

On Algorithmic Portfolio Optimization for a Momentum Investor

a Stochastic Programming Approach with Moment-Matching Scenario Generation

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Problem Statement

The purpose of this thesis is to evaluate the performance of different momentum investment strategies and to assess the possibility to dynamically change from one investment strategy to another.

- Review theory and literature related to momentum factor investing, moment-matching scenario generation and stochastic portfolio optimization
- Evaluate performance of different momentum factor strategies on historical data from Norwegian and US equity markets
- Develop a stochastic portfolio optimization framework with appropriate risk measures that can be implemented algorithmically with portfolios of momentum factor strategies
- Evaluate implications for a momentum investor, provide overall assessment of the implementation, and a discussion of the obtained results

Preface

I submit this thesis in fulfillment of the requirements for my Master of Science degree in Industrial Economics and Technology Management at the Norwegian University of Science and Technology.

This work has been a unique opportunity to explore different sciences, for which I am very grateful.

Special debt of gratitude is owed to my supervisor, Professor Alexei A. Gaivoronski, for steady guidance and excellent supervision throughout the work with this thesis. His insightful comments and creative ideas contributed a great value to this work. Special thanks to Vegard Egeland and Bernt Brun with Fronteer Solutions, for supporting this thesis and feeding me with historical data. Thanks also to Dave Edwards with Quantopian for allowing me to document his cross-sectional momentum strategy.

> Håkon Sverre Rønning Juni, 2016 NTNU, Trondheim.

Abstract

Momentum strategies based on continuation patterns in equity prices have attracted a wide following among money managers and financial investors attempting to exploit anomalies present in the stock market. In this thesis, we first perform out-of-sample tests of four long-only momentum strategies, one contrarian strategy and one low-volatility investment strategy on US and Norwegian equity samples. We find the momentum and contrarian strategies to yield statistically significant abnormal returns on the Norwegian market only, indicating the US market to be more efficient. The highest average monthly returns are found for the individual stock price momentum strategy, and the highest riskreward performance is yielded by a volatility-scaled momentum strategy. The returns to the strategies are found time-varying and not always positive, suggesting that a momentum investor could benefit from periodically changing strategy or investing in a risk-free instrument.

The second part of this thesis studies the problem facing an investor with funds to allocate between investment strategies in the Norwegian market. We build a stochastic portfolio optimization framework with moment-matching scenario generation, and apply it outof-sample on portfolios with momentum and contrarian strategies. We find a significant performance increase in adding a contrarian strategy to a portfolio of momentum strategies. By allowing the investor to allocate wealth portion-wise between these algorithms, we generate higher risk-reward performance than both an equally weighted market index and a buy-and-hold benchmark of the constituent strategies. Finally, by forcing the investor to each month choose between a contrarian and a single momentum strategy, we create investment strategies with superior return performance. In particular, we find a contrarian strategy in combination with an individual stock price momentum strategy, to yield the highest cumulative and average returns among the strategies tested. These findings from stochastic programming suggest that an investor could benefit from periodically changing between contrarian and momentum investing, exploiting both return reversal effects and continuation patterns in equity prices.

Sammendrag

Blant investorer som regelmessig forsøker å kapitalisere på anomalier tilstede i aksjemarkedet, er det flere som benytter investeringsstrategier som baserer seg på fortsettelsesmønster i aksjenes priser. I første del av denne oppgaven implementerer vi fire momentum strategier, en kontrær strategi og en lav-volatilitets investeringsstrategi på historisk prisdata fra det norske og amerikanske aksjemarkedet. Resultatet viser at momentum og kontrær basert investering kun kan gjøres profitabelt i det norske markedet, noe som kan indikere at det amerikanske aksjemarkedet er mer effektivt. Vi finner høyest gjennomsnittlig avkastning for en individuell aksjepris momentum strategi og at en volatilitetsskalert momentum strategi tilbyr det beste forholdet mellom risiko og avkastning. Avkastningen de ulike strategiene gir er tidsvarierende og ikke alltid positiv på årlig basis. Derfor er det sannsynlig at en momentum investor kunne tjent på å periodevis forandre strategi eller investere i et risikofritt instrument.

Den andre delen av denne oppgaven tar for seg problemet til en investor med midler til å fordele mellom forskjellige investeringsstrategier i det norske aksjemarkedet. Vi utvikler et stokastisk porteføljeoptimeringsrammeverk med 'moment-matching' scenariogenerering og anvender det på historiske data med porteføljer bestående av forskjellige investeringsstrategier. Vi finner en signifikant økning i ytelse når en kontrær strategi legges til i en portefølje av momentum strategier. Ved å tillate investoren å fordele midler porsjonsvis mellom ulike strategier, lager vi investeringsstrategier med bedre forhold mellom risiko og avkastning enn både en likevektet marketsindeks og en kjøp-og-hold referanse bestående av porteføljens strategier. Ved å hver måned tvinge investoren til å velge mellom én kontrær og én momentum strategi, utvikler vi investeringsstrategier med overlegen avkastning. Disse funnene fra stokastisk programmering indikerer at en investor kan tjene på å periodevis veksle mellom momentum og kontrær basert investering, noe som innebærer å utnytte både reverseringseffekter i aksjenes avkastning og fortsettelsesmønster i aksjenes priser.

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Chapter 1

Introduction and Overview

The predictability of stock returns has been a controversial topic among academic researchers for a number of years. Investment strategies predicting the cross-section of stock returns based on past equity prices are argued to be at odds with the efficient market hypothesis stating that prices of financial objects fully reflect all available information. Nonetheless, such strategies have been found capable of outperforming traditional benchmarks and many money managers and professional investors attempt to exploit patterns of predictability based on past price history in their investing. There are mainly three ways of exploiting such predictability widely documented in the literature: momentum, contrarian and low-volatility investment strategies. This thesis aims to shed light on momentum investment strategies and their usefulness across different equity markets.

This chapter is organized as follows. Section 1.1 introduces the background and motivation for this thesis. The objective of this work is presented in Section 1.2 and Section 1.3 concerns the overall approach. The academic contributions of the work are clarified in Section 1.4. Finally, in Section 1.5, a detailed overview of the thesis is provided.

1.1 Thesis Background and Motivation

Factor investing is grounded in the existence of factors that have shown abnormal, abovemarket returns over longer periods of time. A factor can be seen as a characteristic that is common for a group of assets that have a statistically significant explanatory power in explaining their risk and return behavior. In this thesis we are concerned with factors based on past price history, namely the momentum, contrarian and low-volatility factors.

A momentum investor buys past winners and (short)sells past losers. Price momentum is the persistence of past price changes. The rationale behind a momentum investment strategy is to exploit continuance in equity prices, i.e. that the high performers will continue to perform well and that the poor performers will continue to perform poorly. A perplexing aspect is that a strategy following the exact opposite investment logic, a contrarian strategy, has been found to work simultaneously. An extensive body of literature provides evidence suggesting that the momentum investment strategy can be done profitably in the medium term (2-12 months holding) and that the contrarian strategy can be done profitably in the short-term and long-term (days, weeks and years of holding). A low-volatility investor, on the other hand, invests in stocks that have either low historic volatility or low forecasted volatility relative to their peers. A familiar axiom in financial theory states that high returns should be associated high risk. Several studies exist that contradict this and suggest that investors may not be rewarded for bearing systematic risk¹[24, 29, 15, 9, 54].

