CONTROLLABILITY ANALYSIS FOR PROCESS AND CONTROL SYSTEM DESIGN

by

Audun Faanes

A Thesis Submitted for the Degree of Dr. Ing.

Department of Chemical Engineering
Norwegian University of Science and Technology

Submitted August 2003

Abstract

Controllability is the ability of a process to achieve acceptable performance, and in this thesis we use controllability analysis in the design of buffer tanks, feedforward controllers, and multivariable controllers such as model predictive control (MPC).

There is still an increasing pressure on the process industry, both from competitors (prize and quality) and the society (safety and pollution), and one important contribution is a smooth and stable production. Thus, it is important to dampen the effect of uncontrolled variations (disturbances) that the process may experience.

The process itself often dampens high-frequency disturbances, and feedback controllers are installed to handle the low-frequency part of the disturbances, including at steady state if integral action is applied. However, there may be an intermediate frequency range where neither of these two dampens the disturbances sufficiently. In the first part of this thesis we present methods for the design of buffer tanks based on this idea. Both mixing tanks for quality disturbances and surge tanks with "slow" level control for flow-rate variations are addressed.

Neutralization is usually performed in one or several mixing tanks, and we give recommendations for tank sizes and the number of tanks. With local PI or PID control, we recommend equal tanks, and give a simple formula for the total volume. We also give recommendations for design of buffer tanks for other types of processes. We propose first to determine the required transfer function to achieve the required performance, and thereafter to find a physical realization of this transfer function.

Alternatively, if measurements of the disturbances are available, one may apply feedforward control to handle the intermediate frequency range. Feedforward control is based mainly on a model, and we study the effect of model errors on the performance. We define feedforward sensitivities. For some model classes we provide rules for when the feedforward controller is effective, and we also design robust controllers such as μ-optimal feedforward controllers.

Multivariable controllers, such as model predictive control (MPC), may use both feedforward and feedback control, and the differences between these two also manifest themselves in a multivariable controller. We use the class of serial processes to gain insight into how a multivariable controller works. For one specific MPC we develop a state space formulation of the controller and its state estimator under the assumption that no constraints are active. Thus, for example the gains of each channel of the MPC (from measurements to the control inputs) can be found, which gives further insight into to the controller. Both a neutralization process example and an experiment are used to illustrate the ideas.

Acknowledgments

I want to thank Sigurd Skogestad for pointing out the directions in which to proceed, and for his support along the way. He is an excellent supervisor. In addition to his knowledge and skills, from which I have learned a lot, I appreciate his focus on reporting and publishing of results. This improves the research and is beneficial for progress.

I thank my fellow students at the institute in Trondheim, my former colleges at the Norsk Hydro Research Centre in Porsgrunn, and my colleges at Statoil Research Centre in Trondheim for always pleasant and often fruitful discussions.

The Norwegian Research Council and my former employer, Norsk Hydro ASA, are gratefully acknowledged for financing this work. I also want to thank my present employer, Statoil ASA, for allowing me time to finish it.

Finally, I want to thank my wife Sophie and our daughters Sarah and Mathilde. Our family has come into being during the same period as this thesis. My parents can be thanked for many things, and of particular relevance is their hospitality when I needed a home in Trondheim during this work.

Contents

Chapter 1

Introduction

We start with some words on the title of the thesis, or more precisely with a definition of what we mean by *controllability* and *controllability analysis* (Skogestad and Postlethwaite, 1996, Chapter 5):

Definition 1.1 (Input-output) controllability *is the ability to achieve acceptable control performance; that is, to keep the outputs (y) within specified bounds or displacements from their references (r), in spite of unknown but bounded variations, such as disturbances (d) and plant changes, using available inputs (u) and available measurements (y_m or d_m).*

A plant is controllable if there *exists* a controller (connecting plant measurements and plant inputs) that yields acceptable performance for all expected plant variations. From this, controllability is independent of the controller, and a property of the process alone. Further, *controllability analysis* is applied to a plant to find out what control performance can be expected.

The definition above is in accordance with the definition given by Ziegler and Nichols (1943) "the ability of the process to achieve and maintain the desired equilibrium value", but must not be confused with the more narrow *state controllability* definition of Kalman from the 60's.

In particular, in this thesis we will apply controllability analysis in the design of processes, namely such processes that are designed for dynamic and control purposes, and in the design and understanding of feedforward and multivariable controllers.

1.1 Motivation

High degree of competition in all branches of the process industry put pressure on each single site and plant to stay competitive. Even within a company there is an internal competition of being the most productive and effective, and delivering the best quality products. The second best risks that investment plans are rejected by the central management, or even that the plant is closed.

There are many important requirements that must be met by a plant organisation

(1) On-site and off-site safety

(2) Discharge shall be below certain limits, both on a long term basis, e.g., total over a year, or on a shorter the period of time, such as on an hourly basis.

(3) Requirements for certain quality parameters to stay within given limits (to obtain maximal prizes)

(4) Minimal production costs, such as energy consummation

(5) Maximal production

Running smoothly without abrupt changes of any kind, will be an important contribution to meet all the above-mentioned requirements. The risk of accidents and undesired discharge is reduced, and a natural consequence is also a more constant product quality. Finally, production cost can be reduced and the production rate increased, because the risk of unplanned stops is reduced and because it is possible to move the operating point closer to the constraints.

On the other hand, within a process, there are many sources that introduce variations of all kinds, namely *disturbances*. This can be such as variations in the quality of the raw materials or the incomming flow rates, inaccurate charging equipment, sticky vales, or badly tuned control loops. Some of these things are, at least in principle, easy to handle, others are more difficult or costly to avoid, and must therefore be treated by other process parts.

It is our experience that the Norwegian process and oil industry has increased the focus on smooth production in recent years, and therefore puts pressure on process control. This is because of the increased competition in the process industry in general (the competitors focus on this), and also because of changes in the oil production in the North Sea, which lead to more disturbances and "new" bottle-necks (primal reasons are increased water and gas production and longer pipes between the wells and the processing units).

In this thesis two basic ideas are elaborated. The first is that high-frequency disturbances are dampened by the process itself (e.g., by inventories like reactor volumes, and liquid hold-ups in distillation columns) whereas low-frequency disturbances can be dampened out with effective single-loop feedback controllers. To handle intermediate frequencies, we look into the design of buffer tanks and more sophisticated controllers (like traditional feedforward control and multivariable control).

As far as we have found in the literature, even though buffer tanks are introduced for control purposes, control theory has not been applied. Further, feedforward control theory is treated by most textbooks on control, but often very briefly, and even a simple analysis of the effect of model errors is often missing (exceptions are Balchen (1968), and the work of Scali and co workers (Lewin and Scali, 1988; Scali *et al.*, 1989)).

Based on our experience from industrial processes, we assume that sinusoidal disturbances of varying frequency are the most important. The disturbances may be caused by oscillations in other parts of the process, for example, from aggressive control, valve stiction etc. However, in the simulations we also consider step disturbances.

The second idea is that within multivariable feedback controllers there may be controller blocks or elements that are similar to feedforward control. Like traditional feedforward controllers, such elements may nominally improve the performance to a large extent. Unfortunately, feedforward controllers rely heavily on a model of the process, and this drawback also applies to the feedforward elements within the multivariable controller.

1.2 Thesis overview

The thesis is composed of six chapters written as independent articles, each with a separate bibliography, and most of them also have their own appendices. In the end of the thesis there is a concluding chapter (Chapter 8) and in addition there are two appendices, A and B, referred to by "Thesis' Appendix A (or B)" to distinguish from the appendices within each chapter.

Chapters 2 and 3 give rules regarding the design of buffer tanks, especially regarding tank sizes (the first specializes on pH-neutralization). Also Chapter 4 can be useful for readers with interest in this, since it looks into different control strategies for serial processes, and one or more buffer tanks are usually placed in series with other process units. In particular, pH neutralization is included as a case study.

Chapters 4 - 7 focus on control design. One may say that Chapters 5 and 6 are theoretical foundations for Chapters 4 and 7.

If the interest is how to handle disturbances, our basic idea is that when neither the process itself, or a simple feedback control system can handle them, either buffer tanks (Chapters 2 and 3) or feedforward controllers (Chapter 6) may be used to handle the resting frequencies.

In **Chapter 2** we provide a simple rule for the size of mixing tanks for pH neutralization processes ensuring that incoming disturbances are dampened such that the outlet pH is kept within given limits. We assume traditional single-loop feedback control, and that the efficiency of the feedback loops are limited by delays and other high order dynamics. Neutralization processes often have large process gains, and it is therefore often convenient to use several stages.

In **Chapter 3** we extend the mixing tank design from Chaper 2 to the design of a broader class of buffer tanks. The aim of the buffer tank is disturbance dampening in the frequency range where neither the process itself nor any feedback loop dampen the disturbances sufficiently. We consider disturbances in both quality and flow rates, for which mixing tanks and surge tanks with slow level control are used, respectively.

Chapter 4 discusses control design for serial processes. As a case study we consider neutralization in several stages, which we also discuss in Chapter 2. We use the structure of serial processes to identify different classes of control blocks of a multivariable controller, and comment, in particular, on feedforward effects and how to obtain integral action.

The multivariable controller we use in Chapter 4 is a model predictive controller (MPC). In **Chapter 5** we assume that no constraints are active, in which case the MPC can be considered as a linear quadratic controller (LQ), and derive a state-space formulation of the resulting controller, including the state estimator. Chapter 5 is mainly a tool for Chapters 4 and 7, but also include a result on how to choose input biases to gain integral action.

One of the control block classes discussed Chapter 4 is feedforward control, and in **Chapter 6** we discuss feedforward control under model uncertainty. In accordance with the sensitivity function defined for feedback control, we introduce feedforward sensitivities, and discuss how this can be used to determine the usefulness of a feedforward controller (or of a feedforward control block).

Chapter 7 verifies some of the results from Chapters 4 and 5 through an experiment. We show that even if simulations indicate that a specific controller gives integral action, when applied to the actual process, steady-state offset is obtained.

Chapter 8 sums up the conclusions from the thesis, and tries to propose some directions for further work.

The thesis' Appendixes A and B are "older" published versions of Chapters 4 (only a part) and 3, respectively. They are included since they contain material that has been removed from the chapters now included (Chapters 4 and 3). Appendix A contains an example where $\mathcal{H}_\infty$ control has been applied (in Chapter 4 model predictive control (MPC) is used). Appendix B is more focused on the short-cut method for buffer tank design than Chapter 3, and contains some more details regarding this.

Preliminary versions or parts of the following chapters have been or will be presented at the following conferences, and versions nearly identical to the chapters have been either submitted to, accepted by or published in the following journals[1]:

Chapter 2: Adchem 2000, June 14-16, 2000, Pisa, Italy (preprints: **1**, pp. 75-80)
Preliminary accepted for publication in Computers and Chemical Engineering

Chapter 3: Nordic Process Control Workshop 9, January 13-15, 2000, Lyngby, Denmark
PSE'2000, 16-21 July, 2000, Keystone, Colorado, USA (Supplement to Computers and Chemical Engng., **24**, pp.1395-1401)
Ind. Eng. Chem. Res., **42**, 10, pp. 2198-2208

Chapter 4: Nordic Process Control Workshop 8, August 23-25, 1998, Stockholm, Sweden
European Control Conference, ECC'99, Aug. 31-Sept. 3, 1999, Karlsruhe, Germany
Submitted to Journal of Process Control

Chapter 5: Submitted to Modeling, Identification and Control, MIC

Chapter 6: Nordic Process Control Workshop 11, January 9-11, 2003, Trondheim
Accepted for presentation at European Control Conference, ECC'03, Sept. 1-4, 2003, Cambridge, UK
Preliminary accepted for publication in European Journal of Control

Chapter 7: Accepted for presentation (poster session) at AIChE, Annual Meeting, Nov. 2003, San Francisco, US

References

Balchen, J. G. (1968). *Reguleringsteknikk Bind 1 (In Norweigan) 1. Ed.*. Tapir. Trondheim, Norway.

Lewin, D. R. and C. Scali (1988). Feedforward control in presence of uncertainty. *Ind. Eng. Chem. Res.* **27**, 2323–2331.

Scali, C., M. Hvala and D. R. Lewin (1989). Robustness issues in feedforward control.. *ACC-89* pp. 577–581.

Skogestad, S. and I. Postlethwaite (1996). *Multivariable Feedback Control*. John Wiley & Sons. Chichester, New York.

Ziegler, J. G. and N. B. Nichols (1943). Process lags in automatic-control circuits. *Trans. ASME* **65**, 433–444.

[1]The difference between the chapters and their corresponding journal article is indicated on the front page of each chapter.

Paper 1.

Faanes, A. and S. Skogestad. pH-Neutralization: Integrated Process and Control Design. Adchem 2000, June 14-16, 2000 Pisa, Italy (preprints: 1, pp.75-80).

Accepted for publication in *Computers and Chemical Engineering*

Paper 2.

Faanes , A. and S. Skogestad. Buffer Tank Design for Acceptable Control Performance. *Ind. Eng. Chem. Res*., 42, 10, pp. 2198-2208.

Paper 3.
Faanes , A. and S. Skogestad. Control Design for Serial Processes
Submitted to *Journal of Process Control*.

Chapter 5

On MPC without active constraints

Audun Faanes* and Sigurd Skogestad†
Department of Chemical Engineering
Norwegian University of Science and Technology
N–7491 Trondheim, Norway

Submitted to *Modeling, Identification and Control, MIC*.

*also affiliated with Statoil ASA, TEK, Process Control, N-7005 Trondheim, Norway
†Author to whom all correspondence should be addressed. E-mail: skoge@chemeng.ntnu.no, Tel.: +47 73 59 41 54, Fax.: +47 73 59 40 80

Abstract

In order to be able to use traditional tools when analysing a multivariable controller as MPC, we develop a state space formulation of the resulting controller for MPC without constraints or assuming that the constraints are not active. Such a derivation was not found in the literature. The state space formulation is used in Chapters 4 and 7. The formulation includes the state estimator.

The MPC algorithm used is a receding horizon controller with infinite horizon based on a state space process model. When no constraints are active, we obtain a state feedback controller, which is modified to achieve either output tracking, or a combination of input and output tracking.

When the states are not available, they need to be estimated from the measurements. It is often recommended to achieve integral action in a MPC by estimating input disturbances and include their effect in the model. We show that to obtain offset free steady state the number of estimated disturbances must equal the number of measurements. The estimator is included in the controller equation to obtain the overall controller with the set-points and measurements as inputs, and which give the manipulated variables.

