Explicit robustness and fragility margins for linear discrete systems with piecewise affine control law ⋆

N. A. Nguyen [a], S. Olaru [a], P. Rodriguez-Ayerbe [a], G. Bitsoris [b], M. Hovd [c]

[a] *Laboratory of Signals and Systems (L2S, UMR CNRS 8506), CentraleSupélec-CNRS-Université Paris Sud, Gif-sur-Yvette, France.*

[b] *Control Systems Laboratory, Department of Electrical and Computer Engineering, University of Patras, Patras, Greece.*

[c] *Department of Engineering Cybernetics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway.*

Abstract

In this paper, we focus on the robustness and fragility problem for piecewise affine (PWA) control laws for discrete-time linear system dynamics in the presence of parametric uncertainty of the state space model. A generic geometrical approach will be used to obtain robustness/fragility margins with respect to the positive invariance properties. For PWA control laws defined over a bounded region in the state space, it is shown that these margins can be described in terms of polyhedral sets in parameter space. The methodology is further extended to the fragility problem with respect to the partition defining the controller. Finally, several computational aspects are presented regarding the transformation from the theoretical formulations to *explicit* representations (vertex/halfspace representation of polytopes) of these sets.

Key words: PWA control laws, explicit robustness/fragility margins.

1 Introduction

When analyzing a control law, both practitioner and theoretician take into account the capacity to cope with disturbances and model uncertainties. This characteristic is classically denoted in control theory as *robustness*. The presence of additive disturbances in the control system structure is due to measurement noises and external perturbation sources. Otherwise, the uncertainty stems from model reduction, linearization of nonlinear elements, imperfect mathematical model or partial information on the parameters. These elements are unavoidable in the control design by the essence of their causes and the practical need of complexity reduction in model-based design, and as a consequence the robustness consideration of the closed-loop is necessary.

This study concentrates on the robustness problem in the presence of model uncertainty for PWA control laws. It is known that in closed loop this class of controllers leads to a hybrid system formulation Heemels et al. (2001). Another motivation for the study of the PWA controllers and their robustness is the recent interest in the optimization-based design via parametric convex programming Bemporad et al. (2002); Tøndel et al. (2003); Seron et al. (2003); Olaru and Dumur (2004); Nguyen et al. (2013) or the approximate explicit solutions in Model Predictive Control (MPC) Johansen and Grancharova (2003). Various types of uncertainties exist, in this paper, our interest is in parametric uncertainties, understood as variations of coefficients of a model with a pre-imposed structure. Unstructured uncertainty will generally lead to an augmented state space and the extension of a predefined controller leads to nonuniqueness and related well-posedness problems which are beyond the scope of this study.

At the same time, from the practical point of view, the implementation of control laws in general leads to numerical round-offs. This may affect closed-loop stability. The maximal admissible set of numerical errors, for which the implemented control law still guarantees the stability, is denoted as the *fragility margin*. This problem has already been investigated in literature Dorato (1998); Keel and Bhattacharyya (1997), but these studies neither provide a constructive procedure to compute such a margin, nor cover our interests in the class of PWA control laws. As far as it concerns the fragility margin of PWA control laws, we will refer to the possible inaccuracy in the coefficients of the PWA con-

⋆ Corresponding author N. A. Nguyen. Tel. + 33 1 69 85 19 70.
Email addresses: Ngocanh.Nguyen@supelec.fr (N. A. Nguyen), Sorin.Olaru@supelec.fr (S. Olaru), Pedro.Rodriguez-Ayerbe@supelec.fr (P. Rodriguez-Ayerbe), bitsoris@ece.upatras.gr (G. Bitsoris), morten.hovd@itk.ntnu.no (M. Hovd).

trollers without assuming any uncertainty on the state space partition. Perturbations in the region description will lead to overlapping regions in the partition with implications on non-uniqueness of the trajectories. All these aspects are addressed for the first time in the literature to our best knowledge.

Based on the preliminary results in Olaru et al. (2013); Nguyen et al. (2014), this paper provides a theoretical framework and mathematical computation for the explicit robustness/fragility margins of a discrete-time linear system, controlled by a given PWA control law. The methodology is centered around the *robust positive invariance* properties which have been studied since the late '80s Bitsoris (1988); Vassilaki et al. (1988); Blanchini (1999); Blanchini and Miani (2008). Note that the robust positive invariance is associated with robust stability since the trajectories are kept inside a subset of the state space, namely a positively invariant set. Guaranteeing robust asymptotic stability is beyond the scope of this paper. Based on the same constructive principle, the problem of finding the biggest set of errors in the description of the regions of the given state space polyhedral partition is also tackled in this study. The main contribution of this paper is to provide a conceptual advance on the determination of the robustness and fragility margins for a PWA controller and a linear system. Aside from this theoretical aspect, for explicit computations of these margins, computational aspects will also be discussed. These computational aspects rely on vertex/facet enumerations and become expensive once the number of critical regions and dimension increase. However, part of the analysis is independent for each region. Also, all these computations are carried out offline, at the design stage. Therefore, is is reasonable to assume that ample computational power, time and memory are available, making computations of substantial complexity acceptable. This situation is in stark contrast to the online controller computations which typically will be performed under strict real time requirements on low cost computational hardware.

Unlike the robust explicit controllers designs which *a priori* take robustness into account (Kerrigan and Maciejowski (2004); Kouramas et al. (2013); Nguyen et al. (2015)), the method presented here allows one to evaluate *a posteriori* the robustness/fragility margins for a given PWA control law. A link can be made between analysis and control design if the fragility/robustness margin is used for retuning PWA controllers to cope with uncertainties while guaranteeing robust positive invariance. However, the robust asymptotic stability should be further elaborated in this case.

Notation and basic definitions

Throughout the paper, $\mathbb{R}, \mathbb{R}_+, \mathbb{N}$ and $\mathbb{N}_+$ denote the field of real numbers, the set of nonnegative real numbers, the set of non-negative integers, the set of positive integer numbers, respectively. For two column vectors: $x, y \in \mathbb{R}^n$, $x = [x_1\ x_2\ \ldots\ x_n]^T$, $y = [y_1\ y_2\ \ldots\ y_n]^T$, the partial order relation $x \leq y$ is equivalent to $x_i \leq y_i, \forall i = 1, \ldots, n$. A vector with its elements equal to one (zero) is denoted by $\mathbf{1}$ ($\mathbf{0}$) or by $\mathbf{1}_n$ ($\mathbf{0}_n$) in case the dimension n must be explicitly stated. Similarly, $\mathbf{I}$ denotes an identity matrix of appropriate dimension, with a subscript when the dimension of this matrix needs to be specified i.e. $\mathbf{I}_n$ means $\mathbf{I} \in \mathbb{R}^{n\times n}$. For a matrix $A \in \mathbb{R}^{m\times n}$, then $\mathrm{vec}(A)$ represents the vector composed of the columns of matrix A as follows: $\mathrm{vec}(A) := \begin{bmatrix} A(\cdot,1)^T & \ldots & A(\cdot,n)^T \end{bmatrix}^T$, where $A(\cdot, i)$ denotes the i^{th} column of matrix A. Given two matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, their Kronecker tensor product, denoted by $A \otimes B \in \mathbb{R}^{mp\times nq}$, is defined as:

$$A \otimes B := \begin{bmatrix} a_{11}B & \ldots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \ldots & a_{mn}B \end{bmatrix}.$$

For an arbitrary set $\mathcal{S} \subseteq \mathbb{R}^n$, $\mathrm{int}(\mathcal{S})$ denotes the interior of $\mathcal{S}$. By $\dim(\mathcal{S})$, we denote the dimension of its affine hull. $\mathcal{V}(\mathcal{S})$ describes the set of vertices whenever $\mathcal{S}$ is a polytope (bounded polyhedral set). If $\mathcal{S} \subset \mathbb{R}^n$ is composed of a finite number of vectors $\mathcal{S} = \{s_1, s_2, \ldots, s_m\}$, then $[\mathcal{S}]$ denotes a matrix for which the columns are the elements of $\mathcal{S}$ in an arbitrary order: $[\mathcal{S}] = \begin{bmatrix} s_1 & s_2 & \ldots & s_m \end{bmatrix}$. Moreover, by $\mathrm{conv}(\mathcal{S})$, we denote the convex hull of $\mathcal{S}$. Given a map $f : \mathbb{R}^m \to \mathbb{R}^n$ and a set $\mathcal{S} \subset \mathbb{R}^m$, $f(\mathcal{S}) = \{y \in \mathbb{R}^n \mid \exists x \in \mathcal{S} \text{ such that } y = f(x)\}$ denotes the image of the set $\mathcal{S}$ via the mapping f. For a linear map $f(x) = Ax$ with $A \in \mathbb{R}^{n\times m}$, the image of a set $S \subset \mathbb{R}^m$ is briefly rewritten as $f(\mathcal{S}) = A\mathcal{S}$. The Minkowski sum of two sets P_1 and P_2, denoted as $P_1 \oplus P_2$, is defined as follows:

$$P_1 \oplus P_2 := \{y \mid \exists x_1 \in P_1, x_2 \in P_2 \text{ such that } y = x_1 + x_2\}.$$

The unit simplex in $\mathbb{R}^L$ is defined as

$$\mathcal{S}_L = \left\{x \in \mathbb{R}^L_+ \mid \mathbf{1}_L^T x = 1\right\}. \tag{1}$$

Finally, for an $N \in \mathbb{N}_+$, $\mathcal{I}_N$ denotes the set of integers: $\mathcal{I}_N := \{i \in \mathbb{N}_+ \mid i \leq N\}$.