The first study to reject martingale behavior in stock prices by documenting profits to a momentum strategy, was Jegadeesh and Titman [59]. Their work reports abnormal returns on the US market with a self-financing momentum strategy that measures past performance based on compounded returns, and initiated long positions in the top decile stocks and short positions in bottom decile stocks. Since their seminal work, several financial academics have reported abnormal returns to similar momentum strategies in US equity markets [60, 31, 20, 46, 69, 41] and in other countries [87, 8, 88, 27, 28]. Other ways in which to capitalize on the momentum anomaly are also reported. George and Hwang [41], for instance, expose that a strategy of purchasing stocks with a price close to their 52-week high price is even more profitable than the strategy proposed by Jegadeesh and Titman [59] on US markets. Marshall and Cahan [79] report similar findings on the Australian stock exchange. Grinblatt and Han [44] find profits to an industry momentum strategy that invests in a certain number of stocks constituent of the past best performing industry. Furthermore, a momentum model found in more recent literature is the residual momentum model by Blitz et al. [16], that measures past performance using the residual in the Fama and French three-factor model. Whichever way the strategy is modelled, abnormal profits over longer periods of time indicate predictability in stock price movements.

Predictability in stock returns is argued to be at odds with the efficient market hypothesis, stating that current prices fully reflect all available information. Consequently, the momentum anomaly has caused heated debate among financial academics over the last decades. There are three prevailing schools of explanations for the sources of abnormal momentum returns: those appertaining to investor behavior[59, 10, 34, 51, 52, 28], those who use rational models where higher returns are merely compensation for higher risk or can be explained by macroeconomic factors[31, 13, 26, 62, 91], and those who claim that market frictions are the explanation[73, 66, 73]. According to the latter two, momentum abnormal returns do not necessarily violate the efficient market hypothesis.

If momentum abnormal returns do not appertain to bearing high systematic risk or can not be explained by market frictions, one might question why it has taken that long for arbitrageurs to act. Jegadeesh and Titman proposed their seminal work over 20 years ago. If markets are well-developed, such opportunities for profits should vanish. Can investors, professional or otherwise, behave irrationally for this long? Limited investability of momentum strategies, as indicated by many underlying assumptions in the commonly applied models, might offer an alternative explanation. Then the momentum

¹Systematic risk is the same as market risk, which generally describe the degree to which securites co-move with the market[12].

effect may theoretically exist in stocks, but capitalizing on this anomaly is not practically straight-forward. In addition to an assumption of no transaction costs, several studies rely heavily on short-selling underperformers. In reality, however, not all stocks are listed for short-selling, and short-selling is often associated with higher costs and risks. This work concerns investing where short-selling is not allowed and adds to the theoretical debate by examining strategies that are closer to the implementable. A similar approach to momentum investing can be found in Israel and Moskowitz [57].

Moreover, recent momentum studies have yielded somewhat different results on US data. Asness et al. [8] expose high profits while Blitz et al. [16] and Hwang and Rubesam [56] claim that the momentum effect has disappeared. This work performs updated out-of-sample momentum tests on US data and compare with results obtained from the Norwegian stock market; a smaller, presumably less efficient market. Our findings support those who claim the momentum effect to have disappeared in US markets, but expose high profits to such strategies in the Norwegian market. However, in line with the findings of [56, 16, 32, 50], we find the returns to be time-varying and not always positive on a yearly basis. Furthermore, with [79, 41, 8, 56] claiming that different momentum strategies may yield profits in different sizes, a reasonable assumption would be that there could be monetary gains for a momentum investor in dynamically changing her ways of investing.

In asset allocation theory, a portfolio may be seen as the mix of financial assets held by an investor. It is then possible, given the nature of a factor investor, to imagine a portfolio to consist of different factor investment strategies. Portfolio optimization is the process of selecting, from a set of available instruments, the subset of those which, in aggregate, best achieve some objective under given constraints. In this work, we take the position of a momentum investor and investigate whether such mathematical techniques could be utilized to dynamically allocate wealth between different momentum strategies in a beneficial way.

However, at the heart of portfolio optimization is the balancing of portfolio risk and reward. With a portfolio consisting of investment strategies trying to exploit the same underlying anomaly, one might question whether there is a sufficient possibility of diversifying risk. The constituent assets are likely to co-move to a significant degree. It is suggested by existing literature that momentum strategies yield poor performance during times of financial turmoil, for instance[56, 32]. This motivates the consideration of contrarian and low-volatility strategies. The contrarian follows the opposite logic of the momentum strategy and the low-volatility strategy is known to perform well when the market is down[24, 29, 15, 9, 54]. Part of this work then aims to find whether such strategies could be utilized as hedging instruments, or instruments of other additional gains, in a portfolio of momentum algorithm assets.

1.2 The Objective

The main objective of this work is twofold. First, this thesis aims to evaluate recent performance of different long-only momentum investment strategies on the Norwegian and US equity markets. Second, the work aims to develop a stochastic portfolio optimization framework with which we can assess the possibility to dynamically change from one investment strategy to another.

The problem domain of the thesis thus concerns empirical tests and stochastic asset allocation, making it an interdisciplinary work intersecting several sciences, including investment theory, empirical finance, mathematical optimization, and statistics. The objective was developed in collaboration with a financial practitioner. Fronteer Solutions is an entity providing investment services based on quantitative methods widely documented in the literature. Part of their investment algorithm uses signals from a momentum factor². Nonetheless, in this work we attempt to obtain results that are of academic value with insights that aim to aid a general momentum investor. No individual actors are considered in particular³.

1.3 The Approach

This work approaches the objective in two stages. In the first, we review related literature and perform out-of-sample tests of the 52weekhigh by George and Hwang [41] and individual stock price momentum by Jegadeesh and Titman [59]. The performance of these strategies are compared to those of a contrarian strategy, a low-volatility strategy and two other momentum strategies not previously documented in the literature. The first is a momentum strategy inspired by the MSCI momentum index, that measures past return performance scaled by historical volatility. The second is a momentum strategy, familiar in quantitative trading[19], that measures past performance based on average deviation from the cross-sectional mean. The back-tests⁴ are conducted on Norwegian and US equity samples from the time period between January 2000 to December 2015. The results of the momentum strategies are compared to those in existing literature, examined for different holding and formation periods, and investigated during different subperiods of time.

In the second part of this thesis we attempt to exploit time-variability in strategy returns by taking the position of an investor with funds to allocate between different investment strategies. We review and derive necessary theory to build a stochastic portfolio optimization framework. Future return distributions are generated with a moment-matching scenario generation heuristic from Høyland et al. [53]. The framework is applied algorithmically out-of-sample on portfolios with momentum and contrarian algorithms. Both

²This means that, if a stock receives a positive sign for momentum over a certain period, there is a certain change that the algorithm issues a buy-order on this stock.

 $^{^{3}}$ To build intuition and understanding of how the results can have practical consequences, we will occasionally provide insights in Fronteers way's of investing.

⁴A back-test is the process of feeding a trading algorithm with historic data to evaluate performance[19].

two-stage and one-stage optimization are considered with mean negative absolute deviation and conditional value at risk as risk measures. To model mean-reversion dependency in the scenario trees, we propose a method of auto-regression Monte Carlo simulation consistent with the moment-matching heuristic by Høyland et al. [53].

1.4 The Contributions

In performing empirical back-tests of four long-only momentum strategies, one contrarian and one low-volatility across the Norwegian and US equity market, we make several academic contributions that may also be of interest for a practitioner. First, we test whether previously reported market inefficiencies are due to data snooping or have become less significant in recent years. New data is a good protection against data snooping[74]. Second, we obtain an assessment of the relative performance of these strategies, and find whether some strategies are more profitable in different periods of time. Finally, by testing the strategies on both US and Norwegian equity samples, we obtain a comparison of a large, presumably more efficient market with a small, presumably less efficient market. As such, one would expect the momentum anomaly to be more prevaling in Norway. Contrary to the majority of existing literature, we avoid making assumptions on shortselling of stocks and apply investment models with long positions only⁵. This makes the implementation more realistic since not all stocks are listed for short-selling, and short-selling is associated with higher costs and risks. Some highlights from our findings are:

- The momentum and contrarian strategies yield statistically significant abnormal returns on the Norwegian market only, indicating the US market to be more efficient.
- The highest average returns are yielded by the individual stock price momentum strategy. However, the volatility-scaled momentum strategy offers the highest risk-reward performance.
- The risk and returns to the strategies are found highly time-varying and not always positive, suggesting that a momentum investor could benefit from periodically changing strategy or investing in a risk free instrument.

