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Goldstone bosons at finite density

Study of spontaneous symmetry breaking in O(N)-symmetric theories at finite chemical potential

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Abstract

Goldstone's theorem states that, in presence of Lorentz invariance, the number of massless modes that appear after spontaneous symmetry breaking coincides with the number of broken generators. These modes are called *Goldstone bosons*. In this thesis, we study spontaneous symmetry breaking in several non-Lorentz invariant theories. This includes both non-relativistic systems and relativistic systems at nonzero density. In particular for the latter, we focus on $O(N)$-symmetric ϕ^4-theories. The method we follow is taking a Lorentz invariant system and explicitly break this invariance by coupling its Hamiltonian to its conserved charges through a finite chemical potential. Once the Lorentz invariance is broken, the remaining symmetry spontaneously breaks, giving rise to massless modes in the spectrum. We see that the number of Goldstone bosons does not always coincide with the number of broken generators.

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Conventions and abbreviations

Abbreviations are explicitly explained in the text, but we will also clarify them here. The following are listed in order of appearance in this thesis.

- QM: Quantum Mechanics.
- QFT: Quantum Field Theory
- SSB: Spontaneous Symmetry Breaking.
- NR: Non-Relativistic.
- vev: Vacuum Expectation Value.
- DR: Dimensional Regularization.
- MS: Minimal Subtraction.
- $\overline{\text{MS}}$: Modified Minimal Subtraction.
- UV: Ultraviolet.
- IR: Infrared.
- LSM: Linear σ-Model.

Conventions

- Natural units are implied, i.e., $\hbar = c = 1$
- We use Minkowski space-time: $\eta^{\mu\nu} = \text{diag}[1, -1, -1, -1]$.
- **Bold text** implies vector.
- *Italic text* is used to introduce new terms.
- For $z \in \mathbb{C}$, z^* means complex conjugate of z.
- For a matrix $M \in \mathbb{C}^n \times \mathbb{C}^m$, M^T means transpose of M and, $M^\dagger$, hermitian conjugate of M.

Chapter 1

Introduction

We here present this master thesis as a requirement in order to complete the International master's programme in Physics at the Norwegian University of Science and Technology (Norges teknisk-naturvitenskapelige universitet). This project corresponds to 60 study points in the European Credit Transfer System (ECTS) and hence, concludes a task performed during the last two semesters of the International master's programme.

1.1 Symmetries and symmetry breaking in modern Physics

Symmetries and symmetry groups play a fundamental role in modern Physics. In 1918, the physicist and mathematician Emmy Noether proved in her *Invariante variationsprobleme* that if a system has a continuous symmetry, there exists an associated quantity whose value is conserved in time [1]. In the following table, we illustrate some examples of this:

Table 1.1: Examples of different symmetries with their corresponding conserved quantity.

Symmetry	**Conserved quantity**
Time-translation	Energy
Space-translation	Linear momentum
Rotation	Angular momentum

In the early 1930s, Noether's theorem was adapted by Weyl and Wigner into the context of Quantum Mechanics in what now is known as the *Weyl and Wigner realization of a symmetry*. The classical conserved quantity becomes an operator on a Hilbert space and it receives the name of *generator*, since it can be proved that it generates the symmetry of the system. At the quantum level, the condition for the quantity to be conserved in time would becomes a zero commutator with the Hamiltonian.

It is however possible to find systems which show invariance under certain transformations, and yet their symmetries are not observed in the spectrum. This is called *spontaneous symmetry breaking* (SSB) [2]. This concept was first introduced by Jeffrey Goldstone in 1961 in his *Field Theories with "Superconductor" Solutions* [3]. A well-known example of such systems is the ferromagnet, which is invariant under rotations in spin-space. The lowest energy configuration, which is described by the ground state of the Hamiltonian, occurs when all spins are parallelly aligned. This configuration is clearly not invariant under rotations. We thus say that the ground state *breaks the symmetry*.

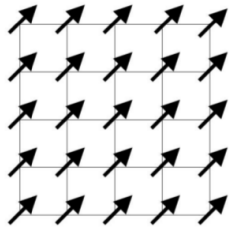


Figure 1.1: Lowest energy configuration for a ferromagnet.

The concept of spontaneous symmetry breaking is used in modern Physics to describe many different phenomena, from phase transitions or superconductivity in condensed matter Physics to supersymmetry in Physics beyond of the Standard Model. It is also worth mentioning that Peter Higgs predicted in 1964 the existence of a massive boson, named after him, which was the result of a spontaneous breaking of a local gauge symmetry [4]. He received a Nobel prize in 2012, when the particle was detected [5].

Higgs' prediction could not have been possible without Goldstone's theorem, which is the main topic of this thesis. In 1962, Jeffrey Goldstone published a paper with Abdus Salam and Steven Weinberg, where they stated and proved the now-called Goldstone's theorem. This says that if a global continuous symmetry of the Hamiltonian is broken by the ground state of the system, there will be a number of massless excitations in the spectrum [6]. These massless modes are known as Goldstone bosons. The theorem requires Lorentz invariance in the system. There will be as many massless modes as broken symmetries.

1.2 Structure of the thesis

The title of this thesis is *Goldstone bosons at finite density*. Our goal is to find these massless modes in such systems. We will be studying $O(N)$-symmetric theories which are used as effective field theories in condensed matter and high-energy physics. The concept of symmetry group will be fundamental in this thesis. We only introduce them in this section, but all of them are explained in detail in their correspondent chapter.

The thesis is organized as follows: we first, in chapter two, introduce the reader to the concept of spontaneous symmetry breaking by using the Schrödinger Lagrangian, that is,

an $E(2)$-symmetric non-relativistic field theory, which is the simplest case where SSB appears. In chapter three, we study an $SO(2)$-symmetric theory, focusing on finding the Goldstone bosons and checking that they remain massless after loop corrections. In chapter four, we introduce a chemical potential to our $SO(2)$-theory. In this way, we explicitly break the Lorentz invariance of the system. The new theory can be used to describe a Bose-Einstein condensate [7]. We will find the Goldstone modes and then use a path integral approach to find some thermodynamic properties of the condensate. This is known as thermal field theory. In chapter five, we study an $SO(3)$-symmetric Lagrangian as example of a non-Abelian theory. As in the previous chapters, we will find the Goldstone modes and then introduce a chemical potential to see how it affects the number of massless modes. Finally, in chapter six, we study the linear σ-model (LSM), as example of an $O(4)$-symmetric field theory. We will follow the same approach as in the previous chapters.

Chapter 2

Spontaneous Symmetry Breaking

The goal of this chapter is to familiarize the reader with the concept of *spontaneous symmetry breaking* (*SSB*). We will start by reminding the reader of some basic concepts from quantum mechanics (QM), like *ground state* and *coherent state*, and the so-called *Fock vacuum* of a *Hilbert* space. This will lead us to a rigorous definition of *spontaneous broken symmetries*. Once we are done with this, we will illustrate this concept using the simplest example where one finds SSB: a free non-relativistic (NR) field theory, i.e. a NR system whose Lagrangian only has a kinetic term.

Our discussions in this chapter follow mainly from Refs. 2, 8, and 9.

2.1 Spontaneous breaking of continuous symmetries

In this section we will introduce and define SSB. In order to do this, we will start by characterizing the *Fock vacuum*, which will require us to define first the *ground state* of a system.

2.1.1 Fock vacuum

In QM, the ground state of a quantum system is defined as the state with the lowest energy. The ground state of the *Fock space* is called *Fock vacuum*. The former is built of tensor products of single-particle states $|n\rangle$, i.e., the Fock space is built from single-particle Hilbert spaces [10]. The Fock vacuum then is the state with no particles, $|0\rangle$. Any Fock state can be built from the vacuum using the *creation operator*, $\hat{a}^\dagger_{\mathbf{k}}$, as

$$|n_1, n_2 \dots\rangle = \prod_i \frac{(\hat{a}^\dagger_{\mathbf{k}_i})^{n_i}}{\sqrt{n_i!}}|0\rangle, \tag{2.1}$$

where the index $\mathbf{k}_i$ labels the i-th state with momentum $\mathbf{k}$. The Hermitian conjugate of the creation operator is called the *annihilation operator*, $\hat{a}$. These two operators obey the

commutation relation

$$[\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'}, \tag{2.2}$$

where $\delta_{\mathbf{k}\mathbf{k}'}$ is the *Kronecker delta*. This discussion concerning the creation and annihilation operators leads to the definition of a *coherent state*. A state $|\alpha\rangle$ is a coherent state if it is an eigenstate of the annihilation operator [11]:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{2.3}$$

where the pre-factor $\alpha = |\alpha|e^{i\theta} \in \mathbb{C}$ is a complex number. We are assuming, for convenience in the notation, single-particle states so that we can drop the momentum index. It is worth mentioning that the coherent states are normalized to one, that is, $\langle\alpha|\alpha\rangle = 1$. One can express a coherent state in terms of the Fock vacuum by using the so-called *displacement operator* $D(\alpha)$ [12]:

$$|\alpha\rangle = D(\alpha)|0\rangle = \exp[\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}]|0\rangle = e^{\frac{|\alpha|^2}{2}} e^{-\alpha^{*}\hat{a}} e^{\alpha\hat{a}^{\dagger}}|0\rangle. \tag{2.4}$$

If we Taylor expand the exponentials and use $a|0\rangle = 0$, we get

$$|\alpha\rangle = e^{\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{2.5}$$

which is equivalent to writing a coherent state $|\alpha\rangle$ in the basis of $|n\rangle$ [11]. The displacement operator plays an important role in quantum optics but, in the context of this thesis, it is just a unitary representation of a symmetry transformation. Using an operator analogous to $D(\alpha)$, we will build, later on in this chapter, coherent states for the Fock vacuum of the theory we will be dealing with. It can be also proved that a coherent state built of a symmetry transformation of the vacuum has exactly the same energy. We will see this using the example of the NR-field theory in section 2.2.

As a final comment, if we have a system which is translationally invariant, we can label its Fock states as eigenstates of the momentum operator. We can then make the substitution $|n\rangle \to |k\rangle$. The exact discussion we made in this section then holds for the $|k\rangle$ states.

2.1.2 Spontaneously broken symmetry

Let us now consider a Lagrangian which is invariant under certain transformation. We then define this transformation to be a *symmetry* of the system. Following *Noether's theorem*, if a system has a continuous symmetry, then there is a quantity which is conserved in time [1]. At the quantum level, this quantity becomes a unitary operator in a Hilbert space (named here, $\hat{Q}$), and it is the generator of the symmetry. This is called the *standard Wigner-Weyl realization* of a symmetry [13]. Following *Heisenberg's equation* [14],

$$\frac{d}{dt}\hat{Q} = i[\hat{H}, \hat{Q}] + \frac{\partial}{\partial t}\hat{Q}, \tag{2.6}$$

if there is no explicit time dependence, the time independence of $\hat{Q}$ becomes, at the quantum level, a zero commutator with the Hamiltonian of the system ($\hat{H}$):

$$[\hat{H}, \hat{Q}] = 0 \tag{2.7}$$

We know from QM that, if an operator commutes with the Hamiltonian, they then share a complete set of eigenstates. In our discussion above we defined the ground state of a system to be the eigenstate of the Hamiltonian with the lowest energy. One would thus expect that, if a system has a symmetry, the ground state would also be an eigenstate of the operator that generates the symmetry.

This assumption we just made takes us to the definition of *spontaneously broken symmetry*. The idea here is that the symmetry generator does not leave the ground state invariant. We then say that a symmetry is *spontaneously broken* when the ground state of the system *is not an eigenstate of the charge operator* $\hat{Q}$.[1] Let $|0\rangle$ be a translationally invariant ground state (e.g. a Fock vacuum). We assume this ground state to be a discrete, non-degenerate eigenstate of the Hamiltonian $\hat{H}$. In this way, it can be used as a one-dimensional representation of the symmetry group that governs the system. The condition for a symmetry to be spontaneously broken can be then written in terms of the existence of a field operator $\hat{\psi}$ such that [2]:

$$\lim_{V\to\infty} \langle 0|[\hat{Q}, \hat{\psi}]|0\rangle \neq 0, \tag{2.8}$$

This vacuum expectation value (vev) is called an *order parameter*. It follows that if the ground state were an eigenstate of the charge operator, the vev in Eq. (2.8) would be zero. Eq. (2.8) also suggests that the ground state is degenerate. Because $\hat{Q}$ commutes with $\hat{H}$, so does a unitary representation of $\hat{Q}$ of the form $\hat{U} = \exp[i\theta\hat{Q}]$. $\hat{U}$ will transform the ground state into another with equivalent energy and, if we have a continuous symmetry group, there will be infinitely many ground states with the same energy. Moreover, we are working here in finite volume V, but, in the limit $V \to \infty$, two ground states connected by a symmetry transformation become orthogonal (We will show this with an example later in this thesis).

Since we can have infinitely many equivalent ground states, the question on how to make the proper choice of the ground state arises. We need here to distinguish between systems with finite and infinite volume. In the latter, the ground state is completely degenerate since the tunneling transition amplitude from one state to another decreases exponentially. On the other hand, in a quantum system with finite volume, the ground state is usually non-degenerate. We would still have infinitely many energy-equivalent ground states, but the exact degeneracy would be lifted by boundary conditions. The "true" ground state then is a superposition of all these states. In order to lift the degeneracy, one introduces an extra symmetry breaking term in the Hamiltonian.[2] The new term in the Hamiltonian acts as a small perturbation, and by choosing the appropriate one, the corresponding ground state can be selected. Taking then the infinite-volume limit, the splitting of the energy levels can be neglected since it is much smaller than the energy difference induced by the perturbation. The explicit symmetry breaking term can then be removed by adiabatically switching it off. This last step will not disturb the vacuum.

[1] The notation *charge operator* comes from Noether's theorem. The expected conserved quantities are called *currents* or *charges*

[2] This is equivalent to a system with a non-exact symmetry. Meaning that the symmetry is perturbed by a small perturbation.

In the following section we will use everything we have explained above in the context of a non-relativistic field theory. We will find symmetries for a Lagrangian and build a Hilbert space out of its Fock vacuum. We will then see how two symmetries will be spontaneously broken by computing their order parameter. Finally, taking a chemical potential μ as small perturbation to the Hamiltonian, we will be able to choose the proper ground state for the system out of the infinitely many there will be.

2.2 The Schrödinger Lagrangian

In this section we will illustrate the simplest case of SSB using the *Schrödiger Lagrangian* as an example of a non-relativistic field theory. We first consider the following Lagrangian for a complex scalar field [15]:

$$\mathcal{L} = -\frac{i}{2}\left[\psi^\dagger \frac{\partial}{\partial t}\psi + (-\frac{\partial}{\partial t}\psi^\dagger)\psi\right] - \frac{1}{2m}(\nabla\psi)^\dagger(\nabla\psi). \tag{2.9}$$

Note that there is not a potential in $\mathcal{L}$ and thus it is a *free theory*. Even though we will work in this simple case of a free theory, adding a potential $V(\psi^\dagger\psi)$ would not affect the derivations that we will perform later on.

Because we are considering a complex field, we can point out that ψ and $\psi^\dagger$ are independent. The reason behind this is the numbers of degrees of freedom. Here, we are dealing with a complex field and thus, we must have two degrees or freedom, the real and the complex part of the field. Then, either we consider

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2),$$

treating ψ_1 and ψ_2 as independent; or, as we are doing here, consider ψ and $\psi^\dagger$ to be independent.

As mentioned before, we are dealing in this section with the Schrödinger Lagrangian. We expect that by computing the *Euler-Lagrange*(*E-L*) equation for this Lagrangian, we will get Schrödinger's equation. Let us check this. For a non-relativistic field, its *E-L* equation has the following form] [16]:

$$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial_t\psi^\dagger)} = \nabla\frac{\partial\mathcal{L}}{\partial(\nabla\psi^\dagger)} + \frac{\partial\mathcal{L}}{\partial\psi^\dagger}. \tag{2.10}$$

We now apply this to Eq. (2.9). Because we are working in Minkowski space-time, we need to remember that there will be an extra minus sign in front of the spatial-derivative operator, $(\nabla\psi)^\dagger(\nabla\psi) = -(\partial_i\psi^\dagger)(\partial^i\psi)$. We then have:

$$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial_t\psi^\dagger)} = \frac{i}{2}\frac{\partial}{\partial t}\psi; \qquad \nabla\frac{\partial\mathcal{L}}{\partial(\nabla\psi^\dagger)} = -\nabla\left(\frac{1}{2m}\nabla\psi\right), \qquad \frac{\partial\mathcal{L}}{\partial\psi^\dagger} = -\frac{i}{2}\frac{\partial}{\partial t}\psi.$$

Plugging this into Eq. (2.10), we obtain Schrödinger's equation:

$$i\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\nabla^2\psi. \tag{2.11}$$

Lastly, because we will be using it later on, we will derive the *Hamiltonian* for this system. We first *Legendre* transform Eq. (2.9):

$$\mathcal{H} = \sum_i \pi_i \dot{\psi}_i - \mathcal{L}, \tag{2.12}$$

where π is the canonically conjugated momentum, $\pi = \partial\mathcal{L}/\partial\dot{\psi}$; i sums for the collection of fields and $\dot{\psi}$ stands for derivative with respect to time of the field. We finally have:

$$\mathcal{H} = \frac{1}{2m}(\nabla\psi)^\dagger(\nabla\psi). \tag{2.13}$$

We will make use the Hamiltonian later in this text to check whether the charge operators are conserved or not.

2.2.1 Symmetries

Continuing our study of SSB in the Schrödinger Lagrangian case, we proceed to look under which transformations the Lagrangian in Eq. (2.9) is invariant, that is, what symmetries the system has. The first, and easiest, one to notice is a $U(1)$ transformation of the form:

$$\begin{aligned} \psi &\to e^{i\theta}\psi, \\ \psi^\dagger &\to e^{-i\theta}\psi^\dagger, \end{aligned} \tag{2.14}$$

which corresponds to a rotation in the complex plane. Here θ is a constant. If we plug the transformations of Eq. (2.14) into the Lagrangian (2.9), we find

$$\mathcal{L}' = -\frac{i}{2}\left[\left(e^{-i\theta}\psi^\dagger\right)\frac{\partial}{\partial t}\left(e^{i\theta}\psi\right) - \frac{\partial}{\partial t}\left(e^{-i\theta}\psi^\dagger\right)\left(e^{i\theta}\psi\right)\right] - \frac{1}{2m}\nabla\left(e^{-i\theta}\psi^\dagger\right)\nabla\left(e^{i\theta}\psi\right),$$

and because θ is a constant with respect the time and space derivatives, the phases cancel and we are left with the Lagrangian (2.9). The next two symmetries we find in the Lagrangian of Eq.(2.9) are *translational* transformations of the form:

$$\psi \to \psi' = \psi + \theta, \tag{2.15}$$

$$\psi \to \psi' = \psi + i\theta, \tag{2.16}$$

where again θ is a constant. We substitute, for example, Eq. (2.15) into (2.9). Again, since θ is a constant with respect the time and space derivatives, we do not need to worry about the terms proportional to $1/2m$ in Eq. (2.9). We then get the following:

$$\begin{aligned} \mathcal{L}' =& -\frac{i}{2}\left[(\psi^\dagger+\theta)\frac{\partial}{\partial t}(\psi+\theta) - \frac{\partial}{\partial t}(\psi^\dagger+\theta)(\psi+\theta)\right] \\ =& -\frac{i}{2}\left[\psi^\dagger\frac{\partial}{\partial t}\psi + (-\frac{\partial}{\partial t}\psi^\dagger)\psi\right] - \frac{i\theta}{2}\frac{\partial}{\partial t}(\psi-\psi^\dagger). \end{aligned} \tag{2.17}$$

Comparing to Eq. (2.9), we notice that we have an extra term proportional to the time derivative of the fields. This term can be explained if we think of θ as an infinitesimal

translation. By doing this transformation, the Lagrangian changes as well as $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}$. We then see that the extra term in Eq. (2.17) corresponds to $\delta\mathcal{L}$ and, since the $\partial_t(\psi - \psi^\dagger)$ term is a total derivative, following the *Principle of least action* [16], it does not affect the current associated with the symmetry.

We have thus seen that the Schrödinger Lagrangian is invariant under the transformations specified in equations (2.14), (2.15) and (2.16). These correspond to a rotation in the complex plane and two translation in this plane defined by the real and complex part of the field ψ. All these transformations generate the *Euclidean* group $E(2)$ of motion in the plane. This group has as subgroups the group of special rotations in two dimensions, $SO(2)$, and the group of translations. Eq. (2.15) is an example of an $E(2)$ transformation taking the identity matrix in two dimensions, $\mathbb{I}_2$, as the rotation. Taking for instance

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad \theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix},$$

Eq. (2.15) can be written as:

$$\begin{pmatrix} \psi_1' \\ \psi_2' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \theta_1 \\ 0 & 1 & \theta_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ 1 \end{pmatrix}, \tag{2.18}$$

where the matrix belongs to $E(2)$. The discussion we made here using vectors applies to the Lagrangian even though we are dealing with scalar fields. The reason behind this is what we mentioned before about the number of degrees of freedom (we need to have two), plus the isomorphism between $U(1)$ and $SO(2)$, which will be explained in more detail in chapter 3.

2.2.2 Conserved currents

From Noether's theorem, we know that for any symmetry there has to be an associated conserved *Noether current* [1], given by:

$$j^\mu = \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\delta\psi_i, \tag{2.19}$$

where μ sums over the Minkowski metric, $\partial_\mu = (\partial_t, \nabla)$, and i sums over the collection of fields, so that we take into account the Hermitian conjugate field. Written explicitly, we have a scalar part of the current and a vector one:

$$j^0 = \frac{\partial\mathcal{L}}{\partial(\partial_t\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_t\psi^\dagger)}\delta\psi^\dagger, \tag{2.20}$$

$$\mathbf{j} = \frac{\partial\mathcal{L}}{\partial(\nabla\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\nabla\psi^\dagger)}\delta\psi^\dagger. \tag{2.21}$$

The term $\delta\psi_i$ corresponds to a small variation of the fields. In order to find these small

deviations, we take transformations such as $\psi \rightarrow \psi + \delta\psi$. We, for instance, take an infinitesimal $U(1)$ transformation to first order in θ:

$$\begin{aligned} \psi \rightarrow &\quad e^{i\theta}\psi \sim (1 + i\theta)\psi, \\ \psi^\dagger \rightarrow &\quad e^{-i\theta}\psi^\dagger \sim (1 - i\theta)\psi^\dagger, \end{aligned}$$

which gives:

$$\delta\psi = i\theta\psi; \qquad \delta\psi^\dagger = -i\theta\psi^\dagger. \tag{2.22}$$

Introducing now Eq. (2.22) into equations (2.20) and (2.21), we get the expressions for the currents associated with the $U(1)$ symmetry. Note that since θ is an arbitrary constant, we can set it to one without loss of generality. We then have:

$$j^0 = \psi^\dagger\psi, \tag{2.23}$$

$$\mathbf{j} = -\frac{i}{2m}[(\nabla\psi^\dagger)\psi - \psi^\dagger(\nabla\psi)]. \tag{2.24}$$

We also saw that the Lagrangian in Eq. (2.9) is invariant under the transformations of equations (2.15) and (2.16). We thus have two more conserved currents. We need again to find the expression for $\delta\psi$ and $\delta\psi^\dagger$. If we take an infinitesimal $\theta \sim \mathcal{O}(\epsilon)$ and compare again to $\psi \rightarrow \psi + \delta\psi$ we find for the transformation (2.15)

$$\delta\psi = \delta\psi^\dagger = \theta. \tag{2.25}$$

Substituting these expressions into equations (2.20) and (2.21), we get:

$$j^0 = \frac{i}{2}(\psi - \psi^\dagger), \tag{2.26}$$

$$\mathbf{j} = -\frac{1}{2m}(\nabla\psi + \nabla\psi^\dagger). \tag{2.27}$$

For the transformation $\psi \rightarrow \psi' = \psi + i\theta$, we find

$$\delta\psi = -\delta\psi^\dagger = i\theta, \tag{2.28}$$

which gives the following currents:

$$j^0 = \frac{1}{2}(\psi + \psi^\dagger), \tag{2.29}$$

$$\mathbf{j} = \frac{i}{2m}(\nabla\psi - \nabla\psi^\dagger). \tag{2.30}$$

And again, as we did with the currents associated to the $U(1)$ symmetry, we have set θ to one without loss of generality.

2.2.3 Total charges

From Noether's theorem, we know that there is a quantity which is conserved in time associated with each symmetry of the system. This quantity is called *total charge* and can be defined by integrating the *charge density*, $j^0(x)$ in Eq. (2.20), in a large volume V

$$Q = \int_V d^3x\, j^0. \tag{2.31}$$

Naming the scalar currents of equations (2.23), (2.26), and (2.29) with subscripts $i = 0, 1, 2$, we have the following total charges:

$$Q_0 = \int_V d^3x\, \psi^\dagger \psi, \tag{2.32}$$

$$Q_1 = \frac{i}{2} \int_V d^3x\, (\psi - \psi^\dagger), \tag{2.33}$$

$$Q_2 = \frac{1}{2} \int_V d^3x\, (\psi + \psi^\dagger). \tag{2.34}$$

Let us check that these charges Q_i are conserved in time. Let us take, for example, Q_2:

$$\frac{d}{dt} Q_2 = \frac{1}{2} \frac{d}{dt} \int_V d^3x\, (\psi + \psi^\dagger).$$

The time derivative is a linear operator which commutes with the spatial integration. We can thus rewrite this expression as:

$$\frac{d}{dt} Q_2 = \int_V d^3x \left[\frac{d\psi}{dt} + \frac{d\psi^\dagger}{dt} \right], \tag{2.35}$$

where we have dropped the one half factor for convenience. Now we recall that these fields have to follow Eq. (2.11). We can thus make the substitution $-i\partial_t \leftrightarrow \frac{1}{2m}\nabla^2$. We then have:

$$\frac{d}{dt} Q_2 = \frac{i}{2m} \int_V d^3x\, (\nabla^2 \psi + \nabla^2 \psi^\dagger).$$

Now, the theorem of divergence says [17]:

$$\int_V d^3x\, \nabla \cdot \mathbf{A} = \int_S d^2x\, \mathbf{A}, \tag{2.36}$$

where $\mathbf{A}$ is a vector field. We can apply Eq. (2.36) to (2.35), using that $\nabla^2 \psi = \nabla \cdot (\nabla \psi)$. We have:

$$\frac{d}{dt} Q_2 = \frac{i}{2m} \int_S d^2x\, (\nabla \psi) + (\nabla \psi)^\dagger. \tag{2.37}$$

We now look at the continuity equation:

$$\begin{aligned} &\partial_\mu j^\mu = \frac{d}{dt} j^0 + \partial_i \mathbf{j}^i = 0 \\ \Rightarrow &\int d^3x\, \nabla \cdot \mathbf{j} + \frac{d}{dt} Q = \\ &\int_S d^2x\, \mathbf{j} + \frac{d}{dt} Q = 0. \end{aligned} \tag{2.38}$$

We thus expect that Q_2 and its correspondent vector current satisfy this relation. Let us check this. Substituting equations (2.37) and (2.30) in (2.38) gives:

$$\frac{i}{2m} \int_S d^2x\, (\nabla \psi) + (\nabla \psi)^\dagger + \frac{i}{2m} \int_S d^2x\, (\nabla \psi - \nabla \psi^\dagger) = \frac{i}{m} \int_S d^2x\, \nabla \psi = 0. \tag{2.39}$$

Q_2 is thus conserved in time. The procedure to show that Q_0 and Q_1 also are conserved in time is analogous to the one we have just performed.

2.2.4 Second Quantization

Until now, we have been dealing with fields at the classical level. Our goal here is to quantize the currents we derived above and compute their commutators. In order to do this, we introduce the representation of the fields in terms of the operators $\hat{a}^\dagger_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ of creation and annihilation (of particle states), respectively. Our complex field can then be expressed as[3] [18]

$$\hat{\psi}(\mathbf{r},t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{a}_{\mathbf{k}} \quad ; \quad \hat{\psi}^\dagger(\mathbf{r},t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\hat{a}^\dagger_{\mathbf{k}}, \tag{2.40}$$

where we restrict ourselves to a system with finite volume V. The field operators obey the canonical equal-time commutation relations:[4]

$$[\hat{\psi}(x), \hat{\psi}^\dagger(x')] = \delta(x-x'), \tag{2.41}$$

where $\delta(x-x')$ is *Dirac's delta* function.