2 Preliminaries

In this section, some basic notions related to the piecewise affine control functions and the discrete dynamics will be introduced to facilitate the problem formulation and the presentation of the main results of the paper.

Definition 2.1 A set of $N \in \mathbb{N}_+$ full-dimensional polyhedra $\mathcal{X}_i \subset \mathbb{R}^n$, i.e. $\mathcal{P}_N(\mathcal{X}) = \{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_N\}$ is called a *polyhedral partition of a polyhedron* $\mathcal{X} \subseteq \mathbb{R}^n$ if:

(1) $\bigcup_{i\in\mathcal{I}_N} \mathcal{X}_i = \mathcal{X}$.
(2) $\mathrm{int}(\mathcal{X}_i) \bigcap \mathrm{int}(\mathcal{X}_j) = \emptyset$ with $i \neq j$, $(i, j) \in \mathcal{I}_N^2$,

Also, $(\mathcal{X}_i, \mathcal{X}_j)$ are called neighbours if $(i, j) \in \mathcal{I}_N^2$, $i \neq j$ and $\dim(\mathcal{X}_i \cap \mathcal{X}_j) = n - 1$. If $\mathcal{X}$ is a polytope, we call $\mathcal{P}_N(\mathcal{X})$ a *polytopic partition*.

Definition 2.2 A function $f_{pwa} : \mathcal{X} \to \mathbb{R}^m$ defined over a polyhedral partition $\mathcal{P}_N(\mathcal{X})$ of the polyhedron $\mathcal{X}$ by the relation $f_{pwa}(x) = A_i x + a_i$ for $x \in \mathcal{X}_i$, $i \in \mathcal{I}_N$, with $A_i \in \mathbb{R}^{m\times n}, a_i \in \mathbb{R}^m$, is said to be a *piecewise affine function* over $\mathcal{P}_N(\mathcal{X})$.

In this paper, we consider discrete linear time–invariant (LTI) systems described by state equations:

$$x_{k+1} = Ax_k + Bu_k, \tag{2}$$

where $x \in \mathbb{R}^n$ represents the state vector, $u \in \mathbb{R}^m$ denotes the control input, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

If the control action is synthesized in terms of a PWA state feedback defined over a polyhedral partition $\mathcal{P}_N(\mathcal{X})$ of a polyhedron $\mathcal{X} \subseteq \mathbb{R}^n$ then it will be described by

$$u(x_k) = f_{pwa}(x_k) = G_i x_k + g_i \text{ for } x_k \in \mathcal{X}_i,\ i \in \mathcal{I}_N, \tag{3}$$

with $G_i \in \mathbb{R}^{m\times n}$ and $g_i \in \mathbb{R}^m$. With this control law, the resulting closed-loop system (2)-(3) is a piecewise affine system described by the state equation:

$$x_{k+1} = (A + BG_i)x_k + Bg_i \text{ for } x_k \in \mathcal{X}_i. \tag{4}$$

Definition 2.3 A set $\mathcal{X} \subset \mathbb{R}^n$ is positively invariant with respect to the system $x_{k+1} = f(x_k)$ if $x \in \mathcal{X}$ implies $f(x) \in \mathcal{X}$.

In the context of robustness analysis for the closed loop PWA dynamics, the introduction of discrete time-varying uncertainty on $[A\ B]$ in the dynamical model (2) is of use. We assume that matrix $[A\ B]$ belongs to a polytopic set Ω:

$$\Omega = \operatorname{conv}\{[A_1\ B_1], \ldots, [A_L\ B_L]\}. \tag{5}$$

Thus, if $[A\ B] \in \Omega$, then there exists nonnegative scalars $\alpha_1, \ldots, \alpha_L, \sum_{i=1}^{L} \alpha_i = 1$ satisfying the relation $[A\ B] = \sum_{i=1}^{L} \alpha_i [A_i\ B_i]$. It is known that a polytope can be described by the convex hull of its vertices, given as vectors in an Euclidean space. Therefore, for the convex hull of matrices, one can exploit the isomorphism between $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{mn}$. With a slight abuse of notation, we call Ω a *parametric uncertainty polytope*, for ease of presentation. Also, a subset of Ω is called *polytope* if its associated set of coefficients $\alpha = [\alpha_1 \ldots \alpha_L]^T$ is a polytope.

The development of the results in this paper is based on a set of hypotheses below.

Assumption: Given a nominal LTI system (2) and a PWA controller $u(x)$ (3), defined over a polyhedral partition $\mathcal{P}_N(\mathcal{X})$ of the set $\mathcal{X} \subset \mathbb{R}^n$, it is assumed that

(1) The set $\mathcal{X}$ is a polytope.
(2) The set $\mathcal{X}$ is positively invariant with respect to the PWA dynamics (4).
(3) The control function $f_{pwa} : \mathcal{X} \to \mathbb{R}^m$ is continuous.
(4) $0 \in \operatorname{int}(\mathcal{X})$

In the most general case, the partition is not convex as for example in case the state/input constraints are not convex. In this context, Assumption 1 implies that we restrict our attention to bounded convex domains and particularly to polytopes. Assumption 2 implies that with the given PWA control law $u(x)$, the trajectories of the nominal linear system (2) are confined in $\mathcal{X}$. According to Assumption 1, the components $\mathcal{X}_i$, $\forall i \in \mathcal{I}_N$ of $\mathcal{P}_N(\mathcal{X})$, can be defined via both the vertex/halfspace representations. The problem of obtaining the vertices of a given polytope from its halfspace representation, is called *vertex enumeration*. Many studies have been dedicated to this problem. A solution in this sense is reported in Avis and Fukuda (1992) with a computation time in $O(ndv)$, where n denotes the number of halfspaces, d denotes the dimension of this polytope, v is its number of vertices. The halfspace representation of the polytopes of interest can be defined as follows for every $i \in \mathcal{I}_N$:

$$\begin{aligned} \mathcal{X} &= \{x : Fx \leq h\}, \ \text{ with } \ F \in \mathbb{R}^{r\times n},\ h \in \mathbb{R}^r \\ \mathcal{X}_i &= \{x : F_i x \leq h_i\}, \ \text{ with } F_i \in \mathbb{R}^{r_i\times n}, h_i \in \mathbb{R}^{r_i}. \end{aligned} \tag{6}$$

The vertex representation of polytopes $\mathcal{X}$ and $\mathcal{X}_i$ with corresponding sets of vertices $\mathcal{V}(\mathcal{X}) = \{v_1, v_2, \ldots, v_q\}$, and $\mathcal{V}(\mathcal{X}_i) = \{w_{i1}, w_{i2}, \ldots, w_{iq_i}\}$ are defined as:

$$\begin{aligned} \mathcal{X} &= \operatorname{conv}\{v_1, v_2, \ldots, v_q\}, \\ \mathcal{X}_i &= \operatorname{conv}\{w_{i1}, w_{i2}, \ldots, w_{iq_i}\}. \end{aligned} \tag{7}$$

For ease of presentation, define the following sets of vertices:

$$\mathcal{W}_i = \mathcal{V}(\mathcal{X}_i),\ \mathcal{W} = \bigcup_{i\in\mathcal{I}_N} \mathcal{V}(\mathcal{X}_i). \tag{8}$$

With respect to an arbitrary order, the following matrices can be defined such that their columns are the elements of their associated sets:

$$\begin{aligned} &V = [\mathcal{V}(\mathcal{X})] \in \mathbb{R}^{n\times q},\ U = [f_{pwa}(\mathcal{W})] \in \mathbb{R}^{m\times p}, \\ &V_i = [\mathcal{W}_i] \in \mathbb{R}^{n\times q_i}, U_i = [f_{pwa}(\mathcal{W}_i)] \in \mathbb{R}^{m\times q_i}, \\ &W = [\mathcal{W}] \in \mathbb{R}^{n\times p}. \end{aligned} \tag{9}$$

3 Explicit robustness margin for PWA control laws

3.1 Problem formulation and structure of the solution

Given a continuous PWA controller (3) and $[A(k)\ B(k)] \in \Omega$ where Ω is defined by (5), the *robustness problem* aims to find the set of coefficients, denoted by $\Omega_{rob}^{\alpha} \subseteq \mathcal{S}_L$, associated with $\Omega_{rob} \subseteq \Omega$ such that the polytope $\mathcal{X}$ is positively invariant with respect to the closed loop system:

$$x_{k+1} = (A(k) + B(k)G_i)x_k + Bg_i \quad \text{for } x \in \mathcal{X}_i, \quad (10)$$

$\forall\,[A(k)\ B(k)] \in \Omega_{rob}$. The set Ω_{rob} can be alternatively called the *robustness margin*.

The set Ω_{rob} can be characterized based on the local structure of the dynamics. The next result shows a strong property that can be obtained despite the global nonlinearity (PWA formulation) of the dynamics.