By applying stochastic portfolio optimization on portfolios of different investment strategies, we add to the existing literature by modelling a setting more realistic for an investor with opportunity to invest in different ways. While existing literature has reported timevariability in returns to momentum strategies[56, 32, 16, 50], and different sizes in returns to different momentum strategies[41, 79, 16], there are none, to our knowledge, that have documented an attempt to exploit it. Some highlights of our findings are:

• The stochastic framework fails to generate superior returns with a portfolio of momentum assets only. However, we find a significant performance increase when adding a contrarian strategy to the mix of available assets.

⁵A similar approach can be found in Israel and Moskowitz [57].

- We find a contrarian strategy in combination with an individual stock price momentum strategy, to yield the highest cumulative and average returns among the strategies tested.
- With our approach we find it easier to constrain mean negative absolute deviation than expected tail-loss, and a two-stage optimization approach is found marginally beneficial in this setting when portfolio expected risk is constrained.

These findings from stochastic programming suggest that an investor could benefit from both contrarian and momentum investing, exploiting both return reversal effects and continuation patterns in equity prices.

1.5 Thesis Organization

This thesis has seven remaining chapters. Their contents are described in the following.

Chapter 2 is confined to concepts relevant for understanding factor investing and financial markets. The momentum factor is emphasized with respect to modelling, usefulness in terms of profits in different markets, and different explanations appertaining to the sources of the profits.

Chapters 3 and 4 detail the theory and derivations underlying the stochastic optimization framework applied in this work. State-of-the art risk measures together with one-stage optimization models are detailed in Chapter 3. These models are subsequently extended to a two-stage setting in Chapter 4. Of particular importance is the moment-matching scenario generation algorithm presented in Section 4.1. Scenario trees are generated with forecasting methods that hinge on this heuristic. The section draws upon theory introduced in Høyland et al. [53] and provides a psedo-code that illustrates the implementation in Python⁶.

Chapter 5 concerns the samples and models applied when back-testing factor investment strategies on US and Norwegian equity markets. The framework applied in algorithmically performing stochastic portfolio optimization with factor model strategies as assets, is also presented together with any key assumption underlying this work. This work presents discussion and results together for the purpose of readability.

Results and discussions of such, are provided in two parts. Chapter 6 details the results to the empirical factor model tests with the focus directed towards the most successful strategies. Chapter 7 concerns results from the algorithmic portfolio optimization. The first part contains results where we attempt to maximize profits, the second where we optimize with respect to both risk and return preferences of an investor. Finally, Chapter 8 presents the conclusion of this thesis and proposes recommendations for future research.

⁶Object oriented programming language. All scripts applied in back-tests, scenario generation and stochastic portfolio optimization are available upon request. Mail: hakonsro@stud.ntnu.no.

Chapter 2

Background - Momentum Factor Investing

Challenging the efficient market hypothesis, the last decades has shown growing academic evidence supporting the possibility of predicting stock returns. A part of these studies focuses on strategies for predicting the cross-section of stock returns based on past equity prices. There are mainly three ways of exploiting such predictability of returns found in the literature; momentum, contrarian and low-volatility strategies. These are based on past price data and are widely documented. A perplexing aspect of this literature is that two of these strategies, momentum and contrarian, have opposite investment logic and tend to work simultaneously. Specifically, contrarian strategies are found profitable in the short-run (days, weeks of holding) and long-term (3-5 years holding) while the momentum strategy is found profitable in the medium-run (2-12 months holding).

Moreover, low-volatility investing is not only at odds with the efficient market hypothesis, it is at odds with the common assumption that higher returns are associated with higher volatility. Strategies based on all three factors will be reviewed in this chapter. Momentum investing is the main focus of this thesis and the strategy most widely found in the literature. As such, this strategy will be reviewed to a greater extent. Section 2.1 introduces relevant theoretic concepts prevailing throughout the rest of this text. The factors will be introduced with relevant theory in subsequent Sections 2.2 and 2.3.

2.1 Underlying Theoretical Concepts

2.1.1 Financial Markets and Equities

An important theoretical concept for any investment or trading activity is the concept of a financial market. In economics, the term market is used to describe the mechanisms by which liquid financial assets can be traded between investors at a relatively low transaction cost. The price of the asset then reflect available supply and demand[5]. An investment is liquid if it easily can be turned into cash by trading in the immediate future[12]. Playing an essential role in a capitalist economy, markets aggregate a financial system through which saving and investments can be conveyed, liquidity and risk transferred and wealth can be stored. One way to divide between different kinds of markets is by the financial asset traded in the market. An overview is provided in table 2.1.

An Overview Over Financial Markets						
Market	Purpose					
Money Markets	Market for short-term borrowing and lending					
Equity Markets	A capital market for trading equities					
Debt Markets	A capital market for borrowing and lending					
	funds					
Foreign Exchange	Market for trading of currencies					
Derivative Markets	Market for trading instruments derived from					
	other underlying assets					
Other Alternative In-	Private equity/hedgefond investments, com-					
vestments	modities, insurance and real estate					

Table 2.1: Financial Markets

Notes: this table provides an overview over different types of financial markets. The focus in this thesis is on the equity market[5].

Equity (stock) price behavior is one of the main focuses of this thesis. Equity is by definition the part of the value of a firm that is not debt. A stock is, simply speaking, a share of the equity in a firm. When acquiring an equity stake in a company, the investor is allowed to partake in the profits of the company. Dividends paid and increase in firm value aggregates the returns to equity investors.

Shares of public companies are traded in stock markets; organized markets providing liquidity to the companies. The main function of capital markets is to provide firms with capital. Capital can thus be raised by issuing debt and by issuing equity stake in the firm. If an investor is to acquire shares directly from the company, she does so in the *primary market*. If a transaction is taking place between investors with no direct involvement in the company, the transaction is said to be done at the *secondary market*. Only existing securities are sold at the secondary market.

A stock exchange is a place or organization by which stock traders (people and companies) can trade stocks. Other stocks may be traded "over the counter" (OTC), that is, through a dealer. A trade in stock market entails the transfer of money for a security from a seller to a buyer[12]. A company may have their stock listed on one or several stock exchanges, normally depending on the size of the company. The exchange can be a physical trading floor (as New York Stock Exchange (NYSE)) or a virtual listed exchange, where all of the trading is done over a computer network (NASDAQ)[5]. In this thesis we consider stocks traded in the secondary market in a stock exchange.

2.1.2 The Efficient Market Hypothesis

Historically, it has been a common assumption in financial theory that information and news spreads very quickly in the market. As a consequence of this, the prices of financial assets or instruments reflect available information without delay. The best prediction of tomorrow's price is then the price today. The Efficient Market Hypotheses states that prices fully reflect all available information[37].

Depending on what type of information that is assumed to be fully reflected in the prices, one can further characterize market efficiency as weak, semi-strong or strong. A weakly efficient market entails that all relevant historical information is incorporated into the current asset price. This includes information on past prices and returns on the asset, as well as all other relevant assets. In a semi-strong efficient market, prices reflect all publically available information. In a market with strong efficiency, all information, private and public, is incorporated into the price with no delay. Thus, in a strongly efficient market insider trading is not possible [37, 12, 17].