One use of the state space formulation is to combine it with the process model to obtain a closed loop model. This can for example be used to check the steady-state solution and see if integral action is obtained.

5.1 Introduction

In this paper, we develop a state-space formulation for a MPC without constraints or assuming that the constraints are not active. This state-space formulation of the controller enables the use of traditional tools to get insight into how the controller behaves (see Chapters 4 and 7). Maciejowski (2002) (independently) use a linear formulation for a MPC controller to analyze its controller tuning for a paper machine headbox. He combines the linear controller formulation with the process model, and calculates the singular values of the sensitivity function and the complementary sensitivity function.

The main idea behind MPC is that a model of the process is used to predict the response of future moves of the control inputs (the inputs that the controller can manipulate to control the process). This prediction is used to find an optimal sequence of the control inputs. Optimal means that a certain criterion containing an output vector and the vector of the control inputs is minimized.

In most MPC implementations the control inputs are assumed to be held constant within a given number of time intervals. At a given time, the first value in the sequence of control inputs is implemented in the process. The prediction depends on the current state of the process, and this will also the optimal sequence do. At the next time step, the state being reached is therefore used in the calculation of a new optimal control input sequence. This sequence will not necessarily be what was computed at the previous time step, due to the effects of model errors and unmodelled disturbances. So, at each time step we only implement the first step in the control input sequence, and discard the rest.

Normally we include constraints in the optimization problem. These are constraints that naturally occur in a process, like the range of control valves and pump speeds (on control inputs), and safety-related constraints on the outputs. One may also restrict the rate of change of the control inputs.

For a review of industrial MPCs we refer to (Qin and Badgwell, 1996; Badgwell and Qin, 2002).

In this chapter, we consider the MPC formulation proposed by Muske and Rawlings (1993). This MPC is based on a state-space model. Our assumption is that *no constraints are active*, and this also covers the case when the same constraints are active all the time and the degree of freedom is reduced. Bemporad *et al.* (2002) (first appeared in (Bemporad *et al.*, 1999)) have shown that the controller also for the case with dynamic constraints is piecewise linear.

Since the models are not perfect, and there always are unmodelled disturbances, the MPC needs some correction from measurements. The most common approach is to estimate some output bias in the measurements, and correct for this bias. However, for integrating processes or processes with long time constants, this method has proved unsatisfactory (Muske and Rawlings, 1993; Lee *et al.*, 1994; Lundström *et al.*, 1995). We therefore estimate input disturbances, which is straight forward using a state-space representation of MPC.

As known, MPC without constraints is a special case of optimal control, and in Sections 5.2, 5.3 and 5.4 we will demonstrate how the control input can be expressed by the current state and the previous control input. The first of these sections, Section 5.2, covers the simple case when the reference for the output vector is zero, while Section 5.3 handles non-zero references. When the number of control inputs exceeds the number of outputs, the extra degree of freedom may also be used to give references to the control inputs (Section 5.4). Since the full state vector normally is not measured, we include a state estimator, which also estimates input disturbances, in Section 5.5. The total controller formulation, i.e., the control inputs, given by the measurements, is given in Section 5.6. In Section 5.7 we find the number of estimated disturbances needed to obtain effective integral action. We develop the closed loop model of the system in Section 5.8.

5.2 Derivation of equivalent controller from receding horizon controller without active constraints

Muske and Rawlings (1993) present a model predictive control algorithm based on the following state-space model:

$$x_{k+1} = Ax_k + Bu_k + E_d d_k \qquad k = 0, 1, 2, \ldots \tag{5.1}$$

$$y_k = Cx_k \tag{5.2}$$

Here x_k is the state vector, u_k the control input vector, d_k the vector of (unmeasured) disturbances and y_k the output vector, all at time k. The model is assumed to be time invariant so A, B, C and E_d are constant matrices. The optimal control input minimizes the following infinite horizon criterion:

$$\min_{u_k^N} \sum_{j=0}^{\infty} \left(y_{k+j}^T Q y_{k+j} + u_{k+j}^T R u_{k+j} + \Delta u_{k+j}^T S \Delta u_{k+j} \right) \tag{5.3}$$

Here $u_k^N = \begin{bmatrix} u_k & u_{k+1} & \ldots & u_{k+N-1} \end{bmatrix}^T$ is a vector of N future moves of the control input, of which only the first is actually implemented. The control input, u_{k+j}, is assumed zero for all $j \geq N$. In the criterion it is assumed that the reference for y is zero. We assume that the process is stable, and Muske and Rawlings (1993) show how this formulation can be transformed into the following finite optimization problem:

$$\min_{u_k^N} \Phi_k = \left(u_k^N\right)^T H u_k^N + 2\left(u_k^N\right)^T (Gx_k - Fu_{k-1}) \tag{5.4}$$

where H, G and F are time independent matrices expressed by the model matrices, A, B and C, and the weight matrices, Q, R and S. Since d_k is unknown in the future, the term $E_d d_k$ from (5.1) is omitted in the derivation of (5.4). For normal use of this MPC algorithm, the control input is found by optimizing (5.4) subject to given constraints on the outputs, the control inputs and changes in the control inputs. Assuming no active constraints, however, the optimum of (5.4) can be found by setting the gradient equal to zero (Halvorsen, 1998):

$$\nabla\Phi_k\left(u_k^N\right) = 2Hu_k^N + 2\left(Gx_k - Fu_{k-1}\right) = 0 \tag{5.5}$$

which implies

$$u_k^N = -H^{-1}Gx_k + H^{-1}Fu_{k-1} \tag{5.6}$$

Only the first vector u_k from u_k^N is applied:

$$u_k = Kx_k + K_u u_{k-1} \tag{5.7}$$

where K and K_u consist of the first r rows in $-H^{-1}G$ and $H^{-1}F$, respectively, and r is the number of control inputs.

Since H, G and F are constant, also K and K_u are constant matrices. The first term can therefore be recognized as state feedback. The second term comes from the weight on the change in control input from the original criterion. The matrix F only contains S and zeros, so when no weight is put on the change in the control input, S is zero, and $K_u = 0$.

5.3 The steady-state solution

Here, we consider tracking of outputs. If the output reference vector, y_r, is nonzero, (5.7) must be shifted to the steady-state values for the states and the control inputs:

$$u_k = K\left(x_k - x_s\right) + K_u\left(u_{k-1} - u_s\right) + u_s \tag{5.8}$$

or

$$u_k = Kx_k + K_u u_{k-1} - \begin{bmatrix} K & K_u - I \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} \tag{5.9}$$

u_s and x_s can be found from the steady-state solver:

$$\min_{[x_s,u_s]} \Psi = \left(u_s - u_r\right)^T R_s \left(u_s - u_r\right) \tag{5.10}$$

subject to

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} E_d d_k \\ y_r \end{bmatrix} \tag{5.11}$$

$$u_{\min} \leq u \leq u_{\max} \tag{5.12}$$

where y_r and u_r are the references for the output and the control input, respectively. Again, we assume that the limitations are never active, and that we have no extra freedom for the control inputs (number of control inputs equals number of outputs), in which case the problem reduces to solving the equation set (5.11).

Assuming square systems (i.e., equal number of control inputs and references), no poles in the origin (which makes $(I - A)$ invertible) and that $C\,(I - A)^{-1}\,B$ is invertible (it is at least quadratic from the first assumption), we get the following solution:

$$\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \Gamma_y y_r + \Gamma_d d_k \tag{5.13}$$

where

$$\Gamma_y = \begin{bmatrix} (I - A)^{-1}\,B\left(C\,(I - A)^{-1}\,B\right)^{-1} \\ \left(C\,(I - A)^{-1}\,B\right)^{-1} \end{bmatrix} \tag{5.14}$$

$$\Gamma_d = \begin{bmatrix} (I - A)^{-1}\left(I - B\left(C\,(I - A)^{-1}\,B\right)^{-1} C\,(I - A)^{-1}\right) E_d \\ -\left(C\,(I - A)^{-1}\,B\right)^{-1} C\,(I - A)^{-1}\,E_d \end{bmatrix} \tag{5.15}$$

Since we have no knowledge of future disturbances, we assume that it will keep its current value, that is $d_s = d_k$. We note that $y_s = Cx_s = y_r$ as desired, and that if we assume that the disturbance enters via the control inputs, i.e., $E_d = B$, the expression for Γ_d simplifies to

$$\Gamma_d' = \begin{bmatrix} 0 \\ -I \end{bmatrix}$$

i.e., $u_s = -d_s$ and $x_s = 0$.

Now (5.9) can be expressed with y_r and d_k:

$$u_k = Kx_k + K_u u_{k-1} + K_y y_r + K_d d_k \tag{5.16}$$

where K and K_u are defined in Section 5.2 and

$$K_y = -[\begin{array}{cc} K & K_u - I \end{array}]\Gamma_y \tag{5.17}$$

$$K_d = -[\begin{array}{cc} K & K_u - I \end{array}]\Gamma_d \tag{5.18}$$

5.4 Generalization with tracking of inputs

In this section, we generalize the steady-state solution to include tracking of both inputs and outputs. The total number of references that it is possible to track is limited by the number of (independent) control inputs.

We collect the inputs that we want to give a reference into the vector u_1, and likewise the outputs we want to give a reference into y_1. The rest of the inputs and outputs are assembled into u_2 and y_2, respectively. The model may now be formulated as

$$\begin{aligned} x_{k+1} &= Ax_k + B_1 u_{k_1} + B_2 u_{k_2} + E_d d_k \\ y_{k_1} &= C_1 x_k \\ y_{k_2} &= C_2 x_k \end{aligned} \tag{5.19}$$

where we have distributed the columns of B into the two matrices B_1 and B_2 corresponding to the division of u_k, and the rows of C is divided into C_1 and C_2 corresponding to the division of y_k. At steady state $y_{k_1} = y_{r_1}$ and $u_{k_1} = u_{r_1}$. Now x_s and u_s can be expressed by y_{r_1}, u_{r_1}, and d_k ($= d_s$):

$$\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} x_s \\ u_{r_1} \\ u_{s_2} \end{bmatrix} = \Gamma_{y_r} y_{r_1} + \Gamma_{u_r} u_{r_1} + \Gamma_d d_k \tag{5.20}$$

where

$$\Gamma_{y_r} = \begin{bmatrix} (I-A)^{-1} B_2 \left(C_1 (I-A)^{-1} B_2\right)^{-1} \\ 0 \\ \left(C_1 (I-A)^{-1} B_2\right)^{-1} \end{bmatrix} \tag{5.21}$$

$$\Gamma_{u_r} = \begin{bmatrix} (I-A)^{-1} \left(I - B_2 \left(C_1 (I-A)^{-1} B_2\right)^{-1} C_1 (I-A)^{-1}\right) B_1 \\ I \\ -\left(C_1 (I-A)^{-1} B_2\right)^{-1} C_1 (I-A)^{-1} B_1 \end{bmatrix} \tag{5.22}$$

$$\Gamma_d = \begin{bmatrix} (I-A)^{-1} \left(I - B_2 \left(C_1 (I-A)^{-1} B_2\right)^{-1} C_1 (I-A)^{-1}\right) E_d \\ 0 \\ -\left(C_1 (I-A)^{-1} B_2\right)^{-1} C_1 (I-A)^{-1} E_d \end{bmatrix} \tag{5.23}$$

provided that $(I-A)$ and $C_1 (I-A)^{-1} B_2$ are invertible. For u_k we obtain

$$u_k = Kx_k + K_u u_{k-1} + K_{y_r} y_{r_1} + K_{u_r} u_{r_1} + K_d d_k \tag{5.24}$$

where

$$K_{y_r} = -\begin{bmatrix} K & K_u - I \end{bmatrix} \Gamma_{y_r} \tag{5.25}$$

$$K_{u_r} = -\begin{bmatrix} K & K_u - I \end{bmatrix} \Gamma_{u_r} \tag{5.26}$$

$$K_d = -\begin{bmatrix} K & K_u - I \end{bmatrix} \Gamma_d \tag{5.27}$$

Introduction of $r = \begin{bmatrix} y_{r_1}^T & u_{r_1}^T \end{bmatrix}^T$ and $K_r = \begin{bmatrix} K_{y_r} & K_{u_r} \end{bmatrix}$ yields

$$u_k = Kx_k + K_u u_{k-1} + K_r r + K_d d_k \tag{5.28}$$

5.5 State and disturbance estimator

To calculate u_k from (5.16) or (5.28) one must know the state, x_k, and if it is not measured, it must be estimated from the measurements. The same applies also to the disturbance vector d_k. If we assume that neither the states nor the disturbances are measured, we extend the state variable with the disturbance vector

$$\tilde{x}_k = \begin{bmatrix} x_k \\ d_k \end{bmatrix} \tag{5.29}$$

As basis for a state estimator the following model based on (5.1) and (5.2) is introduced:

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}u_k + w_k \tag{5.30}$$

$$y_k^m = \tilde{C}\tilde{x}_k + v_k \tag{5.31}$$

where w_k and v_k are zero-mean, uncorrelated, normally distributed white stochastic noise with covariance matrices of Q_w and R_v respectively, and

$$\tilde{A} = \begin{bmatrix} A & E_d \\ 0 & I \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \qquad \tilde{C} = \begin{bmatrix} C^m & 0 \end{bmatrix}$$

y_k^m is the measured output vector, not necessarily the same as the output vector that shall track a reference, and C^m is the corresponding matrix in the estimator model, mapping from the states to the measured output vector. We have modelled the disturbance as constant except for the noise.

