &= \left(\frac{1}{2\pi}\int_{\mathbb{T}} |f(1/2+i\tan(t/2)|^2\,\mathrm{d}t\right)^{\frac{1}{2}} = \left(\frac{1}{\pi}\int_{\mathbb{R}} |f(1/2+ix)|^2\frac{\mathrm{d}x}{1+x^2}\right)^{\frac{1}{2}},\end{aligned}$$

Hence the embedding inequality in equation (3.15) may be restated as

$$\|f\|_{H_i^2} \leq \widetilde{C}\|f\|_{\mathscr{H}^2} \tag{3.16}$$

To prove the embedding inequality 3.9 we begin by estimating the following Hilbert–type inequality.

LEMMA 3.10. *Let* $a = \{a_n\}_{n\geq 1}$, $b = \{b_n\}_{n\geq 1}$ *be sequences in* ℓ^2. *Then*

$$\sum_{m,n\geq 1} a_m\overline{b_n}\frac{\sqrt{mn}}{[\max(m,n)]^2} \leq 2\left(\sum_{m\geq 1}|a_m|^2\right)^{\frac{1}{2}}\left(\sum_{n\geq 1}|b_n|^2\right)^{\frac{1}{2}}, \tag{3.17}$$

where the constant 2 *is optimal.*

PROOF. This is a Hilbert–type (see [20, Ch. IX]) bilinear form, and may be proven in the same fashion as Hilbert's inequality studied in section 2.2. Denote the double sum as

$$B_{a,b} := \sum_{m,n\geq 1} a_m \overline{b_n} \frac{\sqrt{mn}}{[\max(m,n)]^2}$$

By the Cauchy–Schwarz inequality, we find

$$|B_{a,b}| \leq \left(\sum_{m\geq 1} |a_m|^2 \sum_{n\geq 1} \frac{m}{[\max(m,n)]^2} \right)^{\frac{1}{2}} \left(\sum_{n\geq 1} |b_n|^2 \sum_{m\geq 1} \frac{n}{[\max(m,n)]^2} \right)^{\frac{1}{2}}, \quad (3.18)$$

and by symmetry, we only need to consider one of the factors. Since $\max(n,m) = m$ for all $1 \leq n \leq m$ and $\max(n,m) = n$ for all $n \geq m$,

$$\sum_{n\geq 1} \frac{m}{[\max(m,n)]^2} = \sum_{n=1}^{m} \frac{m}{m^2} + \sum_{n=m+1}^{\infty} \frac{m}{n^2} < 1 + m \int_m^\infty \frac{\mathrm{d}x}{x^2} \overset{x\mapsto ym}{=} 1 + \int_1^\infty \frac{\mathrm{d}y}{y^2} = 2\,.$$

Applying this inequality to equation (3.18) gives (3.17). The only thing that remains is to prove that this constant 2 is optimal. Choose $a_n = \overline{b_n} = n^{-(1+\varepsilon)/2}$, then both sides of (3.17) converges for all $\varepsilon > 0$. Trivially,

$$\left(\sum_{m\geq 1} |a_m|^2 \right)^{\frac{1}{2}} \left(\sum_{n\geq 1} |b_n|^2 \right)^{\frac{1}{2}} = \frac{1}{\varepsilon} + O(1)\,, \quad (3.19)$$

by the integral estimate. The left-hand side may be evaluated similarly

$$\begin{aligned} B_{a,b} &> \int_1^\infty \left(\int_1^y + \int_y^\infty \right) (xy)^{-(1+\varepsilon)/2} \frac{\sqrt{xy}}{[\max(x,y)]^2} \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_1^\infty \left(\int_1^y x^{-\varepsilon/2} y^{-2-\varepsilon/2} \,\mathrm{d}x + \int_y^\infty x^{-2-\varepsilon/2} y^{-\varepsilon/2} \,\mathrm{d}x \right) \mathrm{d}y = \frac{4}{\varepsilon(\varepsilon+2)}\,. \end{aligned}$$

Comparing this with equation (3.19) yields

$$\sum_{m,n\geq 1} a_m b_n \frac{\sqrt{mn}}{[\max(m,n)]^2} \geq \frac{4}{\varepsilon+2} \left(\sum_{m\geq 1} |a_m|^2 \right)^{\frac{1}{2}} \left(\sum_{n\geq 1} |b_n| \right)^{\frac{1}{2}} + o(1)\,, \quad (3.20)$$

for $a_n = b_n = n^{-1/2-\varepsilon/2}$, and small enough ε. As equation (3.20) holds for every $\varepsilon > 0$ proves that the constant 2 is optimal. $\square$

PROOF OF THEOREM 3.9. Expanding we find

$$\|f\|_{H_i}^2 = \frac{1}{\pi} \int_{\mathbb{R}} |f(1/2+it)|^2 \frac{\mathrm{d}t}{1+t^2} = \sum_{n,m\geq 1} \frac{a_m \overline{a_n}}{\sqrt{mn}} \frac{1}{\pi} \int_{\mathbb{R}} \frac{(n/m)^{it}}{1+t^2} \,\mathrm{d}t\,. \quad (3.21)$$

Since $x^{it} = e^{it|\log x|}$ it follows from Lemma A.13 that

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{(n/m)^{it}}{1+t^2} \,\mathrm{d}t = e^{-|\log(m/n)|} = \frac{mn}{\max(m,n)^2} = \begin{cases} n/m & \text{if } m \geq n \\ m/n & \text{if } m < n \end{cases}. \quad (3.22)$$

Combining equation (3.21) with (3.22) and applying Lemma 3.10 we find

$$\|f\|_{H_i}^2 = \sum_{m,n\geq 1} a_m \overline{b_n <} \frac{\sqrt{nm}}{[\max(n,m)]^2} \leq 2\Big(\sum_{m\geq 1} |a_m|^2\Big)^{\frac{1}{2}} \Big(\sum_{n\geq 1} |b_n|^2\Big)^{\frac{1}{2}} = 2\|f\|_{\mathscr{H}^2}^2 \,.$$

Taking the square root completes the proof. That the constant $\sqrt{2}$ is optimal follows from equation (3.17). ☐

To conclude this section we will spend some time proving the equivalence of equation (3.15) and (3.14).

Proposition 3.11. *Let $f \in \mathscr{H}^2$. Then equation (3.14) holds if and only if equation (3.15) holds. In particular the constant C in (3.14) satisfies*

$$\frac{2}{\coth \pi} \leq C^2 \leq \frac{5\pi}{2} \,.$$

Proof. We start by proving that (3.14) $\Longrightarrow$ (3.15). Splitting the integral into intervals of length 1 gives

$$\begin{aligned} \|f\|_{H_i^2}^2 &= \frac{1}{\pi} \int_{\mathbb{R}} |f(1/2+it)|^2 \frac{\mathrm{d}t}{1+t^2} \\ &\leq \frac{1}{\pi} \sum_{k\in\mathbb{Z}} \frac{1}{1+k^2} \int_{-k}^{k} |f(1/2+it)|^2 \,\mathrm{d}t \leq \coth(\pi) C^2 \|f\|_{\mathscr{H}^2} \,, \end{aligned}$$

where the embedding constant C from (3.14) was used in the last inequality, and the last sum follows from $\sum_{k\in\mathbb{Z}} 1/(k^2+1^2) = a^{-1}\coth(a\pi)$ in Lemma A.14. To prove the reverse inequality (3.14) $\Longleftarrow$ (3.15) we assume that Theorem 3.9 holds. Then $\|f\|_{H_i}^2 \leq \widetilde{C} \|f\|_{\mathscr{H}^2}^2$, and

$$\frac{1}{\pi} \int_{-1/2}^{1/2} |f(1/2+it)|^2 \frac{\mathrm{d}t}{1+t^2} \leq 2\|f\|_{\mathscr{H}^2}^2 \,, \tag{3.23}$$

where $\widetilde{C} = \sqrt{2}$. By mapping equation (3.14) onto $[-1/2, 1/2]$ we have

$$\int_{\tau}^{\tau+1} |f(1/2+it)|^2 \,\mathrm{d}t = \int_{-1/2}^{1/2} |h_\tau(1/2+it)|^2 \,\mathrm{d}t \,, \tag{3.24}$$

where the shifted function $h_\tau(t) = f(t + i(1/2+\tau))$ was introduced. Thus,

$$\begin{aligned} \frac{1}{\pi} \frac{1}{1+1/2^2} \int_{\tau}^{\tau+1} |f(1/2+it)|^2 \,\mathrm{d}t &= \frac{1}{\pi} \int_{-1/2}^{1/2} \frac{|h_\tau(1/2+it)|^2}{1+(1/2)^2} \,\mathrm{d}t \\ &\leq \frac{1}{\pi} \int_{-1/2}^{1/2} \frac{|h_\tau(1/2+it)|^2}{1+t^2} \,\mathrm{d}t \leq 2\|g_\tau\|_{\mathscr{H}^2}^2 . \end{aligned}$$

Where equation (3.23) was used in the last equality. As $\|g_\tau\|_{\mathscr{H}^2} = \|f\|_{\mathscr{H}^2}$, we have shown that equation (3.15) implies (3.14). In particular we have shown that

$$\int_{\tau}^{\tau+1} |f(1/2+it)|^2 \,\mathrm{d}t \leq \widetilde{C}^2 \frac{5\pi}{4} \|f\|_{\mathscr{H}^2} \,.$$

Combining this with Brevig's result completes the proof. ☐

3.3. The Hardy space $\mathscr{H}^p$

In chapter 3 we studied $\mathscr{H}^2$, the Hilbert space of Dirichlet series with square summable coefficients, and saw this space is a natural analogue of H^2. One of the main difficulties in constructing spaces $\mathscr{H}^p$ $(1 \leq p \leq \infty)$ analogous to H^p is the abscence of any Blaschke factorization: we cannot deduce theorems for $\mathscr{H}^p$ from theorems for $\mathscr{H}^2$ as easily as in the case of H^p.

Bayart [4] extended the definition of $\mathscr{H}^p$ to every $p \geq 1$, by defining $\mathscr{H}^p$ as the closure of all Dirichlet polynomials $f(s) = \sum_{n=1}^{N} a_n n^{-s}$ under the norm[2]

$$\|f\|_{\mathscr{H}^p} := \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^p \, \mathrm{d}t \right)^{\frac{1}{2}} . \tag{3.25}$$

For $p = 2$, this is Carlson's theorem 3.6, and thus gives back the original definition of $\mathscr{H}^2$. However it is far from clear that equation (3.25) is the right one, or that it even yields spaces of convergent Dirichlet series in any right half-plane. The clarification of these matters is provided by the Bohr correspondence.

3.3.1. The Bohr correspondence

In this section, we introduce a new way to view Dirichlet polynomials, which is due to Bohr [7]. Fix N and consider the Dirichlet polynomial $P \in \mathscr{P}$,

$$P(s) = \sum_{n=1}^{N} a_n n^{-s} . \tag{3.26}$$

The fundamental theorem of arithmetic allows us to uniquely factor any integer into prime factors

$$n = \prod_{k=1}^{\pi(n)} p_k^{\alpha_k} , \tag{3.27}$$

where $\pi(n)$ denotes the the prime-counting function. If we now translate each prime number into a variable,

$$z_1 = 2^{-s}, \; z_2 = 3^{-s}, \; \cdots , \; z_k = p_k^{-s}, \; \cdots ,$$

we will have at most $\pi(N)$ variables in the corresponding polynomial. Thus, the factorization (3.27) allows us to bijectively associate each integer to the following multi-index

$$n \;\longleftrightarrow\; \alpha(n) = (\kappa_1, \kappa_2, \ldots, \kappa_{\pi(n)}) . \tag{3.28}$$

Thus, this gives us the Bohr correspondence,

$$P(s) = \sum_{n=1}^{N} a_n n^{-s} \quad \longleftrightarrow \quad \mathscr{B}P(z) = \sum_{n=1}^{N} a_n z^{\kappa(n)} , \tag{3.29}$$

which yields a polynomial of at most $\pi(N)$ variables. From now on, for a given element $P \in \mathscr{H}^2$, we let $\mathscr{B}P$ denote the corresponding power series, and we drop

[2]The norm in is actually a quasi-norm for $0 < p < 1$, however we are only concerned with the case $p \geq 1$.

the relationship between z and s. Letting $N \to \infty$ we see that multiplicative structure of the integers allows us to view an ordinary Dirichlet series as a Fourier series in infinitely many variables.

This transformation — the so-called Bohr correspondence — gives an isometric isomorphism between $\mathscr{H}^p$ and the Hardy space $H^p(\mathbb{T}^\infty)$. In this section we will show that it ensures an unambiguous definition of $\mathscr{H}^p$ for $1 \le p < \infty$.