Theorem 3.1 *The set Ω_{rob} is convex.*

PROOF. Let $[A(k_1)\ B(k_1)]$ and $[A(k_2)\ B(k_2)] \in \Omega_{rob}$. The invariance property of the set $\mathcal{X}$ with respect to (10) implies the following set inclusions:

$$(A(k_1) + B(k_1)G_i)\mathcal{X}_i \oplus B(k_1)g_i \subseteq \mathcal{X}, \forall i \in \mathcal{I}_N,$$
$$(A(k_2) + B(k_2)G_i)\mathcal{X}_i \oplus B(k_2)g_i \subseteq \mathcal{X}, \forall i \in \mathcal{I}_N.$$

Since, by Assumption 1, the set $\mathcal{X}$ is convex, one has:

$$\begin{aligned}&(1-\mu)\left((A(k_2) + B(k_2)G_i)\mathcal{X}_i \oplus B(k_2)g_i\right) \oplus \\ &\mu\left((A(k_1) + B(k_1)G_i)\mathcal{X}_i \oplus B(k_1)g_i\right) \subseteq \mathcal{X},\end{aligned} \quad (11)$$

$\forall i \in \mathcal{I}_N$ and $0 \le \mu \le 1$. Inclusion (11) proves $\mu\,[A(k_1)\ B(k_1)] + (1-\mu)\,[A(k_2)\ B(k_2)] \in \Omega_{rob}$ and consequently the convexity of the set Ω_{rob}. $\square$

As a consequence of the convexity of both Ω_{rob} and Ω, the robustness margin can be expressed by an equivalent set:

$$\begin{aligned}\Omega_{rob}^{\alpha} = \big\{&\alpha \in \mathbb{R}_+^L \mid \forall i \in \mathcal{I}_N,\ \mathbf{1}_L^T\alpha = 1, \\ &\sum_{j=1}^{L} \alpha_j(A_j + B_jG_i)\mathcal{X}_i \oplus \alpha_jB_jg_i \subseteq \mathcal{X}\big\}.\end{aligned}$$

The isomorphic relationship between Ω_{rob} and Ω_{rob}^{α} follows directly from the one-to-one correspondence between the elements of these sets. Consequently, the constructive procedures for the characterization of robustness margins will be expressed in terms of $\Omega_{rob}^{\alpha} \subset \mathbb{R}^L$. If $L < n(m+n)$, this expression is more effective than the one via the elements of $[A\ B]$. However, the paper still handles the latter case.

3.2 Construction based on the vertex representation

With respect to definitions (7)−(9), the first result can be stated as follows:

Theorem 3.2 *Consider the system* (10) *subject to a parametric uncertainty* (5)*. For a given PWA control law* (3) *satisfying Assumptions 1-3, the robustness margin is obtained as the projection*

$$\Omega_{rob}^{\alpha} = \mathrm{Proj}_{\mathbb{R}^L}\mathcal{R} \quad (12)$$

where $\mathcal{R}$ represents the polyhedral set:

$$\begin{aligned}\mathcal{R} = \Big\{&(\alpha, \Gamma) \in \mathcal{S}_L \times \mathbb{R}_+^{q\times p}\ |\mathbf{1}^T\Gamma = \mathbf{1}^T, \\ &\sum_{j=1}^{L} \alpha_j(A_jW + B_jU) = V\Gamma\Big\},\end{aligned} \quad (13)$$

with W, U defined in (9)*, $\mathcal{S}_L$ defined in* (1)*, $p = \mathrm{Card}(\mathcal{W}), q = \mathrm{Card}(\mathcal{V}(\mathcal{X}))$ and Γ represents any matrix with the nonnegative elements, satisfying* (13)*.*

PROOF. If Ω_{rob} describes the robustness margin, then for all $[A\ B] \in \Omega_{rob}$ and $\forall x \in \mathcal{X}_i$, $\forall i \in \mathcal{I}_N$:

$$(A + BG_i)x + Bg_i \in \mathcal{X}. \quad (14)$$

Clearly, (14) can be written by:

$$\sum_{j=1}^{L} \alpha_j(A_j + B_jG_i)x + \alpha_jB_jg_i \in \mathcal{X},\ \forall x \in \mathcal{X}_i \quad (15)$$

with α_j as the elements of a vector $\alpha \in \mathcal{S}_L$. On the other hand, by expressing the state $x \in \mathcal{X}_i$ as a convex combination of the vertices $x = \sum_{l=1}^{q_i}\beta_l w_{il}$ for $\beta_l \in \mathbb{R}_+$ and $\sum_{l=1}^{q_i}\beta_l = 1$, it follows that (15) is equivalent to:

$$\sum_{j=1}^{L} \alpha_j(A_j + B_jG_i)w_{il} + \alpha_jB_jg_i \in \mathcal{X},\ \forall i \in \mathcal{I}_N,\ \forall l \in \mathcal{I}_{q_i}.$$

Further, this inclusion can be explicitly described by the existence of $y_{il} \in \mathcal{X}$ such that:

$$\sum_{j=1}^{L} \alpha_j(A_j + B_jG_i)w_{il} + \alpha_jB_jg_i = y_{il}. \quad (16)$$

y_{il} can be expressed as: $y_{il} = [\mathcal{V}(\mathcal{X})]\gamma_{il}$ for $\gamma_{il} \in \mathcal{S}_q$. By replacing this inclusion in (16) with notation (9), we obtain:

$$\sum_{j=1}^{L} \alpha_j(A_j + B_jG_i)w_{il} + \alpha_jB_jg_i = V\gamma_{il}. \quad (17)$$

Equation (17) holds $\forall i \in \mathcal{I}_N$ and $\forall l \in \mathcal{I}_{q_i}$ which means that it will hold for all the columns of the matrix W as defined in (9). Exploiting the PWA mapping of the columns of W as in (9), equation (17) leads to the matrix formulation of the inclusion: $\sum_{j=1}^{L}\alpha_jA_jW + \alpha_jB_jU = V\Gamma$, wherein each column of Γ is restricted to the simplex $\mathcal{S}_q$, which can be expressed as: $\mathbf{1}^T\Gamma = \mathbf{1}^T$, $\Gamma \in \mathbb{R}_+^{q\times p}$. These elements prove that $\mathcal{R}$ in (12) represents a *parameterized* set of robustness margin over all the model uncertainties guaranteeing the positive invariance of the closed loop. In order to complete the proof, the set $\mathcal{R}$ is projected on the space of the parameters α in (12). $\square$

3.3 Construction based on the halfspace representation

This subsection presents another result related to the robustness margin through the halfspace description of a polytope. The notations of interest are already defined by (6). The main result towards the explicit robustness margin description, is summarized by the next theorem.

Theorem 3.3 *Consider the system* (10) *affected by a parametric uncertainty polytope* (5). *For a given PWA control law* (3) *satisfying Assumptions 1-3, the robustness margin is obtained as the projection*

$$\Omega^{\alpha}_{rob} = \mathrm{Proj}_{\mathbb{R}^L}\mathcal{P} \tag{18}$$

where $\mathcal{P}$ represents the polytope:

$$\begin{aligned}\mathcal{P} = \Big\{&(\alpha, \Gamma_1...\Gamma_N) \in \mathcal{S}_L \times \mathbb{R}_+^{r\times r_1} \times \ldots \times \mathbb{R}_+^{r\times r_N} \mid \\ &\sum_{j=1}^{L} \alpha_j F(A_j + B_j G_i) = \Gamma_i F_i, \\ &\Gamma_i h_i \leq h - F\sum_{j=1}^{L} \alpha_j B_j g_i,\ \forall i \in \mathcal{I}_N\Big\},\end{aligned} \tag{19}$$

where $\Gamma_i, i \in \mathcal{I}_N$ represent suitable matrices with the nonnegative elements, satisfying the above constraints.

PROOF. It is clear that for every $[A\ B] \in \Omega_{rob}$ and $\forall i \in \mathcal{I}_N$: $(A + BG_i)\mathcal{X}_i \oplus Bg_i \subseteq \mathcal{X}$. Note also that $\forall i \in \mathcal{I}_N, \mathcal{X}_i \subseteq \mathcal{X}$, the above inclusion is equivalent to: $\mathcal{X}_i \subseteq \{x \in \mathcal{X} \mid F[(A+BG_i)x + Bg_i] \leq h\}$. In this form, the inclusion has the advantage of an explicit halfspace representation for both terms:

$$\{x \mid F_i x \leq h_i\} \subseteq \{x \in \mathcal{X} \mid F[(A+BG_i)x + Bg_i] \leq h\}.$$

Using the Extended Farkas Lemma Hennet (1995); Schrijver (1998), there exists a matrix Γ_i with nonnegative elements such that:

$$F(A+BG_i) = \Gamma_i F_i,\ \Gamma_i h_i \leq h - FBg_i,\ \forall i \in \mathcal{I}_N. \tag{20}$$

The proof is complete by observing that all the realizations of $[A\ B] \in \Omega_{rob}$ are spanned by convex combinations of the extreme realizations in the polytopic uncertainty set (5):

$$\begin{cases} \sum_{j=1}^{L} \alpha_j F(A_j + B_j G_i) = \Gamma_i F_i \\ \Gamma_i h_i \leq h - F\sum_{i=1}^{L} \alpha_j B_j g_i \end{cases} \forall i \in \mathcal{I}_N. \tag{21}$$

One can observe that (21) defines a polyhedron in the extended space of the elements of α and of the matrices Γ_i, therefore, the set Ω^{α}_{rob} is obtained by the projection onto the space of α as specified by (18). □

3.4 Further properties of the robustness margin

The convexity of the set Ω_{rob} is confirmed by the construction (12) which expresses an isomorphic relation with the set Ω^{α}_{rob}. The following corollary characterizes in a formal manner the structural properties of the robustness margin.