All types of market efficiencies, including the weak form, precludes profitable trading strategies based on past asset prices such as the momentum and contrarian strategy. It is impossible to achieve abnormal returns without being lucky because the same information is available to all. As we shall see, however, much empirical evidence exists, that documents anomalies and strategies with returns that significantly deviate from this assumption.

2.1.3 Factor Investing

Factor investing is grounded in the existence of factors that have shown to abnormal, above-market returns over longer periods of time. The factor investor strategically creates portfolios of assets based on these factor premiums. A factor can be seen as a characteristic that is common for a group of assets that has a statistically significant explanatory power in explaining their risk and return behavior. According to Ang and Longstaff [4], a factor should satisfy four criteria:

- 1. grounded in academic research
- 2. shown significant premiums that are expected to persist in the future
- 3. available history for bad times
- 4. be implementable in liquid, traded instruments

One of the first and most commonly known factors is the market-factor, introduced widely to the public together with the well-known Capital Asset Pricing Model (CAPM) in e.g. Sharpe [93]. Since then researchers have discovered several other factors that have been persistent over time. Connor [30] distinguishes between three different categories of factors; macroeconomical, statistical and fundamental factors. Macroeonomical factors are confined to the space of macroeconomical measures and statistical factors are grounded in statistical techniques. Albeit being interesting, they are not the focus of this thesis. Fundamental factors are characteristics appertaining to individual company attributes such as firm size, dividend yield, book-to-market ratio and other technical indicators[30].

An Overview Over Financial Markets						
Factor	Explanation					
Low Value	Describes equities with low prices relative to					
	their fundamental value					
Low Size	Describes equities with low market capitaliza-					
	tion					
Momentum	Describes instruments with high past perfor-					
	mance relative to self or others					
Contrarian	Describes instruments with low past perfor-					
	mance relative to self or others					
Low-Volatility	A characteristic of equities with low historical					
	or forecasted volatility					
Yield	A characteristic of stocks with high dividend					
	yield					

Table 2.2: Fundamental Factors

Notes: this table gives an overview over fundamental factors that have yielded above-market returns consistently over the last decades and that are grounded in academia[11].

An overview over fundamental factors¹ that have yielded above-market returns consistently over the last decades is provided in table 2.2. These are also heavily documented in the literature[11]. In this thesis we study and model a subset of the factors in table 2.2. The momentum, contrarian and low-volatility factors are based on past price history. Past prices are readily available information, which, unlike most other fundamental data, can be retrieved with relative certainty of high quality, non-erroneous data. This makes past prices suitable for novice back-testing[19]. In the following sections, factors modelled in this thesis will be introduced to a greater extent together with relevant academic literature.

2.2 The Momentum Factor

The momentum investor buys past winners and (short)sells past losers. Price momentum is the persistence of past price changes. Thus, the rationale behind a momentum investment strategy is to exploit continuance in equity prices, that the high performers will continue to perform well and that the poor performers will continue to perform poorly. An extensive body of literature provides evidence suggesting that such an investment strategy can be done profitably in the medium term (2-12 months). The majority of the

 $^{^{1}}$ On this date, the contrarian factor may not be as commonly accepted as the others portrayed in table 2.2.

empirical tests are performed on US data. However, the momentum effect has also been documented in several other markets worldwide.

We review ways in which momentum investing is commonly modeled in Section 2.2.1. Evaluation of the usefulness of an investment strategy requires the associated profits to be measured. Also, one should understand the source of the profits when devising an investment strategy. Therefore, we devote Section 2.2.2 to studies documenting momentum profits in different markets and Section 2.2.3 to different schools of explanations for momentum abnormal returns.

This thesis focuses on equity price momentum. The momentum effect has also been documented in other markets. For momentum in commodity markets see e.g. [94, 81, 42, 36]. For momentum in currency and financial futures markets see e.g. [85, 33, 82].

2.2.1 Momentum Models

Individual Stock Price Momentum - Jegadeesh and Titman (1993)

Jegadeesh and Titman [59], henceforth JT, was one of the first studies to document momentum effects in stock markets. Their model is widely used as a benchmark in the literature [59, 60, 20, 87, 43, 46, 92, 48]. In the following we present the investment methodology and commonly found revisions of the model.

At the beginning of each month t the stocks are ranked in ascending order on basis of the last J month's compounded returns. Based on this performance ranking, 10 equally weighted portfolios are formed. The top 10% portfolio is called the "winner" portfolio, the bottom 10% the "loser" portfolio. The weights on each security $i \in (1, N)$ in the portfolios is given by

$$\omega_i = 1/N \tag{2.1}$$

where N is the number of stocks in each portfolio.

In each month t the investor initiates a long position in the winner portfolio and a short position in the loser portfolio. Each position is held for K months. Thus, after a startup period, we hold K portfolios in each momentum strategy if we allow for overlapping portfolios. With passage of time, the value of each portfolio in the momentum strategy change. JT initially addresses this in two ways;

- 1. Calculating the returns to a series of buy and hold portfolios by averaging the total holding period return of the portfolio
- 2. Calculating returns to a strategy with monthly rebalancing to maintain equal weights on each portfolio and on each constituent

The results cited in their article are based on the second. The rebalancing entails selling a portion of the portfolios that performed over average, and investing the proceeds in those portfolios with worse than average performance. The momentum portfolio can be written as W - L where W is the winner portfolio and L is the loser portfolio. Hence, this describes a zero-cost self-financing portfolio with no regard to trading costs.

Denote by r_{it} the return of stock i in month t. With equally weighting of constituent stocks in each portfolio, the return in each month r_{pt} of each portfolio is given by

$$r_{pt} = \frac{1}{N} \sum_{i=1}^{N} r_{it}$$
(2.2)

with monthly re-balancing of each portfolio to keep equal weights, the monthly return for the zero-cost momentum strategy can be found by

$$r_{tot,m} = \frac{1}{MN} \sum_{m=1}^{M} (\sum_{i=1}^{N} r_{im})_{W} - (\sum_{i=1}^{N} r_{im})_{L})$$
(2.3)

$$\sum_{j=1}^{N} \omega_i = 0 \tag{2.4}$$

where:

 r_{it} = Monthly return to each constituent $r_{tot,m}$ = Monthly return to the strategy M = Number of portfolios in holding, m $\in (1, ..., M)$ N = Number of stocks in each portfolio, i $\in (1, ..., N)$ W,L = Subscripts of the winner and loser portfolio, respectively Positive returns to the portfolio are per dollar (or NOK) invested.

In order to avoid microstructure effects (bid/ask bounce), JT allows for a 1-week skip period (S) between the formation period and the holding period. With JT's notation a momentum strategy can then be defined by the holding period K, the formation period J and the skipping period S. The triplet (J,S,K) defines a momentum strategy that is based on the last J months returns, skips S months and holds for K months. This notation will be used throughout this text. In order to increase the power of their tests (more observations) JT uses overlapping holding periods for the portfolios.

Ranking Criterion JT rank the stocks each month based on the past compounded monthly returns. Then the JT ranking criteria C_{JT} is given by

$$C_{JT} = \prod_{j=t-J}^{t-1} (1+r_{ij}) = (1+r_{ij-1})(1+r_{ij-2})...(1+r_{ij-J})$$
(2.5)

This is a commonly used ranking criteria in the literature. Grundy and Martin [46] however, use the past cumulative monthly return over the ranking period

$$C_{GM} = \sum_{j=t-J}^{t-1} r_{ij}$$
(2.6)

They argue that cumulative returns have two benefits. First, it simplifies theoretical analysis when returns have a factor structure. Second, it is empirically beneficial seeing as errors in estimates of a stock's formation period factor exposure are dependent on the compounded return of that stock over the formation period. Hence, with cumulative returns a stock's winner/loser status is independent of the error in the estimate of its factor loadings. Yet another ranking criteria is found in Marshall and Cahan [79], they simply use the past average monthly return over the formation period J. The MSCI momentum index is furthermore constructed with a measure that is scaled by the volatility such that[11]:

$$C_{MSCI} = \frac{\bar{r}_T}{\sigma_T} \tag{2.7}$$

Where T denotes some historical timespan over which the volatility and return is measured.