PROPOSITION 3.12. *Let P and Q be defined as the following Dirichlet polynomials $P(s) = \sum_{m=1}^N a_m m^{-s}$, $Q(s) = \sum_{n=1}^N \overline{b_n} n^{-s}$. We then have the equality*

$$\langle \mathscr{B}P, \mathscr{B}Q \rangle_{H^2(\mathbb{T}^d)} = \langle P, Q \rangle_{\mathscr{H}^2} . \tag{3.30}$$

PROOF. Recall from Lemma 1.29 that we have

$$\int_{\mathbb{T}^d} z^{\alpha(n)} \cdot \overline{z}^{\alpha(m)} \, \mathrm{d}m_d(z) = \delta_{mn}$$

by orthogonality of the one-dimensional case and $z^{\alpha(r)} \cdot z^{\alpha(s)} = z^{\alpha(rs)}$ from (3.28). A direct computation of the coefficients now yields

$$\langle \mathscr{B}P, \mathscr{B}Q \rangle_{H^2(\mathbb{T}^d)} = \sum_{m,n \ge 1}^{\infty} a_m b_n \int_{\mathbb{T}^d} z^{\alpha(m)} \cdot \overline{z}^{\alpha(n)} \, \mathrm{d}m_d(z) = \sum_{n=1}^{N} a_n b_n$$

To complete the proof recall from section 3.2 the norm of $\mathscr{H}^2$,

$$\langle P, Q \rangle_{\mathscr{H}^2} = \sum_{j=1}^{N} a_j b_j . \qquad \square$$

By taking the closure of the Dirichlet polynomials and Corollary 3.6, we thus have

$$\|f\|_{\mathscr{H}^2} = \|\mathscr{B}f\|_{H^2(\mathbb{D}^\infty)} := \left(\int_{\mathbb{T}^\infty} |\mathscr{B}f|^2 \, \mathrm{d}m_\infty \right)^{\frac{1}{2}} . \tag{3.31}$$

Bayart [4] extended equation (3.31) to hold for $1 \le p < \infty$ using Birkhoff's ergodic theorem and Kronecker's lemma. However Saksman and Seip outlined in [45, Sec. 3] a more elementary approach using an interpolation argument.

PROPOSITION 3.13. *For every $1 \le p < \infty$ then,*

$$\|f\|_{\mathscr{H}^p} = \|\mathscr{B}f\|_{H^p(\mathbb{D}^\infty)} := \left(\int_{\mathbb{T}^\infty} |\mathscr{B}f|^p \, \mathrm{d}m_\infty \right)^{\frac{1}{p}} . \tag{3.32}$$

PROOF. As before it will suffice to prove this for all $f \in \mathscr{P}$ that are Dirichlet polynomials, and then take the closure of the Dirichlet polynomials. It is clear that equation (3.25) holds for every even $p = 2n$,

$$\lim_{T\to\infty} \frac{1}{2T} \left(\int_{-\infty}^{\infty} |P(it)|^{2n} \, \mathrm{d}t \right) = \lim_{T\to\infty} \frac{1}{2T} \left(\int_{-\infty}^{\infty} \left(|P(it)|^n\right)^2 \mathrm{d}t \right) = \|P^n\|_{\mathscr{H}^2}^2 .$$

Thus, for every $n \in \mathbb{N}$ we have

$$\|P\|_{\mathscr{H}^{2n}} = \|P^n\|_{\mathscr{H}^2}^{1/n} = \|\mathscr{B}P^n\|_{H^2(\mathbb{D}^\infty)}^{1/n} = \|\mathscr{B}P\|_{H^{2n}(\mathbb{D}^\infty)}, .$$

Now pick p freely such that $1 \le p < \infty$. By Weierstrass approximation theorem there exists a polynomial $Q(x) = \sum_{j=0}^{J} b_j x^j$ on the interval $[0, L]$ – where $L = \sum_{n=1}^{N} |a_n|^2$ – such that for every $\varepsilon > \infty$,

$$|x^{p/2} - Q(x)| < \varepsilon\,. \tag{3.33}$$

For any $1 \le p < \infty$ we can rewrite the limit as

$$\lim_{T\to\infty} \frac{1}{2T}\left(\int_{-\infty}^{\infty} |P(it)|^p \,\mathrm{d}t\right) = \lim_{T\to\infty} \frac{1}{2T}\left(\int_{-\infty}^{\infty} \left(|P(it)|^2\right)^{p/2} \mathrm{d}t\right),$$

where $P(s) = \sum_{n=1}^{N} a_n n^{-s}$ is some finite Dirichlet polynomial. By using equation (3.33) it is clear that the error in replacing $|P(it)|^p$ with $Q(P(it)^2)$ in the equation above is at most $O(\varepsilon)$.

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{2T}\left(\int_{-\infty}^{\infty} |P(it)|^p \,\mathrm{d}t\right) &= = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} Q(P(it)^2) + O(\varepsilon)\,\mathrm{d}t \\
&= O(\varepsilon) + \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sum_{j=0}^{J} b_j |P(it)|^{2j}\,\mathrm{d}t \\
&= O(\varepsilon) + \sum_{j=0}^{J} b_j \|\mathscr{B}P\|_{H^{2j}(\mathbb{T}^d)}^{2j}\,.
\end{aligned}$$

The last norm is simply $\|P\|_{\mathscr{H}^{2j}}^{2j}$, by the previous discussion and the Bohr correspondence. Thus, letting $\varepsilon \to 0$ wee see that every p can be uniformly approximated by the even p values by taking the closure of the Dirichlet polynomials. □

The fact that we can identify $\mathscr{H}^p$ with $H^p(\mathbb{T}^\infty)$ follows now directly.

THEOREM 3.14. *The mapping $\mathscr{B}\colon \mathscr{P} \to H^p(\mathbb{T}^\infty)$ extends to an isometric isomorphism from $\mathscr{H}^p$ onto $H^p(\mathbb{T}^\infty)$.*

PROOF. From Proposition 3.13, $\mathscr{B}$ is an isometric isomorphism from the Dirichlet polynomials $\mathscr{P}$ onto $\mathscr{B}(\mathscr{P})$. Where the last notation denotes the Dirichlet polynomials on $\mathbb{T}^\infty$, under the norm $\|\mathscr{B}f\|_{H^p(\mathbb{T}^\infty)}$. Since $\mathscr{H}^p$ is the completion of $\mathscr{P}$ under the norm

$$\left(\int_{-T}^{T} |P(it)|^p \,\mathrm{d}t\right)^{\frac{1}{p}},$$

and $H^p(\mathbb{T}^\infty)$ is the completion of $\mathscr{B}(\mathscr{P})$ for $\|\cdot\|_{H^p(\mathbb{T}^\infty)}$ the assertion is proved. □

To complete the picture, we define $\mathscr{H}^\infty$ as the space of Dirichlet series $f(s) = \sum_{n\ge 1}^{\infty} a_n n^{-s}$ that represents bounded analytic functions in the half plane $\mathbb{C}_+$. This space is naturally endowed with the norm

$$\|f\|_{\mathscr{H}^\infty} := \sup_{\sigma>0} |f(s)|, \quad s = \sigma + it\,,$$

and then the Bohr correspondence allows us to associate with $\mathscr{H}^\infty$ with the space $H^\infty(\mathbb{T}^\infty)$. We refer to Queffélec and Queffélec [39] for a further study of this space.

3.3.2. The properties of $\mathscr{H}^p$

We are now ready to try to extend the properties of $\mathscr{H}^2$ from Theorem 3.4 onto $\mathscr{H}^p$. First the domain of definition of an element in $\mathscr{H}^p$ is supplied by the following point estimate.

THEOREM 3.15. *Let $f \in \mathscr{H}^p$. Then, the Dirichlet series which defines f converges in the half-plane $\mathbb{C}_{1/2}$, and if $\mathrm{Re}(s) > 1/2$, then*

$$|f(s)|^p \leq \zeta\big(2\,\mathrm{Re}(s)\big)\|f\|_{\mathscr{H}^p}^p. \tag{3.34}$$

PROOF. We first let $s \in \mathbb{C}_{1/2}$, $F = \mathscr{B}f$ and $z = (2^{-s}, 3^{-s}, \ldots) \in \mathbb{D}^\infty \cap \ell^2$- Since F is in $H^p(\mathbb{T}^\infty)$, we can apply the Cole-Gamelin estimate 1.30

$$|f(s)|^p \leq \prod_{j\geq 1} \frac{1}{1 - |p_j^{-s}|^2}\|f\|_{H^p(\mathbb{D}^\infty)} = \sum_{n\geq 1} n^{-2\,\mathrm{Re}(s)}\|f\|_{\mathscr{H}^p}^p\,.$$

The last equality follows from the Euler product

$$\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}\,,$$

see Corollary A.3 for details. This proves that the abscissa of boundedness for f is less than $1/2$ and Lemma 3.3 implies that f converges in the half-plane $\mathbb{C}_{1/2}$.

To show that $\mathbb{C}_{1/2}$ is the best possible consider the Dirichlet series $f(s) = \zeta(1/2 + \varepsilon + s)^{2/p} \in \mathscr{H}^p$ and rewriting the sum in terms of its Euler product. $\square$

REMARK. From the previous proof and Lemma 3.3 it follows that if $f(s) = \sum_{n\geq 1} a_n n^{-s}$ belongs to $\mathscr{H}^p$, then $\sigma_u(f) \leq 1/2$. Similarly if $p \geq 2$, then $\{a_n\}_{n\geq 1} \in \ell^2$, and by the by the Cauchy-Schwarz inequality gives $\sigma_a \leq 1/2$. Otherwise if $1 < p < 2$, we have by Riesz-Thorin theorem $\{a_n\}_{n\geq 1} \in \ell^q$ where $1/q + 1/p = 1$, for which it follows directly by Hölders inequality that in this case $\sigma_a \leq 1/p$. Note that this also can be shown using the Hausdorff–Young inequality.

This extends the first three properties of Theorem 3.4 onto $\mathscr{H}^p$. The last property to study is the embedding problem for $\mathscr{H}^p$, this was first studied by Bayart [4] and is of primary importance. It will be enough to formulate the question for polynomials, since existence of non-tangential boundary values almost everywhere would be an immediate consequence of a positive answer, and the inequality could then be stated for all Dirichlet series in $\mathscr{H}^p$.

QUESTION 1. *Fix an exponent $p > 2$. Does there exist a constant $1 \leq C_p < +\infty$ such that,*

$$\int_0^1 \big|f(1/2 + it)\big|^p \leq C_p\|f\|_{\mathscr{H}^p}^p, \tag{3.35}$$

holds for every Dirichlet polynomial $f \in \mathscr{P}$?

In the case $p = 2n$ with $n \in \mathbb{N}$ the answer is trivially positive: apply the case $p = 2$ from Theorem 3.8 to the function $f^n \in \mathscr{H}^2$. This provides evidence in favour of a positive answer.

Let us indicate some properties of $\mathscr{H}^p$ that makes Question 1 difficult to answer. It can be shown that for $p > 1$ the isometric subspace $\mathscr{H}^p(\mathbb{D}^\infty) \subset L^p(\mathbb{T}^\infty)$ is not complemented in $L^p(\mathbb{T}^\infty)$ unless $p = 2$. Assume that such a bounded projection existed, then one could apply the same interpolation technique shown in equation (3.32) to prove that the L^2-orthogonal projection is bounded in L^p. In other words, the infinite product of one-dimensional Riesz projections would be bounded in L^p. By considering products of functions each depending on one variable, we see that the only possibility is that the norm of the dimensional projection is 1. However as shown by Hollenbeck and Verbitsky [28] the norm of the dimensional projection is 1 only for $p = 2$. This fact makes it difficult to apply interpolation between the already known values $p = 2, 4, 6, \ldots$.

The reason we only ask whether equation (3.35) holds for $p \geq 2$ and not $p > 0$ is due to Harper [21], who proved that equation (3.35) fails to hold for every $0 < p < 2$. Whether equation (3.35) holds for any $p > 2$ is still unknown.

In analytic number theory there are a couple of famous unsolved conjectures, due to Montgomery regarding norm inequalities for Dirichlet polynomials [32, pp. 129]. One of Montgomery's conjectures states that for every $\varepsilon > 0$ and $p \in (2, 4)$ there exists a constant $C = C(\varepsilon)$ such that for all finite Dirichlet polynomials $f(s) = \sum_{n=1}^{N} a_m n^{-s}$ with $|a_n| \leq 1$ one has

$$\int_0^T |f(it)|^p \, \mathrm{d}t \leq C \cdot N^{p/2+\varepsilon}(T + N^{p/2}) \quad \text{for } T > 1\,.$$

This inequality is known to be true for $p = 2, 4$ and if true would imply the density hypothesis for the zeros of the Riemann zeta function. The similarities suggest for a possible connection between Montgomery's conjectures and our embedding problem. This indicates that answering the embedding problem is likely highly non-trivial.

CHAPTER 4

Multiplicative Hankel forms

We are now ready to introduce the multiplicative Hankel forms. The purpose of this chapter will be to study the most prominent example of an infinite multiplicative Hankel form; the multiplicative Hilbert matrix. With the aid of the Bohr lift, we will see that every multiplicative Hankel matrix can be uniquely associated with either a Hankel form on $H^2(\mathbb{T}^\infty) \times H^2(\mathbb{T}^\infty)$ or equivalently a (small) Hankel operator acting on $H^2(\mathbb{T}^\infty)$. This will be used to determine to which extent Nehari's theorem remains valid in the multiplicative setting.

In two papers, published in 2006 [26] and (posthumously) in 2010 [27], Henry Helson initiated a study of multiplicative Hankel matrices, which are finite or infinite matrices whose entries $a_{m,n}$ only depend on the product nm. For $\psi \in H^2(\mathbb{D}^d)$ the corresponding Hankel form is

$$\rho(a,b) = \sum_{\alpha,\beta \geq 0} a_\alpha b_\beta \rho_{\alpha+\beta}$$

If we now write the positive integers in multi-index notation $mn = p^\alpha p^\beta = p^{\alpha+\beta}$ and let $d \to \infty$ we obtain

$$\varrho(a,b) = \sum_{m,n \geq 1} a_m b_n \varrho_{nm} \tag{4.1}$$

Thus, we see that on the infinite dimensional polydisc, the Hankel forms becomes *multiplicative.*

Definition. Each sequence $\varrho = (\varrho_1, \varrho_2, \cdots)$ induces a (not necessarily bounded) multiplicative Hankel form on $\ell^2 \times \ell^2$,

$$\varrho(a,b) = \sum_{m,n \geq 1} a_m a_n \varrho_{nm}\,, \qquad a, b \in \ell^2. \tag{4.2}$$

We define the multiplicative Hankel form (4.2) to be bounded if there is a constant $C < \infty$ such that

$$|\varrho(a,b)| = \left| \sum_{m,n=1}^{\infty} a_m b_n \varrho_{nm} \right| \leq C \left(\sum_{m=1}^{\infty} |a_m|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |b_n|^2 \right)^{\frac{1}{2}}. \tag{4.3}$$

As in the one dimensional case the smallest number C for which the inequality holds is referred to as the *norm* of ϱ.