Equations (2.40) and (2.41) follow from the *second quantization* formalism. In this formalism, a single particle operator is represented by [18]

$$\hat{B} = \sum_{i=1}^{N}\hat{b}(x_i) \Rightarrow \sum_{\mathbf{kk'}}\langle\mathbf{k'}|\hat{b}|\mathbf{k}\rangle\hat{a}^\dagger_{\mathbf{k'}},\hat{a}_{\mathbf{k}}, \tag{2.42}$$

where

$$\langle\mathbf{k'}|\hat{b}|\mathbf{k}\rangle = \int dr\ \psi^\dagger_{\mathbf{k'}},(\mathbf{r})\hat{b}(\mathbf{r})\psi_{\mathbf{k}}(\mathbf{r}).$$

It now follows to write the Hamiltonian in Eq. (2.13) in this formalism. From Eq. (2.42), we have:

$$\mathcal{H} = \frac{1}{2mV}\sum_{\mathbf{kk'}}\int dr(\mathbf{kk'})e^{i(\mathbf{k}-\mathbf{k'})\mathbf{r}}\hat{a}^\dagger_{\mathbf{k'}}\hat{a}_{\mathbf{k}}. \tag{2.43}$$

We can identify from the expression above as a Kronecker delta [18]:

$$\delta_{\mathbf{kk'}} = \frac{1}{V}\int dr\ e^{i(\mathbf{k}-\mathbf{k'})\mathbf{r}}. \tag{2.44}$$

Then, the sum in $\mathbf{k'}$ in Eq. (2.43) gives zero for $\mathbf{k} \neq \mathbf{k'}$ and we are then left with:

$$\mathcal{H} = \sum_{\mathbf{k}}\frac{\mathbf{k}^2}{2m}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}. \tag{2.45}$$

Continuing towards the main point of this section, we proceed to quantize the Noether currents we calculated before. We start with $\hat{Q}_0$ by inserting the expressions for the fields in Eq. (2.40) into (2.32):

$$\hat{Q}_0 = \int_V d^3x\ \hat{\psi}^\dagger\hat{\psi} \Rightarrow \int dr\frac{1}{V}\sum_{\mathbf{kk'}} e^{i(\mathbf{k}-\mathbf{k'})\mathbf{r}}\hat{a}^\dagger_{\mathbf{k'}}\hat{a}_{\mathbf{k}}, \tag{2.46}$$

[3] We are assuming here periodic boundary conditions, that is $\psi(x+L) = \psi(x)$, where $L^3 = V$

[4] We are assuming here bosonic fields. If the fields were fermionic, the relation (2.41) would be written in terms of anti-commutators.

where we again recognize a Kronecker delta. Inserting then Eq. (2.44), we obtain

$$\hat{Q}_0 = \sum_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}}. \tag{2.47}$$

Proceeding in the same way with $\hat{Q}_1$ and $\hat{Q}_2$, we have:

$$\hat{Q}_1 = \frac{i}{2\sqrt{V}} \sum_{\mathbf{k}} \int dr \left[e^{i\mathbf{k}\cdot\mathbf{r}} \hat{a}_{\mathbf{k}} - e^{-i\mathbf{k}\cdot\mathbf{r}} \hat{a}^\dagger_{\mathbf{k}} \right], \tag{2.48}$$

$$\hat{Q}_2 = \frac{1}{2\sqrt{V}} \sum_{\mathbf{k}} \int dr \left[e^{i\mathbf{k}\cdot\mathbf{r}} \hat{a}_{\mathbf{k}} + e^{-i\mathbf{k}\cdot\mathbf{r}} \hat{a}^\dagger_{\mathbf{k}} \right]. \tag{2.49}$$

Comparing the relations above with the expression for the Kronecker delta in Eq. (2.44), we can recognize in both equations (2.48) and (2.49) a $\delta_{\mathbf{k}0}$. The sums then give zero for all $\mathbf{k} \neq 0$ and we find:

$$\hat{Q}_1 = \frac{i}{2}\sqrt{V} \left(\hat{a}_0 - \hat{a}^\dagger_0 \right), \tag{2.50}$$

$$\hat{Q}_2 = \frac{1}{2}\sqrt{V} \left(\hat{a}_0 + \hat{a}^\dagger_0 \right). \tag{2.51}$$

Now that we have quantized our charges, we are able to compute their commutators. We will use two identities which follow from *Jacobi's identity*. These are [19]:

$$\begin{aligned} [A + B, C] &= [A, C] + [B, C], \\ [A, BC] &= [A, B]C + B[A, C]. \end{aligned} \tag{2.52}$$

For convenience, we will first explicitly compute the commutators between the charges $\hat{Q}_0$ and $\hat{Q}_1$. We have:

$$\begin{aligned} [\hat{a}_0 - \hat{a}^\dagger_0 \,,\, \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}}] &= [\hat{a}_0, \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}}] - [\hat{a}^\dagger_0, \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}}] \\ &= [\hat{a}_0, \hat{a}^\dagger_{\mathbf{k}}] \hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}} [\hat{a}_0, \hat{a}_{\mathbf{k}}] \\ &\quad - [\hat{a}^\dagger_0, \hat{a}^\dagger_{\mathbf{k}}] \hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{\mathbf{k}} [\hat{a}^\dagger_0, \hat{a}_{\mathbf{k}}] \\ &= \hat{a}_{\mathbf{k}} \delta_{\mathbf{k}0} + \hat{a}^\dagger_{\mathbf{k}} \delta_{\mathbf{k}0} \end{aligned}$$

where, in the last step, we used Eq. (2.44) and the fact that $\delta_{ij} = \delta_{ji}$. We now recover the constants and the sum over $\mathbf{k}$ and all terms with $\mathbf{k} \neq 0$ vanish. Finally:

$$[\hat{Q}_1, \hat{Q}_0] = \frac{i}{2}\sqrt{V} \left(\hat{a}_0 + \hat{a}^\dagger_0 \right) = i\hat{Q}_2. \tag{2.53}$$

The procedure is analogous for the commutator between $\hat{Q}_0$ and $\hat{Q}_2$. Skipping here the middle steps, we have

$$[\hat{Q}_2, \hat{Q}_0] = \frac{1}{2}\sqrt{V} \left(\hat{a}_0 - \hat{a}^\dagger_0 \right) = -i\hat{Q}_1. \tag{2.54}$$

Finally, the commutator between the charges $\hat{Q}_1$ and $\hat{Q}_2$ ia given by

$$\left[\hat{Q}_1, \hat{Q}_2\right] = \frac{i}{4} V[\hat{a}_0 - \hat{a}_0^\dagger, \hat{a}_0 + \hat{a}_0^\dagger] = \frac{i}{2} V \hat{\mathbb{I}}. \tag{2.55}$$

The commutators we have computed here should give the Lie algebra of $E(2)$. If we compare these with the commutators we have in Appendix B.4, we see that the commutators between $\hat{Q}_0$ and $\hat{Q}_{1,2}$ are identical to their classical form. The problem arises in Eq. (2.55). Let us not forget that $\hat{Q}_1$ and $\hat{Q}_2$ generate translations and thus, one would expect that their commutator vanishes, since two translations commute at the classical level, as we can see in Appendix B.4. Eq. (2.55) shows that, at quantum level, the algebra of $E(2)$ develops a central charge, that is, an operator that commutes with any element in the symmetry group [2].

Before closing this section, let us check that the total charges that we have derived are the generators of their respective symmetries, as we proved in general in Appendix C.2. We will follow the derivations we made in the general case. Let us for example check Q_0. We build a unitary representation of this generator through its one-parameter family and apply it to the field as follows

$$\psi \to \psi' = U \psi U^{-1} = e^{iQ_0} \psi e^{-iQ_0}. \tag{2.56}$$

If we expand the exponentials of Eq. (2.56), we find:

$$\psi \to \psi' = \psi + i[Q_0, \psi]. \tag{2.57}$$

Following the derivations of Appendix C.2. , the commutator of Eq. (2.57) gives

$$[Q_0, \psi] = -i\delta\psi = \theta\psi,$$

where we used Eq. (2.22). If we now substitute the result of the commutator in Eq. (2.57) we see that we recover a $U(1)$ infinitesimal transformation.

And finally in this part, since we have seen that some properties of the currents do not hold at the quantum level, it is worth to check if the currents $\hat{Q}_i$ still are conserved in time after quantization. We recall ***Heisenberg's equation***, Eq. (2.6), for the time-evolution of an operator. It follows from it that an operator with no explicit time dependence is conserved in time if it commutes with the Hamiltonian, Eq. (2.7). Let us, for example, compute $[\mathcal{H}, \hat{Q}_0]$, where $\mathcal{H}$ and $\hat{Q}_0$ are given by equations (2.45) and (2.47), respectively. Using that

$$[AB, CD] = A[B, C]D + AC[B, D] + [A, C]DB + C[A, D]B, \tag{2.58}$$

and dropping again the sums over $\mathbf{k}$ and $\mathbf{k}'$ in the middle steps, we have:

$$\hat{a}_{\mathbf{k}}^\dagger[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger]\hat{a}_{\mathbf{k}'} + \hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}'}^\dagger[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] + [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{k}'}^\dagger]\hat{a}_{\mathbf{k}}\hat{a}_{\mathbf{k}'} + \hat{a}_{\mathbf{k}'}^\dagger[\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{k}'}]\hat{a}\mathbf{k} = (\hat{a}_{\mathbf{k}}^\dagger\hat{a}_{\mathbf{k}'} - \hat{a}_{\mathbf{k}'}^\dagger\hat{a}_{\mathbf{k}})\delta_{\mathbf{k}\mathbf{k}'}, \tag{2.59}$$

where we again used Eq. (2.44). Recovering the sum, we finally have

$$[\mathcal{H}, \hat{Q}_0] = \frac{1}{2m}\sum_{\mathbf{k}\mathbf{k}'} \mathbf{k}^2(\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}'} - \hat{a}^\dagger_{\mathbf{k}'}\hat{a}_{\mathbf{k}})\delta_{\mathbf{k}\mathbf{k}'}$$
$$= \frac{1}{2m}\sum_{\mathbf{k}} \mathbf{k}^2(\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}) = 0. \tag{2.60}$$

And thus, $\hat{Q}_0$ is still conserved in time after quantization. The derivation with the remaining Q_1 and Q_2 is analogous to the one we have just performed and gives that both are still conserved at the quantum level.

2.2.5 Spontaneous symmetry breaking

In this part of the thesis we will apply all theory explained in section 2.1 to the Schrödinger Lagrangian.

Broken generators As we said before, a symmetry is spontaneously broken when the ground state of the quantum system is not an eigenstate of the symmetry generator. This is equivalent to having a non-zero order parameter. Let us thus compute it for our three generators. Let $|0\rangle$ be a translationally invariant ground state for our system. We start with $[\hat{Q}_0, \hat{\psi}]$ and, again, we will explicitly compute first the commutator between the creation and annihilation operators:

$$[a^\dagger_{\mathbf{k}}a_{\mathbf{k}}, a_{\mathbf{k}'}] = a^\dagger_{\mathbf{k}}[a_{\mathbf{k}}, a_{\mathbf{k}'}] + [a^\dagger_{\mathbf{k}}, a_{\mathbf{k}'}]a_{\mathbf{k}} = -\delta_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}},$$

where we used equations (2.52) and (2.2). We can now compute the order parameter:

$$\lim_{V\to\infty}\langle 0|[\hat{Q}_0, \hat{\psi}]|0\rangle = \lim_{V\to\infty} -\frac{e^{-i\omega t}}{\sqrt{V}}\sum_{\mathbf{k}\mathbf{k}'} e^{i\mathbf{k}'\cdot\mathbf{r}}\langle 0|a_{\mathbf{k}}\delta_{\mathbf{k}\mathbf{k}'}|0\rangle$$
$$= \lim_{V\to\infty} -\frac{e^{-i\omega t}}{\sqrt{V}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\langle 0|a_{\mathbf{k}}|0\rangle = 0, \tag{2.61}$$

where we used $a_{\mathbf{k}}|0\rangle = 0$. Thus, $|0\rangle$ is an eigenstate of $\hat{Q}_0$ and, therefore, the generator is not broken. Let us continue with $[\hat{Q}_1, \hat{\psi}]$. Starting with the commutator:

$$[\hat{a}_0 - \hat{a}^\dagger_0\ ,\ \hat{a}_{\mathbf{k}}] = [\hat{a}_0, \hat{a}_{\mathbf{k}}] - [\hat{a}^\dagger_0, \hat{a}_{\mathbf{k}}] = \delta_{\mathbf{k}0}$$

and thus, the order parameter is

$$\lim_{V\to\infty}\langle 0|[\hat{Q}_1, \hat{\psi}]|0\rangle = \lim_{V\to\infty}\frac{i}{2}e^{-i\omega t}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\langle 0|\delta_{\mathbf{k}0}|0\rangle$$
$$= \lim_{V\to\infty}\frac{i}{2}e^{-i\omega t}\langle 0|0\rangle = \frac{i}{2}e^{-i\omega t} \neq 0. \tag{2.62}$$

This shows that $\hat{Q}_1$ is spontaneously broken. The same happens with $\hat{Q}_2$. We thus have two broken generators. Following *Goldstone's theorem*,[5] there should be, for each broken

[5] To be discussed in appendix D.

generator, a state in the spectrum that couples to the associated current. We must mention here that this statement only applies to Lorentz invariant theories. In our NR case, which is not Lorentz-invariant, it is not true. We will still have massless modes, but the number of modes does not need to coincide with the number of broken generators. In this case, we only have one mode even though two generators are broken [20].

Coherent States Let us now continue our discussion of SSB by building coherent states from the vacuum $|0\rangle$. In order to do this, we introduce a unitary representation of $E(2)$ generated by the charges $\hat{Q}_1$ and $\hat{Q}_2$:

$$U(z) = \exp[i(\alpha_1\hat{Q}_1 + \alpha_2\hat{Q}_2)]. \tag{2.63}$$

Introducing the second-quantized expressions for $\hat{Q}_1$ and $\hat{Q}_2$ in equations (2.50) and (2.51) we have

$$\begin{aligned} U(z) &= \exp\left[i(\alpha_1\hat{Q}_1 + \alpha_2\hat{Q}_2)\right] \\ &= \exp\left[\frac{i\sqrt{V}}{2}\left(i\alpha_1(\hat{a}_0 - \hat{a}_0^\dagger) + \alpha_2(\hat{a}_0 + \hat{a}_0^\dagger)\right)\right] \\ &= \exp\left[\frac{i\sqrt{V}}{2}(-i)\left(\hat{a}_0(\alpha_1 - i\alpha_2) + \hat{a}_0^\dagger(\alpha_1 + i\alpha_2)\right)\right] \\ &= \exp\left[\frac{\sqrt{V}}{2}\left(z\hat{a}_0^\dagger - z^*\hat{a}_0\right)\right], \end{aligned} \tag{2.64}$$

where we introduced the complex number $z = \alpha_1 + i\alpha_2$. The representation $U(z)$ transforms the vacuum $|0\rangle$ into a coherent state $|z\rangle$:

$$|z\rangle = \exp\left[\frac{\sqrt{V}}{2}\left(z\hat{a}_0^\dagger - z^*\hat{a}_0\right)\right]|0\rangle. \tag{2.65}$$

Let us make some comments about Eq. (2.65). First of all, the state $|z\rangle$ is an eigenstate of the annihilation operator $\hat{a}_0$. Let us check this. We start by splitting U into two separate exponentials. By doing this, we have a term like $\exp[-z^*\hat{a}_0]$, which annihilates the vacuum, that is, gives a zero; and another one which we now expand in Taylor series. We have

$$\begin{aligned} \hat{a}_0|z\rangle &= \hat{a}_0 \exp\left[\frac{\sqrt{V}}{2}\left(z\hat{a}_0^\dagger - z^*\hat{a}_0\right)\right]|0\rangle \\ &= \hat{a}_0 \sum_{n=0}^{\infty}\left(\frac{z\sqrt{V}}{2}\right)^n \frac{1}{n!}(\hat{a}_0^\dagger)^n|0\rangle \\ &= \sum_{n=0}^{\infty}\left(\frac{z\sqrt{V}}{2}\right)^n \frac{1}{n!}\hat{a}_0(\hat{a}_0^\dagger)^n|0\rangle. \end{aligned}$$

By using the commutation relation in Eq. (2.2), it can easily be proved by induction that $a_0(a_0^\dagger)^n|0\rangle = n(a_0^\dagger)^{n-1}|0\rangle$. Inserting this result in the sum, we are left with:

$$\sum_{n=0}^{\infty}\left(\frac{z\sqrt{V}}{2}\right)^n \frac{1}{n!}\hat{a}_0(\hat{a}_0^\dagger)^n|0\rangle = \frac{z\sqrt{V}}{2}\sum_{n=0}^{\infty}\left(\frac{z\sqrt{V}}{2}\right)^{n-1}\frac{1}{(n-1)!}(a_0^\dagger)^{n-1}|0\rangle.$$

By making now the substitution $m = n-1$, we recover the exponential. We can also introduce back terms like $\exp[-z^*\hat{a}_0]$, since we would only be adding zeros. By doing this, we recover the operator $U(z)$ and hence, we recover the state $|z\rangle$ on the RHS. We then have

$$\hat{a}_0|z\rangle = \frac{z\sqrt{V}}{2}|z\rangle. \tag{2.66}$$

We have shown that the state $|z\rangle$ is an eigenstate of the annihilation operator $\hat{a}_0$ with eigenvalue $z\sqrt{V}/2$. By performing an analogous procedure, we can show that the state $|z\rangle$ has exactly the same energy as the vacuum $|0\rangle$. We would only need to apply the Hamiltonian in Eq (2.45) to $|z\rangle$ and proceed as we just did.

In section (3.5.2), we mentioned that two arbitrary coherent states are orthogonal in the thermodynamic limit. We will show it here. In order to do this, let us compute the magnitude of the scalar product of two coherent states. We have:

$$\langle z'|z\rangle = \langle 0|\exp\left[\frac{\sqrt{V}}{2}\left(z^{*'}\hat{a}_0 - z'\hat{a}_0^\dagger\right)\right]\exp\left[\frac{\sqrt{V}}{2}\left(z\hat{a}_0^\dagger - z^*\hat{a}_0\right)\right]|0\rangle. \tag{2.67}$$

We start by making use of the Baker-Campbell-Hausdorff formula [19]:

$$e^A e^B = \exp\left[A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + \dots\right]. \tag{2.68}$$

Naming e.g. A to the first exponent and B to the second in Eq. (2.67) and by applying Eq. (2.68), we can write the term inside the bra-ket as two different exponentials.[6] The first one corresponds to the commutator of the two exponents $[A,B]$ whereas the second is just the sum of the exponents $A+B$. We then have

$$\begin{aligned}\langle z'|z\rangle &= \langle 0|\exp\left[-\frac{V}{4}|z-z'|^2\right]\exp\left[\frac{\sqrt{V}}{2}\left(\hat{a}_0(z'-z)^* - \hat{a}_0^\dagger(z'-z)\right)\right]|0\rangle. \\ &= \exp\left[-\frac{V}{4}|z-z'|^2\right]\langle 0|\exp\left[\frac{\sqrt{V}}{2}\left(\hat{a}_0(z'-z)^*\right)\right]\exp\left[\frac{-\sqrt{V}}{2}\left(\hat{a}_0^\dagger(z'-z)\right)\right]|0\rangle \\ &= \exp\left[-\frac{1}{4}V|z-z'|^2\right].\end{aligned} \tag{2.69}$$

This result tells us that any two ground states connected by broken symmetry transformations become orthogonal if we take the infinite volume limit, $V \to \infty$.

[6] We can restrict ourselves to first order in commutators, since the higher-order will vanish when applying them to $|0\rangle$.

Choice of ground state We will conclude this chapter by applying the technique we explained in section 3.5.2 to the Schrödinger field. As we saw before, because the generators $\hat{Q}_1$ and $\hat{Q}_2$ commute with the Hamiltonian, we can have infinitely many degenerate ground states. All these are connected through the symmetry transformation $U(z)$ in Eq. (2.64). This implies that any state $|z\rangle$ defined as in Eq. (2.65) can be chosen as ground state. This will not change the fact that $\hat{Q}_1$ and $\hat{Q}_2$ are spontaneously broken.
We start by introducing an explicit symmetry breaking term in the Hamiltonian in Eq. (2.45). The new one now is:

$$\mathcal{H}_\mu = \mathcal{H} - \mu \hat{Q}_0, \tag{2.70}$$

where μ plays the role of a chemical potential. We can get the energy of a state $|z\rangle$ by computing the expectation value of $\mathcal{H}_\mu$ in that state. In order to do this, we first compute the expectation value of $\hat{Q}_0$ in a state $|z\rangle$. From its expression in Eq. (2.47), we have

$$\langle z|\hat{Q}_0|z\rangle = \sum_{\mathbf{k}} \langle z|\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}|z\rangle = \frac{1}{4}V|z|^2, \tag{2.71}$$

where we used Eq. (2.66). This implies that the energy of a state $|z\rangle$ is:

$$\langle z|\mathcal{H}_\mu|z\rangle = -\frac{1}{4}\mu|z|^2 V, \tag{2.72}$$

which implies that the chemical potential must be negative, in order for the system to have a ground state. We now take the infinite-volume limit, $V \to \infty$. We can clearly see that only the state $|0\rangle$ becomes the single non-degenerate ground state, whereas any other $|z\rangle$ diverges in the thermodynamic limit. We thus conclude that the "true" ground state of the system is the Fock vacuum $|0\rangle$ itself. The last step of this technique is to remove the explicit symmetry breaking term,$-\mu Q_0$, by adiabatically switching it off. This can be achieved by just taking the limit $\mu \to 0$.

2.2.6 Hilbert space

Now that we have found the ground state of the system, we can close this chapter by building the Hilbert space. Let $|n_i\rangle$ denote an excited state with momentum $\mathbf{k}_i$. The state $|n_1, n_2, \dots\rangle$ is then built of tensor direct products of $|n_i\rangle$ as stated in Eq. (2.1).
We have showed before that two states $|z\rangle$ and $|z'\rangle$, connected by symmetry transformations, are orthogonal. Let $|n_1, n_2, \dots\rangle$ and $|n'_1, n'_2, \dots\rangle$ be two states in our Hilbert space. Let us check that they are orthogonal, for $n \geq n'$,

$$(a_{\mathbf{k}'})^{n'}(a^\dagger_{\mathbf{k}})^n|0\rangle = \frac{n}{(n-n')!}\delta_{\mathbf{k}\mathbf{k}'}(a^\dagger_{\mathbf{k}})^{n-n'}|0\rangle. \tag{2.73}$$

We then have

$$\begin{aligned}\langle n_1', n_2', \dots | n_1, n_2, \dots \rangle &= \langle 0 | \prod_i \frac{(\hat{a}^\dagger_{\mathbf{k}_i})^{n_i'}}{\sqrt{n_i'!}} \frac{(\hat{a}^\dagger_{\mathbf{k}_i})^{n_i}}{\sqrt{n_i!}} |0\rangle \\ &= \langle 0 | \prod_i \delta_{n_i n_i'} \frac{n_i!}{\sqrt{n_i! n_i'!}} |0\rangle \\ &= \prod_i \delta_{n_i n_i'}. \end{aligned} \tag{2.74}$$

We have thus checked that they are orthogonal. As we also stated in section 3.5.2, the states $|n_1, n_2, \dots\rangle$ are eigenstates of the Hamiltonian and of the charge operator. Consequently, in our case, they are eigenstates of the shifted Hamiltonian $\mathcal{H}_\mu$. The eigenvalues for $\mathcal{H}$ and $\hat{Q}_0$ can be easily computed by applying equations (2.45) and (2.47) to $|n_1, n_2, \dots\rangle$. We skip here the middle steps and jump straight to $\mathcal{H}_\mu$. We have

$$\mathcal{H}_\mu |n_1, n_2, \dots\rangle = \left[\left(\sum_a \frac{\mathbf{k}_a^2}{2m} - \mu \right) n_a \right] |n_1, n_2, \dots\rangle \tag{2.75}$$

We finally adiabatically turn off the perturbation, i.e., $\mu \to 0$. This gives us the dispersion relation for the mode:

$$E = \frac{\mathbf{k}^2}{2m}. \tag{2.76}$$

When $\mathbf{k} \to 0$, the energy vanishes. This means that the mode is massless. Hence, we have checked that the number of broken symmetry generators does not always coincide with the number of massless modes when the theory is not Lorentz-invariant as we mentioned in section 2.2.5. In this example, we only get one massless mode after SSB, even though we have two broken generators.

Chapter 3

Complex scalar field

In this chapter, we study a charged scalar field as example of an $SO(2)$-symmetric theory. We present the Lagrangian and find its symmetries, which we proceed to spontaneously break. Goldstone's theorem predicts that there will massless modes associated with the broken symmetries. We will check that the modes remain massless after loop corrections.

3.1 Introduction

Consider a complex scalar field Φ, described by the following Lagrangian:

$$\mathcal{L} = (\partial_\mu \Phi^\dagger)(\partial^\mu \Phi) - m^2 \Phi^\dagger \Phi - \frac{\lambda}{6}(\Phi^\dagger \Phi)^2, \tag{3.1}$$

This Lagrangian is invariant under $U(1)$ transformations, which are

$$\Phi \to e^{i\theta}\Phi, \qquad \Phi^\dagger \to e^{-i\theta}\Phi^\dagger.$$

If we plug these transformations into the Lagrangian, we notice that it is still invariant:

$$\mathcal{L} = [\partial_\mu (e^{-i\theta}\Phi^\dagger)][\partial^\mu (e^{i\theta}\Phi)] - m^2 e^{-i\theta}\Phi^\dagger e^{i\theta}\Phi - \frac{\lambda}{6}(e^{-i\theta}\Phi^\dagger e^{i\theta}\Phi)^2.$$

Since the phases commute with the scalar fields, they cancel and we recover the original Lagrangian in Eq. (3.1). As we introduced in the previous chapter, the $U(1)$ symmetry is equivalent, or more accurately *isomorphic*, to an $SO(2)$ symmetry. This can be understood by thinking in terms of rotations. On the one hand, a two-dimensional real vector can be rotated by a two-dimensional matrix $R(\theta) \in SO(2)$ in the real plane. The most well-known representation of an $SO(2)$ matrix is given by:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

On the other hand, a two-dimensional real vector can be written as a complex scalar since both have the same number of degrees of freedom. A $U(1)$ transformation, multiplies the complex scalar by a phase. As a consequence, it rotates it in the complex plane.

Since Φ is a complex scalar field, we can rewrite it as

$$\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2),$$

where ϕ_1 and ϕ_2 are real scalar fields. When plugging this into Eq. (3.1), we find:

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2 - m^2(\phi_1^2 + \phi_2^2)] - \frac{\lambda}{4!}(\phi_1^2 + \phi_2^2)^2, \tag{3.2}$$

which corresponds to two $\lambda\phi^4$ theories for each real field.[1] Here, we recognize the potential $V(\phi_1, \phi_2)$:

$$V(\phi_1, \phi_2) = \frac{1}{2}m^2(\phi_1^2 + \phi_2^2) + \frac{\lambda}{4!}(\phi_1^2 + \phi_2^2)^2. \tag{3.3}$$

This potential is called a *Mexican hat* potential because of its form when we represent it in the $\phi_1 - \phi_2$ plane. We assume $m^2 < 0$, which will make sense in section 3.3:

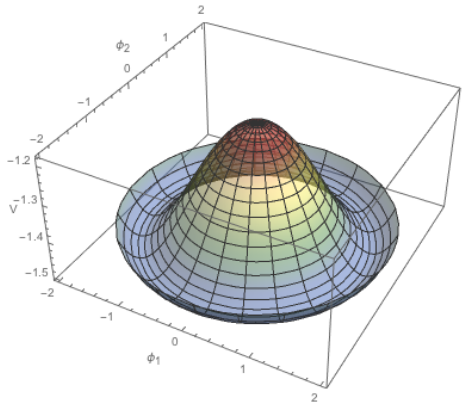


Figure 3.1: Mexican-hat potential. In this figure, the parameters m and λ have been set to one

3.2 Feynman rules

Let us now derive the Feynman rules for the modes ϕ_1 and ϕ_2, i.e., we will specify the propagator for each mode and their vertices.

