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Ikke-kommutative Sobolev-rom

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Non-commutative Sobolev Spaces

Master's Thesis

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Abstract

Sobolev spaces over the Moyal plane and non-commutative tori are introduced via tools from non-commutative geometry. The construction is based on the Laplacian associated to a Hermitian structure we define on these spaces. The main result is based around a relation between this Laplacian and the radial-symmetric weights $v_s(z) = (1 + |x|^2 + |w|^2)^{s/2}$ applied to localization operators, giving an identification between the non-commutative Sobolev spaces and modulation spaces. This thesis presents a new way of considering localization operators, a possible pathway for future research in the field. The work here is a continuation of the results on the relation between non-commutative geometry and time-frequency analysis.

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Summary

First, we introduce some prerequisites on Hilbert C^*-modules, the definition of Morita equivalence for C^*-algebras and the construction of equivalence bimodules along with some consequences. On these bimodules we define some connections and a complex structure. Then we move on to time-frequency analysis, culminating in the definition of $M_m^{p,q}(\mathbb{R}^d)$, weighted modulation spaces, and a description of Feichtinger's algebra $S_0(\mathbb{R}^d)$ and its weighted variants $M_v^1(\mathbb{R}^d)$. We also introduce basic aspects of frame theory, especially Gabor frames.
These structures are used to construct finitely generated projective modules over the Moyal plane and the Moyal plane and a differentiable structure on the non-commutative torus and Moyal plane, respectively. By using the localization operator

$$A_m^{\phi_1,\phi_2} f = \iint_{\mathbb{R}^{2d}} m(z) V_{\phi_1} f(z) \pi(z) \phi_2 \, dz, \quad m \text{ a weight function,}$$

we try to establish the same structure for the weighted case, and succeed with some slight modification. With motivation grounded in the classical Sobolev-theory, we seek to define a non-commutative analogue of $\mathcal{W}_s^{p,q}(\mathbb{R}^d)$. These spaces are shown to have the usual properties, including the compact, bounded embeddings $\mathcal{W}_{s+t}^{p,q}(\mathbb{R}_\theta^d) \hookrightarrow \mathcal{W}_s^{p,q}(\mathbb{R}_\theta^d)$. Differential operators on these spaces are then related to Connes pseudodifferential calculus and Higson's theory of abstract Sobolev spaces.

Oppsummering

Først blir det essensielle av forkunnskaper introdusert. Dette inneholder den grunnleggende Hilbert C^*-modulteorien, definisjonen og konstruksjonen av Morita-ekvivalenser samt noen konsekvenser av dette. På disse bimodulene definer vi derivasjoner og sammenhenger, som lar en kompleks struktur bli etablert. Så tar vi for oss tids-frekvens analyse, som ender i definisjonen av $M_m^{p,q}(\mathbb{R}^d)$, vektede modulajonsrom, og en komplett karakterisering av den vektede versjonen av Feichtingers algebra $S_0(\mathbb{R}^d)$ som vindusklassene $M_v^1(\mathbb{R}^d)$. Vi introduserer ogs rammeteori (frame theory), spesielt Gaborrammer.
Disse strukturene er brukt til å konstruere endelig genererte projektive moduler over Moyal-planet en differensiabel struktur på den ikke-kommutative torusen og Moyal-planet. Ved å bruke lokalisajonsoperatoren

$$A_m^{\phi_1,\phi_2} f = \iint_{\mathbb{R}^{2d}} m(z) V_{\phi_1} f(z) \pi(z) \phi_2 \, dz, \quad m \text{ en vektfunksjon,}$$

prøver vi å etablere den samme strukturen for det vektede tilfellet. Vi lykkes i dette, med noen små modifikasjoner. Med motivasjon grunnet i klassisk Sobolev-teori, definerer vi en ikke-kommutativ analog av $\mathcal{W}_s^{p,q}(\mathbb{R}^d)$. Vi viser at disse rommene har de vanlige egenskapene, inkludert de begrensede, kompakte embeddingene $\mathcal{W}_{s+t}^{p,q}(\mathbb{R}_\theta^d) \hookrightarrow \mathcal{W}_s^{p,q}(\mathbb{R}_\theta^d)$. Differensialoperatorer på disse funksjon-

srommene er da relatert til Connes pseudedifferensiale kalkulus og Higsons teori om abstrakte Sobolev-rom.

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Contents

1 Introduction

The theory of modulation spaces originates from Feichtinger's study of the algebra $S_0(\mathbb{R}^n)$ in the 1980's [14]. Since then, both Feichtinger and K. Gröchenig have been the major proponents of the study, the latter being the author of the standard introductory text on time-frequency analysis and modulation spaces, the book "Foundations of Time-Frequency Analysis" [16]. In essence, it is the study of functions in both time and frequency. The key component of this study is the Short-Time Fourier Transform (the STFT), given by

$$V_g f(x,w) = \int_{\mathbb{R}^n} f(t)e^{-2\pi i w\cdot t}\overline{g(t-x)}\ dt, \text{ for some window-function } g. \tag{1}$$

A central question is about the significance and suitability of the function g. For instance choosing the characteristic function over a compact set Q, will, for every $x \in \mathbb{R}^n$ give the Fourier transform of f restricted to $Q - x$.

Since the information carried in two points of the time-frequency plane, or phase-space as physicists calls it, close together is very similar, a natural question to ask is whether it is possible to discretize the integral into a sum so that one ends up with something more manageable. This question has motivated the field of Gabor analysis and has been shown to be linked with the well-studied Heisenberg group. Applications include audio-representation, scores can be seen as a representation of music into a time and a frequency component, or the study of function spaces on the non-commutative torus.

Gabor analysis and Gabor frames deal with the suitability of the window function g, called the Gabor atom, and the aptness of the following representation of functions.

$$f(t) = \sum_{k,l} a_{k,l}[e^{2\pi i l\cdot t} g(t-\theta k)] \tag{2}$$

with some suitable sampling frequency θ. The systems in the form $\{e^{2\pi i l\cdot t}g(t - \theta k),\ k,l \in >\}$, are overcomplete and non-orthogonal in general. A central result of Gabor analysis is the Balian-Low Theorem, discovered independently by Balian [3] and Low [13]. For $L^2(\mathbb{R}^d)$, it states that any g that gives a Riesz-basis is either not sufficiently smooth or does not decay rapidly enough:

$$\left(\int_{\mathbb{R}^d} |tg(t)|^2\ dt\right)\left(\int_{\mathbb{R}^d} |w\hat{g}(w)|^2\ dw\right) = +\infty, \tag{3}$$

so is not very suitable for our work.

The study of functions on the torus and the real line is well-developed. We would like to develop a similar theory of function spaces for two classes of non-commutative spaces: The Moyal plane and the non-commutative torus.

We start on the Moyal plane, the plane defined by the Moyal identity, or the Orthogonality Lemma in time-frequency analysis,

$$\langle V_{g_1} f_1, V_{g_2} f_2\rangle = \langle f_1, f_2\rangle\overline{\langle g_1, g_2\rangle}. \tag{4}$$

Using this identity and the work of M. A. Rieffel we introduce a Morita equivalence between functions on this plane and $\mathbb{C}$ with the Schwartz functions as connecting module. We also introduce the notion of differentiation on the Moyal plane, which are lifted from the functions it acts on. We find that this represents the Schwartz functions as a finitely generated projected module, simplifying our work immensely. This concept also gives us, for free, the existence of some suitable g's with which to represent our functions, so that we only need to show the correctness of the frames [23].

The same process can be done on the non-commutative torus. We generate the torus by two unitary operators and let it act on the Schwartz functions. Similar to the Moyal plane case we construct a Morita equivalence so that we can introduce differentiation and the projective module-structure. This gives us a connection between the non-commutative torus and the Moyal plane. The importance of this result is precisely that it allows us to connect the continuous integrals of the Moyal plane with the discrete sums of the torus. Since we here are choosing a discrete sampling rate it is more difficult to find suitable frames.

Lastly we introduce localization operators (also known as anti-Wick operators, or Toeplitz operators), important in many numerical aspects. A key observation is that localization operators are related to the Moyal plane viewed as a non-commutative manifold. This motivates our definition of a weighted inner product coinciding with localization operators. By repeating the process of our previous cases we can give the plane some natural differentiable structure with similar, but more restrictive, module properties. This allows us to equate the polynomial weights of the operators to differentiation, and it is this structure that gives the main result of this thesis.

Sobolev spaces are defined in various settings, for instance as weighted L^p-spaces in the frequency domain. One might view the frequency domain as the group C^*-algebra of the real line or the integers. With this intuition we generalize the definition to non-commutative Sobolev spaces $\mathcal{W}_s^{p,q}$. We show some interesting embeddings and relations. Our space $W_s^{p,q}$ has previously been studied by P. Boggiatto and J. Toft in [6] as a generalized Shubin-Sobolev space, where they construct it slightly differently. There are many interesting links to other fields of study, and we mention explicitly this structure as a non-classical example of Higson's abstract pseudodifferential algebra [19], and the compatibility with Connes pseudodifferential calculus. This last compatibility is of significant importance, as it connects our structure with the general theory of pseudo-differential operators and therefore opens up new avenues for future study.

The non-commutative Sobolev spaces have been studied previously, e.g in a recent paper by F. Sukochev & and D. Zanin [30], but these description has been as a space of operators. This thesis however, considers function spaces instead, allowing a more intuitive understanding.

We mention the possibility of a similar structure on the non-commutative torus, which is a definite possibility if one is careful in choosing suitable functions. Here we need to take care in choosing both weight functions and suitable sampling rates. The theory behind the discrete case is less developed than the continuous. However, this seems like a relatively simple extension of our work, and a positive result is surely to appear soon.

The thesis is organized as follows: Section 2 deals with prerequisites and is partitioned into an introduction to Hilbert C^*-modules, modulation spaces and frame theory, a reader familiar with these branches of mathematics can skip it without worrying too much. Section 3 and 4 deals with the construction of the modules and differentiable structure of the Moyal plane and the non-commutative torus respectively and Section 5 introduces the localization operator and ends with the final definition of the non-commutative Sobolev spaces and some properties and application of these spaces.

2 Prerequisites

2.1 Hilbert C^*-modules

We will start the prerequisites by defining Hilbert C^*-modules and presenting the most central results about them. We will mostly be following the exposition given in Landsman's notes [22]. Initially, we can visualise Hilbert C^*-modules by looking at fiber bundles over some locally compact Hausdorff space X. We know that any commutative C^*-algebra $\mathcal{A}$ can be identified with some $C(X)$, the space of continuous $\mathbb{C}$-valued functions over a locally compact Hausdorff space X. This allows us to look at vector-bundles over this X and, endowing it with some sesquilinear product, we can define some Hilbert-like structure on the sections on the bundle.
A more general and precise definition is the following:

Definition A **right Hilbert C^*-module** $\mathcal{E}$ over the C*-algebra $\mathcal{A}$ consists of

- A complex linear space $\mathcal{E}$.
- A right action π_R of $\mathcal{A}$ on $\mathcal{E}$. π_R is a linear antihomomorphism between $\mathcal{A}$ and the bounded operators on $\mathcal{E}$. The reversal of multiplication leads us to write $\psi A = \pi_R(A)\psi$.
- A sesquilinear map $\langle , \rangle_{\mathcal{A}} : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$, conjugate linear in the first coordinate satisfying:

$$\langle \psi, \phi \rangle_{\mathcal{A}} = \langle \phi, \psi \rangle_{\mathcal{A}}^* \tag{5}$$
$$\langle \psi, \pi_R(A)\phi \rangle_{\mathcal{A}} = \langle \psi, \phi \rangle_{\mathcal{A}} A \tag{6}$$
$$\langle \psi, \psi \rangle_{\mathcal{A}} \geq 0 \tag{7}$$
$$\langle \psi, \psi \rangle_{\mathcal{A}} = 0 \iff \psi = 0 \tag{8}$$

  for all $\psi, \phi \in \mathcal{E}$ and $A \in \mathcal{A}$.
- The norm $||\psi||_{\mathcal{E}} := ||\langle \phi, \phi \rangle_{\mathcal{A}}||_{\mathcal{A}}^{\frac{1}{2}}$, with respect to which $\mathcal{E}$ is complete.

We say that $\mathcal{E}$ is a Hilbert $\mathcal{A}$-module and write $\mathcal{E} \rightleftharpoons \mathcal{A}$.
Additionally, a Hilbert $\mathcal{A}$-module is **full** if the set $\{\langle \psi, \phi \rangle_{\mathcal{A}} | \phi, \psi \in \mathcal{E}\}$ is dense in $\mathcal{A}$.

We note that this is a generalisation of the usual Hilbert-spaces, since every Hilbert space is a Hilbert $\mathbb{C}$-module.

As a connection to the previous discussion, let H be a Hilbert bundle over a compact Hausdorff space X, that is the vector bundle over X is complete in its inner product, then $\Gamma(X)$, the space of continuous sections, is a $C(X)$-module. The action and map we use for this is given, respectively, by

$$\pi_R(f)\psi(x) = f(x)\psi(x) \tag{9}$$
$$\langle \psi, \phi \rangle_{C(X)}(x) = \langle \psi(x), \phi(x) \rangle_X \tag{10}$$

By commutativity, the action and (6) are well defined. Since the rest of the requirements hold on the inner product over X, it also holds for (10), including completeness. This is a very natural way to define Hilbert C^*-modules, given any Hilbert bundle, or commutative C^*-algebra in general.

Special Hilbert C^*-modules are vector bundles over non-commutative spaces, the finitely generated projective ones. We will later see two examples of this, the Moyal plane and the continuous functions over the non-commutative torus.

Given, a pre-C^*-algebra $\tilde{\mathcal{A}}$ we can define a pre-Hilbert $\tilde{\mathcal{A}}$-module, by simply removing the completeness requirement. As for C^*-algebras, one often starts with a non-complete space, but luckily, taking completions is not a troublesome activity. We now show some basic norm inequalities for these modules.

Proposition 2.1 *For a pre-Hilbert $\tilde{\mathcal{A}}$-module the following holds:*

$$||\pi_R(A)\psi||_{\mathcal{E}} \leq ||\psi||_{\mathcal{E}}||A||_{\tilde{\mathcal{A}}} \tag{11}$$

$$\langle\psi,\phi\rangle_{\tilde{\mathcal{A}}}\langle\phi,\psi\rangle_{\tilde{\mathcal{A}}} \leq ||\phi||^2_{\mathcal{E}}\langle\psi,\psi\rangle_{\tilde{\mathcal{A}}} \tag{12}$$

$$||\langle\psi,\phi\rangle_{\tilde{\mathcal{A}}}||_{\tilde{\mathcal{A}}} \leq ||\psi||_{\mathcal{E}}||\phi||_{\mathcal{E}} \tag{13}$$

Proof By applying the definition of the norm, both for the module and the C^*-algebra; (5) and (6) we see that

$$||\pi_R(A)\psi||_{\mathcal{E}} = ||\langle\psi A,\psi A\rangle_{\tilde{\mathcal{A}}}||^{\frac{1}{2}}_{\tilde{\mathcal{A}}} = ||\langle\psi,\psi\rangle_{\tilde{\mathcal{A}}}A^*A||^{\frac{1}{2}}_{\tilde{\mathcal{A}}} \leq ||\psi||_{\mathcal{E}}||A||_{\tilde{\mathcal{A}}}$$

The second inequality we omit, in the interest of space, but it follows from expanding $\langle\phi\langle\phi,\psi\rangle_{\tilde{\mathcal{A}}}-\psi,\phi\langle\phi,\psi\rangle_{\tilde{\mathcal{A}}}-\psi\rangle_{\tilde{\mathcal{A}}} \geq 0$ and using the familiar C^*-inequality $B^*A^*AB \leq ||A||^2B^*B$. The third follows from the second, by the simple calculation

$$||\langle\psi,\phi\rangle_{\tilde{\mathcal{A}}}||_{\tilde{\mathcal{A}}} = ||\langle\psi,\phi\rangle_{\tilde{\mathcal{A}}}\langle\psi,\phi\rangle^*_{\tilde{\mathcal{A}}}||^{\frac{1}{2}}_{\tilde{\mathcal{A}}} \leq ||\phi||_{\mathcal{E}}||\langle\psi,\psi\rangle_{\tilde{\mathcal{A}}}||^{\frac{1}{2}}_{\tilde{\mathcal{A}}} = ||\phi\|_{\mathcal{E}}\|\psi||_{\mathcal{E}}$$

∎

This can further be used to complete every pre-Hilbert C^*-module, since we can now extend all our definitions by continuity. We can also move further in trying to make these modules more "Hilbert-like", by giving a definition that mirrors the usual adjoint of maps:

Definition A map $T:\mathcal{E}\to\mathcal{E}$ on a Hilbert $\mathcal{A}$-module is called **adjointable** if there exists a map $T^*:\mathcal{E}\to\mathcal{E}$ such that

$$\langle T^*\psi,\phi\rangle_{\tilde{\mathcal{A}}} = \langle\psi,T\phi\rangle_{\tilde{\mathcal{A}}} \tag{14}$$

For all $\psi,\phi\in\mathcal{E}$. We denote the space of adjointable maps by $C^*(\mathcal{E},\mathcal{A})$. With the usual operator norm, we will show that this space is in fact a C^*-Algebra.

In a Hilbert space setting, all bounded linear maps have unique bounded adjoints, this is not the case for these more general modules. However, we here do not require linearity or boundedness, these properties are more or less immediate from the definition, we gather them in the following theorem:

Theorem 2.2 *An adjointable map T is $\mathbb{C}$-linear, $\mathcal{A}$-linear and bounded. The map $T \to T^*$ defines an involution that is unique on $C^*(\mathcal{E},\mathcal{A})$. With the usual operator norm, this a C^*-algebra. Every adjointable map also satisfies*

$$\langle T\psi, T\psi\rangle_{\mathcal{A}} \leq ||T||^2_{\mathcal{A}}\langle\psi,\psi\rangle_{\mathcal{A}}. \tag{15}$$

Proof $\mathbb{C}$- and $\mathcal{A}$-linearity follow from the linearity of the inner product.

$$\langle\psi, T(\pi_R(A)\phi)\rangle_{\mathcal{A}} = \langle T^*\psi, \pi_R(A)\phi\rangle_{\mathcal{A}} = \langle\psi, T\phi\rangle_{\mathcal{A}}A = \langle\psi, \pi_R(A)T\phi\rangle_{\mathcal{A}},$$

where we have used the property (6). Since $\mathbb{C} \subseteq \mathcal{A}$ for all C^*-algebras, this also shows $\mathbb{C}$-linearity. To show boundedness of T requires that we choose a $\psi \in \mathcal{E}$ and define the operator $S_\psi\phi = \langle T^*T\psi, \phi\rangle_{\mathcal{A}} = \langle\psi, T^*T\phi\rangle_{\mathcal{A}}$. By (13) we have the bounds

$$||S_\psi\phi|| = ||\langle T^*T\psi,\phi\rangle_{\mathcal{A}}|| \leq ||T^*T\psi||\ ||\phi||$$
$$||S_\psi\psi|| = ||\langle\psi, T^*T\phi\rangle_{\mathcal{A}}|| \leq ||T^*T\phi||\ ||\psi||$$

for all $\phi,\psi \in \mathcal{E}$. The first inequality tells us that S_ψ is bounded, and by considering ψ's with $||\psi|| = 1$, the second inequality tells us that $\sup_{||\psi||=1}\{||S_\psi\phi||\} < \infty$. Since the bound we here obtained depends on ϕ, the uniform boundedness principle yields that this holds for the operator-norms as well, $\sup_{||\psi||=1}\{||S_\psi||\} < \infty$. Then we have that

$$||T|| = \sup_{||\psi||=1}||T\phi|| = \sup_{||\psi||=1}||\langle T\psi, T\psi\rangle_{\mathcal{A}}|| = \sup_{||\psi||=1}||T_\psi\psi|| < \infty$$

and T is bounded.

Let S, T^* be two involutions of T. Then clearly $\langle S\psi,\phi\rangle_{\mathcal{A}} = \langle T^*\psi,\phi\rangle_{\mathcal{A}}$, so $S\psi = T^*\phi$ for all $\psi,\phi \in \mathcal{E}$, therefore $S = T^*$.
Involutivity is shown by using the property (5), which is the involution on $\mathcal{A}$,

$$\langle\phi, T\psi\rangle_{\mathcal{A}} = \langle T^*\phi,\psi\rangle_{\mathcal{A}} = \langle\psi, T^*\phi\rangle^*_{\mathcal{A}} = \langle T^{**}\psi,\phi\rangle^*_{\mathcal{A}} = \langle\phi, T^{**}\psi\rangle_{\mathcal{A}} \quad \forall\psi,\phi\in\mathcal{E}$$

We now show norm-closedness. Let $T_n \in C(\mathcal{E},\mathcal{A})$ be a sequence converging to some T, an operator on $\mathcal{E}$. Then, by definition of the norm, we also have that $T_n^* \to S$, for some S, examining further, we find that $S = T^*$, so $T \in C(\mathcal{E},\mathcal{A})$

$$\lim_{n\to\infty}||(T_n - T)\phi|| = ||\langle(T_n - T)\phi, (T_n - T)\phi\rangle_{\mathcal{A}}||$$
$$= ||\langle(T_n - T)^*\phi, (T_n - T)^*\phi\rangle_{\mathcal{A}}|| = ||(T_n^* - T^*)\phi||, \quad \forall\phi\in\mathcal{E}$$

By the properties of involution. $C^*(\mathcal{E},\mathcal{A})$ is therefore a C^*-algebra. For the inequality, we will use the property that for positive elements of a C^*-algebra, we have that $A \leq ||A||$ and the familiar norm-properties. Firstly, we see that for positive elements in $A \in C^*(\mathcal{E},\mathcal{A})$ (which we can write as T^*T) the map $A \to \langle\psi, A\psi\rangle_{\mathcal{A}}$ is positive, for every fixed ψ. This is because $\langle T\psi, T\psi\rangle_{\mathcal{A}} \geq 0$ by the property (7). We then find that

$$\langle\psi, A\psi\rangle_{\mathcal{A}} \leq \langle\psi, ||T^*T||\psi\rangle_{\mathcal{A}} = ||T^*T||\langle\psi,\psi\rangle_{\mathcal{A}} = ||A||\langle\psi,\psi\rangle_{\mathcal{A}}$$

∎

In much the same way as for Hilbert spaces, we wish to construct some analogue of compact operators on general Hilbert C^*-modules. The following definition might seem odd, but rest assured, we will show that it is the correct generalization of compact operators on Hilbert spaces.

Definition The C^*-algebra of **compact operators** $C_0^*(\mathcal{E},\mathcal{A})$ on a Hilbert $\mathcal{A}$-module $\mathcal{E}$ is generated by the adjointable maps of the type $T^{\mathcal{A}}_{\psi,\psi}$ for $\psi,\phi\in\mathcal{E}$ defined by

$$T^{\mathcal{A}}_{\psi,\phi}Z=\psi\langle\phi,Z\rangle_{\mathcal{A}} \tag{16}$$

Where $Z\in\mathcal{E}$. We write $C_0^*(\mathcal{E},\mathcal{A})\rightleftharpoons\mathcal{E}\rightleftharpoons\mathcal{A}$ and call this a dual pair

The right side of (16) is to be understood as the element $\langle\phi,Z\rangle\in\mathcal{A}$ acting on ψ by the normal right action.

Showing that this is indeed a C^*-algebra amounts to showing that it is a closed ideal in $C^*(\mathcal{E},\mathcal{A})$. It is clearly closed under addition and multiplication, so we need only look at the $*$-operation, and composition with $A\in C^*(\mathcal{E},\mathcal{A})$. This is also clear as, for any $\eta\in\mathcal{E}$

$$\langle\eta,T^{\mathcal{A}}_{\psi,\phi}Z\rangle_{\mathcal{A}}=\langle\eta,\psi\langle\phi,Z\rangle_{\mathcal{A}}\rangle_{\mathcal{A}}=\langle\eta,\psi\rangle_{\mathcal{A}}\langle\phi,Z\rangle_{\mathcal{A}}=(\langle Z,\phi\rangle_{\mathcal{A}}\langle\psi,\eta\rangle_{\mathcal{A}})^* \tag{17}$$

$$=\langle Z,\phi\langle\psi,\eta\rangle_{\mathcal{A}}\rangle^*_{\mathcal{A}}=\langle\phi\langle\psi,\eta\rangle_{\mathcal{A}},Z\rangle_{\mathcal{A}}=\langle T^{\mathcal{A}}_{\phi,\psi}\eta,Z\rangle_{\mathcal{A}}=\langle(T^{\mathcal{A}}_{\psi,\phi})^*\eta,Z\rangle_{\mathcal{A}} \tag{18}$$

Also, we have that

$$AT^{\mathcal{A}}_{\psi,\phi}Z=A\psi\langle\phi,Z\rangle_{\mathcal{A}}=T^{\mathcal{A}}_{A\psi,\phi}Z. \tag{19}$$

Composition on the other side is done similarily. Showing the algebra is closed is done in the same way as for $C^*(\mathcal{E},\mathcal{A})$, relying on $\mathcal{E}$ being complete.

When we take our C^*-algebra to be $\mathbb{C}$, the linear span of $T^{\mathbb{C}}_{\psi,\phi}$ are finite sums of some complex coefficients multiplied by some element of H, our Hilbert space. This amounts to transforming the element along some finite subset of the basis, so we can view this span as the set of finite rank operators on H. Taking completion, gives us that $C_0^*(\mathbb{C},H)$ is precisely the C^*-algebra of compact operators on H.

The reason for introducing these notions is to end up with some equivalence of C^*-algebras that is weaker than isomorphism, but still is helpful. We will now give the definition, which will eventually result in a bijective correspondence between non-degenerate representation of these algebras.

Definition The C*-algebras $\mathcal{A}$ and $\mathcal{B}$ are **Morita-equivalent** if there exists a full Hilbert $\mathcal{B}$-module $\mathcal{E}$ such that $\mathcal{A}\simeq C_0^*(\mathcal{E},\mathcal{B})$. This gives rise to the dual pair $\mathcal{A}\rightleftharpoons\mathcal{E}\rightleftharpoons\mathcal{B}$, and we write $\mathcal{A}\overset{M}{\sim}\mathcal{B}$.

Showing that this is an equivalence relation is non-trivial, and we will split the result in three parts.

Lemma 2.3 *When $\mathcal{E}=\mathcal{A}$, one has $C_0^*(\mathcal{E},\mathcal{A})=\mathcal{A}$. This leads to the dual pair*

$$\mathcal{A}\rightleftharpoons\mathcal{A}\rightleftharpoons\mathcal{A} \tag{20}$$

Proof By defining the inner product $\langle A, B\rangle := A^*B$, and the right action of B on A as $\pi_R(B)A = AB$, all the requirements are satisfied, including completeness, since $\mathcal{A}$ is a C^*-algebra. What remains to show is the isomorphism of Hilbert C^*-modules $C_0^*(\mathcal{E}, \mathcal{A}) = \mathcal{A}$. From the general theory on C^*-algebras, we know that $\mathcal{A} \subset \mathfrak{B}_0(\mathcal{A})$ by the map induced by multiplication on the right. This map is an isometric morphism, which can be seen when one explicitly defines the operator $\rho(A)$ as $\rho(A)B = AB$. Then, since $T^{\mathcal{A}}_{A,B} = \rho(AB^*)$, we can define a function from the linear span of $T^{\mathcal{A}}_{A,B}$ to $\mathcal{A}$ by $\varphi(T^{\mathcal{A}}_{A,B}) = AB^*$. This is also an isometric morphism, and can be extended beyond the linear span, where it still will be an isometric morphism, and therefore injective. If $\mathcal{A}$ is unital, choosing $B = \mathbf{1}$ gives us surjectivity as well. In general, there exists some increasing net $\{e_\alpha\}$ such that $||e_\alpha A - A|| \to 0$ for every $A \in \mathcal{A}$. let $A \in \mathcal{A}$ and set $B = A^*$ and $A_\alpha = e_\alpha$. Then $A_\alpha B^* \to A$, so the linear span of $T^{\mathcal{A}}_{A,B}$ is dense in $\mathcal{A}$. Extending ρ as before, it is a morphism between the two C^*-algebras $C_0^*(\mathcal{A}, \mathcal{A})$ and $\mathcal{A}$, we then know from the general theory, see for instance [22] again, that the image of ρ is closed, so it must be a surjection. ρ is then an isomorphism, and the result follows. ∎

This shows reflexivity, $\mathcal{A} \overset{M}{\sim} \mathcal{A}$.

We define $\overline{\mathcal{E}}$ as the complex conjugation of $\mathcal{E}$, then the next result proves the symmetry-condition

Lemma 2.4 *Let $\mathcal{E}$ be a full Hilbert $\mathcal{B}$-module. The inner product defined as*

$$\langle \psi, \phi\rangle_{C_0^*(\mathcal{E},\mathcal{B})} := T^{\mathcal{B}}_{\psi,\phi} \tag{21}$$

in combination with the right action $\pi_R(A)\psi := A^*\psi$, for $A \in C_0^*(\mathcal{E}, \mathcal{B})$ and $\psi, \phi \in \mathcal{E}$, defines $\overline{\mathcal{E}}$ as a full Hilbert $C_0^*(\mathcal{E}, \mathcal{B})$ -module. The left action $\pi_L(B)\psi := \psi B^*$ of $\mathcal{B}$ on $\overline{\mathcal{E}}$ implements the isomorphism $C_0^*(\overline{\mathcal{E}}, C_0^*(\mathcal{E}, \mathcal{B}))$

Proof We begin by checking that $\overline{\mathcal{E}}$ is a full Hilbert $C_0^*(\mathcal{E}, \mathcal{B})$-module. For notational simplicity, we write $C_0^*(\mathcal{E}, \mathcal{B}) = \mathcal{A}$. Clearly $\overline{\mathcal{E}}$ is a complex linear space. Since, for every $A \in \mathcal{A}$, A is an adjointable map, A^* exists and $A^*\psi \in \overline{\mathcal{E}}$. The π_R-map is a linear antihomomorphism, as

$$\pi_R(AB)\psi = (AB^*)\psi = B^*A^*\psi = \pi_R(B)\pi_R(A)\psi$$

The map $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ is well-defined an takes a pair (ψ, ϕ) to an element in $\mathcal{A}$. Sesquilinearity is then found, for all $\mu \in \overline{\mathcal{E}}$ we have

$$\begin{aligned}\pi_R(\langle\psi, \phi + \eta\rangle_{\mathcal{A}})\mu =&(T^{\mathcal{B}}_{\psi,\phi+\eta})^*\mu\\ =&(\phi + \eta)\langle\psi, \mu\rangle_{\mathcal{B}}\\ =&T^{\mathcal{B}}_{\phi,\psi}(\mu) + T^{\mathcal{B}}_{\eta,\phi}(\mu) = \pi_R(\langle\psi, \phi\rangle_{\mathcal{A}} + \langle\psi, \phi\rangle_{\mathcal{A}})(\mu)\end{aligned}$$

We show conjugate linearity in the first variable in the same way. The two properties (5) and (6) are showed directly by using (18) and (19) respectively.