If f and g are Dirichlet series with coefficient sequences a and b, respectively, then (4.3) can be rewritten

$$H_\varphi(fg) = \langle fg, \varphi\rangle_{\mathscr{H}^2} = \sum_{k\geq 1}\Big(\sum_{mn=k} a_m b_n\Big)\varrho_k = \sum_{m,n\geq 1} a_m b_n \varrho_{mn}, .$$

From which we see that the multiplicative Hankel form is bounded if and only if H_φ is a bounded form on $\mathscr{H}^2 \times \mathscr{H}^2$.

Under the Bohr correspondence from section 3.3.1, $\mathscr{H}^p$ corresponds to the infinite-dimensional Hardy space $H^2(\mathbb{T}^\infty)$, which we view as a subspace of $L^2(\mathbb{T}^\infty)$. Following the exact same steps as in Proposition 3.12, a formal computation shows

$$\langle \mathscr{B} f \mathscr{B} g, \mathscr{B}\varphi\rangle_{L^2(\mathbb{T}^d)} = \langle fg, \varphi\rangle_{\mathscr{H}^2},$$

allowing us to compute the multiplicative Hankel form (4.2) on $\mathbb{T}^\infty$ or $\mathscr{H}^2$. This interplay will be used extensively to study Nehari's theorem. We begin by studying the most prominent example of a bounded multiplicative Hankel form using the Dirichlet series.

4.1. The multiplicative Hilbert matrix

In section 2.2 we studied the classical Hilbert matrix

$$M = \Big(\frac{1}{m+n+1}\Big)_{m,n\geq 0}$$

which is the prime example of an (additive) Hankel form. The first bona fide example of a *multiplicative* Hankel form was obtained by Brevig, Perfekt, Seip, Siskakis, and Vukotić, who in [10] found an infinite matrix with entries $\varrho_{m,n}$ that depended only on the product mn and with properties parallel to those of M. The purpose of this section is to study this particular Hankel form. For convergence reasons it will be natural to work with Dirichlet series without constant term

DEFINITION. The subspace $\mathscr{H}_0^2$ consists of those $f \in \mathscr{H}^2$ with $a_1 = f(+\infty) = 0$ and bounded norm

$$\|f\|_{\mathscr{H}_0^2} := \sum_{n\geq 2} |a_n|^2 < \infty\,. \tag{4.4}$$

It follows from Cauchy-Schwarz that every f in $\mathscr{H}_0^2$ is analytic in the half plane $\mathbb{C}_{1/2}$, see Theorem 3.4 for an equivalent proof for $\mathscr{H}^2$. Similar to how the integral in (2.7) induced the Hankel form for the Hilbert matrix, the analog in $\mathscr{H}^2$ is

$$H_\varphi(fg) = \int_{1/2}^\infty f(w)g(w)\,\mathrm{d}w\,, \qquad f, g \in \mathscr{H}_0^2\,. \tag{4.5}$$

THEOREM 4.1. *The bilinear form equation (4.5) is a multiplicative Hankel form with symbol*

$$\varphi(s) = \int_{1/2}^\infty \zeta(s+w) - 1\,\mathrm{d}w = \sum_{n\geq 2}\frac{1}{\sqrt{n}\,(\log n)}n^{-s} \tag{4.6}$$

PROOF. To see that φ is the symbol, we compute $H_\varphi(fg)$ at the level of coefficients:

$$\int_{1/2}^{\infty} f(w)g(w)\,\mathrm{d}w = \int_{1/2}^{\infty} \sum_{m,n\geq 2} a_m a_n (mn)^{-w}\,\mathrm{d}w = \sum_{m,n\geq 2} \frac{a_m a_n}{\sqrt{mn}\log(nm)} \tag{4.7}$$

Comparing this with equation (4.2) we see that

$$\varrho_{mn} = \frac{1}{\sqrt{mn}\log(nm)}$$

Thus the symbol can be written

$$\varphi(s) = \sum_{k\geq 2} \frac{1}{\sqrt{k}\log(k)} k^{-s} = \sum_{k\geq 2} \int_{1/2}^{\infty} k^{-(s+w)}\,\mathrm{d}w = \int_{1/2}^{\infty} \zeta(s+w) - 1\,\mathrm{d}w\,,$$

as wanted. □

Hence, the matrix of $H(fg)$ with respect to the orthonormal basis $\{n^{-s}\}_{n\geq 2}$ is

$$\mathscr{M} := (\varrho_{mn})_{n,m\geq 1} \left(\frac{1}{\sqrt{mn}\log(mn)}\right)_{m,n\geq 2}.$$

We will refer to this matrix as the *multiplicative Hilbert matrix*. Similar to the Hilbert matrix, we will be interested in understanding $\mathscr{M}$ as an symbol on $\ell_0^2 := \ell^2(\mathbb{N}\backslash\{1\})$, which means that, equivalently, we will be concerned with the properties of the Hankel form H acting on $\mathscr{H}_0^2$. That the form H_φ is bounded on $\mathscr{H}_0^2 \times \mathscr{H}_0^2$ is shown in the next theorem.

THEOREM 4.2. *The Hankel form H_φ is a strictly positive and bounded on $\mathscr{H}_0^2 \times \mathscr{H}_0^2$ and $\|H_\varphi\|_{\mathscr{H}_0^2} = \pi$.*

PROOF. The proof relies as in Hardy, Littlewood, and Pólya [20, Chp. IX] on the following Mellin-transformation

$$\int_0^{\infty} x^{-1/p} \frac{\mathrm{d}x}{1+x} = \frac{\pi}{\sin(\pi/p)}\,, \tag{4.8}$$

see Lemma A.18 for details. As before we will prove that $\|\mathscr{M}\|_{\ell_0^2} = \pi$ as this implies that $\|H\|_{\mathscr{H}_0^2}$. By the Cauchy–Schwarz inequality we have

$$\begin{aligned} \sum_{n,m\geq 2} \frac{|a_m||b_n|}{\sqrt{mn}\log(nm)} \leq & \left(\sum_{m\geq 2} |a_m|^2 \sum_{n\geq 2} \frac{1}{n\log(mn)} \sqrt{\frac{\log m}{\log n}}\right)^{\frac{1}{2}} \\ & \left(\sum_{n\geq 2} |b_n|^2 \sum_{m\geq 1} \frac{1}{m\log(mn)} \sqrt{\frac{\log n}{\log m}}\right)^{\frac{1}{2}}, \end{aligned} \tag{4.9}$$

and from symmetry we only need to consider one of the factors. The standard integral estimate gives

$$\begin{aligned}\sum_{n\geq 2}\sqrt{\frac{\log m}{\log n}}\frac{1}{n\log(mn)} &\leq \int_2^\infty \sqrt{\frac{\log m}{\log x}}\frac{1}{\log(mx)}\frac{\mathrm{d}x}{x}\\ &\leq \int_{\frac{\log 2}{\log m}}^\infty u^{-1/2}\frac{\mathrm{d}u}{1+u} \leq \int_0^\infty u^{-1/2}\frac{\mathrm{d}u}{1+u} = \frac{\pi}{\sin\pi/2} = \pi\,,\end{aligned}$$

where $u \mapsto \log x/\log m$ and equation (4.8) were used. Applying this to equation (4.9) gives the estimate

$$\left|\sum_{n,m\geq 2}\frac{a_m b_n}{\sqrt{nm}\log(nm)}\right| \leq \pi\Big(\sum_{n\geq 2}|a_m|^2\Big)^{\frac{1}{2}}\Big(\sum_{n\geq 2}|b_n|^2\Big)^{\frac{1}{2}}$$

This now proves the inequality $\|M\|_{\ell_0^2} \leq \pi$, and thus $\|H\|_{\mathscr{H}_0^2} \leq \pi$. To prove that the norm is bounded below we stress the inequality, with the following sequences

$$a_m = b_m = m^{-1/2}(\log m)^{-(1+\varepsilon)/2}$$

Applying the standard integral estimate again we obtain

$$\|a\|_{\ell_0^2}^2 = \|b\|_{\ell_0^2}^2 = \frac{1}{\varepsilon} + O(1)\,, \tag{4.10}$$

when $\varepsilon \to 0^+$. Inserting this sequence into the left-hand side of (4.9) we find

$$\begin{aligned}\sum_{n,m\geq 2}\frac{a_m b_n}{\sqrt{nm}\log(nm)} &= \sum_{n,m\geq 2}\frac{\big(\log(nm)\big)^{-(1+\varepsilon)/2}}{\sqrt{mn}\log(mn)}\\ &\geq \int_{\log 3}^\infty\int_{\log 3}^\infty \frac{(xy)^{-(1+\varepsilon)/2}}{x+y}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

This iterated integral can computed as the corresponding integral in Hardy, Littlewood, and Pólya [20, p. 233], using Lemma A.18 twice. Thus, we obtain the following estimate

$$\sum_{n,m\geq 2}\frac{a_m b_n}{\sqrt{nm}\log(nm)} = \frac{1}{\varepsilon}\Big(\frac{\pi}{\sin\pi/2} + o(1)\Big) \tag{4.11}$$

For the details of the computation above, see Proposition 2.5, we use the same estimates as done there, only twice.

Comparing equations (4.10) and (4.11) gives

$$\left|\sum_{n,m\geq 2}\frac{a_m b_n}{\sqrt{nm}\log(nm)}\right| \geq \pi\Big(\sum_{n\geq 2}|a_m|^2\Big)^{\frac{1}{2}}\Big(\sum_{n\geq 2}|b_n|^2\Big)^{\frac{1}{2}}$$

thus the inequality is sharp, and the constant π can not be improved. This proves that $\|M\|_{\ell_0^2} = \pi$, and hence completes our proof. ☐

4.1.1. Bounded symbol

DEFINITION. Let H_φ be a multiplicative Hankel form. We say that H_φ has a bounded symbol φ if there exists a symbol $\psi \in L^\infty(\mathbb{T}^\infty)$ such that $P_+\psi = \varphi$, where P_+ is the Riesz-projection.

Note that the existence of such a bounded symbol would imply that the corresponding multiplicative Hankel form is bounded, this is known as the converse of Nehari's theorem. Let ψ be in $L^\infty(\mathbb{T}^\infty)$, then by the Cauchy–Schwarz inequality

$$|H_\psi(fg)| = |\langle fg, \psi\rangle| \leq \|f\|_{H^2\mathbb{D}^\infty}\|g\|_{H^2(\mathbb{D}^\infty)}\|\psi\|_{L^\infty(\mathbb{T}^\infty)}\,,$$

thus proving $\|H_\varphi\| \leq \|\psi\|_{L^\infty(\mathbb{T}^\infty)}$.

A natural question which we are unable to settle is the following: *Does $\mathscr{M}$ have a bounded symbol?* As it was shown in [10] that if the embedding

$$\int_0^1 |P(1/2+it)|\,\mathrm{d}t \leq C\|P\|_{\mathscr{H}^1}\,, \tag{4.12}$$

holds then $\mathscr{M}$ has a bounded symbol. Whether equation (4.12) holds was an open problem for many years, and just solved in the negative Harper [21]. See [45] for more details on the embedding problem. Nevertheless, for completeness sake we will prove that if the following equivalent embedding holds,

$$\|f\|_{H_i^1} := \frac{1}{\pi}\int_{\mathbb{R}} |f(1/2+it)|\frac{\mathrm{d}t}{1+t^2} \leq C\|f\|_{\mathscr{H}^1}, \tag{4.13}$$

then $\mathscr{M}$ has a bounded symbol. Note that proving that $\varphi \in (\mathscr{H}_0^1)^*$ and thus that $\mathscr{M}$ has a bounded symbol is the same as proving that

$$\left|\int_{1/2}^\infty f(s)\,\mathrm{d}s\right| = |\langle f, \varphi\rangle_{\mathscr{H}^2}| \leq C\|f\|_{\mathscr{H}_0^2}\,.$$

From Theorem 3.15 and in particular equation (3.34) we have

$$\left|\int_1^\infty f(s)\,\mathrm{d}s\right| \leq C\|f\|_{\mathscr{H}_0^1},$$

thus we only need to bound the interval $[1/2, 1]$. By Hilbert's classical inequality we have

$$\left|\int_0^1 F(z)\,\mathrm{d}z\right| \leq \pi\|F\|_{H^1(\mathbb{D})}.$$

As before we apply the shifted Cayley transformation $\mathscr{T} = 1/2 + \frac{1-z}{1+z}$ to obtain

$$\left|\int_{1/2}^{3/2} f(s)\,\mathrm{d}s\right| \leq C\|f\|_{H_i^1(\mathbb{C}_{1/2})}.$$

Hence, if $\|f\|_{H_i(\mathbb{C}_{1/2})}$ is bounded this implies that $\mathscr{M}$ admits a bounded symbol. However, as mentioned earlier Harper [21] found a sequence of functions such that the embedding inequality equation (4.12) and thus also equation (4.13) failed to hold for all $0 < p < 2$. Thus, the preceding discussion gives no answer as to whether $\mathscr{M}$ has a bounded symbol.