[1] Plus an interaction between the fields

Let us first rewrite the Lagrangian (3.2) as $\mathcal{L}_Q + \mathcal{L}_{\text{int}}$, where

$$\mathcal{L}_Q = \frac{1}{2}[(\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2 - m^2(\phi_1^2 + \phi_2^2)], \tag{3.4}$$

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}(\phi_1^2 + \phi_2^2)^2. \tag{3.5}$$

The propagator can be derived from the action for the Lagrangian:

$$S = \int_\Omega d^4x \mathcal{L}_Q = \int_\Omega d^4x \frac{1}{2}[(\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2 - m^2(\phi_1^2 + \phi_2^2)]. \tag{3.6}$$

Note that $\mathcal{L}_Q$ describes two independent real scalar field with the same mass and hence their propagator will be the same. We can drop the subscripts and just consider an arbitrary real scalar field ϕ. The kinetic term $\frac{1}{2}\partial_\mu \phi \partial^\mu \phi$ can be rewritten as $\frac{1}{2}[\partial_\mu(\phi \partial^\mu \phi) - \phi \Box \phi])$, with $\Box = \partial_\mu \partial^\mu$. By partial integration, the first term vanishes since it is a total 4-divergence. We are left with:

$$S = \int_\Omega d^4x - \frac{1}{2}\phi(\Box + m^2)\phi. \tag{3.7}$$

The propagator $\Delta(x, x')$ is defined from here as:

$$-(\Box + m^2)\Delta(x, x') = \delta(x - x'). \tag{3.8}$$

By performing a Fourier transformation, we can go to momentum space and find $\Delta(k)$:

$$\Delta(k) = \frac{1}{k^2 - m^2 + i\varepsilon}, \tag{3.9}$$

where we have introduced a factor $i\varepsilon$ to avoid the divergence at $k^2 = m^2$.

We proceed now to find the vertices factors. From the Lagrangian in Eq. (3.5), we can see that we have a self-interaction for each mode ϕ_j and an interaction between the modes ϕ_1 and ϕ_2. The following Feynman diagrams represent them:

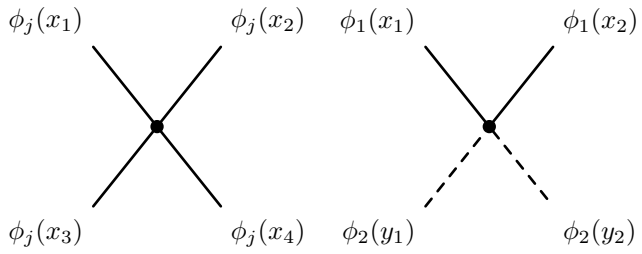


Figure 3.2: Diagramatic representation of the interaction terms in $\mathcal{L}_{\text{int}}$

For the self-interaction, we have 4! equivalent permutations for the fields. This gives a symmetry factor of 4!. For the remaining diagram, we have $2! \times 2!$ equivalent permutations.

This gives a symmetry factor of 4. The vertex functions are given by $i\mathcal{L}_{\text{int}}$:

$$-\frac{i\lambda}{4!}\times 4! = -i\lambda, \tag{3.10}$$

for the self-interaction of ϕ_1 and ϕ_2, and

$$-\frac{i\lambda}{12}\times 4 = -\frac{i\lambda}{3}, \tag{3.11}$$

for the interaction between the two modes.

3.3 Spontaneous symmetry breaking

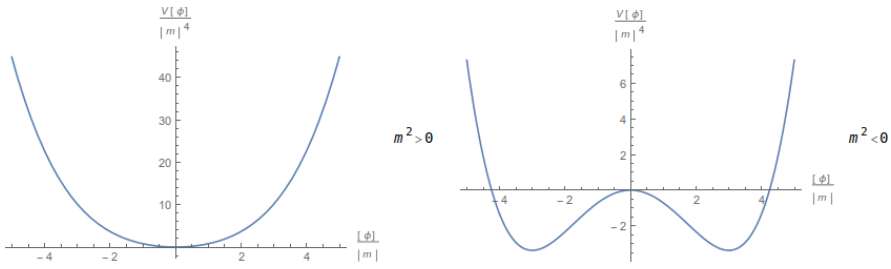
We now want to find the minimum of the potential in Eq. (3.3). We find, in fact, different minima whether we consider a positive mass term in the Lagrangian or a negative one. Independently of the sign of the mass term, the potential (3.3) is spherically symmetric, which implies that we can choose the vacuum to point in any direction and, every possible direction is physically equivalent. We choose the vacuum to point in the ϕ_1 direction and introduce its vacuum expectation value (vev), v, such that:

$$\left.\frac{\partial V}{\partial \phi_1}\right|_{\phi_1=v} = 0 \quad \Rightarrow \quad v\left(m^2+\frac{\lambda}{6}v^2\right) = 0.$$

For $m^2 > 0$, we find the minimum to be at $v = 0$, while for $m^2 < 0$, we find

$$v^2 = -\frac{6m^2}{\lambda}. \tag{3.12}$$

In Fig. 3.3 we represent the potential $V(v, 0)$ with V defined in Eq. (3.3):



(a) **(b)**

Figure 3.3: Comparison of potential (3.3) for positive (a) and negative (b) mass term. The minimum $v = 0$ becomes a maximum when $m^2 < 0$. In order to implement the graphs, we set $m^2 = \pm 1.5$ and $\lambda = 1$

We will now spontaneously symmetry break (SSB) our Lagrangian. This is done by expanding the fields ϕ_j as their vacuum expectation value plus quantum fluctuating fields

χ_j:

$$\begin{aligned} \phi_1 = &\quad \langle\phi_1\rangle + \chi_1 = \quad v + \chi_1, \\ \phi_2 = &\quad \langle\phi_2\rangle + \chi_2 = \qquad \chi_2, \end{aligned} \tag{3.13}$$

with v defined in Eq. (3.12). We now substitute these expression into Eq. (3.2). We get the following Lagrangian:

$$\mathcal{L} = \frac{3}{2}\frac{m^4}{\lambda} + \frac{1}{2}[(\partial_\mu\chi_1)^2 + (\partial_\mu\chi_2)^2] + m^2\chi_1^2 - \frac{\lambda}{6}v\chi_1(\chi_1^2+\chi_2^2) - \frac{\lambda}{4!}(\chi_1^2+\chi_2^2)^2, \tag{3.14}$$

where we can see that there is no mass term for the field ϕ_2. Its propagator is then massless, and so is the mode. It is a Goldstone boson. Our result agrees with Goldstone's theorem since we have a massless mode for one broken symmetry.

3.4 Self-Energy

After SSB, we have two new interaction terms:

$$-\frac{\lambda}{6}v\chi_1(\chi_1^2+\chi_2^2) - \frac{\lambda}{4!}(\chi_1^2+\chi_2^2)^2, \tag{3.15}$$

which are represented by the following Feynman diagrams:

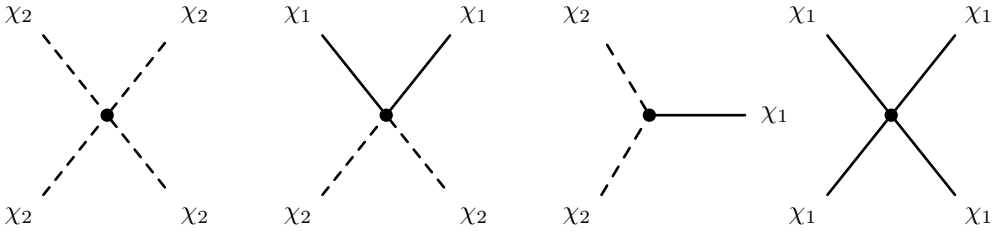
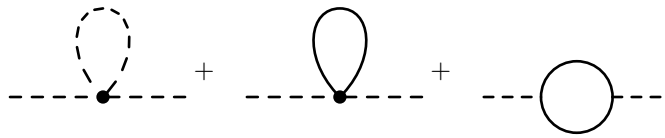


Figure 3.4: Vertices corresponding to interactions like χ_2^4, $\chi_1\chi_2^2$, $\chi_1^2\chi_2^2$ and χ_1^4, respectively. The massless mode is represented by dashed lines.

The new interaction terms have a contribution to the self-energy of the modes, which contributes to the mass. In this section we are checking whether the mode ϕ_2 remains massless after one-loop corrections, or not. In order to do this, we first build the self-energy diagrams corresponding to the mode ϕ_2 from the vertices in Fig. 3.4. We do this by introducing two external lines to each one of the diagrams in Fig. 3.4 and restricting ourselves to one-loop corrections. We get:



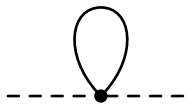
(3.16)

Figure 3.5: One-loop self-energy diagrams for the mode ϕ_2

Let us now evaluate these diagrams. Starting from the left, the first one corresponds to the self-interaction of the mode ϕ_2. Its contribution is given by

$$\Rightarrow \quad -i\lambda \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}, \tag{3.17}$$

which is a massless integral. This type of integrals cannot be regularized, but since they are massless, they give a zero contribution at the UV limit. This diagram vanishes because we are evaluating it at the minimum of the classical potential, i.e. we are choosing the minimum of the effective potential to be where the classical minimum is. The second self-energy diagram is the correspondent to the interaction between ϕ_1 and ϕ_2 (at 1st order). Its contribution is given by



$$\Rightarrow \quad -\frac{i\lambda}{3}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2+2m^2}. \tag{3.18}$$

We see that the integral is quadratically divergent. In order to evaluate this it, we use *dimensional regularization* (DR). The reason for this is that DR preserves the symmetries of the system [21, 22]. DR consists in performing the integral in $d = 4 - \varepsilon$ dimensions and then take the limit $\varepsilon \to 0$ in the result. In this way, we isolate the divergent part by splitting the result into finite and infinite parts. We now proceed to evaluate the integral. We first write $d^nk = d\Omega_n dk\, k^{n-1}$, where $d\Omega_n$ is an n-dimensional differential volume. We have

$$\int_0^\infty \frac{d^nk}{(2\pi)^n}\frac{1}{k^2+M^2} = \int d\Omega_n \int_0^\infty \frac{dk}{(2\pi)^n}\frac{k^{n-1}}{k^2+M^2},$$

where $n = 4 - \varepsilon$. We have also made the substitution $\sqrt{2}m = M$, for convenience. The first integral gives a n-dimensional surface. In order to evaluate the second one, we introduce the substitution $k = M\sqrt{t}$. Then:

$$\int_0^\infty \frac{dk}{(2\pi)^n}\frac{k^{n-1}}{k^2+M^2} = \frac{1}{2}M^{n-2}\frac{1}{(2\pi)^n}\int_0^\infty dt\frac{t^{\frac{n}{2}-1}}{t+1}. \tag{3.19}$$

Here we recognize the integral to be an *Euler's beta* function $B(\frac{n}{2}, 1-\frac{n}{2})$ [23]. Our integral then becomes

$$\frac{1}{2}M^{n-2}\frac{1}{(2\pi)^n}\underbrace{\frac{2\pi^{n/2}}{\Gamma(n/2)}}_{\Omega_n} B\left(\frac{n}{2}, 1-\frac{n}{2}\right) = \frac{M^2}{(4\pi)^{n/2}}M^{n-4}\Gamma(1-\frac{n}{2}), \tag{3.20}$$

where we used that $B\left(\frac{n}{2}, 1-\frac{n}{2}\right) = \Gamma(\frac{n}{2})\Gamma(1-\frac{n}{2})$. We want now to substitute $n = 4-\varepsilon$ and expand the expression near the pole. The pre-factor of Eq. (3.20) cannot be expanded

for small ε since it is not dimensionless. In order to solve this problem, we need to multiply it by a *mass scale* $\tilde{M}^{4-n}$ such that $[\tilde{M}] = [M]$:

$$\frac{M^2}{(4\pi)^{n/2}} \left(\frac{M}{\tilde{M}}\right)^{n-4} \Gamma(1 - \frac{n}{2}). \tag{3.21}$$

The Gamma function can be expanded as follows:

$$\Gamma(1 - \frac{n}{2}) = \Gamma(1 - \frac{4-\varepsilon}{2}) = \Gamma(-1 + \frac{\varepsilon}{2}) = -\frac{2}{\varepsilon} - 1 + \gamma + \mathcal{O}(\varepsilon), \tag{3.22}$$

where we introduced the *Euler-Mascheroni* constant $\gamma = 0.57721\ldots$ [24]. We will also make use of

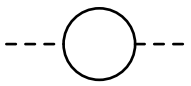
$$M^\varepsilon = e^{\varepsilon \ln M} \approx 1 + \varepsilon \ln M + \mathcal{O}(\varepsilon^2). \tag{3.23}$$

Before introducing the expansions (3.22) and (3.23), we multiply Eq. (3.21) by a factor of $\left(\frac{e^{\gamma-1}}{4\pi}\right)^{\frac{\varepsilon}{2}}$. In this way, we get rid of constants which do not affect to the divergence. This procedure is called *modified minimal subtraction* ($\overline{\text{MS}}$), which is a renormalization scheme that not only removes divergences but also factors proportional to the Euler-Mascheroni constant γ [21]. We then have

$$\frac{1}{12\pi^2\varepsilon}\lambda m^2 \left[1 + \varepsilon \ln\left(\frac{\tilde{M}}{M}\right) + \mathcal{O}(\varepsilon^2)\right], \tag{3.24}$$

where we can see that it diverges in the limit $\varepsilon \to 0$.

The last contribution to the self-energy is the one corresponding to the interaction $\phi_1\phi_2\phi_2$. By looking at its diagram in Fig. 3.4, we see that we have a symmetry factor of 2. The contribution is then given by



$$\Rightarrow \quad \left(\frac{-iv\lambda}{3}\right)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 + 2m^2} \frac{i}{(k+q)^2 + 2m^2}, \tag{3.25}$$

with v^2 defined in Eq. (3.12) Here we have given momentum k to the external lines of the diagram and momenta $k + q$ and q to the upper and lower branch of loop, respectively.

In order to evaluate integrals like the one in Eq. (3.25), we need to use the so-called *Feynman parameterization* [21]:

$$\frac{1}{ab} = \int_0^1 \frac{dz}{[az + b(1-z)]^2}. \tag{3.26}$$

We make $a = k^2 + 2m^2$ and $b = (k+q)^2 + 2m^2$ and find for the denominator

$$\begin{aligned} ab &= [k^2 + 2m^2][(k+q)^2 + 2m^2] \\ &= k^2 + q^2 + 2m^2 + 2kq(1-z) + q^2(1-z) \\ &= p^2 + 2m^2 + q^2 z(1-z), \end{aligned} \tag{3.27}$$

where we have introduced the variable p with $p^2 = [k + q(1-z)]^2$ in order to cancel the term linear in k. The integral in Eq. (3.25) reduces to:

$$\int_0^1 dz \int \frac{d^4p}{[p^2 + 2m^2 + q^2 z(1-z)]^2}, \tag{3.28}$$

which has an analogous structure to Eq. (3.18) if we make the substitution $M^2(z) = 2m^2 + q^2 z(1-z)$. We perform the integral (in p) in $n = 4 - \varepsilon$ dimensions in the same way as before. We get:

$$\int_0^\infty dp \frac{p^{n-1}}{[p^2 + M^2]^2} = \frac{1}{2} M^{n-4} \int_0^\infty dt \frac{t^{\frac{n}{2}-1}}{[t+1]^2} = \frac{1}{2} M^{n-4} B\left(\frac{n}{2}, 2 - \frac{n}{2}\right). \tag{3.29}$$

We now recover all the pre-factors and introduce a mass scale and a factor of $\left(\frac{e^\gamma}{4\pi}\right)^{2-\frac{n}{2}}$.In this way, as we mentioned before, the pre-factor can be expanded and we cancel all constants proportional to γ. We get:

$$\left(\frac{e^\gamma}{4\pi}\right)^{2-\frac{n}{2}} \frac{1}{(4\pi)^{n/2}} \frac{2\pi^{n/2}}{\Gamma(n/2)} \Gamma\left(2 - \frac{n}{2}\right) \int_0^1 dz \left(\frac{M}{M}\right)^{n-4}. \tag{3.30}$$

We plug in the expansions (3.23) and $\Gamma\left(\frac{\varepsilon}{2}\right) \approx \frac{2}{\varepsilon} - \gamma$ and recover the vertex factor of Eq. (3.25). This self-energy contribution gives:

$$-\frac{1}{12\pi^2 \varepsilon} \lambda m^2 \left[1 + \varepsilon \int_0^1 dz \, \ln\left(\frac{\tilde{M}}{M}\right) + \mathcal{O}(\varepsilon^2)\right], \tag{3.31}$$

which, at zeroth order in ε, has the same value as the self-energy contribution in Eq. (3.24) but with opposite sign and hence the contributions cancel. If we go to first order in ε, the contribution in Eq. (3.31) depends on the external momentum q. In order for both contributions to cancel, we need $q^2 = 0$. This is known as an *on-shell* renormalization condition. We have checked that loop-corrections do not give mass to the field ϕ_2 and thus it remains massless after quantum corrections.

In order to complete our discussion on one-loop corrections, we also need to take into consideration one and two-point amplitudes, also known as *tadpole* diagrams. We will do this in section 3.5.3, using a different parameterization for the field Φ.

3.5 Polar parameterization

We now introduce the polar parameterization for our complex scalar field:

$$\Phi = \rho e^{i\theta}, \tag{3.32}$$

where ρ and θ are real fields. Substituting Eq. (3.32) into Eq. (3.1), the Lagrangian becomes:

$$\mathcal{L} = \rho^2 (\partial_\mu \theta)^2 + (\partial_\mu \rho)^2 - m^2 \rho^2 - \frac{\lambda}{6} \rho^4. \tag{3.33}$$

With this parameterization, the $U(1)$ symmetry explained in section 3.1 becomes a shift $\theta \to \theta + \alpha$, being α is a constant. We see that the Lagrangian remains invariant by making this substitution in Eq. (3.33).

Note that there is no mass term for the θ field in the potential. By using the polar parametrization for the complex scalar field, we have made explicit that θ is a Goldstone mode [8].

3.5.1 Feynman rules

We will now derive the Feynman rules for the fields ρ and θ. We first write the Lagrangian in Eq. (3.33) in three parts, $\mathcal{L} = \mathcal{L}_\rho + \mathcal{L}_\theta + \mathcal{L}_{\text{int}}$, where

$$\mathcal{L}_\rho = (\partial_\mu \rho)^2 - m^2 \rho^2, \tag{3.34}$$

$$\mathcal{L}_\theta = \rho^2 (\partial_\mu \theta)^2, \tag{3.35}$$

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{6} \rho^4. \tag{3.36}$$

The Lagrangian in Eq. (3.34) describes a real scalar massive field. Our discussion in section 3.2 applies here and hence the propagator for the ρ field is the same as the one in Eq. (3.9):

$$\Delta_\rho(k) = \frac{1}{k^2 - m^2 + i\varepsilon}. \tag{3.37}$$

The Lagrangian describing the θ field only has kinetic term and thus its propagator is the one for a real scalar massless field:

$$\Delta_\theta(k) = \frac{1}{k^2 + i\varepsilon}. \tag{3.38}$$

Note that in both Eq. (3.37) and (3.38), we have again added a term $i\varepsilon$ to avoid the divergence in the limit $\mathbf{k}^2 \to 0$.

We lastly have $\mathcal{L}_{\text{int}}$, which describes the self-interaction of the ρ field:

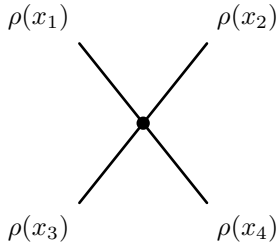


Figure 3.6: Vertex corresponding to interaction ρ^4.

As we saw in section 3.2, the vertex factor is given by $i\mathcal{L}_{\text{int}}$ times a symmetry factor. In this case, we again have 4! equivalent permutations for the ρ field and hence, the symmetry factor is 4!. We find:

$$-\frac{i\lambda}{6} \times 4! = -4i\lambda. \tag{3.39}$$

Note that the θ field does not have interacting term in the Lagrangian.

3.5.2 Spontaneous symmetry breaking

The potential of the Lagrangian in Eq. (3.33) has again two minima depending on whether the mass term is positive or negative. For $m^2 > 0$, we again find the minimum at $v = 0$. For $m^2 < 0$, the potential is minimized at $v^2 = -\frac{3m^2}{\lambda}$. We now proceed as we did in section 3.3 and choose the vacuum in the ρ direction, i.e., $\langle\rho\rangle = v$ and $\langle\theta\rangle = 0$. We expand ρ around its vev plus quantum fluctuating fields:

$$\rho = v + h,$$

and introduce the expansion in the Lagrangian (3.33). We get

$$\mathcal{L} = \frac{3}{2}\frac{m^4}{\lambda} + (\partial_\mu h)^2 + h^2(\partial_\mu\theta)^2 + 2m^2h^2 - \frac{\lambda}{6}h^4 - \frac{2\lambda}{3}vh^3 + (v^2 + 2vh)(\partial_\mu\theta)^2 \quad (3.40)$$

As expected, there is no mass term for the mode θ and hence it is a Goldstone boson.

3.5.3 Self-Energy

In this section, we proceed as we did in section 3.4 and compute the self-energy contributions to the mode θ in order to check if it remains massless after loop corrections. The interacting terms after SSB are:

$$-\frac{\lambda}{6}h^4 - \frac{2\lambda}{3}vh^3 + (h^2 + 2vh)(\partial_\mu\theta)^2, \quad (3.41)$$

which are represented in the following Feynman diagrams:

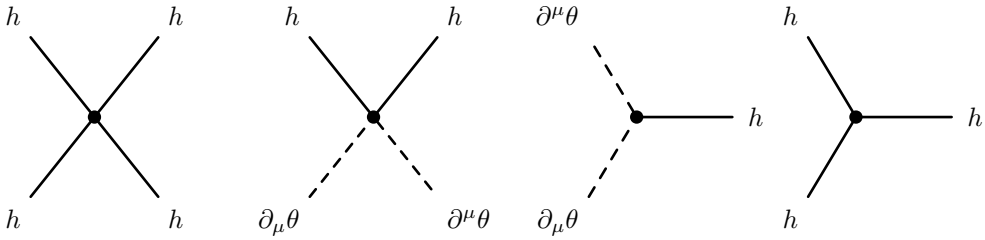
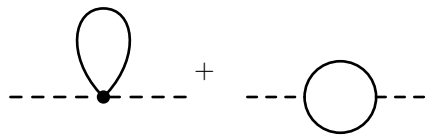


Figure 3.7: Vertices corresponding to interactions like h^4, $h^2(\partial_\mu\theta)^2$, $h(\partial_\mu\theta)^2$ and h^3, respectively. The mode θ is represented with dashed lines.

The only two diagrams contributing to the self-energy of θ are the second and the third in Fig. 3.7 and, they generate the two following self-energy diagrams:



(3.42)

Figure 3.8: One-loop self-energy diagrams for the mode ϕ_2

Note that the diagrams are exactly the same as in section 3.4 but with $\chi_1 \to h$ and $\chi_2 \to \theta$ and different vertex factor. We know that these contributions cancel each other by demanding an on-shell renormalization condition. Let us thus skip the middle steps in this part and use the results we obtained in section 3.4 and, consider directly the tadpole contributions.

Two-point and One-point functions

Because the renormalization conditions we have used in the previous sections, new contributions to the self-energy of the modes appear. We need to check that either all contributions vanish or they cancel each other, so that the mode ϕ_2 (or θ) remains massless.

We can get the one-point contributions are given by:

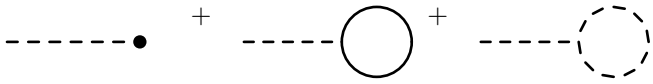
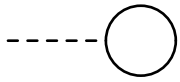


Figure 3.9: One-Point amplitudes.

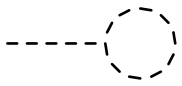
The first diagram in Fig. 3.9 corresponds to the on-shell renormalization condition, which is equivalent to saying that this first term vanishes and does not have any contribution to the self-energy.

The second gives the following contribution



$$\Rightarrow \quad -\frac{iv\lambda}{3}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+2m^2}, \tag{3.43}$$

which is the same integral as Eq. (3.18). We computed it, with a different symmetry factor and we saw it gives a non-zero contribution to the self-energy. We lastly have the third diagram, which gives a contribution of



$$\Rightarrow \quad -i\lambda\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+\mu^2}, \tag{3.44}$$

where we added a regulator μ^2, with $[\mu] = [m]$. In this way we can perform the integral since otherwise the propagator is massless and we cannot regularize it. Once again, the integral is of the form of Eq. (3.18). We know that the result is proportional to the regulator μ^2, hence we know that the integral vanishes in the massless limit. We thus have only one non-zero contribution. Since the mode ϕ_2 (and θ) is massless, we will have to cancel this

contribution with a counter-term [25].

In a general renormalization scheme,the mass of the fields will be also shifted by two-point functions [25]. From the diagrams in figure 3.9, we can build the following ones:

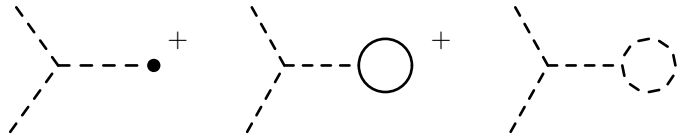


Figure 3.10: Two-Point amplitudes.

We notice that the diagrams in figure 3.10 are essentially the same as the ones in figure. The only difference between these diagrams and the ones in Fig. 3.9 are the external momenta lines. But these will only contribute with a symmetry factor. The two-point amplitudes are proportional to the one-point ones; and since the former contribution vanishes, so does this.

Chapter 4

Spontaneous symmetry breaking in systems at finite temperature

The goal of this part of the thesis is to study a system described with a charged scalar field. These type of fields describe bosons with positive and negative charge, that is, bosons which are each other's antiparticles [26]. We will consider a complex scalar field with a $\lambda\phi^4$ potential. We will first find the conserved Noether current j^μ and then couple them to the Hamiltonian of the system through a finite chemical potential. We will go back to the Lagrangian, work out the SSB and compute the mass spectrum of the theory. Once done with this, we will compute the partition function for the system and derive the thermodynamic potential from it.

Our discussions in this chapter follow mainly from Refs. 26, 27, 28, 29, 30, 31 and 32.

4.1 U(1) Symmetry and Noether current

Let us consider a complex scalar field described by the Lagrangian

$$\mathcal{L}_0 = (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) - m^2 \Phi^\dagger \Phi - \frac{\lambda}{6}(\Phi^\dagger \Phi)^2, \tag{4.1}$$

with $m^2 > 0$. We saw in chapter 3 that $\mathcal{L}_0$ is invariant under the symmetry group $U(1)$, i.e., the transformations

$$\Phi \to e^{i\theta}\Phi,$$

$$\Phi^\dagger \to e^{-i\theta}\Phi^\dagger, \tag{4.2}$$

where θ is space-time independent.