To see (6) note that $\langle\psi, \pi_R(A)\phi\rangle_{\mathcal{A}}A$. To prove positivity, we first show that for all $\mu \in \mathcal{E}$ we have that

$$\langle\eta, T^{\mathcal{B}}_{\psi,\psi}\eta\rangle_{\mathcal{B}} = \langle\eta, \psi\langle\psi, \eta\rangle_{\mathcal{B}}\rangle_{\mathcal{B}} = \langle\eta, \psi\rangle_{\mathcal{B}}\langle\psi, \eta\rangle_{\mathcal{B}} \geq 0$$

The last equality follows from $\mathcal{E}$ being a $\mathcal{B}$-module, and the inequality from the positiveness of $\langle\cdot,\cdot\rangle_{\mathcal{B}}$. By [22] Lemma 3.5.2, this implies that $T^{\mathcal{B}}_{\psi,\psi} = \langle\psi, \psi\rangle_{\mathcal{A}}$ is positive as well. Further, clearly $T^{\mathcal{B}}_{0,0} = 0$. Conversely, let $\langle\psi, \psi\rangle = 0$, which implies $T^{\mathcal{B}}_{\psi,\psi}\psi = \psi\langle\psi, \psi\rangle_{\mathcal{B}} = 0$. We then take norms on our expression and get

$$\begin{aligned} 0 &= ||\langle\psi, \psi\rangle_{\mathcal{A}}\psi|| \\ &= ||T^{\mathcal{B}}_{\psi,\psi}\psi|| \\ &= ||\psi\langle\psi, \psi\rangle_{\mathcal{B}}|| \\ &= ||\langle\psi, \psi\rangle||_{\mathcal{B}}||\psi||_{\mathcal{E}} = ||\langle\psi, \psi\rangle^3||^{\frac{1}{2}}_{\mathcal{B}} \end{aligned}$$

By positivity of $\langle\cdot,\cdot\rangle_{\mathcal{B}}$, we conclude that $\psi = 0$. What remains for the first part is then showing completeness and fullness of $\overline{\mathcal{E}}$ over $\mathcal{A}$, but we will first show the second part of the lemma. Every $\pi_L(B)$ is adjointable with respect to $\langle\cdot,\cdot\rangle_{\mathcal{A}}$, as $\langle\psi, \pi_L(B)\phi\rangle_{\mathcal{A}} = \psi\langle\pi_L(B)\phi, \cdot\rangle_{\mathcal{B}} = \psi B^*\langle\phi, \cdot\rangle_{\mathcal{B}} = \langle\pi_L(B^*)\psi, \phi\rangle_{\mathcal{A}}$. B^* is the adjoint of B in the C^*-algebra $\mathcal{B}$. We will now show that $\pi_L(B)$ is a bounded operator on $\overline{\mathcal{E}}$, but first we show an inequality that will help us on the way,

$$\begin{aligned} ||T^{\mathcal{B}}_{\psi,\phi}||_{\mathcal{A}} &= \sup_{||\mu||=1} ||T^{\mathcal{B}}_{\psi,\phi}\mu||_{\overline{\mathcal{E}}} = \sup ||\langle T^{\mathcal{B}}_{\psi,\phi}\mu, T^{\mathcal{B}}_{\psi,\phi}\rangle_{\mathcal{A}}||^{\frac{1}{2}}_{\mathcal{A}} \\ &\leq \sup ||\psi\langle\phi, \mu\rangle_{\mathcal{B}}||_{\mathcal{E}} \leq ||\psi||\mathcal{E}||\langle\phi, \mu\rangle_{\mathcal{B}}||_{\mathcal{B}} \leq \sup ||\psi||||\phi||||\mu|| = ||\psi||||\phi|| \end{aligned}$$

We used the inequalities (11) and (13). Now we have that for any $\mu \in \overline{\mathcal{E}}$

$$\begin{aligned} ||\pi_L(B)\mu||_{\overline{\mathcal{E}}} = ||\langle\pi_L(B)\mu, \pi_L(B)\mu\rangle_{\mathcal{A}}||^{\frac{1}{2}}_{\mathcal{A}} &= ||T^{\mathcal{B}}_{\pi_L(B)\mu,\pi_L(B)\mu}||^{\frac{1}{2}}_{\mathcal{A}} \\ ||\pi_L(B)\mu|| &= ||\mu B^*|| \leq ||\mu||||B^*||. \end{aligned}$$

Injectivity of the map is showed by considering $\pi_L(B)\psi = 0$ for all $\psi \in \overline{\mathcal{E}}$, then we must have

$$\langle\psi, \pi_L(B)\psi\rangle_{\mathcal{B}} = \langle\psi, \psi B^*\rangle_{\mathcal{B}} = \langle\psi, \psi\rangle_{\mathcal{B}}B^* = 0$$

By positivity of the inner product this implies $B^* = B = 0$, so the kernel of π_L is $\{0\}$. We can therefore extend $\pi_L(B)$ to an operator on the completion of $\overline{\mathcal{E}}$ in the $\mathcal{A}$-norm. Since $T^{\mathcal{A}}_{\psi,\phi}Z = \psi\langle\phi, Z\rangle_{\mathcal{A}} = T^{\mathcal{B}}_{Z,\phi} = Z\langle\phi, \psi\rangle_{\mathcal{B}} = \pi_L(\langle\psi, \phi\rangle_{\mathcal{B}})$ and because $\langle\psi, \phi\rangle_{\mathcal{B}}$ is dense in $\mathcal{B}$ π_L is a surjection. Therefore, we have an isomorphism between $\mathcal{B}$ and $C^*_0(\overline{\mathcal{E}_c}, C^*_0(\mathcal{E}, \mathcal{B}))$ where $\overline{\mathcal{E}_c}$ is the previously mentioned completion. Our goal now, will be to show $\overline{\mathcal{E}_c} = \overline{\mathcal{E}}$. By the definition, it is clear that completeness of $\mathcal{E}$ is equivalent to completeness of $\overline{\mathcal{E}}$ with respect to the same norm. By the same result as above, we have that for $\psi \in \mathcal{E}$

$$||\psi||_{\mathcal{A}} = ||\langle\psi, \psi\rangle_{\mathcal{A}}||^{\frac{1}{2}} = ||T^{\mathcal{B}}_{\psi,\psi}||^{\frac{1}{2}} \leq ||\psi||_{\mathcal{B}}. \tag{22}$$

Since π_L is an injective morphism, it is also an isometry, so we have that

$$||\psi||_{\mathcal{B}} = ||\langle\psi,\psi\rangle_{\mathcal{B}}||_{\mathcal{B}}^{\frac{1}{2}} = ||\pi_L(\langle\psi,\psi\rangle)||_{\mathcal{A}}^{\frac{1}{2}} = ||T^{\mathcal{A}}_{\psi,\psi}||_{\mathcal{A}}^{\frac{1}{2}} \leq ||\psi||_{\mathcal{A}} \tag{23}$$

The norms are equal, and since $\mathcal{E}$ is complete in the $\mathcal{B}$-norm, it is complete in the $\mathcal{A}$-norm. The isomorphism is therefore proved. ∎

Note that this indeed proves symmetry, as if we have the dual pair $\mathcal{A} \rightleftharpoons \mathcal{E} \rightleftharpoons \mathcal{B}$, we automatically have the dual pair $C_0^*(\overline{\mathcal{E}},\mathcal{A}) \rightleftharpoons \overline{\mathcal{E}} \rightleftharpoons \mathcal{A}$, but we showed that $\mathcal{B} \simeq C_0^*(\overline{\mathcal{E}},\mathcal{A})$. Lastly, we must show transitivity.

Lemma 2.5 *When the three C^*-algebras $\mathcal{A},\mathcal{B},\mathcal{C}$ have the properties that $\mathcal{A} \overset{M}{\sim} \mathcal{B}$ and $\mathcal{B} \overset{M}{\sim} \mathcal{C}$, then also $\mathcal{A} \overset{M}{\sim} \mathcal{C}$.*

Proof Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the modules between the Morita-equivalent algebras. In the usual way for tensor product over a C^*-algebras we construct $\mathcal{E}_1 \otimes_{\mathcal{B}} \mathcal{E}_2$, that is the maximal space $\mathcal{E}_1 \otimes \mathcal{E}_2$ collapsed around the ideal $\mathcal{I}_{\mathcal{B}}$ generated by vectors of the form $\psi_1 B \otimes \psi_2 - \psi \otimes B\psi_2$ (remembering that there is a right action of B on $\mathcal{E}_1$ and a left action $\mathcal{E}_2$ from the Morita-equivalences). We define a right action of $\mathcal{C}$ on this space, as well as a sequilinear map:

$$\pi_R^{\otimes}(C)(\psi_1 \otimes_{\mathcal{B}} \psi_2) := \psi_1 \otimes_{\mathcal{B}} (\psi_2 C) \tag{24}$$

$$\langle\psi_1 \otimes_{\mathcal{B}} \psi_2, \phi_1 \otimes_{\mathcal{B}} \phi_2\rangle_{\mathcal{C}}^{\otimes} := \langle\psi_2, \langle\psi_1,\phi_1\rangle_{\mathcal{B}}\phi_2\rangle_{\mathcal{C}} \tag{25}$$

It is clear that (24) defines a right action. Sesquilinearity is omitted, but the fact that (25) is an inner product needs some work. Firstly is is clear that it is a map from the tensor product to $\mathcal{C}$. The properties (5) - (8) all follow from the same properties for the maps $\langle\cdot,\cdot\rangle_{\mathcal{B}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{C}}$, and the similar result $\langle\psi B,\phi\rangle_{\mathcal{C}} = B^*\langle\psi,\phi\rangle_{\mathcal{C}}$.

Involutivity:

$$\begin{aligned}
(\langle\psi_1 \otimes_{\mathcal{B}} \psi_2, \phi_1 \otimes_{\mathcal{B}} \phi_2\rangle_{\mathcal{C}}^{\otimes})^* &= \langle\psi_2, \langle\psi_1,\phi_1\rangle_{\mathcal{B}}\phi_2\rangle_{\mathcal{C}}^* \\
&= \langle\langle\psi_1,\phi_1\rangle_{\mathcal{B}}\phi_2, \psi_2\rangle_{\mathcal{C}} \\
&= \langle\psi_1,\phi_1\rangle_{\mathcal{B}}^*\langle\psi_2,\phi_2\rangle_{\mathcal{C}} \\
&= \langle\langle\psi_1,\phi_1\rangle_{\mathcal{B}}\psi_2,\phi_2\rangle_{\mathcal{C}}^* \\
&= \langle\phi_2, \langle\phi_1,\psi_1\rangle_{\mathcal{B}}\psi_2\rangle_{\mathcal{C}} = \langle\phi_1 \otimes_{\mathcal{B}} \phi_2, \psi_1 \otimes_{\mathcal{B}} \psi_2\rangle_{\mathcal{C}}^{\otimes}
\end{aligned}$$

For all $C \in \mathcal{C}$ it is clear that

$$\begin{aligned}
\langle\psi_1 \otimes_{\mathcal{B}} \psi_2, \pi_R^{\otimes}(C)\,(\phi_1 \otimes_{\mathcal{B}} \phi_2)\rangle_{\mathcal{C}}^{\otimes} &= \langle\psi_2, \langle\psi_1,\phi_1\rangle_{\mathcal{B}}(\phi_2 C)\rangle_{\mathcal{C}} \\
&= \langle\psi_2, \langle\psi_1,\phi_1\rangle_{\mathcal{B}}\phi_2\rangle_{\mathcal{C}}C = \langle\psi_1 \otimes_{\mathcal{B}} \psi_2, \phi_1 \otimes_{\mathcal{B}} \phi_2\rangle_{\mathcal{C}}^{\otimes}C
\end{aligned}$$

Recall that $\langle\psi_1,\psi_1\rangle_{\mathcal{B}}$ is a positive operator on $\mathcal{E}_2$, so we can write $\langle\psi_1,\psi_1\rangle_{\mathcal{B}} = B^*B$, then

$$\begin{aligned}
\langle\psi_1 \otimes_{\mathcal{B}} \psi_2, \psi_1 \otimes_{\mathcal{B}} \psi_2\rangle_{\mathcal{C}}^{\otimes} &= \langle\psi_2, \langle\psi_1,\psi_1\rangle_{\mathcal{B}}\psi_2\rangle_{\mathcal{C}} \\
&= \langle B\psi_2, B\psi_2\rangle_{\mathcal{C}} \geq 0
\end{aligned}$$

As for the last condition, we must here use the quotient. By the properties for the inner products of $\mathcal{B}$ and $\mathcal{C}$, $\langle\psi_1 \otimes_{\mathcal{B}} \psi_2, \phi_1 \otimes_{\mathcal{B}} \phi_2\rangle_{\mathcal{C}}^{\otimes} = 0$ only when either $\langle\psi_1, \psi_1\rangle_{\mathcal{B}} = 0$ or $\langle\psi_2, \psi_2\rangle\mathcal{C} = 0$, this implies that either $\psi_1 = 0$, or $\psi_2 = 0$, but then $\psi_1 \otimes \psi_2 \in \mathcal{I}_{\mathcal{B}}$. This is clear as $0 \otimes \psi_2 = 0 \otimes \psi_2 - 0 \otimes 0(\psi_2)$ for all ψ_2, and similarily for ψ_1.

We therefore have that $\mathcal{E}_{\otimes} = (\mathcal{E}_1 \otimes_{\mathcal{B}} \mathcal{E}_2)_c$ is a Hilbert $\mathcal{C}$-module, where we have completed in the $\langle\cdot,\cdot\rangle_{\mathcal{C}}^{\otimes}$-norm.

Our goal is constructing a dual pair $\mathcal{A} \rightleftharpoons \mathcal{E}_{\otimes} \rightleftharpoons \mathcal{C}$, so we now define a left action $\pi_L^{\otimes}(\mathcal{A})$ by

$$\pi_L^{\otimes}(A)(\psi_1 \otimes_{\mathcal{B}} \psi_2) := (A\psi_1 \otimes_{\mathcal{B}} \psi_2) \tag{26}$$

$\pi_L(A)$ is a bounded operator on $\mathcal{E}_1 \otimes_{\mathcal{B}} \mathcal{E}_2$ and can therefore be extended to $\mathcal{E}_{\otimes}$. Boundedness follows from (15), as

$$\begin{aligned}||\pi_L^{\otimes}(A)(\psi_1 \otimes_{\mathcal{B}} \psi_2)|| &= ||\langle\psi_2, \langle A\psi_1, A\psi_1\rangle_{\mathcal{B}}\psi_2\rangle_{\mathcal{C}}||^{\frac{1}{2}} \\ &\leq ||\langle\psi_2, ||A||^2\langle\psi_1, \psi_1\rangle_{\mathcal{B}}\psi_2\rangle_{\mathcal{C}}||^{\frac{1}{2}} \\ &\leq ||A||||\langle\psi_2, \langle\psi_1, \psi\rangle_{\mathcal{B}}\psi_2\rangle_{\mathcal{C}}||^{\frac{1}{2}} = ||A||_{\mathcal{A}}||\psi_1 \otimes_{\mathcal{B}} \psi_2||_{\mathcal{C}}\end{aligned}$$

We are now hoping for some connection between the action on the first and second coordinate in the tensor product. By the definition of "compact" operators, we have that

$$\begin{aligned}\pi_L^{\otimes}(T^{\mathcal{B}}_{\psi_1\langle\psi_2,\phi_2\rangle_{\mathcal{B}},\phi_1})\,(\mu_1 \otimes_{\mathcal{B}} \mu_2) &= T^{\mathcal{B}}_{\psi_1\langle\psi_2,\phi_2\rangle_{\mathcal{B}},\phi_1}\mu_1 \otimes \mu_2 \\ &= \psi_1\langle\psi_2, \phi_2\rangle_{\mathcal{B}}\langle\phi_1, \mu_1\rangle_{\mathcal{B}} \otimes_{\mathcal{B}} \mu_2 \\ &= \psi_1\langle\psi_2, \phi_2\langle\phi_1, \mu_1\rangle_{\mathcal{B}}\rangle_{\mathcal{B}} \otimes_{\mathcal{B}} \mu_2 \\ &= \psi_1 \otimes_{\mathcal{B}} \langle\psi_2, \phi_2\langle\phi_1, \mu_1\rangle_{\mathcal{B}}\rangle_{\mathcal{B}}\mu_2\end{aligned}$$

Where the last equality follows from the fact that $\psi_1\langle\psi_2, \phi_2\langle\phi_1, \mu_1\rangle_{\mathcal{B}}\rangle_{\mathcal{B}} \otimes_{\mathcal{B}} \mu_2 - \psi_1 \otimes_{\mathcal{B}} \langle\psi_2, \phi_2\langle\phi_1, \mu_1\rangle_{\mathcal{B}}\rangle_{\mathcal{B}}\mu_2 \in \mathcal{I}_{\mathcal{B}}$

Now we use the assumption that $\mathcal{B} = C_0^*(\mathcal{E}_2, \mathcal{C})$, as in $\langle\psi, \phi\rangle_{\mathcal{B}} = T^{\mathcal{C}}_{\psi,\phi}$. This allows us to continue the calculation

$$\begin{aligned}\psi_1 \otimes_{\mathcal{B}} \langle\psi_2, \phi_2\langle\phi_1, \mu_1\rangle_{\mathcal{B}}\rangle_{\mathcal{B}}\mu_2 &= \psi_1 \otimes_{\mathcal{B}} T^{\mathcal{C}}_{\psi_2,\phi_2\langle\psi_1,\mu_1\rangle_{\mathcal{B}}}\mu_2 \\ &= \psi_1 \otimes_{\mathcal{B}} \psi_2\langle\phi_2\langle\phi_1, \mu_1\rangle_{\mathcal{B}}, \mu_2\rangle_{\mathcal{C}}\end{aligned}$$

Now let $\psi_1 \otimes_{\mathcal{B}} \psi_2$ and $\phi_1 \otimes_{\mathcal{B}} \phi_2$ be in $\mathcal{E}_{\otimes}$, then $T^{\mathcal{C}}_{\psi_1\otimes_{\mathcal{B}}\psi_2,\phi_1\otimes_{\mathcal{B}}\phi_2} \in C_0^*(\mathcal{E}_{\otimes}, \mathcal{C})$, and we have that for every $\mu_1 \otimes_{\mathcal{B}} \mu_2$

$$\begin{aligned}T^{\mathcal{C}}_{\psi_1\otimes_{\mathcal{B}}\psi_2,\phi_1\otimes_{\mathcal{B}}\phi_2}(\mu_1 \otimes_{\mathcal{B}} \mu_2) &= (\psi_1 \otimes \psi_2)\,\langle\phi_1 \otimes \phi_2, \mu_1 \otimes \mu_2\rangle_{\mathcal{C}}^{\otimes} \\ &= (\psi_1 \otimes \psi_2)\langle\phi_2, \langle\phi_1, \mu_1\rangle_{\mathcal{B}}\mu_2\rangle_{\mathcal{C}} \\ &= \pi_L^{\otimes}(T^{\mathcal{B}}_{\psi_1\langle\psi_2,\phi_2\rangle_{\mathcal{B}},\phi_1})(\mu_1 \otimes \mu_2)\end{aligned}$$

Since $T^{\mathcal{B}}_{\psi_1\langle\psi_2,\phi_2\rangle_{\mathcal{B}},\phi_1} \in \mathcal{A}$ by definition, we have that $C_0^*(\mathcal{E}_{\otimes}, \mathcal{C}) \subseteq \pi_L^{\otimes}(\mathcal{A})$. Conversely, pick a double sequence $\{\psi_2^i, \phi_2^i\}$ such that $\sum_0^N T^{\mathcal{C}}_{\psi_2^i,\phi_2^i}$ is an approximate unit. Then $\lim_N \sum_0^N T^{\mathcal{C}}_{\psi_2^i,\phi_2^i}\mu = \lim_N \sum_0^N \psi_2^i\langle\phi_2^i,\mu\rangle_{\mathcal{C}} = \mu$. Then, by doing most of the same calculations as before again, we have that since every $A \in \mathcal{A}$ can be written as $T^{\mathcal{B}}_{\psi_1,\phi_1}$

$$\begin{aligned}\pi_L(T^{\mathcal{B}}_{\psi_1,\phi_1})(\mu_1 \otimes_{\mathcal{B}} \mu_2) &= \psi_1 \otimes \langle\phi_1,\mu_1\rangle_{\mathcal{B}}\mu_2 \\ &= \lim_N \sum_0^N \psi_1 \otimes (\psi_2^i\langle\phi_2^i\langle\phi_1,\mu_1\rangle_{\mathcal{B}}\mu_2\rangle_{\mathcal{C}}) \\ &= \lim_N \sum_0^N (\psi_1 \otimes \psi_2^i)\langle\phi_1 \otimes \phi_2^i, \mu_1 \otimes \mu_2\rangle^{\otimes}_{\mathcal{C}} \\ &= \lim_N \sum_0^N T^{\mathcal{C}}_{\psi_1\otimes\psi_2^i,\phi_1\otimes\phi_2^i}(\mu_1 \otimes \mu_2)\end{aligned}$$

This establishes the reverse inclusion, and we have therefore established the dual pair

$$\mathcal{A} \rightleftharpoons \mathcal{E}_{\otimes} \rightleftharpoons \mathcal{C}$$

showing transitivity of Morita equivalence ∎

With these three lemmas in place we have proved the following:

Theorem 2.6 *Morita equivalence is an equivalence relation of C^*-algebras.*

Morita equivalence is a significant useful way to classify C^*-algebras. As an example, we have already shown that the space of compact operators on Hilbert spaces is isomorphic to $C_0^*(\mathbb{C}, H)$, meaning that both the compact operators and all matrix algebras over $\mathbb{C}$ are Morita equivalent to $\mathbb{C}$. This result was the original motivation of introducing Morita equivalence and is due to Rieffel [25].

Constructing a suitable space $\mathcal{E}$ to explicitly establish the Morita equivalence between two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is a rather tedious and challenging process, amounting only to find sesquilinear maps and actions such that we have the equality

$$\langle\psi,\phi\rangle_{\mathcal{A}}\mu = \psi\langle\phi,\mu\rangle_{\mathcal{B}} \qquad \forall\psi,\phi,\mu \in \mathcal{E} \tag{27}$$

relating the structure of the algebras. We use the following to construct dual pairs:

Proposition 2.7 *Suppose one has:*

- *two pre-C^*-algebras $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$;*
- *a full pre-Hilbert $\tilde{\mathcal{B}}$-module $\tilde{\mathcal{E}}$;*
- *a left action of $\tilde{\mathcal{A}}$ on $\tilde{\mathcal{E}}$ such that $\overline{\tilde{\mathcal{E}}}$ can be made into a full pre-Hilbert $\tilde{\mathcal{A}}$-module with respect to the right action $\pi_R(A)\psi = A^*\psi$;*
- *the identity (27) relating the Hilbert C^*-module structures;*

- *the bounds*

$$\langle A\psi, A\psi\rangle_{\tilde{\mathcal{B}}} \leq ||A||^2 \langle \psi, \psi\rangle_{\tilde{\mathcal{B}}} \tag{28}$$

$$\langle \psi B, \psi B\rangle_{\tilde{\mathcal{A}}} \leq ||B||^2 \langle \psi, \psi\rangle_{\tilde{\mathcal{A}}} \tag{29}$$

for all $A \in \tilde{\mathcal{A}}$ *and* $B \in \tilde{\mathcal{B}}$.

Then $A \overset{M}{\sim} B$ *with the connecting space* $\mathcal{E}$ *being the completion of* $\tilde{\mathcal{E}}$ *as a Hilbert* $\mathcal{B}$*-module.*

Proof By applying Proposition 2.1 we can complete $\tilde{\mathcal{E}}$ to a full Hilbert $\mathcal{B}$-module $\mathcal{E}$, by the same logic we can complete $\overline{\tilde{\mathcal{E}}}$ to a full Hilbert $\mathcal{A}$-module $\overline{\mathcal{E}}_c$. In both these cases, we can, by (28) and (29), extend the left action of $\mathcal{A}$ to $\mathcal{E}$ and the right action of $\mathcal{B}$ to $\overline{\mathcal{E}}_c$ respectively. By the same logic as when we showed symmetry of Morita equivalence, we can use the inequalities (22) and (23) to show the norm-equivalences and therefore $\mathcal{E} = \overline{\mathcal{E}}_c$. Since, by the previous, (27) and the fullness of $\overline{\mathcal{E}}$ we can use our lemma showing symmetry of Morita equivalence. This proves that $\mathcal{A} \simeq C_0^*(\mathcal{E}, \mathcal{B})$ and yields the dual pair $\mathcal{A} \rightleftharpoons \mathcal{E} \rightleftharpoons \mathcal{B}$.

This is the preferred way of showing Morita equivalences, and in several cases the only hard part of this process will be showing that (27) holds.

For general C^*-algebras, the goal is often to represent them as bounded operators on some Hilbert space. The GNS construction gives us such a representation and it makes sense to ask if Morita equivalences gives us some relation between the representations of two equivalent C^*-algebras. The Rieffel induction procedure gives us exactly such a relation, and can be defined starting from a state on $\mathcal{B}$, as a generalization of the GNS-construction, or from a representation. The second follows from the first, so we start there.

Theorem 2.8 *Suppose we are given a Hilbert* $\mathcal{B}$*-module* $\mathcal{E}$ *and a state* ω_χ *on* $\mathcal{B}$.

1. *Then we can define a sesquilinear form* $\widetilde{(\cdot,\cdot)}_0^\chi$ *on* $\mathcal{E}$ *by*

$$\widetilde{(\psi,\phi)}_0^\chi := \omega_\chi(\langle\psi,\phi\rangle_{\mathcal{B}}). \tag{30}$$

The null space of the form is

$$\widetilde{\mathcal{N}} := \{\psi \in \mathcal{E} | \widetilde{(\psi,\psi)}_0^\chi = 0\} \tag{31}$$

2. *Let* $\tilde{V}_\chi : \mathcal{E} \to \mathcal{E}/\widetilde{\mathcal{N}}_\chi$ *be the canonical projection. Then the form* $\widetilde{(\cdot,\cdot)}^\chi$ *on the quotient space defined as*

$$\widetilde{(\tilde{V}_\chi\psi, \tilde{V}_\chi\phi)}^\chi := \widetilde{(\psi,\phi)}_0^\chi \tag{32}$$

is an inner product. Taking closure in this inner product gives us the Hilbert space $\widetilde{\mathcal{H}}^\chi$.

3. The representation $\tilde{\pi}^{\chi}(C_0^*(\mathcal{E},\mathcal{B}))$ is defined on $\mathcal{E}/\tilde{N}$ by

$$\tilde{\pi}^{\chi}(A)\tilde{V}_{\chi}\psi := \tilde{V}_{\chi}A\psi. \tag{33}$$

It is clear that $\tilde{\pi}^{\chi}$ is continuous, and can therefore be extended to the whole Hilbert space. This is a representation of $\mathcal{A}$ given a Morita equivalent $\mathcal{B}$ and a state.

The second result starts of with a representation of $\mathcal{B}$ and constructs some Morita-equivalent algebra, but is here omitted as it is not of relevance for our work, and follows from the GNS-construction and Theorem 2.8. We now state The Imprimitivity Theorem, due to Rieffel [25]. This is an important result in this field, but not one we will be using, so we do not prove it.

Theorem 2.9 (The Imprimitivity Theorem) *There exists a bijective correspondence between the non-degenerate representations of Morita-equivalent C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, preserving direct sums and irreducibility.*

2.1.1 Derivations and Connections

The C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ that we will be considering in this paper, will be given some additional structure. Assume that over $\mathcal{A}$ and $\mathcal{B}$ there exists a pair of commuting derivations (linear morphisms) ∂_1 and ∂_2, defined on both algebras. Additionally, there should exist faithful tracial states over both algebras. We will simplify the notation somewhat by denoting the left valued inner product on $\mathcal{A}$ by ${}_\bullet\langle\cdot,\cdot\rangle$, and $\langle\cdot,\cdot\rangle_\bullet$ for the right-valued inner product on $\mathcal{B}$ (note that this implies that the bimodule is constructed as $\mathcal{A} \rightleftharpoons \mathcal{E} \rightleftharpoons \mathcal{B}$). The tracial states, denoted by τ for both algebras, will also satisfy the equations

$$\tau(\partial_j a) = 0, \quad \tau(\partial_j b) = 0 \quad \text{for all } a \in \mathcal{A} \text{ and } b \in \mathcal{B}\ ,\ j = 1, 2 \tag{34}$$

$$\tau({}_\bullet\langle\xi,\eta\rangle) = \tau(\langle\eta,\xi\rangle_\bullet) \quad \text{for all } \xi, \eta \in \mathcal{E}. \tag{35}$$

The connection between the algebras will be a result of an assumed consistent lifting to the connecting space $\mathcal{E}$. That is, we are assuming the existence of covariant derivations (linear maps) ∇_1 and ∇_2, that satisfy the Leibniz rule. Explicitly, we have $\nabla_j : \mathcal{E} \to \mathcal{E}$ for $j = 1, 2$ such that for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $\xi \in \mathcal{E}$, we have

$$\nabla_j(a\xi) = (\partial_j a)\xi + a(\nabla_j \xi) \quad \text{and} \quad \nabla_j(\xi b) = \xi(\partial_j b) + (\nabla_j \xi)b \tag{36}$$

To ensure compatibility between the structure induced by the inner products, and the "right" choice of derivations, we also require that a similar relation holds for the inner products. For all $\xi, \eta \in \mathcal{E}$

$$\partial_j({}_\bullet\langle\xi,\eta\rangle) = {}_\bullet\langle\nabla_j\xi,\eta\rangle + {}_\bullet\langle\xi,\nabla_j\eta\rangle \tag{37}$$

$$\partial_j(\langle\xi,\eta\rangle_\bullet) = \langle\nabla_j\xi,\eta\rangle_\bullet + \langle\xi,\nabla_j\eta\rangle_\bullet \tag{38}$$

As in the paper by Schwarz, [28], we call a Morita equivalence with these operations and properties a **complete Morita equivalence**.

In [26], Rieffel showed that Morita equivalence between $\mathcal{A}$ and some unital $\mathcal{B}$ gives an isomorphism between $\mathcal{B}$ and the compact operators on $\mathcal{E}$ yielding the identification

$$1_{\mathcal{B}} = \sum_j \langle \eta_j, \eta_j \rangle_{\bullet} \quad \text{for some } \{\eta_1, \eta_2 \ldots, \eta_n\} \in \mathcal{E}. \tag{39}$$

We give the definition of a projective module.

Definition A Hilbert $\mathcal{A}$-module $\mathcal{E}$ is **finitely generated, projective** if there exists a projection P in $M_n(\mathcal{A})$, the space of $n \times n$-matrices with values in $\mathcal{A}$, such that $\mathcal{E} = P\mathcal{A}^n$.

We take this opportunity to introduce two notions of frames: *standard module frames* for Hilbert C^*-modules, and the other one for Hilbert spaces.

Definition A **standard module frame** for a finitely generated Hilbert C^*-module $\mathcal{E}$ is a set $\{\eta_1, \eta_2, \ldots, \eta_n\} \subset \mathcal{E}$, such that for $C_1, C_2 > 0$

$$C_1 \; {}_{\bullet}\langle \xi, \xi \rangle \leq \sum_j \; {}_{\bullet}\langle \xi, \eta_j \rangle_{\bullet} \langle \eta_j, \xi \rangle \leq C_2 \; {}_{\bullet}\langle \xi, \xi, \rangle \quad \text{for all } \xi \in \mathcal{E} \tag{40}$$

The frame is called tight if $C_1 = C_2$ and normalized if $C_1 = C_2 = 1$

One can show that any $\xi \in \mathcal{E}$ has a decomposition of the form:

$$\xi = \xi 1_{\mathcal{B}} = \xi \sum_j \langle \eta_j, \eta_j \rangle_{\bullet} = \; {}_{\bullet}\langle \xi, \eta_1 \rangle \eta_1 + \ldots \; {}_{\bullet}\langle \xi, \eta_n \rangle \eta_n, \tag{41}$$

Definition A set $\{e_j \; : \; j \in J\}$ in a separable Hilbert space $\mathcal{H}$ is called a **frame** if there exists positive constants $C_1, C_2 > 0$, such that for all $f \in \mathcal{H}$

$$A||f||^2 \leq \sum_{j \in J} |\langle f, e_j \rangle|^2 \leq B||f||^2$$

A, B are called frame bounds, and if $A = B$ then the frame is tight.

The second definition will be used in the context of L^2-spaces mostly, while the standard module frame is useful for the non-commutative torus and the Moyal plane.

Importantly, if there exists finitely many η_j's such that (39) hold we can construct a projection matrix $P_{jl} = {}_{\bullet}\langle \eta_j, \eta_l \rangle$. It is a projection because

$$\begin{aligned} P_{jl}^2 = \sum_{k=1}^n {}_{\bullet}\langle \eta_j, \eta_k \rangle_{\bullet} \langle \eta_k, \eta_l \rangle &= \sum_k {}_{\bullet}\langle {}_{\bullet}\langle \eta_j, \eta_k \rangle \eta_k, \eta_l \rangle \\ &= \sum_k \; {}_{\bullet}\langle \eta_j \langle \eta_k, \eta_k \rangle_{\bullet}, \eta_l \rangle = {}_{\bullet}\langle \eta_j, \eta_l \rangle \end{aligned}$$

This calculation relies on the $\mathcal{A}$-valued inner product acting on the left, and the associativity condition. P is therefore a projection in $M_n(\mathcal{A})$, the space of $n \times n$

matrices with values in $\mathcal{A}$. This projection, from a standard fact of projections and modules establishes that we may view the connecting space $\mathcal{E}$ as a finitely generated projective left module over $\mathcal{A}$, that is the identification $\mathcal{A}^n P = \mathcal{E}$.