George and Hwang [41] revise the JT model by a ranking criteria based on a readily available piece of information – the 52-week high equity price. The performance measure is the closeness to the 52-week high price denoted by

$$C_{GH} = \frac{P_{it}}{HIGH_{it}} \tag{2.8}$$

where P_{it} is the equity price at the end of month t, and $HIGH_i$, t is the highest price of stock i that ends on the last day of month t. A strategy with this ranking criteria is proven profitable on US and Australian equity markets[41, 79] and is commonly called the **52weekhigh**.

A momentum revision tested by David Edwards with Quantopian² on US stocks, ranks stocks based on the average deviation from the cross-sectional mean. This could be measured on a daily, weekly or monthly basis. The strategy is called **cross-sectional momentum**³ and the ranking criterion is given by the following.

$$C_{DE} = \frac{1}{J} \sum_{t=1}^{J} [r_{it} - \frac{1}{I} \sum_{i=1}^{I} r_{it}]$$
(2.9)

 $^{^{2}}$ A crowd-sourced hedge fund that provides a platform for anyone to build, test, and execute trading algorithms[2].

 $^{^3\}mathrm{Note}$ that this strategy has never before been documented in the literature, but has shown promising results on US data.

where: J = Number of days, weeks or months in the formation period I = Number of equities $r_{it} = Asset return measured on daily, weekly monthly basis$

Portfolio Size The amount (%) of stocks assigned to the winner (W) and loser (L) is commonly noted as the portfolio size. JT forms portfolios with a portfolio size of 10%. Some researches, especially among those focused on other markets than the US equity market, part from this. E.g. [27, 87, 35, 79, 8] form portfolio sizes of 30%. This is because sample sizes are small. For the same reason [43] implement 20% portfolio sizes.

Portfolio Weighting JT weights the stocks in the top and bottom decile portfolios with equal weights. This is the prevailing weighting scheme found in the literature. However, due to the illiquidity of smaller stocks some researchers use a value weighted approach[27, 69, 14]. That is, the weighting is done on basis of market capitalization. Then each weight on each stock is given by

$$\omega_i = \frac{Mcap_i}{\sum Mcap_i} \tag{2.10}$$

The advantage of the market capitalization based weighting scheme is that smaller stocks, which are typically more illiquid and expensive to trade, have smaller weights in the portfolios. On the other hand, large cap stocks dominate the portfolio.

Skipping Period JT form portfolios both with and without a 1-week skipping period. In recent literature, it is more common to use a S = 1-month skipping period. A 1-month gap between formation and investment periods avoids contaminating the momentum strategy with short-term reversal as will be introduced in Section 2.3.1. Also a 1-month skip helps avoid microstructure effects (bid-ask bounce).

The Conrad and Kaul (1998) Methodology

Another widely used model was first introduced to momentum modelling by Conrad and Kaul [31]. This model is based on the short-term⁴ models of Lo and MacKinlay [74] and Lehmann [72], and can easily be applied with a contrarian investment strategy instead. The models aims to capture and mimic the momentum essence of the previously applied models.

In this momentum model the investor buys or sells equities at time t based on the performance in the formation period from t-1 to t. The performance of the stocks in the strategy is determined relative to the average performance. The portfolio weights are

 $^{^{4}}$ Contrarian models, see Section 2.3

based on performance. Denote by J the time length between t and t-1. Let $\omega_{it}(J)$ be the fraction of the momentum portfolio devoted to stock i at time t based on performance from t-1 to t, then

$$\omega_i = \frac{1}{N} (r_{it-1} - \bar{r}_{it-1}) \tag{2.11}$$

$$\bar{r}_{it-1} = \frac{1}{N} \sum r_{it-1}$$
(2.12)

where:

 r_{it-1} = The return on each security in the formation period \bar{r}_{it-1} = The mean return

If a security performs worse than average, we initiate a short position. If the security performs better than the average, we take a long position in the security. The portfolios constructed each month are held for a holding period denoted K. Notice that since the security weights are proportional to the differences between equity return and average returns, the stocks that deviate more from the average/expected cross-sectional return will have larger weights. For any portfolio held over a period K, the profits are given by

$$\pi_t(K) = \sum_{n=1}^{N} \omega_i r_{it}(K)$$
(2.13)

This is also a zero-cost portfolio where $\sum_{n=1}^{N} \omega_i = 0$. Notice that since the weights can be scaled to obtain any level of profit in a frictionless world, it is common to only test if the profits are significantly positive or negative.

Profit Measure To assess the economic significance of the profits from the CK model, beyond the statistical significance, Chan et al. [20] propose a profit measure revision. They divide the profits by the length of the holding period J and amount invested in a long or short position $0.5I_t(K)$.

$$\pi_{CHT,t}(J) = \frac{\pi_t(J)}{0.5JI_t(K)}$$
(2.14)

where

$$I_t(J) = \sum_{i=1}^{N} |\omega_{it}(K)|$$
(2.15)

is the total aggregate long or short investment in the zero-cost strategy at time t. This return could be seen as a per-holding period profits for every dollar (or NOK) invested long or short, or profits of the portfolio W - L.

Residual Momentum - Blitz et al. [16]

A momentum model found in more recent literature is the residual momentum model by Blitz et al. [16]. In the same way as a conventional momentum investment model, the residual momentum ranks the securities in every month t based on past performance. The measure of performance is related to the the monthly residual return estimated using the Fama and French three-factor model

$$R_{it} = \alpha_i + \beta_{1i}RMRF_t + \beta_{2i}SMB_t + \beta_{3i}HML_t + \epsilon_{1,t}$$

$$(2.16)$$

 $RMRF_t$, SMB_t and HML_t are the excess returns on factor-mimicking portfolios for the market, size and value in month t. α_i , β_{1i} , β_{2i} and β_{3i} are the factor loadings that need to be estimated through regression. ϵ_{it} is the residual return for asset I in each month t.

The Fama and French model is estimated for a 'rolling' past 36 months' time period. In the measurement criteria (2.17), the estimated α is not included. This is because α serves as a general control for misspecification in the model of expected returns and is calculated based on the last 36 months, not 12 which the ranking period is. To obtain the measurement criteria, the residual returns (α excluded) are standardized

$$C_{RM} = \frac{\epsilon_{it}}{\sigma_{it}} \tag{2.17}$$

where σ_{it} is the standard deviation for the asset over the formation period. This standardization of the residual return is done to obtain an improved measure, since the raw residual return can be a noisy estimate[48]. These positions are held for a holding period J. Again the momentum portfolio is zero-cost $\sum_{i=1}^{N} \omega_{it} = 0$.

Blitz et al. [16] use a formation period J of the last year excluding the last month (12-1) to avoid microstructure and short term effects. Equally weighted portfolios are formed of every decile. As in the JT model, stocks are sorted in ascending order, the top 10% is the winner portfolio W and bottom 10% is the loser portfolio L. The momentum investor initiates a short position in L and a long position in W as in all the other momentum models described.

Industry Momentum - Grinblatt et al. [45]

An industry momentum strategy first involves the construction of portfolios with constituents from the same industry. Each month t, the industry portfolios are ranked based on the value weighted cumulative returns over the formation period J. Equally weighted portfolios are then formed with the stocks within the top 30% of the ranked industries making up the winner portfolio and the stocks within the bottom 30% of the ranked industries constitute the loser portfolio. The momentum investor initiates a short position in the loser portfolio and a long position in the winner portfolio. Again the momentum portfolio is zero-cost $\sum_{i=1}^{N} \omega_{it} = 0$.