Corollary 3.4 *The robustness margin Ω_{rob} is a polytope.*

PROOF. The sets $\mathcal{S}_L$ and $\mathcal{R}$ used in the construction (12) are polytopes because of their boundedness, as a consequence Ω^{α}_{rob} inherits this structural property. By virtue of the isomorphism, the set Ω_{rob} is also a polytope. □

Theorem 3.2 was stated under Assumptions 1-3 but its formulation can be relaxed if additional properties are considered.

Corollary 3.5 *Under the hypotheses of Theorem 3.2, if in addition Assumption 4 holds, then Ω^{α}_{rob} is obtained as $\Omega^{\alpha}_{rob} = \mathrm{Proj}_{\mathbb{R}^L}\mathcal{R}^*$ with*

$$\begin{aligned}\mathcal{R}^* = \big\{(\alpha, \Gamma) \in &\mathcal{S}_L \times \mathbb{R}_+^{q\times p} \mid \mathbf{1}^T\Gamma \leq \mathbf{1}^T, \\ &\sum_{j=1}^{L} \alpha_j (A_j W + B_j U) = V\Gamma\big\}.\end{aligned} \tag{22}$$

PROOF. Since $0 \in \mathrm{int}(\mathcal{X})$, for $x \in \mathcal{X}_i$, $(A + BG_i)x + Bg_i \in \beta\mathcal{X}$ for some $0 \leq \beta \leq 1$. Following the same line in the proof of Theorem 3.2, there exists a matrix $\widetilde{\Gamma}$ composed of nonnegative elements such that $\sum_{j=1}^{L} \alpha_j (A_j W + B_j U) = V\beta\widetilde{\Gamma}$ and $\mathbf{1}^T\widetilde{\Gamma} = \mathbf{1}^T$. Accordingly, denoting $\Gamma = \beta\widetilde{\Gamma}$, leads to $\mathbf{1}^T\Gamma \leq \mathbf{1}^T$. The proof is complete. □

Note that this corollary may be of help for further development of robustness margin while guaranteeing the asymptotic stability of the origin. More precisely, the contractiveness condition of $\mathcal{X}$ may be required when appropriate constraints are imposed, whereby $\mathbf{1}^T\Gamma \leq \mathbf{1}^T$ is replaced with $\mathbf{1}^T\Gamma \leq \beta\mathbf{1}^T$, with a scalar $0 \leq \beta < 1$.

The continuity can be dropped, as shown in the next result. Accordingly, if Assumption 3 is dropped, we are interested in the class of discontinuous PWA functions defined as follows:

$$f_{pwa}(x) = \begin{cases} G_i x + g_i \text{ for } x \in \mathrm{int}(\mathcal{X}_i), \\ G_i x + g_i \text{ or } G_j x + g_j \text{ for } x \in \mathcal{X}_i \cap \mathcal{X}_j. \end{cases} \tag{23}$$

Corollary 3.6 *Under the hypotheses of Corollary 3.5, if Assumption 3 is dropped and the $f_{pwa}(x)$ is given by* (23), *then Ω^{α}_{rob} is obtained as $\Omega^{\alpha}_{rob} = \mathrm{Proj}_{\mathbb{R}^L}\mathcal{R}^c$ with*

$$\mathcal{R}^c = \big\{(\alpha, \Gamma_1, \ldots, \Gamma_N) \in \mathcal{S}_L \times \mathbb{R}_+^{q\times q_1} \times \cdots \times \mathbb{R}_+^{q\times q_N} \mid \sum_{j=1}^{L} \alpha_j(A_jV_i + B_j(G_iV_i + \mathbf{1}_{q_i}^T \otimes g_i)) = V\Gamma_i, \mathbf{1}^T\Gamma_i \le \mathbf{1}^T, \forall i \in \mathcal{I}_N\big\}.$$

PROOF. The argument follows the same line as the one of Theorem 3.2 with the particularity that the image of the vertices via the forward mapping becomes multi-valued due to the presence of common vertices in the set of generators for neighbor regions, but associated with different control values. This has to be considered consequently in the robustness margin description which contains explicitly the inclusion of the image of each region in the set $\mathcal{X}$. □

4 Explicit fragility margin for PWA control laws

This section aims to provide a measure of the set of admissible variations in the PWA control law coefficients, also denoted as the *fragility margin* such that the positive invariance of $\mathcal{X}$ is guaranteed.

4.1 Problem formulation

Given the nominal system (2) and a continuous PWA control law (3) such that the set $\mathcal{X}$ is positively invariant Benlaoukli et al. (2009); Bitsoris (1988); Blanchini (1999); Hennet (1995); Tahir and Jaimoukha (2012); Athanasopoulos et al. (2014), a *fragility margin* problem aims to characterize the set of admissible parametric variations on the local control gains such that the positive invariance property is preserved. Indeed, due to the characteristic of PWA controllers, the fragility margins of the given PWA controller for each region are independent. Thus, we can consider separately this problem for each region.

Starting from the description of the nominal closed-loop PWA system: $x_{k+1} = (A + BG_i)x_k + Bg_i$, for $x_k \in \mathcal{X}_i$ guaranteeing the positive invariance of $\mathcal{X}$, one considers a set of parametric errors of the PWA control law gains for each region $\mathcal{X}_i \subseteq \mathcal{X}$, denoted as $\Delta_i^G \subset \mathbb{R}^{mn+m}$ such that:

$$x_{k+1} = (A + B(G_i + \delta_{G_i,k}))x_k + B(g_i + \delta_{g_i,k}) \in \mathcal{X} \quad (24)$$

with i such that $x_k \in \mathcal{X}_i$ and $\begin{bmatrix}\mathrm{vec}^T(\delta_{G_i,k}) & \delta_{g_i,k}^T\end{bmatrix}^T \in \Delta_i^G$.

The approach will be similar to the one adopted for the robustness margin. Thus in the preamble, the following theorem can be stated.

Theorem 4.1 *The sets $\Delta_i^G, \forall i \in \mathcal{I}_N$ are convex.*

PROOF. See the proof of Theorem 3.1. □

4.2 Construction based on the vertex representation

The fragility problem can be treated in the same positive invariance framework. The matrix notations in (7)−(9) will be used next.

Theorem 4.2 *Consider a discrete LTI system* (2) *and a PWA state feedback* (3) *over a polytopic partition $\mathcal{P}_N(\mathcal{X})$ of the set $\mathcal{X}$ such that Assumptions 1-3 are fulfilled. The fragility margin of the controller defined over $\mathcal{X}_i$ is obtained as*

$$\Delta_i^G = \mathrm{Proj}_{(\delta_{G_i}, \delta_{g_i})}\, \mathcal{F}_i, \quad (25)$$

where $\mathcal{F}_i$ represents the polyhedron:

$$\mathcal{F}_i = \Bigg\{(\delta_{G_i}, \delta_{g_i}, \Gamma_i) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \mathbb{R}_+^{q\times q_i} \mid \mathbf{1}^T\Gamma_i = \mathbf{1}^T, \begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}V_i\\ U_i\end{bmatrix} + B\delta_{G_i}V_i + B\delta_{g_i}\mathbf{1}^T = V\Gamma_i\Bigg\}. \quad (26)$$

PROOF. By the positive invariance of $\mathcal{X}$,

$$Ax + B((G_i + \delta_{G_i})x + (g_i + \delta_{g_i})) \in \mathcal{X}, \forall x \in \mathcal{X}_i.$$

By a simple transformation, one can obtain

$$\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}x\\ f_{pwa}(x)\end{bmatrix} + B\delta_{G_i}x + B\delta_{g_i} \in \mathcal{X}.$$

From the boundedness and convexity of $\mathcal{X}_i$, it follows that $\forall w_{il} \in \mathcal{W}_i$:

$$\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}w_{il}\\ f_{pwa}(w_{il})\end{bmatrix} + B\delta_{G_i}w_{il} + B\delta_{g_i} = y_{il}. \quad (27)$$

$y_{il} \in \mathcal{X}$ has another description via the generators of $\mathcal{X}$

$$y_{il} = V\gamma_{il} \text{ for } \gamma_{il} \in \mathbb{R}_+^q, \text{ satisfying } \mathbf{1}^T\gamma_{il} = 1. \quad (28)$$

(27), (28) lead directly to the following

$$\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}w_{il}\\ f_{pwa}(w_{il})\end{bmatrix} + B\delta_{G_i}w_{il} + B\delta_{g_i} = V\gamma_{il}. \quad (29)$$

Equation (29) holds $\forall w_{il} \in \mathcal{W}_i$, thus by completing the matrix $V_i = [\mathcal{W}_i]$ which has its columns as the vertices of $\mathcal{X}_i$, and U_i being their image via the map f_{pwa}, one can easily see that $\begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}V_i\\ U_i\end{bmatrix} + B\delta_{G_i}V_i + B\delta_{g_i}\mathbf{1}^T = V\Gamma_i$, where $\mathbf{1}^T\Gamma_i = \mathbf{1}^T$ and $\Gamma_i \in \mathbb{R}_+^{q\times q_i}$, $q_i = \mathrm{Card}(\mathcal{W}_i)$. □

Remark 4.3 The fragility study cannot be extended concomitantly to uncertainties in the state space partition and the associated feedback gains without loosing the linear formulations in (26) and (31). Indeed, to study the impact of the uncertainties in the partition, the matrices F_i and h_i in (6) need to be perturbed and consequently, equations (26), (31) become bilinear in the unknowns. The fragility margin with respect to the state space partition will be studied independently in Section 5.