Since the system has a continuous symmetry, it follows from Noether's theorem that there is a current which is conserved in time. The current is given by:

$$j^\mu = \sum_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_a)} \delta\varphi_a, \tag{4.3}$$

where the subscript a runs over the collection of fields. The infinitesimal transformations $\delta\varphi_a$'s can be computed from Eq. (4.2) by expanding around $\theta = 0$. To first order in θ, we have:

$$\begin{aligned} \Phi \to &\quad e^{i\theta}\Phi \sim (1 + i\theta)\Phi, \\ \Phi^\dagger \to &\quad e^{-i\theta}\Phi^\dagger \sim (1 - i\theta)\Phi^\dagger, \end{aligned}$$

which gives $\delta\Phi = i\theta\Phi$ and $\delta\Phi^\dagger = -i\theta\Phi^\dagger$. We then get for the Lagrangian in Eq. (4.1):

$$j^\mu = \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \Phi)} \delta\Phi + \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \Phi^\dagger)} \delta\Phi^\dagger = i\theta[(\partial^\mu \Phi)^\dagger \Phi - \Phi^\dagger(\partial^\mu \Phi)]. \tag{4.4}$$

Note that we can ignore the θ; the term in the bracket in Eq. (4.4) is still conserved. The total charge Q can be computed by integrating the charge density $\rho = j^0$ over a large volume:

$$Q = \int_\Omega d^3x\, j^0 = i \int_\Omega d^3x\, [(\partial^0 \Phi)^\dagger \Phi - \Phi^\dagger(\partial^0 \Phi)]. \tag{4.5}$$

4.2 Adding a chemical potential

We will now compute the Hamiltonian $\mathcal{H}_0$ for the Lagrangian in Eq. (4.1). $\mathcal{H}_0$ is defined as the Legendre transformation of $\mathcal{L}_0$:

$$\mathcal{H}_0 = \sum_a \pi_a \dot{\varphi}_a - \mathcal{L}_0 = \sum_a \frac{\partial \mathcal{L}_0}{\partial \dot{\varphi}_a} \dot{\varphi}_a - \mathcal{L}_0, \tag{4.6}$$

where π_a is the canonical momentum, $\dot{\varphi}$ means derivative of φ with respect to time and, again, a runs over the collection of fields. From the Lagrangian in Eq. (4.1), it follows that $\pi = \dot{\Phi}^\dagger$ and $\pi^\dagger = \dot{\Phi}$. This gives the following Hamiltonian:

$$\mathcal{H}_0(\pi, \Phi) = \pi^\dagger \pi + (\nabla\Phi)^\dagger(\nabla\Phi) + V(\Phi^\dagger\Phi), \tag{4.7}$$

where $V(\Phi^\dagger\Phi) = m^2\Phi^\dagger\Phi + \frac{\lambda}{6}(\Phi^\dagger\Phi)^2$.

We can now couple $\mathcal{H}_0$ to the conserved charge. This is done by shifting it as $\mathcal{H} \to \mathcal{H}_0 - \mu\rho$, where $\rho = j^0$ is the time-component of the Noether current defined in Eq. (4.4) and μ is a nonzero chemical potential:

$$\mathcal{H}(\pi, \Phi) = \pi^\dagger \pi + (\nabla\Phi)^\dagger(\nabla\Phi) - i\mu(\pi\Phi - \Phi^\dagger\pi^\dagger) + V(\Phi^\dagger\Phi). \tag{4.8}$$

The next step now is to transform back to the Lagrangian. We first invert Eq. (4.6):

$$\mathcal{L} = \sum_a \pi_a \dot{\varphi}_a - \mathcal{H} = \sum_a \pi_a \frac{\partial \mathcal{H}}{\partial \pi_a} - \mathcal{H}, \tag{4.9}$$

where we used the Hamilton equation $\dot{\Phi} = \partial\mathcal{H}/\partial\pi$. For the shifted Hamiltonian we have:

$$\begin{aligned}\dot{\Phi} &= \pi^\dagger - i\mu\Phi \rightarrow \pi^\dagger = \dot{\Phi} + i\mu\Phi, \\ \dot{\Phi}^\dagger &= \pi + i\mu\Phi^\dagger \rightarrow \pi = \dot{\Phi}^\dagger - i\mu\Phi^\dagger.\end{aligned} \tag{4.10}$$

In the following derivation we will, for convenience, omit the potential $V(\Phi^\dagger\Phi)$ since it is not affected by the transformation. Plugging the expressions for π and $\pi^\dagger$ into Eq. (4.9), we have:

$$\begin{aligned}&(\dot{\Phi}^\dagger - i\mu\Phi^\dagger)\dot{\Phi} + (\dot{\Phi} + i\mu\Phi)\dot{\Phi}^\dagger - \pi^\dagger\pi - (\nabla\Phi)^\dagger(\nabla\Phi) + i\mu(\pi\Phi - \pi^\dagger\Phi^\dagger) \\ &= 2\dot{\Phi}^\dagger\dot{\Phi} + i\mu(\Phi\dot{\Phi}^\dagger - \Phi^\dagger\dot{\Phi}) - \dot{\Phi}^\dagger\dot{\Phi} + \mu^2\Phi^\dagger\Phi - (\nabla\Phi)^\dagger(\nabla\Phi) + i\mu(\dot{\Phi}^\dagger\Phi - \dot{\Phi}\Phi^\dagger) \\ &= (\partial_0\Phi)^\dagger(\partial_0\Phi) - i\mu[(\partial_0\Phi)^\dagger\Phi - \Phi^\dagger(\partial_0\Phi)] + \mu^2\Phi^\dagger\Phi - (\nabla\Phi)^\dagger(\nabla\Phi).\end{aligned}$$

We see that the first three terms of the last line of this derivation can be rewritten as $(\partial_0 + i\mu)\Phi^\dagger(\partial_0 - i\mu)\Phi$. Thus, the new Lagrangian is:

$$\mathcal{L} = (\partial_0 + i\mu)\Phi^\dagger(\partial_0 - i\mu)\Phi - (\nabla\Phi)^\dagger(\nabla\Phi) - V(\Phi^\dagger\Phi), \tag{4.11}$$

where we see that adding the chemical potential μ to the Hamiltonian has the same effect as gauging the 0-component of the field ($\partial_0 \rightarrow \partial_0 + i\mu$).

4.3 Spontaneous symmetry breaking

For convenience, we write the Lagrangian of Eq. (4.11) as

$$\mathcal{L} = (\partial_\mu\Phi)^\dagger(\partial^\mu\Phi) - i\mu[(\partial_0\Phi)^\dagger\Phi - \Phi^\dagger(\partial_0\Phi)] - (m^2 - \mu^2)\Phi^\dagger\Phi - \frac{\lambda}{6}(\Phi^\dagger\Phi)^2. \tag{4.12}$$

We can see that the mass term is now shifted by the chemical potential μ. Let us first compute the lowest energy configuration of the field, that is, the value v of the field that minimizes the potential $V(\Phi^\dagger\Phi)$:

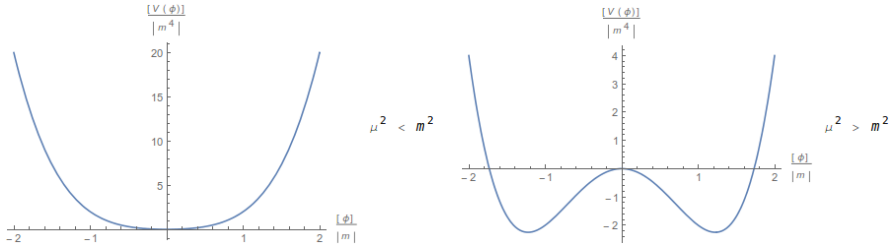
$$\Phi \rightarrow \langle\Phi\rangle = \frac{v}{\sqrt{2}} \;\Rightarrow\; \left.\frac{\partial V}{\partial\Phi}\right|_{\Phi=v/\sqrt{2}} = (\mu^2 - m^2)v + \frac{\lambda}{6}v^3 = 0, \tag{4.13}$$

where the factor $1/\sqrt{2}$ was added due to normalization. We then find two solutions:

$$v = 0 \tag{4.14}$$

$$v^2 = \frac{6}{\lambda}(\mu^2 - m^2). \tag{4.15}$$

In the Fig. 4.1, we compare the two solutions. Not that the solution $v = 0$ only occurs when $\mu^2 < m^2$:



(a) **(b)**

Figure 4.1: Comparison between the two solutions for Eq. (4.13). Note that the minimum $v = 0$ becomes a maximum when $\mu^2 \geqslant m^2$. In order to implement the graphs, we set $m = 1,\ \mu = 0,\ \lambda = 1$ in 4.1a and $m = 1, /\mu = 2,\ \lambda = 1$ in 4.1b

4.3.1 Mass spectrum before SSB

Let us recall that $m^2 > 0$. In the shifted Lagrangian of Eq. (4.11), the symmetry spontaneously breaks when the mass term of $\mathcal{L}$ is positive. We have two phases, a *normal phase*, as shown in Fig. 4.1a, with $\mu^2 < m^2$ and a *broken phase*, illustrated Fig. 4.1b, with $\mu^2 > m^2$. Let us first assume $\mu^2 < m^2$. We only take the part quadratic in the fields in Eq. (4.12). The propagator for $\mathcal{L}$ can be then found from the action S:

$$\begin{aligned} S &= \int d^4x \left[(\partial_\mu \Phi)^\dagger(\partial^\mu \Phi) - i\mu[(\partial_0\Phi)^\dagger\Phi - \Phi^\dagger(\partial_0\Phi)] - (m^2 - \mu^2)\Phi^\dagger\Phi\right] \\ &= \int d^4x \ \left[\Phi^\dagger\left[-\Box - m^2 + \mu^2\right]\Phi - i\mu[(\partial_0\Phi)^\dagger\Phi - \Phi^\dagger(\partial_0\Phi)]\right], \end{aligned} \tag{4.16}$$

where we partial integrated once in order to get $\Box = \partial_\mu\partial^\mu$. If we now write the integrand above as a matrix product like $(\phi_1\ \phi_2)D(\phi_1\ \phi_2)^\dagger$, we see that the propagator D is not diagonal:

$$D = \begin{pmatrix} -\Box - m^2 + \mu^2 & -2i\mu\partial_0 \\ 2i\mu\partial_0 & -\Box - m^2 + \mu^2 \end{pmatrix}. \tag{4.17}$$

We can compute the dispersion relation for the fields from the propagator. We first Fourier transform it to momentum space:

$$D_k = \begin{pmatrix} E^2 - \mathbf{k}^2 - m^2 + \mu^2 & -2iE\mu \\ 2iE\mu & E^2 - \mathbf{k}^2 - m^2 + \mu^2 \end{pmatrix}, \tag{4.18}$$

where we used $\nabla = -i\mathbf{k}$ and renamed the 0-component, k_0, of the 4-momentum to E. The *dispersion relation* for the normal phase can be computed by demanding $\det D_k = 0$. We will have a 4-degree polynomial and therefore four solutions. Taking only the ones that have $E^2 > 0$, we find:

$$(E - \mu)^2 = \mathbf{k}^2 + m^2. \tag{4.19}$$

Note that if we set $\mu = 0$, we recover the usual dispersion relation for a massive field of mass m.

4.3.2 SSB

As we mentioned before, SSB will occur when the mass term of the Lagrangian is positive. Let us assume now that $\mu^2 > m^2$. We first recall that Φ is a complex scalar field. We can then express it using its Euclidean parameterization:

$$\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \tag{4.20}$$

where ϕ_1 and ϕ_2 are real fields. It follows that $\Phi^\dagger\Phi = \frac{1}{2}(\phi_1^2 + \phi_2^2)$, which implies that $(\nabla\Phi)^\dagger(\nabla\Phi) = \frac{1}{2}[(\nabla\phi_1)^2 + (\nabla\phi_2)^2]$. Substituting Eq. (4.20) into (4.11), the remaining term can then be expressed as:

$$\begin{aligned}(\partial_0 + i\mu)\Phi^\dagger(\partial_0 - i\mu)\Phi &= \frac{1}{2}(\partial_0 + i\mu)(\phi_1 - i\phi_2)(\partial_0 - i\mu)(\phi_1 + i\phi_2)\\ &= \frac{1}{2}[\partial_0\phi_1 - i\partial_0\phi_2 + i\mu\phi_1 + \mu\phi_2]\\ &\quad \times [\partial_0\phi_1 + i\partial_0\phi_2 - i\mu\phi_1 + \mu\phi_2]\\ &= \frac{1}{2}\big[(\partial_0\phi_1)^2 + (\partial_0\phi_2)^2\big]\\ &\quad + \mu[\phi_2(\partial_0\phi_1) - \phi_1(\partial_0\phi_2)] + \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2).\end{aligned}$$

Rearranging all terms, we write $\mathcal{L}$ in terms of ϕ_1 and ϕ_2 as

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2] + \mu[\phi_2(\partial_0\phi_1) - \phi_1(\partial_0\phi_2)] - V(\phi_1, \phi_2), \tag{4.21}$$

with

$$V(\phi_1, \phi_2) = \frac{1}{2}(m^2 - \mu^2)(\phi_1^2 + \phi_2^2) + \frac{\lambda}{4!}(\phi_1^2 + \phi_2^2)^2. \tag{4.22}$$

The next step towards SSB is to expand the field around the vacuum, v. We choose the it to be pointing in the ϕ_1 direction:

$$\begin{aligned}\phi_1 &\to v + \chi_1,\\ \phi_2 &\to \chi_2,\end{aligned} \tag{4.23}$$

with v defined in Eq. (4.14). We can now substitute these expressions into the Lagrangian of Eq. (4.21). After rearranging all terms, we obtain

$$\begin{aligned}\mathcal{L}_\chi = \frac{3}{2\lambda}(m^4 - \mu^4) + \frac{1}{2}[(\partial_\mu\chi_1)^2 + (\partial_\mu\chi_2)^2] + \mu\Big[\chi_2(\partial_0\chi_1) - \chi_1(\partial_0\chi_2)\Big]\\ -(\mu^2 - m^2)\chi_1^2 - \frac{\lambda}{6}v\chi_1(\chi_1^2 + \chi_2^2) - \frac{\lambda}{4!}(\chi_1^2 + \chi_2^2)^2.\end{aligned} \tag{4.24}$$

Note that we have omitted the term proportional to $\partial_0\chi_2$. This can be done because it is a total derivative and therefore, it will not affect the action or the equation of motion.

4.3.3 Mass spectrum after SSB

We will now proceed to compute the dispersion relation of the modes after SSB. We start by again taking the quadratic part of the Lagrangian in Eq. (4.24) and writing the action:

$$S_\chi = \frac{1}{2}\int d^4x\,[(\partial_\mu\chi_1)^2 + (\partial_\mu\chi_2)^2] + 2\mu\Big[\chi_2(\partial_0\chi_1) - \chi_1(\partial_0\chi_2)\Big] - 2(\mu^2 - m^2)\chi_1^2. \tag{4.25}$$

We again partial integrate once, so that we can write $(\partial_\mu\chi_j)^2 = -\chi_j\Box\chi_j$. The integrand above can again be written as a matrix product like $(\chi_1\ \chi_2)D_\chi(\chi_1\ \chi_2)^\dagger$, where D_χ is the propagator after SSB. In momentum space, it has the form

$$D_\chi(\mathbf{k}^2) = \begin{pmatrix} E^2 - \mathbf{k}^2 + 2m^2 - 2\mu^2 & 2iE\mu \\ -2iE\mu & E^2 - \mathbf{k}^2 \end{pmatrix}. \tag{4.26}$$

The dispersion relation condition again is $\det D_\chi(\mathbf{k}^2) = 0$

$$\det D_\chi(\mathbf{k}^2) = E^4 - 2E^2\mathbf{k}^2 + \mathbf{k}^4 + 2E^2m^2 - 2\mathbf{k}^2m^2 - 6E^2\mu^2 + 2\mathbf{k}^2\mu^2 = 0, \tag{4.27}$$

and solving for E^2, we have

$$E^2 = \mathbf{k}^2 - m^2 + 3\mu^2 \pm \sqrt{m^4 + 4\mathbf{k}^2\mu^2 - 6m^2\mu^2 + 9\mu^4}. \tag{4.28}$$

We now expand the square root around $\mathbf{k}^2 = 0$

$$E^2 \approx \mathbf{k}^2 - m^2 + 3\mu^2 \pm \left[m^2 - 3\mu^2 + \frac{2\mu^2}{m^2 - 3\mu^2}\mathbf{k}^2\right] + \mathcal{O}(\mathbf{k}^4). \tag{4.29}$$

We thus have two dispersion relations:

$$E_+^2 = \frac{\mu^2 - m^2}{3\mu^2 - m^2}\mathbf{k}^2 + \mathcal{O}(\mathbf{k}^4) \tag{4.30}$$

$$E_-^2 = 6\mu^2 - 2m^2 + \frac{5\mu^2 - m^2}{3\mu^2 - m^2}\mathbf{k}^2 + \mathcal{O}(\mathbf{k}^4) \tag{4.31}$$

We can see that the dispersion relation in Eq. (4.30) is linear in k. This corresponds to a Goldstone boson of type-I,[1]. Note that the mode becomes massless at $\mu = m$. On the other hand, if we take the limit of $\mathbf{k}^2 \to 0$ in Eq. (4.31), we see that the mode has a mass gap. Hence, this dispersion relation corresponds to the mode χ_1, which remains massive after SSB.

In Fig. 4.2, we plot the mass term of $\mathcal{L}$, named here M, as a function of the chemical potential μ. We see how SSB occurs at a critical value for the chemical potential, $\mu^c = m$, and how the mode ϕ_1 acquires mass, while the mode ϕ_2 becomes massless.

[1] The energy of the mode goes as $E \sim k^{2n+1}$.

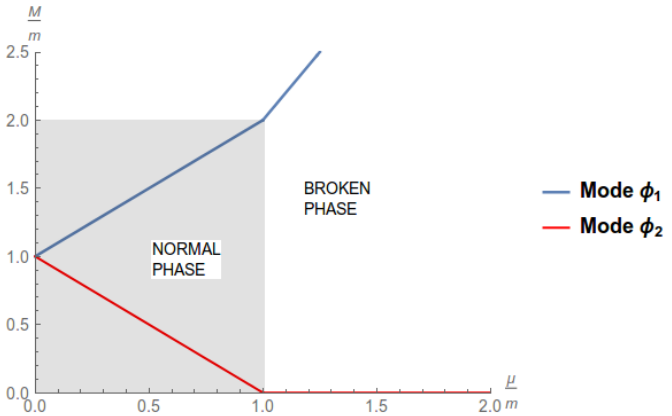


Figure 4.2: Mass spectrum of $\mathcal{L}$ as a function of μ. We have shaded the phase before SSB. The axes are normalized by the mass m. After SSB, the mode ϕ_2 becomes a massless mode.

4.4 Partition function and thermodynamic potential

Following now appendix F.1, we write the partition function Z, for the Lagrangian in Eq. (4.21). Once again, we only take the terms which are quadratic in the fields. The action, named here S_E, can be computed from $\mathcal{L}$ by integrating it over space-time volume. This time though, we integrate over imaginary time $\tau = it$:

$$S_E = \frac{1}{2}\int_0^\beta d\tau \int d^3x \Big[[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2] + 2\mu[\phi_2(\partial_0\phi_1) - \phi_1(\partial_0\phi_2)] \\ -(m^2-\mu^2)(\phi_1^2+\phi_2^2)\Big]. \tag{4.32}$$

The partition function is then given by:

$$Z = \int \mathcal{D}[\phi_1]\mathcal{D}[\phi_2]\ \exp\big[-S_E\big], \tag{4.33}$$

where the fields ϕ_1 and ϕ_2 satisfy that $\phi_i(\mathbf{x},0) = \phi_i(\mathbf{x},\beta)$. We now take Eq. (4.23) and Fourier expand the quantum fluctuations of each real field with respect to the vacuum as follows:

$$\phi_1 = \sqrt{2}\zeta\cos\theta + \sqrt{\frac{\beta}{V}}\sum_{n=-\infty}^{\infty}\sum_{\mathbf{k}}\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega_n\tau)]\phi_{1n}(\mathbf{k}), \tag{4.34}$$

$$\phi_2 = \sqrt{2}\zeta\sin\theta + \sqrt{\frac{\beta}{V}}\sum_{n=-\infty}^{\infty}\sum_{\mathbf{k}}\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega_n\tau)]\phi_{2n}(\mathbf{k}), \tag{4.35}$$

where the parameters ζ and θ are constants which determine the low-energy behavior of the fields, $\beta = 1/T$ is the inverse of the temperature and, V is the volume. Comparing

both Eq. (4.23) and (4.34), we notice that $v = \sqrt{2}\zeta\cos\theta$ and $\sqrt{2}\zeta\sin\theta = 0$, which implies $\theta = 0$. This shows that the $U(1)$ symmetry is broken by choosing a specific angular direction. We have also introduced as zeroth component of the 4-momentum, the *Matsubara frequencies*, ω_n. The reason for this is that ω_n are periodic and we want to preserve the boundary condition of periodicity of the fields, $\phi(\mathbf{x}, 0) = \phi(\mathbf{x}, \beta)$ [26]. Given $n \in \mathbb{Z}$, these frequencies are given by:

$$\omega_n = \begin{cases} 2\pi n/\beta & \text{for bosons,} \\ (2n+1)\pi/\beta & \text{for fermions.} \end{cases} \tag{4.36}$$

If we partial integrate once Eq. (4.32), we can then express the integrand as:

$$\begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} -\Box - m^2 + \mu^2 & -2i\mu\partial_0 \\ 2i\mu\partial_0 & -\Box - m^2 + \mu^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{4.37}$$

We now substitute the expressions for the fields in equations (4.34) and (4.35). The partition function then is

$$Z = \exp[\zeta^2\beta V(\mu^2 - m^2)] \int \mathcal{D}[\phi_1]\mathcal{D}[\phi_2] \; \exp\left[-\frac{1}{2}\sum_n \sum_{\mathbf{k}} \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} D \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\right], \tag{4.38}$$

where D is:

$$D = \beta^2 \begin{pmatrix} \omega_n^2 + \omega^2 - \mu^2 & -2\mu\omega_n \\ 2\mu\omega_n & \omega_n^2 + \omega^2 - \mu^2 \end{pmatrix}, \tag{4.39}$$

and where we have introduced $\omega = \sqrt{\mathbf{k}^2 + m^2}$. In order to perform the integrations over volume and over τ, we also used that:

$$\delta_{\mathbf{k}\mathbf{k}'} = \frac{1}{V}\int d^3x \; e^{i(\mathbf{k}-\mathbf{k}')\mathbf{x}}. \tag{4.40}$$

The next step is to compute the thermodynamic potential. We make use of Eq. (F.15). Note that the partition function of Eq. (4.38) is a simple Gaussian integral in two dimensions [13]:

$$\int d^2x \; \exp\left(-\frac{1}{2}x^i A_{ij} x^j\right) = \sqrt{\frac{(2\pi)^2}{\det A}}, \tag{4.41}$$

where A is a symmetric matrix. Putting all together, we express the partition function of Eq. (4.38) as:

$$Z = \exp\left[\zeta^2\beta V(\mu^2 - m^2)\right] \exp\left[-\frac{1}{2}\sum_n \sum_{\mathbf{k}} \ln\det D\right]. \tag{4.42}$$

The thermodynamic potential, here named Ω, can be computed from Z by taking the logarithm:

$$\Omega = \frac{1}{\beta}\ln Z = \frac{1}{\beta}\left[\zeta^2\beta V(\mu^2 - m^2) - \frac{1}{2}\ln\left(\prod_n \prod_{\mathbf{k}} \det D\right)\right]. \tag{4.43}$$

We now proceed to evaluate the last term in Eq. (4.43):

$$\begin{aligned}\ln\det D &= \ln\left[\prod_n\prod_{\mathbf{k}}\beta^4\left[(\omega_n^2+\omega^2-\mu^2)^2+4\mu^2\omega_2^2\right]\right]\\ &= \ln\left[\prod_n\prod_{\mathbf{k}}\beta^4\left[\omega_n^4+\omega^4+\mu^4+2\mu^2\omega_n^2-2\mu^2\omega^2+2\omega^2\omega_n^2\right]\right].\end{aligned}$$

We now add and subtract $2\omega\mu\omega_n$, so we can complete a total square. In this way, the terms inside the bracket can be rearranged like:

$$\begin{aligned}\ln\det D &= \ln\left[\prod_n\prod_{\mathbf{k}}\beta^4[\omega_n^2+(\omega-\mu)^2][\omega_n^2+(\omega+\mu)^2]\right]\\ &= \ln\left[\prod_n\prod_{\mathbf{k}}\beta^2[\omega_n^2+(\omega-\mu)^2]\right]+\ln\left[\prod_n\prod_{\mathbf{k}}\beta^2[\omega_n^2+(\omega+\mu)^2]\right], \quad (4.44)\end{aligned}$$

where we used that $\ln(a\cdot b)=\ln a+\ln b$. The potential then reduces to:

$$\Omega=\zeta^2V(\mu^2-m^2)-\frac{1}{2\beta}\sum_n^{\mathbf{k}}\ln\left[\beta^2[\omega_n^2+(\omega-\mu)^2]\right]-\frac{1}{2\beta}\sum_n^{\mathbf{k}}\ln\left[\beta^2[\omega_n^2+(\omega+\mu)^2]\right]. \quad (4.45)$$

We now recall that $\omega_n=2\pi nT$ and make use of Ref. 26 for the following expressions

$$\ln[(2\pi n)^2++\beta^2\omega^2]=\int_1^{\beta^2\omega^2}\left[\frac{dx^2}{x^2+(2\pi n)^2}+\ln[1+(2\pi n)^2]\right], \quad (4.46)$$

and

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+(x/2\pi)^2}=\frac{2\pi^2}{x}\left[\frac{e^x+1}{e^x-1}\right], \quad (4.47)$$

in order to perform the sum over the Matsubara frequencies. We first make the substitutions $\omega-\mu\to\omega$ and $\omega+\mu\to\omega$ in the two last terms of Eq. (4.45). Substituting then equations (4.46) and (4.47), each logarithm in Eq. (4.45) can be written as $\frac{1}{2}\beta\omega-\ln(1-\exp[-\beta\omega])$. Performing the sum over ω_n makes $\omega=E_k$, where E_k are the dispersion relations defined in Eq. (4.28) . We finally take the continuum limit $\sum_{\mathbf{k}}\to\int d^3k$. The thermodynamic potential Ω then is:

$$\begin{aligned}\Omega=\zeta^2V(\mu^2-m^2)-V\int\frac{d^3k}{(2\pi)^3}\Big[&E_k+T\ln\left(1-\exp[-\beta(E_k-\mu)]\right)\\ &+T\ln\left(1-\exp[-\beta(E_k+\mu)]\right)\Big]. \quad (4.48)\end{aligned}$$

Note that the first term in the integrand is independent of the temperature and leads to a divergent integral, which we will proceed to renormalize in the next section. Note also

that the angular dependence θ does not appear in the result, which makes sense since the theory is $U(1)$ symmetric. The two other terms in the integrand represent the particle and antiparticle contribution, respectively. This can be easily seen if we compute the charge density $\rho = Q/V$ of the condensate. We apply Eq. (F.16) to (4.48) and then divide by the volume V. We have:

$$\rho = \frac{1}{V}\frac{\partial}{\partial\mu}\Omega = 2\mu\zeta^2 + \int \frac{d^3k}{(2\pi)^3}\left(\frac{1}{\exp[\beta(E_k-\mu)]-1} - \frac{1}{\exp[\beta(E_k+\mu)]-1}\right). \tag{4.49}$$

The first term of the integrand corresponds to the particle contribution to the condensate, while the second term corresponds to the antiparticle contribution. Note that for $\mu = 0$, both terms are the same and thus, they cancel, i.e., both particles and antiparticles make the same contribution. We thus see that charge condensation requires a nonzero chemical potential.

As a last comment before closing this section, it is worth mentioning that the free parameter ζ, which appears both in Eq. (4.48) and (4.49), is related to the charge carried by condensed particles and, it can only be computed when $|\mu| = m$ since at fixed temperature and chemical potential, Ω has an extremum with respect to variation of ζ, hence it is zero otherwise:

$$\frac{\partial\Omega}{\partial\zeta} \sim (\mu^2 - m^2)\zeta. \tag{4.50}$$

The parameter ζ can then be obtained from Eq. (4.49) by setting $|\mu| = m$.