As a Hilbert $\mathcal{A}$-module, $\mathcal{E}$ is self-dual. For any *-homomorphism $\phi : {}_{\mathcal{A}}\mathcal{E} \to {}_{\mathcal{A}}\mathcal{A}$, which is the generalization of linear functionals to a Hilbert $\mathcal{A}$-module, we have

$$\begin{aligned}\phi(\xi) &= \phi\Big(\sum_{k=1}^{n} {}_{\bullet}\langle \xi, \eta_k\rangle \eta_k\Big) = \sum_k {}_{\bullet}\langle \xi, \eta_k\rangle \phi(\eta_k) \\ &= \Big(\sum_k \phi(\eta_k)^* {}_{\bullet}\langle \eta_k, \xi\rangle\Big)^* = \sum_k {}_{\bullet}\langle \xi, \phi(\eta_k)^* \eta_k\rangle = {}_{\bullet}\langle \xi, \sum_k \phi(\eta_k)^* \eta_k\rangle,\end{aligned}$$

for every $\xi \in \mathcal{E}$. Defining $\zeta_\phi = \sum_k \phi(\eta_k)^* \eta_k \in \mathcal{E}$, it is clear that $\phi(\xi) = {}_{\bullet}\langle \xi, \zeta_\phi\rangle$, and that this holds for all such ϕ. Conversely, let $\xi \in \mathcal{E}$, then we cam define the corresponding "functional" to be $\phi_\xi(\mu) = {}_{\bullet}\langle \mu, \xi\rangle$, which is in $\mathrm{Hom}({}_{\mathcal{A}}\mathcal{E}, {}_{\mathcal{A}}\mathcal{A})$ by the properties of the inner product.

We gather these results in the following proposition originally due to Rieffel [27].

Proposition 2.10 *Let $\mathcal{A} \rightleftharpoons \mathcal{E} \rightleftharpoons \mathcal{B}$ be a dual pair. Assume $\mathcal{B}$ is unital, so that there exists a Parseval standard module frame $\{\eta_k\}_{k=1}^n$. Then $\mathcal{E}$ is a projective $\mathcal{A}$-module such that $\mathcal{E} \simeq \mathcal{A}^n P$ isometrically, where P is the matrix $(P_{jk}) = ({}_{\bullet}\langle \eta_j, \eta_k\rangle)$. Furthermore, $\mathcal{E}$ is self-dual as a Hilbert C^*-module over $\mathcal{A}$. There is a unique $\zeta_\phi \in \mathcal{E}$ for every $\phi \in Hom({}_{\mathcal{A}}\mathcal{E}, {}_{\mathcal{A}}\mathcal{A})$*

Proof We have already established the second part of the proposition, so we focus on the first part.

P is clearly a projection, $P^2 = P$ as previously shown, and $P^* = P$ by the involution properties of the inner product. Define $\psi : \mathcal{E} \to \mathcal{A}^n$ by

$$(\psi\xi)_j = {}_{\bullet}\langle \xi, \eta_j\rangle.$$

By linearity of the inner product, this is an $\mathcal{A}$-module homomorphism. It is injective as a result of the reconstruction formula (41). If $(\psi\xi)_j = (\psi\mu)$ for all j in the frame, then

$$\sum_k {}_{\bullet}\langle \xi, \eta_k\rangle = \sum_k {}_{\bullet}\langle \mu, \eta_k\rangle$$

Multiplying by an element of the frame on the right side allows us to conclude that $\xi = \mu$.

For every $\xi \in \mathcal{E}$ we have

$$\begin{aligned}((\psi\xi)P)_j &= \sum_{k=1}^{n} {}_{\bullet}\langle \xi, \eta_k\rangle {}_{\bullet}\langle \eta_j, \eta_k\rangle = \sum_k {}_{\bullet}\langle \xi, {}_{\bullet}\langle \eta_j, \eta_k\rangle \eta_k\rangle \\ &= {}_{\bullet}\langle \xi, \sum_k \eta_j \langle \eta_k, \eta_k\rangle_{\bullet}\rangle = {}_{\bullet}\langle \xi, \eta_j\rangle = (\psi\xi)_j.\end{aligned}$$

Since multiplication of P on $\psi(\xi)$ is the identity, we can conclude that the range of P contains the range of ψ. Conversely, let v be in the range of P, that is $v \in \mathcal{A}^n$ and $v = vP$. Then every coordinate of v is of the form

$$v_j = \sum_k v_k {}_\bullet\langle \eta_k, \eta_j \rangle = {}_\bullet\langle \sum_k v_k \eta_k, \eta_j \rangle.$$

Let $\xi = \sum_k \eta_k v_k$. Then $\psi(\xi)_j = {}_\bullet\langle \sum_k \eta_k v_k, \eta_j \rangle$, and thus $\phi(\xi) = v$. The ranges therefore coincide and P projects exactly onto the range of ψ. We have constructed an isomorphism between $\mathcal{E}$ and $\mathcal{A}^n P$, and $\mathcal{E}$ is therefore a finitely generated projective $\mathcal{A}$-module. Lastly we show that ψ is also an isometry, where we are using the standard inner product for vectors in $\mathcal{A}^n$. Let $\xi, \mu \in \mathcal{E}$. Then

$$\begin{aligned}\langle \psi\xi, \psi\mu \rangle &= \sum_k (\psi\xi)_k \cdot (\psi\mu)_k^* = \sum_k {}_\bullet\langle \xi, \eta_k \rangle {}_\bullet\langle \mu, \eta_k \rangle^* \\ &= \sum_k {}_\bullet\langle \xi, \eta_k \rangle {}_\bullet\langle \eta_k, \mu \rangle = {}_\bullet\langle \sum_k {}_\bullet\langle \xi, \eta_k \rangle \eta_k, \mu \rangle = {}_\bullet\langle \xi, \mu \rangle.\end{aligned}$$

∎

We can use this result to construct a dual frame to $\{\eta_1, \eta_2, \cdots, \eta_n\}$. $\mathcal{A}$ and $\mathcal{B}$ have already been given left- and right-linear structures respectively, by being dual pairs over $\mathcal{E}$. We now define opposite actions for these spaces. Let the right action of $\mathcal{A}$ on $\mathcal{E} \simeq \mathrm{Hom}({}_\mathcal{A}\mathcal{E}, {}_\mathcal{A}\mathcal{A})$ be

$$\phi_\xi \cdot a = R_a \circ \phi_\xi = \phi_{a^*\xi} \quad \text{for all } a \in \mathcal{A},$$

and the left action of $\mathcal{B}$ be

$$b \cdot \phi_\xi = \phi_\xi \circ R_b \quad \text{for all } b \in \mathcal{B}.$$

Explicitly this can be viewed as

$$(\phi_\xi \cdot a)(\mu) = {}_\bullet\langle \mu, a^*\xi \rangle = {}_\bullet\langle a^*\xi, \mu \rangle^* = {}_\bullet\langle \mu, \xi \rangle a$$

. We use the given left-action of $\mathcal{A}$ to pull a out of the inner product. In the same manner the action of $\mathcal{B}$ is

$$(b \cdot \phi_\xi)(\mu) = (\phi_\xi \circ R_b)(\mu) = \phi_\xi(\mu b) = {}_\bullet\langle \mu b, \xi \rangle.$$

Note that we cannot do the same for the $\mathcal{B}$-action because we are using the inner product over $\mathcal{A}$.

Now we compute, by using (41)

$$\begin{aligned}\phi_\xi(\mu) = \phi_{\sum_k {}_\bullet\langle \xi, \eta_k \rangle \eta_k}(\mu) &= \sum_k {}_\bullet\langle \mu, {}_\bullet\langle \xi, \eta_k \rangle \eta_k \rangle \\ &= \sum_k {}_\bullet\langle \mu, \eta_k \rangle {}_\bullet\langle \eta_k, \xi \rangle = \sum_k \big(\phi_{\eta_k} \cdot {}_\bullet\langle \eta_k, \xi \rangle\big)(\mu).\end{aligned}$$

Where we have used the right action of $\mathcal{A}$ to get from the first to the second line. We have decomposed ϕ_ξ as

$$\phi_\xi = \phi_{\eta_1} \cdot {}_\bullet\langle \eta_1, \xi\rangle + \cdots + \phi_{\eta_n} \cdot {}_\bullet\langle \eta_n, \xi\rangle, \tag{42}$$

which is dual to the decomposition (41), and we denote ϕ_{η_k} as the dual frame.

Central to this work is the notion of projections and their connections to the Morita equivalence of $\mathcal{A}$ and $\mathcal{B}$. We have already seen that for a Parseval frame (η_j), the values ${}_\bullet\langle \eta_j, \eta_k\rangle$ define a projection $P \in \mathcal{A}^n$. This result can be generalized somewhat.

Lemma 2.11 *Let $\eta \in \mathcal{E}$ a single element defining a standard module frame for the bimodule $\mathcal{A} \rightleftharpoons \mathcal{E} \rightleftharpoons \mathcal{B}$. Then $p := {}_\bullet\langle \eta, \eta\rangle$ is a projection in $\mathcal{A}$ if and only if*

$$\eta\langle \eta, \eta\rangle_\bullet = \eta$$

Note that for a Parseval frame with one element η, by definition $\langle \eta, \eta\rangle_\bullet = 1_\mathcal{B}$, so the second part is trivially satisfied.

Proof Assume $\eta\langle \eta, \eta\rangle_\bullet = \eta$. Clearly $p^* = p$, so we need only show $p^2 = p$.

$${}_\bullet\langle \eta, \eta\rangle {}_\bullet\langle \eta, \eta\rangle = {}_\bullet\langle {}_\bullet\langle \eta, \eta\rangle \eta, \eta\rangle = {}_\bullet\langle \eta\langle \eta, \eta, \rangle_\bullet, \eta\rangle = {}_\bullet\langle \eta, \eta\rangle$$

Where we use left-linearity of the inner product over $\mathcal{A}$, the associativity condition and the assumption respectively.

Assume now that $p = {}_\bullet\langle \eta, \eta\rangle$ is a projection, to arrive at the conclusion we prove that $\eta\langle \eta, \eta\rangle_\bullet - \eta = 0$. To do this we use the positivity of the inner products and show that ${}_\bullet\langle \eta\langle \eta, \eta\rangle_\bullet - \eta, \eta\langle \eta, \eta\rangle_\bullet - \eta\rangle = 0$. This is a straightforward calculation, but we will for clarity show a part of it. Note firstly that by linearity we can split this into two inner products, with left side $\eta\langle \eta, \eta\rangle_\bullet$ and $-\eta$ respectively. Then use the associativity condition, now

$$\begin{aligned}
{}_\bullet\langle {}_\bullet\langle \eta, \eta\rangle \eta, {}_\bullet\langle \eta, \eta\rangle \eta - \eta\rangle &= {}_\bullet\langle \eta, \eta\rangle {}_\bullet\langle \eta, {}_\bullet\langle \eta, \eta\rangle \eta - \eta\rangle \\
&= {}_\bullet\langle \eta, \eta\rangle \big({}_\bullet\langle {}_\bullet\langle \eta, \eta\rangle \eta - \eta, \eta\rangle\big)^* \\
&= {}_\bullet\langle \eta, \eta\rangle \big({}_\bullet\langle \eta, \eta\rangle {}_\bullet\langle \eta, \eta\rangle - {}_\bullet\langle \eta, \eta\rangle\big)^* \\
&= {}_\bullet\langle \eta, \eta\rangle^3 - {}_\bullet\langle \eta, \eta\rangle^2 \\
&= 0.
\end{aligned}$$

Doing the same process on the other inner product yields the same answer. We therefore have that $\eta\langle \eta, \eta\rangle_\bullet = \eta$ ∎

We take this opportunity to also introduce the holomorphic and anti-holomorphic connections on $\mathcal{E}$,

$$\nabla = \nabla_1 - i\nabla_2 \quad \text{and} \quad \overline{\nabla} = \nabla_1 + i\nabla_2. \tag{43}$$

Clearly they are lifts of the natural complex derivatives on $\mathcal{A}$ and $\mathcal{B}$. These are also known as creation and annihilation operators from quantum mechanics on the Moyal plane. We use the same notation for the complex derivatives, that is $\partial = \partial_1 - i\partial_2$, and $\overline{\partial} = \partial_1 + i\partial_2$.

Proposition 2.12 *Let $\eta \in \mathcal{E}$ be such that $\langle \eta, \eta, \rangle_\bullet = 1_\mathcal{B}$. Then the corresponding projection $p_\eta = {}_\bullet\langle \eta, \eta\rangle \in \mathcal{A}$ is a solution of the "self-duality" equation*

$$\overline{\partial}(p_\eta)p_\eta = 0 \tag{44}$$

if and only if η satisfies

$$\overline{\nabla}\eta = \eta\lambda \quad \textit{for some } \lambda \in \mathcal{B}$$

Proof We will express $\overline{\partial}(p_\eta)p_\eta$ in a different way. By using the equation (37) and the associativity condition

$$\overline{\partial}(p_\eta)p_\eta = {}_\bullet\langle\overline{\nabla}\eta, \eta\rangle_\bullet\langle\eta,\eta\rangle + {}_\bullet\langle\eta, \nabla\eta\rangle_\bullet\langle\eta,\eta\rangle.$$

Since ${}_\bullet\langle\cdot,\cdot\rangle$ is linear in the first argument and conjugate in the second, we will have first the holomorphic and then the anti-holomorphic covariant derivatives. With the same argument one concludes that $\overline{\partial}(\langle\eta,\eta\rangle_\bullet) = \langle\nabla\eta,\eta\rangle_\bullet + \langle\eta,\overline{\nabla}\eta\rangle_\bullet$. Since $\overline{\partial}(1_\mathcal{B}) = 0$ we have $\langle\nabla\eta,\eta\rangle_\bullet = -\langle\eta,\overline{\nabla}\eta\rangle$. Continuing on:

$$\begin{aligned}\overline{\partial}(p_\eta)p_\eta &= {}_\bullet\langle{}_\bullet\langle\overline{\nabla}\eta,\eta\rangle\eta,\eta\rangle + {}_\bullet\langle{}_\bullet\langle\eta,\nabla\eta\rangle\eta,\eta\rangle\\ &= {}_\bullet\langle\overline{\nabla}\eta\langle\eta,\eta\rangle_\bullet,\eta\rangle + {}_\bullet\langle\eta\langle\nabla\eta,\eta\rangle_\bullet,\eta\rangle\\ &= {}_\bullet\langle\overline{\nabla}\eta - \eta\langle\eta,\overline{\nabla}\eta\rangle_\bullet,\eta\rangle.\end{aligned}$$

This is an element of $\mathcal{A}$ and can therefore be applied to $\eta \in \mathcal{E}$,

$$\begin{aligned}{}_\bullet\langle\overline{\nabla}\eta - \eta\langle\eta,\overline{\nabla}\eta\rangle_\bullet,\eta\rangle\eta &= \big(\overline{\nabla}\eta - \eta\langle\eta,\overline{\nabla}\eta\rangle_\bullet\big)\langle\eta,\eta\rangle_\bullet\\ &= \overline{\nabla}\eta - \eta\langle\eta,\overline{\nabla}\eta\rangle_\bullet.\end{aligned}$$

We conclude that if $\overline{\partial}(p_\eta)p_\eta = 0$, then $\overline{\nabla}\eta = \eta\langle\eta,\overline{\nabla}\eta\rangle_\bullet$, and since $\langle\eta,\overline{\nabla}\eta\rangle_\bullet \in \mathcal{B}$, we call it λ. The other implication follows by doing the calculations the opposite way. ∎

We lastly introduce the idea of curvature for our bidual, and define it as

$$F_{1,2} := [\nabla_1, \nabla_2] = \nabla_1\nabla_2 - \nabla_2\nabla_1 \tag{45}$$

which is also known as the Heisenberg commutation relations in quantum mechanics. We note that in our case we have that

$$F_{1,2}\xi(t) = (\nabla_1\nabla_2 - \nabla_2\nabla_1)\xi(t) = it\xi'(t) - (i\xi(t) + it\xi'(t)) = -i\xi(t),$$

so that $F_{1,2} = -iId_\mathcal{E}$. It is from this connection to the Moyal plane that the terms annihilation and creation operators originate. The curvature being constant gives us the term constant curvature connection for these structures.

2.2 Modulation Spaces

2.2.1 Weighted mixed-norm and modulation spaces

We will assume that the reader is familiar with general L^p-spaces, and some basic functional analysis, especially the Hahn-Banach Theorem and its consequences. See any introductory book on the subject, for instance [21], for details. This section will aim to construct and present the most important aspects of modulation spaces with radial symmetric weights. With this in mind, we will introduce mixed-norm spaces, the short time Fourier transform and the concept of moderate weight functions. See [7], [16] for a more complete introduction of frame-theory and modulation spaces.

The short time Fourier transform of f with respect to the window function g, both functions on $\mathbb{R}^d$, $V_g f$ is formally defined as

$$V_g f(x, w) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i t\cdot\omega} \overline{g(t-x)} dt \tag{46}$$

By defining the unitary operators, and their adjoints,

$$M_\omega f(t) = e^{2\pi i t\cdot\omega} f(t) \qquad M_w^* = M_{-w} \tag{47}$$

$$T_x f(t) = f(t-x) \qquad T_x^* = T_{-x} \tag{48}$$

We can shorten the definition to, in the distributional sense

$$V_g f(x, w) = \langle f, M_\omega T_x g\rangle \tag{49}$$

Here we have assumed that f and g are in appropriate spaces, for instance $f \in \mathcal{S}(\mathbb{R}^d)$ and $g \in \mathcal{S}'(\mathbb{R}^d)$. There exists many equivalent expressions of the STFT, too many to mention here, they will be introduced as they come of use.

The short time Fourier transform can, as the name suggests, be regarded as the Fourier transform of the function f over a short interval. Using some g with fast decay around the origin, $g(t-x)$ will give us the Fourier transform of f restricted to some interval around x. We will see that $V_g f(x, \omega)$ can, in some sense, be viewed as a measure of the amplitude of the frequency near ω at time x. From the theory of Fourier analysis, we know that $V_g f$ contains all possible information of f, and a central question of frame theory is if it possible to turn the integral into a discrete sum and still retain the properties of f, or being able to reconstruct the function.

2.2.2 Moderate weight functions

A weight function will in our case be a non-negative, locally integrable function on $\mathbb{R}^{2d}$. We will mostly working with the following types of weights

Definition A weight function v is called **submultiplicative** if

$$v(z_1 + z_2) \le v(z_1)v(z_2) \quad \text{for all } z_1, z_2 \in \mathbb{R}^{2d}. \tag{50}$$

A weight function m is called **v-moderate** if there exists some $C > 0$ such that

$$m(z_1 + z_2) \leq Cv(z_1)m(z_2) \quad \text{for all } z_1, z_2 \in \mathbb{R}^{2d}. \tag{51}$$

Two weights m_1, m_2 are **equivalent** if

$$C^{-1}m_1(z) \leq m_2(z) \leq Cm_1(z) \quad \text{for all } z \in \mathbb{R}^{2d} \tag{52}$$

we will write $m_1 \asymp m_2$ for equivalent weights, and do the same for equivalent norms.

In general we will denote a submultiplicative weight by v and a v-moderate weight by m, we will be also assume that v is symmetric in both coordinates. The standard weight we will be using is the radial symmetric polynomial $v_s(z) = (1 + |z|^2)^{s/2} = (1 + (x^2 + \omega^2))^{\frac{s}{2}}$.

To show some nice properties of weighted mixed norm-spaces we will require the following lemma.

Lemma 2.13 *1. If m is a v-moderate weight function, then*

$$\frac{m(z)}{Cv(t)} \leq m(z - t) \leq Cv(t)m(z). \tag{53}$$

2. For $0 \leq t \leq s$ the weight v_t is submultiplicative and both v_t and v_t^{-1} are v_s-moderate.

3. If $s \geq 2d$, then

$$\frac{1}{v_s} * \frac{1}{v_s}(z) \leq C_s \frac{1}{v_s}(z). \tag{54}$$

Proof 1. Since m is v-moderate, $m(z-t+t) \leq Cv(t)m(z-t)$. Immediately this gives us the left inequality, then since v is symmetric $m(z - t) \leq Cv(t)m(z)$ and we are done.

2. We have

$$v_t(z_1 + z_2) = (1 + |z_1 + z_2|)^{t/2} \leq (1 + |z_1|)^{t/2}(1 + |z_2|)^{t/2} = v_t(z_1)v_t(z_2),$$

showing submultiplicativity. Then since $(1 + |z_1|) \geq 1$

$$v_t(z_1)v_t(z_2) \leq Cv_s(z_1)v_t(z_2)$$

for some constant $C \geq 1$. Setting $z_2 = w_1 + w_2$ and $z_1 = -w_1$ and dividing by $v_t(w_1 + w_2)$ in the previous calculation we see that

$$v_t(w_1 + w_2)^{-1} \leq v_t(w_1)v_t(w_2)^{-1} \leq Cv_s(w_1)v_t(w_2)^{-1},$$

showing v_s-moderateness. Note that this is okay as $v_t \geq 1$.

3. By the definition of convolution, we need to show that

$$\int_{\mathbb{R}^{2d}} (1+|t|)^{-s}(1+|x-t|)^{-s}dt \leq C_s(1+|x|)^{-s}.$$

We will split $\mathbb{R}^{2d}$ into two regions and find good estimates for both, showing convergence. For every $x \in \mathbb{R}^{2d}$ we can define

$$\mathcal{N}_x = \{t| \quad |t-x| \leq \frac{|x|}{2}\} \text{ and } \mathcal{N}_x^c = \{t| \quad |t-x| > \frac{|x|}{2}\}.$$

If $t \in \mathcal{N}_x$, then $|t| \geq \frac{x}{2}$. Conversely, if t is not, then $|t-x| \geq \frac{|x|}{2}$. Therefore we have the inequalities

$$\begin{aligned}\int_{\mathcal{N}_x} (1+|t|)^{-s}(1+|x-t|)^{-s}dt &\leq (1+\frac{|x|}{2})\int_{\mathcal{N}_x}(1+|x-t|)^{-s}dt \\ &\leq 2^s(1+|x|)^{-s}\int_{\mathcal{N}_x}(1+|t-x|)^{-s}dt \\ \int_{\mathcal{N}_x^c} (1+|t|)^{-s}(1+|x-t|)^{-s}dt &\leq 2^s(1+|x|)^{-s}\int_{\mathcal{N}_x^c}(1+|t|)^{-s}dt\end{aligned}$$

By doing a change in variables, integrating over the whole $\mathbb{R}^{2d}$, and adding them together, we see that the initial integral is less than

$$2^{s+1}(1+|x|)^{-s}\int_{\mathbb{R}^{2d}}(1+|t|)^{-s}dt$$

Which clearly converges if $s > 2d$. The lemma is therefore proved with $C_s = 2^{s+1}\int_{\mathbb{R}^{2d}}(1+|t|)^{-s}dt$

These results hold in a similar way for the discrete case, but we omit the proofs.

2.2.3 Weighted mixed-norm spaces

Definition Let m be a weight function on $\mathbb{R}^{2d}$ and let $1 \leq p, q < \infty$. Then the **weighted mixed-norm space** $L_m^{p,q}(\mathbb{R}^{2d})$ consists of all measurable functions on $\mathbb{R}^{2d}$, such that the norm

$$||F||_{L_m^{p,q}} = \Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|F(x,w)|^p m(x,w)^p dx\Big)^{\frac{q}{p}}dw\Big)^{\frac{1}{q}} \tag{55}$$

is finite. If p or q is infinite, the corresponding integral is replaced by the essential supremum. This is essentially a weighted L^p-norm with respect to x, and an L^q-norm with respect to w.

We write $L_m^p = L_m^{p,p}$ and $L^{p,q} = L_m^{p,q}$ if $m \equiv 1$. $L_m^{p,q}$-spaces behave mostly in the way one would expect, and enjoy most of the same properties as regular L^p-spaces.

Lemma 2.14 *Let m be v-moderate and $1 \leq p, q \leq \infty$. Then*

1. *$L^{p,q}_m(\mathbb{R}^{2d})$ is a Banach space.*

2. *$L^{p,q}_m(\mathbb{R}^{2d})$ is invariant under the translations $T_tF(z) := F(z-t), t = (u,\eta) \in \mathbb{R}^{2d}$, and*
$$||T_tF||_{L^{p,q}_m} \leq Cv(t)||F||_{L^{p,q}_m} \tag{56}$$

3. ***Hölder's inequality** holds. That is, for $F \in L^{p,q}_m(\mathbb{R}^{2d})$ and $H \in L^{p',q'}_{1/m}(\mathbb{R}^{2d})$, with the usual $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$, $F \cdot H \in L^1(\mathbb{R}^{2d})$, and*
$$\left| \int_{\mathbb{R}^{2d}} F(z)\overline{H(z)}dz \right| \leq ||F||_{L^{p,q}_m} ||H||_{L^{p',q'}_{1/m}} \tag{57}$$

4. ***Duality**: If $p, q < \infty$ then $(L^{p,q}_m(\mathbb{R}^{2d}))^* = L^{p',q'}_{1/m}(\mathbb{R}^{2d})$, and the duality is given in the normal distributional sense,*
$$\langle F, H \rangle = \int_{\mathbb{R}^{2d}} F(z)\overline{H(z)}dz \tag{58}$$
with F and H as above.

Proof 1. The proof is time- and space-consuming and offers little to reader already familiar with L^p-spaces, so we omit it here. The more general space L^P for $P = (p_1, p_2, \dots)$ is even a Banach space, see [15] for the proof and a more in-depth introduction of all the properties of general mixed norm spaces, or [4] for the finite case.

2. By doing a change of variables and by v-moderateness of m we have
$$\begin{aligned}
||T_tF|| &= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x-u, w-\eta)|^p m(x,w)^p dx \right)^{\frac{q}{p}} dw \right)^{\frac{1}{q}} \\
&= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x, w)|^p m(x+u,w+\eta)^p dx \right)^{\frac{q}{p}} dw \right)^{\frac{1}{q}} \\
&\leq C \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x, w)|^p v(u,\eta)^p m(x,w)^p dx \right)^{\frac{q}{p}} dw \right)^{\frac{1}{q}} \\
&= Cv(t)||F||.
\end{aligned}$$

3. Let $F \in L^{p,q}_m(\mathbb{R}^{2d})$ and $H \in L^{p',q'}_{1/m}(\mathbb{R}^{2d})$. Then
$$\left| \int_{\mathbb{R}^{2d}} F(z)\overline{H(z)}dz \right| \leq \int_{\mathbb{R}^{2d}} |F(z)||H(z)|dz = \int_{\mathbb{R}^{2d}} |F(z)m(z)||H(z)\frac{1}{m(z)}|dz.$$

For every w, $F(x,w) \in L^p_m(\mathbb{R}^d)$, and similarly $H(x,w) \in L^{p'}_{1/m}(\mathbb{R}^d)$, so we can apply Hölder's inequality for $L^p(\mathbb{R}^d)$,

$$\leq \int_{\mathbb{R}^d} ||F(\cdot,w)||_{L^p_m} ||H(\cdot,w)||_{L^{p'}_{1/m}} dw.$$

By definition of $L^{p,q}_m(\mathbb{R}^{2d})$, we have that $||F(\cdot,w)||_{L^p_m}|| \in L^q(\mathbb{R}^d)$, so we can apply Hölder's once more,

$$\leq ||(||F(\cdot,\cdot)||_{L^p_m})||_{L^q} \cdot ||(||H(\cdot,\cdot)||_{L^{p'}_{1/m}})||_{L^{q'}} = ||F||_{L^{p,q}_m} ||H||_{L^{p',q'}_{1/m}}$$

4. The proof is similar to the case for regular L^p-spaces and gives little new insight. See [4] for details.

∎

For general L^p-spaces we have the important convolution relation $L^1 * L^p \subseteq L^p$, which we show extends to mixed-norm spaces. Although, we will not consider the case when $p, q = \infty$, we mention the following result too, without proof.

Lemma 2.15 *If m is v-moderate, $F \in L^1_v(\mathbb{R}^{2d})$, and $G \in L^{p,q}_m(\mathbb{R}^{2d})$, then*

$$||F * G||_{L^{p,q}_m} \leq C||F||_{L^1_v} ||G||_{L^{p,q}_m}, \tag{59}$$

thus, $L^1_v * L^{p,q}_m \subseteq L^{p,q}_m$. Additionally, if $s > 2d$, then $L^\infty_{v_s} * L^\infty_{v_s} \subseteq L^\infty_{v_s}$, and we have the same relation as (59)*, with C depending on s.*

Proof Let $H \in L^{p',q'}_{1/m}(\mathbb{R}^{2d})$. Then by Hölder's inequality $F \cdot H \in L^1(\mathbb{R}^{2d})$, and we see that $|F(w)G(z-w)\overline{H(z)}| \in L^1(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$. Since we can change the order of integration in the following, by Fubini's theorem, and then apply (57) and (56).

$$\begin{aligned}
|\langle F * G, H \rangle| &= \left| \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} F(w)G(z-w)\overline{H(z)} dw dz \right| \\
&\leq \int_{\mathbb{R}^{2d}} |F(w)| \int_{\mathbb{R}^{2d}} |G(z-w)||H(z)| dz dw \\
&\leq \int_{\mathbb{R}^{2d}} |F(w)| ||T_w G||_{L^{p,q}_m} ||H||_{L^{p',q'}_{1/m}} dw \\
&\leq C||G||_{L^{p,q}_m} ||H||_{L^{p',q'}_{1/m}} \int_{\mathbb{R}^{2d}} |F(w)|v(w) dw \\
&\leq C||G||_{L^{p,q}_m} ||H||_{L^{p',q'}_{1/m}} ||F||_{L^1_v}.
\end{aligned}$$

By duality, we then find the estimate of the norm

$$||F * G||_{L^{p,q}_m} = \sup_{||H|| \leq 1} \{|\langle F * G, H \rangle|\} \leq C||G||_{L^{p,q}_m} ||F||_{L^1_v} \leq \infty.$$

∎

Before we can finally get to modulation spaces, we must first define the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. We will try to give the necessary introduction to the theory, but for a complete understanding of distribtions and differential operators see for instance [18].

The Schwartz functions are rapidly decreasing continuous functions on $\mathbb{R}^d$, and it's distribution space is therefore the space of functions that are increasing at a moderate (temperate) pace. To define this more rigorously, we introduce the standard multi-index notation. Let $\alpha = (\alpha_1, \alpha_2 \ldots, \alpha_d)$ for $\alpha_i \in \mathbb{N}$, $|\alpha| = \sum_{i=1}^d \alpha_i$, and define $\alpha \leq \beta$ means $\alpha_i \leq \beta_i$ for all $1 \leq i \leq d$. Then the differentiation and multiplication operators can be defined as

$$D^\alpha f := \prod_{i=1}^d \frac{\partial^{\alpha_i} f}{\partial x_i^{\alpha_i}} \qquad X^\alpha f(t) := \prod_{i=1}^d t_i^{\alpha_i} f(t). \tag{60}$$

This leads us to define a collection of seminorms $||f||_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} |X^\alpha D^\beta f(x)|$, and we define the Schwartz space as

Definition The **Schwartz space** $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing continuous functions is the function space

$$\mathcal{S}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) \;\mid\; ||f||_{\alpha,\beta} < \infty \;\text{ for all } \alpha, \beta \in \mathbb{N}^d\}. \tag{61}$$

The topology of this space is induced by the metric induced by the collection of seminorms, which is given (in general for Fréchet Spaces) by

$$d(f,g) = \sum_{\alpha,\beta\in\mathbb{N}^d} \frac{1}{2^{|\alpha|+|\beta|}} \frac{||f-g||_{\alpha,\beta}}{1+||f-g||_{\alpha,\beta}} \tag{62}$$

Note that the condition for convergence $f_n \to f$, has two equivalent forms, either $d(f_n, f) \to 0$ or $||f_n - f||_{\alpha,\beta} \to 0$ for all $\alpha, \beta \in \mathbb{N}^d$.

It should be clear that $\mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$, it is in fact true for all $L^p(\mathbb{R}^d)$-spaces, $1 \leq p < \infty$. The dual of this space is the function space of tempered distribution defined in the usual distributional sense.

Definition $\mathcal{S}'(\mathbb{R}^d)$ is the subspace of all distributions, $\mathcal{D}'(\mathbb{R}^d)$, given by

$$\mathcal{S}'(\mathbb{R}^d) = \{f \in \mathcal{D}'(\mathbb{R}^d) \;\mid\; |\langle \phi, f\rangle| < \infty \;\text{ for all } \phi \in \mathcal{S}(\mathbb{R}^d)\} \tag{63}$$

Continuity (coming from the natural norm) of $f \in \mathcal{S}'(\mathbb{R}^d)$ means that there exists a $C > 0$ and integers $M, N > 0$ such that

$$|\langle f, \phi\rangle| \leq C \sum_{|\alpha|\leq M} \sum_{|\beta|\leq N} ||\phi||_{\alpha,\beta} \tag{64}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^d)$.

Our current objective is finding for what choice of f, g the $V_g f$ should be defined. A natural assumption, by considering the distributional definition in (49) would be to let $f \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$, as they are dual spaces. Firstly, however, we will need some technicalities in order.