It can be observed that the sets $\Delta_i^G, \forall i \in \mathcal{I}_N$ in (25), are polyhedra. This property is related to the linearity of the constraints in the set description and can be officially stated as follows:

Corollary 4.4 *The set Δ_i^G in* (25) *is a polyhedron $\forall i \in \mathcal{I}_N$.*

PROOF. See the proof of Corollary 3.4. $\square$

Corollary 4.5 *Under the hypotheses of Theorem 4.2, if Assumption 4 holds, then the fragility margin of the controller associated with the region $\mathcal{X}_i, i \in \mathcal{I}_N$, can be obtained as $\Delta_i^{G*} = \mathrm{Proj}_{(\delta_{G_i},\delta_{g_i})}\mathcal{F}_i^*$ whose definition is below*

$$\mathcal{F}_i^* = \big\{(\delta_{G_i}, \delta_{g_i}, \Gamma_i) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \mathbb{R}_+^{q\times q_i} \mid \mathbf{1}^T\Gamma_i \leq \mathbf{1}^T, \\ \begin{bmatrix} A & B \end{bmatrix}\begin{bmatrix} V_i \\ U_i \end{bmatrix} + B\delta_{G_i}V_i + B\delta_{g_i}\mathbf{1}^T = V\Gamma_i\big\}.$$

PROOF. See the proof of Corollary 3.5. $\square$

Remark 4.6 Corollary 4.5 describes a relaxation in the formulation of the set Γ_i. Analyzing exclusively the constraints, it naturally leads to a larger set Δ_i^{G*} as the result of Corollary 4.5 relative to Δ_i^G in Theorem 4.2. Note however that under Assumptions 1–4 these sets are equivalent. Also, the fragility margin obtained by the above results can be used in the context of explicit MPC design under finite precision arithmetic discussed in Suardi et al. (2014).

4.3 Construction based on the halfspace representation

Using the halfspace representation of the polytopes in the partition, the following result can be stated:

Theorem 4.7 *Consider a discrete LTI system* (2) *and a PWA control law* (3) *satisfying Assumptions 1-3. For each region $\mathcal{X}_i$ of the partition $\mathcal{P}_N(\mathcal{X})$ in the controller definition, the fragility margin is defined by the set:*

$$\Delta_i^G = \mathrm{Proj}_{(\delta_{G_i},\delta_{g_i})}\mathcal{Q}_i \tag{30}$$

where $\mathcal{Q}_i$ represents the polyhedron:

$$\mathcal{Q}_i = \big\{(\delta_{G_i},\delta_{g_i},\Gamma_i) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \mathbb{R}_+^{r\times r_i} \mid \\ F(A + B(G_i + \delta_{G_i})) = \Gamma_i F_i, \\ \Gamma_i h_i \leq h - FB(g_i + \delta_{g_i})\big\}. \tag{31}$$

PROOF. For $i \in \mathcal{I}_N$ and $\forall x \in \mathcal{X}_i$

$$(A + B(G_i + \delta_{G_i}))x + B(g_i + \delta_{g_i}) \in \mathcal{X}.$$

From the halfspace representation of the polytope $\mathcal{X}$, it follows that $\forall x \in \mathcal{X}_i$

$$F((A + B(G_i + \delta_{G_i}))x + B(g_i + \delta_{g_i})) \leq h.$$

In other words, $\mathcal{X}_i = \{x \in \mathbb{R}^n \mid F_i x \leq h_i\} \subseteq \mathcal{H}_i = \big\{x \in \mathbb{R}^n \mid F(A + B(G_i + \delta_{G_i}))x \leq h - FB(g_i + \delta_{g_i})\big\}$. The Extended Farkas Lemma leads directly to the result

$$F(A + B(G_i + \delta_{G_i})) = \Gamma_i F_i,\ \Gamma_i h_i \leq h - FB(g_i + \delta_{g_i}).$$

This inclusion completes our proof. $\square$

5 Explicit fragility of state space partition

In this section, a so-called *explicit fragility of the state space partition* problem stemming from the implementation of a piecewice affine controller, is tackled. It aims to compute the set of tolerable errors for the description of the regions in the polytopic partition $\mathcal{P}_N(\mathcal{X})$ of the set $\mathcal{X}$ provided the positive invariance property of $\mathcal{X}$ is preserved. Note that if the halfspace representation is considered, the linearity of imposed constraints will be lost. Instead, we compute this margin via the vertex representation, whereby the errors on the halfspace description are implicitly deduced.

Consider an LTI dynamic (2) and a continuous PWA control law (3), this state feedback controller is defined over a polytopic partition $\mathcal{P}_N(\mathcal{X})$ of the state space $\mathcal{X}$. Consider the vertex representation of $\mathcal{X}_i$ as in (7), the description of $\mathcal{X}_i$ in the presence of coefficient errors can be presented as follows $\widetilde{\mathcal{X}}_i := \mathrm{conv}\{w_{i1} + \delta_{i1}, \ldots, w_{iq_i} + \delta_{iq_i}\}$. A solution to the explicit fragility margin of the components in the polytopic partition $\mathcal{P}_N(\mathcal{X})$ will be provided next in terms of the admissible errors $\delta_{il}, l \in \mathcal{I}_{q_i}$ for each region $\mathcal{X}_i$. The polytope $\mathcal{X}$ is under the following assumption:
Assumption
(5) The boundary of the polytope $\mathcal{X} = \bigcup_{i\in\mathcal{I}_N}\mathcal{X}_i$ is not subject to uncertainty which is equivalent to $\mathcal{X} = \bigcup_{i\in\mathcal{I}_N}\widetilde{\mathcal{X}}_i$.

This assumption ensures that the positive invariance can be stated and analyzed in terms of an explicit inclusion:

$$(A + BG_i)x + Bg_i \in \mathcal{X}, \quad \forall x \in \widetilde{\mathcal{X}}_i \subseteq \mathcal{X}, \tag{32}$$

with the right hand side represented by a set $\mathcal{X}$ free of uncertainties. The set of admissible errors $\delta_i = \left[\delta_{i1}^T \dots \delta_{iq_i}^T\right]^T \in \mathbb{R}^{nq_i}$ of the vertices of $\mathcal{X}_i$ can be computed through the following result with respect to the notations in (8), (9):

Theorem 5.1 *Consider a polytopic partition $\mathcal{P}_N(\mathcal{X})$ of $\mathcal{X}$ over which a PWA controller* (3) *is defined. The controller, designed with respect to a nominal LTI dynamic* (2)*, satisfies Assumptions 1-3 and 5. The fragility margin of the vertex representation of the polytopic partition $\mathcal{P}_N(\mathcal{X})$ is described for each region $\mathcal{X}_i$ as follows:*

$$\Delta_i^v = \Big\{\delta_i \in \mathbb{R}^{nq_i} \mid (\mathbf{I} \otimes F)\, \delta_i \leq \mathbf{1} \otimes h - (\mathbf{I} \otimes F)\mathrm{vec}(V_i),$$
$$(\mathbf{I} \otimes F(A + BG_i))\, \delta_i \leq \mathbf{1} \otimes h - (\mathbf{I} \otimes F\, [A\ B])\mathrm{vec}(\widehat{V}_i)\Big\},$$

where $\mathbf{1} \in \mathbb{R}^{q_i}$ and $\mathbf{I} \in \mathbb{R}^{q_i \times q_i}$, $\widehat{V}_i = \left[V_i^T\ U_i^T\right]^T$.