The augmented state estimator is then formulated as

$$\overline{\tilde{x}}_{k+1} = \tilde{A}\widehat{\tilde{x}}_k + \tilde{B}u_k \tag{5.32}$$

$$\widehat{\tilde{x}}_k = \overline{\tilde{x}}_k + L\left(y_k^m - \tilde{C}\overline{\tilde{x}}_k\right) \tag{5.33}$$

where L is the estimator gain matrix, for example the Kalman filter gain. $\overline{\tilde{x}}_{k+1}$ is called the *a priori* estimate (since it is prior to the measurement), and $\widehat{\tilde{x}}_k$ the *a posteriori* estimate (after the measurement is available). For a Kalman filter, L is given by the solution of a Ricatti equation:

$$P = \tilde{A}\left[P - P\tilde{C}^T\left(\tilde{C}P\tilde{C}^T + R_v\right)^{-1}\tilde{C}P\right]\tilde{A}^T + Q_w \tag{5.34}$$

$$L = P\tilde{C}^T\left(\tilde{C}P\tilde{C}^T + R_v\right)^{-1} \tag{5.35}$$

We want to express the estimator in a single expression, and this can be done in two ways, depending on which of the two estimates one prefers to use. Alternative 1: *A posteriori estimate*, $\widehat{\tilde{x}}_k$:

$$\widehat{\tilde{x}}_{k+1} = \left(I - L\tilde{C}\right)\tilde{A}\widehat{\tilde{x}}_k + \left(I - L\tilde{C}\right)\tilde{B}u_k + Ly_{k+1}^m \tag{5.36}$$

Alternative 2: *A priori estimate*, $\bar{\tilde{x}}_{k+1}$:

$$\bar{\tilde{x}}_{k+1} = \tilde{A}\left(I - L\tilde{C}\right)\bar{\tilde{x}}_k + \tilde{B}u_k + \tilde{A}Ly_k^m \tag{5.37}$$

Remark 1 *Muske and Rawlings (1993) refer to Åström (1970) who used* a priori *estimate (Alternative 2), (noting that their L corresponds to our $\tilde{A}L$). However, according to (Rawlings, 1999) they actually used Alternative 1 (*a posteriori*) in their work. Normally the control input is implemented directly after a new measurement has been sampled, in which case the* a posteriori *estimate is preferred since it utilizes this new measurement. Thus, in this paper we will use Alternative 1, the* a posteriori *estimate.*

5.6 State-space representation of the overall controller

In this section, we will form the overall controller, containing the state feedback, the steady-state solution and the estimator on state-space form.

With the extended state vector $\tilde{x}_k$ from (5.29) and

$$\tilde{K} = [K, K_d] \tag{5.38}$$

the controller equations (5.16) and (5.28) can both be expressed by

$$u_k = \tilde{K}\tilde{x}_k + K_u u_{k-1} + K_r r \tag{5.39}$$

For (5.16) (without input resetting) $r = y_r$ and $K_r = K_y$. Since $\tilde{x}_k$ generally is not available, we use the estimate $\hat{\tilde{x}}_k$. Combination of the controller equation (5.39) with the estimator difference equation (5.36) yields

$$\hat{\tilde{x}}_{k+1} = \bar{A}\hat{\tilde{x}}_k + \left(I - L\tilde{C}\right)\tilde{B}K_u u_{k-1} + \left(I - L\tilde{C}\right)\tilde{B}K_r r + Ly_{k+1}^m \tag{5.40}$$

$$u_k = \tilde{K}\hat{\tilde{x}}_k + K_u u_{k-1} + K_r r \tag{5.41}$$

where $\bar{A} = \left(I - L\tilde{C}\right)\left(\tilde{A} + \tilde{B}\tilde{K}\right)$. This is not an ordinary discrete state-space formulation. First, y_{k+1} and $\hat{\tilde{x}}_k$ do not have the same index on the right side of (5.40). To overcome this we introduce the artificial state variable $z_k = \hat{\tilde{x}}_k - Ly_k^m$:

$$z_{k+1} = \bar{A}z_k + \bar{A}Ly_k^m + \left(I - L\tilde{C}\right)\tilde{B}K_u u_{k-1} + \left(I - L\tilde{C}\right)\tilde{B}K_r r \tag{5.42}$$

$$u_k = \tilde{K}z_k + \tilde{K}Ly_k^m + K_u u_{k-1} + K_r r \tag{5.43}$$

Next, the term u_{k-1} is a problem. We first assume that in the optimization criterion (5.3) $S = 0$. Then $K_u = 0$, and we get an ordinary discrete state-space system with z_k as the states, y_k^m as the input and u_k as the output. The reference, r, can be seen as a ”disturbance” to the controller. We may express the controller as

$$\begin{aligned} z_{k+1} &= A_K z_k + B_K y_k^m + E_K r \\ u_k &= C_K z_k + D_K y_k^m + F_K r \end{aligned} \tag{5.44}$$

where $A_K = \bar{A}$, $B_K = \bar{A}L$, $C_K = \tilde{K}$, $D_K = \tilde{K}L$, $E_K = \left(I - L\tilde{C}\right)\tilde{B}K_r$ and $F_K = K_r$.

For $S \neq 0$ we have not yet obtained the controller on ordinary state-space form. We first express the controller as

$$\begin{aligned} z_{k+1} &= A_k z_k + B_K y_k^m + E_K r + G_K u_{k-1} \\ u_k &= C_K z_k + D_K y_k^m + F_K r + H_K u_{k-1} \end{aligned} \tag{5.45}$$

where in addition to the definitions above $G_K = \left(I - L\tilde{C}\right)\tilde{B}K_u$ and $H_K = K_u$. We repeat u_k shifted one time step,

$$\begin{aligned} z_{k+1} &= A_k z_k + B_K y_k^m + E_K r + G_K u_{k-1} \\ u_{k+1} &= C_K z_{k+1} + D_K y_{k+1}^m + F_K r + H_K u_k \\ u_k &= C_K z_k + D_K y_k^m + F_K r + H_K u_{k-1} \end{aligned} \tag{5.46}$$

insert for z_{k+1} in the expression for u_{k+1} and re-arrange:

$$\begin{aligned} u_{k+1} &= H_K u_k + C_K G_K u_{k-1} + C_K A_K z_k \\ &+ C_K B_K y_k^m + D_K y_{k+1}^m + C_K E_K r + F_K r \\ u_k &= H_K u_{k-1} + C_K z_k + D_K y_k^m + F_K r \\ z_{k+1} &= G_K u_{k-1} + A_k z_k + B_K y_k^m + E_K r \end{aligned} \tag{5.47}$$

We now introduce the state vector

$$\hat{z}_k = \begin{bmatrix} u_k \\ u_{k-1} \\ z_k \end{bmatrix} \tag{5.48}$$

and obtain

$$\begin{aligned} \hat{z}_{k+1} &= \begin{bmatrix} H_K & C_K G_K & C_K A_K \\ 0 & H_K & C_K \\ 0 & G_K & A_K \end{bmatrix} \hat{z}_k + \begin{bmatrix} C_K B_K \\ D_K \\ B_K \end{bmatrix} y_k^m \\ &+ \begin{bmatrix} D_K \\ 0 \\ 0 \end{bmatrix} y_{k+1}^m + \begin{bmatrix} C_K E_K + F_K \\ F_K \\ E_K \end{bmatrix} r \end{aligned} \tag{5.49}$$

Again, we have y^m_{k+1} in the expression for $\hat{z}_{k+1}$, and introduce

$$\tilde{z}_k = \hat{z}_k - \begin{bmatrix} D_K \\ 0 \\ 0 \end{bmatrix} y^m_k \tag{5.50}$$

which yields

$$\begin{aligned} \tilde{z}_{k+1} &= \begin{bmatrix} H_K & C_K G_K & C_K A_K \\ 0 & H_K & C_K \\ 0 & G_K & A_K \end{bmatrix} \tilde{z}_k + \begin{bmatrix} H_K D_K + C_K B_K \\ D_K \\ B_K \end{bmatrix} y^m_k \\ &+ \begin{bmatrix} C_K E_K + F_K \\ F_K \\ E_K \end{bmatrix} r \end{aligned} \tag{5.51}$$

For u_k we obtain

$$u_k = \begin{bmatrix} I & 0 & 0 \end{bmatrix} \hat{z}_k = \begin{bmatrix} I & 0 & 0 \end{bmatrix} \tilde{z}_k + D_K y^m_k \tag{5.52}$$

which yields the following expression for the total controller:

$$\begin{aligned} \tilde{z}_{k+1} &= \tilde{A}_K \tilde{z}_k + \tilde{B}_K y^m_k + \tilde{E}_K r \\ u_k &= \tilde{C}_K \tilde{z}_k + \tilde{D}_K y^m_k \end{aligned} \tag{5.53}$$

where

$$\tilde{A}_K = \begin{bmatrix} H_K & C_K G_K & C_K A_K \\ 0 & H_K & C_K \\ 0 & G_K & A_K \end{bmatrix}; \quad \tilde{B}_K = \begin{bmatrix} H_K D_K + C_K B_K \\ D_K \\ B_K \end{bmatrix}$$

$$\tilde{C}_K = \begin{bmatrix} I & 0 & 0 \end{bmatrix}; \quad \tilde{D}_K = D_K; \quad \tilde{E}_K = \begin{bmatrix} C_K E_K + F_K \\ F_K \\ E_K \end{bmatrix}$$

In summary, we have shown that with no active constraints, the MPC controller with augmented state estimator can be expressed on discrete state-space form.

If we instead use the *a priori* estimate (Alternative 2), we get a different controller with other poles.

5.7 On the number of estimated disturbances

In this section, we will discuss the number of estimated disturbances (the dimension of $\widehat{d}_k$) necessary to avoid steady-state offset. According to Muske and Rawlings (1993), the number of elements in $\widehat{d}_s$ can not exceed the number of measurements if observability of the estimator shall be achieved. But what is the smallest number required?

We first have to specify clearer what ”no steady-state offset” means. If the process is perturbed by measurement noise and disturbances that change their value from time step to time step, the control will never be offset free, and no steady state will be obtained. Thus, we will consider the response when the noise, the model error and the disturbances are constant. (Alternatively, one may model noise, model error and disturbances as stochastic processes and consider a large number of experiments.)

Using as before the *a posteriori* estimate, the estimate of the measurement is

$$\widehat{y}_k^m = C^m \widehat{x}_k \tag{5.54}$$

In order to obtain a offset free steady state, the estimator must provide a correct state estimate for the MPC despite model errors, constant measurement errors or noise and a constant input disturbance at steady state. More precisely, the prediction of the measured output must equal the actual one:

$$\widehat{y}_s^m = y_s^m \tag{5.55}$$

We let index s to denote steady state.

We want to see what this condition means for our MPC and estimator, and first we extract the expression for x_{k+1} from the estimator equation (5.36):

$$\widehat{x}_{k+1} = \left(I - L_x C^m\right) A \widehat{x}_k + \left(I - L_x C^m\right) B u_k + \left(I - L_x C^m\right) E_d \widehat{d}_k + L_x y_{k+1}^m \tag{5.56}$$

where L_x is the upper part of L, corresponding to the dimension of $\widehat{x}_k$. At steady state

$$\widehat{x}_{k+1} = \widehat{x}_k = \widehat{x}_s \tag{5.57}$$

which yields

$$\left(I - \left(I - L_x C^m\right) A\right) \widehat{x}_s = \left(I - L_x C^m\right) B u_s + \left(I - L_x C^m\right) E_d \widehat{d}_s + L_x y_s^m \tag{5.58}$$

To find u_s we cannot use (5.13) or (5.20) since these include the actual state and disturbance vectors and not their estimates. Instead we apply (5.39) which yields for the steady-state control input

$$u_s = \left(I + K_u\right)^{-1} K \widehat{x}_s + \left(I + K_u\right)^{-1} K_d \widehat{d}_s + \left(I + K_u\right)^{-1} K_r r \tag{5.59}$$

We insert this into (5.58) and obtain

$$\begin{aligned}\Big(I - \left(I - L_x C^m\right) \Big(A + B \left(I + K_u\right)^{-1} K\Big)\Big) \widehat{x}_s = \\ \left(I - L_x C^m\right) \Big(B \left(I + K_u\right)^{-1} K_d + E_d\Big) \widehat{d}_s \\ + \left(I - L_x C^m\right) B \left(I + K_u\right)^{-1} K_r r + L_x y_s^m\end{aligned} \tag{5.60}$$

To simplify the notation we introduce the matrices

$$M_1 = (I - M_2 (A + M_3 K))^{-1} \tag{5.61}$$
$$M_2 = (I - L_x C^m) \tag{5.62}$$
$$M_3 = B (I + K_u)^{-1} \tag{5.63}$$

and obtain for the *a posteriori* state estimate

$$\widehat{x}_s = M_1 M_2 M_3 K_r r + M_1 M_2 (M_3 K_d + E_d) \widehat{d}_s + M_1 L_x y_s^m \tag{5.64}$$

Thus (5.54) and (5.55) yields

$$y_s^m = C^m \widehat{x}_s = C^m M_1 M_2 M_3 K_r r + C^m M_1 M_2 (M_3 K_d + E_d) \widehat{d}_s + C^m M_1 L_x y_s^m \tag{5.65}$$

which leads to the following matrix equation

$$C^m M_1 M_2 M_3 K_r r + C^m M_1 M_2 (M_3 K_d + E_d) \widehat{d}_s + (C^m M_1 L_x - I) y_s^m = 0 \tag{5.66}$$

In (5.66) the number of scalar equations equals the number of measurements (the number of rows in C^m). The only free variables are the elements of $\widehat{d}_s$. To obtain an offset free steady-state solution of the control problem there must exist a solution of (5.66), which implies that the number of elements in $\widehat{d}_s$ must be equal or greater than the number of measurements (independent of the size of the reference, r, and the number of control inputs, u). Thus, since the number of estimated disturbances cannot exceed the number of measurements (see above), we may conclude that:

> If offset free steady state shall be obtained, the number of estimated disturbances must be equal to the number of measurements.

This was, independently, also found by Muske and Badwell (2002), except that they do not distinguish between outputs to be controlled by the MPC and the measurements. Such a distinction proves to be useful in Chapter 7, where an experimental illustration is given.