Lemma 2.16 *If $g \in \mathcal{S}(\mathbb{R}^d)$, then*

$$D^\alpha X^\beta (M_w T_x g) = \sum_{\gamma_1 \leq \alpha} \sum_{\gamma_2 \leq \beta} \binom{\alpha}{\gamma_1} \binom{\beta}{\gamma_2} x^{\gamma_2} (2\pi i w)^{\gamma_1} M_w T_x (D^{\alpha-\gamma_1} X^{\beta-\gamma_2} g)$$

for all $(x, w) \in \mathbb{R}^{2d}$ and multi-indices α, β.

Proof By applying the definitions, we see that $X^\beta T_x g(t) = t^\beta g(t-x) = T_x[(t+x)^\beta g(t)] = T_x(x+X)^\beta g(t)$. Applying the binomial identity to this expression yields

$$X^\beta T_x g = \sum_{\gamma_2 \leq \beta} \binom{\beta}{\gamma_2} x^{\gamma_2} T_x X^{\beta-\gamma_2} g.$$

A general theorem of Fourier analysis states that $\mathscr{F}(D^\alpha g) = (2\pi i)^{|\alpha|} X^\alpha \mathscr{F}(g)$ and that $\mathscr{F}(M_w g) = T_w \mathscr{F}(g)$, and can be easily verified. This can be used to see that

$$\begin{aligned} \mathscr{F}(D^\alpha M_w g) &= (2\pi i)^{|\alpha|} X^\alpha T_w \mathscr{F}(g) \\ &= \sum_{\gamma_1 \leq \alpha} \binom{\alpha}{\gamma_1} (2\pi i w)^{|\alpha|+\gamma_1} T_w X^{\alpha-\gamma_1} \mathscr{F}(g) \\ &= \sum_{\gamma_1 \leq \alpha} \binom{\alpha}{\gamma_1} (2\pi i w)^{\gamma_1} \mathscr{F}\big(M_w D^{\alpha-\gamma_1} g\big) \end{aligned}$$

This immediately implies that the left and right side are equal even when we remove the Fourier operator, as the transform is a homeomorphism on Schwartz space. Combining our equalities, the lemma is shown if M_w and T_x commutes with X^β and D^α respectively. This is clear however, two multiplication operators commute trivially, and the time-shift and differentiation requires only a small calculation. The lemma is therefore proven. ∎

Corollary 2.17 *The operator-valued map $(x, w) \to M_w T_x$ is strongly continuous on $\mathcal{S}(\mathbb{R}^d)$ and weak*-continuous on $\mathcal{S}'(\mathbb{R}^d)$.*

Proof As mentioned in the definition of $\mathcal{S}'(\mathbb{R}^d)$, we must have that for all $g \in \mathcal{S}(\mathbb{R}^d)$

$$\lim_{|x|,|w| \to 0} ||D^\alpha X^\beta (M_w T_x g - g)||_\infty = 0$$

for all $\alpha, \beta \in \mathbb{N}^d$. We can apply the previous lemma, and split the norm into the case where $\gamma_1 = \gamma_2 = 0$ and the rest, so that

$$||D^\alpha X^\beta (M_w T_x g - g)||_\infty \leq ||(M_w T_x (D^\alpha X^\beta g) - g||_\infty +$$
$$\sum_{\substack{\gamma_1 \leq \alpha \\ (\gamma_1,\gamma_2)\neq 0}} \sum_{\gamma_2 \leq \beta} \binom{\alpha}{\gamma_1}\binom{\beta}{\gamma_2} |x^{\gamma_2}(2\pi i w)^{\gamma_1}| \, ||(D^{\alpha-\gamma_1} X^{\beta-\gamma_2} g)||_\infty$$

We know that $C_0^\infty(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$, since it is dense in $C^\infty(\mathbb{R}^d)$. Convergence in the first term for $g \in C_0^\infty$ is clear, simply from the continuity of g. The second term converges to 0 because g, all its derivatives and multiplication by a polynomial are bounded. Then by a standard argument convergence for the whole Schwartz space is proven.

To show weak*-continuity, we recall the adjoint of our operators, (47) and (48). Then, for $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$

$$\lim_{|x|,|w|\to 0} \langle M_w T_x f, \phi\rangle = \lim_{|x|,|w|\to 0} \langle f, T_{-x} M_{-w} \phi\rangle = \langle f, \phi\rangle$$

by strong continuity in $\mathcal{S}(\mathbb{R}^d)$. ∎

By this result, and the distributional definition of the STFT, we can see that $V_g f$ is a continuous function for f a tempered distribution, it is in fact a very well-behaved function.

Theorem 2.18 *Let $g \in \mathcal{S}(\mathbb{R}^d)\backslash\{0\}$ and $f \in \mathcal{S}'(\mathbb{R}^d)$. Then $V_g f$ is continuous and there exists constants $C > 0$ and $N \geq 0$ such that*

$$|V_g f(x,w)| \leq C(1 + |x| + |w|)^N \quad \textit{for all } x, w \in \mathbb{R}^d \tag{65}$$

Proof Since $V_g f(x,w) = \langle f, M_w T_x g\rangle$ and the right-hand side is continuous by Corollary 2.17, the first statement is proved.

From the definition of continuity, (64), and (2.16), we have

$$|\langle f, M_w T_x g\rangle| \leq C \sum_{|\alpha|\leq M_1} \sum_{|\beta|\leq M_2} ||D^\alpha X^\beta (M_w T_x g)||_\infty$$
$$\leq C \sum_{\alpha,\beta} \sum_{\gamma_1,\gamma_2} \binom{\alpha}{\gamma_1}\binom{\beta}{\gamma_2} |x^{\gamma_2}| \, |(2\pi i w)^{\gamma_1}| \, ||D^{\alpha-\gamma_1} X^{\beta-\gamma_2}(g)||_\infty .$$

The norm $||D^\alpha X^\beta g||_\infty$ is bounded for all α and β, so we can pull it out of the sum.

$$\leq C \max_{(\alpha,\beta)\leq(M_1,M_2)} ||D^\alpha X^\beta g||_\infty \sum_{\alpha,\beta} \sum_{\gamma_1,\gamma_2} \binom{\alpha}{\gamma_1}\binom{\beta}{\gamma_2} |x^{\gamma_2}| \, |(2\pi i w)^{\gamma_1}|$$

This is then a polynomial in the variables $|x|$ and $|w|$, both in $\mathbb{R}^d$. The maximum degree of this polynomial is $\max(M_1, M_2) := N$, and any such polynomial is bounded by $C(1 + |x| + |w|)^N$ for a suitable C. ∎

This last result is not quite sufficient for our needs, remembering the work we did with weights, we would ideally have $V_g f$'s that had a faster decay than some weight function. Then $V_g f$ would be in the weighted space $L^{p,q}_v(\mathbb{R}^{2d})$.
Having rapid decay comes with many upsides, as we will se in the next theorem, where we are giving the definition for a possible inverse for the STFT.

Theorem 2.19 *Fix $g \in \mathcal{S}(\mathbb{R}^d)\backslash\{0\}$ and let $F(x,w)$ have rapid decay. That is, for every $n \geq 0$ there exists a $C_n > 0$ such that*

$$|F(x,w)| \leq C_n(1+|x|+|w|)^{-n}. \tag{66}$$

Then the integral

$$f(t) = \iint_{\mathbb{R}^{2d}} F(x,w) M_w T_x g(t)\ dw\ dx \tag{67}$$

defines a function f in $\mathcal{S}(\mathbb{R}^d)$

Proof Since $F(x,w)$ is bounded by a polynomial and g is a Schwartz function, the integral (67) is absolutely convergent, and we can therefore differentiate with respect to t as long as the answer is also absolutely convergent. Since our goal is to show that $f(t)$ is a Schwartz function, we try to differentiate and multiply by a polynomial. We know that since $F(x,w)$ is bounded by (66) for every $n \geq 0$, replacing F by $F \cdot P$ for any polynomial P will not change the convergence of f. By (2.16) we have

$$D^\alpha X^\beta f(t) = \iint_{\mathbb{R}^{2d}} F(x,w) D^\alpha X^\beta (M_w T_x g)(t)\ dw\ dx$$
$$= \sum_{\gamma_1 \leq \alpha} \sum_{\gamma_2 \leq \beta} \binom{\alpha}{\gamma_1}\binom{\beta}{\gamma_2} \iint_{\mathbb{R}^{2d}} F(x,w) x^{\gamma_2}(2\pi i w)^{\gamma_1} M_w T_x D^{\alpha-\gamma_1} X^{\beta-\gamma_2} g(t)\ dw\ dx.$$

Set $C = \max\{||M_w T_x D^{\gamma_1} X^{\gamma_2} g||_\infty \mid 0 \leq \gamma_1 \leq \alpha, 0 \leq \gamma_2 \leq \beta\}$. Then C is finite, since $g \in \mathcal{S}(\mathbb{R}^d)$, and we have

$$||D^\alpha X^\beta f||_\infty \leq C \iint_{\mathbb{R}^{2d}} |F(x,w)| \cdot |P(x,w)| dw dx < \infty. \tag{68}$$

The integral is finite because of the assumption on F. We have then showed that $f \in \mathcal{S}(\mathbb{R}^d)$ ∎

Before we embark on a classification of $\mathcal{S}(\mathbb{R}^d)$ via the STFT we show some properties of $V_g f$ which hold for all f and g in the larger space $L^2(\mathbb{R}^d)$.

Theorem 2.20 (The Orthogonality Lemma/The Moyal Identity) *Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$. Then both $V_{g_1} f_1$ and $V_{g_2} f_2$ are in $L^2(\mathbb{R}^{2d})$ and we have the relation*

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle = \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle} \tag{69}$$

Proof The proof is based on the famous Parseval Formula $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$, for all $f, g \in L^2(\mathbb{R}^d)$, where $\hat{f}$ denotes the Fourier transform of f. We will first show the equality and boundedness for $g \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$. By a denseness argument of $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$ it will also hold for $g \in L^2(\mathbb{R}^d)$. This is because both the mappings $g_1 \to \langle V_{g_1} f_1, V_{g_2} f_2\rangle$ and $g_2 \to \langle V_{g_1} f_1, V_{g_2} f_2\rangle$ are bounded linear functionals on the dense subspace that coincide with $\langle f_1, f_2\rangle\overline{\langle g_1, g_2\rangle}$. They can therefore be extended to the whole of $L^2(\mathbb{R}^d)$.

Let $g_1, g_2 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. We know that $f_j \cdot T_x g_j \in L^2(\mathbb{R}^d)$ for all $x \in \mathbb{R}^d$, so we can apply Parseval's formula in the fourth step of this calculation with some rearrangement of terms,

$$\begin{aligned}
\langle V_{g_1} f_1, V_{g_2} f_2\rangle &= \iint_{\mathbb{R}^{2d}} V_{g_1} f_1(x,w)\overline{V_{g_2} f_2(x,w)}dx\ dw \\
&= \iint_{\mathbb{R}^{2d}} \left[\int_{\mathbb{R}^d} f_1(t) e^{-2\pi i t\cdot w}\overline{g_1(t-x)}dt\right] \\
&\qquad \cdot \overline{\left[\int_{\mathbb{R}^d} f_2(t) e^{-2\pi i t\cdot w}\overline{g_2(t-x)}dt\right]}dx\ dw \\
&= \iint_{\mathbb{R}^{2d}} (f_1 \cdot T_x\overline{g_1})\hat{}\,(w)\overline{(f_2 \cdot T_x\overline{g_2})\ \hat{}\,(w)}dx\ dw \\
&= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f_1(t)\overline{f_2(t) g_1(t-x)} g_2(t-x)dt\right)dx.
\end{aligned}$$

By Fubini, we can then change the order of integration, yielding

$$= \int_{\mathbb{R}^d} f_1(t)\overline{f_2(t)}\left(\int_{\mathbb{R}^d}\overline{g_1(t-x)} g_2(t-x)dx\right)dt = \langle f_1, f_2\rangle\overline{\langle g_1, g_2\rangle}.$$

∎

This result will be very important in the main part of this paper, and is one of the main reasons we are able to construct dual pairs between our spaces. For now, note that this immediately implies the following.

Corollary 2.21 *If $f, g \in L^2(\mathbb{R}^d)$, then*

$$||V_g f||_2 = ||f||_2 ||g||_2$$

Note also that if $||g|| = 1$ this is a partial isometry from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{2d})$.
The last result we shall need before continuing with the Schwartz spaces is the following inversion formula. This is the result that allows us to represent functions via the frames constructed from time-frequency shifts of g, and is vitally important for our future work.

Corollary 2.22 *Let $g, \gamma \in L^2(\mathbb{R}^d)$ and let $\langle g, \gamma\rangle \neq 0$. Then, for all $f \in L^2(\mathbb{R}^d)$*

$$f = \frac{1}{\langle \gamma, g\rangle}\iint_{\mathbb{R}^{2d}} V_g f(x,w) M_w T_x \gamma\ dw\ dx \tag{70}$$

Proof Firstly, by the Corollary 2.21, $V_g f \in L^2(\mathbb{R}^{2d})$ and the integral in (70) is a well-defined function in $L^2(\mathbb{R}^d)$. We call it $\tilde{f}$ and show it to be equal to f by duality. For all $h \in L^2(\mathbb{R}^d)$ we have

$$\begin{aligned}
\langle \tilde{f}, h\rangle &= \frac{1}{\langle g,\gamma\rangle} \iint_{\mathbb{R}^{2d}} V_g f(x,w)\langle M_w T_x \gamma, h\rangle \; dw \; dx \\
&= \frac{1}{\langle \gamma, g\rangle} \iint_{\mathbb{R}^{2d}} V_g f(x,w)\overline{\langle h, M_w T_x \gamma\rangle} \; dw \; dx \\
&= \frac{1}{\langle \gamma, g\rangle}\langle V_g f, V_\gamma h\rangle = \frac{\overline{\langle g,\gamma\rangle}}{\langle \gamma, g\rangle}\langle f, h\rangle = \langle f, h\rangle.
\end{aligned}$$

Thus $\tilde{f} = f$. ∎

We now return to the space $\mathcal{S}(\mathbb{R}^d)$, where we can apply the inversion formula (70).

Theorem 2.23 *Fix $g \in \mathcal{S}(\mathbb{R}^d)\backslash\{0\}$. Then for $f \in \mathcal{S}'(\mathbb{R}^d)$ the following are equivalent:*

1. $f \in \mathcal{S}(\mathbb{R}^d)$.
2. $V_g f \in \mathcal{S}(\mathbb{R}^{2d})$
3. *for all $n \geq 0$, there is a $C_n \geq 0$ such that*

$$|V_g f(x,w)| \leq C_n(1+|x|+|w|)^{-n} \qquad \text{for all } x, w \in \mathbb{R}^d \tag{71}$$

Proof $1 \implies 2$: We use the standard definition of $(f\otimes g)(x,w) = f(x)g(t)$ along with the operators $\mathscr{F}_2 F(x,t) = \int_{\mathbb{R}^d} F(x,t)e^{-2\pi i t\cdot w}dw$, and $\mathcal{T}_a F(x,t) = F(t, t-x)$. Then $V_g f = \mathscr{F}_2\mathcal{T}_a(f\otimes \overline{g})$, and we state that if $f, g \in \mathcal{S}(\mathbb{R}^d)$ then $f\otimes g \in \mathcal{S}(\mathbb{R}^{2d})$. This can be shown by considering $C_0^\infty(\mathbb{R}^d)\otimes C_0^\infty(\mathbb{R}^d) = C_0^\infty(\mathbb{R}^{2d})$, and the completion of $C_0^\infty(\mathbb{R}^d)$ being $\mathcal{S}(\mathbb{R}^d)$. $\mathcal{S}(\mathbb{R}^{2d})$, being a subspace of $L^2(\mathbb{R}^{2d})$ is invariant under $\mathscr{F}_2$, and also under $\mathcal{T}_a$, so it follows that $V_g f \in \mathcal{S}(\mathbb{R}^{2d})$.
$2 \implies 3$ is clear by definition of rapidly decreasing functions.
$3 \implies 1$: Set

$$\tilde{f} = \frac{1}{||g||_2^2}\iint_{\mathbb{R}^{2d}} V_g f(x,w) M_w T_x g \; dx \; dw.$$

By 2.19, $\tilde{f} \in \mathcal{S}(\mathbb{R}^d)$, and by (70) $f = \tilde{f}$. ∎

Corollary 2.24 *If $g \in \mathcal{S}(\mathbb{R}^d)$, then the collection of seminorms*

$$||V_g f||_{L^\infty_{v_s}} = \sup_{z\in\mathbb{R}^{2d}} (1+|z|)^s |V_g f(z)|, \quad z \geq 0$$

forms an equivalent collection of seminorms for $\mathcal{S}(\mathbb{R}^d)$.

Proof Set

$$\widetilde{\mathcal{S}}(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d) \mid \sup_{z\in\mathbb{R}^{2d}} (1+|z|)^s |V_g f(z)| < \infty \text{ for all } s \geq 0\}.$$

Then we have an equality of sets $\widetilde{\mathcal{S}}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d)$ by Corollary 2.23. If $f \in \mathcal{S}(\mathbb{R}^d)$, then $V_g f \in \mathcal{S}(\mathbb{R}^{2d})$ so f is also in $\widetilde{S}(\mathbb{R}^d)$. Conversely, if $f \in \widetilde{S}(\mathbb{R}^d)$, (71) holds, so $f \in \mathcal{S}(\mathbb{R}^d)$.
We then need to show an equality of topology between these two spaces, by showing that their norms are equivalent. Since the norm over $\mathcal{S}(\mathbb{R}^d)$ is defined by the collection of multiplication by all polynomials and application of derivatives, and there is an equivalence between $f \in \mathcal{S}(\mathbb{R}^d)$ and $V_g f \in \mathcal{S}(\mathbb{R}^{2d})$ it should be clear that $C||f||_{\widetilde{\mathcal{S}}(\mathbb{R}^d)} \leq ||f||_{\mathcal{S}(\mathbb{R}^d)} = ||D^\alpha X^\beta f||_\infty$. Recalling (68), using it along with the inversion formula, and setting the polynomial $P(x,w)$ to be $v_n(z) = (1+|z|)^n$ for large enough n, we have

$$\begin{aligned} ||D^\alpha X^\beta f||_\infty &\leq C \int_{\mathbb{R}^{2d}} |V_g f(z)| P(z) dz \\ &\leq C' \int_{\mathbb{R}^{2d}} |V_g f(z)| (1+|z|)^n dz \\ &\leq \sup_{z\in\mathbb{R}^{2d}} |V_g f(z)| (1+|z|)^{n+2d+1} C' \int_{\mathbb{R}^{2d}} (1+|z|)^{-2d-1} dz \leq \infty \end{aligned}$$

Where we have constructed the last integral to be finite. It is simply a constant. Therefore there is an equivalence of norms, so the topologies coincide.
∎

Corollary 2.25 *Let $g, \gamma \in \mathcal{S}(\mathbb{R}^d)\backslash\{0\}$.*

1. *If $|F(x,w)| \leq C(1+|x|+|w|)^N$ for some constants $C, N \geq 0$, then the integral $\iint_{\mathbb{R}^{2d}} F(x,w) M_w T_x g \; dx \; dw$ defines a tempered distribution f. That is, for all $\phi \in \mathcal{S}(\mathbb{R}^d)$*

$$\langle f, \phi \rangle = \iint_{\mathbb{R}^{2d}} F(x,w) \langle M_w T_x g, \phi \rangle \; dx \; dw. \tag{72}$$

2. *If $F = V_g f$ for some $f \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$f = \frac{1}{\langle \gamma, g \rangle} \iint_{\mathbb{R}^{2d}} V_g f(x,w) M_w T_x \gamma \; dw \; dx. \tag{73}$$

Therefore, by having an inverse, the short-time Fourier Transform is injective on $\mathcal{S}(\mathbb{R}^d)$.

Proof 1. Since $|F(z)|$ has growth less than $|z|^N$ and $|V_g \phi(z)|$ has faster than polynomial decay (by 2.23), (72) converges absolutely. Therefore we can do as in the last proof,

$$
\begin{aligned}
|\langle f, \phi \rangle| &\leq C \iint_{\mathbb{R}^{2d}} (1+|z|)^N |V_g \phi(z)| \; dz \\
&\leq C \sup_{z \in \mathbb{R}^{2d}} |V_g \phi(z)| (1+|z|)^{N+2d+1} \iint_{\mathbb{R}^{2d}} (1+|z|)^{-2d-1} \; dz < \infty.
\end{aligned}
$$

This implies that f defines a continuous (bounded) linear functional on $\tilde{\mathcal{S}}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d)$, so is a tempered distribution. $f \in \mathcal{S}'(\mathbb{R}^d)$.

2. (73) defines a tempered distribution $\tilde{f}$, by *1.*, by the formula

$$
\langle \tilde{f}, \phi \rangle = \frac{1}{\langle \gamma, g \rangle} \iint_{\mathbb{R}^{2d}} V_g f(x,w) \langle M_w T_x \gamma, \phi \rangle \; dx \; dw.
$$

By the inversion formula for $\mathcal{S}(\mathbb{R}^d)$, we also have that

$$
\phi = \frac{1}{\langle g, \gamma \rangle} \iint_{\mathbb{R}^{2d}} V_\gamma \phi(x,w) M_w T_x g \; dx \; dw.
$$

Using the standard inner product, we then calculate

$$
\begin{aligned}
\langle f, \phi \rangle &= \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}^d} f(t) \overline{\iint_{\mathbb{R}^{2d}} V_\gamma \phi(x,w) M_w T_x g(t)} \; dx \; dw \; dt \\
&= \frac{1}{\langle \gamma, g \rangle} \iint_{\mathbb{R}^{2d}} \overline{V_\gamma \phi(x,w)} \int_{\mathbb{R}^d} f(t) \overline{M_w T_x g(t)} \; dt \; dx \; dw \\
&= \frac{1}{\langle \gamma, g \rangle} \iint_{\mathbb{R}^{2d}} \langle M_w T_x \gamma, \phi \rangle \langle f, M_w T_x g \rangle \; dx \; dw \\
&= \langle \tilde{f}, \phi \rangle.
\end{aligned}
$$

So $\tilde{f} = f$ and the inversion formula holds also on $\mathcal{S}'(\mathbb{R}^d)$. ∎

We have finally reached the point of being able to define the modulation spaces. As before the example to keep in mind of submultiplicative v will be the radial symmetric $v_s(z) = (1+|z|^2)^{s/2}$. We will initially be using the normalized Gaussian $g_0(t) = e^{-x^2}$ as window-function. Then, we can define the modulation space over $\mathbb{R}^d$, still depending on the choice of g_0.

Definition The **modulation space** $M_m^{p,q}(\mathbb{R}^d)$, for a v-moderate weight function m on $\mathbb{R}^{2d}$ and $1 \leq p, q \leq \infty$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_{g_0} f \in L_m^{p,q}(\mathbb{R}^{2d})$. The norm on $M_m^{p,q}$ is the natural one

$$
||f||_{M_m^{p,q}} = ||V_{g_0} f||_{L_m^{p,q}}. \tag{74}
$$

While this definition depends on the choice of g_0, we will soon see that different choices of $g \in \mathcal{S}(\mathbb{R}^d)$ all define equivalent norms. Hence we have a lot of freedom in selecting appropriate window functions $g \in \mathcal{S}(\mathbb{R}^d) \backslash \{0\}$. As with the mixed-weight normed spaces, we write $M^{p,p} = M^p$ and $M_m^{p,q} = M^{p,q}$ if $m \equiv 1$. The following proposition will make it clearer what the modulation space "measures", in the variables x and w, and its connection with generalized Sobolev Spaces (or Bessel potential spaces).

Proposition 2.26 *Let g_0 be the normalized Gaussian.*

1. *If $|f(x)| \leq C(1+|x|)^{-s}$ and $s > d$, then $|V_{g_0}f(x,w)| \leq C'(1+|x|)^{-s}$. Similarily, if $|\hat{f}(w)| \leq C(1+|w|)^{-s}$ and $s > d$, then $|V_{g_0}f(x,w)| \leq C'(1+|w|)^{-s}$.*
2. *If $m(x,w) = m(x)$, then $M^2_m(\mathbb{R}^d) = L^2_m(\mathbb{R}^d)$.*
3. *If $m(x,w) = m(w)$, then $M^2_m(\mathbb{R}^d) = \mathscr{F}L^2_m(\mathbb{R}^d)$.*
4. *for $v_{-s}(z) = (1+|z|)^{-s}$, we have*

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{s\geq 0} M^{\infty}_{v_{-s}}(\mathbb{R}^d) \quad \text{and} \quad \mathcal{S}'(\mathbb{R}^d) = \bigcup_{s\geq 0} M^{\infty}_{1/v_{-s}}(\mathbb{R}^d)$$

In the proof of this proposition, we must recall similar properties for the mixed-weight normed spaces, as well as Plancherel's theorem, which states that if $f \in L^1 \cap L^2$ then $||f||_2 = ||\hat{f}||_2$.

Proof 1. Let $\mathcal{I}g_0(t) = g_0(-t)$ be the involution operator. Then

$$|V_{g_0}f(x,w)| \leq \int_{\mathbb{R}^d} |f(t)\ |g_0(t-x)\ dt = |f| * |\mathcal{I}g_0|(x).$$

Now $g_0 \in L^{\infty}_{v_s}(\mathbb{R}^d)$, and by assumption f is also in this space. This is because the assumption immediately implies that for all $x \in \mathbb{R}^d$, $(1+|x|)^s|f(x)| \leq C$, the condition for being in $L^{\infty}_{v_s}(\mathbb{R}^d)$. Now we can apply Lemma 2.15, stating that $||\ |f| * |\mathcal{I}g_0|\ ||_{L^{\infty}_{v_s}} \leq C||f||_{L^{\infty}_{v_s}}||g_0||_{L^{\infty}_{v_s}}$. We therefore have that $V_{g_0}f \in L^{\infty}_{v_s}(\mathbb{R}^{2d})$, which means that $|V_{g_0}f(x,w)| \leq C'v_s(x)^{-1}$, where $C' \leq C||f||_{L^{\infty}_{v_s}}||g_0||_{L^{\infty}_{v_s}}$. The other case follows in exactly the same way on w instead.

2. Let $f \in M^2_m(\mathbb{R}^d)$. Then $||f||_{M^2_m} = \iint_{\mathbb{R}^{2d}} |V_{g_0}f(x,w)|^2 m(x)^2\ dw\ dx < \infty$. Since we can pull $m(x)$ out of the inner integral $V_{g_0}f(x,w) \in L^2(\mathbb{R}^d)$ as a function on w. We write $V_{g_0}f(x,w) = (f \cdot T_x g_0)\hat{}(w)$. Then we apply Plancherel's theorem and do some change of variables in the following calculation

$$\begin{aligned} ||f||^2_{M^2_m} &= \int_{\mathbb{R}^d} m(x)^2 \int_{\mathbb{R}^d} |(\widehat{f \cdot T_x g_0})(w)|^2\ dw\ dx \\ &= \int_{\mathbb{R}^d} m(x)^2 \int_{\mathbb{R}^d} |f(t)|^2 |g_0(t-x)|^2\ dt\ dx \\ &= \iint_{\mathbb{R}^{2d}} |f(t)|^2 |g_0(u)|^2 m(t-u)^2\ du\ dt. \end{aligned}$$

Using (53), we find the inequalities

$$C^{-2} \iint_{\mathbb{R}^d} |f(t)|^2 |g_0(u)|^2 m(t)^2 v(u)^{-2}\ du\ dt \leq ||f||_{M^2_m}$$
$$C^2 \iint_{\mathbb{R}^d} |f(t)|^2 |g_0(u)|^2 m(t)^2 v(u)^2\ du\ dt \geq ||f||_{M^2_m}.$$

Since we can split up these integrals, we have that

$$C^{-2}||f||_{L^2_m}||g||_{L^2_{1/v}} \leq ||f||_{M^2_m} \leq C^2||f||^2_{L^2_m}||g_0||_{L^2_v}, \tag{75}$$

so the norms are equivalent.

3. Follows in the same way as in 2.

4. This follows from our previous results. If $f \in \mathcal{S}'(\mathbb{R}^d)$, then by 2.18 we have that there exists $C > 0$ and $N > 0$ such that $|V_{g_0}f(x,w)| \leq C(1+|x|+|w|)^N$. Consequently $f \in M^{\infty}_{1/v_s}(\mathbb{R}^d)$ for some $s \geq 0$, and therefore we have equalities as sets, norm equivalences are shown in the usual manner. By 2.23, if $f \in \mathcal{S}(\mathbb{R}^d)$, then, for every $n \geq 0$, there exists a C_n such that $|V_{g_0}f(z)| \leq C_n(1+|x|+|w|)^{-n}$ for all $z \in \mathbb{R}^{2d}$. So $f \in M^{\infty}_{v_s}(\mathbb{R}^d)$ for every $s \geq 0$. We have the same set equality again. ∎

Defining a space is often not enough, we want function spaces to behave in a nice manner and have nice properties. In this case we would like to show that $M^{p,q}_m(\mathbb{R}^d)$ is a Banach space and find its dual space.

Definition Given a $\gamma \in \mathcal{S}(\mathbb{R}^d)\backslash\{0\}$, and a function F, let

$$V^*_\gamma F = \iint_{\mathbb{R}^{2d}} F(x,w)M_wT_x\gamma \; dx \; dw.$$

This is the adjoint of the short-time Fourier transform in the sense that

$$\begin{aligned}\langle V^*_\gamma F, f\rangle &= \iint_{\mathbb{R}^{2d}} F(x,w)\langle M_wT_x\gamma, f\rangle \; dx \; dw \\ &= \iint_{\mathbb{R}^{2d}} F(x,w)\overline{V_\gamma f(x,w)} \; dx \; dw \\ &= \langle F, V_\gamma f\rangle.\end{aligned}$$

Proposition 2.27 *Let m be v-moderate, and $\gamma \in \mathcal{S}(\mathbb{R}^d)$. Then*

1. *V^*_γ maps $L^{p,q}_m(\mathbb{R}^{2d})$ into $M^{p,q}_m(\mathbb{R}^d)$ and satisfies*

$$||V^*_\gamma F||_{M^{p,q}_m} \leq C||V_{g_0}\gamma||_{L^1_v}||F||_{L^{p,q}_m} \tag{76}$$

2. *In particular, if $F = V_g f$ then the inversion formula*

$$f = \frac{1}{\langle\gamma, g\rangle}\iint_{\mathbb{R}^{2d}} V_g f(x,w)M_wT_x\gamma \; dw \; dx \tag{77}$$

holds on $M^{p,q}_m(\mathbb{R}^d)$.

3. *$M^{p,q}_m(\mathbb{R}^d)$ is independent on choice of window $g_0 \in \mathcal{S}(\mathbb{R}^d)\backslash\{0\}$.*

Proof 1. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then, by using Hölder's inequality for $L^{p,q}_m(\mathbb{R}^{2d})$ and the same method as in the proof of 2.24, we have

$$\begin{aligned}|\langle V_\gamma^* F, \phi\rangle| &= |\langle F, V_\gamma \phi\rangle| \\ &\leq ||F||_{L^{p,q}_m} ||V_\gamma \phi||_{L^{p,q}_{1/m}} \\ &\leq ||F||_{L^{p,q}_m} ||(1+|z|)^n V_\gamma \phi||_\infty ||(1+|z|^{-n})||_{L^{p,q}_{1/m}}.\end{aligned}$$

This is finite for large enough n. Since by 2.24, the middle term is an equivalent seminorm on $\mathcal{S}(\mathbb{R}^d)$, $V_\gamma^* F$ is well-defined and bounded and is contained within the set of distributions on $\mathcal{S}(\mathbb{R}^d)$, $\mathcal{S}'(\mathbb{R}^d)$. Then we know that the short-time Fourier transform is well-defined and continuous on $V_\gamma^* F$. We calculate

$$\begin{aligned}V_{g_0} V_\gamma^* F(\mu,\eta) &= \langle V_\gamma^* F, M_\eta T_\mu g_0\rangle \\ &= \iint_{\mathbb{R}^{2d}} F(x,w)\overline{V_\gamma(M_\eta T_\mu g_0)(x,w)}\; dx\; dw \\ &= \iint_{\mathbb{R}^{2d}} F(x,w) V_{g_0}\gamma(\mu - x, \eta - w) e^{-2\pi i x\cdot(\eta - w)}\; dx\; dw.\end{aligned}$$

Since the exponential has absolute value 1, we have the estimate

$$|V_{g_0} V_\gamma^* F(\mu,\eta)| \leq (|F| * |V_{g_0}\gamma|)(\mu,\eta), \quad \text{for every}(\mu,\eta) \in \mathbb{R}^{2d}. \tag{78}$$

By assumption, $F \in L^{p,q}_m(\mathbb{R}^{2d})$, and since $\gamma \in \mathcal{S}(\mathbb{R})$, we know that $V_{g_0}\gamma \in \mathcal{S}(\mathbb{R}^{2d})$, so specifically also in $L^1_v(\mathbb{R}^{2d})$. Using 2.15, we get that

$$||V_{g_0} V_\gamma^* F||_{L^{p,q}_m} \leq ||\; |F| * |V_{g_0}\gamma|\; ||_{L^{p,q}_m} \leq C||F||_{L^{p,q}_m} ||V_{g_0}\gamma||_{L^1_v}$$

So $V_\gamma^* F \in M^{p,q}_m(\mathbb{R}^d)$.