4.2. Nehari's theorem

We now come to the question of to which extent Nehari's theorem remains valid in the multiplicative setting. Under the Bohr correspondence, $\mathscr{H}^p$ corresponds to the infinite dimensional Hardy space $H^2(T^\infty)$, which we view as a subspace of $L^2(\mathbb{T}^\infty)$. As a reminder we have

$$\langle \mathscr{B}f\mathscr{B}g, \mathscr{B}\varphi\rangle_{L^2(\mathbb{T}^d)} = \langle fg, \varphi\rangle_{\mathscr{H}^2}\,.$$

allowing us to compute the multiplicative Hankel form (4.1) on $\mathbb{T}^\infty$ or $\mathscr{H}^2$. In the remainder of this section we work exclusively in the polydisc, with no reference to Dirichlet series. We therefore drop the notation $\mathscr{B}$ and study Hankel forms

$$H_\varphi = \langle fg, \varphi\rangle_{L^2(\mathbb{T}^d)}\,, \qquad f, g \in H^2(\mathbb{T}^d).$$

The Hankel form H_φ may be realized as a small Hankel operator $\mathbf{H}_\varphi$ on the polydisk. When this operator is bounded it acts as an operator from $H^2(\mathbb{D}^\infty)$ to the antianalytic space $\big(L^2(\mathbb{T}^\infty) \ominus H^2(\mathbb{T}^\infty)\big)$ which consists of all antianalytic elements in $L^2(\mathbb{T}^\infty)$, i.e., all functions in L^2 for which all Fourier coefficients with at least one positive index vanish.

Lemma 4.3. *Let $\varphi \in H^2(\mathbb{D}^d)$, where $d \in \mathbb{N} \cup \{\infty\}$, and define the operator $\mathbf{H}_\varphi$ by*

$$\mathbf{H}_\varphi(f) := \overline{P}(\overline{\varphi} f)\,.$$

where $\overline{P}$ denotes the orthogonal projection of $L^2(\mathbb{T}^\infty)$, onto $\overline{H^2}(\mathbb{D}^\infty)$. Then the Hankel form $H_\varphi(fg) = \langle fg, \varphi\rangle_{H^2(\mathbb{D}^2)}$ has the same norm as $\mathbf{H}_\varphi$.

Proof. For the ease of notation we assume that g is a normalized function such that $\|g\|_{\overline{H^2}(\mathbb{D}^d)} = 1$. Then

$$\|\mathbf{H}_\varphi f\|_{\overline{H^2(\mathbb{D}^d)}} \;=\; \sup_{g\in\overline{H^2(\mathbb{D}^d)}} |\langle \overline{P}(\overline{\varphi} f), g\rangle| \;=\; \sup_{g\in\overline{H^2(\mathbb{D}^d)}} |\langle \overline{\varphi} f, g\rangle|\,,$$

since $\overline{P}$ is an orthogonal projection and $f \in \overline{H^2}(\mathbb{D}^d)$. If $g \in \overline{H^2}(\mathbb{D}^d)$, then $\overline{g} \in H^2(\mathbb{D}^d)$, thus

$$\|\mathbf{H}_\varphi f\|_{\overline{H^2(\mathbb{D}^d)}} = \sup_{g\in H^2(\mathbb{D}^d)} |\langle \overline{\varphi} f, \overline{g}\rangle| = \sup_{g\in H^2(\mathbb{D}^d)} |\langle fg, \varphi\rangle| = \|H_\varphi(fg)\|\,,$$

where the second equality follows from the integral representation

$$\langle fg, \varphi\rangle = \int_{\mathbb{T}^d} fg\cdot\overline{\varphi}\,\mathrm{d}m_d = \int_{\mathbb{T}^d} \overline{\varphi} f\cdot\overline{\overline{g}}\,\mathrm{d}m_d = \langle \overline{\varphi} f, \overline{g}\rangle\,. \qquad \square$$

Lemma 4.4. *Suppose that $\varphi_1,\varphi_2,\cdots,\varphi_m$ are symbols that depend on mutually separate variables and which generate multiplicative Hankel forms H_{φ_j}, $1 \le j \le m$. Then*

$$\|H_\varphi\| = \prod_{j=1}^m \|H_{\varphi_j}\|\,,$$

where $\varphi = \prod \varphi_j$.

PROOF. For ease of notation we let $\varphi(z_1, z_2) = \varphi_1(z_1)\varphi_2(z_2)$ and note that the proof may be extended to as many variables as necessary. In two variables we have $\overline{P} = \overline{P_1 P_2}$ where $\overline{P_1}$ exclusively work on z_1 and $\overline{P_2}$ on z_2. Thus, by the orthogonality of P and the linearity of φ

$$\mathbf{H}_\varphi(f) = \overline{P}(\overline{\varphi} f) = \overline{P_2 P_1}(\overline{\varphi_2 \varphi_1} f) = \overline{P_2}\big(\overline{\varphi_2}\,\overline{P_1}(\overline{\varphi_1} f)\big) = H_{\varphi_2}\big(\mathbf{H}_{\varphi_1}(f)\big)$$

Introduce the function $g(z_2) = \mathbf{H}_{\varphi_1}\big(f(z_1, z_2)\big)$. Then the norm of $\mathbf{H}_\varphi$ becomes

$$\begin{aligned}
\|\mathbf{H}_\varphi\|^2 = \|\mathbf{H}_{\varphi_2}(g)\|^2 &= \int_{\mathbb{T}^2} |\mathbf{H}_{\varphi_2}(g)|^2 \,\mathrm{d}m_2 \\
&\leq \|\mathbf{H}_{\varphi_2}\|^2 \int_{\mathbb{T}^2} |g|^2 \,\mathrm{d}m_2 \\
&= \|\mathbf{H}_{\varphi_2}\|^2 \int_{\mathbb{T}^2} \|\mathbf{H}_{\varphi_1}(f)\|^2 \,\mathrm{d}m_2 \\
&\leq \|\mathbf{H}_{\varphi_2}\|^2 \|H_{\varphi_1}\|^2 \int_{\mathbb{T}^2} |f|^2 \,\mathrm{d}m_2 \,.
\end{aligned}$$

This proves the inequality $\|\mathbf{H}_\varphi\| \leq \|\mathbf{H}_{\varphi_2}\|\|\mathbf{H}_{\varphi_1}\|$.

To prove the reverse inequality is slightly simpler, let f_j, and g_j be the functions such that $H_{\varphi_j}(f_j, g_j)$ is maximized. This means that $H_\varphi(fg)$ is optimal with $f = \prod f_j$, $g = \prod g_j$,

$$H_\varphi(fg) = \prod_{j=1}^m H_{\varphi_j}(f_j g_j)$$

hence

$$\begin{aligned}
\|H_\varphi\| &= \sup_{\|f\|=\|g\|=1} |H_\varphi(fg)| \\
&\geq \sup_{\|f_j\|=\|g_j\|=1} |H_\varphi(\prod_{j=1}^m f_j g_j)| \\
&= \sup_{\|f_j\|=\|g_j\|=1} \prod_{j=1}^m H_{\varphi_j}|\big(f_j g_j\big)| = \prod_{j=1}^m \|H_{\varphi_j}\|
\end{aligned}$$

Recalling $\|\mathbf{H}_\varphi\| = \|H_\varphi\|$ from Lemma 4.3 completes the proof. □

We are now ready to prove that Nehari's theorem holds for multiplicative Hankel forms in the special case where the symbol is completely multiplicative.

THEOREM 4.5. *Suppose that $a(n)$ is a multiplicative function and define*

$$\varphi(s) = \sum_{n\geq 1} a(n) n^{-s}$$

such that $\varphi \in \mathscr{H}^2$. If $H_{\mathscr{B}\varphi}$ is a bounded Hankel form on $H^2(\mathbb{T}^\infty) \times H^2(\mathbb{T}^\infty)$, then there exists a $\Psi \in L^\infty(\mathbb{T}^\infty)$ such that $\mathscr{B}\varphi = P_+\Psi$.

Moreover, if the function $a(n)$ is completely multiplicative, then the Hankel form $H_{\mathscr{B}\varphi}$ is always bounded on $H^2(\mathbb{T}^\infty) \times H^2(\mathbb{T}^\infty)$.

PROOF. Assume that $\alpha(n)$ is a multiplicative function, then it follows from Theorem A.1 that the symbol $\varphi(s) = \sum_{n\geq 1} \alpha(n) n^{-s}$ may be factored into the following Euler product

$$\varphi(s) = \prod_{j\geq 1}\left(1 + \sum_{k\geq 1} \alpha(p_j^k) p_j^{-ks}\right) := \prod_{j\geq 1} \varphi_j(s).$$

Since each φ_j only depends on the variable z_j we may apply the Bohr correspondence

$$\Phi(z) := \mathscr{B}\varphi(z) = \prod_{j\geq 1} \Phi_j(z_j)\,,$$

where $\Phi_j = \mathscr{B}\varphi_j$. For each j, H_{φ_j} is a one variable Hankel form, and so Nehari's theorem applies. Thus, there exists some Ψ_j in L^∞ such that $H_{\Psi_j} = H_{\Psi_j}$ and $\|H_{\Phi_j}\| = \|H_{\Psi_j}\|_\infty$. Define $\Psi(z) := \prod_{j\geq 1} \Psi_j(z_j)$.

It now follows directly from the one variable case and Lemma 4.4 that there exists some $\Phi = P_+\Psi$ such that $\|H_\Phi\| = \|H_\Psi\|_\infty$. This shows that the multiplicative Hankel form is bounded given that the symbol is multiplicative.

The second statement of Theorem 4.5 is just a reformulation of the fact that the set of bounded point evaluations for $H^1(\mathbb{T}^\infty)$ is in $\mathbb{D}^\infty \cap \ell^2$. In details, since φ_j is completely multiplicative we have

$$\varphi(z) = \prod_{j\geq 1} \frac{1}{1-\alpha(p_j)z_j}\,.$$

Using the formula for the geometric series again the norm can be written

$$\|\varphi\|_{\mathscr{H}^2}^2 = \sum_{n\geq 1} a(n)^2 = \prod_{j\geq 1} \frac{1}{1-\alpha(p_j)^2} < \infty$$

where again the Bohr correspondence was applied. From Theorem 3.15 we have the following point-estimate

$$|f(\alpha(p_1), \alpha(p_2), \cdots)| \leq \left(\prod_{j=1} \frac{1}{1-|\alpha(p_j|^2}\right) \|f\|_{\mathscr{H}^1} = \|\varphi\|_{\mathscr{H}^2}^2 \cdot \|f\|_{\mathscr{H}^1}$$

where again the Euler-product A.3 was used. Since $\|\varphi\|_{(H^1)^*} \leq \|\varphi\|_{\mathscr{H}^2}^2$, the calculation above shows that $H_{\mathscr{B}\varphi}$ is always bounded on $H^2(\mathbb{T}^\infty) \times H^2(\mathbb{T}^\infty)$ when the symbol is completely multiplicative. $\square$

We now turn to proving that Nehari's theorem does not hold in full generality by doing a proof by contradiction.

LEMMA 4.6. *Given that Nehari's theorem holds on $\mathbb{T}^\infty$ then there exists a constant C such that*

$$\inf_{P_+\psi=\varphi} \|\psi\|_{L^\infty(\mathbb{T}^\infty)} \leq C\|H_\varphi\|_{(H^2\odot H^2)^*} \tag{4.14}$$

holds for all $\varphi \in (H^2 \odot H^2)^*$.

PROOF. We define

$$X = L^\infty(\mathbb{T}^\infty)/\big(L^2(\mathbb{T}^\infty) \ominus H^2(\mathbb{T}^\infty)\big)\,,$$

where $L^2(\mathbb{T}^\infty) \ominus H^2(\mathbb{T}^\infty)$ denotes the orthogonal compliment of H^2 in $L^2(\mathbb{T}^\infty)$. The space X was chosen such that

$$\|\varphi\|_X = \inf_{P_+\psi=\varphi} \|\varphi\|_{L^\infty(\mathbb{T}^\infty)}$$

By assumption Nehari's theorem holds and thus H_φ has a symbol $\varphi \in L^2(\mathbb{T}^\infty)$, hence the map $T\colon (H^2 \odot H^2) \to X$ is well defined. The graph of T is closed in $H^2 \odot H^2$, and thus T is continuous and bounded. In other words there exists a constant C such that equation (4.14) holds, since

$$\|T\varphi\|_X \le C\|H_\varphi\|_{(H^2\odot H^2)^*}\,. \qquad \square$$

As noted earlier Ferguson and Lacey [17] and Lacey and Terwilleger [30] was able to prove the remarkable fact that Nehari's theorem extends to $1 \le d < \infty$. In other words H_ψ extends to a bounded form on $H^2(\mathbb{D}^d) \times H^2(\mathbb{D}^d)$ if and only if $\psi = P_+\varphi$ for some bounded function on φ on $\mathbb{T}^d$; here P_+ is the Riesz projection on $\mathbb{T}^d$. We define C_d as the smallest constant C that can be chosen in the estimate

$$\inf_{P_+\psi=\varphi} \|\psi\|_{L^\infty(\mathbb{T}^d)} \le C_d\|H_\varphi\|_{(H^2(\mathbb{T}^d)\odot H^2(\mathbb{T}^d))^*} \tag{4.15}$$

For $d = 1$, this is Nehari's original theorem and so $C_1 = 1$. Ortega-Cerdà and Seip was the first to prove that there is no analogue of Nehari's theorem on the infinite-dimensional polydisc [36].

THEOREM 4.7 (Ortega-Cerdà and Seip). *For every integer $d \ge 2$, the constant C_d in equation (4.15) is at least $(\pi^2/8)^{d/4}$.*

This gives a nontrivial lower bound for the constant appearing in the norm estimate in Nehari's theorem. As $(\pi^2/8)^{1/4} > 1$, we see that $C_d \to \infty$ when $d \to \infty$. Thus, by Lemma 4.6 the theorem above proves that Nehari's theorem does not extend to the infinite-dimensional polydisc.