PROOF. From Assumption 5 we have that $\forall x \in \mathcal{X}$, there exists $\gamma_i \in \mathcal{S}_{q_i}$ such that $\forall x \in \widetilde{\mathcal{X}}_i \subseteq \mathcal{X}$ and subsequently: $x = \sum_{l=1}^{q_i} \gamma_{il}(w_{il} + \delta_{il})$. Then we can easily see due to the halfspace representation of $\mathcal{X}$ that: $F(w_{il} + \delta_{il}) \leq h$, $\forall l \in \mathcal{I}_{q_i}$. It follows that:

$$(\mathbf{I} \otimes F)\delta_i \leq \mathbf{1} \otimes h - (\mathbf{I} \otimes F)\mathrm{vec}(V_i). \tag{33}$$

In addition, (32) holds true only if it holds also $\forall w_{il} \in \mathcal{V}(\mathcal{X}_i)$. More clearly,

$$(A + BG_i)(w_{il} + \delta_{il}) + Bg_i \in \mathcal{X},\ \forall l \in \mathcal{I}_{q_i}. \tag{34}$$

From the halfspace representation of $\mathcal{X}$, (34) amounts to:

$$F(A + BG_i)\delta_{il} \leq h - F\,[A\ B] \begin{bmatrix} w_{il} \\ f_{pwa}(w_{il}) \end{bmatrix}, \forall l \in \mathcal{I}_{q_i}.$$

The above inclusion leads directly to the following:

$$(\mathbf{I} \otimes F(A + BG_i))\delta_i \leq \mathbf{1} \otimes h - (\mathbf{I} \otimes F\,[A\ B])\mathrm{vec}\left(\begin{bmatrix} V_i \\ U_i \end{bmatrix}\right).$$

Finally, Δ_i^v is found by the concomitant satisfaction of (33) and the above inclusion. $\square$

From the above result, the following set:

$$\widehat{\mathcal{X}}_i = \mathrm{conv}\left\{\bigcup_{l \in \mathcal{I}_{q_i}} w_{il} \oplus \mathrm{Proj}_{\delta_{il}} \Delta_i^v\right\}, \tag{35}$$

represents the maximal erroneous halfspace representation of $\mathcal{X}_i$. More clearly, if $\widetilde{\mathcal{X}}_i$ stands for the implemented halfspace representation of $\mathcal{X}_i$, then any implemented $\widetilde{\mathcal{X}}_i \subseteq \widehat{\mathcal{X}}_i$ can guarantee the positive invariance of $\mathcal{X}$ with respect to the given PWA control law.

6 Computational aspects

The above formulations for computation of the robustness and fragility margins are not in the canonical representations (vertex/halfspace representations). Therefore, to explicitly compute these margins, transformations from these matrix equalities/inequalites into canonical representations will be discussed in this section.

6.1 Explicit robustness margin of PWA controller

6.1.1 The vertex representation

Let us consider (13) element by element for $l \in \mathcal{I}_p$:

$$\Omega_l^\alpha = \big\{\alpha \in \mathcal{S}_L \mid \mathbf{1}^T\Gamma(\cdot, l) = 1, \Gamma(\cdot, l) \in \mathbb{R}_+^q, \\ \sum_{j=1}^{L} \alpha_j(A_j W(\cdot, l) + B_j f_{pwa}(W(\cdot, l))) = V\Gamma(\cdot, l)\big\}. \tag{36}$$

Then the robustness margin can also be defined: $\Omega_{rob}^\alpha = \bigcap_{l \in \mathcal{I}_p} \Omega_l^\alpha$. Recall that V is the matrix having the columns composed of the vertices of $\mathcal{X}$. If $\widehat{w}_l = \left[W^T(\cdot, l)\ f_{pwa}^T(W(\cdot, l))\right]^T$, then (36) can be rewritten in the form of a matrix equation where the variable is $\beta_l = [\alpha_1 \dots \alpha_L\ \ \Gamma^T(\cdot, l)]^T \geq 0$

$$\begin{bmatrix} \begin{bmatrix} A_1 & B_1 \end{bmatrix} \widehat{w}_l \dots \begin{bmatrix} A_L & B_L \end{bmatrix} \widehat{w}_l & -V \\ \mathbf{0}_L^T & \mathbf{1}_q^T \\ \mathbf{1}_L^T & \mathbf{0}_q^T \end{bmatrix} \beta_l = \begin{bmatrix} \mathbf{0}_n \\ 1 \\ 1 \end{bmatrix}. \tag{37}$$

This system of equations in the form $\mathcal{A}\beta_l = \mathcal{B}$, has a family of solutions: $\beta_l = \mathcal{A}_s t + \mathcal{B}_s$, where $\mathcal{A}_s$ is an orthonormal basis for the null space of $\mathcal{A}$ (satisfying $\mathcal{A}\mathcal{A}_s = 0$), $\mathcal{B}_s$ denotes a feasible solution of equation (37) and t stands for a vector of appropriate dimension. Due to the nonnegativity of all elements in β_l, we obtain the admissible set of t, denoted by Φ_t i.e. $\Phi_t = \{t \mid -\mathcal{A}_s t \leq \mathcal{B}_s\}$. It is observed that $\Phi_{\beta_l} := \{\beta_l \mid (37) \text{ holds}\} = \mathcal{A}_s \Phi_t \oplus \mathcal{B}_s$ represents a polytope. Therefore, due to the above relation, Φ_t also represents a polytope. So one needs to calculate all vertices of Φ_{β_l} by applying the transformation to the vertices of Φ_t. Finally, the set Ω_l^α of coefficients α for which (36) holds is obtained via the orthogonal projection of Φ_{β_l} on the space of α: $\Omega_l^\alpha = \mathrm{Proj}_{\mathbb{R}^L} \Phi_{\beta_l}$.

6.1.2 The halfspace representation

From equation (18), it follows that $\Omega_{rob}^{\alpha} = \bigcap_{i\in\mathcal{I}_N} \mathrm{Proj}_{\mathbb{R}^L}\mathcal{P}_i$, where $\mathcal{P}_i \subset \mathbb{R}_+^L \times \mathbb{R}_+^{r\times r_i}$ are derived from the definition of $\mathcal{P}$ in (19) for each $i \in \mathcal{I}_N$:

$$\begin{aligned}\mathcal{P}_i = \Bigg\{(\alpha,\Gamma_i)\Big|\sum_{j=1}^{L}\alpha_j F(A_j + B_j G_i) = \Gamma_i F_i,\\ \Gamma_i h_i \le h - F\sum_{j=1}^{L}\alpha_j B_j g_i\Bigg\}.\end{aligned} \tag{38}$$

To facilitate the computation, one needs to transform the above conditions into a polyhedral form with the meaningful variables for each region. Indeed, the equation in (38) needs to be decoupled row by row $\forall k \in \mathcal{I}_r$:

$$\Gamma_i(k,\cdot)F_i = [\alpha_1 ... \alpha_{L-1}]Z_k + F(k,\cdot)(A_L + B_L G_i)$$
$$Z_k = \begin{bmatrix} F(k,\cdot)(A_1 - A_L + B_1 G_i - B_L G_i) \\ \dots \\ F(k,\cdot)(A_{L-1} - A_L + B_{L-1}G_i - B_L G_i)\end{bmatrix}.$$

Denote the following vector: $z = \left[\mathrm{vec}^T(\Gamma_i^T)\,\alpha_1 \dots \alpha_{L-1}\right]^T$, then the following can be obtained:

$$D_1 z = E_1,\ D_1 = \begin{bmatrix} F_i & \dots & \mathbf{0}_{r_i\times n} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{r_i\times n} & \dots & F_i \\ -Z_1 & \dots & -Z_r \end{bmatrix}^T, \tag{39}$$
$$E_1 = (\mathbf{I}_r \otimes (A_L + B_L G_i)^T)\mathrm{vec}(F^T).$$

In the same way, an equivalent representation of the inequality in (38) can be presented below:

$$\begin{aligned}D_2 z \le E_2,\\ E_2 = h - F B_L g_i,\end{aligned}\ D_2 = \begin{bmatrix} h_i & \dots & \mathbf{0}_{r_i} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{r_i} & \dots & h_i \\ Y_1 & \dots & Y_r \end{bmatrix}^T, \tag{40}$$

with $Y_k = [F(k,\cdot)(B_1 - B_L)g_i \dots F(k,\cdot)(B_{L-1} - B_L)g_i]^T$ $\forall k \in \mathcal{I}_r$. The solution of (39) is a set of z which depends on t s.t. $z = \mathcal{D}_1 t + \mathcal{E}_1$, where $\mathcal{D}_1$ is an orthonormal basis for the null space of D_1 and $\mathcal{E}_1$ is a feasible solution of (39). Due to the nonnegativity of z, the values of t satisfy $-\mathcal{D}_1 t \le \mathcal{E}_1$. Also, due to (40), the set of t denoted by Φ_t, can be described by: $\Phi_t = \{t \mid -\mathcal{D}_1 t \le \mathcal{E}_1,\ D_2\mathcal{D}_1 t \le E_2 - D_2\mathcal{E}_1\}$. Recall that the set of z denoted by Φ_z, can be described via Φ_t as: $\Phi_z = \mathcal{D}_1\Phi_t \oplus \mathcal{E}_1$. Then $\mathrm{Proj}_{\mathbb{R}^L}\mathcal{P}_i$ can be computed from $\mathrm{Proj}_{\mathbb{R}^{L-1}}\Phi_z$.

6.2 Explicit fragility margin of PWA controller

For simplicity, without loss of generality, variations in G_i are exclusively considered.