Remark 2 *In the general case (5.66) cannot be used to determine $\widehat{d}_s$ given r and y_s^m. It will often be many $\widehat{d}_s$ that fulfills (5.66), and the value of $\widehat{d}_s$ will depend on the disturbance, measurement or model error that is present.*

Example 5.1 *For the neutralization example in Chapter 4 we use three measurements, and thus estimation of three disturbances is required. For the "original" MPC we only estimate two input disturbances, and the result is insufficient integral action, as expected. The modified MPC with three disturbance estimates gets full integral action.*

5.8 Closed loop model

The combination of the process model with the controller yields the closed loop model of the system. The process is expressed by the discrete model (5.1) and (5.2) which we repeat for the actual process, marked with a prime:

$$x_{k+1} = A'x_k + B'u_k + E'_d d_k \qquad k = 0, 1, 2, \dots \tag{5.67}$$

$$y_k = C'x_k \tag{5.68}$$

The vector of measurements, y^m, is expressed by

$$y^m = C'^m x_k + m_k \tag{5.69}$$

where C'^m is the matrix mapping from the states to the measured output vector and m_k is the measurement error. The controller is expressed by (5.44) or (5.39). u_k and y^m are then eliminated from the equations by combining the controller with (5.67), (5.68) and (5.69). We then get the following closed loop model (where we have omitted the tilde in the controller matrices from (5.39)):

$$x_{k+1} = (A' + B'D_K C^m)\, x_k + B'C_K z_k + B'D_K m_k + B'F_K r + E'_d d_k \tag{5.70}$$

$$z_{k+1} = B_K C'^m x_k + A_K z_k + B_K m_k + E_K r \tag{5.71}$$

$$y_k = C'x_k \tag{5.72}$$

We combine the process states, x_k, and the controller states, z_k into $\psi_k = \left[\begin{array}{cc} x_k^T & z_k^T \end{array} \right]^T$ and obtain the following model

$$\psi_{k+1} = \Omega\psi_k + \Delta r + \Lambda d_k + \Gamma m_k \tag{5.73}$$

$$y_k = \Psi\psi_k \tag{5.74}$$

where

$$\Omega = \left[\begin{array}{cc} A' + B'D_K C'^m & B'C_K \\ B_K C'^m & A_K \end{array} \right] \tag{5.75}$$

$$\Delta = \left[\begin{array}{c} B'F_K \\ E_K \end{array} \right] \tag{5.76}$$

$$\Lambda = \left[\begin{array}{c} E'_d \\ 0 \end{array} \right] \tag{5.77}$$

$$\Gamma = \left[\begin{array}{c} B'D_K \\ B_K \end{array} \right] \tag{5.78}$$

$$\Psi = \left[\begin{array}{cc} C' & 0 \end{array} \right] \tag{5.79}$$

One possible use of the closed loop model is to study the steady state of input steps. Introducing the time-shift operator $z = e^{Ts}$ where T is the time step, gives

$$y(z) = \Psi (zI - \Omega)^{-1} \Delta r(z) + \Psi (zI - \Omega)^{-1} \Lambda d(z) + \Psi (zI - \Omega)^{-1} \Gamma m(z) \tag{5.80}$$

The z-transform of a unit step is $z/(z-1)$. We apply a unit step on one of the inputs at a time. This may be formulated as

$$r(z) = \rho \frac{z}{z-1}; \quad d(z) = \delta \frac{z}{z-1}; \quad m \frac{z}{z-1} \tag{5.81}$$

where ρ, δ and μ are vectors with zeros except one element equal 1. From see e.g., (Phillips and Harbor, 1991, p. 452), we have

$$\lim_{k\to\infty} y_k = \lim_{z\to 1} (z-1)\, y(z) \tag{5.82}$$

and thus

$$\begin{aligned} \lim_{k\to\infty} y_k &= \lim_{z\to 1} z \left[\Psi (zI - \Omega)^{-1} \Delta\rho + \Psi (zI - \Omega)^{-1} \Lambda\delta + \Psi (zI - \Omega)^{-1} \Gamma\mu \right] \\ &= \Psi (I - \Omega)^{-1} \Delta\rho + \Psi (I - \Omega)^{-1} \Lambda\delta + \Psi (I - \Omega)^{-1} \Gamma\mu \end{aligned} \tag{5.83}$$

Thus the matrices $\Psi (I - \Omega)^{-1} \Delta$, $\Psi (I - \Omega)^{-1} \Lambda$ and $\Psi (I - \Omega)^{-1} \Gamma$ reveal the steady-state effect of a unit step in each of the inputs on each of the outputs. For example, element $(2, 3)$ in matrix $\Psi (I - \Omega)^{-1} \Lambda$ gives the steady-state effect of a unit step in disturbance no.3 on output no. 2 (when the controller is applied).

Example 5.2 *For the neutralization example in Chapter 4 we get for the "original" MPC with estimation of disturbances into first tank only (resulting in insufficient integral action):*

$$\Psi (I - \Omega)^{-1} \Delta = I + \begin{bmatrix} 7\cdot 10^{-8} & 2\cdot 10^{-6} & -8\cdot 10^{-9} \\ 4\cdot 10^{-8} & 1\cdot 10^{-6} & -5\cdot 10^{-9} \\ 6\cdot 10^{-8} & 2\cdot 10^{-6} & -8\cdot 10^{-9} \end{bmatrix} \tag{5.84}$$

$$\Psi (I - \Omega)^{-1} \Lambda = \begin{bmatrix} 8\cdot 10^{-7} \\ 1\cdot 10^{-6} \\ 1\cdot 10^{-4} \end{bmatrix} \tag{5.85}$$

$$\Psi (I - \Omega)^{-1} \Gamma = -I + \begin{bmatrix} -2\cdot 10^{-2} & 2\cdot 10^{-2} & -1\cdot 10^{-4} \\ -0.3 & 0.3 & -2\cdot 10^{-3} \\ 1 & -1 & 8\cdot 10^{-3} \end{bmatrix} \tag{5.86}$$

We see that we get significant deviations from set point when measurement errors are present. For example, a measurement error of 1 in measurement no.1 gives a *deviation from set-point of 1 in output 3 (element $(3,1)$ in the matrix in (5.86)). With disturbances in all outputs (and full integral action), we obtain*

$$\Psi\left(I-\Omega\right)^{-1}\Delta = I + \begin{bmatrix} -1\cdot 10^{-6} & -9\cdot 10^{-6} & 3\cdot 10^{-5} \\ -5\cdot 10^{-7} & -4\cdot 10^{-6} & 1\cdot 10^{-5} \\ -9\cdot 10^{-8} & -8\cdot 10^{-7} & 3\cdot 10^{-6} \end{bmatrix} \tag{5.87}$$

$$\Psi\left(I-\Omega\right)^{-1}\Lambda \approx \begin{bmatrix} 5\cdot 10^{-6} \\ 8\cdot 10^{-7} \\ -2\cdot 10^{-4} \end{bmatrix} \tag{5.88}$$

$$\Psi\left(I-\Omega\right)^{-1}\Gamma \approx -I + \begin{bmatrix} 4\cdot 10^{-8} & 8\cdot 10^{-10} & -6\cdot 10^{-12} \\ -2\cdot 10^{-7} & 8\cdot 10^{-8} & -6\cdot 10^{-10} \\ -3\cdot 10^{-5} & 1\cdot 10^{-5} & -8\cdot 10^{-8} \end{bmatrix} \tag{5.89}$$

and there are no significant steady-state errors.

5.9 Conclusions

In this paper, we have developed a state-space formulation for a MPC (for stable processes) without constraints or assuming that the constraints are not active. This state-space formulation of the controller makes it possible to use traditional tools to get insight into how the controller behave (see Chapters 4 and 7). The controller can be extended with tracking of inputs, and also include the state estimator necessary if not all the states are measured. To obtain offset-free tracking, estimates of the input disturbances are included in the estimator and in the calculation of steady state. We show that the length of this estimated disturbance vector must equal the number of measurements available to the estimator.

Finally, a closed loop state-space formulation is derived, assuming a state-space formulation of the process model.

References

Åström, K. J. (1970). *Introduction to Stochastic Control Theory*. Mathematics in Science and Engineering. Academic Press. New York San Francisco London.

Badgwell, T. A. and S. J. Qin (2002). Industrial model predictive control - an updated overview. *Process Control Consortium, University of California (UCSB), March 8 - 9, 2002*.

Bemporad, A., M. Morari, V. Dua and E. N. Pistikopoulus (1999). The explicit linear quadratic regulator for constrained systems. Technical Report AUT99-16. ETH, Zürich.

Bemporad, A., M. Morari, V. Dua and E. N. Pistikopoulus (2002). The explicit linear quadratic regulator for constrained systems. *Automatica* **38**, 3–20.

Halvorsen, I. (1998). Private communication.

Lee, J. H., M. Morari and C. E. Garcia (1994). State-space interpretation of model predictive control. *Automatica* **30**(4), 707–717.

Lundström, P., J. H. Lee, M. Morari and S. Skogestad (1995). Limitations of dynamic matrix control. *Comp. Chem. Engng.* **19**(4), 409–421.

Maciejowski, J. M. (2002). *Predictive Control with Contraints*. Prentice Hall. Harlow, England.

Muske, K. R. and J. B. Rawlings (1993). Model predictive control with linear models. *AIChE Journal* **39**(2), 262–287.

Muske, K. R. and T. A. Badwell (2002). Disturbance modeling for offset-free linear model predictive control. *J. Proc. Cont.* **12**, 617–632.

Phillips, C. L. and R. D. Harbor (1991). *Feedback Control Systems*. Prentice-Hall International, Inc.. London.

Qin, S. J. and T. A. Badgwell (1996). An overview of industrial model predictive control technology.. *Presented at Chemical Process Control-V, Jan 7-12, 1996, Tahoe, CA. Proceedings AIChE Symposium Series 316* **93**, 232 – 256.

Rawlings, J. B. (1999). Private communication. *European Control Conference, ECC'99, Karlsruhe, September 2, 1999.*

Paper 5.
Faanes , A. and S. Skogestad. Feedforward Control under the Presence of Uncertainty.Nordic Process Control Workshop 11, January 9-11, 2003, Trondheim
Accepted for publication in *European Journal of Control*.

Paper 6.

Faanes , A. and S. Skogestad. Offset free tracking with MPC under uncertainty: Experimental verification. Will be submitted to *Journal of Process Control*.

Chapter 8

Conclusions and directions for future work

8.1 Conclusions

8.1.1 Buffer tank design

The first part of this thesis treats the design of buffer tanks for control purposes. The basic idea is that the buffer tank shall handle disturbances in the frequency range where neither the (original) process nor the basic feedback control system dampens them sufficiently. **Chapter 2** addresses control-related design for neutralization plants. One or several mixing tanks are usually installed to smoothen disturbances that cannot be handled by the control system. Control theory has been applied to determine the required number of mixing tanks and their volumes, assuming strong acids and bases. Skogestad (1996) derived a minimum required total volume, $V_0 = qn\theta\sqrt{k_d^{2/n} - 1}$, where q is the flow rate, n is the number of tanks, θ is the delay in each tank and k_d is the scaled disturbance gain. With PI or PID control in each tank, we compute numerically the required volume for different tunings, and based on this we recommend a total volume of $V_{tot} = 2V_0$. We recommend identical tank sizes (in contrast to Shinskey (1973) and McMillan (1984)).

Chapter 3 extends the ideas from Chapter 2 to buffer tanks for all kind of processes. We first find the required buffer tank transfer function such that (with scaled variables) the gain from the disturbance to the output (including the process, the feedback loop, and the buffer tank) is less than 1. We realize this transfer function with either one or several mixing tanks (for quality disturbances) or a surge tank with “slow” level control (for flow-rate disturbances).

The work is based on (Skogestad, 1994). In the present work more “accurate” numerical and graphical methods have been included, and we have distinguished between the case when the feedback loop of the original plant is given (such that the sensitivity function, S, is known), and the case when it is not. Aspects regarding the buffer tank placement (before or after the process) are discussed. A literature survey and several process examples are included.

8.1.2 Feedforward control under the presence of uncertainty

In **Chapter 6** feedforward control under the presence of model uncertainty is discussed, and we define the *feedforward sensitivity functions*, S_{ff} and $S_{\text{ff},r}$ for the disturbance and the reference, respectively. For "ideal" feedforward controllers, we find that S_{ff} is equal to the relative error in G/G_d, and $S_{\text{ff},r}$ is equal to the relative error in G (except for the signs). A simple frequency domain analysis of $|S_{\text{ff}}|$ and $|S_{\text{ff},r}|$ shows for which frequencies feedforward control has a dampening effect when some common model errors are present (in gain, delay, dominant time constant, or a common combination of gain and time constant). The effect of more complex uncertainties is also discussed.

Feedforward is needed when the bandwidth, ω_B, of the feedback controller is below the frequency ω_d for which $|G_d|$ becomes less than one (with appropriate scaling). We must require $|S_{\text{ff}}(j\omega)| < 1$ in the frequency region between ω_B and ω_d, or if it is known, for all frequencies where the magnitude of the closed loop disturbance response, $|S(j\omega)\, G_d(j\omega)|$, is above 1.

To make the feedforward controller more robust, two methods have been proposed: 1) Adding a low-pass filter to the nominal design and 2) μ-optimal feedforward controller design.

8.1.3 Multivariable control under the presence of uncertainty

Serial processes are very common in the process industry, and in **Chapter 4** we use this class of processes to illustrate that a multivariable controller may actually use the two basic principles of "feedforward" action (based mainly on the model), and feedback correction (based mainly on measurements) simultaneously. The feedforward action may improve the performance significantly, but is sensitive to uncertainty, in particular at low frequencies. Therefore it is important to include efficient feedback control by using measurements late in the process, and to include integral action if offset-free steady-state is important.

In Chapter 4 we see that testing the process on a too idealistic process model may give the impression that the control is better than it actually is. This is confirmed by the experiments reported in **Chapter 7** (Model predictive control, MPC, is used for temperature control of a process with two tanks in series). Simulations may indicate that integral action is present and that disturbances are handled well, but unmodelled phenomena may give a poor result in the actual plant, also at steady state. It should be verified that integral action (feedback) is actually present and not an apparent effect of “ideal feedforward control”.

Estimates of input disturbances have been described in the literature as efficient for a quick response back to the desired steady state. The experiments confirm this provided that it is correctly done. Care must be taken when choosing which input disturbance estimates to include. It is not enough to estimate a disturbance or bias in the control input(s), even if the control input(s) are sufficient to control the process. The number of disturbance estimates must equal the number of measurements (as found theoretically in Chapter 5).

When designing the controller, one must also consider which of the outputs that are really important. If the number of inputs exceed the number of (important) outputs, one may either give set-points to other (less important) outputs, or one may let the controller bring some of the inputs back to ideal resting positions (Chapter 4).

As a tool to understand the model predictive controller (MPC), in **Chapter 5** we derive a (linear, discrete) state-space realization of a MPC controller (Muske and Rawlings, 1993) under the assumption that it is operated with no active constraints. A generalization to tracking of both inputs and outputs is derived. The final controller expression also includes a state estimator that is extended with input disturbance states. We have not found such a derivation of a MPC controller on state-space form elsewhere.

A direct result is that to obtain integral action with input bias estimation, it is required to include the same number of input biases as measurements. Combined with the process model (also on state-space form), the closed loop model is determined, and this can, for example, be used to check the steady-state solution.

The state-space MPC formulation has been applied (in **Chapters 4 and 7**) to obtain the frequency dependent gain for each controller channel and the magnitude of each of the elements in the sensitivity function matrix. The frequency dependent gain in each channel may give insight into how the controller utilizes each measurement and the magnitude of the control actions for each input. The steady-state behaviour can be seen from the low-frequency gains. But, often more than one channel in a row have high gain at low frequencies, and then it is difficult to interpret the result. It is then better to consider the elements of the sensitivity function matrix. An offset-free, steady-state control for a specific output requires that all the elements in the corresponding row have low gain at low frequencies.