2. By the above $\tilde{f} = \frac{1}{\langle\gamma, g_0\rangle} V_\gamma^* V_{g_0} f \in M^{p,q}_m(\mathbb{R}^d)$, so $f = \tilde{f}$ on $\mathcal{S}'(\mathbb{R}^d)$. (73), gives equality on $M^{p,q}_m(\mathbb{R}^d)$ in the same way.

3. With some clever choice of g, norm equivalence follows from 1 and 2. Let $g = \gamma$ and $||g||_2 = 1$. Then

$$||f||_{M^{p,q}_m} = ||V_{g_0} f||_{L^{p,q}_m} = ||V_{g_0}(V_g^* V_g f)||_{L^{p,q}_m} \leq C||V_{g_0} g||_{L^1_v} ||V_g f||_{L^{p,q}_m}.$$

Doing the same, but swapping g and g_0, we get an inequality the other way, so that the norms are equivalent. Therefore $f \in M^{p,q}_m(\mathbb{R}^d)$ if and only if $V_g f \in L^{p,q}_m(\mathbb{R}^{2d})$ for all $g \in \mathcal{S}(\mathbb{R}^d \backslash\{0\})$ with $||g||_2 = 1$. Since this is just a change of a constant, the conclusion holds for all $g \in \mathcal{S}(\mathbb{R}^d)\backslash\{0\}$.

∎

We will from now on, for simplicity, assume that $g \in \mathcal{S}(\mathbb{R})\backslash\{0\}$, with $||g||_2 = 1$.

Proposition 2.28 *If $m(z) \leq C(1+|z|)^N$ for some $N > 0$ and $1 \leq p, q < \infty$, then $\mathcal{S}(\mathbb{R}^d)$ is a dense subspace of $M^{p,q}_m(\mathbb{R}^d)$.*

Proof Once again we employ the strategy of 2.24 and estimate the norm of f by

$$||f||_{M_m^{p,q}} = ||V_g f||_{L_m^{p,q}} \leq ||V_g f v_s||_\infty ||v_s^{-1}||_{L^1}.$$

For large enough s and $f \in \mathcal{S}(\mathbb{R}^d)$ this is clearly finite, so $\mathcal{S}(\mathbb{R}^d) \subset M_m^{p,q}(\mathbb{R}^d)$

Now we show the actual denseness by a standard compact set-argument. Let K_n be an increasing sequence of compact sets and set $F_n = V_g f \cdot \chi_{K_n}$. χ_{K_n} being the characteristic function of the set K_n. Define

$$f_n = V_g^* F_n = \iint_{\mathbb{R}^{2d}} F_n(x,w) M_w T_x g \; dx \; dw = \iint_{K_n} V_g f(x,w) M_w T_x g \; dx \; dw,$$

because f decays rapidly (is in $\mathcal{S}(\mathbb{R}^d)$), $V_g f \in \mathcal{S}(\mathbb{R}^{2d})$. Then we can apply fubini's Theorem to the middle term and see that this coincides with the integral in Theorem 2.19, so $f_n \in \mathcal{S}(\mathbb{R}^d)$. Using the isometric properties of $V_g^* V_g$ and (76), we calculate

$$||f - f_n||_{M_m^{p,q}} = ||V^*(V_g f - V_g f_n)||_{M_m^{p,q}} \leq C ||V_g g||_{L_v^1} ||V_g f - F_n||_{L_m^{p,q}}.$$

The right term goes to zero as long as $p, q < \infty$, by definition of F_n. This proves convergence, so $\mathcal{S}(\mathbb{R}^d)$ is dense in $M_m^{p,q}(\mathbb{R}^d)$. ∎

The following theorems will give us the desired properties of modulation spaces, most importantly completeness and duality.

Theorem 2.29 *1. $M_m^{p,q}(\mathbb{R}^d)$ is a Banach space for $1 \leq p, q < \infty$.*

2. $M_m^{p,q}(\mathbb{R}^d)$ is invariant under time-frequency shifts, by the boundedness of v and the formula $||T_x M_w f||_{M_m^{p,q}} \leq C v(x,w) ||f||_{M_m^{p,q}}$

3. If $m(w,-x) \leq C m(x,w)$ then $M_m^p(\mathbb{R}^d)$ is invariant under the Fourier transform.

Note that the last point corresponds to our intuition that the Fourier transform changes time-shifts to frequency-shifts and vice versa.

Proof 1. We will be using the equivalence of norms for g's in the STFT. Define $V = \{F \in L_m^{p,q}(\mathbb{R}^{2d}) \mid F = V_{g_0} f\}$ as a subspace of $L_m^{p,q}(\mathbb{R}^{2d})$. It is linearly closed by inheriting the structure of $L_m^{p,q}(\mathbb{R}^{2d})$. By the norm-equivalences of Proposition 2.27, V is isometrically isomorphic to $M_m^{p,q}(\mathbb{R}^d)$. To show completeness of $M_m^{p,q}(\mathbb{R}^d)$ we now only need to show that V is a closed subspace of $L_m^{p,q}(\mathbb{R}^{2d})$. Let $\{f_n\}$ be a Cauchy sequence in $M_m^{p,q}(\mathbb{R}^d)$. Then $\{V_{g_0} f_n\}$ is a Cauchy sequence in $V \subset L_m^{p,q}(\mathbb{R}^{2d})$, so it converges to some $F \in L_m^{p,q}(\mathbb{R}^{2d})$. We need to show that $F = V_{g_0} f$ for some $f \in M_m^{p,q}(\mathbb{R}^d)$. We define

$$f = \frac{1}{||g_0||_2^2} V_{g_0}^* F.$$

Since $F \in L^{p,q}_m(\mathbb{R}^{2d})$ and $g_0 \in \mathcal{S}(\mathbb{R}^d)$, we have by Proposition 2.27 Part 1 that $f \in M^{p,q}_m(\mathbb{R}^d)$, since the norm is finite. Using the inversion formula on f_n and the estimate (76) again, yields

$$||f - f_n||_{M^{p,q}_m} = \frac{1}{||g_0||_2^2} ||V^*_{g_0}(V_{g_0} f - V_{g_0} f_n)||_{M^{p,q}_m} = C \frac{1}{||g_0||_2^2} ||F - V_{g_0} f_n||_{L^{p,q}_m}.$$

We know that there is convergence for $L^{p,q}_m(\mathbb{R}^{2d})$, so the last term goes to zero, therefore there is also convergence in $M^{p,q}_m(\mathbb{R}^d)$, and since $f \in M^{p,q}_m(\mathbb{R}^d)$, we conclude that the modulation spaces are complete.

2. Firstly we show the remarkable equality $|V_g(T_x M_w f)| = |T_{(x,w)} V_g f|$.

$$\begin{aligned} V_g\big(T_x M_w f(x_1, w_1)\big) &= \int T_x M_w f(t) e^{-2\pi i t w_1} \overline{g(t - x_1)}\, dt \\ &= \int e^{2\pi i (t-x) w} f(t - x) e^{-i t w_1} \overline{g(t - x_1)}\, dt \\ &= \int f(t) e^{2\pi i (wt - wx + wx - w_1 t + w_1 x)} \overline{g(t + x - x_1)}\, dt \\ &= \int f(t) e^{2\pi i (wt - w_1 t + w_1 x)} \overline{g(t - (x_1 - x))}\, dt, \end{aligned}$$

while

$$T_{x,w}\big(V_g f(x_1, w_1)\big) = V_g f(x_1 - x, w_1 - w) = \int f(t) e^{-2\pi i t (w_1 - w)} \overline{g(t - (x_1 - x))}\, dt.$$

Taking absolute values gives the desired equality. From one of our first results (56), we then have

$$\begin{aligned} ||T_x M_w f||_{M^{p,q}_m} &= ||V_g\big(T_x M_w f\big)||_{L^{p,q}_m} \\ &= ||T_{(x,w)} V_g f||_{L^{p,q}_m} \\ &\leq C v(x, w) ||V_g f||_{L^{p,q}_m} \\ &= C v(x, w) ||f||_{M^{p,q}_m}. \end{aligned}$$

Since $v(x, w)$ is bounded, this is finite, so $M^{p,q}_m(\mathbb{R}^d)$ is invariant under time-frequency shifts.

3. In much the same way as in Part *2.*, we have the equality $V_g f(x, w) = e^{-2\pi i x w} V_{\hat{g}} \hat{f}(w, -x)$ (the reader is encouraged to prove this). This allows us to do a straightforward calculation, using the equivalence of norms of

g and $\hat{g}$,

$$\begin{aligned}||\hat{f}||^p_{M^p_m} &= ||V_g\hat{f}||^p_{L^p_m}\\ &\leq C||V_{\hat{g}}\hat{f}||^p_{L^p_m}\\ &= C\iint_{\mathbb{R}^{2d}} |V_{\hat{g}}\hat{f}(x,w)|^p m(x,w)^p\ dx\ dw\\ &= C\iint_{\mathbb{R}^{2d}} |V_g f(-w,x)|^p m(x,w)^p\ dx\ dw.\end{aligned}$$

Changing the roles of $-w$ and x, and applying the assumption yields

$$\begin{aligned}&= C\iint_{\mathbb{R}^{2d}} |V_g f(x,w)|^p m(w,-x)^p\ dx\ dw\\ &\leq C'\iint_{\mathbb{R}^{2d}} |V_g f(x,w)|^p m(x,w)\ dx\ dw = C'||f||^p_{M^p_m}.\end{aligned}$$

So if $f \in M^p_m(\mathbb{R}^d)$, $\hat{f} \in M^p_m(\mathbb{R}^d)$, so $M^p_m(\mathbb{R}^d)$ is invariant under the Fourier transform. ∎

Theorem 2.30 *If $1 \leq p, q < \infty$, then $(M^{p,q}_m(\mathbb{R}^d))^* = M^{p',q'}_{1/m}(\mathbb{R}^d)$, under the duality*

$$\langle f, h\rangle = \iint_{\mathbb{R}^{2d}} V_g f(x,w)\overline{V_g h(x,w)}\ dx\ dw$$

for $f \in M^{p,q}_m(\mathbb{R}^d)$ and $h \in M^{p',q'}_{1/m}(\mathbb{R}^d)$. This is for the usual p', q' given by $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Proof Let $h \in M^{p',q'}_{1/m}(\mathbb{R}^d)$. Since $V_g f \in L^{p,q}_m(\mathbb{R}^{2d})$ and $V_g h \in L^{p',q'}_{1/m}(\mathbb{R}^{2d})$, we have by Hölder's inequality (57) that

$$|\langle f, h\rangle| = |\iint_{\mathbb{R}^{2d}} V_g f(z)\overline{V_g h(z)}\ dz| \leq ||f||_{M^{p,q}_m}||h||_{M^{p',q'}_{1/m}}.$$

By the linearity of the integral, we have that the functional defined by h, namely $l_h(f) = \langle f, h\rangle$, is both linear and bounded.

Assume now that l is a bounded linear functional on $M^{p,q}_m(\mathbb{R}^d)$. We will consider the functional on the subspace V of $L^{p,q}_m(\mathbb{R}^{2d})$, as in the previous proof. Since V is isometrically isomorphic to $M^{p,q}_m(\mathbb{R}^d)$, there is some functional $\tilde{l}$ on V that coincides with l, so that $l(f) = \tilde{l}(V_g f)$. Hahn-Banach allows us to extend $\tilde{l}$ to the whole of $L^{p,q}_m(\mathbb{R}^{2d})$. By the duality of $L^{p,q}_m(\mathbb{R}^{2d})$, (58), we know there exists some function $H \in L^{p',q'}_{1/m}(\mathbb{R}^{2d})$ such that $\tilde{l}(V_g f) = \langle V_g f, H\rangle$. We must show that H is of the form $V_g h$ for some $h \in M^{p',q'}_{1/m}(\mathbb{R}^d)$. Once again by the inversion formula, we have that by defining $h = V^*_g H$, $h \in M^{p',q'}_{1/m}(\mathbb{R}^d)$ and

$$\begin{aligned}\langle f, h\rangle &= \iint_{\mathbb{R}^{2d}} V_g f(z)\overline{V_g V^*_g H(z)}\ dz\\ &= \iint_{\mathbb{R}^{2d}} V_g f(z)\overline{H(z)}\ dz = \tilde{l}(V_g f) = l(f).\end{aligned}$$

$\blacksquare$

As a last result in this section, we find the proper class of window functions. They are Feichtinger's algebra $S_0(\mathbb{R}^d)$,[14], for the unweighed case or the modulation spaces M_v^1 otherwise, for suitable weights.

Theorem 2.31 *Assume that m is v-moderate and let $g, \gamma \in M_v^1(\mathbb{R}^d)\backslash\{0\}$. Then*

1. *V_γ^* is a bounded map from $L_m^{p,q}(\mathbb{R}^{2d})$ into $M_m^{p,q}(\mathbb{R}^d)$. The same estimate holds for $\gamma \in \mathcal{S}(\mathbb{R}^d)$.*
2. *The inversion formula* (70) *holds for $f \in M_m^{p,q}(\mathbb{R}^d)$*
3. *$||V_g f||_{L_m^{p,q}}$ is an equivalent norm on $M_m^{p,q}(\mathbb{R}^d)$. That is, any $g \in M_v^1(\mathbb{R}^d)$ give an equivalent norm.*

Proof 1. Let $\gamma \in \mathcal{S}(\mathbb{R}^d)$, and define the mapping $\gamma \to V_\gamma^* F$, for a given $F \in L_m^{p,q}(\mathbb{R}^{2d})$. This is then a map from $\mathcal{S}(\mathbb{R}^d)$ into $M_m^{p,q}(\mathbb{R}^d)$, and the estimate (76) holds, so

$$||V_\gamma^* F||_{M_m^{p,q}} \leq C||F||_{L_m^{p,q}}||V_{g_0}\gamma||_{L_v^1} = C||F||_{L_m^{p,q}}||\gamma||_{M_v^1}.$$

We see that it is a bounded map, and since $\mathcal{S}(\mathbb{R}^d)$ is dense in $M_v^1(\mathbb{R}^d)$, we can extend it here, so that all $\gamma \in M_v^1(\mathbb{R}^d)$ defines such a map.

2. Let $f \in M_m^{p,q}(\mathbb{R}^d)$. Consider the inequality (78), with $F = V_g f$ and $\gamma = g_0$. We have by the inversion formula that

$$|V_g f(x,w)| \leq \frac{1}{||g_0||_2^2}\big(|V_{g_0} f| * |V_g g_0|\big)(x,w).$$

Taking the $L_m^{p,q}(\mathbb{R}^{2d})$-norm yields

$$||V_g f||_{L_m^{p,q}} \leq \frac{C}{||g_0||_2^2}||V_{g_0} f||_{L_m^{p,q}}||V_g g_0||_{L_v^1} = \frac{C}{||g_0||_2^2}||f||_{M_m^{p,q}}||g||_{M_v^1}.$$

By boundedness and denseness again, the map $g \to V_g f$ for a fixed $f \in M_m^{p,q}(\mathbb{R}^d)$ extends from $\mathcal{S}(\mathbb{R}^d)$ to $M_v^1(\mathbb{R}^d)$. By this and Part *1*, we have that for sequences $g_n, \gamma_n \in \mathcal{S}(\mathbb{R}^d)$ converging to g, γ respectively

$$||V_{g_n} f - V_g f||_{L_m^{p,q}} \to 0 \text{ and } ||V_{\gamma_n}^* F - V_\gamma^* F||_{M_m^{p,q}} \to 0.$$

There is still a problem with the constant $\langle\gamma, g\rangle^{-1}$, so we first do an estimation to see that $M_v^1(\mathbb{R}^d)$ is embedded in $L^2(\mathbb{R}^d)$. This allows us to conclude that we also have the convergence $\langle\gamma_n, g_n\rangle \to \langle\gamma, g\rangle$. We have

$$\begin{aligned} ||f||_2^2 &= ||g_0||_2^{-2}\iint_{\mathbb{R}^{2d}} |V_{g_0} f(x,w)|^2 \, dx \, dw \\ &\leq ||g_0||_2^{-2}||V_{g_0} f||_\infty \iint_{\mathbb{R}^{2d}} |V_{g_0} f(x,w)|v(x,w) \, dx \, dw \\ &\leq ||g_0||_2^{-1}||f||_2||f||_{M_v^1}. \end{aligned}$$

The last equality comes from the familiar norm-estimate $||V_g f|| \leq ||g||\,||f||$. Since every part in $\langle \gamma_n, g_n\rangle^{-1} V^*_{\gamma_n} V_{g_n} f$ converges, the whole formula converges to $\langle \gamma, g\rangle^{-1} V^*_\gamma V_g f$. By the fact that the inversion formula holds on the left side, it must also do so on the right, so it holds for all $f \in M^{p,q}_m(\mathbb{R}^d)$.

3. Since the inversion formula holds, we can do as in the proof of 2.27 and choose in the inversion formula $g = \gamma$ and $||g||_2 = 1$. Then the same calculation holds and every $g \in M^1_v(\mathbb{R}^d)$ yields an equivalent norm for $M^{p,q}_m(\mathbb{R}^d)$. ∎

Before we move on to a discussion on frames for Hilbert spaces, we will give a characterization of the important space $M^1_v(\mathbb{R}^d)$ that illustrates its connection to the short-time Fourier transform.

Lemma 2.32 *Assume that $f, g \in L^2(\mathbb{R}^d)$. If $V_g f \in L^1_v(\mathbb{R}^{2d})$, then both $f \in M^1_v(\mathbb{R}^d)$ and $g \in M^1_v(\mathbb{R}^d)$.*

Proof Fix $g_0 \in \mathcal{S}(\mathbb{R}^d)$ to measure the modulation space-norms. By 2.27 and the formula (78) we have that

$$|V_{g_0} f(x, w)| \leq \frac{1}{|\langle g_0, g\rangle|} (|V_g f| * |V_{g_0} g_0|)(x, w).$$

This is the same formula, with $\gamma = g_0$. Note that we do do not require $g \in \mathcal{S}(\mathbb{R}^d)$ as the true requirement is $V_g f$ being in the appropriate mixed norm space, in this case $L^1_v(\mathbb{R}^{2d})$. Taking the L^1_v-norm, the left side transforms to the modulation space norm, and

$$||f||_{M^1_v} \leq |\langle g_0, g\rangle|^{-1} ||V_g f||_{L^1_v} ||V_{g_0} g_0||_{L^1_v} < \infty.$$

Here we have used the convolution estimate 2.15. We have shown that $f \in M^1_v(\mathbb{R}^d)$. The same method gives us the estimate for g. Changing all mentions of g to f and vice versa, gives the result when one recognizes that $||V_g f||_{L^1_v} = ||V_f g||_{L^1_v}$. ∎

Proposition 2.33 *The following are equivalent:*

1. $f \in M^1_v(\mathbb{R}^d)$
2. $f \in L^2(\mathbb{R}^d)$ *and for one* $g \in \mathcal{S}(\mathbb{R}^d)$, $V_g f \in L^1_v(\mathbb{R}^{2d})$.
3. $f \in L^2(\mathbb{R}^d)$ *and for one* $g \in M^1_v(\mathbb{R}^d)$, $V_g f \in L^1_v(\mathbb{R}^{2d})$.
4. $V_f f \in L^1_v(\mathbb{R}^{2d})$

Proof Since $M^1_v(\mathbb{R}^d)$ is embedded in $L^2(\mathbb{R}^d)$, and by definition of $M^1_v(\mathbb{R}^d)$ we have $1 \implies 2$.
$2 \implies 3$ by the equivalence of norms for $g \in M^1_v(\mathbb{R}^d)$.
If we set $f = g$ in Lemma 2.32, then $3 \implies 4$ since $M^1_v(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. Explicitly, since $f \in L^2(\mathbb{R}^d)$ and $V_g f \in L^1_v(\mathbb{R}^{2d})$ for one $g \in M^1_v(\mathbb{R}^d)$, by equivalence

of norms, $V_f f \in L^1_v(\mathbb{R}^{2d})$.
$4 \implies 1$ by Lemma 2.32. Assume $V_f f \in L^1_v(\mathbb{R}^{2d})$ and $f \notin M^1_v(\mathbb{R}^d)$, then f cannot be in $L^2(\mathbb{R}^d)$ either. Because the L^1-norm of $|f|^2$ is less than or equal to $V_f f$ since M_w and T_x are unitary operators, and we have that the L^1-norm of $|f|^2$ is equal to the L^2-norm of f, this is a contradiction, and we have the desired conclusion. ∎

We now have a characterization of the weighted case of Feichtinger's algebra, and an understanding of how they work. So we will immediately jump into frame theory for Hilbert spaces.

2.3 Frame Theory

Frame theory can naively be explained as finding representation of functions as a discrete sum with redundancies (so a frame is in general not a basis). We recall the inversion formula for $f \in M_m^{p,q}(\mathbb{R}^{2d})$ as $V_g^* V_g f = f$, where $||g||_2 = 1$, which we can write as

$$f = \iint_{\mathbb{R}^{2d}} V_g f(x, w) M_w T_x g \; dw \; dx = \iint_{\mathbb{R}^{2d}} \langle f, M_w T_x g \rangle M_w T_x g \; dw \; dx.$$

A natural way of discretization would simply be replacing the integral with a sum. Sampling at some rate $(\alpha, \beta) \in \mathbb{R}^{2d}$,

$$f = \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{Z}^d} \langle f, M_{\alpha k} T_{\beta n} g \rangle M_{\alpha k} T_{\beta n} g. \tag{79}$$

Equality should hold for some choice of sample rate and $g \in M_v^1(\mathbb{R}^d)$. It now should become clear why we refer to g as a window function, and why we would want it to only have values on a "small" domain. This allows us to choose a suitable sample rate as the STFT will only take large values where g is non-zero, so by time-frequency shifts of g we are able to cover the whole phase-space. Very small sampling constants (and indeed the integral itself) is very redundant, as the values $\langle f, M_{\alpha k} T_{\beta n} g \rangle$ will change very little for increasing (n, k) if (α, β) are very small and f and g sufficiently "nice".

A measure of the quality our representation is how much it differs in norm from the original function. Recall the definition we gave in the Hilbert C^*-module chapter, Definition 2.1.1 for separable Hilbert spaces. This is the general case, we will mostly work with $\{e_j\} = \{M_{\alpha k} T_{\beta n} g\}$, but we introduce the theory more generally.

The most important operators associated to frame theory are the analysis and synthesis operators, allowing us to move from functions to sequences and back again.

Definition For **any** subset of a separable Hilbert space $\mathcal{H}$, $\{e_j : j \in J\}$ for an index set J, the **analysis operator** C is given by

$$Cf = \{\langle f, e_j \rangle : j \in J\}. \tag{80}$$

The **synthesis operator** D for a finite sequence $c = (c_j)_{j \in J}$ is defined by

$$Dc = \sum_{j \in J} c_j e_j. \tag{81}$$

The **frame operator** S is defined on $\mathcal{H}$ by

$$Sf = \sum_{j \in J} \langle f, e_j \rangle e_j \tag{82}$$

It should be clear that the frame operator is the composition of C and D and that $Dc \in \mathcal{H}$, since it is an expansion along the subset $\{e_j\} \in \mathcal{H}$. Do not take too much note of the finiteness condition of the synthesis operator, we will soon extend it to infinite sequences. For a frame, these operators have many nice properties which are natural to check first. We gather them in the following proposition.

Proposition 2.34 *Suppose $\{e_j : j \in J\}$ is a frame for $\mathcal{H}$.*

1. *C is a bounded operator from $\mathcal{H}$ to $\ell^2(J)$ with closed range.*
2. *$C = D^*$, so we can extend D to be defined on $\ell^2(J)$.*
3. *$S = DC = C^*C$ is surjective onto $\mathcal{H}$ and is invertible satisfying $AI_{\mathcal{H}} \leq S \leq BI_{\mathcal{H}}$.*

The inequalities in Part *3* are in the usual positive operator sense from general operator theory. In the proof some background in operator theory is useful, as we are using some well-known properties of operators and their adjoints.

Proof

1. Since $\{e_j\}$ is a frame we have the inequalities $A||f||^2 \leq \sum |\langle f, e_j\rangle|^2 \leq B||f||^2$, so $Cf \in \ell^2(J)$. Slightly less obvious is showing that C has closed range, but this is in fact equivalent to the left inequality. This is an operator-theoretic result and we refer to [18] for proof.

2. If we show that $C^* = D$ on a finite sequence, then it follows that D is bounded and can therefore be extended to all sequences $(c_j) \in \ell^2(J)$. Additionally, it has the same bound as C, so since $||C|| \leq B^{1/2}$ by the frame bounds, we have that

$$|| \sum_{j \in J} c_j e_j || \leq B^{1/2} ||c||_2 \tag{83}$$

since $||Dc||^2 = ||C^*c||^2 \leq ||C^*||^2||c||_2^2 \leq B||c||_2^2$. We consider a finite sequence (d_j), and calculate

$$\langle C^*d, f\rangle = \langle d, Cf\rangle = \sum_{j \in J} d_j \overline{\langle f, e_j\rangle} = \langle \sum_{j \in J} d_j e_j, f\rangle = \langle Dd, f\rangle.$$

Here it is important to keep in mind where the inner products are taken, some are in ℓ^2 while some are in $\mathcal{H}$.

3. It is clear that $S = DC$, so by Part *2*, also $S = C^*C = DD^*$. It is also then positive and self-adjoint. By the equality

$$\langle Sf, f\rangle = \langle \sum \langle f, e_j\rangle e_j, f\rangle = \sum \langle f, e_j\rangle\langle e_j, f\rangle = \sum |\langle f, e_j\rangle|^2$$

we have that

$$\langle AI_{\mathcal{H}} f, f\rangle \leq \langle Sf, f\rangle \leq \langle BI_{\mathcal{H}}, f\rangle$$

so in the sense of positive operators, we have $AI_{\mathcal{H}} \leq S \leq BI_{\mathcal{H}}$. Since all positive operators that are bounded from below are invertible, S is invertible. The inequalities are preserved as long as one is multiplying by a commuting operator, since S^{-1} commutes with both $I_{\mathcal{H}}$ and S we can multiply by this operator, and invert the chain of inequalities. This yields

$$B^{-1}I_{\mathcal{H}} \leq S^{-1} \leq A^{-1}I_{\mathcal{H}},$$

which will also become useful. ∎

Since we are in general not dealing with orthogonal sequences, we must specify our method of convergence of frames. Norm convergence can be somewhat misleading here. Unconditional convergence is what we will be using, and ensures convergence for all infinite subsets of J. This is not just to make our jobs easier in dealing with these sequences, but is a necessary distinction to make, as there are examples of Fourier series that converge in norm, but not for every subset.

We show a convergence result while defining more rigorously this notion

Proposition 2.35 *Let $\{e_j : j \in J\}$ be a frame for $\mathcal{H}$. If $f = \sum_{j\in J} c_j e_j$ for some $c \in \ell^2(J)$, then for every $\epsilon > 0$ there exists a finite subset $F_0 = F_0(\epsilon) \subseteq J$ such that*

$$||f - \sum_{j\in F} c_j e_j|| < \epsilon \quad \textit{for all finite subsets } F \supseteq F_0$$

We say that the series $\sum_{j\in J} c_j e_j$ converges to $f \in \mathcal{H}$ unconditionally.

Proof We will be using properties from Proposition 2.34. This result could rightly be called a corollary, but since it includes a definition we separate it.

For every ϵ we can choose a set $F_0 \subseteq J$ such that $\sum_{n\notin F} |c_j|^2 < \frac{\epsilon}{B^{1/2}}$ for all finite $F \supseteq F_0$. This is possible simply by making F_0 large enough since $c \in \ell^2(J)$, the tail must go to 0. Define a new sequence for every such F by $c_F = c\chi_F$. Then, since c_F is a finite sequence, we have that $\sum_{j\in F} c_j e_j = Dc_F$, and we know from the last proposition that also $f = \sum_{j\in J} c_j e_j = Dc$. Then we can use the bound (83) and calculate

$$\begin{aligned} ||f - \sum_{j\in F} c_j e_j|| &= ||Dc - Dc_F|| \\ &= ||D(c - c_F)|| \leq B^{1/2}||c - c_F||_{\ell^2(J)} < \epsilon \end{aligned}$$

∎

Since our professed main goal of introducing frame theory was to find a reconstruction formula, the reader will be delighted to hear that already now we can prove such a formula. Of course, there is still much to be done, since we have until now been working with modulation spaces, and not a separable Hilbert space, but a small celebration would be in order.

Proposition 2.36 *Let $\{e_j : j \in J\}$ be a frame for $\mathcal{H}$ with bounds $A, B > 0$. Then the set $\{S^{-1}e_j : j \in J\}$ is also a frame, with bounds $B^{-1}, A^{-1} > 0$. This is called the dual frame and for every $f \in \mathcal{H}$ there are two (non-)orthogonal expansions*

$$f = \sum_{j \in J} \langle f, S^{-1}e_j \rangle e_j \tag{84}$$

and

$$f = \sum_{j \in J} \langle f, e_j \rangle S^{-1}e_j. \tag{85}$$

Where both the convergences are unconditional.

Proof S^{-1} giving a frame is based on the last section of the proof of Proposition 2.34, we only need to show that $\sum_{j \in J} |\langle f, S^{-1}e_j \rangle|^2 = \langle S^{-1}f, e_j \rangle$, by the same logic as used in the proof. Keeping in mind the definition, self-adjointedness and invertibility of S (and therefore also S^{-1}) we see that

$$\begin{aligned}
\sum_{j \in J} |\langle f, S^{-1}e_j \rangle|^2 &= \sum_{j \in J} |\langle S^{-1}f, e_j \rangle|^2 \\
&= \sum_{j \in J} \langle S^{-1}f, e_j \rangle \langle e_j S^{-1}f \rangle \\
&= \langle \sum_{j \in J} \langle S^{-1}f, e_j \rangle e_j, S^{-1}f \rangle \\
&= \langle S(S^{-1}f), S^{-1}f \rangle = \langle S^{-1}f, f \rangle.
\end{aligned}$$

So we have a frame with bounds B^{-1}, A^{-1}.

The basis of the second part of the proof relies on the equalities $I_{\mathcal{H}} = S^{-1}S = SS^{-1}$. Clearly $f = SS^{-1}f$, so

$$f = S(S^{-1}f) = \sum_{j \in J} \langle S^{-1}f, e_j \rangle e_j = \sum_{j \in J} \langle f, S^{-1}e_j \rangle e_j.$$

In the same way

$$f = S^{-1}Sf = S^{-1}\big(\sum_{j \in J} \langle f, e_j \rangle e_j\big) = \sum_{j \in J} \langle f, e_j \rangle S^{-1}e_j$$

These are then expansions along the frame and the dual frame respectively ∎

Interestingly enough the possible coefficients in these expansions are not unique, in contrast to the orthonormal case. This should be clear, as an example of a frame is simply an orthonormal basis with every element repeated twice, there are an infinite possible expansions in this case. The coefficients we have chosen are canonical in the following sense.