4.5 Renormalization

Let us now take a closer look at Eq. (4.48). If we take the limit $T \to 0$, the integral corresponds to the 1-loop correction to the classical potential, V. As we mention in section 4.4, the integral diverges and we therefore need to renormalize it. Instead of renormalizing Eq. (4.48), which is only evaluated at the minimum v, defined in Eq. (4.14), we want to renormalize a general expression. We hence go back to the expansions in Eq. (4.23) and, instead expanding around v, we expand around a background field, ϕ_0, which is equal to v at the minimum. We make the substitutions $\phi_1 \to \phi_0 + \chi_1$ and $\phi_2 \to \chi_2$. The resulting Lagrangian is

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}[(\partial_\mu\chi_1)^2 + (\partial_\mu\chi_2)^2] + \mu\Big[\chi_2(\partial_0\chi_1) - \chi_1(\partial_0\chi_2)\Big] \\ &- \frac{1}{2}(m^2 - \mu^2 + \frac{\lambda}{2}\phi_0)\chi_1^2 + \frac{1}{2}(m^2 - \mu^2 + \frac{\lambda}{6}\phi_0)\chi_2^2 + \dots,\end{aligned} \tag{4.51}$$

where we have only included terms quadratic in the fields since they are the ones contributing to the actions and hence, the terms we need in order to find the dispersion relations. As usual, we get the propagator from the action by partial integrating once. We skip the middle steps. In momentum space, the propagator for the Lagrangian in Eq. (4.51) is given

by:

$$\mathcal{D} = \begin{pmatrix} E^2 - \mathbf{k}^2 - m^2 + \mu^2 - \frac{\lambda}{2}\phi_0^2 & 2iE\mu \\ -2iE\mu & E^2 - \mathbf{k}^2 - m^2 + \mu^2 - \frac{\lambda}{6}\phi_0^2 \end{pmatrix}. \tag{4.52}$$

Demanding now $\det \mathcal{D} = 0$ and solving for E^2, we get the general dispersion relation:

$$E_\pm^2 = \mathbf{k}^2 + m^2 + \mu^2 + \frac{\lambda}{3}\phi_0^2 \pm \frac{1}{6}\sqrt{144\mathbf{k}^2\,\mu^2 + 144m^2\mu^2 + 48\lambda\mu^2\phi_0^2 + \lambda^2\phi_0^4}. \tag{4.53}$$

Note that for $\phi_0 = v$, we recover the dispersion relations in Eq. (4.28). The whole potential now looks like

$$V(\phi_0) = \underbrace{\frac{1}{2}(m^2 - \mu^2)\phi_0^2 + \frac{\lambda}{4!}\phi_0^4}_{tree-level} + \underbrace{\sum_{i=\pm}\int \frac{d^3k}{(2\pi)^3}\, E_i}_{1-loop}. \tag{4.54}$$

We now proceed to evaluate the integral in Eq. (4.54). Since the dispersion relation is not analytical, we need to first approximate it. In order to do this, we expand in powers of k around ∞. The easiest way to do this is to define $x = \frac{1}{k}$ and expand around $x = 0$. We first redefine the constants in the integral as

$$a = \frac{1}{144\mu^2}[144m^2\mu^2 + 48\lambda\mu^2\phi_0^2 + \lambda^2{\phi_0}^4], \tag{4.55}$$

$$b = m^2 + \mu^2 + \frac{\lambda}{3}\phi_0^2. \tag{4.56}$$

The expressions to expand then are:

$$E_\pm = \frac{1}{x}\sqrt{1 + bx^2 \pm 2x\mu\sqrt{1 + ax^2}}. \tag{4.57}$$

We expand the two expressions around $x = 0$ and then perform the sum. The result gives the contributions to the 1-loop corrections. Let us rewrite $\mathbf{k}$ as k for simplicity. We have

$$\int \frac{d^3k}{(2\pi)^3}\left[2k + \left(m^2 + \frac{1}{3}\lambda\phi_0^2\right)\frac{1}{k} - \left(\frac{m^4}{4} + \frac{1}{6}m^2\lambda\phi_0^2 + \frac{5\,\lambda^2}{144}\phi_0^4\right)\frac{1}{k^3}\right]. \tag{4.58}$$

If we integrate this expression until a cutoff of momentum Λ, in the limit $\Lambda \to \infty$ the divergent ultraviolet (UV) behavior of each term appears. We can see that they are quartic, quadratic and logarithmic, respectively. We now proceed to regularize each integral.

The first two terms, proportional to k and k^{-1}, respectively are massless integrals, i.e. there is no mass term in the denominator. Such integrals cannot be regularized. In fact the lack of mass term makes them have zero contribution at the UV limit. A less physical but more mathematical way of justifying this is the so-called *'t Hooft and Veltman* conjecture [33].

Let us now look at the logarithmic divergent term. The problem with this type of integral is that it is both UV and infrared[2] (IR) divergent. In order to regularize this type of integrals, we need to use two regulators. The UV divergence can be regulated by using dimensional regularization (DR) and, by giving mass to the particle and then taking the massless limit, we can take care of the IR divergence. Let us then make the substitution

$$\frac{1}{k^3} \to \frac{1}{\left[k^2+M^2\right]^{\frac{3}{2}}}, \tag{4.59}$$

which has the same UV behavior and is not IR divergent. Note that the parameter M has dimensions of mass. We regularize the integral by performing it in $n = 3-\varepsilon$ dimensions. We have

$$\int \frac{d^n k}{(2\pi)^n} \frac{1}{\left[k^2+M^2\right]^{\frac{3}{2}}} = \frac{1}{(2\pi)^n}\frac{1}{M^3}\int d\Omega_n \int dk \frac{k^{n-1}}{\left[1+\frac{k^2}{M^2}\right]^{\frac{3}{2}}}, \tag{4.60}$$

where we used $d^n k = d\Omega_n dk\ k^{n-1}$. Making now the substitution $k = \sqrt{M}t$, we can rewrite this expression in a way where we can recognize a *Beta* function [23]:

$$\frac{1}{2}M^{n-3}\frac{\Omega_n}{(2\pi)^n}\int dt \frac{t^{\frac{n}{2}-1}}{(1+t)^{\frac{3}{2}}} = \frac{1}{2}M^{n-3}\frac{\Omega_n}{(2\pi)^n} B\left(\frac{n}{2}, \frac{3-n}{2}\right). \tag{4.61}$$

We cannot proceed to expand the expression around the pole ε since it is not dimensionless. In order to fix this, we multiply the expressions by a parameter $\tilde{M}^{3-n}$, such that $[M] = [\tilde{M}]$. We then have

$$\frac{1}{2}\left(\frac{e^\gamma}{4\pi}\right)^{\frac{\varepsilon}{2}}\left(\frac{M}{\tilde{M}}\right)^{n-3}\frac{\Omega_n}{(2\pi)^n}B\left(\frac{n}{2}, \frac{3-n}{2}\right) = \frac{1}{2}\left(\frac{e^\gamma}{4\pi}\right)^{\frac{\varepsilon}{2}}\left(\frac{M}{\tilde{M}}\right)^{n-3}\frac{\Omega_n}{(2\pi)^n}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{3-n}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}. \tag{4.62}$$

where we have added a factor $[\exp(\gamma)/4\pi]^{\varepsilon/2}$. The reasons behind this is the renormalization condition we are using. In the modified minimal substraction scheme ($\overline{\text{MS}}$), we do not only subtract the divergent poles but also factors proportional to the Euler-Mascheroni constant γ. We now make use of:

$$M^\varepsilon = e^{\varepsilon \ln M} \approx 1 + \varepsilon \ln M + \mathcal{O}(\varepsilon^2), \tag{4.63}$$

$$\Gamma\left(\frac{\varepsilon}{2}\right) \approx \frac{2}{\varepsilon} - \gamma + \mathcal{O}(\varepsilon^2), \tag{4.64}$$

and the final result including the expansion around ε and the pre-factors is

$$-\frac{2}{4\pi^2\varepsilon}\left(\frac{m^4}{4}+\frac{1}{6}m^2\lambda\phi_0^2+\frac{5\,\lambda^2}{144}\phi_0^4\right)\left[1+\varepsilon\ln\left(\frac{\tilde{M}}{M}\right)+\mathcal{O}(\varepsilon^2)\right]. \tag{4.65}$$

The parameter M was added to fix the IR divergence during the regularization. Since it is arbitrary we can choose it to be equal to $\tilde{M}$. In this way, the loop-correction is independent

[2] Divergence at $k^2 \to 0$

of the chosen IR regulator.

We thus have three divergent terms proportional to $\frac{1}{\varepsilon}$. We need now to find counter-terms which cancel these three divergences. The term proportional to m^4 has no classical counter part. Thus in order to fix this divergence, we introduce a counter-term, i.e. the same term $\Delta\varepsilon \sim \frac{m^4}{\varepsilon}$ but with opposite sign. In this way, the divergences cancel. Let us now look at the other two divergent terms. Let us express the mass and coupling constant as

$$m^2 = m_0^2 + \delta m^2, \tag{4.66}$$

$$\lambda = \lambda_0 + \delta\lambda. \tag{4.67}$$

By dimensional analysis, we can see that the two remaining terms in the result of the integral correspond to corrections to the mass and coupling constant, respectively:

$$\delta m^2 = \frac{\lambda_0}{24\pi^2\varepsilon} m_0^2, \tag{4.68}$$

$$\delta\lambda = \frac{5}{72\pi^2\varepsilon}\lambda_0^2. \tag{4.69}$$

Now that we have isolated the divergence, we can write the full effective potential $\Gamma[\phi_0, m_0^2, \lambda_0]$ as:

$$\Gamma[\phi_0, m_0^2, \lambda_0] = \frac{1}{2}(m_0^2 + \delta m^2 - \mu^2)\phi_0^2 + \frac{1}{4}(\lambda_0 + \delta\lambda)\phi_0^4 + \sum_{i=\pm}\int \frac{d^3k}{(2\pi)^3}\, E_i(\phi_0). \tag{4.70}$$

The integral part is now finite, but still not analytical. If we were to evaluate it, we would have to do it numerically.

Chapter 5

Spontaneous symmetry breaking in non-Abelian theories

In this chapter, we consider a Lagrangian which is invariant under non-Abelian, i.e. non-commutative, transformations. We will use the group of proper rotations in 3-dimensional space $SO(3)$ as an example of a non-Abelian group.

Our discussions in this chapter follow mainly from Refs. 34, 35, 36. In particular, our discussion about $SO(3)$ follows from Ref. 19.

5.1 The rotation group SO(3)

Let R be a 3×3 real matrix. We say that $R \in O(3)$, that is, the *orthogonal group* of dimension 3, if and only if

$$R^T = R^{-1}, \tag{5.1}$$

that is, R is orthogonal if its transpose is equal to its inverse. If, in addition, $\det R = 1$, we say that $R \in SO(3)$, that is, the *special* orthogonal group of dimension 3. Any matrix in $SO(3)$ can be expressed as $\exp L$, where, L is the Lie algebra of the group, $\mathfrak{so}(3)$. If $R^T = R^{-1}$, it follows that $L^T = -L$. If we now take the trace of L, we have Tr $L^T\big($ = Tr $L\big)$ = Tr$[-L]$, which implies that the trace has to be zero. The Lie algebra of $SO(3)$ then is all 3×3 traceless antisymmetric matrices. The three linearly independent matrices fulfilling these conditions are

$$\lambda_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.2}$$

The most general form for L then is

$$L = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} = a\lambda_1 + b\lambda_2 + c\lambda_3. \tag{5.3}$$

The Lie algebra of $SO(3)$ is 3-dimensional vector space where the matrices λ_i's are the basis. We then call them *generators* of the $SO(3)$, and they obey the commutation relations

$$[\lambda_i, \lambda_j] = \varepsilon_{ijk}\lambda_k, \tag{5.4}$$

where ε_{ijk} is the *Levi-Civita* tensor. Let us now define $\vec{n} = (n_1, n_2, n_3) = \frac{1}{\theta}(a, b, c)$. We then have $L = \theta\vec{n} \cdot \vec{\lambda}$. If we apply L to a 3-dimensional vector $\vec{u} \in \mathbb{R}^3$, we find a useful result :

$$\begin{aligned} L\vec{u} = \theta\vec{n} \cdot \vec{\lambda} &\sim \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \\ &= u_3n_2 - u_2n_3 + u_1n_3 - u_3n_1 + u_2n_1 - u_1n_2 = \vec{n} \times \vec{u}. \end{aligned} \tag{5.5}$$

Let us finally relate a matrix belonging to $SO(3)$ to rotations in coordinate space. In order to do this, we start by computing L^2 and L^3:

$$L^2 = -\theta^2\mathbb{I} + \begin{pmatrix} a \\ b \\ c \end{pmatrix} (a\ b\ c), \tag{5.6}$$

$$L^3 = -\theta^2 L. \tag{5.7}$$

We now Taylor expand $R = \exp L$:

$$\begin{aligned} \exp L &= \mathbb{I} + (1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots)L + (\frac{1}{2} - \frac{\theta^2}{4!} + \frac{\theta^4}{6!} - \cdots)L^2 \\ &= \mathbb{I} + \frac{1}{\theta}\sin\theta L + \frac{1}{\theta^2}(1 - \cos\theta)L^2. \end{aligned} \tag{5.8}$$

Using finally equations (5.5) and (5.8), we can write the action of a matrix $R \in SO(3)$ to a vector $u \in \mathbb{R}^3$ as

$$R\vec{u} = \vec{u} + \sin\theta(\vec{n} \times \vec{u}) + (1 - \cos\theta)(\vec{n} \times (\vec{n} \times \vec{u})). \tag{5.9}$$

In terms of rotations, the vector $\vec{n}$ is the rotation axis and θ the rotation angle.

5.2 Noether currents

We now consider the Lagrangian for a real vector field:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - \frac{1}{2}m^2\phi_i\phi_i - \frac{\lambda}{4!}(\phi_i\phi_i)^2, \tag{5.10}$$

where $i = 1, 2, 3$ and $m^2 > 0$. $\mathcal{L}$ is invariant under rotation in 3-dimensional space. We can see this by performing the transformation $\phi_i \to R_{ij}\phi_j$ in Eq. (5.10) with $R_{ij} \in SO(3)$:

$$\begin{aligned}\mathcal{L}' &= \frac{1}{2}(\partial_\mu R_{ji}\phi_i)(\partial^\mu R_{ij}\phi_i) - \frac{1}{2}m^2 R_{ji}\phi_i R_{ij}\phi_i - \frac{\lambda}{4!}(R_{ji}\phi_i R_{ij}\phi_i)^2 \\ &= \frac{1}{2}R_{ji}R_{ij}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - \frac{1}{2}m^2 R_{ji}R_{ij}\phi_i\phi_i - \frac{\lambda}{4!}(R_{ji}R_{ij}\phi_i\phi_i)^2. \end{aligned} \tag{5.11}$$

Since $R_{ij} \in SO(3)$, we have that $R_{ji} = R_{ij}^{-1}$ and hence $R_{ji}R_{ij} = \mathbb{I}$. Therefore $\mathcal{L}' = \mathcal{L}$. Our theory has a continuous symmetry so, by Noether's theorem, there is a corresponding quantity whose value is conserved in time. As we saw in the previous section, $SO(3)$ has three generators, and thus, we expect to have the same number of conserved Noether charges. We know from previous chapters that the Noether currents are given by

$$j^\mu = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a. \tag{5.12}$$

We can compute the $\delta\phi_i$ for our theory by applying transformations like $\phi_i \to \phi_i + \delta\phi_i$. If we use vector notation, we have:

$$\vec{\phi} \to R\vec{\phi} \approx (\mathbb{I} + \theta\vec{n}\cdot\vec{\lambda})\vec{\phi},$$

which implies $\delta\vec{\phi} = \theta(\vec{n}\cdot\vec{\lambda})\vec{\phi} = \theta(\vec{n}\times\vec{\phi})$. We have used here that $R = \exp L = \exp(\theta\vec{n}\cdot\vec{\lambda})$ and the relation in Eq. (5.5). We substitute the expression of $\delta\phi_i$ into Eq. (5.12) and we see that we have the following Noether currents:

$$j_1^\mu = \begin{pmatrix}\partial^\mu\phi_1 & \partial^\mu\phi_2 & \partial^\mu\phi_3\end{pmatrix}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -n_1\\ 0 & n_1 & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = n_1(\partial^\mu\phi_2)\phi_3 - n_1(\partial^\mu\phi_3)\phi_2, \tag{5.13}$$

$$j_2^\mu = \begin{pmatrix}\partial^\mu\phi_1 & \partial^\mu\phi_2 & \partial^\mu\phi_3\end{pmatrix}\begin{pmatrix}0 & 0 & n_2\\ 0 & 0 & 0\\ -n_2 & 0 & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = n_2(\partial^\mu\phi_1)\phi_3 - n_2(\partial^\mu\phi_3)\phi_1, \tag{5.14}$$

$$j_3^\mu = \begin{pmatrix}\partial^\mu\phi_1 & \partial^\mu\phi_2 & \partial^\mu\phi_3\end{pmatrix}\begin{pmatrix}0 & -n_3 & 0\\ n_3 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = n_3(\partial^\mu\phi_2)\phi_1 - n_3(\partial^\mu\phi_1)\phi_2, \tag{5.15}$$

which written in a more elegant way looks like:

$$j^\mu = (\partial^\mu\phi_a)\delta\phi_a = \varepsilon_{ijk}n_i\partial^\mu\phi_j\phi_k. \tag{5.16}$$

Note that we again can ignore the θ and the current is still conserved. The total (conserved) charges, Q_i, can be computed from Eq. (5.16) by integrating the zeroth component over a large volume Ω:

$$Q_i = \int_\Omega d^3x\; j_i^0 = \int_\Omega d^3x\; \varepsilon_{ijk}n_i\dot{\phi_j}\phi_k, \tag{5.17}$$

where we used the notation $\partial^0\phi = \dot{\phi}$. We recall now that the currents Q_i generate the symmetry and hence, they are a fundamental representation, i.e. irreducible and finite dimensional, of the Lie algebra of $SO(3)$. We thus expect that the Q_i's follow the commutation relations in Eq. (5.4) after quantization. Let us see this. For this purpose and for simplicity in the derivation, let us assume that the vector $\vec{n}$ is a unit vector, so that we can set every component n_j to 1. We will compute the commutator for the charges Q_1 and Q_2, which can be computed from equations (5.13) and (5.14), respectively. We will also make use of the equal-time canonical commutation relations between a field and its conjugate momentum

$$\big[\phi_a(x), \dot{\phi}_b(x')\big] = \delta_{ab}\, \delta^{(3)}(x - x'). \tag{5.18}$$

We have:

$$\begin{aligned}\big[Q_1, Q_2\big] &= \int_\Omega d^3x \int_\Omega d^3x' \, [\phi_2(x)\dot{\phi_3}(x) - \phi_3(x)\dot{\phi_2}(x), \phi_3(x')\dot{\phi_1}(x') - \phi_1(x')\dot{\phi_3}(x')] \\ &= \int_\Omega d^3x \int_\Omega d^3x' \, \Big[[\phi_2(x)\dot{\phi_3}(x), \phi_3(x')\dot{\phi_1}(x')] - [\phi_2(x)\dot{\phi_3}(x), \phi_1(x')\dot{\phi_3}(x')] \\ &- [\phi_3(x)\dot{\phi_2}(x), \phi_3(x')\dot{\phi_1}(x')] + [\phi_3(x)\dot{\phi_2}(x), \phi_1(x')\dot{\phi_3}(x')]\Big]. \end{aligned} \tag{5.19}$$

Making now use of Eq. (5.18) and

$$[AB, CD] = A[B, C]D + [A, C]BD + CA[B, D] + C[A, D]B, \tag{5.20}$$

we can compute the commutators of Eq. (5.19) and rearrange terms as

$$\begin{aligned}\big[Q_1, Q_2\big] &= \int_\Omega d^3x \int_\Omega d^3x' \big[\phi_1(x)\dot{\phi_2}(x') - \phi_2(x)\dot{\phi_1}(x')\big]\delta^{(3)}(x - x') \\ &= \int_\Omega d^3x \big[\phi_1\dot{\phi_2} - \phi_2\dot{\phi_1}\big] = Q_3. \end{aligned} \tag{5.21}$$

The rest of the commutators can be computed in the same way as we just showed. We thus check that the charges follow the commutation relations in Eq. (5.4).

5.3 Mass spectrum

Consider now the action S for the Lagrangian in Eq. (5.10):

$$\begin{aligned} S &= \frac{1}{2}\int d^4x \, \mathcal{L} \\ &= \frac{1}{2}\int d^4x \Big[(\partial_\mu\phi_i)(\partial^\mu\phi_i) - m^2\phi_i\phi_i \\ &= \frac{1}{2}\int d^4x \, \phi_j \Big[-\Box - m^2\Big]\delta_{ij}\phi_j. \end{aligned} \tag{5.22}$$

The last term between brackets in the equation above is the propagator, Δ_{ij}. By writing it in matrix form, we can see that it is diagonal:

$$\Delta = \begin{pmatrix} -\Box - m^2 & 0 & 0 \\ 0 & -\Box - m^2 & 0 \\ 0 & 0 & -\Box - m^2 \end{pmatrix}. \tag{5.23}$$

The dispersion relation for the fields ϕ_i can be computed from Δ by Fourier transforming to momentum space and demanding $\det \Delta = 0$. The Fourier transformation gives $\Delta = \text{diag}[E^2 - \mathbf{k}^2 - m^2, E^2 - \mathbf{k}^2 - m^2, E^2 - \mathbf{k}^2 - m^2]$. Solving then for E^2, we have:

$$\det \Delta = 0 \quad \Rightarrow \quad E^2 = \mathbf{k}^2 - m^2, \tag{5.24}$$

for each field ϕ_i $(i = 1, 2, 3)$.

5.4 Spontaneous symmetry breaking

Let us now consider the potential $V(\phi_i \phi_i)$ for the Lagrangian in Eq. (5.10):

$$V(\phi_i \phi_i) = \frac{1}{2} m^2 \phi_i \phi_i + \frac{\lambda}{4!} (\phi_i \phi_i)^2, \tag{5.25}$$

where we recall that $m^2 < 0$. The potential is minimized for $v^2 = |\langle \vec{\phi} \rangle|^2 = -6m^2/\lambda$. Let us now choose the direction of the vacuum so that it points along one of the components of the field. Without loss of generality we can make $\langle \phi_1 \rangle = \langle \phi_2 \rangle = 0$ and $\langle \phi_3 \rangle = v$. Written in vector form, we have

$$\begin{pmatrix} \langle \phi_1 \rangle \\ \langle \phi_2 \rangle \\ \langle \phi_3 \rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}. \tag{5.26}$$

This vector is invariant under $SO(2) \subset SO(3)$, i.e. rotations around the ϕ_3-axis which rotate ϕ_1 and ϕ_2 . The $SO(3)$ symmetry spontaneously breaks down to $SO(2)$. From Goldstone's theorem, we know that the number of massless modes that will appear after SSB equals the dimension of quotient group $\dim[SO(3)/SO(2)] = \dim SO(3) - \dim SO(2)$. Since the latter has one generator and the former three, there will be two massless modes. Let us see this. As usual, we expand each component of the vector field around their vacuum expectation value:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \to \begin{pmatrix} \langle \phi_1 \rangle + \chi_1 \\ \langle \phi_2 \rangle + \chi_2 \\ \langle \phi_3 \rangle + \chi_3 \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \chi_2 \\ v + \chi_3 \end{pmatrix}, \tag{5.27}$$

where the χ_i's are the quantum fluctuating fields. Plugging these expressions in the Lagrangian of Eq. (5.10) results into the following:

$$\mathcal{L}_\chi = \frac{3}{2} \frac{m^4}{\lambda} + \frac{1}{2} (\partial_\mu \vec{\chi}) \cdot (\partial^\mu \vec{\chi}) + m^2 \chi_3^2 - \frac{\lambda}{6} v \chi_3 (\vec{\chi} \cdot \vec{\chi}) + \frac{\lambda}{4!} (\vec{\chi} \cdot \vec{\chi})^2, \tag{5.28}$$

where we can see that the mass terms for the modes ϕ_1 and ϕ_2 have disappeared. Taking now only the terms which are quadratic in fields, we can compute the new action, S_χ, from which we can get the new propagator Δ_χ:

$$S_\chi = \int d^4x\, \mathcal{L} = \frac{1}{2}\int d^4x \Big[(\partial_\mu \vec{\chi}) \cdot (\partial^\mu \vec{\chi}) + 2m^2 \chi_3\Big]$$

$$= \frac{1}{2}\int d^4x \begin{pmatrix}\chi_1 & \chi_2 & \chi_3\end{pmatrix} \Delta_\chi \begin{pmatrix}\chi_1 \\ \chi_2 \\ \chi_3\end{pmatrix}, \tag{5.29}$$

with

$$\Delta_\chi = \begin{pmatrix}-\Box & 0 & 0 \\ 0 & -\Box & 0 \\ 0 & 0 & -\Box + 2m^2\end{pmatrix}. \tag{5.30}$$

We can compute the dispersion relation after SSB. We Fourier transform the propagator in Eq. (5.30) to momentum space and demand $\det \Delta_\chi = 0$. We get the following for each mode:

$$E^2_{\chi_1} = \mathbf{k}^2, \tag{5.31}$$
$$E^2_{\chi_2} = \mathbf{k}^2, \tag{5.32}$$
$$E^2_{\chi_3} = \mathbf{k}^2 - 2m^2. \tag{5.33}$$

If we compare these dispersion relations to the one in Eq. (5.24), we can see that the modes ϕ_1 and ϕ_2 have become massless, whereas the mode ϕ_3 has acquired a mass. We can see that the dispersion relations for both ϕ_1 and ϕ_2 are linear in momentum. This implies that they are Goldstone modes of type-I.

5.5 Adding a chemical potential

We now want to couple our $SO(3)$-symmetric theory to one of its conserved currents through a chemical potential. Let us fist rename the Lagrangian in Eq. (5.10) as $\mathcal{L}_0$. We can compute the Hamiltonian density, $\mathcal{H}_0$ by Legendre transforming $\mathcal{L}_0$. We first compute the conjugated momentum for each field ϕ_a:

$$\pi_a = \frac{\partial \mathcal{L}_0}{\partial \dot{\phi}_a} = \dot{\phi}_a. \tag{5.34}$$

The usual Legendre transformation of $\mathcal{L}_0$ gives the Hamiltonian density

$$\mathcal{H}_0 = \frac{1}{2}\big[\vec{\pi} \cdot \dot{\pi} + \nabla\vec{\phi} \cdot \nabla\vec{\phi}\big] + V(\vec{\phi} \cdot \vec{\phi}), \tag{5.35}$$

where $\vec{\pi} = (\pi_1\ \pi_2\ \pi_3)$, and $V(\vec{\phi} \cdot \vec{\phi})$ is given in Eq. (5.25). We can couple one of the conserved currents to our theory by making the transformation $\mathcal{H}_0 \to \mathcal{H} = \mathcal{H}_0 - \mu_3 j^0_3$,

where μ_3 is a chemical potential and j_3^0 is the zeroth component of the Noether current in Eq. (5.15). We assume $n_3 = 1$ for simplicity in the derivation. The new Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\left[\vec{\pi}\cdot\dot{\pi} + \nabla\vec{\phi}\cdot\nabla\vec{\phi}\right] - \mu_3(\pi_2\phi_1 - \pi_1\phi_2) + V(\vec{\phi}\cdot\vec{\phi}). \tag{5.36}$$

The next step is to Legendre transform $\mathcal{H}$ back to a new Lagrangian $\mathcal{L}$. The new $\dot{\phi}_a$'s are given by the following:

$$\dot{\phi}_1 = \frac{\partial\mathcal{H}}{\partial\pi_1} = \pi_1 + \mu_3\phi_2 \Rightarrow \pi_1 = \dot{\phi}_1 - \mu_3\phi_2, \tag{5.37}$$

$$\dot{\phi}_2 = \frac{\partial\mathcal{H}}{\partial\pi_2} = \pi_2 - \mu_3\phi_1 \Rightarrow \pi_2 = \dot{\phi}_2 + \mu_3\phi_1, \tag{5.38}$$

$$\dot{\phi}_3 = \frac{\partial\mathcal{H}}{\partial\pi_3} = \pi_3 \Rightarrow \pi_3 = \dot{\phi}_3 \tag{5.39}$$

We now substitute these new π_a's into the Legendre transformation of $\mathcal{H}$. The resulting Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) + \mu_3(\dot{\phi}_2\phi_1 - \dot{\phi}_1\phi_2) + \frac{1}{2}\mu_3^2(\phi_1^2+\phi_2^2) - \frac{1}{2}m^2\phi_i\phi_i - \frac{\lambda}{4!}(\phi_i\phi_i)^2, \tag{5.40}$$

where, again, $(i = 1, 2, 3)$. We can now write the action S for this new Lagrangian density. As usual, we partial integrate once so that we can express $(\partial_\mu\phi_a)^2$ as $-\phi_a\Box\phi_a$. The action can then be expressed as the following matrix product:

$$S = \int d^4x\,\mathcal{L} = \frac{1}{2}\int d^4x \begin{pmatrix}\phi_1 & \phi_2 & \phi_3\end{pmatrix}\Delta_\mu \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}, \tag{5.41}$$

with Δ_μ being the new propagator and given by

$$\Delta_\mu = \begin{pmatrix} -\Box - m^2 + \mu_3^2 & 2\mu_3\partial_0 & 0\\ -2\mu_3\partial_0 & -\Box - m^2 + \mu_3^2 & 0\\ 0 & 0 & -\Box - m^2\end{pmatrix}. \tag{5.42}$$

Once again, we can compute the dispersion relation for our new theory by Fourier transforming to momentum space and demanding $\det\Delta_\mu = 0$. We have

$$\det\begin{pmatrix} E^2 - \mathbf{k}^2 - m^2 + \mu_3^2 & 2iE\mu_3 & 0\\ -2iE\mu_3 & E^2 - \mathbf{k}^2 - m^2 + \mu_3^2 & 0\\ 0 & 0 & E^2 - \mathbf{k}^2 - m^2\end{pmatrix} = 0, \tag{5.43}$$

which gives three solution, i.e, three dispersion relations for the three modes ϕ_a, if we solve for E^2:

$$(E_{\phi_1} + \mu_3)^2 = \mathbf{k}^2 + m^2, \tag{5.44}$$

$$(E_{\phi_2} - \mu_3)^2 = \mathbf{k}^2 + m^2, \tag{5.45}$$

$$E_{\phi_3}^2 = \mathbf{k}^2 + m^2. \tag{5.46}$$

We see that the chemical potential shifts the dispersion relations of the modes ϕ_1 and ϕ_2, whereas the mode ϕ_3 remains decoupled.