2.2.2 Evidence of Momentum Profits

An extensive body of finance literature documents that stock returns are predictable based on past price history. In this section we present momentum documenting studies in a geographical split. The purpose is to illustrate the usefulness of the momentum investment strategy in terms of profits. An overview over the findings is reported in table 2.3.

US Equity Market

One of the first studies to document the momentum anomaly was Jegadeesh and Titman [59]. They implement strategies with (months) formation period J = [3,6,9,12], skip S = [0,0.25] and holding period K = [3,6,9,12]. The test was conducted with daily CRSP data from NYSE and AMEX in the time period from 1965 to 1989. Of the 32 strategies tested, 31 strategies showed positive returns at a significant level. The 12-month/1-week/3-month strategy yielded the best result with an average monthly return of 1,49%. Similar results can be found in Jegadeesh and Titman [60] when they extend their sampling period and test a 6-month/1-week/6-month strategy on daily CRSP data from 1989 to 1998. This was done to exclude data snooping biases. The momentum strategy yielded average monthly returns of 1,39% at a significant level.

Conrad and Kaul [31] investigate 8 different momentum strategies with holding and formation periods K=H=[1 week, 3, 6, 9, 12, 18, 24, 36] months. Their analysis was done with daily data from NYSE/AMEX in the period from 1926 to 1989 divided in 5 different subperiods. Momentum profits are found for holding (formation) periods for up to 18 months, except for the 1-week/1-week strategy. Their findings are consistent with Jegadeesh and Titman and indicate that a momentum strategy is profitable at the medium (3-to 12-month) horizon.

With daily data from CRSP NYSE/AMEX in the time period from 1973 to 1993, Chan et al. [21] test a 6-month/5-day/6-month momentum strategy. They report an average monthly return of 1,47% for the portfolios formed. Interestingly they also find that the momentum payoff turns negative after 1-2 years. Seeing as a momentum strategy is essentially opposite of the contrarian strategy previously discussed, this evidence is consistent with the long-term contrarian findings. Furthermore, in Chan et al. [22] they extend the sampling period to include daily data from 1994 to 1998. They find a monthly average return of 7.8% for the 6-month/5-day/6-month strategy during these five years. In addition they report larger abnormal returns for short-selling "loser" portfolios than for "winner" portfolios.

Lee and Swaminathan [71] also provide evidence suggesting positive price momentum profits involving NYSE/AMEX stocks for the time period from 1965 to 1995. As Jagadesh and Titman they form equally weighted momentum portfolios consisting of the top decile "winner" stocks, and the bottom decile "loser" stocks with formation period J = [3,6,9,12], skip S = [0,0.25] and holding period K = [3,6,9,12] (months). They find the 12-month/3-month to be the best performing portfolio with an average monthly return of 1,54%. All strategies show statistically significant positive returns.

Grundy and Martin [46] also serve a study providing evidence of momentum profits in US markets in the time period form 1926 to 1995. They use the same methodology as Jegadeesh and Titman, longing the top 10% performing stocks of the market and shorting the worst 10% performing stocks, measured in cumulative total return. They find that a total return momentum strategy would have earned a statistically significant monthly return in excess of 1,3% (risk-adjusted) over the entire period.

Another study to confirm momentum excess returns is Korajczyk and Sadka [69]. Their analysis is conducted on NYSE, AMEX and NASDAQ for the time period from 1967 to 1999. With J = [2,5,11], S = [1] and K = [1,3,6,12] they find that all strategies show positive excess monthly returns (relative to the risk free rate).

European Markets

Even though Asness et al. [7] study return patters across European markets at the country index level, Rouwenhorst [87] is the first study to analyse momentum evidence at country level outside the US market. They perform tests with data from 12 European countries in the time period from 1978 to 1995. The study is conducted with the JT methodology with J = [3,6,9,12] S = [0,1] and K = [3,6,9,12] (months). Their findings are remarkably similar to those of Jegadeesh and Titman (1993). All 32 zero-cost portfolios show statistically significant positive results. The best performing strategy is the 12/0/3 without skip with an average monthly return of 1,35%. Furthermore, the momentum strategy yield positive returns for all 12 individual countries in the study, including Norway. Interestingly they also find the momentum effects to be bigger for smaller firms. For stocks on Oslo stock exchange they expose average monthly returns of 1% with a 6-month formation and 6-month holding strategy of 30% portfolio size.

In an attempt to replicate the tests of Rouwenhorst [87], Van Dijk and Huibers [95] use data from 15 European countries in the time period from 1987 to 1999 and a momentum strategy with equal weighting and holding periods of 3, 6, 9 and 12 months with a 1 year formation period. The findings do in fact confirm that mid-term momentum strategies generate risk-corrected returns in excess of an equally weighted European market index in the full sample period.

Bird and Whitaker [14] evaluate price momentum strategies in seven major European markets over the time period from January 1990 to June 2002. With formation periods J = [6,12] and holding periods K = [3,6,9,12,24,36,48], they find statistically significant positive returns for both equally weighted and value weighted portfolios for holding periods up to 9 months. The equally weighted momentum portfolio with the greatest return is the 12-month formation and 1-month holding, yielding an average of 1.5% per month. Interestingly, they find that while value weighted portfolios show smaller returns for short holding periods than equally weighted portfolios, they yield greater returns for longer than 3 months holding.

Asia and Australia

Relatively few studies have been conducted on Asian markets. One of them, Chui et al. [27], examine 8 different Asian markets with data from 1976 to 2000. For their tests to be comparable with Jegadeesh and Titman [59] and Rouwenhorst [87], they form 6/1/6 zero-cost portfolios. However, the winner portfolio consists of stocks from the monthly top 30% performing stocks and the loser portfolio consists of the bottom 30%. Also, they use a value-weighted approach since the Asian stocks are smaller and more illiquid. The study finds significantly positive returns for Asian stock markets outside Japan, with average monthly returns of 1,45% per month prior to the financial crisis in 1997, and 0,54% after. Furthermore they see a tendency; the momentum effect is stronger for firms with smaller market capitalization. This is also consistent with the findings in the US equity market.

Demir et al. [35] investigates the returns to short-term and mid-term momentum strategies on the Australian Stock Exchange (ASX) in the time period from 1990 to 2001. They use formation and holding periods of 30, 60, 90 and 120 days for a total of 16 strategies. As in Chui et al. [27], the top and bottom 30% are assigned to the winner and loser portfolio each month, respectively. They find that momentum is indeed prevalent in the Australian market at a statistically significant level, and that the returns are of greater magnitude than in the US equity market. The highest returns are found for a 180-day/ 30-day portfolio, with average monthly returns of 5,34% per month.

Marshall and Cahan [79] confirm the findings of Demir et al. [35] in Australian markets. In addition to conducting analysis with the conventional JT momentum strategy, they also test the 52weekhigh momentum strategy previously described. Both strategies are conducted with a 6 months holding period. Winner portfolios consist of the top 30% and loser portfolios bottom 30% each month. The study finds that both strategies prove profitable, but conclude that the 52weekhigh momentum strategy is highly profitable on Australian equity markets with an average monthly return of 2,14%.

Worldwide Studies

Chan et al. [20] examine the profitability of momentum strategies in international equity markets. They use data from 23 countries from Asia-Pacific (9), Europe (11), North America (2) and Africa (1) in the time period from 1980 to 1995. Their momentum strategy is similar to that of Conrad and Kaul (1998), and is implemented with country indices and equal holding and formation periods of 1, 2, 4, 12 and 26 weeks. Their findings confirm statistically significant evidence of momentum profits for holding periods over 4 weeks. They also find that if they implement the strategy on markets that experience increases in volume in the previous period, the profits are higher.