We will offer a more instructive proof of Theorem 4.7 here, using the multiplicative nature of the Hankel form. To show that Nehari's theorem fails on $\mathbb{T}^\infty$, it will be enough to find a polynomial φ such at the constant C in equation (4.14) exceeds 1. Assume that such a symbol exists, in other words

$$\frac{\|\varphi\|_{(H^1(\mathbb{T}^d))^*}}{\|H_\varphi\|} \ge 1 + \delta\,, \qquad \delta > 0\,. \tag{4.16}$$

Where $\varphi(z) = \varphi(z_1, z_2, \ldots, z_d)$. Further we define φ_n as the product of n such symbols with distinct variables,

$$\varphi_n(z) = \varphi(z^{(1)})\varphi(z^{(2)})\cdots\varphi(z^{(n)})$$

Here $z^{(k)}$ is simply the short-hand notation for d-distinct variables, $z^{(1)} = z_1, \ldots, z_d$, similarly $z^{(2)} = z_{d+1}, \ldots, z_{2d}$ and so forth. From Lemma 4.4 it now follows that

$$\|H_{\varphi_n}\| = \|H_\varphi\|^n$$

and similarly by taking the norm of $\varphi_n(z)$ the inequality $\|\varphi_n\|_{(H^1)^*} \geq (\|\varphi\|_{(H^1)^*})^n$ is clear. Thus, equation (4.16) becomes

$$\frac{\|\varphi_n\|_{(H^1(\mathbb{T}^{nd}))^*}}{\|H_{\varphi_n}\|} \geq (1+\delta)^n\,, \qquad \delta > 0\,.$$

This contradicts Theorem 4.5 as $(1+\delta)^n \to \infty$ as $n \to \infty$, and gives a non-trivial lower bound for the constant in Nehari's theorem. Thus, finding such a symbol would prove that Nehari's theorem fails to hold on $\mathbb{T}^\infty$. The exsistence of such a symbol is shown in the following lemma.

LEMMA 4.8. *Let* $\varphi(z_1, z_2) = z_1 + z_2$ *then*

$$\frac{\|\varphi\|_{(H^1)^*}}{\|H_\varphi\|} \geq \frac{\pi}{2\sqrt{2}} > 1$$

PROOF. A rough but sufficient estimate shows

$$\|\varphi\|_{(H^1)^*} = \sup_{f\neq 0} \frac{|\langle f, \varphi\rangle|}{\|f\|_{H^1}} \geq \frac{|\langle \varphi, \varphi\rangle|}{\|\varphi\|_{H^1}} = \frac{\|\varphi\|_{H^2}^2}{\|\varphi\|_{H^1}}$$

Where we have estimated the $\big(H^1(\mathbb{T})\big)^*$-norm by testing φ against itself. As $\varphi(z > 1, z_2) = z_1 + z_2$, we clearly have $\|\varphi\|_{H_2}^2 = 2$, see Theorem 1.10 for details. The matrix of H_φ with respect to the standard basis of $H^2(\mathbb{T})$ is

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

As the spectral norm of a symmetric matrix is $\|M\| = \max_j |\lambda_j|$, where λ_j are the eigenvalues of A, a straightforward computation gives

$$\|H_\varphi\| = \|M\| = \sqrt{2}$$

That the largest eigenvalue is $\sqrt{2}$ can be seen from the characteristic polynomial $\lambda(\lambda^2 - 2)$ of M. While a computation at the level of coefficients shows

$$\|\varphi\|_{H^1} = \int_{\mathbb{T}^2} |z_1 + z_2|\,\mathrm{d}m_2 = \int_{\mathbb{T}} |1 + z|\,\mathrm{d}m = \frac{4}{\pi}\,, \tag{4.17}$$

where the second equality follows by symmetry, see Lemma A.12 for details. We are now done, since

$$C_1 \geq \frac{\|\varphi\|_{(H^1)^*}}{\|H_\varphi\|} \geq \frac{\|\varphi\|_{H^2}^2}{\|\varphi\|_{H^1}} \geq \frac{\pi}{2\sqrt{2}}\,. \qquad \square$$

Notice that the proof above also shows that the factorization in the polydisc is not norm-preserving, and therefore the weak factorization

$$H^1(\mathbb{T}^\infty) = H^2(\mathbb{T}^\infty) \odot H^2(\mathbb{T}^\infty)$$

does not hold.

4.3. Hilbert-Schmidt forms

While Nehari's theorem fails to hold in full generality on $\mathbb{T}^\infty$, we will show that it holds for a restrictive class of Hankel forms. By applying Cauchy–Schwarz on the multiplicative Hankel form we obtain the following crude estimate

$$\left|\sum_{m,n\geq 1} \varrho_{mn} a_m b_n\right|^2 \leq \sum_{n,m\geq 1} |\rho_{mn}|^2 \sum_{m\geq 1} |a_m|^2 \sum_{n\geq 1} |b_n|^2 .$$

From which it is clear that if

$$\sum_{m,n\geq 1} \left|\varrho_{mn}\right|^2 < \infty \,, \tag{4.18}$$

then the multiplicative hankel form is bounded, with bound at most the square root of the sum. We say that a multiplicative Hankel form with kernel ϱ is of *Hilbert-Schmidt* type if equation (4.18) holds. The study of the Hilbert–Schmidt class $\mathcal{S}^2$, was initiated by Helson in [25] where he asked the following question:

Does every multiplicative Hankel form in $\mathcal{S}^2$ have a bounded symbol?

Helson himself gave a positive answer to this in [26], and this section is dedicated to a reformulation of his proof. The terms of the sum in (4.18) are the same for all pairs (m, n) such that the product mn has a given value k. The number of ways to write k as a product of two integers is precisely the number of divisors, since $k = a \cdot k/a$ for every divisor a. Thus (4.18) becomes

$$\sum_{m,n\geq 1} \left|\varrho_{mn}\right|^2 = \sum_{n\geq 1} \left|\varrho_n\right|^2 d(n) \,, \tag{4.19}$$

where $d(n)$ is the number of divisors of n. The next is to prove that the last sum is related to Helson's inequality 1.31.

LEMMA 4.9. *Given $f \in H^1(\mathbb{D}^\infty)$ with coefficients a_n, then*

$$\sum_{\alpha\geq 0} \frac{|a_\alpha|^2}{(1+\alpha_1)(1+\alpha_2)\cdots} = \sum_{n\geq 1} |a_n|^2/d(n) \,. \tag{4.20}$$

PROOF. Let n have the prime factoring $n = \prod p_j^{\alpha_j}$. The divisors of n are obtained by replacing each α_j by all β_j satisfying $0 \leq \beta_j \leq \alpha_j$. Thus, the number of divisors is

$$d(n) = \prod_j (1+\alpha_j) \,,$$

where only a finite number of the numbers α_j differ from 0. Using this representation of $d(n)$ on the right-hand side of equation (4.20) completes the proof. □

Thus, using the Cauchy–Schwarz on (4.19) and Lemma 4.9 we obtain

$$|\langle f, \varphi\rangle| \leq \left(\sum_{n\geq 1} \frac{|a_n|^2}{d(n)}\right)^{1/2} \left(\sum_{n\geq 1} |\rho_n|^2 d(n)\right)^{1/2} \leq C\|f\|_{\mathscr{H}^1}$$

This shows that $\|\varphi\|_{(\mathscr{H}^1)^*} \leq C$, thus every multiplicative Hankel form in $\mathcal{S}^2$ indeed have a bounded symbol.

Remark. The restriction in equation (4.18) is very strict. Let $\lambda = \{\lambda_n\}_{n\geq 1}$ be the eigenvalues to the matrix $(\varrho_{mn})_{m,n\geq 1}$, then an Hankel form is bounded if λ is bounded – in fact the matrix norm is $\max_n |\lambda_n|$. In comparison Hankel forms in the Hilbert–Schmidt class $\mathcal{S}^2$ requires $\lambda \in \ell^2$.

4.4. Some related open problems

We end this chapter by including some remarks on a few open problems related to the topics in this thesis.

The Embedding constant for $p = 2$. *Determine the smallest constant, such that*

$$\left(\int_0^1 |f(1/2+it)|^2\right)^{\frac{1}{2}} \leq C\|f\|_{\mathscr{H}^2} \tag{4.21}$$

is sharp for all $f \in \mathscr{H}^2$. A rough estimate as done in [35] shows that

$$\begin{aligned}\|f\|_{H_i^2}^2 &= \frac{1}{\pi}\int_{\mathbb{R}} |f(1/2+it)|^2 \frac{\mathrm{d}t}{1+t^2} \\ &\leq \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \frac{1}{1+k^2}\int_k^{k+1} |f(1/2+it)|^2 \,\mathrm{d}t \leq \coth(\pi) C^2 \|f\|_{\mathscr{H}^2}\,,\end{aligned}$$

thus giving the estimate $\sqrt{2}/\sqrt{\coth(\pi)} \leq C$. Since $\sqrt{\coth(\pi)} \approx 1.0018$ one might conjecture that the bound can be improved to 1 such that equation (4.21) is sharp with constant $\sqrt{2}$. However this is still an open problem.

The Embedding problem. *Fix an exponent $p > 2$. Does there exists a constant $C_p < \infty$ such that*

$$\left(\int_0^1 |f(1/2+it)|^p\right)^{\frac{1}{p}} \leq C_p\|f\|_{\mathscr{H}^p} \tag{4.22}$$

for every Dirichlet polynomial f? In the case $p = 2k$ the answer is trivially true: just apply the case $p = 2$ to the function f^k in $\mathscr{H}^2$. This provides some evidence in favour of a positive answer, however due note that a recent result by Harper [21] proved that for all $0 < p < 2$ there exists functions such that equation (4.22) fails to hold.

This problem was discussed in more depth in section 3.3.2 and it seems likely that further progress will require novel and unconventional combinations of tools from functional, and complex analysis, as well as from analytic number theory. See [45] for a further discussion on the problem.

Bounded Hankel forms without a bounded symbol. The proof showed in section 4.2 first proven by Ortega-Cerdà and Seip [36] is non-constructive. Meaning that no concrete example of a bounded Hankel form without a bounded has been found.

Bounded symbol for the multiplicative Hankel matrix *Is the symbol* $\varphi(s) = \sum_{n\geq 2}(\log n)^{-1}n^{-1/2-s}$ *is the Riesz projection of a function in* $L^\infty(\mathbb{T}^\infty)$*?*

This is a natural question as it was shown to hold true for the Hilbert matrix, see section 2.3.3. As $\varphi(s)$ neither is completely multiplicative or has square summable coefficients neither Theorem 4.5 nor section 4.3 answers the question above. An equivalent formulation is to ask whether there exists a constant $C < +\infty$ such that

$$\left| a_1 + \sum_{n\geq 2} \frac{a_n}{\sqrt{n}\log n} \right| \leq C\|f\|_{\mathscr{H}^p}, \tag{4.23}$$

holds for every Dirichlet polynomial $f(s) = \sum_{n=1}^{N} a_n n^{-s}$ when $p = 1$? Clearly equation (4.23) holds for $p = 2$, and it was shown recently by Bayart and Brevig [5] that equation (4.23) holds for all $p > 1$. Whether equation (4.23) holds for $p = 1$ is still unclear.

APPENDIX A

Preliminaries

1.1. Euler products

The next theorem, discovered by Euler in 1737, is sometimes called the analytic version of the fundamental theorem of arithmetic.

THEOREM A.1. *Let f be a multiplicative arithmetical function such that $\sum f(n)$ is absolutely convergent. Then the sum of the series can be expressed as an absolutely convergent infinite product,*

$$\sum_{n\geq 1} f(n) = \prod_{p} \{\, 1 + f(p) + f(p^2) + \cdots \,\}\,, \tag{A.1}$$

extended over all primes. If f is completely multiplicative, the product simplifies to

$$\sum_{n\geq 1} f(n) = \prod_{p} \frac{1}{1 - f(p)}\,. \tag{A.2}$$

REMARK. In each case the product is called the *Euler product* of the series.

PROOF. Consider the product

$$P(x) = \prod_{p\leq x} 1 + f(p) + f(p^2) + \cdots$$

extended over all primes $p \leq x$. Since this is the product of a finite number of absolutely convergent series we may rearrange the terms in any fashion without altering the sum. By the fundamental theorem of arithmetic we can write

$$P(x) = \sum_{n\in A} f(n)$$

where A consists of those n having all their prime factors less than x. Hence

$$\sum_{n\geq 1} f(n) - P(x) = \sum_{n\in B} f(n)\,,$$

where B is the set of n having at least one prime factor $> x$. Therefore

$$\left|\sum_{n\geq 1} f(n) - P(x)\right| \leq \sum_{n\in B} |f(n)| \leq \sum_{n\geq x} f(n)$$

and thus $P(x) \to \sum f(n)$ as $x \to \infty$. This follows since $f(n)$ converges absolutely and the sum on the right tends to zero as $n \to \infty$. As seen earlier $\prod 1+a_n$ converges absolutely whenever $\sum a_n$ converges absolutely. In this case,

$$\sum_{p\le x} \big|f(p)+f(p^2)\cdots\big| \le \sum_{p\le x}\big|f(p)\big| + \big|f(p^2)\big|\cdots \le \sum_{n=2}|f(n)|$$

which follows from the triangle inequality. Since all the partial sums are bounded, the series of positive terms

$$\sum_p \big|f(p)+f(p^2)+\cdots\big|$$

converges, and this implies absolute convergence of the product in equation (A.1). When f is a completely multiplicative function we have $f(p^n)=f(p)^n$ and each series on the right of (A.1) is a convergent geometric series, in detail

$$\begin{aligned}\sum_{n\ge1} f(n) &= \prod_p \{\,1+f(p)+f(p^2)+\cdots\} \\ &= \prod_p \sum_{n\ge0} f(p)^n = \prod_p \frac{1}{1-f(p)}\end{aligned}$$ □

In particular if we apply Theorem A.1 to absolutely convergent Dirichlet series we immediately obtain

Theorem A.2. *Assume $\sum_{n\ge1} f(n)n^{-s}$ converges absolutely for $\sigma>\sigma_a$. If f is multiplicative we have*

$$\sum_{n\ge1}\frac{f(n)}{n^s} = \prod_p\left\{\,1+\frac{f(p)}{p^s}+\frac{f(p^2)}{p^{2p}}+\cdots\right\} \quad \text{if } \sigma>\sigma_a\,,$$

and if f is completely multiplicative we have

$$\sum_{n\ge1}\frac{f(n)}{n^s} = \prod_p \frac{1}{1-f(p)p^{-s}} \qquad \text{if } \sigma>\sigma_a$$

Taking $f(n)=1$ we immediately obtain the following Euler product.