5.6 Spontaneous symmetry breaking

Let us now consider the Lagrangian in Eq. (5.40). We first rewrite $\mathcal{L}$ as $\mathcal{L} = \mathcal{L}_{12} + \mathcal{L}_3 + \mathcal{L}_{\text{int}}$, where:

$$\begin{aligned}\mathcal{L}_{12} = &\frac{1}{2}[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2] + \mu[\phi_2(\partial_0\phi_1) - \phi_1(\partial_0\phi_2)] \\ &- \frac{1}{2}(m^2 - \mu^2)(\phi_1^2 + \phi_2^2) + \frac{\lambda}{4!}(\phi_1^2 + \phi_2^2)^2, \end{aligned} \tag{5.47}$$

$$\mathcal{L}_3 = \frac{1}{2}(\partial_\mu\phi_3)^2 - \frac{1}{2}m^2\phi_3^3 - \frac{\lambda}{4!}\phi_3^4, \tag{5.48}$$

$$\mathcal{L}_{int} = -\frac{\lambda}{12}(\phi_1^2\phi_3^3 + \phi_2^2\phi_3^2). \tag{5.49}$$

By writing $\mathcal{L}$ in this way, we notice that it is invariant under $SO(2)$ transformations, that is, rotations of the $\phi_1\phi_2$-plane around the ϕ_3-axis.[1] By having coupled our theory to one of the currents, we have explicitly broken the previous $SO(3)$ symmetry into $SO(2)$. Our purpose in this section is to spontaneously break this remaining symmetry and find the corresponding Goldstone modes. Since $SO(N)$ has $\frac{1}{2}N(N-1)$ generators, $SO(2)$ has only one. Therefore we can only expect one massless mode.

The potential density of $\mathcal{L}_{12}$ is minimized for $\rho^2 = \frac{6}{\lambda}(\mu_3^2 - m^2)$. Without loss of generality let us choose the vacuum in the ϕ_1 direction.

$$\begin{pmatrix}\langle\phi_1\rangle \\ \langle\phi_2\rangle \\ \langle\phi_3\rangle\end{pmatrix} = \begin{pmatrix}\rho \\ 0 \\ 0\end{pmatrix}. \tag{5.50}$$

In order for the symmetry to be spontaneously broken, we need to have a positive "mass term", which implies that $m^2 < \mu_3^2$. We now make the substitutions $\phi_i \to \langle\phi_i\rangle + \chi_i$, where χ_i are, once again, quantum fluctuating fields. The resulting Lagrangian is

$$\begin{aligned}\mathcal{L}_\chi = &\frac{3}{2\lambda}(m^4 - \mu^4) + \frac{1}{2}(\partial_\mu\vec{\chi})\cdot(\partial^\mu\vec{\chi}) + \mu_3(\dot{\chi}_2\chi_1 - \dot{\chi}_1\chi_2) \\ &+ (m^2 - \mu_3^2)\chi_1^2 - \frac{1}{2}\mu_3^2\chi_3^2 - \frac{\lambda}{6}v\chi_1(\vec{\chi}\cdot\vec{\chi}) + \frac{\lambda}{4!}(\vec{\chi}\cdot\vec{\chi})^2. \end{aligned} \tag{5.51}$$

which is analogous to the Lagrangian that we obtained in the previous chapter in the case of a $U(1)$ symmetry. This is expected since $SO(2) \cong U(1)$.

From this new Lagrangian we can write the new action, and from it, compute the new dispersion relations after SSB. Taking once again only the terms quadratic in the fields, the action is

$$S = \int d^4x\, \mathcal{L} = \frac{1}{2}\int d^4x \begin{pmatrix}\chi_1 & \chi_2 & \chi_3\end{pmatrix} \Delta_{\mu 1} \begin{pmatrix}\chi_1 \\ \chi_2 \\ \chi_3\end{pmatrix}, \tag{5.52}$$

[1] The charge operator Q_3 generates rotation around the ϕ_3-axis.

where $\Delta_{\mu 1}$ is the new propagator, defined as

$$\Delta_{\mu 1} = \begin{pmatrix} -\Box + 2m^2 - 2\mu_3^2 & 2\mu_3\partial_0 & 0 \\ -2\mu_3\partial_0 & -\Box & 0 \\ 0 & 0 & -\Box - \mu_3^2 \end{pmatrix}. \tag{5.53}$$

The new dispersion relations are obtained by Fourier transforming $\Delta_{\mu 1}$ to momentum space and demanding $\det \Delta_{\mu 1} = 0$. We then have the following three relations:

$$E^2 = \mathbf{k}^2 + \mu_3^2, \tag{5.54}$$

$$E_\pm^2 = \mathbf{k}^2 - m^2 + 3\mu_3^2 \pm \sqrt{m^4 + 4\mathbf{k}^2\mu_3^2 - 6m^2\mu_3^2 + 9\mu_3^4}. \tag{5.55}$$

We see that, in the limit $\mu_3 \to 0$, we recover the dispersion relation in equations (5.31), (5.32) and (5.33). Let us finally expand the dispersion relations in Eq. (5.55) for small momenta. The square root gives, to fourth order in $\mathbf{k}$,

$$\sqrt{m^4 + 4\mathbf{k}^2\mu_3^2 - 6m^2\mu_3^2 + 9\mu_3^4} \approx m^2 - 3\mu_3^2 + \frac{2\mu^2}{m^2 - 3\mu^2}\mathbf{k}^2 + \mathcal{O}(\mathbf{k}^4). \tag{5.56}$$

The two resulting dispersion relations hence are:

$$E_+^2 = \frac{\mu^2 - m^2}{3\mu^2 - m^2}\mathbf{k}^2 + \mathcal{O}(\mathbf{k}^4) \tag{5.57}$$

$$E_-^2 = 6\mu^2 - 2m^2 + \frac{5\mu^2 - m^2}{3\mu^2 - m^2}\mathbf{k}^2 + \mathcal{O}(\mathbf{k}^4) \tag{5.58}$$

We see that only one of the modes has become massless. This was expected since $SO(2)$ only has one generator that can be spontaneously broken. We also see that the dispersion relation (5.57) is linear in momentum, this implies a type-I Goldstone mode. In addition, we see that if we take the limit $\mathbf{k}^2 \to 0$ in Eq. (5.58), we have a mass gap.

Chapter 6

The linear σ-model

In this chapter, we will study the linear σ-model (LSM) introduced by Gell-Mann and Lévy in 1960 [37]. The model was originally introduced to describe pions and nucleons and, in addition, a scalar meson σ-field. Gell-Man [38] and Zweig [39] would (independently) introduce the *quark model* in 1964. The LSM would later be modified so that it would include the quark contribution. It would also be renamed as the *linear σ-model with quarks* (LSMq) or just *quark-meson model* (QM). In this chapter, we will not consider the fermion terms and only focus on the LSM, which consists of the σ and the three π^k-fields. Our model is then constructed in terms of four real scalars σ and $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$.

Our discussions in this chapter is based on Refs. 7, 28, 31, 40 and 41. In particular, the discussion on $SU(2)$ follows from Ref. 19.

6.1 The complex group $SU(2)$

Let U be a 2×2 complex matrix,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

We say that $U \in SU(2)$, i.e., the special unitary group of dimension two, if and only if,

$$U^\dagger = U^{-1} \text{ and } \det U = 1. \tag{6.1}$$

The condition $\det U = 1$ implies $\alpha\delta - \beta\gamma = 1$, while $U^\dagger = U^{-1}$ implies

$$\begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \cdot \Rightarrow \begin{array}{c} \delta = \alpha^*, \\ \gamma = -\beta^*. \end{array}$$

The general form for U then is

$$U = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \text{ with } |\alpha|^2 + |\beta|^2 = 1. \tag{6.2}$$

In order to find the generators of $SU(2)$, we proceed as in the previous chapter and express U as $\exp N$, where N is the Lie algebra of the group. By imposing the conditions of Eq. (6.1), we find that N are 2×2 anti-Hermitian traceless matrices. The three linearly independent matrices fulfilling these conditions are the *Pauli matrices*, τ_i:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.3}$$

which follow the commutation relations

$$[\tau_i, \tau_j] = \varepsilon_{ijk}\tau_k. \tag{6.4}$$

Defining an axis vector $\vec{n} = (n_1, n_2, n_3)$, we find the most general expression for the matrix U:

$$U = \exp N = \exp[-\frac{i}{2}\theta\ (\vec{n} \cdot \vec{\tau})]. \tag{6.5}$$

If we compare the algebras of $SO(3)$ and $SU(2)$, we see that they look similar. We can actually make a correspondence between the generators λ_j of $SO(3)$ and $i\tau_j$. This correspondence is an isomorphism between the algebras of the two groups. Following the *Campbell-Baker-Hausdorff* theorem, the two groups are locally isomorphic within a finite region around the identity [19].

6.2 Lagrangian density and symmetries

Let us consider the Lagrangian for an $O(4)$-multiplet field $\Phi^T = (\phi_1, \phi_2, \phi_3, \phi_4)$:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_i)(\partial^\mu\phi_i) - \frac{1}{2}m^2\phi_i\phi_i - \frac{\lambda}{4!}(\phi_i\phi_i)^2, \tag{6.6}$$

with $i = 1, 2, 3, 4$ and let us rename the components of the multiplet as $\Phi = (\sigma, \boldsymbol{\pi})$, so that we can identify them with the scalar σ meson and the pion fields. The Lagrangian above can then be rewritten as the known one for the linear σ-model.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\boldsymbol{\pi})^2 - \frac{1}{2}m^2(\sigma^2 + \boldsymbol{\pi}^2) - \frac{\lambda}{4!}(\sigma^2 + \boldsymbol{\pi}^2)^2. \tag{6.7}$$

The σ-model is invariant under $O(4)$. Let $R \in O(4)$, i.e. R is a 4×4 real matrix such that $R^T = R^{-1}$. If we apply the transformation $\Phi \to R\Phi$ and $\Phi^T \to \Phi^T R^T$ in Eq. (6.6), we have

$$\begin{aligned} \mathcal{L}' &= \frac{1}{2}[\partial_\mu(R\Phi)]^T[\partial^\mu(R\Phi)] - \frac{1}{2}m^2(R\Phi)^T(R\Phi) - \frac{\lambda}{4!}[(R\Phi)^T(R\Phi)]^2 \\ &= \frac{1}{2}[\partial_\mu\Phi]^T R^T R[\partial^\mu\Phi] - \frac{1}{2}m^2(\Phi^T R^T R\Phi) - \frac{\lambda}{4!}[\Phi^T R^T R\Phi]^2. \end{aligned} \tag{6.8}$$

Since $R^T = R^{-1}$, we see that $\mathcal{L}$ remains invariant.

Another way of seeing this $O(4)$ symmetry in the Lagrangian is to use that local isomorphism between $O(4)$ and $SU(2) \times SU(2)$. In order to do this we redefine Φ as a complex doublet in terms of the complex scalar fields χ_i as follows

$$\Phi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma - i\pi_3 \\ \pi_2 - i\pi_1 \end{pmatrix}. \tag{6.9}$$

If we apply transformations like $\Phi \to U\Phi$, with $U = \exp\exp[-\frac{i}{2}\theta\ (\vec{n} \cdot \vec{\tau})]$, we have:

$$\begin{aligned} \mathcal{L}' &= \frac{1}{2}[\partial_\mu (U\Phi)]^\dagger [\partial^\mu (U\Phi)] - \frac{1}{2}m^2 (U\Phi)^\dagger (U\Phi) - \frac{\lambda}{4!}[(U\Phi)^\dagger (U\Phi)]^2 \\ &= \frac{1}{2}[\partial_\mu \Phi]^\dagger U^\dagger U [\partial^\mu \Phi] - \frac{1}{2}m^2 (\Phi^\dagger U^\dagger U \Phi) - \frac{\lambda}{4!}[\Phi^\dagger U^\dagger U \Phi]^2, \end{aligned} \tag{6.10}$$

and since $U \in SU(2)$, we have $U^\dagger = U^{-1}$. We then see that $\mathcal{L}$ still remains invariant. The $SU(2) \subset O(4)$ symmetry of the Lagrangian is clear.[1] The next step to obtain the full $O(4)$ symmetry is to define the Σ matrices as

$$\Sigma = \frac{1}{\sqrt{2}}(\sigma - i\vec{\tau} \cdot \boldsymbol{\pi}) = \begin{pmatrix} \chi_1 & -\chi_2^\dagger \\ \chi_2 & \chi_1^\dagger \end{pmatrix}, \tag{6.11}$$

which fulfill $\mathrm{Tr}[\Sigma^\dagger \Sigma] = \frac{1}{2}(\sigma^2 + \boldsymbol{\pi}^2)$. Note that the original fields can be expressed in terms of the trace of Σ:

$$\sigma = \frac{1}{\sqrt{2}}\mathrm{Tr}\Sigma, \qquad \pi^k = \frac{i}{\sqrt{2}}\mathrm{Tr}[\tau^k \Sigma]. \tag{6.12}$$

The Lagrangian for the σ-model in terms of these matrices is:

$$\mathcal{L}_\Sigma = \frac{1}{4}\mathrm{Tr}[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma] - \frac{1}{4}m^2 \mathrm{Tr}[\Sigma^\dagger \Sigma] - \frac{\lambda}{16}(\mathrm{Tr}[\Sigma^\dagger \Sigma])^2. \tag{6.13}$$

$\mathcal{L}_\Sigma$ is invariant under transformations like $\Sigma \to U\Sigma V^\dagger$, with $U, V \in SU(2)$ and $U \neq V$. We then have a $SU(2) \times SU(2)$ invariance, which is essentially the same as $O(4)$.

NB Since U and V are independent, each matrix has a set of generators which are independent of each other. As we saw in section 6.1, U and V have three generators each. $SU(2) \times SU(2)$ then has six; two independent sets of three generators each. This is expected since $SU(2) \times SU(2)$ is isomorphic to $O(4)$ which also has six generators [19].

6.3 Noether currents

Since our theory is symmetric under transformations like $\Sigma \to U\Sigma V^\dagger$, Noether's theorem predicts that there are associated conserved currents. Let $U, V \in SU(2)$ be now

[1] $\mathcal{L}$ is now, in fact, invariant under $U(2) \subset O(4)$.

infinitesimal transformations. We have:

$$\begin{aligned}\Sigma \to U\Sigma V^\dagger &\approx \left(\mathbb{I} - \frac{i}{2}\varepsilon_1 \vec{n}\cdot\vec{\tau}\right)\Sigma\left(\mathbb{I} + \frac{i}{2}\varepsilon_2\vec{m}\cdot\vec{\tau}\right)\\ &\approx \Sigma + \frac{i}{2}\varepsilon_2(\vec{m}\cdot\vec{\tau})\Sigma - \frac{i}{2}\varepsilon_1(\vec{n}\cdot\vec{\tau})\Sigma + \mathcal{O}(\varepsilon^2),\end{aligned} \tag{6.14}$$

which comparing to $\Sigma \to \Sigma + \delta\Sigma$ gives $\delta\Sigma = \frac{i}{2}\varepsilon_2(\vec{m}\cdot\vec{\tau})\Sigma - \frac{i}{2}\varepsilon_1(\vec{n}\cdot\vec{\tau})\Sigma$. We can explicitly see her that the matrices U and V have each an independent set of generators. It is possible from here to get the corresponding infinitesimal transformations for the σ and π^k-fields. For the former, we just need to multiply the last term of Eq. (6.14) by a factor $\frac{1}{\sqrt{2}}$ and take the trace. We have:

$$\frac{1}{\sqrt{2}}\mathrm{Tr}\Sigma + \frac{1}{2}\vec{m}\mathrm{Tr}[\frac{i}{\sqrt{2}}\vec{\tau}\,\Sigma] = \sigma + \frac{1}{2}\vec{m}\cdot\boldsymbol{\pi}. \tag{6.15}$$

For each π^j field, we multiply the last term of Eq. (6.14) by $\frac{i}{\sqrt{2}}\tau_j$ and make use of $\tau_j\tau_k = \delta_{jk} + i\varepsilon^{jkl}\tau^l$ and then take the trace. We find:

$$\begin{aligned}\frac{i}{\sqrt{2}}\mathrm{Tr}[\tau_j\Sigma] + \frac{i}{2}m_k\mathrm{Tr}\left[\frac{i}{\sqrt{2}}\tau_j\tau_k\,\Sigma\right] &= \pi^j - \frac{i}{2}m_k\mathrm{Tr}\left[\frac{i}{\sqrt{2}}[\delta_{jk} + i\varepsilon^{jkl}\tau^l]\,\Sigma\right]\\ &= \pi^k - \frac{1}{2}\vec{m}_j\frac{1}{\sqrt{2}}\mathrm{Tr}\Sigma - \frac{1}{2}m_k\varepsilon^{jkl}\mathrm{Tr}\left[\frac{i}{\sqrt{2}}\tau^l\Sigma\right]\\ &= \pi^j - \frac{1}{2}\vec{m}_j\sigma - \frac{1}{2}\varepsilon^{jkl}m_k\pi^l.\end{aligned} \tag{6.16}$$

Note that we have performed the derivation using $\vec{m}$ but the same applies for $\vec{n}$. We thus have contributions from both $\vec{m}$ and $\vec{n}$ to the infinitesimal transformations for the fields σ and π^k:

$$\delta\sigma_m = \frac{1}{2}\vec{m}\cdot\boldsymbol{\pi}, \tag{6.17}$$

$$\delta\sigma_n = -\frac{1}{2}\vec{n}\cdot\boldsymbol{\pi}, \tag{6.18}$$

$$\delta\boldsymbol{\pi}_m = -\frac{1}{2}\left[\vec{m}\sigma + \vec{m}\times\boldsymbol{\pi}\right], \tag{6.19}$$

$$\delta\boldsymbol{\pi}_n = \frac{1}{2}\left[\vec{n}\sigma + \vec{n}\times\boldsymbol{\pi}\right]. \tag{6.20}$$

We now have all we need in order to compute the Noether currents. Each infinitesimal transformation gives rise to a current. From the general expression, we have:

$$j^\mu_M = \frac{1}{2}(\vec{m}\cdot\vec{\pi})\partial^\mu\sigma - \frac{1}{2}\sigma(\partial^\mu\boldsymbol{\pi}\cdot\vec{m}) - \frac{1}{2}(\partial^\mu\boldsymbol{\pi})\cdot(\vec{m}\times\boldsymbol{\pi}), \tag{6.21}$$

$$j^\mu_N = -\frac{1}{2}(\vec{n}\cdot\vec{\pi})\partial^\mu\sigma + \frac{1}{2}\sigma(\partial^\mu\boldsymbol{\pi}\cdot\vec{n}) - \frac{1}{2}(\partial^\mu\boldsymbol{\pi})\cdot(\vec{n}\times\boldsymbol{\pi}). \tag{6.22}$$

Note that the vectors $\vec{m}$ and $\vec{n}$ are arbitrary. As we did in previous chapters, we can choose them to be unit vectors, and the currents will still be conserved. We will identify each

of them with axis vectors. We find a total of six currents by combining equations (6.21) and (6.22); one set of three conserved currents like

$$V_k^\mu = j_k^{M\mu} + j_k^{N\mu} = \varepsilon_{klm}\pi_l\partial^\mu\pi_m, \tag{6.23}$$

and another set currents like

$$A_k^\mu = j_k^{M\mu} - j_k^{N\mu} = \pi_k\partial^\mu\sigma - \sigma\partial^\mu\pi_k, \tag{6.24}$$

which give the total conserved charges

$$Q_k^A = \int_\Omega d^3x\ A_k^0 = \int_\Omega d^3x[\pi_k\partial^0\sigma - \sigma\partial^0\pi_k], \tag{6.25}$$

$$Q_k^V = \int_\Omega d^3x\ V_k^0 = \int_\Omega d^3x[\varepsilon_{klm}\pi_l\partial^0\pi_m]. \tag{6.26}$$

We have then checked that each of the six generators of $SU(2) \times SU(2)$ give rise to one Noether current. Another way of understanding the reason for which we have gotten six conserved charges is thinking in terms of $O(4)$-rotations. In 4 dimensions, we have 4 axes $\{\hat{w}, \hat{x}, \hat{y}, \hat{z}\}$, which implies that we have six different planes, $\{wx, wy, wz, xy, xz, yz\}$. We can perform rotations in each of these planes, and the system will remain invariant. In fact we can see that the currents Q_k^A generate rotations in the $\sigma\pi^k$-planes, while Q_k^V generate rotations in the $\pi^i\pi^j$-planes. We can also check that these transformations are independent of each other by checking whether they commute or not. Let us see this. We first explicitly compute the commutator of the integrands:

$$\begin{aligned}\left[\pi_k\partial^0\sigma - \sigma\partial^0\pi_k, \varepsilon_{abc}\pi_b\partial^0\pi_c\right] &= \partial^0\sigma\left[\pi_k, \varepsilon_{abc}\pi_b\partial^0\pi_c\right] - \sigma\left[\partial^0\pi_k, \varepsilon_{abc}\pi_b\partial^0\pi_c\right] \\ &= \Big[\partial^0\sigma\left(\varepsilon_{abc}\delta_{kb}\pi_c\right) + \sigma\left(\varepsilon_{abc}\delta_{kb}\partial^0\pi_c\right)\Big]\delta(x-x'),\end{aligned} \tag{6.27}$$

where we have used

$$[\pi_i(x), \partial^0\pi_j(x')] = \delta_{ij}\delta(x-x'). \tag{6.28}$$

The rest of the commutators vanish. Performing the integral, we have

$$\begin{aligned}[Q_k^A(x), Q_a^V(x')] &= \int_\Omega d^3x \int_\Omega d^3x'\ \left[\partial^0\sigma\pi_k + \sigma\partial^0\pi_k\right]\delta(x-x') \\ &= \int_\Omega d^3x\ \left[\partial^0\sigma\pi_k + \sigma\partial^0\pi_k\right] \\ &= \int_\Omega d^3x\ \partial^0[\sigma\pi_k] = 0.\end{aligned} \tag{6.29}$$

The last integral vanishes since the integrand is a surface term. We thus have checked that the charges commute. This agrees with the fact that in $O(2N)$-symmetric theories there is a maximum of N commuting charges [7, 31].

6.4 Spontaneous symmetry breaking

Consider now the potential for the σ-model, $V(\sigma, \boldsymbol{\pi})$:

$$V(\sigma, \boldsymbol{\pi}) = \frac{1}{2}m^2(\sigma^2 + \boldsymbol{\pi}^2) + \frac{\lambda}{4!}(\sigma^2 + \boldsymbol{\pi}^2)^2. \tag{6.30}$$

As usual, the potential is minimized for different values depending on the sign of m^2. The vacuum can be chosen in any direction, since all of them are physically equivalent. Without loss of generality, we make

$$\langle\sigma\rangle = v \qquad \langle\pi^k\rangle = 0. \tag{6.31}$$

For $m^2 > 0$, we find the minimum at $v = 0$, while for $m^2 < 0$, the minimum is at $v^2 = -\frac{6m^2}{\lambda}$. We now proceed to break the symmetry.

With this choice of vacuum, the $O(4) \cong SU(2) \times SU(2)$ symmetry is broken down to $SU(2)$. $SU(n)$ has $n^2 - 1$ generators [19], which implies that $SU(2)$ has three. Since $O(4)$ has six generators, we expect to have three broken and three unbroken ones. Perhaps a more intuitive way of understanding this symmetry breaking is to again think in terms of rotations. $O(4)$ generates 4-dimensional rotations. By choosing the vacuum in a specific direction, we are making 3-dimensional rotations around an axis. The $O(4)$ symmetry then breaks into $O(3)$, which also has three generators, as we saw in the last chapter. This is related to the homomorphism between $SU(2)$ and $SO(3)$. Following then Goldstone's theorem, we expect to have three massless modes. Let us see this. As usual, we expand the fields around their minimum:

$$\sigma \to \langle\sigma\rangle + \alpha \qquad \pi^k \to \beta^k, \tag{6.32}$$

where α and β^k are the respective quantum fluctuating fields. Plugging these expansions in the Lagrangian in Eq. (6.7) results in

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\alpha)^2 + \frac{1}{2}(\partial_\mu\boldsymbol{\beta})^2 + m^2\alpha^2 - \frac{\lambda}{6}v\alpha(\alpha^2 + \boldsymbol{\beta}^2) - \frac{\lambda}{4!}(\alpha^2 + \boldsymbol{\beta}^2)^2. \tag{6.33}$$

We see that the mass term related to the $\boldsymbol{\pi}$-fields has disappeared. Taking only the terms quadratic in the fields, we can write down the action and compute the dispersion relations for our modes α and $\boldsymbol{\beta}$. Defining $\boldsymbol{\chi} = (\alpha, \boldsymbol{\beta})$:

$$S = \int d^4x\, \boldsymbol{\chi}^\dagger \Delta_{\alpha\beta} \boldsymbol{\chi}, \tag{6.34}$$

with

$$\Delta_{\alpha\beta} = \begin{pmatrix} -\Box + 2m^2 & 0 & 0 & 0 \\ 0 & -\Box & 0 & 0 \\ 0 & 0 & -\Box & 0 \\ 0 & 0 & 0 & -\Box \end{pmatrix}. \tag{6.35}$$

Fourier transforming to momentum space and demanding $\det \Delta_{\alpha\beta} = 0$ gives the dispersion relations:

$$E_\alpha^2 = \mathbf{k}^2 + 2m^2, \tag{6.36}$$

$$E_{\beta_k}^2 = \mathbf{k}^2. \tag{6.37}$$

These dispersion relations show that the three π^k-fields have become massless after SSB. We thus have, as Goldstone's theorem predicted, three Goldstone bosons.

6.5 Adding a chemical potential

In this section we will couple the conserved charges that we derived in section 6.3 to our theory through a chemical potential. As we mentioned before, for $O(4)$-symmetric theories, the maximum number of commuting charges is two. For each conserved charge, we can introduce a nonzero chemical potential. We will couple the Lagrangian to, in particular, Q_3^A and Q_3^V through two chemical potentials μ_A and μ_V and study the cases $\mu_A = \mu_V$ and $\mu_A \neq \mu_V$. This corresponds to the third component of the isospin chemical potential.