6.2.1 The vertex representation

Consider the fragility margin for the controller of the region $\mathcal{X}_i$. Define also the following set for $l \in \mathcal{I}_{q_i}$

$$\begin{aligned}\Delta_{il}^G = \big\{\delta_{G_i} \in \mathbb{R}^{m\times n} \mid \mathbf{1}^T\Gamma_i(\cdot,l) = 1, \Gamma_i(\cdot,l) \in \mathbb{R}_+^q\\ \begin{bmatrix} A & B\end{bmatrix}\begin{bmatrix} V_i(\cdot,l) \\ U_i(\cdot,l)\end{bmatrix} + B\delta_{G_i}V_i(\cdot,l) = V\Gamma_i(\cdot,l)\big\}.\end{aligned} \tag{41}$$

The fragility margin can also be defined as follows: $\Delta_i^G = \bigcap_{l\in\mathcal{I}_{q_i}} \Delta_{il}^G$. If we denote $\widehat{w}_{il} = \left[V_i^T(\cdot,l)\,U_i^T(\cdot,l)\right]^T$, then (41) can be rewritten as a system of linear equations where the variable is $\beta_{il} = \left[\mathrm{vec}^T(\delta_{G_i})\ \Gamma_i^T(1:q-1,l)\right]^T \in \mathbb{R}^{nm+q-1}$ ($\Gamma_i(q,l) = 1 - \mathbf{1}_{q-1}^T\Gamma_i(1:q-1,l)$):

$$\begin{bmatrix} V_i^T(\cdot,l)(\mathbf{I}_n \otimes B(1,\cdot)) & \\ \vdots & -\widetilde{V} \\ V_i^T(\cdot,l)(\mathbf{I}_n \otimes B(n,\cdot)) & \end{bmatrix}\beta_{il} = v_q - \begin{bmatrix} A & B\end{bmatrix}\widehat{w}_{il}, \tag{42}$$

with $\widetilde{V} = [v_1 - v_q \dots v_{q-1} - v_q]$ (recall that $V = [\mathcal{V}(\mathcal{X})] = [v_1 \dots v_q]$.) Equation (42) in the form $\mathcal{A}\beta_{il} = \mathcal{B}$, has a family of solutions: $\beta_{il} = \mathcal{A}_s t + \mathcal{B}_s$, where $\mathcal{A}_s$ is an orthonormal basis for the null space of $\mathcal{A}$ and $\mathcal{B}_s$ denotes a feasible solution of (42). Due to the nonnegativity $\beta_{il}(nm+1 : nm+q-1) = \Gamma_i(1 : q-1, l) \ge 0$, the values of t satisfy: $-\mathcal{A}_s^{(2)}t \le \mathcal{B}_s^{(2)}$ where the matrices $\mathcal{A}_s^{(1)}, \mathcal{B}_s^{(1)}, \mathcal{A}_s^{(2)}, \mathcal{B}_s^{(2)}$ are defined below:

$$\begin{bmatrix}\mathcal{A}_s^{(1)} & \mathcal{B}_s^{(1)}\end{bmatrix} = \begin{bmatrix}\mathcal{A}_s & \mathcal{B}_s\end{bmatrix}(1:nm,\cdot),$$
$$\begin{bmatrix}\mathcal{A}_s^{(2)} & \mathcal{B}_s^{(2)}\end{bmatrix} = \begin{bmatrix}\mathcal{A}_s & \mathcal{B}_s\end{bmatrix}(nm+1:nm+q-1,\cdot).$$

Also, $\Gamma_i(1 : q-1, l)$ satisfies the constraint: $\mathbf{1}^T\Gamma_i(1 : q-1, l) \le 1$. Thus, the set of t denoted by Φ_t can be represented as: $\Phi_t = \left\{t \mid -\mathcal{A}_s^{(2)}t \le \mathcal{B}_s^{(2)}, \mathbf{1}^T\mathcal{A}_s^{(2)}t \le 1 - \mathbf{1}^T\mathcal{B}_s^{(2)}\right\}$, with the remark that $\Phi_{\beta_{il}} = \{\beta_{il} \mid (42)\text{ holds}\} = \mathcal{A}_s\Phi_t \oplus \mathcal{B}_s$ represents a polyhedral set. Therefore, due to the boundedness of $\Gamma_i(1 : q-1, l)$, Φ_t is a polytope, meaning so is $\Delta_{il}^G = \mathcal{A}_s^{(1)}\Phi_t \oplus \mathcal{B}_s^{(1)}$. Repeat the same computation for all $l \in \mathcal{I}_{q_i}$, then the fragility margin Δ_i^G for G_i is obtained.

6.2.2 The halfspace representation

From equation (31), it follows that for each $i \in \mathcal{I}_N$ the fragility margin can be described in terms of a set:

$$\begin{aligned}\mathcal{Q}_i = \big\{(\delta_{G_i}, \Gamma_i) \in \mathbb{R}^{m\times n} \times \mathbb{R}_+^{r\times r_i} \mid \\ \Gamma_i h_i \le h - FBg_i, F(A + B(G_i + \delta_{G_i})) = \Gamma_i F_i\big\}.\end{aligned} \tag{43}$$

In order to facilitate the computation, one has to transform the above conditions into a polytope formulation with a reduced set of meaningful variables for each region. Define z as: $z_1 = \mathrm{vec}(\Gamma_i^T), z_2 = \mathrm{vec}(\delta_{G_i}), z = \begin{bmatrix} z_1^T & z_2^T \end{bmatrix}^T$. The equality in (43) allows the iterative elimination (step by step for each row) of dependent variables:

$$\Gamma_i(k,\cdot)F_i = F(k,\cdot)B\delta_{G_i} + F(k,\cdot)(A + BG_i).$$

and leads to the following set of relationships:

$$\begin{aligned} D_1 z = E_1, \quad D_1 = \begin{bmatrix} F_i & \dots & \mathbf{0}_{r_i\times n} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{r_i\times n} & \dots & F_i \\ Z_1 & \dots & Z_r \end{bmatrix}^T, \\ E_1 = (\mathbf{I}_r \otimes (A + BG_i)^T)\mathrm{vec}\left(F^T\right), \\ Z_k = \mathbf{I}_n \otimes (-B^T F^T(k,\cdot)),\ \forall k \in \mathcal{I}_r. \end{aligned} \tag{44}$$

Similarly, the inequality in (43) is equivalent to:

$$\begin{aligned} D_2 z_1 \le E_2, \\ E_2 = h - FBg_i, \end{aligned} \quad D_2 = \begin{bmatrix} h_i & \dots & \mathbf{0}_{r_i} \\ \vdots & \ddots & \vdots \\ \mathbf{0}_{r_i} & \dots & h_i \end{bmatrix}^T. \tag{45}$$

The family of solutions in (44) has the following form: $z = \mathcal{A}_s\tilde{z} + \mathcal{B}_s$, where $\mathcal{A}_s$ is an orthonormal basis for the null space of D_1, $\mathcal{B}_s$ is a feasible solution of $D_1 z = E_1$. Define the following matrices:

$$\begin{aligned} \mathcal{A}_s^{(1)} = \mathcal{A}_s(1 : rr_i, \cdot), \quad \mathcal{A}_s^{(2)} = \mathcal{A}_s(rr_i + 1 : rr_i + nm, \cdot) \\ \mathcal{B}_s^{(1)} = \mathcal{B}_s(1 : rr_i), \quad \mathcal{B}_s^{(2)} = \mathcal{B}_s(rr_i + 1 : rr_i + nm). \end{aligned}$$

Due to the nonnegativity of $z_1 = \mathrm{vec}(\Gamma_i^T)$ and (45), the set of $\tilde{z}$ denoted by $\Phi_{\tilde{z}}$, can be described as: $\Phi_{\tilde{z}} = \left\{\tilde{z} \mid -\mathcal{A}_s^{(1)}\tilde{z} \le \mathcal{B}_s^{(1)},\ D_2\mathcal{A}_s^{(1)}\tilde{z} \le E_2 - D_2\mathcal{B}_s^{(1)}\right\}$. Consequently, Δ_i^G can be obtained as: $\Delta_i^G = \mathcal{A}_s^{(2)}\Phi_{\tilde{z}} \oplus \mathcal{B}_s^{(2)}$.

7 Numerical example

Several examples allow the previous theoretical results to be illustrated. Note that all simulations in this article have been carried out in MPT 3.0 (see Herceg et al. (2013)).

7.1 Explicit robustness margin of PWA controllers

An illustration is carried out on a linear system with uncertainty set described by:

$$[A_1\ B_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 1 & 1.5 \end{bmatrix}, [A_2\ B_2] = \begin{bmatrix} 1 & 0 & 1.5 \\ 0.5 & 1.5 & 1 \end{bmatrix},$$
$$[A_3\ B_3] = \begin{bmatrix} 1.5 & 0 & 1 \\ 3.8 & 1 & 1 \end{bmatrix},$$

in the presence of constraints on the control variable and the output variable: $-5 \le u_k \le 5, -5 \le y_k \le 5$, with the nominal model chosen to synthesize a PWA control law:

$$\begin{aligned} A = 0.3A_1 + 0.2A_2 + 0.5A_3, \\ B = 0.3B_1 + 0.2B_2 + 0.5B_3,\ C = [1\ 0]. \end{aligned}$$

A continuous PWA control law is designed with prediction horizon 2, weighting matrices $Q = \mathbf{I}_2, R = 1$ and the terminal constraint chosen as the maximal output admissible set Gilbert and Tan (1991). The state space partition is presented in Figure 1. Figure 2 shows the image of Ω_{rob}^{α} via the orthogonal projection on the plane $[\alpha_1\ \ \alpha_2]$. Note that the shaded violet region presents the whole region of α_1, α_2. The blue point denotes the considered nominal system, this point coincides with a vertex of this robustness margin set. It is observed that this robustness margin differs from the classical notion, because the given control law cannot guarantee the positive invariance of the feasible region $\mathcal{X}$ if the nominal system is perturbed away from the robustness margin.