Proposition 2.37 *Let* $\{e_j : j \in J\}$ *be a frame for* $\mathcal{H}$ *and let* $f = \sum_{j\in J} c_j e_j$ *for some choice of coefficients* $c \in \ell^2(J)$*. Then*

$$\sum_{j\in J} |\langle f, S^{-1}e_j\rangle|^2 \leq \sum_{j\in J} |c_j|^2$$

with equality only if $c_j = \langle f, S^{-1}e_j\rangle$ *for all* $j \in J$*.*

Proof By our expansion formula (84), $f = \sum_{j\in J} a_j e_j$ for $a_j = \langle f, S^{-1}e_j\rangle$ and

$$\langle f, S^{-1}f\rangle = \sum_{j\in J} a_j\langle e_j, S^{-1}f\rangle = \sum_{j\in J} a_j\langle S^{-1}e_j, f\rangle = \sum_{j\in J} a_j\overline{a_j} = \sum_{j\in J} |a_j|^2.$$

Similarly, choosing to expand f using c_j in the first argument yields

$$\langle f, S^{-1}f\rangle = \sum_{j\in J} c_j\overline{a_j} = \langle c, a\rangle.$$

This means that $||a||_2^2 = \langle c, a\rangle$, and we can do some norm-calculations using general Hilbert-space equalities ($||x||^2 = \langle x, x\rangle$).

$$\begin{aligned}
||c||_2^2 &= ||c - a + a||_2^2 \\
&= ||c - a||_2^2 + ||a||_2^2 + \langle c - a, a\rangle + \langle a, c - a\rangle \\
&= ||c - a||_2^2 + ||a||_2^2 + \langle c, a\rangle - \langle a, a\rangle + \overline{\langle c, a\rangle} - \overline{\langle a, a\rangle} \\
&= ||c - a||_2^2 + ||a||_2^2 + ||a|_2^2 - ||a||_2^2 + ||a||_2^2 - ||a||_2^2 \\
&= ||c - a||_2^2 + ||a||_2^2 \geq ||a||_2^2.
\end{aligned}$$

By the positivity of norms, there can only be equality if $c = a$. ∎

There are significant differences in properties between tight frames, dual frames and orthonormal bases derived from the use of the inverse frame operators. Along with an explicit formula for S^{-1} the next result sheds some light on these differences.

Lemma 2.38 1. *If* $\{e_j : j \in J\}$ *is a tight frame for* $\mathcal{H}$ *with* $A = B = 1$ *and* $||e_j|| = 1$ *for all* j*, then it is an orthonormal basis.*

2. *If* $\{e_j : j \in J\}$ *is a frame for* $\mathcal{H}$*, then* $\{S^{-1/2}e_j\}$ *is a tight frame with bounds* $A = B = 1$*.*

3. *If* $\{e_j : j \in J\}$ *is a frame for* $\mathcal{H}$*, then* S^{-1} *is the frame operator of the dual frame* $\{S^{-1}e_j\}$*, and is given by*

$$S^{-1}f = \sum_{j\in J}\langle f, S^{-1}e_j\rangle S^{-1}e_j \tag{86}$$

Proof 1. Since the frame is tight and has bounds 1, we have that for every frame element e_m,

$$1 \leq \sum_{j\in J} |\langle e_m, e_j\rangle|^2 = ||e_m||^2 + \sum_{j\in J\setminus\{m\}} |\langle e_m, e_j\rangle|^2 \leq 1.$$

Since $||e_m|| = 1$, the sum $\sum_{j\in J\setminus\{m\}}\langle e_m, e_j\rangle = 0$, $\{e_j\}$ is therefore orthogonal, and is also normalized by assumption.

2. Since we can write $f = S^{-1/2}SS^{-1/2}f$, we can do the calculation

$$||f||^2 = \langle f, f\rangle = \langle S^{-1/2}SS^{-1/2}f, f\rangle = \langle S^{-1/2}\sum_{j\in J}\langle S^{-1/2}f, e_j\rangle e_j, f\rangle$$
$$= \sum_{j\in J}\langle S^{-1/2}f, e_j\rangle\langle S^{-1/2}e_j, f\rangle = \sum_{j\in J}|\langle f, S^{-1/2}e_j\rangle|^2.$$

So $\{S^{-1/2}e_j\}$ is a frame with bounds $A = B = 1$.

3. We see that

$$S^{-1}f = S^{-1}SS^{-1}f = S^{-1}\sum_{j\in J}\langle S^{-1}f, e_j\rangle e_j = \sum_{j\in J}\langle f, S^{-1}e_j\rangle S^{-1}e_j.$$

∎

One might think that we would want our coefficients to be unique, but such a requirement is not feasible in our case, as it can be shown that this is equivalent to the frame being a Riesz basis. The implications of this, is that a frame based on a window-function g is badly localized, it has low decay in either time or frequency. This result is known as the Balian-Low Theorem, proven independently by R. Balian [3] and F.Low [13].

2.3.1 Gabor Frames

We will from now on be concerned with a special subset of frames for $L^2(\mathbb{R}^d)$, the Gabor Frames. It is a frame defined from time-frequency shifts of a single window-function, called a Gabor atom.

Definition Assume given a function $g \in L^2(\mathbb{R}^d)\setminus\{0\}$, and positive lattice constants $\alpha, \beta \in \mathbb{R}^d$. Then the set

$$\mathcal{G}(g, \alpha, \beta) = \{T_{\alpha k}M_{\beta n}g : k, n \in \mathcal{Z}^d\}$$

is called a **Gabor system**. If it is a frame for $L^2(\mathbb{R}^d)$, then it is known as a **Gabor frame**.

We can give an explicit formula for the associated frame operator,

$$Sf = \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d}\langle f, T_{\alpha k}M_{\beta n}g\rangle T_{\alpha k}M_{\beta n}g.$$

We will write $S_{g,g}^{\alpha,\beta}$ when necessary.

The first question that needs to be resolved relates to the dual frame of this system. A good starting point would be the function $\gamma = S^{-1}g$ by our previous discussion.

Proposition 2.39 *If $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame, then there exists a $\gamma \in L^2(\mathbb{R}^d)$ such that the dual frame is $\mathcal{G}(\gamma,\alpha,\beta)$. Since S^{-1} commutes with time-frequency shifts this yields, for every $f \in L^2(\mathbb{R}^d)$, the expansions*

$$f = \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} \langle f, T_{\alpha k}M_{\beta n}g\rangle T_{\alpha k}M_{\beta n}\gamma$$
$$= \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} \langle f, T_{\alpha k}M_{\beta n}\gamma\rangle T_{\alpha k}M_{\beta n}g$$

with unconditional convergence. If the frame bounds for g is A, B, then the bounds for γ are B^{-1}, A^{-1}.

Proof If we can show that $S_{g,g}^{\alpha,\beta} = S$ commutes with time-frequency shifts $T_{\alpha k}M_{\beta n}$, the rest of the proposition follows from the last chapter. The dual frame is given by $S^{-1}\big(T_{\alpha k}M_{\beta n}g\big) = T_{\alpha k}M_{\beta n}S^{-1}g = T_{\alpha k}M_{\beta n}\gamma$, and the bounds follows from Proposition 2.34.

We then do the actual calculation. Let $f \in L^2(\mathbb{R}^d)$ and $r, s \in \mathbb{R}^d$. Then

$$\begin{aligned}
&\big(T_{\alpha r}M_{\beta s}\big)^{-1}S\big(T_{\alpha r}M_{\beta s}\big)f \\
&= \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} \langle T_{\alpha r}M_{\beta s}f, T_{\alpha k}M_{\beta n}g\rangle \big(T_{\alpha r}M_{\beta s}\big)^{-1}T_{\alpha k}M_{\beta n}g \\
&= \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} \langle f, \big(T_{\alpha r}M_{\beta s}\big)^{*}T_{\alpha k}M_{\beta n}g\rangle \big(T_{\alpha r}M_{\beta s}\big)^{-1}T_{\alpha k}M_{\beta n}g \\
&= \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} \langle f, e^{-2\pi i\alpha\beta(k-r)s}T_{\alpha(k-r)}M_{\beta(n-s)}g\rangle e^{-2\pi i\alpha\beta(k-r)s}T_{\alpha(k-r)}M_{\beta(n-s)}g \\
&= \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}^d} \langle f, T_{\alpha(k-r)}M_{\beta(n-s)}g\rangle T_{\alpha(k-r)}M_{\beta(n-s)}g \\
&= Sf.
\end{aligned}$$

Where we have used the unitariness of T_x and M_w and the commutation relation $T_xM_w = e^{-2\pi i x\cdot w}M_wT_x$. The function $\gamma = S^{-1}g$ defines the dual Gabor frame. ∎

Immediately, we can find an explicit formula for the inverse frame operator of $S_{g,g}^{\alpha,\beta}$, and see how it relates to the dual frame. By (86), we have that

$$\big(S_{g,g}^{\alpha,\beta}\big)^{-1}f = \sum_{k,n\in\mathbb{Z}^d} \langle f, T_{\alpha k}M_{\beta n}\gamma\rangle T_{\alpha k}M_{\beta n}\gamma = S_{\gamma,\gamma}^{\alpha,\beta}f. \tag{87}$$

So the inverse frame operator of a Gabor frame is the operator associated to its dual frame.

We have now discovered a possible way of discretizising the Short-Time Fourier transform, as was our goal in this section. By using some suitable window function g and a dual window γ we have a representation

$$f = \sum_{k,n \in \mathbb{Z}^d} V_g f(\alpha k, \beta n) T_{\alpha k} M_{\beta n} \gamma. \tag{88}$$

This reduces our problem to finding solutions for $S\gamma = g$. Numerically, it is also interesting to find a suitable sample rate used in real-world application, but this is outside of the scope of this thesis.

3 The Moyal Plane

As a continuation of our work on modulation spaces we will be considering the twisted Banach algebra $L^1(\mathbb{R}^{2d}, c)$ from [12]. We will from now on be using the definition from partial differential equations of the Fourier transform and similar, that is we will be removing all instances of 2π to make our calculations easier. We are considering $L^1(\mathbb{R}^{2d})$ with respect to twisted convolution. If $f, g \in L^1(\mathbb{R}^{2d})$ and $z = (x, w)$, $z' = (x', w')$ then

$$(f\natural g)(z) = \iint_{\mathbb{R}^{2d}} f(z')g(z-z')c(z, z-z')dz' \quad \text{for } c(z, z') = e^{-ixw'}. \tag{89}$$

This is a natural choice for us in the sense that for our time-frequency operators, we have the relation $M_w T_x M_{w'} T_{x'} = c(z, z') M_{w+w'} T_{x+x'}$. This is a great help in our further calculations, like the following lemma. We will sometimes use the notation $\pi(z) = M_w T_x$ for simplicity.

Lemma 3.1 *Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R}^d)$. Then*

$$V_{g_1} f_1 \natural V_{g_2} f_2(z) = \langle f_2, g_1 \rangle V_{g_2} f_1(z). \tag{90}$$

Proof This is just a slightly arduous calculation, but it allows one to accustom oneself to work with these operations. Firstly we find that

$$\begin{aligned}
\langle \pi(z')f, \pi(z)g \rangle &= \int_{\mathbb{R}^d} M_{w'} T_{x'} f(t) \overline{M_w T_x g(t)} dt \\
&= \int_{\mathbb{R}^d} e^{it(w'-w)} f(t-x') \overline{g(t-x)} dt \\
&= \int_{\mathbb{R}^d} e^{-i(t+x')(w-w')} f(t) g(t-(x-x')) dt \\
&= c(z', z-z') \langle f, \pi(z-z')g \rangle. \\
&= c(z', z-z') V_g f(z-z')
\end{aligned}$$

The right side helps to simplify the expressions in the next calculations

$$\begin{aligned}
V_{g_1} f_1 \natural V_{g_2} f_2(z) &= \iint_{\mathbb{R}^{2d}} V_{g_1} f_1(z) V_{g_2} f_2(z'-z) c(z', z-z') \; dz' \\
&= \iint_{\mathbb{R}^{2d}} V_{g_1} f_1(z) \langle \pi(z') f_2, \pi(z) g_2 \rangle \; dz' \\
&= \iint_{\mathbb{R}^{2d}} \langle f_1, \pi(z') g_1 \rangle \langle \pi(z') f_2, \pi(z) g_2 \rangle \; dz',
\end{aligned}$$

gathering the appropriate terms, we see that we can simplify this to

$$\langle f_1, \pi(z) g_2 \rangle \langle f_2, g_1 \rangle = \langle f_2, g_1 \rangle V_{g_2} f_1(z).$$

∎

We note that this implies that for $f \in L^2(\mathbb{R}^d)$ with $||f||_2 = 1$, we have $V_f f \sharp V_f f = V_f f$.

To make $L^1(\mathbb{R}^{2d})$ into an involutive Banach algebra we also need some involution operators. It will be no surprise that we define the twisted involution of $f \in L^1(\mathbb{R}^{2d})$ as

$$f^*(z) = c(z,z)\overline{f(-z)} = e^{-ixw}\overline{f(-z)}. \tag{91}$$

A similar result as Lemma 3.1 holds here.

Lemma 3.2 *Let g, f be such that $V_g f \in L^1(\mathbb{R}^{2d})$. Then $\big(V_g f\big)^* = V_f g$.*

Proof We simply calculate

$$\begin{aligned}
\big(V_g f(z)\big)^* &= e^{-ixw}\overline{\langle f, \pi(-z)g\rangle} \\
&= e^{-ixw}\langle M_{-w}T_{-x}g, f\rangle \\
&= e^{-ixw}\langle g, T_x M_w f\rangle = e^{-ixw}\langle g, e^{-ixw}M_w T_x\rangle = V_f g(z)
\end{aligned}$$

∎

This shows specifically that $\big(V_f f\big)^* = V_f f$, so we have proved the following.

Lemma 3.3 *Let $f \in L^2(\mathbb{R}^d)$ with $||f||_2 = 1$. Then $p_f = V_f f$ is a projection. For $f \in \mathcal{S}(\mathbb{R}^d)$, p_f is a projection in $L^1(\mathbb{R}^{2d}, c) \supseteq \mathcal{S}(\mathbb{R}^{2d}, c)$*

To make use of our theory on Hilbert C^*-modules, we would like to represent $L^1(\mathbb{R}^d, c)$ as a space operators. To make it easier for ourselves, so that we need not worry about convergences and integrability, we restrict our work to functions in the Schwartz class.

Definition Denote by $\mathcal{A}$ the class of operators

$$K = \pi(k) = \iint k(z)\pi(z)\ dz \quad \text{for } k \in \mathcal{S}(\mathbb{R}^{2d}) \tag{92}$$

with composition and adjoint given by

$$\begin{aligned}
KL &= \iint_{\mathbb{R}^2} (k\sharp l)(z)\pi(z)\ dz \quad k, l \in \mathcal{S}(\mathbb{R}^{2d}) \\
K^* = \pi(k^*) &= \iint_{\mathbb{R}^2} e^{-ixw}k(z)\pi(z)\ dz
\end{aligned}$$

The norm-completion of $\mathcal{A}$ are the compact operators on $L^2(\mathbb{R}^d)$, which we refer to as the Moyal plane. We then have the result

Proposition 3.4 *The space $\mathcal{S}(\mathbb{R}^d)$ is a complete equivalence bimodule between $\mathcal{A}$ and $\mathbb{C}$ with respect to the actions*

$$K \cdot f = \iint_{\mathbb{R}^2} k(z)\pi(z)f\ dz, \quad \textit{and} \quad f \cdot \lambda = f\overline{\lambda}, \tag{93}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$*,* $k \in \mathcal{S}(\mathbb{R}^{2d})$ *and* $\lambda \in \mathbb{C}$*; and* $\mathcal{A}$*-valued left inner product and* $\mathbb{C}$*-valued right inner products:*

$${}_\bullet\langle f, g\rangle = \iint_{\mathbb{R}^2} \langle f, \pi(z)g\rangle \pi(z)\, dz = \iint_{\mathbb{R}^2} V_g f(z)\pi(z)\, dz \text{ and } \langle f, g\rangle_\bullet = \langle g, f\rangle_{L^2(\mathbb{R}^d)}$$

for $g \in \mathcal{S}(\mathbb{R}^d)$*. Derivations on* $\mathcal{A}$ *are given by*

$$\partial_1(K) = \iint_{\mathbb{R}^{2d}} ixk(z)\pi(z)\, dz$$
$$\partial_2(K) = \iint_{\mathbb{R}^{2d}} iwk(z)\pi(z)\, dz,$$

which can be lifted to operations on $\mathcal{S}(\mathbb{R}^d)$ *given by*

$$\nabla_1 f(t) = itf(t), \quad \nabla_2 f(t) = f'(t).$$

Any $g \in \mathcal{S}(\mathbb{R}^d)$ *with* $||g||_2 = 1$ *generates a standard module frame for* $\mathcal{S}(\mathbb{R})$*. Hence* $\mathcal{S}(\mathbb{R}^d)$ *is a line bundle over the Moyal plane.*

Note that ∇_1, ∇_2 define a constant curvature connection on $\mathcal{S}(\mathbb{R})$:

$$F_{1,2} = [\nabla_1, \nabla_2]f(t) = \nabla_1\nabla_2 f(t) - \nabla_2\nabla_1 f(t) = itf'(t) - if(t) - itf'(t) = -if(t).$$

Proof The action of $\mathcal{S}(\mathbb{R}^2)$ can be written as the adjoint operator of the short-time Fourier transform, that is $K \cdot f = V_f^* k$. Since $k \in \mathcal{S}(\mathbb{R}^{2d})$ and $f \in \mathcal{S}(\mathbb{R}^{2d})$, clearly $K \cdot f \in \mathcal{S}(\mathbb{R}^d)$.

By Theorem 2.23, since $f, g \in \mathcal{S}(\mathbb{R}^d)$ we have that $V_g f \in \mathcal{S}(\mathbb{R}^{2d})$ and therefore ${}_\bullet\langle f, g\rangle \in \mathcal{A}$.

The right action and inner product are compatible, we omit the calculations.

Let $h \in \mathcal{S}(\mathbb{R}^d)$. Showing the associativity condition is trickier, first we introduce an additional function $k \in \mathcal{S}(\mathbb{R}^d)$, and take the $L^2(\mathbb{R}^d)$ inner product. If $\langle {}_\bullet\langle f, g\rangle h, k\rangle = \langle f\langle g, h\rangle_\bullet, k\rangle$ for all such k, then we have the desired equality. Note that the right side is easy,

$$\langle f\langle g, h\rangle_\bullet, k\rangle = \int f(t)\langle h, g\rangle\overline{k(t)}\, dt = \int f(t)\overline{k(t)}\, dt\langle h, g\rangle = \langle f, k\rangle\langle h, g\rangle.$$

The left side requires some work, and changing the order of integration.

$$\begin{aligned}
\langle {}_\bullet\langle f, g\rangle h, k\rangle &= \int_{\mathbb{R}^d} \iint_{\mathbb{R}^{2d}} V_g f(z)\pi(z)h(t)\, dz\, \overline{k(t)}\, dt \\
&= \int_{\mathbb{R}^d} \iint_{\mathbb{R}^{2d}} V_g f(z)\overline{k(t)\overline{\pi(z)h(t)}}\, dz\, dt \\
&= \iint_{\mathbb{R}^{2d}} V_g f(z)\overline{\int_{\mathbb{R}^d} k(t)\overline{\pi(z)h(t)}\, dt}\, dz \\
&= \iint_{\mathbb{R}^{2d}} V_g f(z)\overline{V_h k(z)}\, dz \\
&= \langle V_g f, V_h k\rangle_{L^2(\mathbb{R}^{2d})},
\end{aligned}$$

By the Moyal Identity, Theorem 2.20, we have

$$\langle V_g f, V_h k \rangle_{L^2(\mathbb{R}^{2d})} = \langle f, h \rangle \langle k, g \rangle,$$

completing the equality. The norm-estimates of Proposition 2.7 are easy to show, so we can complete the algebras to a Morita Equivalence. That these are appropriate derivations will be shown in Section 5 as a special case of the weighted forms.

The last statement is equivalent to the reconstruction formula for such a g, which we have shown in the chapter on frame-theory. $\blacksquare$

The description of $\mathcal{A}$ in terms of continuous functions over $\mathbb{R}^{2d}$ under twisted convolution allows us to denote $\mathcal{A} = C(\mathbb{R}^{2d}_\theta)$ for some $\theta \in \mathbb{R}$. The same applies for the non-commutative torus, which can be denoted $C(\mathbb{T}^2_\theta)$. This notation will be used later to distinguish between the classical and non-commutative function spaces.

On the Moyal plane it is now possible to introduce a complex structure using the derivations ∂_1, ∂_2:

$$\partial = \partial_1 - i\partial_2, \quad \overline{\partial} = \partial_1 + i\partial_2.$$

We then have the constant curvature connection and the canonical commutation relation of quantum theory. The Laplace operator from the classical theory can be extended to this case as well, by letting

$$\Delta = \partial\overline{\partial} = \partial_1^2 + \partial_2^2. \tag{94}$$

The associated structure for the connections ∇_1, ∇_2 induces a complex structure on $\mathcal{S}(\mathbb{R})$:

$$\nabla = \nabla_1 - i\nabla_2, \quad \overline{\nabla} = \nabla_1 + i\nabla_2$$

and a Laplacian

$$\Delta = \nabla_1^2 + \nabla_2^2. \tag{95}$$

The complex structure on $\mathcal{S}(\mathbb{R})$ is related to the quantum harmonic oscillator due to the fact that the constant curvature condition in this case amounts to the canonical commutation relations of quantum mechanics. The operators ∇ and $\overline{\nabla}$ are the annihilation and creation operators, and Δ is the Hamiltonian of the quantum harmonic oscillator, the operator $\nabla\overline{\nabla}$ is also known as the number operator.

4 The Non-Commutative torus

Given $\theta \in \mathbb{R}$, we will be considering the universal C^*-algebra generated by unitary operators U and V such that

$$UV = e^{i\theta}VU.$$

Then for $\theta \notin \mathbb{Z}$ we have a non-commutative space, the **noncommutative torus**. Hence, the C^*-algebra A_θ is the norm closure of the span of $\{U^kV^l : k,l \in \mathbb{Z}\}$. We restrict our discussion to the smooth subalgebra of A_θ functions.

Definition The **smooth non-commutative torus** $\mathcal{A}_\theta$ is the subalgebra of A_θ consisting of operators

$$\pi(\mathbf{a}) = \sum_{k,l\in\mathbb{Z}} a_{kl}U^kV^l, \quad for \;\; \mathbf{a} = (a_{kl}) \in \mathcal{S}(\mathbb{Z}^2) \tag{96}$$

We are using the familiar notation of π being a representation of some algebra as operators. Explicitly, we will be using this as a representation of $\mathcal{S}(\mathbb{Z}^2)$ on $\mathcal{S}(\mathbb{R})$ to create a bimodule. Already, we can see the connection between this space and modulation spaces, where $U = M_w$ and $V = T_x$. we could of course have generalized the definition, by using several unitary operators U_i, but we have no need for them in this thesis.
We note that

$$\begin{aligned}\pi(\mathbf{a})\pi(\mathbf{b}) &= \Big(\sum_{k,l\in\mathbb{Z}} a_{kl}U^kV^l\Big)\Big(\sum_{p,q\in\mathbb{Z}} b_{kl}U^pV^q\Big)\\ &= \sum_{k,l\in\mathbb{Z}}\sum_{p,q\in\mathbb{Z}} a_{p,q}b_{k-p,q-l}e^{-i\theta q(k-p)}U^kV^l\\ &= \pi(\mathbf{a}\natural\mathbf{b})\end{aligned}$$

Where we denote the twisted convolution of $\mathbf{a},\mathbf{b} \in L^1(\mathbb{Z})$ by

$$\mathbf{a}\natural\mathbf{b} = \sum_{m,n\in\mathbb{Z}} a_{m,n}b_{k-m,n-l}e^{-i\theta n(k-m)}.$$

This is then the natural multiplication operation associated to the representation. Similarly, we would want the relation $\pi(\mathbf{a})^* = \pi(\mathbf{a}^*)$ to hold, for some appropriate "twisted" involution.

$$\pi(\mathbf{a})^* = \Big(\sum_{k,l\in\mathbb{Z}} a_{k,l}U^kV^l\Big)^* = \sum_{k,l\in\mathbb{Z}} \overline{a_{k,l}}V^{-l}U^{-k}$$
$$= \sum_{k,l\in\mathbb{Z}} \overline{a_{-k,-l}}V^lU^k = \sum_{k,l\in\mathbb{Z}} \overline{a_{-k,-l}}e^{-i\theta kl}U^kV^l = \pi(\mathbf{a}^*).$$

We have here defined $(a_{k,l})^* = e^{-i\theta kl}\overline{a_{-k,-l}}$

4.1 Derivations and connections

We wish to construct the same structure on the non-commutative torus as we have defined for the dual pairs $\mathcal{A} \rightleftharpoons \mathcal{E} \rightleftharpoons \mathcal{B}$ in the chapter on Hilbert C^*-modules. To make everything explicit we will be using the operators $U^l = M_l$, $V^k = T_{\theta k}$, using the modulation and translation operators M_1 and T_θ.

The C^*-algebra of the norm closure of the span of these operators is a realization of $\mathcal{A}_\theta$ on $L^2(\mathbb{R}^d)$. The smooth non-commutative torus can be represented as elements of the form

$$\pi(\mathbf{a}) = \sum_{k,l\in\mathbb{Z}} a_{kl} M_l T_{\theta k} \quad \text{for } (a_{kl}) \in \mathcal{S}(\mathbb{Z}^2) \tag{97}$$

This is a Fréchet algebra with the a set of semi-norms $\{\|a\|_{\theta,s} : s \geq 0\}$:

$$\|a\|_{\theta,s} := \sup_{k,l\in\mathbb{Z}} |a_{kl}|(1 + |k|^2 + |l|^2)^{s/2}.$$

In much the same way we can define the smooth non-commutative torus $\mathcal{A}_{-1/\theta}$ consisting of elements

$$\pi(\mathbf{b}) = \sum_{k,l\in\mathbb{Z}} b_{kl} M_{l/\theta} T_k \quad \text{for } (b_{kl}) \in \mathcal{S}(\mathbb{Z}^2), \tag{98}$$

with a similar collection of semi-norms. The minus is there to represent that this will be acting from the right, the opposite way of $\mathcal{A}_\theta$.

To construct an equivalence bimodule between $\mathcal{A}_\theta$ and $\mathcal{A}_{-1/\theta}$ we need to define derivations, inner products, traces and actions. In the following, let $\mathbf{a} \in \mathcal{A}_\theta$, $\mathbf{b} \in \mathcal{A}_{-1/\theta}$, and $f, g \in \mathcal{S}(\mathbb{R})$. Then define

$$\begin{aligned} {}_\bullet\langle f, g\rangle &= \theta \sum_{k,l\in\mathbb{Z}} V_g f(\theta k, l) M_l T_{\theta k}, \\ \text{and } \langle f, g\rangle_\bullet &= \sum_{k,l\in\mathbb{Z}} V_f g(k, l\theta^{-1}) M_{l\theta^{-1}} T_k. \end{aligned}$$

Note the similarity to the Gabor frame representation of functions.

The natural action of $\mathcal{A}_\theta$ and $\mathcal{A}_{-1/\theta}$ is

$$\begin{aligned} \mathbf{a} \cdot f &= \sum_{k,l\in\mathbb{Z}} a_{kl} M_l T_{\theta k} f, \\ \text{and } f \cdot \mathbf{b} &= \sum_{k,l\in\mathbb{Z}} b_{kl} (M_{l\theta^{-1}} T_k)^* f. \end{aligned}$$

The derivations on $\mathcal{A}_\theta$ are given by

$$\begin{aligned} \partial_1(\mathbf{a}) &= i \sum_{k,l\in\mathbb{Z}} k a_{kl} M_l T_{\theta k} \\ \text{and } \partial_2(\mathbf{a}) &= i \sum_{k,l\in\mathbb{Z}} l a_{kl} M_l T_{\theta k}, \end{aligned}$$

and on $\mathcal{A}_{-1/\theta}$

$$\partial_1(\mathbf{b}) = -i\theta^{-1} \sum_{k,l\in\mathbb{Z}} k b_{kl} (M_{l\theta^{-1}} T_k)^*$$
$$\text{and } \partial_2(\mathbf{b}) = -i\theta^{-1} \sum_{k,l\in\mathbb{Z}} l b_{kl} (M_{l\theta^{-1}} T_k)^*.$$

These should lift to derivations on the equivalence bimodule we have chosen, $\mathcal{S}(\mathbb{R})$. We define the operations here as

$$(\nabla_1 f)(t) = i\theta^{-1} t f(t), \text{ and } (\nabla_2 f)(t) = f'(t).$$

Lastly, we define the trace as $\tau(\mathbf{a}) = a_{00}$ and $\tau(\mathbf{b}) = \theta b_{00}$ on $\mathcal{A}_\theta$ and $\mathcal{A}_{-1/\theta}$ respectively.

Theorem 4.1 *With the operations and actions defined above, the space $\mathcal{S}(\mathbb{R})$ acts as an equivalence bimodule between $\mathcal{A}_\theta$ and $\mathcal{A}_{-1/\theta}$, and the properties of the derivations and connections are satisfied. That is $\mathcal{A}_\theta$ and $\mathcal{A}_{-1/\theta}$ are completely Morita equivalent.*

Proof As discussed in the chapter on derivations and connections and by using Proposition 2.7 we need to check the equalities (34),(35),(36), (37), (38) and (27).

(34) is quite simple, as $\tau(\partial_j \mathbf{a}) = 0 a_{00} = 0$. The same holds for $\mathbf{b}$.

(35) follows from a short calculation. By definition of the inner product we see that

$$\tau({}_\bullet\langle f, g\rangle) = \theta V_g f(0,0) \text{ and } \tau(\langle g, f\rangle_\bullet) = \theta V_g f(0,0).$$

(36) requires somewhat more work, but is not difficult as long as one has good control of the definitions. There are four different cases to consider here, $j = 1, 2$ for both $\mathcal{A}_\theta$ and $\mathcal{A}_{-1/\theta}$. We show equality for $\mathcal{A}_\theta$, the other case follows in the same way. Let first $j = 1$, then the right side reads

$$\begin{aligned}(\partial_1 a)f(t) + a(\nabla_1 f(t)) &= i\sum_{k,l} k a_{kl} M_l T_{\theta k} f(t) + i\theta^{-1} \sum_{k,l} a_{kl} M_l T_{\theta k}(t f(t)) \\ &= i\sum_{k,l} k a_{kl} M_l T_{\theta k} f(t) + i\theta^{-1} \sum_{k,l} (t - \theta k) a_{kl} M_l T_{\theta k}(f(t)) \\ &= i\theta^{-1} \sum_{k,l} t a_{kl} M_l T_{\theta k}(f(t)).\end{aligned}$$

The left side is a much easier calculation,

$$\nabla_1(a f(t)) = \nabla_1(\sum_{k,l} a_{kl} M_l T_{\theta k} f(t)) = i\theta^{-1} t \sum_{k,l} a_{kl} M_l T_{\theta k} f(t),$$

so we have the desired equality. Now let $j = 2$. Then the left side is

$$\nabla_2(a \cdot f(t)) = \frac{d}{dt} \sum_{k,l} a_{kl} M_l T_{\theta k} f(t) = \sum_{k,l} a_{kl} \frac{d}{dt}(M_l T_{\theta k} f(t)).$$

The right side is, in much the same way as for the first case

$$
\begin{aligned}
(\partial_2 a)f(t) + a(\nabla_2 f(t)) &= i\sum_{k,l} l a_{kl} M_l T_{\theta k} f(t) + \sum_{k,l} M_l T_{\theta k} f'(t) \\
&= i\sum_{k,l} l a_{kl} M_l T_{\theta k} f(t) - \sum_{k,l} a_{kl} i l e^{ilt} f(t-\theta k) + \\
&\sum_{k,l} a_{kl} \frac{d}{dt}\big(M_l T_{\theta k}\big) \\
&= \sum_{k,l} a_{kl} \frac{d}{dt}\big(M_l T_{\theta k} f(t).
\end{aligned}
$$

We show (37), and omit (38).

$$
\partial_1({}_\bullet\langle f,g\rangle) = \partial_1(\theta \sum_{k,l} V_g f(\theta k, l) M_l T_{\theta k}) = \theta i \sum_{k,l} k V_g f(\theta k, l) M_l T_{\theta k}.
$$

The left side is somewhat trickier, we first note that

$$
\begin{aligned}
V_{i\theta^{-1}tg} f(\theta k, l) &= -i\theta^{-1} V_g((t-\theta k)f)(\theta k, l) \\
\text{and} \quad V_g(i\theta^{-1}tf)(\theta k, l) &= i\theta^{-1} V_g(tf)(\theta k, l).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
{}_\bullet\langle \nabla_1 f, g\rangle +_\bullet \langle f, \nabla_1 g\rangle &= \theta \sum_{k,l} V_g(i\theta^{-1}tf)(\theta k, l) M_l T_{\theta k} + \theta \sum_{k,l} V_{i\theta^{-1}tg} f(\theta k, l) M_l T_{\theta k} \\
&= i\sum_{k,l} V_g(tf)(\theta k, l) M_l T_{\theta k} - i\sum_{k,l} V_g((t-\theta k)f)(\theta k, l) M_l T_{\theta k} \\
&= \theta i \sum_{k,l} k V_g f(\theta k, l) M_l T_{\theta k}.
\end{aligned}
$$

The calculation is similar for $j = 1$.