We know from previous chapters that coupling the charge density to the Hamiltonian by shifting it like $\mathcal{H} \to \mathcal{H} - \mu j^0$, has the same result as "gauging" the kinetic term in the Lagrangian, $\partial_0 \Phi \to (\partial_0 + i\mu)\Phi$. When coupling the currents Q_3^A and Q_3^V, the gauging in the Lagrangian is made as $\partial_0 \Phi \to (\partial_0 + i\mu\delta_{03})\Phi$ for Q_3^A and $\partial_0 \Phi \to (\partial_0 + i\mu\delta_{12})\Phi$ for Q_3^V, where $\Phi = (\sigma, \boldsymbol{\pi})$ is implied. Let us then skip the Hamiltonian part of the derivation and directly consider the following Lagrangian.

$$\begin{aligned}\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \boldsymbol{\pi})^2 - \frac{1}{2}m^2(\sigma^2 + \boldsymbol{\pi}^2) - \frac{\lambda}{4!}(\sigma^2 + \boldsymbol{\pi}^2)^2 \\ -i\mu_A[\pi_3 \partial^0 \sigma - \sigma \partial^0 \pi_3] - \mu_V[\pi_1 \partial^0 \pi_2 - \pi_2 \partial^0 \pi_1].\end{aligned} \tag{6.38}$$

We will get a different number of massless modes depending on whether $\mu_A = \mu_V$ or not. Let us assume first that both chemical potentials are equal, $\mu_A = \mu_V = \mu$. We can write the Lagrangian of Eq. (6.38) in terms of the complex doublets Φ defined in Eq. (6.9) as

$$\mathcal{L} = (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) - i\mu[(\partial_0 \Phi)^\dagger \Phi - \Phi^\dagger (\partial_0 \Phi)] - (m^2 - \mu^2)\Phi^\dagger \Phi - \lambda(\Phi^\dagger \Phi)^2. \tag{6.39}$$

As in the $SO(2)$ and $SO(3)$ cases, the new kinetic term breaks the global symmetry of the system. The breaking of $O(4)$ is clearer if we express the Lagrangian in terms of the Σ matrices:

$$\begin{aligned}\mathcal{L}_\Sigma = \frac{1}{2}\mathrm{Tr}[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma] - \frac{1}{2}(m^2 - \mu^2)\mathrm{Tr}[\Sigma^\dagger \Sigma] - \frac{\lambda}{4}(\mathrm{Tr}[\Sigma^\dagger \Sigma])^2 \\ + \frac{i}{2}\mu\ \mathrm{Tr}\left[\Sigma^\dagger \partial_0 \Sigma \frac{\mathbb{I} + \tau_3}{2} - \Sigma \frac{\mathbb{I} + \tau_3}{2}(\partial_0 \Sigma)^\dagger\right].\end{aligned} \tag{6.40}$$

From here we can see that the first three terms still are $SU(2) \times SU(2)$ invariant, while the last term is not. The symmetry is broken by the last term to $SU(2) \times U(1)$.

Note that once again, we deal with a symmetric normal phase for $m^2 > \mu^2$ and a broken phase for $m^2 < \mu^2$. The global symmetry of the Lagrangian now is $SU(2) \times U(1)$. After SSB, the symmetry will break down to $U(1)$. $SU(2) \times U(1)$ is isomorphic to $U(2)$. $U(n)$ has n^2 generators [19]. This implies that $U(2)$ has four and $U(1)$ has one. We thus have three broken generators. Note that our theory is not Lorentz invariant anymore. This implies that we cannot ensure that the number of broken generators will be equal to the number of massless modes [42].

The potential in Eq. (6.38) is minimized for $v^2 = \frac{6}{\lambda}(\mu^2 - m^2)$. As usual, all choices for the direction of the vacuum are physically equivalent. So for convenience we take it in the σ-direction:

$$\begin{pmatrix} \sigma \\ \boldsymbol{\pi} \end{pmatrix} \to \begin{pmatrix} \langle\sigma\rangle + h \\ \langle\boldsymbol{\pi}\rangle + \boldsymbol{\pi} \end{pmatrix} = \begin{pmatrix} v + h \\ \boldsymbol{\pi} \end{pmatrix}. \tag{6.41}$$

Substituting these expansions in the Lagrangian of Eq. (6.38) gives

$$\begin{aligned} \mathcal{L} = &\frac{1}{2}(\partial_\mu h)^2 + \frac{1}{2}(\partial_\mu \boldsymbol{\pi})^2 - i\mu[\pi_3\partial^0 h - h\partial^0\pi_3 - \pi_1\partial^0\pi_2 + \pi_2\partial^0\pi_1] \\ &+ \frac{3}{2\lambda}(m^4 - \mu^4) + (m^2 - \mu^2)h^2 - \frac{\lambda}{6}vh(h^2 + \boldsymbol{\pi}^2) - \frac{\lambda}{4!}(h^2 + \boldsymbol{\pi}^2)^2. \end{aligned} \tag{6.42}$$

We see that there is no mass term for the $\boldsymbol{\pi}$-fields. The action for this Lagrangian is

$$S = \int_\Omega d^4x \ \begin{pmatrix} h & \boldsymbol{\pi} \end{pmatrix} \Delta_h \begin{pmatrix} h \\ \boldsymbol{\pi} \end{pmatrix}, \tag{6.43}$$

where Δ_h is in momentum space given by

$$\Delta_h = \begin{pmatrix} E^2 - \mathbf{k}^2 + 2m^2 - 2\mu^2 & 0 & 0 & 2iE\mu \\ 0 & E^2 - \mathbf{k}^2 & 2iE\mu & 0 \\ 0 & -2iE\mu & E^2 - \mathbf{k}^2 & 0 \\ -2iE\mu & 0 & 0 & E^2 - \mathbf{k}^2 \end{pmatrix}. \tag{6.44}$$

Note that we may now find two different conditions for the spectrum of the sectors (σ, π_3) and (π_1, π_2). For the former, we have:

$$\Delta_{\sigma\pi_3} = \begin{pmatrix} E^2 - \mathbf{k}^2 + 2m^2 - 2\mu^2 & 2iE\mu \\ -2iE\mu & E^2 - \mathbf{k}^2 \end{pmatrix}. \tag{6.45}$$

Demanding its determinant to vanish and solving for E gives the following dispersion relations:

$$E^2_{\sigma\pi_3} = \mathbf{k}^2 - m^2 + 3\mu^2 \pm \sqrt{m^4 + 4\mathbf{k}^2\mu^2 - 6m^2\mu^2 + 9\mu^4}. \tag{6.46}$$

We can now proceed to expand these dispersion relations for small momenta. Since the relation (6.46) is already known from previous chapters, let us use those results. We have

$$E^2_{\sigma\pi_{3+}} = \frac{\mu^2 - m^2}{3\mu^2 - m^2}\mathbf{k}^2 + \mathcal{O}(\mathbf{k}^4), \tag{6.47}$$

$$E^2_{\sigma\pi_{3-}} = 6\mu^2 - 2m^2 + \frac{5\mu^2 - m^2}{3\mu^2 - m^2}\mathbf{k}^2 + \mathcal{O}(\mathbf{k}^4). \tag{6.48}$$

And for the (π_1, π_2) sector, we have

$$\Delta_{\pi_1\pi_2} = \begin{pmatrix} E^2 - \mathbf{k}^2 & 2iE\mu \\ -2iE\mu & E^2 - \mathbf{k}^2 \end{pmatrix}, \tag{6.49}$$

which, again demanding $\det \Delta_{\pi_1\pi_2} = 0$ and solving for E^2, gives the four following dispersion relations:

$$E^2_{\pi_1\pi_2} = \left(\mu \pm \sqrt{\mathbf{k}^2 + \mu^2}\right)^2. \tag{6.50}$$

Expanding for small momenta to the first not-vanishing order gives:

$$E^2_{\pi_1\pi_{2+}} = 4\mu^2 + 2\mathbf{k}^2 + \mathcal{O}(\mathbf{k}^4), \tag{6.51}$$

$$E^2_{\pi_1\pi_{2-}} = \frac{\mathbf{k}^4}{4\mu^2} + \mathcal{O}(\mathbf{k}^6). \tag{6.52}$$

The first dispersion relations are the same as for the $SO(2)$ cases. We see that we have one massive mode and a massless one which is linear in momentum. In the second case, we again have one massless mode and one massive. In this case though, the massless mode is quadratic in momentum, which implies that it is a type-II Goldstone boson.[2] This agrees with having three broken generators.

We also want to point out that even though the fields π_1 and π_2 do not have mass terms in the Lagrangian (6.42), i.e. they correspond to two flat directions of the potential, they only describe one massless mode. We may explain this by taking the limit of very small frequencies. In this limit, the terms $(\partial_0\pi_k)^2$ are much smaller than $\pi_1\partial^0\pi_2 - \pi_2\partial^0\pi_1$ and thus we can neglect their contribution. We then notice that π_1 and π_2 are, in fact, two canonically conjugate variables, which describe one massless field and its time-derivative [28].

We will now check the case $\mu_A \neq \mu_V$. In general, in theories $O(2N)$-symmetric, introducing k chemical potentials breaks the global symmetry group to $[O(2)]^k \times O(2N-2k)$ [7]. We have an $O(4)$-symmetric theory. By coupling it to Q_3^A and Q_3^V, the global symmetry will then be broken down to $O(2) \times O(2)$, which has two generators, one for each $O(2)$. The Lagrangian in Eq. (6.38) can now be written in three parts, $\mathcal{L} = \mathcal{L}_{\pi_1\pi_2} + \mathcal{L}_{\sigma\pi_3} + \mathcal{L}_{\text{int}}$, with:

$$\begin{aligned} \mathcal{L}_{\pi_1\pi_2} = &\frac{1}{2}[(\partial_\mu\pi_1)^2 + (\partial_\mu\pi_2)^2] + \mu_V[\pi_2(\partial_0\pi_1) - \pi_1(\partial_0\pi_2)] \\ &- \frac{1}{2}(m^2 - \mu_V^2)(\pi_1^2 + \pi_2^2) - \frac{\lambda}{4!}(\pi_1^2 + \pi_2^2)^2, \end{aligned} \tag{6.53}$$

$$\begin{aligned} \mathcal{L}_{\sigma\pi_3} = &\frac{1}{2}[(\partial_\mu\sigma)^2 + (\partial_\mu\pi_3)^2] + \mu_A[\pi_3(\partial_0\sigma) - \sigma(\partial_0\pi_3)] \\ &- \frac{1}{2}(m^2 - \mu_A^2)(\sigma^2 + \pi_3^2) - \frac{\lambda}{4!}(\sigma^2 + \pi_3^2)^2, \end{aligned} \tag{6.54}$$

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{12}\left[\sigma^2\pi_1^2 + \sigma^2\pi_2^2 + \pi_1^2\pi_3^2 + \pi_2^2\pi_3^2\right]. \tag{6.55}$$

[2] The energy as $E \sim k^{2n}$.

$\mathcal{L}_{\pi_1\pi_2}$ and $\mathcal{L}_{\sigma\pi_3}$ describe the $\pi_1\pi_2$-sector and $\sigma\pi_3$-sector, respectively. $\mathcal{L}_{\text{int}}$ describes the interaction between the two sectors. The $O(2) \times O(2)$ symmetry is now clear. The Lagrangian in Eq. (6.53) is invariant under rotations in the $\pi_1\pi_2$-plane, whereas $\mathcal{L}_{\sigma\pi_3}$ remains invariant under rotations in the $\sigma\pi_3$-plane. As we mentioned before, each $O(2)$ has only one generator. Thus when breaking the symmetry, we can only expect one massless mode from each $O(2)$. We also expect that the dispersion relations for the modes should be the same ones as in equations (6.47) and (6.48) because of the local isomorphism between $SO(2)$ and $U(1)$.

Now, not only the σ-field, but also either the π_1 or π_2-field acquires a non-zero vacuum expectation value. The potentials for the Lagrangians in equations (6.53) and (6.54) have their minimum at $\rho_V^2 = \frac{6}{\lambda}(\mu_V^2 - m^2)$ and $\rho_A^2 = \frac{6}{\lambda}(\mu_A^2 - m^2)$, respectively. As usual, we will expand the fields around their vev, and all choices for the direction of the vacuum are physically equivalent. Our choices will be in the π_1 and σ-directions, respectively:

$$\begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix} \to \begin{pmatrix}\rho_V + h_V\\ \pi_2\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix}\sigma\\ \pi_3\end{pmatrix} \to \begin{pmatrix}\rho_A + h_A\\ \pi_3\end{pmatrix} \tag{6.56}$$

Substituting these expansions in equations (6.47) and (6.48) gives

$$\begin{aligned}\mathcal{L}_{\pi_1\pi_2} &= \frac{3}{2\lambda}(m^4 - \mu_V^4) + \frac{1}{2}(\partial_\mu h_V)^2 + \frac{1}{2}(\partial_\mu \pi_2)^2 - i\mu_V[\pi_2\partial^0 h_V - h_V\partial^0\pi_2] \\ &\quad + (m^2 - \mu_V^2)h_V^2 - \frac{\lambda}{6}\rho_V h_V(h_V^2 + \pi_2^2) - \frac{\lambda}{4!}(h_V^2 + \pi_2^2)^2, \qquad (6.57)\\ \mathcal{L}_{\sigma\pi_3} &= \frac{3}{2\lambda}(m^4 - \mu_A^4) + \frac{1}{2}(\partial_\mu h_A)^2 + \frac{1}{2}(\partial_\mu \pi_3)^2 - i\mu_A[\pi_3\partial^0 h_A - h_A\partial^0\pi_3] \\ &\quad + (m^2 - \mu_A^2)h_A^2 - \frac{\lambda}{6}\rho_A h_A(h_A^2 + \pi_3^2) - \frac{\lambda}{4!}(h_A^2 + \pi_3^2)^2. \qquad (6.58)\end{aligned}$$

We see that there is no mass term for the modes π_2 and π_3. As usual, the propagator for each Lagrangian can be obtained from the action. We skip this step and directly write the two propagators in momentum space:

$$\Delta_{\pi_1\pi_2} = \begin{pmatrix}E^2 - \mathbf{k}^2 + 2m^2 - 2\mu_V^2 & 2iE\mu_V\\ -2iE\mu_V & E^2 - \mathbf{k}^2\end{pmatrix}, \tag{6.59}$$

$$\Delta_{\sigma\pi_3} = \begin{pmatrix}E^2 - \mathbf{k}^2 + 2m^2 - 2\mu_A^2 & 2iE\mu_A\\ -2iE\mu_A & E^2 - \mathbf{k}^2\end{pmatrix}. \tag{6.60}$$

We see that both propagators have the same form as the one in Eq. (6.45). The dispersion relations for the modes will thus be the same as in Eq. (6.46). Let us, for completeness,

write them , already expanded for small momenta:

$$E^2_{\sigma\pi_{3+}} = \frac{\mu_A^2 - m^2}{3\mu_A^2 - m^2}\mathbf{k}^2 + \mathcal{O}(\mathbf{k}^4), \tag{6.61}$$

$$E^2_{\pi_1\pi_{2+}} = \frac{\mu_V^2 - m^2}{3\mu_V^2 - m^2}\mathbf{k}^2 + \mathcal{O}(\mathbf{k}^4), \tag{6.62}$$

$$E^2_{\sigma\pi_{3-}} = 6\mu_A^2 - 2m^2 + \frac{5\mu_A^2 - m^2}{3\mu_A^2 - m^2}\mathbf{k}^2 + \mathcal{O}(\mathbf{k}^4), \tag{6.63}$$

$$E^2_{\pi_1\pi_{2-}} = 6\mu_V^2 - 2m^2 + \frac{5\mu_V^2 - m^2}{3\mu_V^2 - m^2}\mathbf{k}^2 + \mathcal{O}(\mathbf{k}^4). \tag{6.64}$$

As predicted, we have gotten one massless mode for each broken $O(2)$-symmetry. As we saw in previous chapters, the modes described in equations (6.61) and (6.62) are linear in momentum, which implies a type-I Goldstone boson. Note that we have broken the same symmetry as in chapters 4 and 5 and thus, the same discussion applies here. We see that for both equations (6.63)and (6.64), we have a massless gap in the limit $\mathbf{k}^2 \to 0$.

Chapter 7

Conclusions and outlook

7.1 Conclusions

In this thesis, we have studied in detail the concept of spontaneous symmetry breaking in several $O(N)$-symmetric theories. We focused on those theories that are not Lorentz invariant since in such theories we cannot ensure that the number the massless modes, which are predicted to appear after SSB, coincide with the number of broken generators. Our goal has been to find and describe these massless modes.

We started by considering the simplest case where one can find SSB. This was a non-relativistic $E(2)$-symmetric theory. We found that the ground state of the Hamiltonian was not an eigenstate of the symmetry generator and thus broke the symmetry. The group $E(2)$ has three generators, two of which were spontaneously broken, yet we only found one massless mode which was quadratic in momentum.

Once we had understood the concept of SSB, we started applying it to relativistic $O(N)$-symmetric theories. We started, in chapter two, with an $SO(2)$-symmetric Lagrangian. Because of its Lorentz invariance, we expected that the number of massless modes would coincide with the number of broken generators. We found one massless mode which agreed with the number of broken symmetries. In this chapter we focused on checking that the mode remained massless after loop corrections. We then, in chapter four, broke the Lorentz invariance by coupling the conserved charge to the Lagrangian through a chemical potential. Even though the theory is not Lorentz invariant, there was only one possible symmetry to be spontaneously broken. We thus found one massless mode which was linear in momentum. We then derived some thermodynamic properties using the formalism of thermal field theory and renormalized afterwards the thermodynamic potential at one loop.

In chapter five, we applied the techniques we introduced in the previous chapter to an $SO(3)$ non-Abelian symmetric theory. We saw that $SO(3)$ spontaneously breaks to $SO(2)$. The former has three generators while the latter has only one. We thus expected, and

then confirmed, two massless modes which were linear in momentum. We then explicitly broke $SO(3)$ down to $SO(2)$ by coupling one of the three conserved charges to the theory through a chemical potential. After SSB, we again got one massless mode. This agreed with our result in chapter four, since the Lagrangian is invariant under the same symmetry group. We got the same result as in the $SO(2)$ case with an extra massive decoupled mode, which agreed with having an extra degree of freedom.

Finally, in chapter six, we studied the LSM, which is described by an $O(4)$-symmetric Lagrangian. We made use of the local isomorphism between $O(4)$ and $SU(2) \otimes SU(2)$, in order to find the conserved charges of the theory and in order to predict the remaining symmetry after SSB. We saw that the $O(4)$ symmetry spontaneously broke down to $O(3)$. Since the former has six generators and the latter three, we checked that after SSB, we got three massless modes. We also saw that $O(4)$ has six conserved charges. Since only two of them commute, we could only couple a maximum of two currents through chemical potentials. We then studied two cases: i) coupling the two currents through the same chemical potential and ii) through different chemical potentials. In the former case, the $O(4)$ symmetry was broken down to $SU(2) \otimes U(1)$, which spontaneously broke down to $U(1)$. Since $SU(2) \otimes U(1)$ has four generators and $U(1)$ one, we would expect three massless modes. But, again, because of a lack of Lorentz invariance, we could not ensure this. We got in fact two massless modes, one quadratic and one linear in momentum. We then study the case when we couple the Lagrangian to two currents through different chemical potentials. The $O(4)$ symmetry was then broken down to $O(2) \otimes O(2)$. We could thus expect only one massless mode from each $O(2)$. In fact, this was the same case as in the previous chapters, so we could expect two linear modes.

7.2 Outlook

In this thesis we have discussed the process of SSB and Goldstone's theorem in systems at finite density, but we have not gone into further detail in its applications in condensed matter Physics. We have given the basis for the interested reader to further investigate. In chapter four for instance, we briefly discussed thermal field theory, but one could go deeper. As we can see in appendix F, more thermodynamic properties, as entropy or pressure, can be derived from the thermodynamic potential. We have also restricted ourselves to one-loop corrections. Higher order corrections and more detailed discussions can be found, for example, in Ref. 43.

It is also interesting to further study the characterization and number of massless modes which appear in non-Lorentz invariant systems after spontaneous symmetry breaking. We have seen in this thesis that while the number of broken generators coincide with the number of massless modes in Lorentz invariant systems, as Goldstone's theorem states, this does not necessarily happen otherwise. Nielsen and Chadha found an inequality between the number of massless bosons and the number of broken symmetry generators [42]. They distinguish two types of Goldstone bosons: type-I with their energy going as $E \sim k^{2n+1}$ and type-II, where the energy goes as $E \sim k^{2n}$. They formulated a theorem, which states that one has to count each type-II Goldstone mode type twice. This theorem also explains

our final result in chapter 2. There are several examples where this inequality can be more precise, and a improved relation has been proposed, leading to a different classification of Goldstone bosons [20].

In chapter 6 we provided an introduction for the reader to further investigate when studying the LSM. One of the first things one can do in order to continue the study of the LSM is to *explicitly break* the $O(4)$ symmetry by introducing a term $\mathcal{L} = -h\sigma$. The minimum of the potential is then shifted, which will make the pions acquire a mass [41].

In chapter 6, we considered the bosonic part of the model and ignored the quark contribution. When the fermion terms are also considered, the LSM is called LSMq. The Lagrangian would become:

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + g\bar{\psi}[\sigma + i\gamma_5(\vec{\tau}\cdot\boldsymbol{\pi})]\psi + \frac{1}{2}(\partial_{\mu}\sigma)^2 + \frac{1}{2}(\partial_{\mu}\boldsymbol{\pi})^2 \\ -\frac{1}{2}m^2(\sigma^2 + \boldsymbol{\pi}^2) - \frac{\lambda}{4!}(\sigma^2 + \boldsymbol{\pi}^2)^2, \tag{7.1}$$

where $\bar{\psi} = (\bar{u}\ \bar{d})$, being u and d the *up* and *down* quarks, respectively. In this case, in order to take into account the left and right-handed fermions (chirality), the symmetry group would be $O(4) \simeq SU_L(2) \times SU_R(2)$. The term $g\bar{\psi}[\sigma + i\gamma_5(\vec{\tau}\cdot\boldsymbol{\pi})]\psi$ in the Lagrangian affects the conserved currents that we computed in chapter 6. In particular, the transformations $\delta\sigma$ and $\delta\pi^k$ include change of parity. This results in vector current becoming a *pseudovector* (or *axial vector*) and, its corresponding conserved charge becoming a *pseudoscalar* [41].

LSMq has several applications in modern and condensed matter Physics. Because of the local isomorphism between $O(4)$ and $SU_L(2) \times SU_R(2)$, the LSMq is of particular interest in high-energy quantum field theory, since it is used as low-energy effective field theory of quantum chromodynamics [7]. LSMq can also be used as a toy model for the description of *kaon condensates* [7, 28, 44]

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Appendices

Appendix A

Group theory

In this appendix, we review some introductory concepts on group theory in order. Discussions about these topics can be found in several algebra books, e.g. Refs. 19, 45 or 46.

A.1 Group axioms

Let G be a nonempty set of elements and $\cdot \ : \ G \times G \ \rightarrow \ G$ a binary operation. The combination $(G, \cdot)$ is called a **group** if for all element in G the following properties are satisfied, under the action of $\cdot$:

$$\textbf{Closure}: \quad \forall\, x, y \in G \ \rightarrow \ x \cdot y \in G. \tag{A.1}$$

$$\textbf{Associativity}: \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z \ \ \forall x, y, z \in G. \tag{A.2}$$

$$\textbf{Existence of identity}: \quad \exists\, e \in G \mid \forall\, x \in G : x \cdot e = x. \tag{A.3}$$

$$\textbf{Existence of inverse}: \quad \forall\, x \in G\, \exists\, x^{-1} \mid x \cdot x^{-1} = e \tag{A.4}$$

One example of a group is the set of real numbers excluding , under multiplication, $(\mathbb{R} - \{0\}, \cdot)$. Let us check the four properties written above using this example. Obviously, the multiplication of two real numbers gives another real number, and this multiplication law is associative. The identity element for the multiplication of real numbers is the number 1. Finally every non-zero real number has an inverse defined as $g^{-1} = 1/g, \ g \in \mathbb{R} - \{0\}$.

A.2 Maps and homomorphisms

Let A and B be (in general) two different sets. A map between A and B implies that every element $a \in A$ is related to, at least, one element $b \in B$:

$$\begin{aligned} f: \quad A \ &\rightarrow \ B \\ a \ &\rightarrow \ f(a). \end{aligned} \tag{A.5}$$

An element $b \in B \mid b = f(a)$ is called an *image* of a under f. We can define different

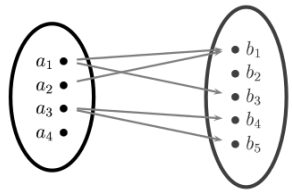
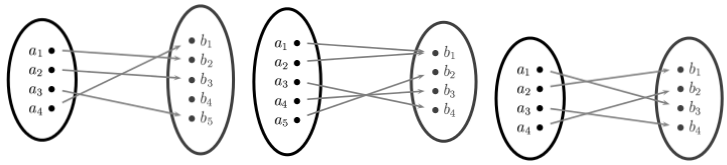


Figure A.1: Graphical representation of a map

types of maps depending on how we associate the elements of two sets. Let $f : A \to B$ be a map between the sets A and B. We will say that f is,

- **Injective**: if different elements in A have different images in B. This is equivalent to say that if $f(a) = f(b)$ then $a = b$.
- **Surjective**: if every element $b \in B$ is the image of at least one element in A. This is equivalent to $f(A) = B$.
- **Bijective**: if the map is injective and surjective. That is, every element in B is the image of one and only one element in A, $\forall b \in B \; \exists! a \in A \mid f(a) = b$.



(a) Injective map **(b)** Surjective map **(c)** Bijective map

Figure A.2: Graphical representation of maps.

A map between two group $(G, \cdot)$ and $(H, *)$, $f : G \to H$, is called a **homomorphism** between G and H if the following is satisfied:

$$f(g \cdot h) = f(g) * f(h) \;\; \forall g, h \in G. \tag{A.6}$$

An *injective* homomorphism is called a **Monomorphism**, a *surjective* homomorphism is called an **Epimorphism**, and a *bijective* homomorphism is called a **Isomorphism**.

A.3 Lie groups and Lie algebras

Formally, a **Lie group** G of dimension n is characterized by the property that it has a group structure and a differentiable manifold [19]. Going on in this direction would require to introduce and define terms such as *Topological space* or *smoothness*. Since that is not our purpose here, let us give a user-friendly definition for Lie group. A Lie group is a group

whose group elements depend on continuous parameters. In this way, the group multiplication and the inversion operation become continuous and differentiable. An example of a Lie group is the group of proper rotations, $SO(3)$. This is formed by all 3×3 matrices, with determinant equal to 1, which fulfill that $A^T A = \mathbb{I}$.

Let us now define **Lie algebra**. In order to do this, let us consider a matrix group called a *one-parameter subgroup*. A matrix $A(t)$ belonging to this group is defined as

$$A(t) = \exp[tL], \tag{A.7}$$

where the matrix L is interpreted here as a tangent vector at the identity. We say that L belongs to the *tangent spaces* of $A(t)$, $L \in T(\mathbb{I})$.

Consider the parameter t to be infinitesimal, $t = \epsilon$, then we can Taylor expand $A(\epsilon)$ as

$$A(\epsilon) \approx \mathbb{I} + \epsilon L, \tag{A.8}$$

where we have used $\frac{d}{dt}A(t) = LA(t)$. If we take another matrix $B(t) = \exp[\epsilon M]$, the product can be expanded by applying Leibniz' rule:

$$C(t) = A(t)B(t) \Rightarrow \dot{C} = \dot{A}B + B\dot{B}, \tag{A.9}$$

where we have used the notation $\dot{A} = \frac{d}{dt}A(t)$. We then have:

$$A(\epsilon)B(\epsilon) \approx \mathbb{I} + \epsilon(L + M) + \mathcal{O}(\epsilon^2). \tag{A.10}$$

Recall that we are dealing with a matrix group. The non-commutative part of the group product arises when we expand to second order in ϵ. It is described by a commutator product between the generators. This turns the n-dimensional tangent space $T(\mathbb{I})$ into the **Lie algebra** of the group.