8.2 Directions for further work

8.2.1 Serial processes: Selection of manipulated inputs and measurements

A general question related to control structure design is the choice of manipulated inputs and measurements. In Section 4.4 we study a serial process with three units, and with one candidate measurement (pH) and one candidate manipulated input (addition of a reactant) in each unit. To save installation and operational costs, one may omit one or more of the instruments or actuators. From Table 8.1 we see there are 49 possible combinations. Often one would like to monitor the final output, in which case the number of possible combinations is 28.

Table 8.1: Possible combinations of inputs and measurements for the example in Section 4.4. The last column is for the case with a measurement in the last unit.

Inputs	**Measurements**	**No of combinations**	**No of combinations pH in last tank used**
3	3	1	1
3	2	3	2
3	1	3	1
2	3	3	3
2	2	9	6
2	1	9	3
1	3	3	3
1	2	9	6
1	1	9	3
Total		**49**	**28**

In general, if one may choose from 1 to M inputs and from 1 to L possible measurements, the number of combinations is given by (Nett, 1989):

$$\sum_{m=1}^{M}\sum_{l=1}^{L}\binom{L}{l}\binom{M}{m} \tag{8.1}$$

In the example $M = L = 3$.

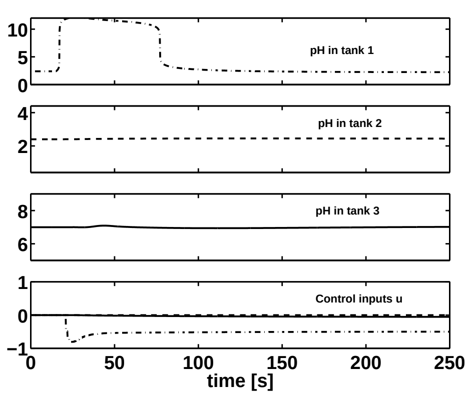
To illustrate the problem, we will here compare two realistic combinations from the example:

(1) pH measurement and reactant addition in tanks 1 and 3.

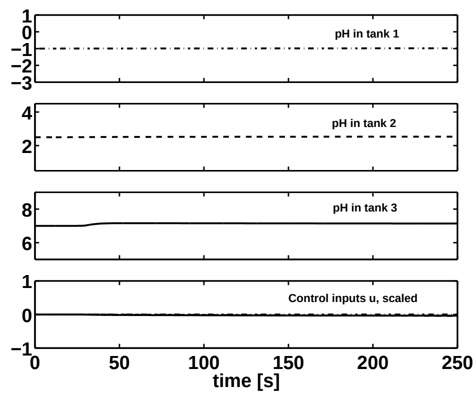
(2) pH measurement and reactant addition in tanks 2 and 3.

In both cases we keep the measurement and reactant addition in the last tank, since normally we want to measure the product quality, and the late reactant addition minimizes the delay in the last control loop. When we omit reactant addition to a tank, the steady-state pH will be the same as the inflow pH. From the simulations in Figure 8.1 we see that the resulting pH-response in the last tank is similar to the full instrumentation case (compare with Figure 4.7(a)). We see that the small deviation in the pH of last tank has a shorter duration for case 1 (with no instrumentation in tank 2). In case 2 (Figure 8.1(b)) the control inputs have not reached their steady state after $250\,\mathrm{s}$ (u_2 reaches -0.34).

The simulations indicate that with a multivariable controller one may omit the instrumentation in one of the three tanks.



(a) Instrumentation is removed from tank 2. pH set-point in tanks 1 and 2 are both set to 2.4.



(b) Instrumentation is removed from tank 1. At steady state the pH in tank 1 is equal to the influent pH. pH set-point is 2.5 in tank 2.

Figure 8.1: pH measurement in and reactant addition to two tanks only. $Q = I$ (not $Q = diag(100, 1, 1)$ as with full instrumentation)

Even if the final results for the two cases are similar, one may point out some important distinctions: In case 1, the total control loop includes all three tanks, whereas in case 2, only the two last tanks are included. In case 1, therefore the feedback loop from the last tank to the first is slower, but on the other hand, the "feedforward" controller element can be made close to "ideal", in contrast to case 2 (because of the delays).

A further analysis of the differences between different control configurations would be useful, both as a basis for recommendations to process designers, but also to get a deeper understanding of the process and the controller.

8.2.2 MIMO feedforward controllers under the presence of uncertainty

MPC vendors often offer feedforward control from measured disturbances (e.g., Honeywell (1999) and ABB (2003)), and therefore the study of multivariable feedforward controllers (from multiple measurements to multiple control inputs) has become more interesting. The theory of Chapter 6 covers multiple-input, multiple-output (MIMO) feedforward controllers, but the application of the theory to MIMO examples is still remaining.

8.2.3 Effect of model uncertainty on the performance of multivariable controllers

In this thesis we have studied some aspects of multivariable control under the presence of uncertainty. The basic idea is that a multivariable controller consists of both "feedforward" and feedback control elements, and these two types of elements respond differently to model error. We believe that a closer look into some of the following thoughts might be useful

- Identify elements or blocks of a multivariable controller that may degrade the performance, and redesign the controller to avoid this. In principle, it should be possible to identify such elements from the process model. One way to change (or remove) a controller element is to change the corresponding part of the model, for example, by removing the relationship in the model between the control input and the output.

  One method to investigate, is to consider feedforward elements (i, j) (either manually or automatically detected) and compute $\left|S'_{\mathrm{ff}i,j}\left(j\omega\right)\right|$ for expected model errors to determine the frequency range for which the controller element is effective. If there are any feedback element (e.g., (i, k)) that also controls output i, one may compute $\left|S_{i,k}\left(j\omega\right)\right|$ to see if this control element remove errors introduced by the feedforward branch. If the frequency range for which the feedforward element is effective is not overlapping with the range where it is needed, it is better to leave this controller element out. A simple example using this method has been presented (Faanes and Skogestad, 2003).

- Automatically detect feedforward control elements. Sometimes this is not an easy thing to do manually. One possible automatic method is (from the process model) to determine which outputs depend on which inputs when all the loops are closed. A control element from measurement y_i to manipulated variable u_j is feedforward control if 1) y_i is (closed loop) independent of u_j, 2) there is another output y_r which depends on u_j, and 3) there is another input that both y_i and y_r depend on. An output is *(closed loop) dependent of* an input if a change in the input leads to a change in the output (when all the loops are closed).

Due to other feedback loops or weak dependencies in the process, a control element may fail to fulfil the criteria for a feedforward controller, even though it has many similarities with feedforward control. This is seen in the case study in Chapter 4. For such cases it may be better to find an appropriate definition for the "degree of feedforward action" for a (total) controller or its control elements. This may for example be a number between 0 and 1 where 1 corresponds to pure feedforward control and 0 corresponds to pure feedback control.

8.2.4 MPC with integral action

There are many ways of obtaining integral action with mode predictive controllers (MPC). Output bias estimation is the most popular. Another is input disturbance or bias estimation (which we have used). Alternatively, integration may be introduced in the process model itself (for example by integrating the control input) with the disadvantage that the MPC optimization problem has grown, and also that the "new process" includes poles at the imaginary axis. For example, this means that the state-space formulation we derived in Chapter 5 must be modified since it only applies to stable processes.

We believe that a comparison of the different methods would be useful. The recent paper by Muske and Badwell (2002) is a good starting point. It is also interesting to consider the methods proposed for integral action for linear qudratic (LQ) controllers, since a criterion for obtaining offset-free steady state is that none of the constraints are active (Muske and Badwell, 2002).

References

ABB (2003). IndustrialIT Solutions for advanced process control and optimization. *Brochure*.

Faanes, A. and S. Skogestad (2003). Feedforward control under the presence of uncertainty. *Nordic Process Control Workshop 11, January 9-11, 2003, Trondheim*.

Honeywell, Hi-Spec Solutions (1999). RMPCT implementation course.

McMillan, G. K. (1984). *pH Control*. Instrument Society of America. Research Triangle Park, NC, USA.

Muske, K. R. and J. B. Rawlings (1993). Model predictive control with linear models. *AIChE Journal* **39**(2), 262–287.

Muske, K. R. and T. A. Badwell (2002). Disturbance modeling for offset-free linear model predictive control. *J. Proc. Cont.* **12**, 617–632.

Nett, C. N. (1989). A quantitative approach to the selection and partitioning of measurements and manipulations for the control of complex systems. *Presentation at Caltech Control Workshop, Pasadena, USA*.

Shinskey, F. G. (1973). *pH and pIon Control in Process and Waste Streams*. John Wiley & Sons. New York.

Skogestad, S. (1994). Design modifications for improved controllability - with application to design of buffer tanks. *AIChE Annual Meeting, San Francisco, Nov. 1994*.

Skogestad, S. (1996). A procedure for SISO controllability analysis - with application to design of pH neutralization processes. *Computers Chem. Engng.* **20**(4), 373–386.

Appendix A

Control Structure Selection for Serial Processes with Application to pH-Neutralization

Audun Faanes and Sigurd Skogestad

Extract from paper presented at European Control Conferance, ECC'99,
Aug.31-Sept.3, 1999, Karlsruhe, Germany.

Abstract

In this paper we aim at obtaining insight into how a multivariable feedback controller works, with special attention to serial processes.

Keywords: Control structure, Serial process, Multivariable control, Feedforward, Feedback

A.1 Example: pH neutralization

Neutralization of strong acids or bases is often performed in several steps. The reason for this is mainly that the pH control in one tank cannot be quick enough to compensate for disturbances (Skogestad, 1996). In (McMillan, 1984), an analogy from golf is used: the difficulty of controlling the pH in one tank is compared to getting a hole in one. Using several tanks, and smaller valves for addition of reagent for each tank, is compared to the easier task of reaching the hole with a series of shorter and shorter strokes.

In this example, control structures for neutralization of a strong acid by use of three tanks in series are discussed. The aim of the control is to keep the outlet pH from last tank constant despite changes in inlet pH or flow. This is obviously a serial process, since the flow goes from one tank to another. For each tank, the pH can be measured, and the reagent can also be added to each tank. Referring to Figure 4.1, the three units (i-1, i and i+1) correspond to the three tanks (1, 2 and 3).

To study this process we model each tank as described in (Skogestad, 1996). In each tank we model the excess H^+ concentrations, that is $c = c_{H^+} - c_{OH^-}$. This gives bilinear models, which are further linearized around a stationary working point so that methods from linear control theory can be used. We get two states in each process unit (tank), namely the concentration, c, and the level. The disturbances enter tank 1 only. We here assume that there is a delay of 5 seconds for the effect of a change in inlet acid or base flow or inlet concentration to reach the outflow of the tank, e.g. due to incomplete mixing, and a further delay of 5 seconds until the change can be measured. In the linear state space model these transportation delays are modelled by Padé-approximations of 4th order. There is assumed no further delay in the pipes between the tanks. We assume that the levels are controlled by the outflows using a P controller such that the time constant for the level is about 1/10 the time constants for the concentrations.

The volumes of the tanks were chosen to $13.6m^3$, the smallest possible volumes according to the discussion in (Skogestad, 1996). The acid inflow (disturbance) has $pH = -1$. The pH of the final product in tank 3 should be $pH = 7 \pm 1$, and we selected the set-points in tank 1 as 1.65 and in tank 2 as 3.8. The concentrations are scaled so that a variation of $\pm 1\, pH$ around these set-points corresponds to a scaled value of ± 1. The control inputs and the disturbances are also scaled appropriately. The linear model was used for multivariable controller design, while the simulations are performed on the nonlinear model.

A conventional way of controlling this process is to use local control of the pH in each tank using PID-controllers. Figure A.1 shows the response of pH in each tank when the acid concentration in the inflow is decreased from 10mol/l to 5mol/l. As expected from (Skogestad, 1996), this control system is barely able

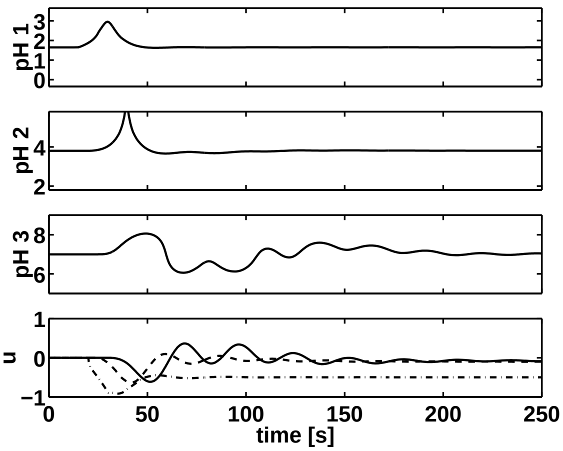


Figure A.1: With only local control, PID controllers must be agressively tuned to keep the pH in the last tank within 7 ± 1. (Disturbance in inlet concentration occurs at $t = 10$.)

to give acceptable control. However, the nominal response can be significantly improved with multivariable control.

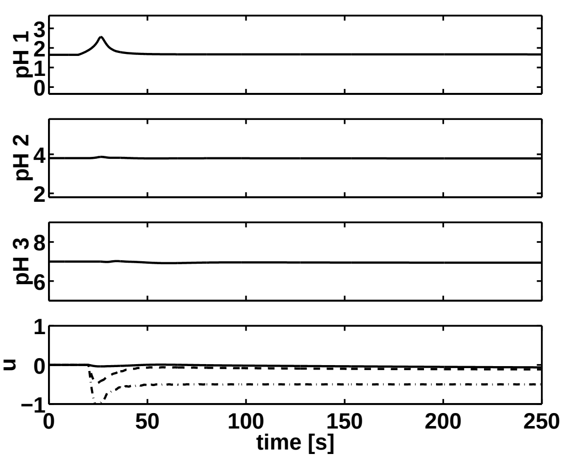


Figure A.2: A large improvement in nominal performance is possible with multivariable control. (Disturbance in inlet concentration occurs at $t = 10$)

Figure A.2 shows the response with a 3×3 multivariable $\mathcal{H}_\infty$ controller designed with performance weights on the outputs and on the control inputs in all tanks, and with composition into tank 1 as a disturbance. The main reason for the large improvement is the feedforward effect discussed in section 4.3.