Corollary A.3. *If* $\mathrm{Re}(s)>1$ *then*

$$\zeta(s)=\sum_{n\ge1}\frac{1}{n^s}=\prod_p\frac{1}{1-p^{-s}}\,.$$

1.2. Inequalites

Three of the most famous “classical inequalitues” are those of Cauchy, Hölder and Minkowski. These inequalities are omitted from the main part due to their general or elementary nature, but are used so frequently that a short treatise is justified.

Our main reference is Hardy, Littlewood, and Pólya [20], for an excellent introduction to these inequalities see the monograph by Steele [47]. Throughout this section, we assume that $p > 1$ and p, q are real constants satisfying

$$\frac{1}{p} + \frac{1}{q} = 1\,,$$

unless otherwise stated. We denote p and q as the *Hölder conjugates.*

LEMMA A.4 (Young's inequality). *Let a and b be real non-negative constants. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}\,, \tag{A.3}$$

where equality holds if and only if $a^p = b^q$. Here p and q are the Hölder conjugates.

PROOF. As $(a-b)^2 = a^2 + b^2 - 2ab \geq 0$, the claim is true for $p = q = 2$. This is often used as the start in proving Cauchy-Schwarz inequality. Similarly the claim is certainly true if $b = 0$ or $a = 0$, we therefore assume that $a > 0$ and $b > 0$. Since $(\log x)'' = -1/x^2$, the logarithm is convex for all $x > 0$, thus for all $\theta \in [0,1]$ and $x, y \in \mathbb{R}_+$ the following inequality always hold

$$\log\big[(1-\theta)x + \theta y\big] \geq (1-\theta)\log x + \theta \log y\,. \tag{A.4}$$

Set $x = a^p$ and $y = b^q$, if $1 - \theta = 1/p$ then $\theta = 1/q$. So (A.4) becomes

$$\log\Big(\frac{a^p}{p} + \frac{b^q}{q}\Big) \geq \frac{1}{p}\log a^p + \frac{1}{q}\log b^q = \log a + \log b = \log(ab)\,, \tag{A.5}$$

with equality if and only if $a^p = b^q$. Young's inequality follows by exponentiation. □

THEOREM A.5 (Hölder's inequality). *Let p, q be the Hölder conjugates, and $a \in \ell^p$, $b \in \ell^q$ be sequences. Then*

$$\sum_{n=0}^{\infty} |a_n b_n| \leq \Big(\sum_{n\geq 0} |a_n|^p\Big)^{1/p} \Big(\sum_{n\geq 0} |b_n|^q\Big)^{1/q} \tag{A.6}$$

PROOF. For simplicity we begin by defining

$$S = \Big(\sum_{n\geq 0} |a_n|^p\Big)^{1/p} \quad , T = \Big(\sum_{n\geq 0} |b_n|^q\Big)^{1/q}\,. \tag{A.7}$$

By replacing a and b in (A.3) by $|a_n|/S$ and $|b_n|/T$ respectively, we get the inequalities

$$\frac{|a_n|}{S}\frac{|b_n|}{T} \leq \frac{1}{p}\left(\frac{|a_n|}{S}\right)^p + \frac{1}{q}\left(\frac{|b_n|}{T}\right)^q \quad (n = 1, 2, \ldots)$$

Adding up the right and left-hand sides of the inequalities for all $n \in \mathbb{N}$, using (A.7) and $1/p + 1/q = 1$, we get

$$\frac{1}{ST}\sum_{n=0}^{\infty}|a_n b_n| \leq \frac{1}{p}\frac{\sum |a_n|^p}{S^p} + \frac{1}{q}\frac{\sum |b_n|^p}{T^q} = \frac{1}{p} + \frac{1}{q} = 1$$

Finally multiplying the equation above by ST, we obtain Hölder's inequality. □

COROLLARY A.6 (Cauchy–Schwarz inequality). *Let $a = \{a_m\}_{m\geq 0}$, $b = \{b_n\}_{n\geq 0}$ be families of complex numbers. If $a, b \in \ell^2$ then*

$$\Big|\sum_{n=0}^{\infty} a_n b_n\Big|^2 \leq \Big(\sum_{n=0}^{\infty}|a_n|^2\Big)\Big(\sum_{n=0}^{\infty}|b_n|^2\Big)$$

THEOREM A.7 (Hölders inequality). *Let (X, Σ, μ) be a measure space and suppose f and g are Σ-measurable complex valued functions on X. Then*

$$\int_X |fg|\,\mathrm{d}\mu \leq \left(\int_X |f|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}}\left(\int_X |g|^q\,\mathrm{d}\mu\right)^{\frac{1}{q}}$$

where p, q are the Hölder conjugates. Equality holds when $|f(x)|^p = |g(x)|^q$ holds for almost every $x \in X$.

PROOF. Similar to our proof for the discrete case we introduce the normalized functions $\widetilde{f} = f/\|f\|_p$ and $\widetilde{g} = g/\|g\|_p$, such that

$$\int_X \big|\widetilde{f}\big|^p\,\mathrm{d}\mu = \int_X \big|\widetilde{g}\big|^q\,\mathrm{d}\mu = 1\,. \tag{A.8}$$

Applying Young's inequality A.4 to $\widetilde{f}$ and $\widetilde{g}$ yields

$$\left|\widetilde{f}\cdot\widetilde{g}\right| \leq \frac{1}{p}\big|\widetilde{f}\big|^p + \frac{1}{q}\big|\widetilde{g}\big|^q$$

Since inequalities are preserved under integration, we can integrate

$$\int_X \left|\widetilde{f}\cdot\widetilde{g}\right|\mathrm{d}\mu \leq \frac{1}{p} + \frac{1}{q} = 1$$

where equation (A.8) was used in the first inequality. Multiplying by $\|f\|_p\|g\|_q$ and using the definition of $\widetilde{f}$ and $\widetilde{g}$ completes the proof. □

COROLLARY A.8 (Cauchy–Schwarz inequality). *Let (X, Σ, μ) be a measure space and suppose f and g are Σ-measurable complex valued functions on X. Then*

$$\int_X |fg|\,\mathrm{d}\mu \leq \left(\int_X |f|^2\,\mathrm{d}\mu\right)^{\frac{1}{2}}\left(\int_X |g|^2\,\mathrm{d}\mu\right)^{\frac{1}{2}}$$

Equality holds when $|f(x)|^2 = |g(x)|^2$ holds for almost every $x \in X$.

THEOREM A.9 (Minkowski's inequality). *Let (X, Σ, μ) be a measure space and suppose f and g are Σ-measurable complex valued functions on X. Then*

$$\left(\int_X |f+g|^p\,\mathrm{d}\mu\right)^p \leq \left(\int_X |f|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p\,\mathrm{d}\mu\right)^{\frac{1}{p}} \tag{A.9}$$

where p, q are the Hölder conjugates. Equality holds when $|f(x)|^p = |g(x)|^p$ holds for almost every $x \in X$.

PROOF. For short the inequality above can be written $\|f+g\|_p \leq \|f\|_p + \|g\|_p$. The case $p = 1$ follows directly from the triangle inequality. Assume therefore that $p > 1$, since $|f+g|^p = |f+g||f+g|^{p-1}$ we can again use the triangle inequality,

$$\int_X |f+g|^p \, \mathrm{d}\mu \leq \int_X |f||f+g|^{p-1} \, \mathrm{d}\mu + \int_X |g||f+g|^{p-1} \, \mathrm{d}\mu \ . \tag{A.10}$$

If we then apply Hölder's inequality A.7 separately to each of the bounding sums, we find that

$$\int |f||f+g|^{p-1} \, \mathrm{d}\mu \leq \left(\int_X |f|^p \right)^{\frac{1}{p}} \left(\int_X |f+g|^{q(p-1)} \, \mathrm{d}\mu \right)^{(p-1)/p}$$
$$\int |g||f+g|^{p-1} \, \mathrm{d}\mu \leq \left(\int_X |g|^p \right)^{\frac{1}{p}} \left(\int_X |f+g|^{q(p-1)} \, \mathrm{d}\mu \right)^{(p-1)/p} .$$

Thus, in our shorthand notation (A.10) an be written

$$\|f+g\|_p^p \leq \|f\|_p \|f+g\|_p^{p-1} + \|g\|_p \|f+g\|_p^{p-1} \, . \tag{A.11}$$

Since Minkowski's inequality (A.9) is trivial when $\|f+g\|_p = 0$, we can assume without loss of generality that $\|f+g\|_p > 0$. Thus, we can divide both sides of the bound (A.11) by $\|f+g\|_p^{p-1}$ to complete the proof. □

THEOREM A.10 (Minkowski's continuous Inequality). *Let (X, Σ, μ) and (Y, Ω, ν) be σ-finite measure spaces, and assume that $f \colon X \times Y \to \mathbb{C}$ is a measurable function. For $1 \leq p < \infty$ we have*

$$\left[\int_X \left(\int_Y |f(x,y)| \mathrm{d}\nu(y) \right)^p \mathrm{d}\mu(x) \right]^{\frac{1}{p}} \leq \int_Y \left(\int_X |f(x,y)|^p \right)^{\frac{1}{p}} \mathrm{d}\nu(y) \tag{A.12}$$

Informally we can repeatedly apply Minkowski's inequality A.9 to get

$$\left\| \sum_k f_k \right\|_p \leq \sum_k \|f_k\|_p$$

Then using the scaling property of $\|\cdot\|_p$, we can write

$$\left\| \sum_k f_k \Delta_k \right\|_p \leq \sum_k \|f_k\|_p \Delta_k$$

So we can write a Riemann sum for the integral inequality (A.12) and pass to the limit to get the desired integral inequality. A more formal proof is shown below.

PROOF. Since our measure space are σ-finite and $|f(x,y)| \leq 0$ we can apply Tonelli's theorem where needed. In particular, the case $p = 1$ follows directly from Tonelli's theorem with equality. Assume therefore that $p > 1$. Now, if

$$\int_X \left(\int_Y |f(x,y)| \mathrm{d}\nu(y) \right)^p \mathrm{d}\mu(x) = 0 \, ,$$

we are done. Assume therefore that it is strictly positive and define

$$g(x) := \left(\int_Y |f(x,y)| \mathrm{d}\nu(y)\right)^{p-1}.$$

By observing that $(p-1)q = p$ the norm of g can be computed

$$\|g\|_{L^q} = \left[\int_X g(x)^q \mathrm{d}\mu(x)\right]^{\frac{1}{q}} = \left[\int_X \left(\int_Y |f(x,y)| \mathrm{d}\nu(y)\right)^p \mathrm{d}\mu(x)\right]^{\frac{1}{q}} =: I^{1/q}$$

The inequality now follows by applying Tonelli's theorem and Hölder's inequality to

$$\begin{aligned}
I &= \int_X \int_Y |f(x,y)| g(x) \mathrm{d}\nu(y)\, \mathrm{d}\mu(x) = \int_Y \int_X |f(x,y)| g(x) \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) \\
&\leq \int_Y \left[\left(\int_X |f(x,y)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} \left(\int_X g(x)^q \mathrm{d}\mu(x)\right)^{\frac{1}{q}}\right] \mathrm{d}\nu(y) \\
&= \int_Y \left(\int_X |f(x,y)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} \|g\|_{L^q}\, \mathrm{d}\nu(y) = I^{\frac{1}{q}} \int_Y \left(\int_X |f(x,y)|^p \mathrm{d}\mu(x)\right)^{\frac{1}{p}} \mathrm{d}\nu(y)
\end{aligned}$$

Dividing by $I^{1/q}$ and using that $I/I^{1/q} = I^{1-1/q} = I^{1/p}$ completes the proof. □

1.3. Integrals and sums

LEMMA A.11.