Lastly we show the associativity connection. First, assume that the Fundamental Identity of Gabor Analysis holds, we will provide a proof later. It states that for $f_1, f_2, g_1, g_2 \in M^1(\mathbb{R}) \supset \mathcal{S}(\mathbb{R})$, we have

$$
\sum_{\lambda\in\Lambda} V_{g_1} f_1(\lambda) \cdot \overline{V_{g_2} f_2(\lambda)} = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} V_{g_1} g_2(\lambda^0) \overline{V_{f_1} f_2(\lambda^0)}
$$

for appropriate lattices Λ and Λ^0, we will also show that our choice of lattice is correct. Using this for our case, we find that

$$
\begin{aligned}
\langle {}_\bullet\langle f_1, g_1\rangle g_2, f_2\rangle &= \theta \sum_{k,l} V_g f(\theta k, l) \cdot \overline{V_{g_2} f_2(\theta k, l)} \\
&= \sum_{k,l} V_{g_1} g_2(k, \theta^{-1} l) \overline{V_{f_1} f_2(k, \theta^{-1} l)} = \langle f_1 \langle g_1, g_2\rangle_\bullet, f_2\rangle.
\end{aligned}
$$

∎

Since $\mathcal{A}_{-1/\theta}$ is unital, the sequence with $b_{00} = 1$ and $b_{kl} = 0$ otherwise gives the identity, we can apply Proposition 2.10. By this, there exists some finite standard module frame $\{g_1, g_2, \cdots, g_n\} \subset \mathcal{A}_\theta$ such that $\mathcal{S}(\mathbb{R})$ is a finitely generated projective $\mathcal{A}_\theta$-module and is self-dual over $\mathcal{A}_\theta$.

Our choice of lattices ($\mathbb{Z}\theta \times \mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Z}\theta^{-1}$), might seem arbitrary, but it allows the Fundamental Identity of Gabor Analysis to hold, by ensuring that we have commutativity of $\pi(\lambda)$ and $\pi(\lambda^0)$. This is apparent from the fact that our symplectic form c yields $c((\theta k, l), (k, \theta^{-1} l) = 1$. We state the Poisson Summation Formula for a lattice Λ and its adjoint lattice Λ^0 given by

$$\Lambda^0 = \{z \in \mathbb{R}^2 \ : \ \pi(\lambda)\pi(z) = \pi(z)\pi(\lambda) \text{ for all } \lambda \in \Lambda\}$$

Theorem 4.2 *Let $F \in M^1(\mathbb{R}^2)$, then*

$$\sum_{\lambda \in \Lambda} F(\lambda) = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} \hat{F}^s(\lambda^0),$$

Where $\hat{F}^s$ denotes the Symplectic Fourier transform

$$\hat{F}^s(z) = \iint_{\mathbb{R}^2} e^{i\Omega(z,z')} F(z')\ dz = \iint_{\mathbb{R}^2} e^{i(x\rho - yw)} F(z') dz'$$

for $z = (x, w)$ and $z' = (y, \rho)$.

Using this theorem allows us to prove the Fundamental Identity of Gabor Analyis, finishing the proof of the construction of the equivalence bimodule

Theorem 4.3 *Let $f_1, f_2, g_1, g_2 \in M^1(\mathbb{R})$ and Λ, Λ^0 be lattices in $\mathbb{R}^2$, then*

$$\sum_{\lambda \in \Lambda} V_{g_1} f_1(\lambda) \cdot \overline{V_{g_2} f_2(\lambda)} = |\Lambda|^{-1} \sum_{\lambda^0 \in \Lambda^0} V_{g_1} g_2(\lambda^0) \overline{V_{f_1} f_2(\lambda^0)} \tag{99}$$

Proof Denote by $F(\lambda)$ the product $V_{g_1} f_1(\lambda)\overline{V_{g_2} f_2(\lambda)}$. Then we compute $\hat{F}^s$.

$$\begin{aligned}
\hat{F}^s(z) = \hat{F}^s(x, w) &= \iint_{\mathbb{R}^2} e^{i(yw - x\rho)} \langle f_1, \pi(y, \rho) g_1 \rangle \overline{\langle f_2, \pi(y, \rho)\rangle g_2}\ dy\ d\rho \\
&= \iint_{\mathbb{R}^2} e^{i(yw - x\rho)} \langle \pi(x, w) f_1, \pi(x, w)\pi(y, \rho) g_1 \rangle \overline{\langle f_2, \pi(y, \rho) g_2}\ dy\ d\rho \\
&= \iint_{\mathbb{R}^2} e^{yw - x\rho} \langle \pi(z) f_1, e^{i(yw - x\rho)} \pi(z')\pi(z) g_1 \rangle \overline{\langle f_2, \pi(z') g_2 \rangle}\ dz' \\
&= \iint_{\mathbb{R}^2} \langle \pi(z) f_1, \pi(z')\pi(z) g_1 \rangle \overline{\langle f_2, \pi(z') g_2 \rangle}\ dz' \\
&= \iint_{\mathbb{R}^2} V_{\pi(z) g_2}[\pi(z) f_1](z') \overline{V_{g_2} f_2(z')}\ dz' \\
&= \langle V_{\pi(z) g_1}[\pi(z) f_1], V_{g_2} f_2 \rangle \\
&= \langle g_2, \pi(z) g_1 \rangle \overline{\langle f_2, \pi(z) f_1 \rangle} \\
&= V_{g_1} g_2(z) \overline{V_{f_1} f_2}(z).
\end{aligned}$$

The second to last line is the Orthogonality Lemma, Theorem 2.20, from our chapter on modulation spaces (also known as the Moyal Identity.)

Since $V_{g_1}f_1\widehat{(z)\overline{V_{g_2}f_2(z)}}^s = V_{g_1}g_2(z)\overline{V_{f_1}f_2}$, the Poisson summation formula gives us the result. This relies also on the result that $M^1(\mathbb{R}^2)$ is a Banach algebra with respect to pointwise multiplication, otherwise we could not assume that the product F is in $M^1(\mathbb{R}^2)$. $\blacksquare$

By [2], we have that that $0 < \theta < 1$ is the necessary and sufficient condition for the Gaussians $g \in \mathcal{S}(\mathbb{R})$ to generate a Gabor frame. This implies that as a left $\mathcal{A}_\theta$-module it is generated by a single projection. The one-element Parseval frame g gives, for any $f \in \mathcal{S}(\mathbb{R})$ the equality $f = {}_\bullet\langle f, g\rangle g$, by associativity $\langle g, g\rangle_\bullet = 1_{\mathcal{A}_{-1/\theta}}$ and by Lemma 2.11, ${}_\bullet\langle g, g\rangle$ is a projection in $\mathcal{A}_\theta$. Lastly we have that $\mathcal{S}(\mathbb{R}) = ({}_\bullet\langle g, g\rangle)\mathcal{A}_\theta$. For Gabor systems generated by a lattice in $\mathbb{R}^{2d}$, we can only say that there exist a finite set of functions $g_1, ..., g_n$ that generate a frame for $L^2(\mathbb{R}^d)$, aka multi-window Gabor frame. Luef has deduced the existence of such functions in Feichtinger's algebra using the theory developed in this section [23].

5 Non-Commutative Sobolev Spaces

5.1 Localization Operators

We will now introduce the localization operator $A_m^{\phi_1,\phi_2}$ on the modulation spaces $M_m^{p,q}(\mathbb{R}^d)$. They are also known as anti-Wick operators or Toeplitz operators, and are well studied. For a complete introduction see for instance [8] or [17], where they study the special case $\phi_1 = \phi_2$ and denote it A_m^g.

Definition The **localization operator** $A_m^{\phi_1,\phi_2}$ for a function m and window-functions ϕ_1, ϕ_2 is formally defined by

$$A_m^{\phi_1,\phi_2} f = \iint_{\mathbb{R}^{2d}} m(z) V_{\phi_1} f(z) \pi(z) \phi_2 \, dz. \tag{100}$$

Where $\pi(z) = M_w T_x$.

Choosing for instance a cut-off function $m(z) = \chi_Q(z)$ for some compact set $Q \subseteq \mathbb{R}^{2d}$ will localize the integral to only a compact set, making it easier to work with. This is helpful numerically, but the non-smoothness of χ_Q makes it difficult to work with in theory. So we will mostly be considering v_s-moderate weight functions m, where we recall that $v_s(z) = (1 + |x|^2 + |w|^2)^{s/2}$.

The localization operators have a strong relation to the short-time Fourier transform, and we have some interesting equalities that follow easily from the definitions.

Lemma 5.1 *Let $g, f \in M_v^1(\mathbb{R}^d)$ and $m \in M^\infty(\mathbb{R}^{2d})$ be v-moderate, v is as usual a submultiplicative weight function. Then we have that*

$$\langle A_m^{\phi_1,\phi_2} f, k \rangle_{L^2(\mathbb{R}^{2d})} = \langle m V_{\phi_1} f, V_{\phi_2} k \rangle_{L^2(\mathbb{R}^d)} = \langle m, \overline{V_{\phi_1} f} V_{\phi_2} k \rangle_{L^2(\mathbb{R}^{2d})},$$

and

$$V_g(A_m^{g,g} f)(w) = \iint_{\mathbb{R}^{2d}} m(z) V_g f(z) \langle \pi(z) g, \pi(w) g \rangle \, dz = \left((m V_g f) \natural V_g g \right)(w)$$

for the usual twisted convolution.

Proof We do the calculation for the second case formally, and then show boundedness. Let $g, f \in M_v^1(\mathbb{R}^d)$ and $m \in M^\infty(\mathbb{R}^{2d})$ be v-moderate. Then

$$\begin{aligned} V_g \left(A_m^{g,g} f \right)(w) &= \int_{\mathbb{R}^d} A_m^{g,g} f(t) \overline{\pi(w) g(t)} \, dt \\ &= \int_{\mathbb{R}^d} \iint_{\mathbb{R}^{2d}} m(z) V_g f(z) \pi(z) g(t) \, dz \overline{\pi(w) g(t)} \, dt \\ &= \iint_{\mathbb{R}^{2d}} \int_{\mathbb{R}^d} m(z) V_g f(z) \pi(z) g(t) \overline{\pi(w) g(t)} \, dt \, dz \\ &= \iint_{\mathbb{R}^{2d}} m(z) V_g f(z) \langle \pi(z) g, \pi(w) g(t) \, dz, \end{aligned}$$

showing the first equality. Now using the calculations in Lemma 3.1, we have that

$$\begin{aligned}
&= \iint_{\mathbb{R}^{2d}} m(z)V_g f(z)\langle \pi(z)g, \pi(w)g(t)\ dz \\
&= \iint_{\mathbb{R}^{2d}} m(z)V_g f(z)c(z, w-z)V_g g(w-z)\ dz \\
&= ((mV_g f)\natural V_g g)\,(w).
\end{aligned}$$

Since localization operators can be seen as the image of the multilinear map $(m, \phi_1, \phi_2) \to A_m^{\phi_1,\phi_2}$, we have the bound $||A_m^{\phi_1,\phi_2}|| \leq C||\phi_1||\ ||\phi_2||\ ||m||$ for their respective spaces. So $||A_m^{\phi_1,\phi_2} f||_{M_v^1} \leq C||\phi_1||_{M_v^1}||\phi_2||_{M_v^1}||m||_{M^\infty}||f|| < \infty$, so we can take the STFT of $A_m^{\phi_1,\phi_2} f$ and the rest follows. ∎

Interestingly enough, the localization operators allows us to construct isomorphism between differently weighted modulation spaces. The main result of [17] states:

Theorem 5.2 *Let $g \in M^1_{v^2w}(\mathbb{R}^d)$, μ, m be w- and v-moderate respectively and m be radial in each coordinate for submultiplicative v and w such that*

$$\lim_{n\to\infty} v(nz)^{1/n} = 1. \tag{101}$$

Then

$$A_m^g : M^{p,q}_\mu(\mathbb{R}^d) \to M^{p,q}_{\mu/m}(\mathbb{R}^d)$$

is an isomorphism for every $1 \leq p, q \leq \infty$.

For our work, we mostly need boundedness of the operator, but it is nice to know that these spaces are so well-behaved and the relation they have with their weights.

This result also holds for the discrete localization operator, known as the Gabor multiplier because of its similarity with Fourier multipliers. For some suitable lattice Λ, we define it formally as

$$G_m^{g,\gamma,\Lambda} f = \sum_{\lambda\in\Lambda} m(\lambda)\langle f, \pi(\lambda)g\rangle \pi(\lambda)\gamma.$$

Optimally we would like to construct a similar equivalence bimodule as we have done for the Moyal plane and the non-commutative torus using these operators, but such a result is likely not possible. Considering that such a bimodule would result in Proposition 1.10 holding, our difficulty is clear, as by introducing the weight function m we loose our ability to extract information from the entire phase-space in general. By having a function that disappears somewhere on $\mathbb{R}^2$ we loose information in this area, so we would have to be very careful at least. What we can have is one side of the module for some carefully chosen weights, going back to the simple Hilbert C^*-module structure. We can define a candidate for an inner product over $M^1_m(\mathbb{R}^d)$, for a v-moderate m and submultiplicative v as

$$\langle f, g\rangle_m = \iint_{\mathbb{R}^{2d}} m(z) V_g f(z)\pi(z)\ dz \tag{102}$$

Then $\langle f, g\rangle_m$ acts on $M^1_m(\mathbb{R}^d)$ from the left by $\langle f, g\rangle h = A^{g,h}_m f$. The whole space $\mathcal{A}$ of elements that acts from the left consists of operators

$$K = \pi(k) = \iint_{\mathbb{R}^{2d}} m(z)k(z)\pi(z)\ dz \quad \text{for } k \in L^1_v(\mathbb{R}^{2d})$$

acting on $M^1_m(\mathbb{R}^d)$ by

$$K \cdot f = \iint_{\mathbb{R}^{2d}} m(z)k(z)\pi(z) f\ dz. \tag{103}$$

Multiplication (and involution similarly) is defined by

$$K \cdot L = \iint_{\mathbb{R}^{2d}} m(z)(mk\natural l)\pi(z)\ dz \tag{104}$$

and similarly for the discrete case. The image of the map $\langle \cdot, \cdot\rangle_m$ is then dense in $\mathcal{A}$.

Note that $\langle f, g\rangle_1 = {}_\bullet\langle f, g\rangle$ from our previous structure on the Moyal plane and non-commutative torus.

We need to check that this is in fact an $\mathcal{A}$-valued inner product and completeness.

We go through equations (5) - (8) (replacing right by left in the actions) in order,

1. Since the weight functions are strictly positive and real we have $m^*(z) = m(-z)$, and by using the commutation relations for $\pi(z) = M_w T_x$ we have

$$\begin{aligned}
\langle f, g\rangle_m^* &= \left(\iint_{\mathbb{R}^{2d}} m(z) V_g f(z)\pi(z)\ dz\right)^* \\
&= \iint_{\mathbb{R}^{2d}} m(-z)\langle f, \pi(z)g\rangle^*_{L^2}\pi^*(z)\ dz \\
&= \iint_{\mathbb{R}^{2d}} m(-z)\langle \pi(z)g, f\rangle\pi^*(z)\ dz \\
&= \iint_{\mathbb{R}^{2d}} m(-z)\langle g, T_{-x}M_{-w}f\rangle T_{-x}M_{-w}\ dz \\
&= \iint_{\mathbb{R}^{2d}} m(-z)\langle g, M_{-w}T_x f\rangle e^{iwx}e^{-iwx}M_{-w}T_{-x}\ dz \\
&= \iint_{\mathbb{R}^{2d}} m(z)V_f g(z)\pi(z)\ dz = \langle g, f\rangle_m,
\end{aligned}$$

by doing the change of coordinate $z \to -z$.

2. Since all functions are in L^1 either over $\mathbb{R}^d$ or $\mathbb{R}^{2d}$ we can use Fubini's theorem to freely change the order of integration, therefore

$$\begin{aligned}\langle K \cdot f, g\rangle_m &= \iint_{\mathbb{R}^{2d}} m(z) V_g \left(\iint_{\mathbb{R}^{2d}} m(z')k(z')\pi(z')f \; dz' \right)(z)\pi(z)\; dz \\ &= \iint_{\mathbb{R}^{2d}} m(z) \int_{\mathbb{R}^d} \iint_{\mathbb{R}^{2d}} m(z')k(z')\pi(z')f(t)\; dz'\overline{\pi(z)g(t)}\; dt\; \pi(z)\; dz \\ &= \iint_{\mathbb{R}^{2d}} m(z) \iint_{\mathbb{R}^{2d}} \int_{\mathbb{R}^d} m(z')k(z')\pi(z')f(t)\overline{\pi(z)g(t)}\; dt\; dz'\; \pi(z)\; dz \\ &= \iint_{\mathbb{R}^{2d}} m(z) \iint_{\mathbb{R}^{2d}} m(z')k(z')V_g f(z-z')e^{-ix(w-w')}\; dz'\; \pi(z)\; dz \\ &= \iint_{\mathbb{R}^{2d}} m(z)(mk\natural V_g f)(z)\pi(z)\; dz \\ &= K \cdot \langle f, g\rangle_m\end{aligned}$$

3. Positivity is in many ways the limiting factor of our inner product, we will show that it requires some extra conditions on the weights. By using the continuous version of Theorem 4.2 and the convolution properties of the symplectic Fourier Transform (similar to the equality for the euclidean Fourier transform) we have

$$\begin{aligned}\langle\langle f, f\rangle_m g, g\rangle &= \langle m V_f f, V_g g\rangle \\ &= \iint_{\mathbb{R}^{2d}} m(z) V_f f \overline{V_g g}(z)\; dz \\ &= \mathcal{F}^s(m * V_f f \overline{V_g g})(0)\end{aligned}$$

We have already calculated the symplectic Fourier Transform of $V_f f\overline{V_g g}$ in the proof of Theorem 4.3. This combined with the convolution properties of the Fourier transform gives us that

$$\begin{aligned}\langle\langle f, f\rangle_m g, g\rangle &= \mathcal{F}^s(m)(0)\mathcal{F}^s(V_f f\overline{V_g g})(0) \\ &= \mathcal{F}^s(m)(0)V_f g\overline{V_f g}(0) \\ &= \mathcal{F}^s(m)(0)\langle f, g\rangle\overline{\langle f, g\rangle} \\ &= \mathcal{F}^s(m)(0)|\langle f, g\rangle|^2.\end{aligned}$$

Because of the general fact that an operator A is positive if and only if $\langle Ax, x\rangle \geq 0$, we conclude that $\langle f, f\rangle_m$ is a positive operator if and only if the symplectic Fourier transform of m is positive at 0. We have by Proposition 2.7 in [24], with some rearrangement of terms, that v_s for $s < 0$ is an example of such a function.

4. If $f \equiv 0$, clearly $\langle f, f\rangle_m = 0$. If $\langle f, f\rangle_m g = 0$ for all $g \in M_v^1(\mathbb{R})$, then specifically for $f = g$ we have $\langle f, f\rangle_m f = A_m^f = 0$, but we know from Theorem 5.2 that A_m^f is an isomorphism, so $f = 0$. This assumes that v satisfies the equation (101), so if we assume that we are using v_s-moderate weights we are fine.

We now claim that the space $M^1_m(\mathbb{R}^d)$ is a pre-Hilbert $\mathcal{A}$-module, and that we can complete it with respect to the $||\cdot||_{\mathcal{A}}$-norm. Firstly, the image of $\langle\cdot,\cdot\rangle_m$ defines the dense $*$-subalgebra $\mathcal{A}_0$ such that $M^1_m(\mathbb{R}^d)$ is a projective (pre)-$\mathcal{A}_0$-module. This is clear, since for any $g \in \mathcal{S}(\mathbb{R}^d)$ and $f \in M^1_m(\mathbb{R}^d)$ we have that $f = \langle f, g\rangle_m g$. Since $\mathcal{A}$ is a C^*-algebra under twisted convolution in the same way as for the Moyal Plane we can apply Rieffel's result from [26] Proposition 3.7 to complete $M^1_m(\mathbb{R}^d)$ under the action from the left of $\mathcal{A}$. We will in future make no distinction between $M^1_m(\mathbb{R}^d)$ and its completion however. This allows us to conclude that the completion of $M^1_m(\mathbb{R}^d)$ is a projective $\mathcal{A}$-module. This result is summarized in the following proposition.

Proposition 5.3 *Let m be a v_s-moderate function, $v_s(z) = (1 + |x| + |w|)^{s/2}$, such that its symplectic Fourier transform evaluated at the origin is positive. Then the action of $\mathcal{A}$ on $M^1_m(\mathbb{R}^d)$ defined by (103) along with the inner product (102) defines $M^1_m(\mathbb{R}^d)$ as a pre-Hilbert $\mathcal{A}$-module, and can therefore be completed to a proper Hilbert $\mathcal{A}$-module, for simplicity also denoted by $M^1_m(\mathbb{R}^d)$*

Note that we could generalize this result further, by requiring v to only satisfy the conditions of Theorem 5.2. Similarly, the same holds for the discrete case, if the weights satisfy the condition of K. Gröchenig & J. Toft's result, Proposition 5.2 in [17]. In what follows, we will frequently write $\langle f, g\rangle_v$, even when we have not shown the positivity of the symplectic Fourier transform of v. Here we do not, a priori, have the same Hilbert $\mathcal{A}$-module structure, but we do not use these properties anyway. We will define derivatives and connections as for the Moyal plane and non-commutative torus, so it is convenient to use the same notation. If at any point, the Hilbert structure is required, we define it not using the inner product $\langle f, g\rangle_v$, but rather by using $\langle f, g\rangle_1 = {}_\bullet\langle f, g\rangle$ and apply derivations.
We have the following interesting norm-estimation:

Lemma 5.4 *For any $f \in M^1_m(\mathbb{R}^d)$, where the assumptions of Proposition 5.3 is satisfied, the $\mathcal{A}$-norm, denoted $||\cdot||_m$ for v_s-moderate weight m is bounded by*

$$||f||^2_m \leq C||f||_{M^1_{mv}} \leq C||f||_{M^1_{v^2}} \tag{105}$$

Proof The proof is an exercise in rearranging terms and using twisted convolution. We quickly summarize the calculations.

$$
\begin{aligned}
||f||_m^2 &= ||\langle f, f\rangle_m||_{op} = || \iint_{\mathbb{R}^{2d}} m(z)V_f f(z)\pi(z)\ dz||_{op} \\
&= \sup_{||g||_{M^1_m}=1} \iint_{\mathbb{R}^{2d}} m(z')|V_\phi \left(\iint_{\mathbb{R}^{2d}} m(z)V_f f(z)\pi(z)g\ dz \right)(z')|\ dz' \\
&= \sup_{||g||_{M^1_m}=1} \iint_{\mathbb{R}^{2d}} m(z')|(mV_f f)\natural V_\phi g|(z')dz' \\
&= \sup_{||g||_{M^1_m}=1} ||(mV_f f)\natural V_\phi g||_{L^1_m} \\
&\le \sup_{||g||_{M^1_m}=1} ||mV_f f||_{L^1_v} ||V_\phi g||_{L^1_m} \\
&= C||f||_{M^1_{mv}} \le C||f||_{M^1_{v^2}},
\end{aligned}
$$

completing the proof ∎

We can create derivations and connections also here, even though this is just a simple module. We would like the same relations to hold, (34)-(38), but with only one derivative this time. We define the covariant derivatives on $M^1_m(\mathbb{R}^d)$ to be

$$\nabla_1(f(t)) = itf(t), \quad \nabla_2(f(t)) = f'(t), \tag{106}$$

which are liftings of the derivations on $\mathcal{A}$

$$\partial_1(K) = i \iint_{\mathbb{R}^{2d}} xm(z)k(z)\pi(z)\ dz \tag{107}$$

$$\partial_2(K) = i \iint_{\mathbb{R}^{2d}} wm(z)k(z)\pi(z)\ dz. \tag{108}$$

Proposition 5.5 *The definitions* (106)-(108) *establishes the Hilbert $\mathcal{A}$-module $M^1_m(\mathbb{R}^d)$ as a complete module, with the appropriate connections.*

We will from now on denote the algebra with weight function m by $\mathcal{A}_m$.

Proof We first show that they satisfy the Leibniz Rule (36), the calculation is very similar to the one in the section on the non-commutative torus.

$$
\begin{aligned}
&(\partial_1 K)f(t) + K(\nabla_1 f(t)) \\
=& \iint_{\mathbb{R}^{2d}} ixm(z)k(z)\pi(z)f(t) + im(z)k(z)\pi(z)(tf(t))\ dz \\
=& \iint_{\mathbb{R}^{2d}} ixm(z)k(z)\pi(z)f(t) + itm(z)k(z)\pi(z)f(t) - ixm(z)k(z)\pi(z)f(t)\ dz \\
=& \nabla_1(Kf(t)).
\end{aligned}
$$

Then we check for ∇_2,

$$\begin{aligned}
&(\partial_2 K)f(t) + K(\nabla_2 f(t)) \\
=&\iint_{\mathbb{R}^{2d}} iwm(z)k(z)\pi(z)f(t) + m(z)k(z)\pi(z)f'(t)\ dz \\
=&\iint_{\mathbb{R}^{2d}} iwm(z)k(z)\pi(z)f(t) - iwm(z)k(z)\pi(z)f(t) + m(z)k(z)\frac{d}{dt}\left(\pi(z)f(t)\right)\ dz \\
=&\iint_{\mathbb{R}^{2d}} m(z)k(z)\frac{d}{dt}\left(\pi(z)f(t)\right)\ dz \\
=&\nabla_2(Kf(t)).
\end{aligned}$$

We can then check the compatibility of the inner product, equation (37):

$$\begin{aligned}
&\langle \nabla_1 f, g\rangle_m + \langle f, \nabla_1 g\rangle_m \\
=&\iint_{\mathbb{R}^{2d}} m(z)V_g[itf](z)\pi(z)\ dz + \iint_{\mathbb{R}^{2d}} m(z)V_{itg}f(z)\pi(z)\ dz \\
=&i\iint_{\mathbb{R}^{2d}} m(z)\left(V_g[tf](z) - V_{tg}f(z)\right)\pi(z)\ dz.
\end{aligned}$$

Considering only the middle part of the integral,

$$\begin{aligned}
V_g[tf](z) - V_{tg}f(z) &= \int_{\mathbb{R}^d} tf(t)\overline{\pi(z)g(t)} - f(t)\overline{\pi(z)tg(t)}\ dt \\
&= \int_{\mathbb{R}^d} tf(t)\overline{\pi(z)g(t)} - (t-x)f(t)\overline{\pi(z)g(t)}\ dt \\
&= \int_{\mathbb{R}^d} xf(t)\overline{\pi(z)g(t)}\ dt = xV_gf(z).
\end{aligned}$$

So we have finally

$$\begin{aligned}
\langle \nabla_1 f, g\rangle_m + \langle f, \nabla_1 g\rangle_m &= \iint_{\mathbb{R}^{2d}} xm(z)V_gf(z)\pi(z)\ dz \\
&= \partial_1(\langle f, g\rangle_m).
\end{aligned}$$

The case for ∇_2 is slightly more intricate,

$$\langle \nabla_2 f, g\rangle_m + \langle f, \nabla_2 g\rangle_m = \iint_{\mathbb{R}^{2d}} m(z)\left(V_gf'(z) + V_{g'}f(z)\right)\pi(z)\ dz.$$

We consider once again the middle term,

$$\begin{aligned}
V_gf'(z) + V_{g'}f(z) &= \int_{\mathbb{R}^d} f'(t)\overline{\pi(z)g(t)} + f(t)\overline{\pi(z)g(t)}\ dz \\
&= \int_{\mathbb{R}^d} f'(t)\overline{\pi(z)g(t)} + f(t)\frac{d}{dt}(\overline{\pi(z)g(t)}) + iwf(t)\overline{\pi(z)g(t)}\ dz \\
&= \langle \frac{d}{dt}f, \pi(z)g\rangle_2 + \langle f, \frac{d}{dt}\pi(z)g\rangle_2 + iwV_gf(z).
\end{aligned}$$

The two first terms cancel out by the rule $\langle f, g' \rangle = -\langle f', g \rangle$ (Integration by parts), and we are left with

$$\begin{aligned}\langle \nabla_2 f, g \rangle_m + \langle f, \nabla_2 g \rangle_m &= \iint_{\mathbb{R}^{2d}} iwm(z) V_g f(z) \pi(z) \, dz \\ &= \partial_2(\langle f, g \rangle_m).\end{aligned}$$

Lastly we can define a tracial state on $\mathcal{A}$, τ by

$$\tau(K) = k(0)m(0).$$

It trivially satisfies $\tau(\partial_j K) = 0$, since

$$\tau(\partial_1 K) = \tau \left(\iint_{\mathbb{R}^{2d}} ixm(z)k(z)\pi(z) \, dz \right) = i0m(0)k(0) = 0$$

and the same for ∂_2. ∎

5.2 Function spaces

We have already shown that $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, but of central importance are also the Sobolev spaces, or the Bessel Potential spaces.

Definition Let Ω be an open subset of $\mathbb{R}^d$, and $m \in \mathbb{N}_0$. Then the **Sobolev space** $H^m(\Omega)$ is defined by

$$H^m(\Omega) = \{f \in L^2(\Omega) \;:\; D^\alpha f \in L^2(\Omega) \text{ for } |\alpha| \leq m\} \tag{109}$$

with D^α being applied in the distributional sense.

The Bessel potential spaces are a generalization of these,

Definition For the weight $v_s(z) = v_s(w) = (1 + |w|^2)^{s/2}$ for $s \in \mathbb{R}$, the **Bessel potential space** $W_s^2(\mathbb{R}^d)$ is defined by

$$W_s^2(\mathbb{R}^d) = \{f \;:\; \hat{f}(w)v_s(w) \in L^2(\mathbb{R}^d)\} \tag{110}$$

The first thing to note is that by the familiar Fourier transform equality

$$\mathcal{F}(D^\alpha f)(w) = w^\alpha (\mathcal{F}f)(w)$$

we have that $W_s^2(\mathbb{R}^d) = H^s(\mathbb{R}^d)$ for $s \in \mathbb{N}_0$, so we can generalize Sobolev spaces to general $s \in \mathbb{R}$. An equivalent definition of generalized Sobolev spaces or potential spaces over L^p is therefore given by

$$W_s^p(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) \;:\; ||\mathcal{F}^{-1}[1 + |\cdot|^2]^{s/2} \mathcal{F} f||_p < \infty\}, \tag{111}$$

an equivalence between using the differential operator $(1 - \Delta)^s$ and v_s.

By the results of Proposition 2.26 we immediately have the following equalities between the modulation spaces and these other spaces.

1. If $v_s(z) = v_s(x)$, then
$$M^2_{v_s}(\mathbb{R}^d) = L^2_{v_s}(\mathbb{R}^d).$$
2. If $v_s(z) = v_s(w)$, then
$$M^2_{v_s}(\mathbb{R}^d) = H^s(\mathbb{R}^d) = W^2_s(\mathbb{R}^d)$$

The last equality is established by the equality $M^2_m(\mathbb{R}^d) = \mathcal{F}L^2_m(\mathbb{R}^d)$ from Proposition 2.26. By the definition of the Bessel potential space $W^2_s(\mathbb{R}^d) = \mathcal{F}L^2_{v_s}(\mathbb{R}^d)$, so we have $M^2_{v_s}(\mathbb{R}^d) = W^2_s(\mathbb{R}^d)$, and we already know $W^2_s(\mathbb{R}^d) = H^s(\mathbb{R}^d)$.

Note also that the equality $H^s(\mathbb{R}^d) = W^2_s(\mathbb{R}^d)$ allows one to define Sobolev spaces by the set
$$H^s(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) \ : \ (1-\Delta)^{s/2} f \in L^2(\mathbb{R}^d)\}.$$

The final space we define is the Shubin-Sobolev space, first introduced in the study of pseudodifferential equations [29].

Definition Let $s \in \mathbb{R}$ and $\phi(t)$ be the Gaussian, $\phi(t) = 2^{n/4} e^{-\pi t^2}$. Then the **Shubin-Sobolev space** Q_s is defined by
$$Q_s(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d) \ : \ A^{\phi,\phi}_{v_s} f \in L^2(\mathbb{R}^d)\} = A^{-1}_{v_s} L^2(\mathbb{R}^d)\}$$
with norm $||f||_{Q_{v_s}} := ||A^{\phi,\phi}_{v_s} f||_2$

The Shubin-Sobolev spaces may be identified with modulation spaces, by a result of P. Boggiatto, E. Cordero and K. Gröchenig [5].

Lemma 5.6 *For all $s \in \mathbb{R}$, we have*
$$M^2_{v_s}(\mathbb{R}^d) = Q_s(\mathbb{R}^d)$$
with equivalent norms.