The linear action of a Lie group on its tangent space is called *adjoint action* and it is defined as [19]

$$Ad(A)L = ALA^{-1}. \tag{A.11}$$

Let us see what happens when an element L of the tangent space $T(\mathbb{I})$ acts onto another element $M \in T(\mathbb{I})$. Using that $L = \dot{A}(t = 0)$, we have

$$\begin{aligned} Ad(L)M &= LML^{-1} \\ &= \frac{d}{dt}\exp[tL]\bigg|_{t=0} M \, \frac{d}{dt}\exp[-tL]\bigg|_{t=0} \\ &= \frac{d}{dt}\Big[\exp[tL]M\exp[-tL]\Big]_{t=0}. \end{aligned} \tag{A.12}$$

Applying again Leibniz' rule, $D = ABC \rightarrow \dot{D} = \dot{A}BC + A\dot{B}C + AB\dot{C}$, we find:

$$\frac{d}{dt}\Big[\exp[tL]M\exp[-tL]\Big]_{t=0} = LM - ML. \tag{A.13}$$

We conclude from here that L acts on M by commutation. Moreover, if L and M belong to the Lie algebra, so does $[L, M]$. This commutation product defines the Lie algebra of the group. Every commutator satisfies the following three properties:

$$[L, M] = -[M, L], \tag{A.14}$$

$$[L, aM + bN] = a[L, M] + b[L, N], \tag{A.15}$$

$$[L, [M, N]] + [M, [N, L]] + [N, [L, M]] = 0. \tag{A.16}$$

The relation in Eq. (A.16) is called *Jacobi's Identity*.

Appendix B

The Euclidean group $E(n)$

In this appendix, we will define the **Euclidean group**. Aiming for this, we first introduce the concept of isometry. After that, we will give a proper definition and work with some of its properties.

Our discussions in this appendix follow from Refs. 47, 48, 49 and 50.

B.1 Isometries

A **metric space** is a set M together with a distance function d:

$$d : M \times M \to \mathbb{R}, \tag{B.1}$$

such that d satisfies the following properties:

- $d(p, q) \geq 0,\ \forall p, q$ and only $d(p, p) = 0$ for $p = q$.
- It is symmetric, that is $d(p, q) = d(q, p)$.
- The triangle inequality, $d(p, r) \leq d(p, q) + d(q, r)$, is satisfied $\forall q, p, r \in M$.

Between two metric spaces M and N, we define **isometry** as a map $f : M \to N$ which preserves the distance between two points, that is:

$$d_N(f(p), f(q)) = d_M(p, q) \qquad \forall p, q \in M. \tag{B.2}$$

If we can find a bijective isometry between two metric spaces, we will call them **isometric**; and the set of bijective isometries from a metric space to itself has group structure. This is called the **isometry group**. Two isometry groups have particular relevance in the field of Physics; these are the *Poincaré group* (isometry group of *Minkowski space*) and the *Euclidean group*.

B.2 Definition of the Euclidean Group

We introduce the **Euclidean space** E^n, as the metric space with distance function defined as:

$$\forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \qquad d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||, \tag{B.3}$$

where we introduced the notation $||\mathbf{x}|| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ as the norm of the vector $\mathbf{x}$.[1] The **Euclidean group** $E(n)$ is just the isometry group of an n-dimensional Euclidean space. We define the elements of $E(n)$ as a pair (R, a) with $R \in O(n)$ and $u \in \mathbb{R}^n$. As we said before, the Euclidean group is an example of an isometry group. That means that any element on $E(n)$ can be seen as a map $f_{(R,a)} : \mathbb{R}^n \to \mathbb{R}^n$ on a Euclidean space built from an orthogonal transformation and a translation:

$$f_{(R,a)}(x) = Rx + a. \tag{B.4}$$

Using the fact that $E(n)$ is a set of maps, we can easily define the group multiplication and inversion operation. Let us look first at the group multiplication. Let $f_{(R,a)}$ and $f_{(R',a')}$ be two elements of $E(n)$. Based on the group axioms, the group multiplication $\diamond$ of two elements of a group must give another element of this group (so that we want to ensure closure):

$$\begin{aligned} f_{(R,a)} \diamond f_{(R',a')} &= f(R,a)(R'x + a') \\ &= R(R'x + a') + a \\ &= RR'x + Ra' + a. \end{aligned} \tag{B.5}$$

Note that the last line of eq. (B.5) has the same form as eq. (B.4) by rewriting $RR' = R'' \in O(n)$ and $Ra' + a = a'' \in \mathbb{R}^n$, and thus the closure property is satisfied.

It comes straightforward from here to define the identity element in $E(n)$; that is, the identity $\mathbb{I}_{n\times n}$ in $O(n)$ and no translations. In the *pair notation*, introduced before, it would be written like $(\mathbb{I}, 0)$.
Having the identity in $E(n)$, we can easily find the inverse element $f_{(R',a')}$, defined as:

$$f_{(R',a')} \mid f_{(R,a)} \diamond f_{(R',a')} = f_{(\mathbb{I},0)}. \tag{B.6}$$

Doing a simple calculation, analogous to the one we did for the multiplication law, we get the inverse element of an arbitrary (R, a): $(R^T, -R^T a)$.

Note that by having done these simple derivations, we have almost shown that the Euclidean group is, in deed, a group. Only the associativity of $\diamond$ is left to prove.

As an example of this we look to the Euclidean group of $\mathbb{R}^3$, $E(3)$. We rewrite as $f_{(R,a)}(x) = Rx + a = x'$. In a 3-dimensional space, the map $f_{(R,a)}$ has a 4×4 matrix

[1] Here, we used the standard inner product of $\mathbb{R}^n$.

form. Eq. (B.4) is then rewriten for $\mathbb{R}^3$ as:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & a_1 \\ R_{21} & R_{22} & R_{23} & a_2 \\ R_{31} & R_{32} & R_{33} & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}. \tag{B.7}$$

B.3 Rigid-body motions

In this section, we will focus on two of the most important Lie groups in classical mechanics. Let us introduce a subgroup of $E(n)$, the Special Euclidean group, $SE(N)$, consisting of all rigid body transformations, in n-dimensional Euclidean space, which preserve the distance between points. A matrix which belongs to $SE(N)$ has the following form:

$$A = \begin{pmatrix} R & z \\ \vec{0}^T & 1 \end{pmatrix}, \tag{B.8}$$

where $R \in SO(N)$, $z \in \mathbb{R}^n$ and $\vec{0}$ is an n-dimensional column vector whose elements are zero. Note that eq. (B.8) has the same form as eq. (B.7).
It is of particular interest to look closer at the cases $n = 2$ and $n = 3$. Those are the Euclidean motions in 2 (plane) and 3 (space) dimensions. In the case of, for example, $SE(2)$, $R \in SO(2)$ is a well-known (and used) common rotation matrix:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{B.9}$$

Obviously, $a \in \mathbb{R}^2$ and $\vec{0}^T$ is 2-dimensional row vector. Thus, $SE(2)$ is the group of 3×3 matrices of the form:

$$\begin{pmatrix} \cos\theta & -\sin\theta & a_1 \\ \sin\theta & \cos\theta & a_2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{B.10}$$

where a_1 and a_2 are the elements of a.
In the case if $SE(3)$, $R \in SO(3)$ is a common rotation matrix in 3 dimensions and $a \in \mathbb{R}^3$ is a translation in 3 dimensions.

B.4 The Euclidean group $E(2)$

In the past section, we introduced in Eq. (B.10) one of the possible representations for $SE(2)$. In this section, we will justify this representation from the perspective of *group theory* and *representation theory*. It is worth mentioning before going into detail, that Eq. (B.10) is what is called a *matrix representation* of $SE(2)$. It is not unique but we will focus and work with this one.

From group theory, we know that $SO(n)$ has $n(n-1)/2$ generators. Roughly speaking, $SE(2)$ is a combination of $SO(2)$ and translations in two dimensions. Thus, it has

three generators, one corresponding to rotations in the plane and two translations. We will name this generator of rotations as J_3 and the generators of translations as T_1 and T_2. These generators commute as follows:

$$[J_3, T_1] = T_2 \tag{B.11}$$
$$[J_3, T_1] = -T_1 \tag{B.12}$$
$$[T_1, T_2] = 0. \tag{B.13}$$

These three commutators define the Lie algebra of $E(2)$ and an element g of it can be expressed as:

$$g = \theta J_3 + a_1 T_1 + a_2 T_2, \tag{B.14}$$

where $\theta, a_i \in \mathbb{R}$. Now we take the following representation:

$$\rho(J_3) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(T_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(T_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{B.15}$$

These matrices correspond to infinitesimal rotations about an axis perpendicular to the two-dimensional plane and translations along two axes, respectively. This can be seen by taking the exponential map:

$$\exp[\theta\rho(J_3)] = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \exp[a_1\rho(T_1)] = \begin{pmatrix} 1 & 0 & a_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\exp[a_2\rho(T_2)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{B.16}$$

As a final comment, one can extend this representation to the complex plane by redefining the generators as $J_3' = -iJ_3$ and $T_\pm = T_1 \mp iT_2$. The commutators now satisfy:

$$[J_3', T_\pm] = \pm T_\pm \ , \quad [T_+, T_-] = 0. \tag{B.17}$$

Appendix C

Noether's theorem

In this section we will introduce and prove *Noether's theorem*. The version of the theorem that we consider in this section applies to fields. The theorem was first introduced in Ref. 1, while the proof, for fields, can be found in several QFT books, such as Refs. 13, 21 or 51.

C.1 Noether currents and total conserved charges

Noether's theorem can be stated as follows: *If a system has a continuous symmetry, then there exist associated quantities whose values are conserved in time.* [1]

In practice, this means that if we find a Lagrangian to be invariant under a certain transformation, there has to be a conserved quantity. This quantity is called *Noether's current* or simply *current*. We will show now how to derive a mathematical expression for this current. Let us consider a Lagrangian that does not depend explicitly on space-time coordinates but only on a collection of fields ψ_i and their derivatives $\partial_\mu\psi_i$, $\mathcal{L}(\psi_i, \partial_\mu\psi_i)$. We start by considering an infinitesimal change of this Lagrangian, $\delta\mathcal{L}$. By *Hamilton's principle*, we have the following:

$$0 = \delta\mathcal{L} = \sum_i \frac{\partial\mathcal{L}}{\partial\psi_i}\delta\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\delta(\partial_\mu\psi_i), \tag{C.1}$$

where the subindex i runs over the collection of fields. Now by using the *Euler-Lagrange* equation we can identify the first term in eq. (C.1) with $\partial_\mu[\partial\mathcal{L}/\partial(\partial_\mu\psi_i)]$. If we now exchange $\delta\partial_\mu$ by $\partial_\mu\delta$, we are able to identify the result as a total derivative:

$$0 = \sum_i \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\right)\delta\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\partial_\mu(\delta\psi_i) = \sum_i \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\delta\psi_i\right). \tag{C.2}$$

The last term between brackets in eq. (C.2) is thus constant and it is what we will call Noether's current:

$$j^\mu = \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\delta\psi_i \tag{C.3}$$

The total charge can be computed from j^μ by integrating the zeroth component j^0 over a large volume:

$$Q = \int_V d^3x\ j^0. \tag{C.4}$$

The total charge Q is the quantity which is conserved in time. This can be seen by integrating Eq. (C.2):

$$0 = \int_V d^3x\ \partial_\mu j^\mu = \partial_t \int_V d^3x\ j^0 - \int_V d^3x\ \nabla \cdot \mathbf{j}. \tag{C.5}$$

The last term in Eq. (C.5) is a volume integral which can be rewritten as a surface integral, $\int_S d^2x\ \mathbf{j}$ by applying Gauss' theorem. By assuming now that $\mathbf{j}(x) = 0$ on the surface, this term vanishes and we are left with:

$$\partial_t \int_V d^3x\ j^0 = \frac{d}{dt} Q\ = 0. \tag{C.6}$$

And, thus, the total charge Q is conserved in time.

C.2 Noether charges as symmetry generators

In general, when we perform an infinitesimal transformation like $\psi \to \psi + \delta\psi$ for a field ψ under a group G, what we are actually doing is applying an element of this group to the field: $\psi \to g\psi,\ g \in G$. More precisely, we apply a unitary representation of the group element acting under conjugation on the field. In this section we will clarify this statement and give a connection between this and total conserved charges.

Let G be a Lie group and let $g \in G$ be an element of this group. We build a unitary representation like $U = \exp[tX]$, where X is the generator of the group and t is a parameter (this is called a *one-parameter family*). Now we apply U to a field:

$$\psi \to g\psi :\ \ U\psi U^{-1} = e^{tX}\psi e^{-tX}. \tag{C.7}$$

Since the transformation is infinitesimal, we can expand the exponentials to first order as $\exp[tX] \sim 1 + tX$. Having in mind that ψ and X do not necessarily commute, this leads to:

$$\psi \to \psi + t[X, \psi]. \tag{C.8}$$

And thus, we can relate

$$\delta\psi = t[X, \psi]. \tag{C.9}$$

On the other hand, let us look at the expression of the total charge in Eq. (C.4). From the explicit form of j^0, we can identify the canonical conjugated momentum π:

$$Q = \int_V d^3x\ j^0 = \int_V d^3x\ \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \delta\psi = \int_V d^3x\ \pi\delta\psi. \tag{C.10}$$

If we now compute the commutator between the charge Q and the field ψ, we get the following:

$$[Q, \psi] = \int_V d^3x \, [\dot{\psi}(x)\delta\psi(x), \psi(x')] = -i\delta\psi, \tag{C.11}$$

where we used the canonical equal-time commutation rules:

$$[\psi(x,t), \psi(x',t)] = 0 \tag{C.12}$$

$$[\pi(x,t), \psi(x',t)] = -i\delta(x-x'). \tag{C.13}$$

Thus, we can relate the total conserved charges to the generators of the symmetries. In fact, the charges generate the symmetry transformations and they form a representation of the generators of the symmetry group.

Appendix D

Goldstone's theorem

D.1 Classical level

Let $\mathcal{L}$ be a Lorentz-invariant Lagrangian of the form $\mathcal{L} = T(\phi_j) - V(\phi_j)$, where T and V are the kinetic and potential part of the Lagrangian, respectively. Assume that $\mathcal{L}$ is invariant, i.e. symmetric, under a certain set of transformations. Let G denote the symmetry group of the Lagrangian with n_G generators, and $H \subseteq G$ the subgroup with n_H generators which leaves the ground state invariant after spontaneous symmetry breaking. For each generator which does not annihilate the vacuum one obtains a massless **Goldstone boson**, i.e., the total number of Goldstone bosons equals the dimension of the quotient group G/H, $\dim G/H = n_G - n_H$.

Proof. Let us denote the collection of fields that minimize the potential $V(\phi_i)$ as $\phi_0 = \{\phi_i\}_0$. The Taylor expansion of V around its minimum is

$$V(\phi_i) \approx V(\phi_0) + \frac{1}{2}(\phi_i - \phi_0)(\phi_j - \phi_0) \underbrace{\left.\frac{\partial^2 V}{\partial\phi_i \partial\phi_j}\right|_{\phi_0}}_{M_{ij}} + \dots, \tag{D.1}$$

which shows that the mass matrix elements M_{ij} are given by the second derivative of the potential with respect to the fields.

Let us in addition denote $U(g)$ as a representation of G acting on the fields ϕ_i and $U(h)$ to the representation of $H \subseteq G$. The invariance of V under G implies that.

$$V(U(g)\phi_i) = V(\phi_i). \tag{D.2}$$

Let us now substitute this invariance, for an infinitesimal group action, in the expansion of the potential around its minimum:

$$V(\phi_0) = V(U(g)\phi_0) \approx V(\phi_0) + \frac{1}{2} M_{ij}\delta\phi_i\delta\phi_j, \tag{D.3}$$

which by comparison with Eq. (D.2) gives

$$M_{ij}\delta\phi_i\delta\phi_j = 0. \tag{D.4}$$

The variation $\delta\phi_k$ depends on whether the transformation belongs to $U(h)$ or not.

1. If $g \in H$, then the vacuum remains invariant, i.e., $\delta\phi_j = 0$. The generators of G are then *unbroken*. The condition (D.4) is satisfied.

2. If $g \notin H$, then $\delta\phi_j \neq 0$. The generators are *broken*, and we say that the symmetry has been spontaneously broken. Eq. (D.4) then is an eigenvalue equation which shows that the mass matrix M has a zero eigenvalue.

In the latter case, g must be in the left coset (or quotient) G/H. The number of massless modes is equal to the dimension of G/H. □

Appendix E

Functional integrals

A *functional* is a mapping from a vector space V to a field of scalars (either $\mathbb{R}$ or $\mathbb{C}$). V is usually a space of functions and thus, a functional is usually referred to as a *function of functions* [52]. A deeper discussion about functional integrals in QFT can be found in several books, e.g. Refs 13, 21 or 51

E.1 Quantum Mechanics

Consider the *transition amplitude*, A, from an initial state $|x_i\rangle$ at initial time t_i to a final state $|x_f\rangle$ at time t_f:

$$A = \langle x_f | \exp[-i(t_f - t_i)H(\hat{p}, \hat{x})] | x_i \rangle, \tag{E.1}$$

where $H(\hat{p}, \hat{x})$ is the Hamiltonian of a one-particle system.

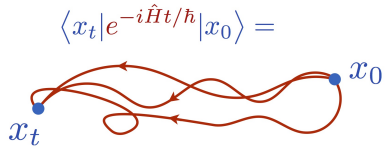


Figure E.1: Graphical representation of a transition amplitude from a state $|x_0\rangle$ to $|x_t\rangle$.

Let us first split the time interval into N subintervals of length ϵ. Eq. (E.1) becomes:

$$\langle x_f | \exp[-i\epsilon H(\hat{p}, \hat{x})] \dots \exp[-i\epsilon H(\hat{p}, \hat{x})] | x_i \rangle. \tag{E.2}$$

The states $|x\rangle$ form a complete set, that is, $\int dx |x\rangle\langle x| = 1$ and thus, we can insert it between each exponential. This gives

$$A = \int \left(\prod_{j=1}^{N-1} dx_j \right) \langle x_f | \exp[-i\epsilon H(\hat{p}, \hat{x})] | x_{N-1} \rangle \cdots \langle x_1 | \exp[-i\epsilon H(\hat{p}, \hat{x})] | x_i \rangle. \tag{E.3}$$

We now insert between each element a complete set of momentum eigenstates, $\int dp|p\rangle\langle p| = 1$. For each element, we have:

$$\langle x_j|\exp[-i\epsilon H(\hat{p},\hat{x})]|x_{j-1}\rangle = \int dp_j \langle x_j|p_j\rangle\langle p_j|\exp[-i\epsilon H(\hat{p},\hat{x})]|x_{j-1}\rangle, \tag{E.4}$$

with

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi}}\exp[ipx]. \tag{E.5}$$

Substituting this scalar product into Eq. (E.4) and computing to first order in ϵ, we have

$$\langle x_j|\exp[-i\epsilon H(\hat{p},\hat{x})]|x_{j-1}\rangle = \frac{1}{2\pi}\int dp_j \exp\left[ip_j(x_j - x_{j-1}) - i\epsilon H(p_j, x_{j-1})\right]. \tag{E.6}$$

We now assume a non-relativistic particle and plug $H = \frac{p^2}{2m} + V(x)$ into Eq. (E.6):

$$\langle x_j|\exp[-i\epsilon H(\hat{p},\hat{x})]|x_{j-1}\rangle = \frac{1}{2\pi}\int dp_j \exp\left[ip_j(x_j - x_{j-1}) - i\epsilon\frac{p_j^2}{2m} - i\epsilon V(x_{j-1})\right], \tag{E.7}$$

which is a Gaussian integral in p_j. The result is:

$$\langle x_j|\exp[-i\epsilon H(\hat{p},\hat{x})]|x_{j-1}\rangle = \sqrt{\frac{m}{2i\pi\epsilon}}\exp\left[i\epsilon\left(\frac{m^2}{2\epsilon^2}(x_j - x_{j-1})^2 - V(x_{j-1})\right)\right]. \tag{E.8}$$

We finally substitute Eq. (E.8) into (E.3) and take the limit of $N \to \infty$ subintervals, which is equivalent to taking the limit $\epsilon \to 0$:

$$A = \lim_{\substack{N\to\infty \\ \epsilon\to 0}} \int \left(\prod_{j=1}^{N-1} dx_j \sqrt{\frac{m}{2i\pi\epsilon}}\right) exp\left[i\epsilon\sum_{j=1}^{N-1}\left(\frac{m^2}{2\epsilon^2}(x_j - x_{j-1})^2 - V(x_{j-1})\right)\right]. \tag{E.9}$$

Using now that

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}(x_j - x_{j-1}) = \frac{d}{dt}x,$$
$$\lim_{\epsilon\to 0}\ \epsilon\sum_{j=1}^{N-1} = \int_{t_i}^{t_f},$$

we can identify the exponent of Eq. (E.9) as the action $S[x(t)]$:

$$S[x(t)] = \int_{t_i}^{t_f} dt\left[\frac{1}{2}m\left(\frac{d}{dt}x\right)^2 - V(x(t))\right] = \int_{t_i}^{t_f} dt\ L[x(t)], \tag{E.10}$$

where $L[x(t)]$ is the Lagrangian of the system. Lastly, we introduce the notation

$$\int\left(\prod_{j=1}^{N-1} dx_j\sqrt{\frac{m}{2i\pi\epsilon}}\right) = \int_{t_i}^{t_f}\mathcal{D}[x(t)]. \tag{E.11}$$

Rearranging all terms together, we express the transition amplitude A as

$$A = \int_{t_i}^{t_f} \mathcal{D}[x(t)] \ \exp[iS[x(t)]]. \tag{E.12}$$

Because $L[x(t)]$ and $S[x(t)]$ are functionals of $x(t)$, the transition amplitude A is called *functional integral* and it integrates over all possible paths that the particle can take.

E.2 Quantum field theory

The path integral formalism we have presented in the previous section can be extended to field theory. We now consider the transition amplitude from an initial state $|\phi_i\rangle$ at time t_i to a final state $|\phi_f\rangle$ at time t_f:

$$A_\phi = \langle \phi_f | \exp[-i(t_f - t_i)H] | \phi_i \rangle. \tag{E.13}$$

The same discussion we made in section E.1 can be used here by replacing the paths $x(t)$ with the fields $\phi(\mathbf{x}, t)$. In the context of field theory "integrating over all possible paths" translates into integrating over all possible field configurations of the system during the transition. The action $S[\phi(\mathbf{x}, t)]$ is now expressed in terms of the Lagrangian density $\mathcal{L}[\phi(\mathbf{x}, t)]$ as

$$S[\phi(\mathbf{x}, t)] = \int_{t_i}^{t_f} dt \ L = \int d^4x \ \mathcal{L}[\phi(\mathbf{x}, t)], \tag{E.14}$$

where we took the limits $t_{i,f} \to \pm\infty$. The transition amplitude A_ϕ, for a real scalar field ϕ can then be written as a functional integral

$$A_\phi = \langle \phi_f | \exp[-i(t_f - t_i)H] | \phi_i \rangle = \int_{\phi_i}^{\phi_f} \mathcal{D}[\phi] \exp[iS[\phi]], \tag{E.15}$$

where the theory is described by a Lagrangian

$$\mathcal{L}[\phi] = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi), \tag{E.16}$$

where $V(\phi)$ is a *potential density*.

Appendix F

Functional integral representation of the partition function

In this appendix we will use the path integral approach explained in appendix E to introduce *thermal field theory*, that is, quantum field theory including the effects of finite temperature and density. Our discussion in this appendix is mainly based on Ref. [26].

F.1 Partition function

Consider the partition function Z for a canonical ensemble:

$$Z = \text{Tr } \exp[-\beta H], \tag{F.1}$$

where H is the Hamiltonian of the system and $\beta = 1/T$ is the inverse of the temperature. Let us assume a single-particle system. Because the trace is independent of the choice of basis, we can express Eq. (F.1) in a position-state basis:

$$Z = \text{Tr } \exp[-\beta H] = \int dx \, \langle x| \exp[-\beta H]|x\rangle. \tag{F.2}$$

The integrand of Eq. (F.2) can be interpreted as a functional integral. This can be done because both Eq. (F.2) and (E.12) are similar if we integrate the latter over imaginary time $\tau = it$ from 0 to β. This is called *Wick rotation*:

$$Z = \int dx \, \langle x| \exp[-\beta H]|x\rangle = \int dx \int_{x(0)=x(\beta)} \mathcal{D}[x(\tau)] \, \exp\left[-\int_0^\beta d\tau \, L_E\right], \tag{F.3}$$

where the states $|x\rangle$ satisfy periodic boundary conditions and where the subscript L_E stands for *Euclidean Lagrangian* and is defined as $L_E = -L(t \to -i\tau)$.[1] The exponent is then called *Euclidean action*, S_E. Recalling now the periodicity of the states, i.e.

[1] We have Wick rotated from Minkowski to Euclidean space

$x(0) = x(\beta) = x$, we have:

$$Z = \int \mathcal{D}[x(\tau)] \ \exp\big[- S_E[x(\tau)]\big]. \tag{F.4}$$

Here, we made use of the Quantum Mechanics formalism and, once again, we can go to field theory by replacing the states $|x\rangle$ by the fields $\phi(\mathbf{x}, t)$. For a scalar field theory, we have:

$$Z = \int \mathcal{D}[\phi] \ \exp\left[-S_E[\phi]\right], \tag{F.5}$$

where the fields satisfy $\phi(\mathbf{x}, 0) = \phi(\mathbf{x}, \beta)$. Here the Euclidean action is again defined in terms of the (Euclidean) Lagrangian density:

$$S_E[\phi(\mathbf{x}, \tau)] = \int_0^\beta d\tau \int d^3x \ \mathcal{L}_E[(\mathbf{x}, \tau)], \tag{F.6}$$

where, as in the QM case, $\mathcal{L}_E = -\mathcal{L}(t \to -i\tau)$.

Until now, we have considered the canonical ensemble, which does not consider exchange of particles and energy with a heat bath. In order to take this into account, we need to make use of the *grand canonical ensemble*. For this ensemble, the partition function is defined as

$$Z = \mathrm{Tr} \exp[-\beta(H - \mu_i N_i)], \tag{F.7}$$

where the μ_i are chemical potentials and the N_i are number operators. Each N_i has a μ_i associated. In general, instead of using N_i, we can couple each chemical potential to each conserved charge Q_i of the system. Eq. (F.7) becomes:

$$Z = \mathrm{Tr} \exp[-\beta(H - \mu_i Q_i)]. \tag{F.8}$$

The way of introducing the grand canonical ensemble in field theory is just coupling the Hamiltonian density $\mathcal{H}_0$ of the system to the charge density $j_i^0 = \rho_i$ of the Noether current:

$$\mathcal{H} = \mathcal{H}_0 - \mu_i \rho_i, \tag{F.9}$$

and one can then compute the new Lagrangian from it. Note that by introducing a chemical potential, the Lorentz invariance of the theory is broken.

F.2 Thermodynamic quantities

In the following we only consider the grand canonical ensemble. The partition function plays a essential role in statistical mechanics because all thermodynamic properties can be computed from it. For instance, the *thermal average* of an operator $\hat{A}$ is defined as:

$$\langle A \rangle = \frac{1}{Z} \mathrm{Tr} \left[\hat{A} \exp[-\beta(H - \mu_i Q_i)] \right], \tag{F.10}$$

which can be also expressed, using the path integral approach, as

$$\langle A \rangle = \frac{1}{Z} \int \mathcal{D}[\phi] \ \hat{A} \ \exp\left[-S[\phi]\right]. \tag{F.11}$$

Other thermodynamic properties such as pressure, particle number and entropy can also be computed from the partition function as follows:

$$P = \frac{\partial}{\partial V}(T \ln Z), \tag{F.12}$$

$$N_i = \frac{\partial}{\partial \mu_i}(T \ln Z), \tag{F.13}$$

$$S = \frac{\partial}{\partial T}(T \ln Z). \tag{F.14}$$

We see that the quantity $T \ln Z$ is present in all three equations. Let us then define the *thermodynamic potential as*:

$$\Omega(T, V, \mu_i) = -\frac{1}{\beta} \ln Z, \tag{F.15}$$

where again $T = \frac{1}{\beta}$. Let us take the partition function of Eq. (F.8), then every conserved charge of the system can be computed from the thermodynamic potential as

$$Q_i = -\frac{\partial}{\partial \mu_i} \Omega(T, V, \mu_i). \tag{F.16}$$