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|1-e^{i\theta}\right| \mathrm{d}\theta = 0\,.$$

PROOF. Expanding the logarithm as a Taylor series gives a quick proof

$$\int_0^{2\pi} \log\left|1-e^{i\theta}\right| \mathrm{d}\theta = -\int_0^{2\pi}\left(\sum_{n\geq 1}\frac{e^{i\theta}}{n}\right)\mathrm{d}\theta = i\sum_{n\geq 1}\frac{e^{2\pi i n}-1}{n} = 0\,.$$

However, the justification of termwise integration is not entirely trivial as the series is not absolutely convergent. For a more formal proof using Cauchy's residue theorem see [44, p. 307]. A slightly longer solution would be to note that

$$\int_0^{2\pi} \log\left|1-e^{i\theta}\right| \mathrm{d}\theta = \int_0^{2\pi} \log\left|2\sin(\theta/2)\right| \mathrm{d}\theta = 2\pi\log 2 + 2\int_0^{\pi}\log\sin x\,\mathrm{d}x \quad \text{(A.13)}$$

The last integral is quite famous, see [1, p. 206]. More elementary

$$\begin{aligned}\int_0^{\pi}\log\sin x\,\mathrm{d}x &= \left(\int_0^{\pi/2}+\int_{\pi/2}^{\pi}\right)\log\sin x\,\mathrm{d}x\\ &= \int_0^{\pi/2}\log\sin x + \int_0^{\pi/2}\log\sin\left(\frac{\pi}{2}+x\right)\mathrm{d}x\\ &= \int_0^{\pi/2}\log\frac{1}{2}\sin(2x)\,\mathrm{d}x = -\frac{\pi}{2}\log 2 + \frac{1}{2}\int_0^{\pi}\log\sin x\,\mathrm{d}x\end{aligned}$$

Where $u \mapsto x-\pi/2$, $\sin(\pi/2+x) = \cos x$ and $\sin(2x) = 2\sin x\cos x$ were used. Thus, $\int_0^{\pi}\log\sin x\,\mathrm{d}x = -\pi\log 2$, meaning equation (A.13) is zero and we are done. $\square$

LEMMA A.12. *Let* $\varphi(z) = z_1 + z_2$

$$\int_{\mathbb{T}}\int_{\mathbb{T}}|z_1+z_2|\,\mathrm{d}m(z_1)\,\mathrm{d}m(z_2) = \int_{\mathbb{T}}|1+z|\,\mathrm{d}m(z) = \frac{4}{\pi}\,, \quad \text{(A.14)}$$

PROOF. The first equality follows by symmetry. Since $|e^{-iy}| = 1$, we have the equality $|e^{-ix}+e^{-iy}| = |e^{-i(x-y)}+1|$. Thus,

$$\begin{aligned}\int_{\mathbb{T}}\int_{\mathbb{T}}|z_1+z_2|\,\mathrm{d}m(z_1)\,\mathrm{d}m(z_2) &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}|1+e^{-i(x-y)}|\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{1}{(2\pi)^2}\int_y^{y+2\pi}\int_0^{2\pi}|1+e^{-ix}|\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{1}{2\pi}\int_0^{2\pi}|1+e^{-ix}|\,\mathrm{d}x = \int_{\mathbb{T}}|1+z|\,\mathrm{d}m(z)\,,\end{aligned}$$

by the linear substitution $x \mapsto x-y$. For the last equality in (A.14) we have,

$$\int_{\mathbb{T}}|1+z|\,\mathrm{d}m(z) = \frac{1}{2\pi}\int_0^{2\pi}\left|2\cos\left(\frac{x}{2}\right)\right|\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi}\cos\left(\frac{x}{2}\right)\mathrm{d}x = \frac{4}{\pi}\,. \qquad \square$$

LEMMA A.13. *Let $a \in \mathbb{R}/\{0\}$, and $g(x) = 1/(x^2 + a^2)$, then*

$$\hat{g}(s) = \int_{\mathbb{R}} \frac{e^{-ixs}}{a^2 + x^2}\,\mathrm{d}x = \frac{\pi}{|a|} e^{-|as|}\,. \tag{A.15}$$

PROOF. This integral is often used to show the benefits of contour integration. Since $a \neq 0$ we can remove the variable a, through the substitution $x \mapsto u|a|$,

$$\hat{g}(k) = \frac{1}{|a|} \int_{\mathbb{R}} \frac{e^{-iuk}}{1 + u^2}\,\mathrm{d}u\ . \tag{A.16}$$

Where the variable $k = |a|s$ was introduced. Suppose $s > 0$ and define the contour C_R that goes along the real line from $-R$ to R and then counterclockwise along a semicircle centered at 0 from R to $-R$. See figure 1. Take R to be greater than 1, so

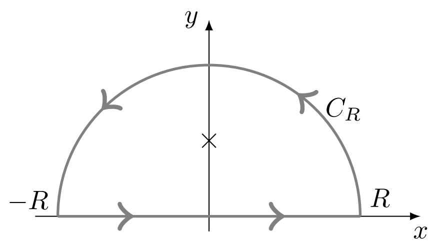

FIGURE 1. The half-circle contour

that the imaginary unit i is enclosed within the curve. Since $1+z^2 = (1-iz)(1+iz)$ the only singularity within C_R is at $z = i$, and by the Cauchy residue theorem

$$\int_{C_R} f(z)\,\mathrm{d}z = \int_{C_R} \frac{e^{-ikz}}{1+z^2}\,\mathrm{d}z = 2\pi i \cdot \operatorname*{Res}_{z=i} f(z) = 2\pi i \cdot \frac{e^{-k}}{2i} = \pi e^{-k}\,.$$

The contour C_R may be split into two parts,

$$\int_{C_R} f(z) = \int_{-a}^{a} f(z)\,\mathrm{d}z + \int_{\text{arc}} f(z)\,\mathrm{d}z\,, \tag{A.17}$$

and using some simple estimations, we have

$$\left| \int_{\text{arc}} \frac{e^{ikz}}{1+z^2}\,\mathrm{d}z \right| \leq \int_{\text{arc}} \frac{1}{|1+z^2|}\,\mathrm{d}z \leq \int_{\text{arc}} \frac{\mathrm{d}z}{R^2 - 1} = \frac{\pi R}{R^2 - 1}\,.$$

Thus, letting $R \to \infty$ in equation (A.17) we obtain

$$\int_{\mathbb{R}} \frac{e^{-ixs}}{1+x^2}\,\mathrm{d}x = \pi e^{-k}\,.$$

If $s < 0$ then a similar argument with an arc $C_{R'}$ that winds around $-i$ rather than i shows that

$$\int_{\mathbb{R}} \frac{e^{-ixk}}{1+x^2}\,\mathrm{d}x = \pi e^{k}$$

Combining the last two equations with equation (A.16) completes the proof.

PROOF: 2. This integral may also be evaluated without the use of complex analysis. Note by symmetry that

$$\int_{\mathbb{R}} \frac{e^{-ixt}}{1+t^2}\,\mathrm{d}t = \int_{\mathbb{R}} \frac{\cos(xt)+i\sin(xt)}{1+t^2}\,\mathrm{d}t = \int_{\mathbb{R}} \frac{\cos(xt)}{1+t^2}\,\mathrm{d}t$$

since $\sin(at)/(1+t^2)$ is odd. Using the Laplace-transform $\mathscr{L}(f) = \int_0^\infty f(x)e^{-sx}\,\mathrm{d}x$, on the last integral we obtain

$$\begin{aligned}\mathscr{L}\left(\int_0^\infty \frac{\cos(xt)}{1+t^2}\,\mathrm{d}t\right) &= \int_0^\infty \left(\int_0^\infty \cos(xt)e^{-sx}\,\mathrm{d}x\right)\frac{\mathrm{d}t}{1+t^2} \\ &= \int_0^\infty \frac{s}{s^2+t^2}\frac{\mathrm{d}t}{1+t^2} = \frac{\pi}{2}\frac{1}{1+s}\,.\end{aligned}$$

Since $\mathscr{L}(e^{-|t|}) = \int_0^\infty e^{-|t|}e^{-ts}\,\mathrm{d}t = 1/(1+s)$, taking the inverse $\mathscr{L}$-transform yields

$$\int_{\mathbb{R}} \frac{\cos xt}{1+t^2}\,\mathrm{d}x = 2\mathscr{L}^{-1}\left(\frac{\pi}{2}\frac{1}{1+s}\right) = \pi e^{-|x|}\,. \tag{A.18}$$

To evaluate the $\mathscr{L}$-transform the order of integration was interchanged, the justification follows from Fubini since the integrand is absolutely integrable

$$\left|\int_{\mathbb{R}_+^2} \frac{\cos(xt)e^{-sx}}{1+t^2}\mathrm{d}(x,t)\right| \leq \int_{\mathbb{R}_+^2} \frac{e^{-x}}{1+t^2}\mathrm{d}(x,t) = \frac{\pi}{2} < \infty\,.$$

Where the short-hand notation $\mathbb{R}_+^2 = [0,\infty)\times[0,\infty)$ was used. □

LEMMA A.14. *Let $a \in \mathbb{R}/\{0\}$ then*

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth(\pi a)$$

PROOF. This follows from the Poisson summation formula (Rudin [44, p. 194])

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{w\in\mathbb{Z}} \hat{f}(2\pi w)\,,$$

and Lemma A.13. Then

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^2+a^2} = \frac{\pi}{|a|}\sum_{w\in\mathbb{Z}} e^{-2\pi|aw|} = \frac{\pi}{|a|}\frac{e^{2\pi|a|}+1}{e^{2\pi|a|}-1} = \frac{\pi}{a}\coth(\pi a)$$

where we used the extended geometric formula

$$\sum_{n\in\mathbb{Z}} r^{|n|} = 1 + 2\sum_{n\geq 0} r^n = \frac{1+r}{1-r}\,,$$

and the absolute value was dropped since $\coth(-a) = -\coth(a)$. □

Proposition A.15. *For all $\theta \in (0, 2\pi)$ we have,*

$$\sum_{n\geq 1} \frac{\sin(n\theta)}{n} = \frac{\pi - x}{2} \tag{A.19}$$

Proof. This result follows immediately from using the Poisson summation formula on $f(\theta) = \mathbf{1}_{[-\pi,\pi]}(\pi-\theta)/2$. However it is possible to calculate the sum without knowing the result in advance. By the Abel's theorem

$$f(\theta) := \sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = \lim_{s\to 0^+} \sum_{n=1}^{\infty} \frac{\sin n\theta}{n} e^{-ns}.$$

Utilizing the Taylor expansion of the logarithm, gives

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\sin n\theta}{n} e^{-ns} &= \operatorname{Im} \sum_{n=1}^{\infty} \frac{e^{n(i\theta - s)}}{n} = -\operatorname{Im}\log(1 - e^{i\theta - s}) \\ &= -\operatorname{Im}\log(1 - e^{-s}\cos\theta - ie^{-s}\sin\theta) = \arctan\left(\frac{e^{-s}\sin\theta}{1 - e^{-s}\cos\theta}\right).\end{aligned}$$

Thus letting $s \to 0^+$,

$$f(\theta) = \arctan\left(\frac{\sin\theta}{1-\cos\theta}\right) = \arctan\left(\cot\frac{\theta}{2}\right) = \arctan\left(\tan\frac{\pi-\theta}{2}\right).$$

As $\theta \in (0, 2\pi)$ this completes our proof. □

The Gamma function. Let $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$. We define the Γ-function for $\sigma > 0$ by

$$\Gamma(s) := \int_0^{\infty} e^{-t} t^{s-1}\, \mathrm{d}t$$

The Γ function is analytic for $\sigma > 0$, and it's most important properties are shown in the following theorem.

Theorem A.16 (Bohr-Mollerup). *Let $f : (0,\infty) \to (0,\infty)$ satisfy the following properties*

(1) $f(1) = 1$
(2) $f(x+1) = f(x)$
(3) $\log f$ *is convex.*

Then $f(x) = \Gamma(x)$ for all $x \in (0,\infty)$.

Proposition A.17 (Euler's reflection formula). *For every $0 < s < 1$, then*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

The standard way to prove A.17 is to use the Weierstrass product formula for $\sin(\pi x)$ and $\Gamma(x)$ see Artin [3] for details. While this computation is very straightforward, the derivation of these product formulas are cumbersome, instead we will rewrite $\Gamma(s)\Gamma(1-s)$ as the following integral.

$$\begin{aligned}\int_0^\infty \frac{x^{s-1}}{1+x}\,\mathrm{d}x &= \int_0^\infty\int_0^\infty e^{-(1+x)y}x^{s-1}\,\mathrm{d}x\,\mathrm{d}y\\ &= \int_0^\infty\int_0^\infty e^{-y-u}\left(\frac{u}{y}\right)^{s-1}\frac{\mathrm{d}u}{y}\,\mathrm{d}y\\ &= \left(\int_0^\infty e^{-y}y^{-s}\,\mathrm{d}y\right)\left(\int_0^\infty e^{-u}u^{s-1}\,\mathrm{d}u\right) = \Gamma(s)\Gamma(1-s)\end{aligned}$$

where $\int_0^\infty \exp(-(x+1)y)\,\mathrm{d}y = 1/(1+x)$ and $x \mapsto u/y$ was used to rewrite the integral. We note that the integral in question is the Mellin transform of $1/(1+x)$.

Definition. Suppose $f\colon [1,\infty)$ is locally Lebesgue integrable, and satisfies the growth condition $|f(x)| \leq Ax^B$. We define *Mellin transformation* of f to be

$$\mathcal{M}_f(s) = \int_0^\infty \frac{f(x)}{x^{1-s}}\,\mathrm{d}x\ .$$

Hence Proposition A.17 is proven by the following lemma.