This is significant because it gives us a larger class of weights and shows the naturality of introducing the localization operator.

Proof Proposition 2.27 tells us that V^* is bounded, and therefore we also have the boundedness of V_g as an operator. By the definition of $A^{\phi,\phi}_{v_s}$, we have that
$$||A^{\phi,\phi}_{v_s} f||_2 = || \iint_{\mathbb{R}^{2d}} v_s(z) V_\phi f(z) \pi(z)\phi \, dz||_2 \leq ||\phi||_2 ||v_s V_\phi f||_2 \asymp ||\phi||_2 ||f||_{M^2_{v_s}} \asymp ||f||_{M^2_{v_s}}$$
so we have the inclusion $M^2_{v_s} \subset Q_s$

The other inclusion is a standard duality argument, we have already shown that $(M^2_{v_s})^* = M^2_{v_{-s}}$, so if we can show $(Q_s)^* = Q_{-s}$ we would have
$$Q_s = (Q_{-s})^* \subseteq (M^2_{v_{-s}})^* = M^2_{v_s}$$
completing the proof. This is shown in [29], so we have the result ∎

Note that both the definition of the Shubin-Sobolev space and Lemma 5.6 require only $s \in \mathbb{R}$ and thus gives us hope for a similar generalization of our case.

5.3 Differential Operators

We have an interesting relation relating the Laplacian operator to the radial weights. Consider first $\mathcal{A}_1$, then by applying $1 - \partial_1^2 - \partial_2^2 = 1 - \Delta$ to our operator-valued inner product we have

$$\begin{aligned} &(1-\partial_1^2-\partial_2^2)({}_\bullet\langle f,g\rangle) \\ =&\iint_{\mathbb{R}^{2d}} V_gf(z)\pi(z)+x^2V_gf(z)\pi(z)+w^2V_gf(z)\pi(z)\ dz \\ =&\iint_{\mathbb{R}^{2d}}(1+x^2+w^2)V_gf(z)\pi(z)\ dz \\ =&\langle f,g\rangle_{v_2}. \end{aligned}$$

So the Laplacian gives a(n unbounded) map from $\mathcal{A}_1$ to $\mathcal{A}_{v_2}$, where v_s are the familiar radial weights. We have then found a relation between derivations on the Hilbert C^*-module on the Moyal plane and weighted modulation spaces. Of course the same holds for different weights and derivations. A slightly tedious calculation shows that for even s, we have

$$(1-\partial_1^2-\partial_2^2)^{s/2}(\langle f,g\rangle_{v_t}) = \iint_{\mathbb{R}^{2d}} v_{s+t}(z)V_gf(z)\pi(z)\ dz \qquad (112)$$

thus relating $(1-\Delta)^{s/2}$ and $v_s(z)$. We would then like to know for what s does this equality hold? In the same vein as the definition of the Sobolev spaces, we will define fractional differential operators so that they coincide with the action of the Laplacian operator. For any $s \in \mathbb{R}$, sub-multiplicative v and $f,g \in M^1(\mathbb{R}^d)$ we can define

$$(1-\Delta)^{s/2}({}_\bullet\langle f,g\rangle) = \iint_{\mathbb{R}^{2d}} v_s(s)V_gf(z)\pi(z)\ dz. \qquad (113)$$

Note that $(1-\Delta)^{s/2}$ is not bounded. If $s>0$ the weight-function increases so that the norm increases and can be unbounded. Note that allowing negative s coincides with the inverse of v_s up to a compact perturbation. To increase our class of operators further we can give the definition that for any $s \in \mathbb{R}$ we have

$$\partial_1^s(\langle f,g\rangle_v) = \iint_{\mathbb{R}^{2d}}(ix)^s v(z)V_gf(z)\pi(z)\ dz, \qquad (114)$$

$$\partial_2^s(\langle f,g\rangle_v) = \iint_{\mathbb{R}^{2d}}(iw)^s v(z)V_gf(z)\pi(z)\ dz. \qquad (115)$$

By extending this definition to any polynomial $P(\partial_1,\partial_2)$ we obtain (113). We can also here see that for $s \in \mathbb{Z}$ the operator ∂_j^{-s} is clearly the inverse of ∂_j^s up to some finite-rank operator.

It now makes sense to reconsider the importance of the radial weights v_s and v_s-moderate weight functions m. Assume we have a v_s-moderate function corresponding in the sense of (114) and (115) to a (pseudo)-differential operator $P(\partial_1, \partial_2)$. Then

$$P(\partial_1, \partial_2)(\langle f, g\rangle_{v_t}) = \iint_{\mathbb{R}^{2d}} m(z) v_t(z) V_g f(z) \pi(z)\ dz$$
$$\leq C \iint_{\mathbb{R}^{2d}} v_s(z) v_t(z) V_g f(z) \pi(z)\ dz = (1-\Delta)^{s/2}(\langle f, g\rangle_{v_t}),$$

showing once again that the radial-symmetric weights v_s are the "right" choice of submultiplicative function. The Laplacian and v_s coincides with the natural notion of the "biggest" polynomial weight both as differential operator and weight-function. This is a powerful result already, since any polynomial m of equal or less degree than s, for $s \in \mathbb{N}$, is already v_s-moderate, but if we can find some way of representing any (pseudo)-differential as a weight function we have greatly increased our class of acceptable operators. Since we are mostly working with polynomial weights however, we will not delve too deep into this theory. A theory for exponentially increasing weights is dealt with for instance in [16]. We formalize the definition of the action of the differentiation.

Definition For any polynomial of $P(\partial_1, \partial_2)$ of (possibly fractional) order less than s, it coincides with the the v_s-moderate weight $P(x, w)$ in the following way:

$$P(\partial_1, \partial_2)\langle f, g\rangle_{v_t} := \iint P(x, w) v_t(z) V_g f(z) \pi(z)\ dz = \langle f, g\rangle_{P \cdot v_t} \tag{116}$$

This definition can be extended, in the natural way, to all v_s-moderate functions

This gives us the inverse of these differential operators.

Lemma 5.7 *Given any positive v_s-moderate weight function m of polynomial growth corresponding to a differential P then the inverse m^{-1} is also v_s-moderate and corresponds to the inverse operator P^{-1} up to a finite rank operator.*

Proof By (53) m^{-1} is v_s-moderate. Since m is a polynomial in (x, w) so is m^{-1}, so we have a correspondence to the operator P^{-1}. By a special case of the main theorem of [9] we have that the localization operator with window v_s is a Fredholm operator and that the localization operator with window $1/v_s$ is its parametrix. Thus we conclude that P and P^{-1} are inverses up to some compact (finite rank) operator. ∎

We are with these definitions motivated to make a similar description as Sobolev spaces for our function spaces. Let the generalized non-commutative Sobolev space of rank $s \in \mathbb{R}$ over the Moyal plane be defined by the set

$$\mathcal{W}^1_s(\mathbb{R}^d_\theta) = \{f \in \mathcal{S}'(\mathbb{R}^d) \ : \ (1-\Delta)^{s/2} \bullet \langle f, f\rangle \in \mathcal{A}_1\} \tag{117}$$

which is equivalent to $V_f f \in L^1_{v_s}(\mathbb{R}^d)$, or, by Proposition 2.33 $f \in M^1_{v_s}(\mathbb{R}^d)$. We have therefore

$$\mathcal{W}^1_s(\mathbb{R}^d_\theta) = M^1_{v_s}(\mathbb{R}^d). \tag{118}$$

By v_s-moderateness of all polynomials of degree t for $-s \leq t \leq s$ this includes all temperate distribution with derivatives up to s degrees in $L^1(\mathbb{R}^d)$. We have here an analogue of theorem 5.2, namely that we have a map from $\mathcal{W}^1_{t+s}(\mathbb{R}^d)$ to $\mathcal{W}^1_t(\mathbb{R}^d)$ (corresponding to a map from $\mathcal{W}^1_{v_t v_s}(\mathbb{R}^d)$ to $\mathcal{W}^1_{v_t}(\mathbb{R}^d)$) given by applying $(1-\Delta)^{s/2}$ to the space. This map is not an isomorphism, but we will show that is bounded and compact.

Because of a lack of analogue to Theorem 2.31 pertaining to acceptable window classes, it is difficult to generalize the definition to general M^p-spaces. A possible way is Proposition 3.4 from [11] giving an equivalence between modulation spaces.

Proposition 5.8 *Let $g \in M^1(\mathbb{R}^d)$ and $1 \leq p \leq \infty$ be given. Then $f \in M^p(\mathbb{R}^d)$ if and only if $V_g f \in M^p(\mathbb{R}^{2d})$ with*

$$||V_g f||_{M^p} \asymp ||g||_{M^p} ||f||_{M^p} \tag{119}$$

A recent result by E. Cordero & F. Nicola, Theorem 5.2 of [10] also states that

Theorem 5.9 *Assume $s \geq 0$, the indices $p_i, q_i, p, q \in [1, \infty], i = 1, 2$ satisfy the relations*

$$\min\{\frac{1}{p_1} + \frac{1}{p_2'}, \frac{1}{q_1} + \frac{1}{q_2'}\} \geq \frac{1}{p'} + \frac{1}{q'}$$
$$\text{and } q \leq \min\{p_1', q_1', p_2, q_2\}$$

Let $r \in [1, 2]$, if $m \in M^{p,q}_{1 \otimes v_{-s}}(\mathbb{R}^{2d})$ and $\phi_1, \phi_2 \in M^r_{v_{2s}}(\mathbb{R}^d)$, then the localization operator A^{ϕ_1,ϕ_2}_m is continuous from $M^{p_1,q_1}_{v_s}(\mathbb{R}^d)$ to $M^{p_2,q_2}_{v_s}(\mathbb{R}^d)$ with

$$||A^{\phi_1,\phi_2}_m||_{op} \leq C ||m||_{M^{p,q}_{1 \otimes v_{-s}}} ||\phi_1||_{M^r_{v_{2s}}} ||\phi_2||_{M^r_{v_{2s}}} \tag{120}$$

Note that the requirements on the p_i, q_i's are satisfied for every number $1 \leq p_i, q_i \leq \infty$ in our case, since we can choose $p = q = 1$. The v_s-moderate weights are in this class. We reformulate this theorem as a definition.

Definition The **non-commutative Sobolev space** $\mathcal{W}^{p,q}_s(\mathbb{R}^d_\theta)$, for $p, q > 1$ and $s > 0$ is defined by

$$\begin{aligned} \mathcal{W}^{p,q}_s(\mathbb{R}^d_\theta) = \{f \in \mathcal{S}'(\mathbb{R}^d) \mid (1-\Delta)^{\frac{s}{2}} \langle f, g \rangle_1 h \in M^{p,q}_1(\mathbb{R}^d) \\ \text{for all } h, g \in M^r_{v_{2s}}(\mathbb{R}^d) \text{and } r \in [1, 2]\} \end{aligned} \tag{121}$$

${}_\bullet\langle f, g \rangle h \in M^{p,q}(\mathbb{R}^d)$ by Theorem 5.9 everywhere the localization operator is defined, and the requirement of $(1-\Delta)^{\frac{s}{2}}{}_\bullet\langle f, g \rangle h \in M^{p,q}(\mathbb{R}^d)$ mirrors the usual Sobolev definition, (111). For every t such that $|t| \leq s$, v_t is a v_s-moderate weight. This implies that

$$||(1-\Delta)^{\frac{t}{2}}{}_\bullet\langle f, g \rangle h||_{M^{p,q}_{v_t}} \leq ||(1-\Delta)^{\frac{t}{2}}{}_\bullet\langle f, g \rangle h||_{M^{p,q}_{v_s}} \leq ||(1-\Delta)^{\frac{s}{2}}{}_\bullet\langle f, g \rangle h||_{M^{p,q}_{v_s}},$$

which leads us to the following conclusion.

Theorem 5.10 *For $1 \le p, q \le \infty$ and $s, t \ge 0$, we have the embedding*

$$W^{p,q}_{s+t}(\mathbb{R}^d_\theta) \hookrightarrow W^{p,q}_s(\mathbb{R}^d_\theta).$$

Multiplication by the radial-symmetric weight v_t is a bounded map from $\mathcal{W}^{p,q}_{s+t}(\mathbb{R}^d_\theta)$ to $\mathcal{W}^{p,q}_s(\mathbb{R}^d_\theta)$, as for Sobolev spaces on $\mathbb{R}^d$.

Proof This should conceptually be clear, but we do the calculations anyway. Let $f \in \mathcal{W}^{p,q}_{s+t}(\mathbb{R}^d_\theta)$ Then $||(1-\Delta)^{\frac{s+t}{2}}{}_\bullet\langle f, g\rangle h||_{M^{p,q}}$ is finite, and by v_{s+t}-moderateness of v_s we have

$$\begin{aligned} ||(1-\Delta)^{\frac{s}{2}}{}_\bullet\langle f, g\rangle h||_{M^{p,q}} &= \\ ||{}_\bullet\langle v_s f, g\rangle h||_{M^{p,q}} &\le \\ \text{(Since } v_s \le v_{s+t}) \qquad ||{}_\bullet\langle v_{s+t} f, g\rangle h||_{M^{p,q}} &\le \\ ||(1-\Delta)^{\frac{s+t}{2}}{}_\bullet\langle f, g\rangle h||_{M^{p,q}} &< \infty. \end{aligned}$$

This shows the first statement. Now, we apply $v_t(z)$ to $f \in \mathcal{W}^{p,q}_{s+t}(\mathbb{R}^d_\theta)$ and estimate its norm

$$||(1-\Delta)^{\frac{s}{2}}{}_\bullet\langle v_t f, g\rangle h||_{M^{p,q}} \asymp ||(1-\Delta)^{\frac{s+t}{2}}{}_\bullet\langle f, g\rangle h||_{M^{p,q}} < \infty,$$

showing that the map is bounded ∎

We also have a different inclusion, which follows directly from Theorem 5.9:

Proposition 5.11 *If $p, q \in [1, \infty]$ and $s > 0$, then $M^{p,q}_{v_s}(\mathbb{R}^d) \subseteq \mathcal{W}^{p,q}_s(\mathbb{R}^d_\theta)$*

Proof Since $(1-\Delta)^{\frac{s}{2}}{}_\bullet\langle f, g\rangle h = A^{h,g}_{v_s} f$ we can restate the bounds of Theorem 5.9 as $||(1-\Delta)^{\frac{s}{2}}{}_\bullet\langle f, g\rangle h||_{M^{p,q}} \le C||v_s||_{M^1_{1\otimes v_{-s}}}||h||_{M^r_{v_{2s}}}||g||_{M^r_{v_{2s}}}||f||_{M^{p,q}_{v_s}}$. This is bounded for $f \in M^{p,q}_{v_s}(\mathbb{R}^d)$, so we have the desired inclusion. ∎

As further evidence of the "correctness" of the definition of the non-commutative Sobolev space, we show that the embeddings from Theorem 5.10 are compact. This notion was first shown by Shubin in [29] as the compactness of the inclusion map $i : Q_{s_1} \to Q_{s_2}$ whenever $s_2 < s_1$. Since the Shubin-Sobolev space closely resembles the modulation spaces, even agrees under some circumstances (Lemma 5.6), it comes as little surprise that we can extend this result. Theorem 2.2 of [6] states:

Theorem 5.12 *Let w_1, w_2 be positive, bounded, locally integrable functions on $\mathbb{R}^{2d}$ that are v_s-moderate for some $s > 0$, and $p, q \in [1, \infty]$. Then the embedding*

$$i : M^{p,q}_{w_1}(\mathbb{R}^d) \to M^{p,q}_{w_2}(\mathbb{R}^d) \tag{122}$$

is compact if and only if $w_2/w_1 \in L^\infty_0(\mathbb{R}^{2d})$.

$L_0^\infty(\mathbb{R}^{2d})$ are bounded functions that go to 0 when approaching infinity.
We wish to extend this result to our Sobolev space. As it turns our, this extension is trivial under our assumption. In the same paper, [6], P. Boggiatto and J. Toft define the generalized Shubin-Sobolev space $Q^{p,q}_{(g,m)}(\mathbb{R}^d)$ as all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$||f||_{Q^{p,q}_{(g,m)}} = ||A^g_m f||_{M^{p,q}} < \infty. \tag{123}$$

The only difference between this space and $\mathcal{W}^{p,q}_s(\mathbb{R}^d_\theta)$ is the choice of weight function. The fact that they consider only $A^{g,g}_m = A^g_m$ is of no concern, since our general theory of modulation spaces have already concluded that these will yield equivalent norms. A small refinement of Theorem 3.5 from this paper states

Theorem 5.13 *Assume that $g \in \mathcal{S}(\mathbb{R}^d)\backslash\{0\}$. Then for $s > 0$ and every $p, q \in [1,\infty]$ we have that $M^{p,q}_{v_s}(\mathbb{R}^d) = Q^{p,q}_{(g,v_s)}(\mathbb{R}^d)$. In particular, $Q^{p,q}_{(g,v_s)}(\mathbb{R}^d)$ is independent of g and we can therefore extend the choice of g to the whole space $M^1(\mathbb{R}^d)$.*

Inserting v_s for the more general weights in [6] yields this result.
Now since $\mathcal{W}^{p,q}_s(\mathbb{R}^d_\theta) = Q^{p,q}_{(g,v_s)}(\mathbb{R}^d) = M^{p,q}_{v_s}(\mathbb{R}^d)$, the embedding of Theorem 5.12 is compact from $\mathcal{W}^{p,q}_{s+t}(\mathbb{R}^d_\theta)$ to $\mathcal{W}^{p,q}_s(\mathbb{R}^d_\theta)$.

Theorem 5.14 *Assume $s, t > 0$ and $p, q \in [1,\infty]$, then there is a compact embedding*

$$i : \mathcal{W}^{p,q}_{s+t}(\mathbb{R}^d_\theta) \to \mathcal{W}^{p,q}_s(\mathbb{R}^d_\theta). \tag{124}$$

We now take the time to introduce the theory of abstract differential operators by N. Higson in [19]. In non-commutative geometry abstract Sobolev spaces based on generalized differential operators are of relevance in the theory of spectral triples [20, 31]. Let D be an invertible, positive, selfadjoint operator on a Hilbert space $\mathcal{H}$. Then the D-Sobolev space of order $s \in \mathbb{R}$, denoted $W^s = W^s(D)$ is the Hilbert space completion of $\mathrm{dom}(D^{s/2})$ with respect to the innerproduct given by

$$\langle f, g\rangle_{W^s} = \langle D^{s/2} f, D^{s/2} g\rangle_{\mathcal{H}} \qquad \text{for} \;\; f, g \in \mathrm{dom}(D^{s/2}).$$

The associated $\|.\|_{W^s}$ norm is non-degenerate due to the invertibility of D. The space of D-smooth vectors W^∞ is

$$W^\infty := \cap^\infty_{s\geq 0} W^s = \cap^\infty_{k=0} W^{2k} = \cap^\infty_{k=0} \mathrm{dom}(D^k) \subseteq \mathcal{H}.$$

Lemma 5.15 *1. For $s \geq t$, we have a continuous inclusion of W^s into W^t.*

2. For $s \geq 0$ we have that $W^s = \mathrm{dom}(D^{s/2})$ and that W^∞ is dense in W^s for any $s \in \mathbb{R}$.

3. The space of D-smooth vectors $W^\infty \subset \mathcal{H}$ is a common core for the operators D^z for $z \in \mathbb{C}$ and hence D^z is essentially selfadjoint on $W^\infty \subseteq \mathcal{H}$.

Then we can create an $\mathbb{N}$-filtered algebra, an increasing union of linear subspaces,

$$\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \cdots \subseteq \mathcal{D}$$

such that $\mathcal{D}_p \cdot \mathcal{D}_q \subseteq \mathcal{D}_{p+q}$. We say that for $X \in \mathcal{D}$, order$(X) \leq q$ if $X \in \mathcal{D}_q$.

An $\mathbb{N}$-filtered subalgebra $\mathcal{D}$ of $\mathcal{B}(\mathcal{H})$ is called an algebra of generalized differential operators if $\mathcal{D}$ is closed under the derivation $[D,-]$ and satisfies $[D,\mathcal{D}^k] \subseteq D^{k+1}$ for $k \in \mathbb{N}$.

An example for an operator D is the closure of the quantum harmonic oscillator:

$$H = 1 + x^2 - \tfrac{d^2}{dx^2}$$

with the domain $\mathcal{S}(\mathbb{R}^d)$, Example 2.3 in [31]. It is well-known that H is a essentially selfadjoint and positive operator with compact resolvent. We consider the algebra of polynomial differential operators of order at most n on $\mathbb{R}^d$ action on $L^2(\mathbb{R}^d)$ which is generated by x and $\frac{d}{dx}$. We introduce a filtration on $\mathcal{D}_n$: order$(x) = 1$ and order$\frac{d}{dx} = 1$. Then D_n is an algebra of generalized differential operators and H is an element of $\mathcal{D} = \cup_{n\geq 0}\mathcal{D}_n$.

The main contribution of this manuscript is to relate the Laplacian associated to the complex structure of the Moyal plane, abstract Sobolev spaces with modulation spaces with the help of localization operators. Observe that $H = 1 - \Delta$ and

$$H(A_m^{\phi_1,\phi_2} f) = [(1 - \partial_1^2 - \partial_2^2)\langle f, \phi_1\rangle_m] \cdot f + \langle f, \phi_1\rangle_m (H\phi_2).$$

Let us view $L^2(\mathbb{R}^d)$ as $M^2(\mathbb{R}^d)$, where the $\|.\|_{L^2}$-norm is defined with respect to the short-time Fourier transform with a Gaussian window and innerproduct:

$$\langle f, g\rangle_{M^2} = \iint_{\mathbb{R}^2} V_\phi f(z)\overline{V_\phi g(z)}\, dz.$$

The Sobolev space $W^s(H)$ associated to the quantum harmonic oscillator on $M^2(\mathbb{R}^d)$ is the subspace of $L^2(\mathbb{R}^d)$ defined by

$$\langle f, g\rangle_{W^s} = \iint_{\mathbb{R}^2} (1 + |z|^2)^{s/2} V_\phi f(z)\overline{V_\phi g(z)}\, dz$$

and hence $W^s(H) = Q_s(\mathbb{R}^d)$, the Shubin-Sobolev spaces. Now, since we already have found that

$$Q_s(\mathbb{R}) = M^2_{v_s}(\mathbb{R}) = Q^2_{g,v_s}(\mathbb{R}) = \mathcal{W}^2_s(\mathbb{R}_\theta),$$

the linear subspaces $\mathcal{D}_s$ consists of the weight functions that are v_s-moderate. The central result of N. Higson's theory for our use is the following lemma.

Lemma 5.16 *Let the pair $(\mathcal{D}, H)$ be an algebra of generalized differential operators, and $X \in \mathcal{D}_t$. Then for every $s \geq 0$, X extends to a bounded linear operator from $W^{s+t}(H)$ to $W^s(H)$*

This allows us to conclude that:

Proposition 5.17 *Let $t, s > 0$ and $1 \leq p, q < \infty$. Then every differential operator, M, corresponding to a v_t-moderate weight function m, in the sense of Definition 5.3 can be extended to a bounded linear operator*

$$M : \mathcal{W}^{p,q}_{s+t}(\mathbb{R}^d_\theta) \to \mathcal{W}^{p,q}_{s}(\mathbb{R}^d_\theta) \tag{125}$$

This is then an extension of Theorem 5.10, showing boundedness of all polynomial $P(\partial_1, \partial_2)$ with $\deg(P) \leq t$. Since we already have that these polynomials also have an inverse up to some compact operator, we conclude that they are very well-behaved on the non-commutative Sobolev spaces. In addition, this is important because non-commutative Sobolev spaces are the first explicit example (as far as the author knows) of a non-classical example of N. Higson's theoretical work.

As a final remark before we move on to the non-commutative torus we show a remarkable compatibility of this theory to A. Connes' pseudodifferential calculus which could in the future open for many interesting new avenues of research. We follow the construction given in [1] and adapt it to our case.

5.3.1 A. Connes Pseudodifferential Calculus

In what follows we will not give any thorough introduction of the subject, we simply give the definitions and justify the compatibility for our case of Connes's pseudodifferential calculus, the Moyal plane instead of the noncommutative torus. Let $\{U^w : w \in \mathbb{R}^d\}$ and $\{U^x : x \in \mathbb{R}^d\}$ be two strongly continuous groups of unitary operators such that $U^w V^x = e^{2\pi i w x} V^x U^w$. Suppose $\mathcal{A}$ consists of elements of the form

$$\mathbf{a} = \iint_{\mathbb{R}^{2d}} a(x, w) U^w V^x \; dw \; dx,$$

where a is a bounded function on $\mathbb{R}^{2d}$, with twisted convolution as the algebra operation. With the derivations determined by

$$\begin{aligned} \delta_1(U) &= U, & \delta_1(V) &= 0 \\ \delta_2(U) &= 0, & \delta_2(V) &= 0 \end{aligned}$$

we have an action

$$\alpha_\zeta(U^w V^x) = e^{i\zeta\cdot(w,x)} U^w V^x \tag{126}$$

that is generated by these derivations. (the δ_i's correspond to our normal derivations on the Moyal plane by the relation $\delta_1 = -i\partial_1$ and $\delta_2 = -i\partial_2$.)

Then we can define the smooth subalgebra $\mathcal{A}^\infty$ of $\mathcal{A}$ consisting of all $\mathbf{a} \in \mathcal{A}$ such that the map

$$\zeta \in \mathbb{R}^{2d} \to \alpha_\zeta(\mathbf{a}) \in \mathcal{A} \tag{127}$$

is smooth. We can endow these spaces with inner product (and therefore norms) and tracial states in the same way as for the non-commutative torus and Moyal plane, this is clear since they are constructed in the exact same way.

Differential operators of order d are polynomial expressions of the form:

$$P = \sum_{\substack{I \in \mathbb{N}^2 \\ |I| \leq d}} \mathbf{a}_I \delta^I, \quad \mathbf{a}_I \in \mathcal{A}_\theta^\infty, \quad \delta^I := \delta_1^{i_1} \delta_2^{i_2}.$$

We wish to associate to every such operator P a symbol ρ. Let $\hat{\mathbb{R}^{2d}}$ denote the dual of $\mathbb{R}^{2d}$, using coordinates (f_1, f_2) and corresponding derivatives $(\partial_1, \partial_2) := (\frac{\partial}{\partial f_1}, \frac{\partial}{\partial f_2})$ where we denote $\partial = \partial_1 \partial_2$. A $\mathcal{A}^\infty$-valued function ρ, is called a symbol of order $d \in \mathbb{Z}$ if the following conditions hold:

1. ρ is smooth in the sense of equation (127)
2. For all $I, J \in \mathbb{N}^2$

$$||\delta^I \partial^J \rho(f)|| \leq C_{\rho,I,J} (1 + |f|)^{d - |J|}$$

   for some constant $C_{\rho,I,J}$ depending on ρ, I, J.

3. There exists a function $\rho_d \in C^\infty(\hat{\mathbb{R}^{2d}} \backslash \{0,0\}; \mathcal{A}^\infty)$ such that

$$\lim_{\mu \to \infty} \mu^{-d} \rho(\mu f) = \rho_d(f).$$

We denote the space of all symbols of order d by S^d. Then it is well known that the union $S = \bigcup_{d \in \mathbb{Z}} S^d$ form an algebra. Every symbol ρ can now be associated to an operator P_ρ acting on $\mathcal{A}^\infty$ by the formula

$$P_\rho(\mathbf{a}) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^{2d} \times \hat{\mathbb{R}^{2d}}} e^{-ix \cdot f} \rho(f) \alpha_\zeta(\mathbf{a}) \; d\zeta \; df$$

The $(2\pi)^2$ factor is there for convenience, as we must apply the well known identity that the Fourier transform of the constant function is Dirac's delta, δ, so we need some factor of 2π either in the exponential or outside.
We apply the definition of α_x and calculate:

$$\begin{aligned}
P_\rho(\mathbf{a}) &= \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d} \times \hat{\mathbb{R}^{2d}}} e^{-i\zeta \cdot f} \rho(f) \mathbf{a}(x,w) \alpha_\zeta(U^w V^x) \; d\zeta \; df \; dw \; dx \\
&= \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^{2d}} \iint_{\mathbb{R}^{2d} \times \hat{\mathbb{R}^{2d}}} e^{-i\zeta \cdot f} \rho(f) e^{i\zeta \cdot (x,w)} \mathbf{a}(x,w) U^w V^x \; d\zeta \; df \; dw \; dx \\
&= \iint_{\mathbb{R}^{2d}} \int_{\hat{\mathbb{R}^{2d}}} \rho(f) \left(\int_{\mathbb{R}^{2d}} \frac{1}{(2\pi)^2} e^{-i\zeta \cdot [f - (x,w)]} d\zeta \right) \mathbf{a}(x,w) U^w V^x \; df \; dw \; dx \\
&= \iint_{\mathbb{R}^{2d}} \left(\int_{\hat{\mathbb{R}^{2d}}} \rho(f) \delta(f - (x,w)) \; df \right) \mathbf{a}(x,w) U^w V^x \; dw \; dx \\
&= \iint_{\mathbb{R}^{2d}} \rho(x,w) \mathbf{a}(x,w) U^w V^x \; dx \; dw
\end{aligned}$$

Restricting ourselves to the case where $U^w V^x = \pi(z)$, $\rho(x, w)$ is a weight function and $\mathbf{a}$ can be written in the form $V_g f$ we see that we have regained our inner product. Explicitly, if $\mathbf{a} = V_g f$, we have that $P_\rho(\mathbf{a}) = \langle f, g\rangle_\rho$. By Theorem 2.31 every $\mathbf{a} \in L^{p,q}_m(\mathbb{R}^{2d})$ can be written as $V_g f$ for $g \in M^1_v(\mathbb{R}^d)$ and $f \in M^{p,q}_m(\mathbb{R}^d)$. This leads to the following theorem:

Theorem 5.18 *Assume $s \in \mathbb{N}$, $p, q \in [1, \infty]$ and that ρ is a v_s-moderate weight function. Then the pseudo-differential operator P_ρ applied to the subalgebra of $\mathcal{A}^\infty$ defined by $\mathbf{a} \in L^{p,q}_m(\mathbb{R}^d)$ can be written*

$$P_\rho(\mathbf{a}) = \langle f, g\rangle_\rho \tag{128}$$

This is important in two ways, firstly it connects our work with the wider scheme of the pseudo-differential calculus, and secondly it automatically gives us a possible structure and boundedness of the pseudo-differential operators from the Moyal plane as it acts on functions.

5.4 The Non-commutative Torus Case

We wish to do the same process for the non-commutative torus, in the same vein as in section 5. We try to define the inner product over $M^1_m(\mathbb{R}^d)$ as the discrete version of the localization operator, The Gabor multiplier.

$$\langle f, g\rangle_m = \theta \sum_{k,l\in\mathbb{Z}} m(\theta k, l) V_g f(\theta k, l)\pi(\theta k, l) \tag{129}$$

This will be an element of the algebra of operators on $M^1_m(\mathbb{R}^d)$ defined by

$$\mathcal{A}_{\theta m} = \{ \sum_{k,l\in\mathbb{Z}} m(\theta k, l) a_{kl}\pi(\theta k, l) \ : \ (a_{kl}) \in \ell^1_m(\mathbb{Z}^2)\} \tag{130}$$

We do a similar calculation as the continuous case, and since both the Isomorphism Theorem of [17] and the Fundamental Identity of Gabor Analysis holds, we end up with the same requirements for the permissible weights.
We then define the derivations and trace in the same way also, for $f \in M^1_m(\mathbb{R}^d)$ and $\mathbf{a} \in \mathcal{A}_{\theta k}$ we have

$$\begin{aligned}
\partial_1(\mathbf{a}) &= i \sum_{k,l} k a_{kl}\pi(\theta k, l) \\
\partial_2(\mathbf{a}) &= i \sum_{k,l} l a_{kl}\pi(\theta k, l) \\
\nabla_1(f)(t) &= i\theta^{-1} t f(t) \\
\nabla_2(f)(t) &= f'(t)\tau(\mathbf{a}) = a_{00}
\end{aligned}$$

All these maps satisfy the usual requirements, ∇_j is a lift of ∂_j and $\tau(\partial A) = 0$. The same equality (112) holds also here, at least for even s. We then extend the definition in the same way as for the Moyal plane and end up with the same relationships here. The same differential structure holds here, by the previously mentioned results and boundedness of the Gabor multipliers for some weights and window functions. We could then define some Sobolev spaces, but not all the results leading up to this definition have been found for the discrete case. The author has no doubt that such theorems as 5.9 and 5.8 hold however, so the discrete case is surely not far away.

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