The gain of the elements in the multivariable controller as a function of fre-

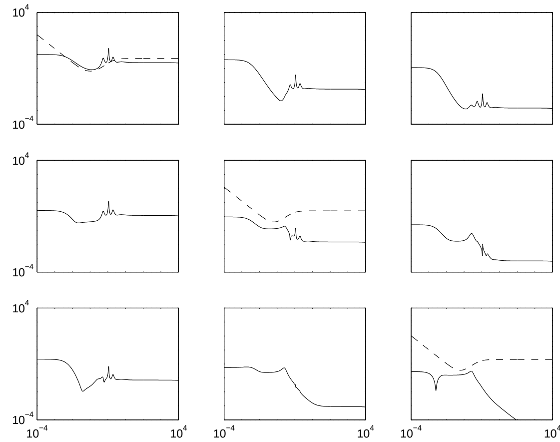


Figure A.3: Gain of the control elements of the original 3×3 $\mathcal{H}_\infty$ controller. (Local PID controllers are dashed.)

quency are shown in Figure A.3. The diagonal control elements are the local controllers in each tank, whereas the elements below the diagonal represent the ”feedforward” elements. From such plots we get an idea of how the multivariable controller works. For example, we see that the control input to tank 1 (row 1) is primarily determined by local feedback, while in tank 2 it seems that ”feedforward” from tank 1 is most decisive for the control input. In tank 3 the control actions are smaller. This is also seen from the simulation in Figure A.2 (the solid line in the plot of u).

We observe that none of the control elements have any integrators, even though the simulation in Figure A.2 show no steady-state offset. However, if some model error is introduced (20% reduced gain in tank 2 and 3), we do get a steady-state offset. Figure A.4 shows the start of the response, it finally ends up slightly above $pH = 8$. Local PID controllers give no such steady-state offset.

We subsequently redesigned the controller to get three integrators in the control loop shape (Figure A.5). The simulation in this case gives no steady-state offset. This illustrates one of the problems of the ”feedforward” control block, namely the sensitivity to static uncertainty. Simulations on the perfect model may lead the designer to believe that no integrator is necessary.

To study the feed forward effect separately, a $\mathcal{H}_\infty$ controller was designed using the measurement in tank 1, and control inputs in all tanks. The result is local control in tank 1 and feed forward from tank 1 to tanks 2 and 3. Simulation on the linear model gives the same result as for the 3×3 controller (Figure A.2), whereas nonlinear simulation gives steady-state offset due to static model error and no feedback in tanks 2 and 3.

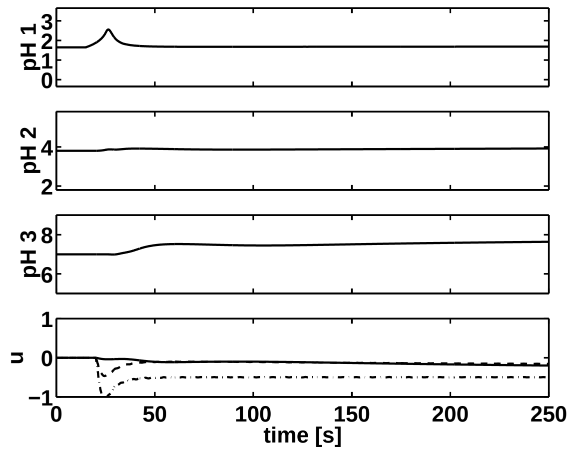


Figure A.4: Model error gives steady-state offset with original 3×3 controller.

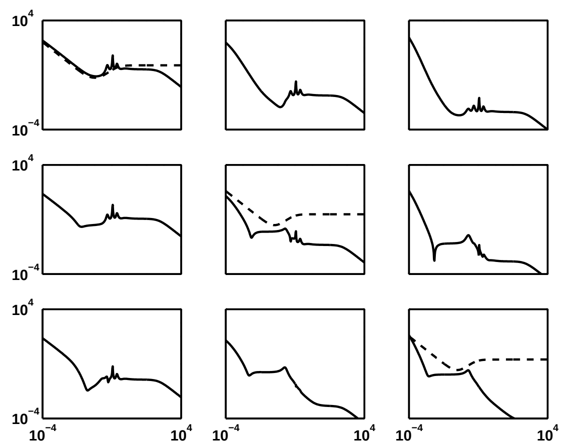


Figure A.5: Gain of the control elements of the redesigned 3×3 $\mathcal{H}_\infty$ controller. (Local PID controllers are dashed).

The effect of feedback from downstream tanks, i.e. the blocks above the diagonal from the discussion in section 4.3, is illustrated through the following simulations. We introduce a static measurement noise in tank 2 of 1 pH unit. In Figure A.6 we see the response for the process with local control with PID. We can see that the pH in tank 3 relatively quickly returns to a pH of 7. The problem is the control input in tank 3, which stabilizes at a level away from the point in the middle of the range (0), which we consider as the ideal resting position. Since we really are interested in the pH in only the last tank, we get two extra degrees of freedom, which can be used for resetting the control inputs of the last two tanks. Figure A.7 shows the simulation for the multivariable controller. Here we see that both the pH and the control input in tank 3 go to their desired values. The actual pH in tank 2 is increased to the correct value to obtain this. This illustrates that the elements above the diagonal in the multivariable controller give input resetting.

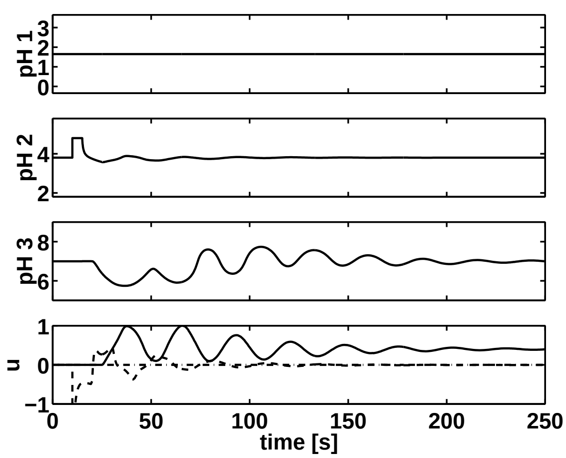


Figure A.6: Steady-state measurement noise in tank 2: Local control with PID do not bring the control input for tank 3, u_3, back to the ideal resting position. (u-plot: solid line.)

To summarize the example we can say that the multivariable controller gives significant improvements compared to local control based on PID. This is especially due to the feedforward effect, and with large model errors, the feedforward may lead to worse performance. Integral action is important in the controllers, even if the feedforward effect may give no stationary deviation for the nominal case. The inputs in the last two tanks are reset to their ideal resting position with the multivariable controller, because of the feedback from downstream tanks.

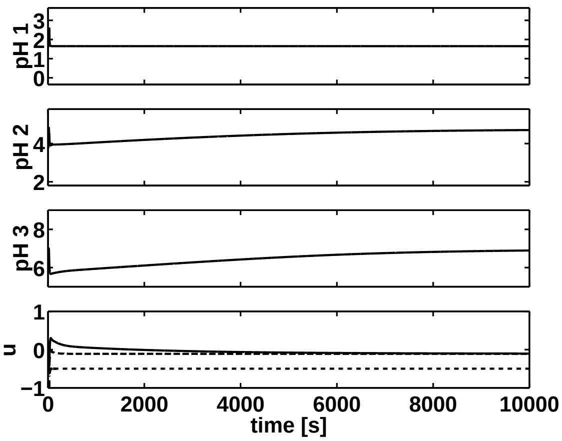


Figure A.7: Steady-state measurement noise in tank 2: The multivariable controller has built in input resetting, and brings u_3 back to the ideal resting position (u-plot: solid). Note that the timescale differs from the other plots.

A.2 Conclusion

An example of neutralization of a strong acid with base in a series of three tanks is used to illustrate some of the ideas in the paper. This process is obviously serial. The example illustrates that the multivariable controller yields significant nominal improvements compared to local control based on PID. But this is especially due to feedforward, and with model errors, the feedforward may in fact lead to worse performance. Integral action or strong gain in the local controllers at low frequencies is important to obtain no steady-state offset, even if the feedforward effect itself may nominally give no steady-state. Feedback to upstream tanks brings the inputs to their ideal resting positions, also when a wrong pH measurement give problems in an upstream tank. The example indicates that it is possible to get a good performance with careful use of a multivariable controller or a combination of local control, feed forward from tank 1 and input resetting.

In this study we used a $\mathcal{H}_\infty$-contoller, but similar results have also been found for a MPC controller.

References

McMillan, G. K. (1984). *pH Control*. Instrument Society of America. Research Triangle Park, NC, USA.

Skogestad, S. (1996). A procedure for SISO controllability analysis - with application to design of pH neutralization processes. *Computers Chem. Engng.* **20**(4), 373–386.

Appendix B

A Systematic Approach to the Design of Buffer Tanks

Audun Faanes and Sigurd Skogestad

Presented[1] at PSE'2000, July 16-21, 2000, Keystone, Colorado, USA, Supplement to Computers and Chemical Engineering, **24**, pp. 1395-1401

Abstract

Buffer tanks are often designed and implemented for control purposes, yet control theory is rarely used when sizing and designing buffer tanks and their control system. Instead, rules of thumb such as "10 min residence time" are used. The objective of this paper is to provide a systematic approach. We consider mainly the case where the objective of the buffer tank is to dampen ("average out") the fast (i.e. high frequency) disturbances, e.g. in flow and concentration, which cannot be handled by the feedback control system.

Keywords: Process control, process design, buffer tanks

[1] In the present version some corrections and clarifying modifications from the original text have been made. The most important error was step 3 in Table B.1. Some missing values have been provided for the examples, and equation (B.26) has been modified. A concluding section that was omitted due to spatial limitations has been included.

B.1 Introduction

The objective of this paper is to provide a systematic approach to the design of buffer tanks based on control theory. The background for this approach is that buffer tanks often are implemented for control purposes. Even so, control theory is rarely used when sizing and designing the tanks. Instead, rules of thumb are used.

Text books on chemical process design seem to agree that a half-full residence time of 5-10 minutes is appropriate for reflux drums and that this also applies for other buffer tanks. For tanks between distillation columns a half-full residence time of 10-20 minutes is recommended. ((Lieberman, 1983), (Sandler and Luckiewicz, 1987), (Ulrich, 1984), (Walas, 1987) and (Wells, 1986)). Sigales (1975) is more specific concerning what follows after the drum. None of these references give any justifications for their choice. (Watkins, 1967) gives a reflux drum volume dependent on instrumentation and labor factors (both related to operational use of the buffer tank), reflux and product rates, and a factor dependent on how well external units are operated. The method gives half full hold-up times from 1.5 to 32 min.

Design of vessels to dampen flow variations is presented by Harriott (1964) using a specification of outlet flow rate change given a certain step in inlet flow. This method has similarities with the one presented for flow variations in the present paper.

Another related class of process equipment is neutralization tanks. The main problems for this process are large and varying process gain and delays in the control loop. Design is described in (Shinskey, 1973) and (McMillan, 1984). Another design method and a critical review is found in (Walsh, 1993).

Zheng and Mahajanam (1999) find the necessary buffer tank volume by optimization and use it as a controllability measure.

A stated above, due to limitations in the control system, there is a limitation in frequencies above which the control system is not effective. The process itself must dampen the disturbances in this area. If it initially does not, addition of one or more buffer tanks is necessary. In this paper we present design methods for buffer tanks based on this fundamental understanding.

B.2 Transfer functions for buffer tanks

Consider the effect of a disturbance, d, on the controlled variable, y. The linearized model in terms of deviation variables may be written as

$$y(s) = G_d(s)\, d(s) \tag{B.1}$$

To illustrate the effect of the buffer tank, we express the dynamic model of the tank with the transfer function $h(s)$. The disturbances passes through the buffer tank (e.g. see Figure B.1), so that the process with a buffer tank may be expressed by

$$G_d(s) = G_{d_0}(s)\, h(s) \tag{B.2}$$

where $G_{d_0}(s)$ is the disturbance transfer function of the original plant, and $G_d(s)$ is the modified disturbance transfer function. A typical buffer tank transfer function is

$$h(s) = 1/(\tau s + 1) \tag{B.3}$$

Note that $h(0) = 1$ so that the buffer tank has no steady state effect.

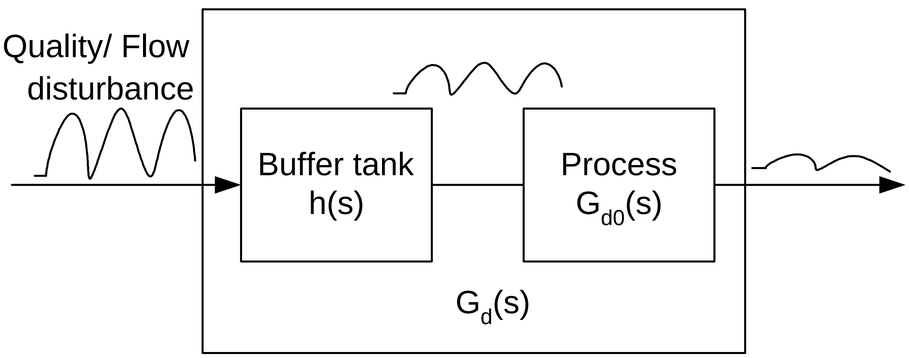


Figure B.1: Example of how a buffer tank dampens disturbances.

We consider a buffer tank with liquid volume $V\ [m^3]$, inlet flow-rate $q_{in}\ [m^3/s]$, outlet flow-rate q. Further we let c_{in} and c denote the inlet and outlet quality (concentration or temperature), respectively. A component or simplified energy balance for a perfectly mixed tank yields

$$d(Vc)/dt = q_{in}c_{in} - qc \tag{B.4}$$

In addition we have the total mass balance (assuming constant density):

$$dV/dt = q_{in} - q \tag{B.5}$$

B.2.1 Quality disturbance

For quality disturbances the objective of the buffer tank is to smoothen the quality response, $c(s) = h(s)\, c_{in}(s)$, so that the variations in c are smaller than in c_{in}.

Combining (B.4) and (B.5) yields $V\frac{dc}{dt} = q_{in}\left(c_{in} - c\right)$ and for a single buffer tank linearization yields

$$c\left(s\right) = \frac{1}{\frac{V^*}{q^*}s + 1}\left[c_{in}\left(s\right) + \frac{c_{in}^* - c^*}{q^*}q_{in}\left(s\right)\right] \tag{B.6}$$

where * denotes the nominal (steady state) values. Note that the dynamics of V (level control) have no effect on the linearized response of c. Furthermore for the case with a single feed stream $c_{in}^* = c^*$ and the dynamics of q_{in} have no effect on the response of c. In any case we find that the transfer function for quality is

$$h\left(s\right) = 1/\left(\tau_h s + 1\right) \tag{B.7}$$

where $\tau_h = V^*/q^*\ [s]$ is called the residence time (steady state). We can see that the buffer tank works as a first order filter. Similarly for n buffer tanks in series we have

$$h\left(s\right) = 1/\left(\frac{\tau_h}{n}s + 1\right)^n \tag{B.8}$$

where τ_h is the total residence time.