Lemma A.18. *Let $f(x,y) = (x+y)^{-1}$ be the homogenity kernel. Then*

$$\mathcal{M}_f(s) := \int_0^\infty \frac{f(x)}{x^{1-s}}\,\mathrm{d}x = \frac{1}{y^{1-s}}\frac{\pi}{\sin\pi s}\,, \tag{A.20}$$

for every $0 < \mathrm{Re}(s) < 1$. If $y=1$ then there exists constants $1/p + 1/q = 1$ such that,

$$\mathcal{M}_f(1/p) = \mathcal{M}_f(1/q)\,. \tag{A.21}$$

Proof. Equation (A.21) follows directly from (A.20), since

$$\sin(\pi/p) = \sin(\pi(1-1/q) = \sin\pi\cos\pi/q - \cos\pi\sin\pi/q = \sin(\pi/q)\,,$$

and the fact that $y=1$. Using the substitution $x \mapsto y\cdot t$, we obtain

$$\mathcal{M}_f(s) = \frac{1}{y^{1-s}}\int_0^\infty \frac{1}{1+t}\frac{\mathrm{d}t}{t^{s-1}} := \frac{1}{y^{1-s}}\mathcal{M}_g(s)\,,$$

and the Mellin transform of $g(x) = 1/(1+x)$ will be evaluated using complex analysis. Consider the branch of $z^{s-1}/(z+1)$ defined on the slit plane $C\backslash[0,\infty)$ by

$$f(z) = \frac{r^{s-1}e^{i(s-1)\theta}}{z+1}\,,$$

where $z = re^{i\theta}$, and $\theta \in (0,2\pi)$. For small ε and $R > 1$, we consider the keyhole domain C, consisting of z in the slit plane $\mathbb{C}\backslash[0,\infty)$ satisfying $\varepsilon < |z| < R$, see figure 2. Since f has a simple pole at $= -1$, Cauchy's residue theorem yields

$$\int_C f(z) = 2\pi i \operatorname*{Res}_{z=-1} f(z) = -2\pi i e^{\pi i s}\,. \tag{A.22}$$

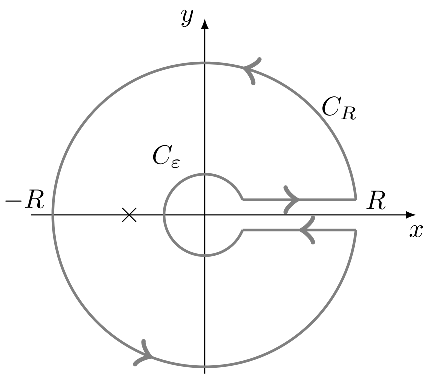

FIGURE 2. The keyhole contour

The integral over C breaks into the sum of 4 integrals.

$$\begin{aligned}\int_C f(z)\,\mathrm{d}z = \int_\varepsilon^R \frac{x^{s-1}}{x+1}\,\mathrm{d}x + \int_{C_R} \frac{z^{s-1}}{z+1}\,\mathrm{d}z \\ + \int_R^\varepsilon \frac{e^{2\pi i(s-1)}x^{s-1}}{1+x}\,\mathrm{d}x + \int_{C_\varepsilon} \frac{z^{s-1}}{z+1}\,\mathrm{d}z\,,\end{aligned} \tag{A.23}$$

and for the integrals over C_R and C_ε we obtain the following estimates

$$\left|\int_{C_R} \frac{z^{s-1}}{z+1}\,\mathrm{d}z\right| \le \frac{R^{s-1}}{R-1} 2\pi R = O(R^s)\,, \tag{A.24}$$

$$\left|\int_{C_\varepsilon} \frac{z^{s-1}}{z+1}\,\mathrm{d}z\ .\right| \le \frac{\varepsilon^{s-1}}{1-\varepsilon} 2\pi\varepsilon = O(\varepsilon^s) \tag{A.25}$$

Letting $R \to \infty$ and $\varepsilon \to 0$ in equation (A.23) gives

$$\left(1 - e^{2\pi i(s-1)}\right) \int_0^\infty \frac{x^{s-1}}{x+1}\,\mathrm{d}x = -2\pi i e^{\pi i s}\,,$$

since (A.24) and (A.25) vanish as $R \to \infty$ and $\varepsilon \to 0$, for all $0 < s < 1$. Thus

$$\mathcal{M}_g(s) = \int_0^\infty \frac{x^{s-1}}{1+x}\,\mathrm{d}x = \pi \frac{2i}{e^{\pi i s - 2\pi i} - e^{-\pi i s}} = \frac{\pi}{\sin(\pi s)}\,,$$

and since $\mathcal{M}_f(s) = y^{s-1}\mathcal{M}_g(s)$ we are done. $\square$

THE BETA FUNCTION. Let x, y be complex variables. We define the β-function[1] as

$$\beta(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t\ , \quad \mathrm{Re}(x), \mathrm{Re}(y) > 0\,. \tag{A.26}$$

Note that by using the substitution $1+x \mapsto t^{-1}$, equation (A.20) can be written as

$$\int_0^\infty \frac{x^{-s}}{1+x}\,\mathrm{d}x = \int_0^1 t^{s-1}(1-t)^{-s}\,\mathrm{d}t = \beta(s, 1-s)\,.$$

Using Lemma A.18, this shows that $\beta(s, 1-s) = \Gamma(s)\Gamma(1-s)/(\Gamma(s+[1-s])$. That this result extends to all $x, y \in \mathbb{R}$ is proven in the next proposition.

PROPOSITION A.19. *Let $u, v \in \mathbb{C}$, such that $\mathrm{Re}(u), \mathrm{Re}(v) > 0$. Then*

$$\beta(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}\,. \tag{A.27}$$

PROOF. Let $f(u) = \beta(u,v)\Gamma(u+v)/\Gamma(v)$. By proving that f satisfies the three conditions in the Bohr–Mullerup theorem A.16, it follows that $f(u) = \Gamma(u)$. Thus proving the claim, see Rudin [43, p. 194] for details.

PROOF 2: Let $f * g = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau$. By using the convolution theorem $\mathscr{L}(f*g) = \mathscr{L}(f)\cdot\mathscr{L}(g)$, on $t^u * t^v$ we have

$$\mathscr{L}\left(\int_0^t s^{u-1}(t-s)^{v-1}\,\mathrm{d}s\right) = \mathscr{L}(t^{u-1})\mathscr{L}(t^{v-1}) = \frac{\Gamma(u)}{s^u}\frac{\Gamma(v)}{s^v}\,,$$

where $\mathscr{L}(t^a) = \Gamma(a+1)/s^{a+1}$ was used. Thus

$$\int_0^t s^{u-1}(t-s)^{v-1}\,\mathrm{d}s = \mathscr{L}^{-1}\left(\frac{\Gamma(u)\Gamma(v)}{s^{u+v}}\right) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}t^{u+v-1}\,.$$

Setting $t = 1$ completes the proof.

PROOF 3: We apply the change of variables $t = xy$ and $s = x(1-y)$ to the integral

$$\Gamma(u)\Gamma(v) = \int_0^\infty\int_0^\infty e^{-(t+s)}t^{u-1}s^{v-1}\,\mathrm{d}t\,\mathrm{d}s\ .$$

Note that $t+s = x$, $0 < t < \infty$ and $0 < s < \infty$ imply that $0 < x < \infty$ and $0 < y < 1$. The Jacobian is

$$J(x,y) = \frac{\partial(t,s)}{\partial(x,y)} = \begin{vmatrix} y & x \\ 1-y & -x \end{vmatrix} = -x\,,$$

and since $x > 0$ we conclude that $\mathrm{d}t\,\mathrm{d}s = |J(x,y)|\,\mathrm{d}x\,\mathrm{d}y = x\,\mathrm{d}x\,\mathrm{d}y$. Thus,

$$\begin{aligned}\Gamma(u)\Gamma(v) &= \int_0^1\int_0^\infty e^{-x}x^{u-1}y^{u-1}x^{v-1}(1-y)^{v-1}x\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_0^\infty e^{-x}x^{v+u-1}\,\mathrm{d}x\int_0^1 y^{u-1}(1-y)^{v-1}\,\mathrm{d}y = \Gamma(u+v)\beta(u,v)\,. \qquad \square\end{aligned}$$

[1]The notation β was choosen for the Beta-function to avoid confusion with the Blaschke product.

COROLLARY A.20. *For every* $n, k \in \mathbb{C}$*, such that* $\mathrm{Re}(n), \mathrm{Re}(k) > 0$*, we have*

$$\beta(n,k) = \frac{1}{k}\binom{n+k-1}{n-1}^{-1}. \tag{A.28}$$

PROOF. We begin by using $n! = \Gamma(n+1) = n\Gamma(n)$ and expanding the binomial

$$\binom{\eta}{\kappa} = \frac{\eta!}{\kappa!(\eta-\kappa)!} = \frac{\Gamma(\eta+1)}{\Gamma(\kappa+1)\Gamma(\eta-\kappa)}\frac{1}{(\eta-\kappa)} = \frac{1}{\beta(\eta-\kappa,\kappa+1)}\frac{1}{(\eta-\kappa)},$$

where the $\beta(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ was used. Solving the equation with respect to the β-function and setting $\eta = k+n-1$ and $\kappa = n-1$ proves the claim. $\square$

1.4. Functional analysis and measure theory

DEFINITION. Let $(X, \|.\|)$ be a normed linear space. A linear functional ψ on X is said to be *bounded* if

$$\sup\{|\psi(x)| \colon x \in X, \|x\| \leq 1\} \tag{A.29}$$

is finite. When this is the case, the above quantity is called the *norm* of ψ and denoted by $\|\psi\|$.

THEOREM A.21 (Hahn-Banach theorem). *Let $(X, \|.\|)$ be a normed linear space, Y a subspace of X and ψ a bounded linear functional defined on Y. Then there exists a bounded linear functional ψ defined on X such that $\phi(y) = \psi(y)$ for all $y \in Y$ and $\|\psi\| = \|\varphi\|$*

DEFINITION. For any operator $T \colon X \to Y$, we define the graph of T as the set

$$\{(x, y) \in X \times y \colon Tx = y\} .$$

THEOREM A.22 (Closed graph theorem). *Let X and Y be Banach spaces, and $T \colon X \to Y$ a linear operator. Then T is continuous if and only if its graph is closed in $X \times Y$.*

PROPOSITION A.23 (Mean value theorem). *Let $f(z)$ be analytic in a disk $\mathbb{D}$, let $a \in \mathbb{D}$ and $0 \leq r < 1$ then*

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta})\, d\theta. \tag{A.30}$$

PROOF. As $f(z)$ is analytic we get by the Cauchy integral formula

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_{|z-a|=r} \frac{f(z)}{z-a} \,\mathrm{d}z \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{i\theta})}{(a+re^{i\theta})-a} \,\mathrm{d}(a+re^{i\theta}) \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} rie^{i\theta} \,\mathrm{d}\theta = \frac{1}{2\pi} \int_0^{2\pi} f(a+re^{i\theta}) \,\mathrm{d}\theta \,. \end{aligned}$$

For the parametrised circle with centre a, we have $\mathrm{d}\theta = \mathrm{d}z \,/ i(z-a)$, so the integral of $\mathrm{d}z$ over a circle vanishes, while the integral of $\mathrm{d}\theta$ does not. □

THEOREM A.24 (Weierstrass Approximation Theorem). *The set of polynomial functions on a closed interval $[a, b]$ are dense in the set of continuous functions $C([a, b])$.*

For every continuous function $f \colon [a, b] \to \mathbb{R}$ and $\varepsilon > 0$, there exists a polynomial $p \colon [a, b] \to \mathbb{R}$ such that

$$\|f(x) - p(x)\|_\infty < \varepsilon ,$$

for all $x \in [a, b]$.

THEOREM A.25 (Lebesgue's Monotone Convergence Theorem). *Let $\{f_n\}$ be a sequence of monotonely increasing sequence functions, such that $f_n(x) \to f(x)$ as $n \to \infty$ for every X. Then f is measurable, and*

$$\int_X f \,\mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu$$

LEMMA A.26 (Fatou's Lemma). *If $f_n : X \to [0, \infty]$ is measurable for every $n \in \mathbb{N}$ then*

$$\int_X \liminf_{n\to\infty} f_n \,\mathrm{d}\mu \le \liminf_{n\to\infty} \int_X f_n \,\mathrm{d}\mu \tag{A.31}$$

THEOREM A.27 (Lebesgue's Dominated Convergence Theorem). *Let f_n be a sequence of real-valued measurable functions on a measure space (X, Σ, μ). Suppose that the sequence converges pointwise to a function f and is dominated by some integrable function g in the sence that $\left|f_n(x)\right| \le g(x)$ for all $n \in \mathbb{N}$, and all $x \in X$. Then f is lebesgue integrable and*

$$\lim_{n\to\infty} \int_X |f_n - f| \,\mathrm{d}\mu = 0$$

this also implies

$$\lim_{n\to\infty} \int_X f_n \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu$$

THEOREM A.28 (Tonelli's theorem). *Let (X, A, μ) and (Y, A, v) be σ-finite measure space. If f from $X \times Y \to [0, \infty)$ is non-negative and measurable, then*

$$\int_X \left(\int_Y f(x,y) \,\mathrm{d}y \right) \mathrm{d}x = \int_Y \left(\int_X f(x,y) \,\mathrm{d}x \right) \mathrm{d}y = \int_{X\times Y} f(x,y) \,\mathrm{d}(x,y) \,.$$

THEOREM A.29 (Fubini's theorem). *Let (X, A, μ) and (Y, A, v) be σ-finite measure space, and $X \times Y$ is the given product measure. If $f(x,y)$ is measurable and if any of the three integrals*

$$\int_X \left(\int_Y |f(x,y)| \,\mathrm{d}y \right) \mathrm{d}x, \quad \int_Y \left(\int_X |f(x,y)| \,\mathrm{d}x \right) \mathrm{d}y, \quad \int_{X\times Y} |f(x,y)| \,\mathrm{d}(x,y) \,,$$

is finite, then

$$\int_X \left(\int_Y f(x,y) \,\mathrm{d}y \right) \mathrm{d}x = \int_Y \left(\int_X f(x,y) \,\mathrm{d}x \right) \mathrm{d}y = \int_{X\times Y} f(x,y) \,\mathrm{d}(x,y) \,.$$

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