Rouwenhorst [88] is one of the first studies to emphasize emerging markets in a momentum study. Rouwenhorst examine 20 countries from the Emerging Markets Database (EMDB) in the time period from 1980 to 1996. With a 6-month/1-month/6-month momentum strategy with equal weighting sorted based on past best and worst 30% performance, he finds momentum profits in 17/20 countries. Taiwan, Indonesia, and Argentina did not

seem to display momentum effects in the time period. If the strategy is implemented in all countries simultaneously, the average monthly return is 0,39% when stocks are equally weighted and 0,58% when countries are market weighted. The study concludes that momentum effects are present in both emerging and developed markets, yet stronger in developed countries.

Griffin et al. [43] investigates a J=6 S=1 K=6 momentum strategy with portfolio sizes of 20% in 40 countries in five regions; Africa, America, Asia, Europe and the US. The time period under which the back-tests are performed vary, but all time periods end in 2000. The study finds positive returns in 2/2 African countries, 7/7 American countries, 10/14 Asian countries and 14/17 European countries. The average monthly momentum profit is 1.63%, 0.78%, 0.32%, and 0.77% in Africa, Americas (excluding the United States), Asia, and Europe, respectively. The overall conclusion is that momentum investment strategies are profitable worldwide, but more so for developed than emerging markets. This is in line with the findings of Rouwenhorst [88].

Asness et al. [8] examine momentum portfolios of individual stocks globally across four equity markets: the United States, the United Kingdom, continental Europe, and Japan. They find consistent evidence of momentum return premia in all markets. The momentum portfolios are formed on basis of the past 12-month cumulative return with a 1 month skip to avoid microstructure effects. The portfolios are formed with a size of 33%. For the total set of global stocks they report an average monthly excess return over the risk free rate of 5.8% (3.18) in the time period from 1972 to 2011.

While investigating how cultural differences influence the returns of momentum strategies, Chui et al. [28] consider individual stock samples from 41 markets around the world in the time period from February 1980 to June 2003. In their samples they exclude stocks whose market capitalization is below the fifth percentile. Their momentum strategy form portfolios based on stocks' past 6 month returns that hold for 6 months. Portfolio sizes are 30%. All but four countries exhibit profits. A strategy that includes all stocks yield monthly average returns 0.93%. Interestingly they also find an average monthly return of around 1% for stocks listed on the Oslo Stock Exchange in the time period from March 1983 to June 2003.

2.2.3 Sources of Momentum Profits

When devising an investment strategy, it is important to understand the source of the returns. In the absence of a reasonable explanation, the return-patterns observed could be a statistical error. The investment strategy is then unlikely to be useful in the future. Despite short-term continuation of returns being well documented in the literature, it exists different and somewhat opposing theories as to the cause. Throughout the academic literature there are three prevailing schools of explanations for the sources of momentum returns; those appertaining to investor behaviour, those who use rational models where higher returns merely is compensation for higher risk or can be explained by macroeconomic factors, and those who claim that market frictions is the explanation.

Study	Sample	Model*	Return	Tstat
US Equity Market				
Jegadeesh and Titman [59]	1965-1989	ISPM/12/3/10%	1.49%	4.28
Jegadeesh and Titman [60]	1989 - 1998	$\mathrm{ISPM}/6/6/10\%$	1.39%	4.7
Conrad and Kaul [31]	1926-1989	CK	Positive	Significant
Chan et al. [21]	1973 - 1993	ISPM/6/6/10%	1.47%	-
Chan et al. [22]	1994 - 1998	ISPM/6/6/10%	7.8%	-
Lee and Swaminathan [71]	1965 - 1995	ISPM/12/3/10%	1.54%	5.63
Grundy and Martin [46]	1926 - 1995	ISPM/6/6/10%	$1.3\%^{***}$	3.19
Korajczyk and Sadka [69]	1967 - 1999	ISPM/11/1/10%	1.58%	5.08
Grinblatt et al. [45]	1963 - 1995	ISPM/6/6/30%	0.43%	4.65
Fama and French [38]	1963 - 1993	ISPM/12/2/10%	1.31%	-
George and Hwang [41]	1963 - 2001	52WH/-/6/10%	1.23%	7.06
European Markets				
Rouwenhorst [87]	1978-1995	ISPM/12/3/10%	1.35%	3.29
Van Dijk and Huibers [95]	1897 - 1999	ISPM/12/1/10%	1.5%	-
Bird and Whitaker $[14]$	1990-2002	ISPM/12/1/4-6%	1.5%	-
Asia and Australia				
Chui et al. [27]	1976-1997	ISPM/6/6/30%	1.45%	-
Demir et al. [35]	1990-2001	$\mathrm{ISPM}/6/1/30\%$	5.34%	10.68
Marshall and Cahan [79]	1990-2003	52 WH/-/6/30%	2.14%	10.39
Worldwide Studies				
Chan et al. [20]	1980-1995	CK	Positive	Significant
Griffin et al. $[43]$	1926-2000	$\mathrm{ISPM}/6/6/20\%$	0.49%	2.95
Rouwenhorst [88]	1980 - 1996	$\mathrm{ISPM}/6/6/30\%$	0.58%	-
Chui et al. [28]	1981 - 2003	$\mathrm{ISPM}/6/6/30\%$	0.93%	5.73
Asness et al. [8]	1972 - 2011	$\mathrm{ISPM}/12/6/33\%$	$5.8\%^{**}$	3.18

 Table 2.3: Profitability of The Momentum Investment Strategy - Overview

Model: ISPM = individual stock price momentum; CK = Conrad and Kauls methodology; 52WH = 52weekhigh. This table reports average monthly returns to the momentum investment strategies in markets worldwide. The returns documented are the highest statistically significant returns obtain in the respective studies. The list is not exhaustive, but illustrates of the usefulness of the momentum strategy in several markets and time periods. Other country-level studies have been conducted.

*) The model column contains information on the /Model used/Formation Period/Holding Period/ Portfolio size. For a complete description of the models applied, please see Section 2.2

**) Excess the risk free rate

***)Risk Adjusted

Behavioural Explanations

The behavioural-based explanations build on inherent investor biases. Jegadeesh and Titman [59] was one of the first studies to claim that investors are systematically biased, and that biases in part explain momentum returns. However, they didn't explain the biases to any further extend. Barberis et al. [10] specifically suggest investor conservatism as an explanatory bias. Conservatism makes the investor reluctant to update his believes immediately when facing new information. Consequently, there is an overlap period where prices don't reflect current information and some predictability exist. If the investor is facing good news, prices will rise slowly. If facing bad new, prices will fall slowly. The authors claim that momentum trading can be done profitably in the medium term.

In much the same way as Bondt and Thaler [18] use overreaction to explain the abnormal return in to the contrarian strategy, under-reaction to new information is one of the behavioural-based explanation as to why momentum investing creates abnormal returns. Daniel et al. [34] claim that investor self-attribution causes the investor to be overconfident in own private information, rather than public information. This in term causes the investor to over-react to new private information, but under-react to public news. Chan et al. [21] also support the under-reaction hypothesis. However, they claim that markets respond slowly to new information because the financial analysts covering the firms respond slowly to earnings announcements. I.e. their financial forecasts are slowly updated. They also find that this sluggish response to earnings announcements is especially true for the past worst performing firms. Hence, momentum is strongest for past losers.