B.2.2 Flow rate disturbance

For flow rate disturbances the objective of the buffer tank is to smoothen the flow response, $q\left(s\right) = h\left(s\right)q_{in}\left(s\right)$. Note that we need to use a “slow” level controller, as tight level control yields $q \approx q_{in}$. Let $k\left(s\right)$ denote the transfer function for the level controller including measurement and actuator dynamics and the possible dynamics of an inner flow control loop. Then $q\left(s\right) = k\left(s\right)\left(V\left(s\right) - V_s\right)$, where V_s is the set-point for the volume. Combining this with the total mass balance (B.5) yields

$$q = \frac{k\left(s\right)}{s + k\left(s\right)}q_{in}\left(s\right) - \frac{sk\left(s\right)}{s + k\left(s\right)}V_s \tag{B.9}$$

The buffer tank transfer function is thus given by

$$h\left(s\right) = \frac{k\left(s\right)}{s + k\left(s\right)} = \frac{1}{\frac{s}{k(s)} + 1} \tag{B.10}$$

In this case we have more freedom in selecting $h\left(s\right)$ since we can select the controller $k(s)$. With a proportional controller $k\left(s\right) = K$, we get that $h\left(s\right)$ is a first order filter with $\tau = 1/K$. For a given $h\left(s\right)$ the controller is

$$k\left(s\right) = sh\left(s\right)/\left(1 - h\left(s\right)\right) \tag{B.11}$$

B.3 Controllability analysis

We here provide a review of some controllability results which are subsequently used for buffer tank design. We consider SISO (single input-single output) systems. Consider a linear process in terms of deviation variables

$$y(s) = G(s)\,u(s) + G_d(s)\,d(s) \tag{B.12}$$

Here y denotes the output, u the manipulated input and d the disturbance (including disturbances entering at the input which are frequently referred to as "load changes"). We assume throughout this paper that the model has been scaled such that expected disturbances make the magnitude of the elements of d lie within ± 1 for all frequencies and the requirement for the scaled output vector, y, is that the magnitude of each element in y shall lie between -1 and 1 for all frequencies, and u is scaled so that the manipulated input range corresponds to a variation of ± 1 in u.

Feedback control yields $u(s) = K(s)\,(y_s(s) - y(s))$, and from this we eliminate u to get

$$\begin{aligned} y(s) &= \frac{G(s)\,K(s)}{1 + G(s)\,K(s)} y_s(s) + \frac{G_d(s)}{1 + G(s)\,K(s)} d(s) \\ &= T(s)\,r(s) + S(s)\,G_d(s)\,d(s) \end{aligned} \tag{B.13}$$

y_s is the set-point, and $S(s)$ and $T(s)$ are the sensitivity function and the complementary sensitivity function, respectively. We ignore set-point changes and get the following expression for the effect of disturbances

$$y(s) = S(s)\,G_d(s)\,d(s) \tag{B.14}$$

Two different requirements must be fulfilled to get acceptable control performance. The first relates to the speed of response to reject disturbances. From (B.14) we see that to keep $|y| < 1$ when $|d| = 1$, we must require

$$|S(j\omega)\,G_d(j\omega)| \le 1; \quad \forall \omega \tag{B.15}$$

We define ω_B as the frequency where $|S(j\omega)| = 1$. At higher frequencies we cannot rely on feedback control for disturbance rejection, so that

$$|G_d(j\omega)| \le 1; \quad \omega \ge \omega_B \tag{B.16}$$

For acceptable performance and robustness we have the following maximum value of the bandwidth (Skogestad, 1999), (Skogestad and Postlethwaite, 1996):

$$\omega_B = 1/\theta_{\text{eff}} \tag{B.17}$$

where θ_{eff} is the effective delay. With PI or PID control we have (Skogestad, 1999):

$$\theta_{\text{eff}} = \theta + \tau_z + \frac{\tau_j}{2} + \sum_{i>j} \tau_i; \quad \begin{array}{l} j = 2 \text{ for PI} \\ j = 3 \text{ for PID} \end{array} \tag{B.18}$$

where θ is the delay, $\tau_z = 1/z$, where z is a right half plane zero, and τ_i is lag number i ordered by size so that τ_1 is the largest time constant. For more realistic PI controllers, ω_B must be reduced compared to (B.17). Ziegler-Nichols tuning gives $\omega_B = 1/(1.31\theta_{\text{eff}})$, while a more robust tuning (Skogestad, 1999) gives

$$\omega_B = 1/(2\theta_{\text{eff}}) \tag{B.19}$$

Note that (B.16) is only a necessary requirement, as (B.15) needs to be satisfied for $\omega < \omega_B$. In particular, (B.15) may impose additional requirements if G_d is of high order; this is discussed later.

In words (B.16) tells us that at sufficiently high frequencies the process must be "self-regulating". If (B.16) is not satisfied then we need to modify the process. One commonly used approach is to add buffer tanks as illustrated in Figure B.1, such that the "new" disturbance response becomes as in equation (B.2).

The second limitation relates to input constraints for disturbances, but will not be covered by this article.

B.3.1 Additional requirements due to high order G_d

As mentioned, (B.16) is only a necessary requirement as (B.15) needs to be satisfied also for $\omega < \omega_B$. To investigate this further we make the following approximation of the sensitivity function, $S(j\omega)$, with the loop transfer function, $L(j\omega)$ $(= G(j\omega)K(j\omega))$:

$$S(j\omega) = 1/(1 + L(j\omega)) \approx 1/L(j\omega) \tag{B.20}$$

Inserting this approximation into (B.15), we obtain

$$|G_d(j\omega)| \leq |L(j\omega)|; \quad \forall\omega \tag{B.21}$$

Now it may be difficult to have sufficiently high roll-off (slope) in the loop transfer function $L(s)$ to get $|L(j\omega)| \geq |G_d(j\omega)|$ at frequencies below the bandwidth (even though we satisfy it at the bandwidth). The problem is that a high roll-off in $L(s)$ yields a large phase lag, and we get instability problems. For reasonable robustness and performance we must have that the slope for $|L|$ is about -1 near the bandwidth ω_B. In this case it is difficult to make general formulas for the buffer tank design. Graphical or optimization based solutions are probably simplest. One particular case is studied later.

We can get a steeper slope around the bandwidth, however, with multiple control loops. E.g. with a series of n buffer tanks and control in each tank, the total slope of $|L|$ is $-n$ (even though it is -1 for each individual tank).

B.4 Quality variations

When the main source of disturbances are variations in the inflow quality (temperature or concentration) they may be smoothened by a mixing tank. With perfect mixing and a residence time of τ_h (h denotes hold-up), the outflow quality is roughly speaking the sliding mean of the input quality within a time window of length τ_h. The transfer function for one buffer tank is given by (B.7). We may also consider using a series of buffer tanks. For n equal tanks in series with a total residence time of τ_h, and total volume V, the transfer function is given by (B.8).

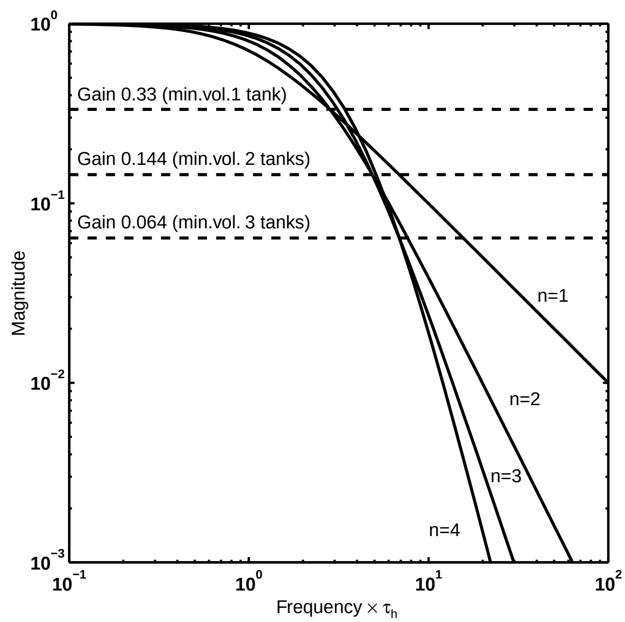


Figure B.2: Quality disturbance: Frequency responses for n tanks in series with total residence time τ_h, $h(s) = 1/\left(\frac{\tau_h}{n}s + 1\right)^n$.

In Figure B.2 we show the amplitude plot of $h(s)$ for $n = 1, 2, 3, 4$ equal tanks in series with a given total residence time τ_h. Physically, on the x-axis is shown the normalized frequency, $\omega\tau$, of the sinusoidal varying input concentration,

$$c_{in}(t) = c_{in,0}(t)sin(\omega t)$$

into the first tank, and on the y-axis is shown the normalized output concentration from tank n, $c_0/c_{in,0}$, where $c_{in,0}$ and c_0 denote the magnitude of the sinusoidal variations. Note that both axis are logarithmic.

At low frequencies, $\omega \ll 1/\tau$, we have $c_0/c_{in,0} \approx 1$, which means that slow sinusoidal variations are unaffected when they pass through the tanks. However, fast variations (with high frequencies) are dampened by the tanks which tend to "average out" the variations. At sufficiently high frequencies, $\omega \gg 1/\tau$, we find that $c_0/c_{in,0}$ (log-scale) as a function of frequency (log-scale) approaches a straight line. This follows because the high-frequency asymptote is $|h(j\omega)| = \tau^{-n} \cdot \omega^{-n}$ (in words, "the slope is $-n$" at high frequencies for n tanks in series). Thus, at high frequencies the use of many tanks is "better", in terms of providing more dampening for a given total volume. On the other hand, the frequency where the asymptote crosses magnitude 1 (its "break" or "corner" frequency) is $\omega = 1/\tau = n/\tau_h$, which is at a lower frequency when n is smaller, so at lower frequencies fewer tanks is better. This is also seen from the more exact plot in Figure B.2.

The plot may be used to obtain the total required volume of the buffer tanks if we at a given frequency specify the factor f by which we want to reduce the disturbance. The required "gain" of the buffer transfer function is then $1/f$ and we can read off $\omega\tau_h$ and with a given value of ω obtain the total residence time τ_h. Typically, the given frequency is the achievable closed-loop bandwidth of the feedback control system, $\omega = 1/\theta_{\text{eff}}$, and f is the value of G_{d0} at this frequency.

We see that one tank is "best" if we want to reduce the effect of the disturbance at a given frequency by a factor $f = 3 = 1/0.33$ or less; two tanks is "best" if the factor is between 3 and about $7 = 1/0.144$, and three tanks is "best" if the factor is between about 7 and $15 = 1/0.064$. The word "best" has been put in quotes because we here only consider the total combined volume of the tanks. In practice, there are several other factors that favor using as few tanks as possible; this includes the scaling law for cost (typically, cost scales with $V^{0.7}$), the cost of additional equipment like pipes, pumps, sensors, control systems, etc. as well as other controllability considerations (slope condition on L). Therefore, one would probably consider using only one tank also when we want to reduce the effect of the disturbance by a factor $f = 100$, even though in this case the volume of one tank is about 5 times larger than the total volume of two tanks, and more than 7 times larger than the total volume of three tanks (this is seen from Figure B.2 by reading off the value of $\omega\tau_h$ that corresponds to magnitude 10^{-2}).

To satisfy the necessary condition (B.16) we need to select $h\left(s\right)$ such that

$$|h\left(j\omega_B\right)|\,|G_{d_0}\left(j\omega_B\right)| \leq 1 \tag{B.22}$$

We introduce the factor by which the effect of the disturbance must be reduced

$$f = |G_{d_0}\left(j\omega_B\right)| \tag{B.23}$$

We must at least require $|h(j\omega_B)| = 1/f$. As mentioned this may be solved graphically using Figure B.2, but alternatively we can find the analytical solution from (B.8) and (B.17):

$$\tau_h > \theta_{\text{eff}} n \sqrt{f^{2/n} - 1} \tag{B.24}$$

For one tank and $f \gg 1$ we have the appropriate formula $\tau_h > f\theta_{\text{eff}}$. For $n \geq 2$ the use of (B.24) assumes that the total slope of $|L|$ around ω_B can be $-n$. This can be achieved with local quality control in each tank, e.g. for a neutralization plant, it must be possible to measure the concentration and automatically add a reactant in each tank.

To find the optimal number of tanks one must then take into account equipment, piping, control systems (each tanks may require a level controller), etc. as mentioned above. Normally the optimal number of tanks will not be large, so that the cost calculations has to be made for a limited number of cases.

Example B.1 *Consider mixing of two process streams, A and B as illustrated in Figure B.3. The concentration and flow rate of stream A are denoted c_A and q_A, and for stream B they are called c_B and q_B (c_A and c_B may also be temperatures). The two streams with total flowrate $q = 1m^3/s$, are mixed in a mixing tank of $1m^3$, and the concentration of the outlet flow is denoted c_0. The concentrations represent the difference between component 1 and 2. $c_A \geq 0$ since stream A never has less of component 1, whereas c_B is negative. The objective is to mix equal amounts of the components such that $c_0 = c_{0_1} - c_{0_2}$ is zero. This concentration c_0 is controlled by manipulating the flow rate of B. First we check if this controller, together with the mixing tank, is sufficient for suppressing disturbances in the concentration of stream A. Combination of component balance and total material balance gives the following model:*

$$\frac{dc_0}{dt} = \frac{1}{V}\left[(c_A - c)\, q_A + (c_B - c)\, q_B\right] \tag{B.25}$$

This model is linearized and scaled (as described in the controllability section). We require a variation in c less than $1/10$ of the variation in c_A. The scaled deviation variables are marked with a prime and we get the following model after Laplace transformation

$$c'(s) = \frac{1}{1+s}\left[10c'_A(s) - 20q'_B(s)\right] \tag{B.26}$$

where we have assumed constant c_B. We study concentration disturbances, leading to $G_{d_0}(s) = 10/(1+s)$ and further $G_0(s) = -20/(1+s)$. Mainly due to the measurement, the control loop has an effective delay of $1s$. With a robust controller tuning, (B.19) gives a bandwidth of $0.5 rad/s$.