7.2 Explicit fragility margin of PWA controllers

Region 6 has the halfspace representation and its corresponding controller as follows:

$$\begin{aligned} F_6 &= \begin{bmatrix} -1 & 1 & -0.2073 & 0.2073 \\ 0 & 0 & -0.9783 & 0.9783 \end{bmatrix}^T, \\ h_6 &= \begin{bmatrix} -0.8 & 5 & 23.6177 & -17.9116 \end{bmatrix}^T, \\ u(x) &= \begin{bmatrix} -1.5625 & 0 \end{bmatrix} x + 6.25. \end{aligned}$$

The fragility margin for the control law of region $\mathcal{X}_6$ is illustrated in Figure 3. Note that this margin via two different approaches is theoretically identical. It can be seen that the slope gain G_6 without parametric error of the control law associated with this region is pointed out at point $(0, 0)$ in blue which is a vertex of the fragility margin set. It is easy to see that this control law is fragile since if the control law gain G_6 is perturbed away from the fragility set, then closed loop stability may be lost.

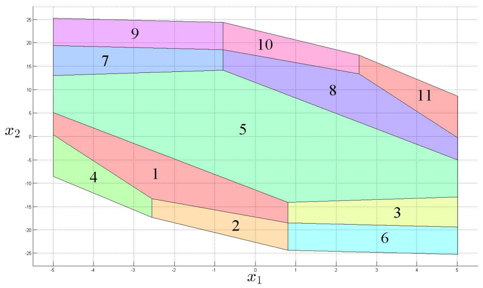

Fig. 1. State space partition.

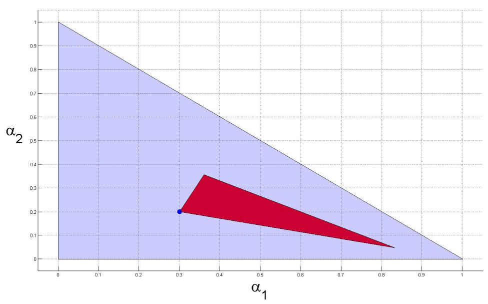

Fig. 2. Robustness margin in the plane of α_1, α_2.

7.3 *Explicit fragility of state space partition*

Again, the state space partition and the PWA control law designed above, are considered. The outer blank polytope in Figure 4, represents $\mathcal{X}$. For illustration, we focus on the unconstrained region $\mathcal{X}_5$, which is the orange polytope. The pink polytope represents $\widehat{\mathcal{X}}_5$, defined in (35). It implies that for any implemented representation $\widetilde{\mathcal{X}}_5$ of $\mathcal{X}_5$, satisfying $\widetilde{\mathcal{X}}_5 \subseteq \widehat{\mathcal{X}}_5$, the positive invariance of $\mathcal{X}$ is ensured with respect to the above PWA control law.

8 Conclusions

A measure of the robustness and fragility of the positive invariance for a piecewise affine system has been introduced in this paper. Two points of view have been presented with respect to the closed-loop dynamics of a linear system with a PWA control law: the robustness with respect to parametric model uncertainties and the fragility of this PWA controller. For both cases it has been shown that these margins are represented by convex sets of admissible parameter variations. Following this idea, the extension to the explicit fragility margin of the state space partition has been also tackled. This problem also leads to polyhedral set descriptions. The approach allows one to have a generic vision about the margins related to PWA control laws and also provides new insight in the implementation limitations for this class of controllers.

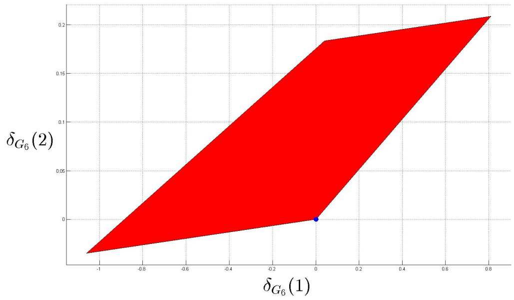

Fig. 3. Fragility margin of the controller in region $\mathcal{X}_6$.

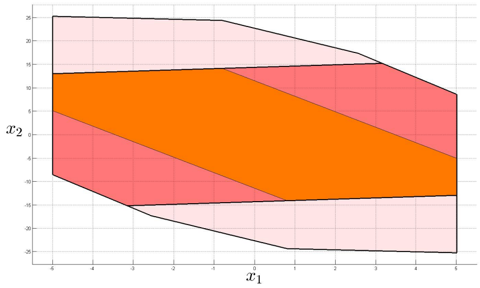

Fig. 4. The shaded pink region is $\widehat{\mathcal{X}}_5$, defined in (35).

References

Athanasopoulos, N., Bitsoris, G., Lazar, M., 2014. Construction of invariant polytopic sets with specified complexity. International Journal of Control 87 (8), 1681–1693.

Avis, D., Fukuda, K., 1992. A pivoting algorithm for convex hulls and vertex enumeration of arrangements and polyhedra. Discrete & Computational Geometry 8 (1), 295–313.

Bemporad, A., Morari, M., Dua, V., Pistikopoulos, E. N., 2002. The explicit linear quadratic regulator for constrained systems. Automatica 38 (1), 3–20.

Benlaoukli, H., Hovd, M., Olaru, S., Boucher, P., 2009. On the construction of invariant sets for piecewise affine systems using the transition graph. In: ICCA, International Conference on Control and Automation. IEEE, pp. 122–127.

Bitsoris, G., 1988. Positively invariant polyhedral sets of discrete-time linear systems. International Journal of Control 47 (6), 1713–1726.

Blanchini, F., 1999. Set invariance in control. Automatica 35 (11), 1747–1767.

Blanchini, F., Miani, S., 2008. Set-theoretic methods in control. Springer.

Dorato, P., 1998. Non-fragile controller design: an overview. In: Proceedings of the American Control Conference. Vol. 5. IEEE, pp. 2829–2831.

Gilbert, E. G., Tan, K. T., 1991. Linear systems with state and control constraints: The theory and application of maximal output admissible sets. Automatic Control, IEEE Transactions on 36 (9), 1008–1020.

Heemels, W. P. M. H., De Schutter, B., Bemporad, A., 2001. Equivalence of hybrid dynamical models. Automat-

ica 37 (7), 1085–1091.
Hennet, J. C., 1995. Discrete time constrained linear systems. Control and Dynamic Systems 71, 157–213.
Herceg, M., Kvasnica, M., Jones, C., Morari, M., July 17–19 2013. Multi-Parametric Toolbox 3.0. In: Proc. of the European Control Conference. Zürich, Switzerland, pp. 502–510.
Johansen, T. A., Grancharova, A., 2003. Approximate explicit constrained linear model predictive control via orthogonal search tree. Automatic Control, IEEE Transactions on 48 (5), 810–815.
Keel, L. H., Bhattacharyya, S. P., 1997. Robust, fragile, or optimal? Transactions on Automatic Control, 1098–1105.
Kerrigan, E. C., Maciejowski, J. M., 2004. Feedback min-max model predictive control using a single linear program: robust stability and the explicit solution. International Journal of Robust and Nonlinear Control 14 (4), 395–413.
Kouramas, K. I., Panos, C., Faísca, N. P., Pistikopoulos, E. N., 2013. An algorithm for robust explicit/multi-parametric model predictive control. Automatica 49 (2), 381–389.
Nguyen, H.-N., Gutman, P.-O., Olaru, S., Hovd, M., 2013. Implicit improved vertex control for uncertain, time-varying linear discrete-time systems with state and control constraints. Automatica 49 (9), 2754–2759.
Nguyen, N. A., Olaru, S., Bitsoris, G., Rodriguez-Ayerbe, P., Hovd, M., 2014. Explicit fragility margins for pwa control laws of discrete-time linear systems. In: 13th European Control Conference, Strasbourg, France.
Nguyen, N. A., Olaru, S., Rodriguez-Ayerbe, 2015. Robust control design based on convex liftings. In: the 8th IFAC Symposium on Robust Control Design.
Olaru, S., Dumur, D., 2004. A parameterized polyhedra approach for explicit constrained predictive control. In: 43rd Conference on Decision and Control. Vol. 2. pp. 1580–1585.
Olaru, S., Nguyen, N. A., Bitsoris, G., Rodriguez-Ayerbe, P., Hovd, M., 2013. Explicit robustness margins for discrete-time linear systems with pwa control. In: 17th ICSTCC, Sinaia, Romania.
Schrijver, A., 1998. Theory of linear and integer programming. Wiley.
Seron, M. M., Goodwin, G. C., Doná, J. A., 2003. Characterisation of receding horizon control for constrained linear systems. Asian Journal of Control 5 (2), 271–286.
Suardi, A., Longo, S., Kerrigan, E. C., Constantinides, G. A., 2014. Robust explicit mpc design under finite precision arithmetic. In: Proceedings of the 19th World Congress IFAC, Cape Town, South Africa.
Tahir, F., Jaimoukha, I. M., 2012. Robust positively invariant sets for linear systems subject to model-uncertainty and disturbances. In: Nonlinear Model Predictive Control. Vol. 4. pp. 213–217.
Tøndel, P., Johansen, T. A., Bemporad, A., 2003. An algorithm for multi-parametric quadratic programming and explicit mpc solutions. Automatica 39 (3), 489–497.
Vassilaki, M., Hennet, J. C., Bitsoris, G., 1988. Feedback control of linear discrete-time systems under state and control constraints. International Journal of control 47 (6), 1727–1735.