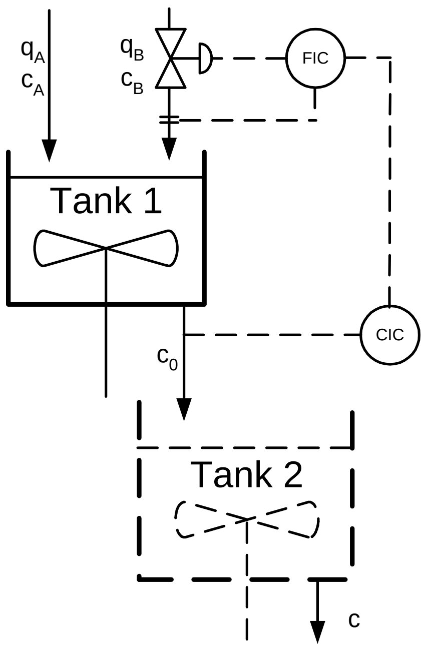


Figure B.3: Extra buffer tank for a mixing process. Concentration is controlled by manipulating flow rate of stream B. Nominal data: $q_A = 1\,\mathrm{m}^3\,/\,\mathrm{s}$, $q_B = 0\,\mathrm{m}^3\,/\,\mathrm{s}$, $c_A = 0\,\mathrm{mol}\,/\,\mathrm{m}^3$, $c_B = -2\,\mathrm{mol}\,/\,\mathrm{m}^3$, $c_0 = 0\,\mathrm{mol}\,/\,\mathrm{m}^3$. Range, used for scaling: Expected variations in c_A: $\pm 1\,\mathrm{mol}\,/\,\mathrm{m}^3$. Range for q_B: $\pm 1\,\mathrm{m}^3\,/\,\mathrm{s}$. Allowed range for c: $\pm 0.1\,\mathrm{mol}\,/\,\mathrm{m}^3$.

$|G_0(j\omega)|$ and $|G_{d_0}(j\omega)|$ are shown in Figure B.4 (dashed lines). We see that $|G_0| > |G_{d_0}|$ for all frequencies, so that input constraints pose no problems in this case. In the figure the bandwidth frequency, ω_B, is also marked. We see that $|G_{d_0}| > 1$ at frequencies above the bandwidth, so a standard (robust) control system is not sufficient to fulfil the requirements on the outlet concentration. To solve this problem, we may either improve the control system (e.g. feedforward control), increase the volume of the mixing tank, or install an extra buffer tank. In this case we assume that the latter alternative is the best, and introduce a new tank after the mixing tank (dashed in Figure B.3). We see from Figure B.4 that the gain must be reduced with 10 at the bandwidth ($f = 10$), and obtain from (B.24) ($n = 1$) a required residence time of the buffer tank of $20s$, corresponding to a volume of $V = q\tau = 20\,\mathrm{m}^3$. The modified disturbance transfer function gain, $|G_d|$, is shown with a solid line in Figure B.4. The slope is -1 or smaller below the bandwidth, so that we need not consider the problem discussed in section B.3.1. $|L(j\omega)|$ is plotted (dash-dotted) to illustrate this ($|G_d| < |L|$). $|SG_d|$ is below 1 for all frequencies (dashed). Figure B.5 shows the response of a unit step in concentration of stream A with (solid) and without (dashed) the extra buffer tank. We see that it is kept below 0.1 with the extra buffer tank present.

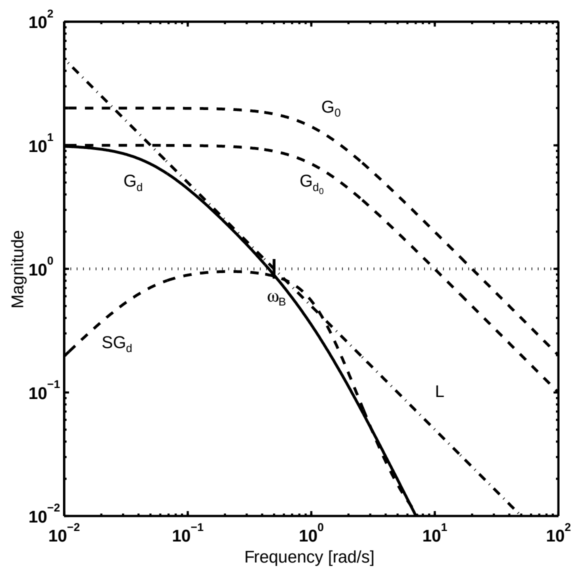


Figure B.4: With an extra buffer tank, $|G_d|$ is brought below 1 for all frequencies above the bandwidth.

If the slope of $|G_d|$ is steeper than the slope of $|L|$, τ_h is too optimistic. We will however analyze one case. We assume $|G_{d_0}|$ has slope -1 so that $|G_d|$ has slope

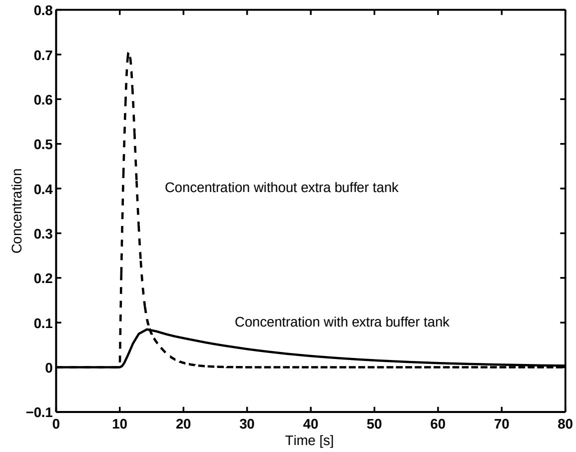


Figure B.5: With an extra buffer tank the outlet concentration is kept within 0.1 from set-point despite a unit step in disturbance. This is not the case without the extra buffer tank.

-2 above the frequency $1/\tau_h$, where τ_h is the buffer tank residence time. Further we assume that $|L|$ has slope -1 near the bandwidth and that it increases to -2 due to an integrator in the controller below $\omega = 1/\tau_I$, where τ_I is the integral time. A robust choice of τ_I is $8\theta_{\text{eff}}$ (Skogestad, 1999). Using geometry it is easy to show that in this case $\tau_h = 8f\theta_{\text{eff}}$. Compared to (B.24) for one tank we see that the residence time for this case is increased by a factor of 8.

Example B.2 *Consider the process from example B.1, modified so that the measurement delay is $0.1s$, the volume of the first tank is $5m^3$ and the variation requirements for the outlet concentration is 0.01. The concentration in the first tank is controlled with a robust PI controller (Skogestad, 1999). In this case the slope of $|G_d(j\omega)|$ is -2 around the bandwidth, and (B.24) leads to a residence time of $0.39s$, which is insufficient. In Figure B.6 a residence time of $\tau_h = 8f\theta_{\text{eff}} = 3.2s$ is applied. The method uses asymptotes, and we see that $|G_d(j\omega)|$ is just touching the asymptote of $|L(j\omega)|$. $|L(j\omega)|$ itself is a distance above $|G_d|$ so the result here is slightly conservative. By optimization one find a minimum residence time of $2.4s$ required to fulfil (B.21) for this controller tuning.*

B.5 Flow variations

By exploiting the volume of the buffer tank, flow variations in the outflow may be dampened using a slow level control. The outflow will then be dependent on the

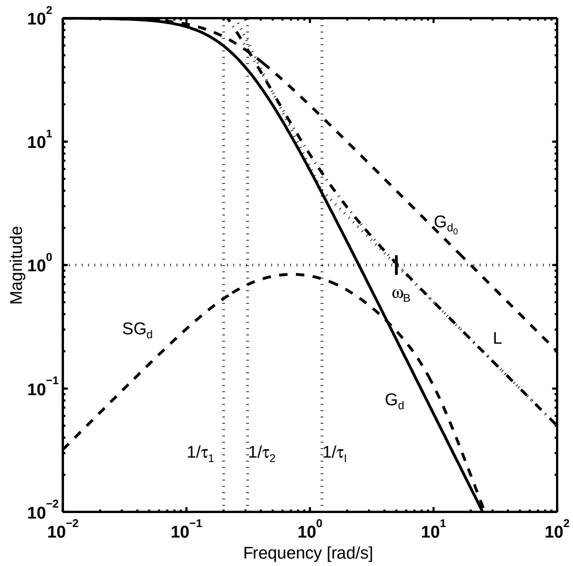


Figure B.6: With a residence time of $\tau_h = 8f\theta_{\text{eff}}$ in the second tank, $|L(j\omega)| > |G_d(j\omega)|$ for all frequencies, and disturbances are rejected.

chosen controller. Denote the tank volume V $[m^3]$ and the inlet and outlet flow-rates q_{in} and q respectively. The transfer function for the buffer tank is then given by (B.10). Compared to the quality disturbance case, we have more freedom in selecting h, since we can select the controller $k(s)$. But the level will vary, so the size of the tank must be chosen so that the level remains between its limits. The volume variation is given by $V(s)/q_{in}(s) = 1/(s + k(s))$, and combination with (B.11) yields:

$$V(s)/q_{in}(s) = (1 - h(s))/s \tag{B.27}$$

which is used to find the required tank volume. The tank size design consists of the following steps:

(1) Select $h(s)$ such that if has the desired shape, that is such that (B.16) is satisfied.

(2) Find the corresponding controller from (B.11) (is it realizable?)

(3) Find the largest effect of q_{in} on V from (B.27) (usually at steady state, $s = 0$).

(4) Obtain the required total volume from the expected range of q_{in} (denoted Δq_{in}).

In table B.1 we have applied the method for first and second order filtering.

Table B.1: Flowrate disturbance: Procedure for buffer tank design applied to first and second order filtering

Step	1st order	2nd order
1. Desired $h(s)$	$1/(\tau_1 s+1)$	$1/(\tau_2 s+1)^2$
2. $k(s)$ from (B.11)	$1/\tau_1$	$\frac{1}{2\tau_2}\frac{1}{\frac{\tau_2}{2}s+1}$
3. $V(0)/q_{in}(0)$ from (B.27)	τ_1	$2\tau_2$
4. V_{tot}	$\tau_1 \Delta q_{max}$	$2\tau_2 \Delta q_{max}$

B.5.1 First-order filtering

With $h(s) = \frac{1}{\tau_1 s+1}$ the required controller is a P-controller with gain $K_c = 1/\tau_1$. From (B.27), $V(s) = \frac{\tau_1}{\tau_1 s+1} q_{in}(s)$. The maximum value of this transfer function occurs at low frequencies ($s = 0$), and the required volume of the tank is $V_{tot} = \tau_1 \Delta q_{max}$. Adding a slow integral action to the controller will not affect these results considerably. Such an integral action will reset the volume to its nominal value. This is not always desired, however. If e.g. q_{in} is at its maximum, we may want the volume to stay at a large value to anticipate a possible large reduction in q_{in}.

B.5.2 Second-order filtering

With $h(s) = \frac{1}{(\tau_2 s+1)^2}$ we get from (B.11) that the required controller is a lag

$$k(s) = \frac{1}{2\tau_2} \frac{1}{\frac{\tau_2}{2}s+1} \tag{B.28}$$

and from (B.27) the response of the volume deviation is

$$V(s) = 2\tau_2 \frac{(\tau_2/2)s+1}{(\tau_2 s+1)^2} q_{in}(s) \tag{B.29}$$

This has its largest value equal to $2\tau_2$ at low frequencies ($s = 0$), and the required volume is $2\tau_2 \Delta q_{max}$.

B.6 Conclusions

The objective of the control system is to counteract disturbances. However, the maximum achievable control bandwidth is approximately equal to the inverse of the effective process delay, i.e. $\omega_B \approx 1/\theta_{eff}$. For “fast” disturbances, above the bandwidth frequency, one must rely on the process itself, including any buffer tanks, to dampen the disturbances. The requirement is that the effect of disturbances on the controlled variable (usually concentration), should be less than 1 (in scaled units) at frequencies above the bandwidth. Specifically, if the magnitude of the original disturbance transfer function $G_{d0}(s)$ is larger than 1 at frequencies above the bandwidth, then we must add one or more buffer tanks, with overall transfer function $h(s)$, such that $G_d = G_{d0}h$ is less than 1. In the paper we present design methods for sizing buffer tanks based on this fundamental insight.

The two fundamentally different sources of disturbances are variations in flowrate and variations in quality (concentration, temperature). Quality variations are dampened by mixing, and it may be adventageous to use several smaller rather than a single large buffer tank. Figure B.2 shows how $h(s)$ depends on the number of tanks n and total residence time τ_h. If we define f as the value of $|G_{d0}|$ at the bandwidth frequency ω_B, then the design objective is that $|h|$ should be less than 1/f at this frequency, and we derive in (B.24) the required value for τ_h. The volume in each buffer tank is then $V = q\tau_h/n$ where q is the total flowrate. If the resulting slope of G_d around the bandwidth is steeper than -1, then we need to increase the volume or add local feedback loops. The design method is illustrated in Examples B.1 and B.2.

Flowrate variations are dampened using a slow level controller $k(s)$ in the buffer tank, and there is no advantage of using several tanks as we may include dynamics in $k(s)$. Table B.1 gives a design procedure for flowrate disturbances.

In conclusion, buffer tanks are designed and implemented for control purposes, yet control theory is rarely used when sizing and designing buffer tanks and their control system. In this paper we have presented a systematic approach for design of buffer tanks to dampen disturbances in quality and flowrate.

References

Harriott, P. (1964). *Process Control*. McGraw-Hill. New York.

Lieberman, N. P. (1983). *Process Design for Reliable Operations*. Gulf Publishing Company. Houston.

McMillan, G. K. (1984). *pH Control*. Instrument Society of America. Research Triangle Park, NC, USA.

Sandler, H. J. and E. T. Luckiewicz (1987). *Practical Process Engineering*. McGraw-Hill Book Company. New York.

Shinskey, F. G. (1973). *pH and pIon Control in Process and Waste Streams*. John Wiley & Sons. New York.

Sigales, B. (1975). How to design reflux drums. *Chem. Eng.* **82**(5), 157–160.

Skogestad, S. (1999). Lecture notes for the course Process control (Tillegg til fag 52041 Prosessregulering, in Norwegian).

Skogestad, S. and I. Postlethwaite (1996). *Multivariable Feedback Control*. John Wiley & Sons. Chichester, New York.

Ulrich, G. D. (1984). *A Guide to Chemical Engingeering Process Design and Economics*. John Wiley & Sons. New York.

Walas, S. M. (1987). Rules of thumb, selecting and designing equipment. *Chem. Eng.* **94**(4), 75–81.

Walsh, S. (1993). Integrated Design of Chemical Waste Water Treatment Systems. PhD thesis. Imperial College, UK.

Watkins, R. N. (1967). Sizing separators and accumulators. *Hydrocarbon Processing* **46**(11), 253–256.

Wells, G. L. (1986). *The Art of Chemical Process Design*. Elsevier. Amsterdam.

Zheng, A. and R. V. Mahajanam (1999). A quantitative controllability index. *Ind. Eng. Chem. Res.* **38**, 999–1006.