Hong and Stein [51] develop a theoretical behavioural model around gradual diffusion of information. They imagine a market consisting of two types of investors; "news watchers" and "momentum traders". The "news watchers" ignore past price information and invest based on private information. The momentum traders invest solely based on past price information. The idea is that since private information only gradually diffuses in a population, "news watchers" causes a short-term delay in the price paths of the assets. Prices under-react and this causes momentum abnormal returns options. Moreover, "the momentum traders" invests based solely on past price information; this causes the prices to be pushed beyond/below their fundamental values, and thus "momentum traders" causes equilibrium through over reaction in the long run. A similar result can be found in Swaminathan and Lee (2001), where short-term price under-reaction is followed by long-term price over-reaction. Chui, Titman and Wei (2010) also find return reversal after 9-10 months that support these studies.

Hong et al. [52] empirically test the gradual information diffusion model. They find that firms with low analyst coverage are particularly exposed to slow diffusion. Thus, weak analyst coverage may lead to stronger momentum. They also find that bad firm specific information is more likely to diffuse slowly in the population. People and firms are more likely to share good news than bad.

Grinblatt and Han [44] empirically find that investors are prone to hold on to losing stocks and to sell winning stocks. This is called the disposition effect. Holding on to losers makes the investor under react to new information, and hence the study support the under-reaction hypothesis. Furthermore, they manage to capture this behaviour through unrealized capital gains, the difference between the share price and its cost base, as a variable. Adjusting for the unrealized capital gains, the momentum return disappears.

Seeing as the behavioural models claim that prices to some extent are predictable, they all oppose the market efficiency hypothesis (even the weak form). However, as is seen in the next section, abnormal momentum returns need not imply an inefficient market.

Factor-Based Explanations

Some academics claim that the presence of momentum in stock returns can be explained by different exposure to risk- and macroeconomic factors and is not inconsistent with rational pricing theories. Conrad and Kaul [31] for instance, present empirical evidence suggesting that the abnormal momentum payoffs are related to cross-sectional differences in expected returns. Since the expected return is dependent on relevant risk factors, the excess momentum return is merely a result of higher risk. Their conclusions are however contingent on the mean returns being constant during the periods in which the trading strategies are implemented. Berk et al. [13] develop a dynamic model of expected returns. In this model, risk factor variation creates time dependent cross-sectional differences in expected returns. Their results suggest that stocks with high-realized return will be those that have low expected return and vice versa. This way, time-dependent variation in expected returns is used to explain much of the abnormal momentum returns, and profitability of momentum strategies represents compensation for bearing time-varying risk.

Another study that supports time-varying expected returns as an explanation of momentum payoffs is Chordia and Shivakumar [26]. They argue that consistent and persistent under-reaction would provide low risk arbitrage opportunities to more rational investors. Their study finds that momentum return can be attributed to a set of macroeconomic variables that are related to business cycle. Dividend yield, default spread, yield on threemonth Treasury bills and term structure spread can predict time-varying cross-sectional differences in expected returns that is directly linked to past realized returns. In other words, the article finds evidence of systematic variation in momentum profits with respect to the above-mentioned macroeconomic variables.

Johnson [62] find a significant relationship between expected stock growth rates and recent performance. Seeing as expected growth rate is directly related to risk, this study also argues that the momentum anomaly (at least in part) can be explained by crosssectional variation in expected return because of exposure to risk factors. Again, this is because a momentum trader invests on basis of recent performance. The article doesn't aim to prove market inefficiency, just to present an alternative that does not hinge on the opposite. Along the same lines, Sagi and Seasholes [91] claim that firm specific attributes such as revenues, costs and growth options, combine to determine how the firms returns are auto-correlated. In other words, future return can be predicted through past returns and firm-specific attributes. A momentum strategy that incorporate this knowledge, reaps greater returns than the traditional momentum strategy. It should be noted that Jegadeesh and Titman (2001) attribute the findings of Conrad and Kaul (1998) to small sample biases in their empirical tests and bootstrap simulations. The results of Jegadeesh and Titman (2001) show that differences in expected returns don't explain momentum payoffs to a significant extent. Furthermore, Grundy and Martin (2001) look at CRSP data from 1924 to 1994 and find that adjusted for risk, a momentum strategy still proves abnormal returns. Hence, they provide evidence that the risk-based explanations are insufficient.

Market Friction Based Explanations

As previously mentioned, a lot of the literature related to momentum strategies builds on the models of Jegadesh and Titman (1993). Consequently, few momentum research studies include market frictions such as transaction costs in their models.

In their study, Lesmond et al. [73] expose that momentum trading involves high trading frequency and stocks with high associated transaction cost (returns are higher for high cost equities). As a result, the abnormal returns generated by a momentum strategy does not significantly exceed trading costs. Keim [66] also find that momentum strategies will generate trading costs close to the excess momentum payoffs. Furthermore, Keim claims that in order for the efficient market hypothesis to be challanged, the momentum strategy must generate abnormal returns in excess of the market after accounting for transaction costs. In this way, his findings support the efficient market hypothesis.

On the other hand, Korajczyk and Sadka [69] find that momentum strategies remain profitable even after considering trading costs. They also develop a trading model that includes liquidity as a factor, and show that the liquidity based momentum strategy performs better than the traditional equally weighted and value-weighted strategies after introducing trading costs.

Moreover, when Blitz et al. [16] examine the residual momentum strategy on US data in the time period from 1926 to 2009, they find the traditional momentum strategy to be unprofitable in the time period from 2000 to 2009. Similar results can be found in Hwang and Rubesam [56] where they investigate momentum strategies with structural breaks and expose that after the break in 2000, the profitability of momentum profits has disappeared.

2.3 The Contrarian and Low-Volatility Strategies

2.3.1 The Contrarian Strategy

Following a contrarian investment strategy, the investor (short)sells past winners and buys past losers. The idea is that the trend will reverse, and that the sign of the returns will become contrarian. The literature shows that a contrarian strategy can be done profitably both in the short-term (days, weeks of holding) and in the long-term (3-5 year holding).
One of the first studies to document long-term reversal was Bondt and Thaler [18]. They found that long term past losers outperform long term past winners over the subsequent 3-5 year period. With data from NYSE they formed two portfolios based on past returns; extreme winners and extreme losers. Over the next three year period, the losers outperformed the winners by 25% per year. This outperformance lasted for up to five years. Similar long term results can be found in Chopra et al. [25]. Looking at monthly CRSP NYSE data from 1926 to 1986, they found that in portfolios formed on the basis of prior five-year returns, extreme prior losers outperform extreme prior winners by 5% to 10% per year during the subsequent five years.

Jegadeesh [58] showed abnormal returns following a short term reversal strategy after finding serial correlation in monthly returns of CRSP individual securities data from 1934 to 1987. Through a strategy that buys and sell stocks on the basis of their prior month returns and holds them for a month, the article exposes profits of around 25% per year over the 53-year period. Lehmann [72] also finds what he claim to be evidence of market inefficiency through a short term reversal (weekly rebalancing, weekly holding) strategy on the CRSP data from 1962 to 1986.

Moreover, Knez and Ready [68] built a short term reversal strategy that switches between small and large cap firms based on previous weeks return on CRSP NYSE. This strategy was based on the findings of Lo and MacKinlay [74], namely that the return of a portfolio of small firm stock is strongly correlated with its own previous weeks return and with the previous weeks return on a portfolio of large-firm stock.

Overreaction to information is a common theory as to why contrarian strategies are profitable. The overreaction hypothesis states that individual investors overreact to news and consequently, the price of an asset moves away from its' fundamental value following new information. This initial movement caused by overreaction is then followed by a price reversal, a movement in the opposite direction. Profitable strategies can then be devised by buying past losers and selling past winners when the market is ready to correct (that is, when mean reversal takes place).

Modelling the contrarian strategy is commonly done with the method of Jegadeesh and Titman (1993), only with opposite investment logic and shorter/longer rebalancing periods.

2.3.2 Low-Volatility Investing

Few recent studies have been conducted with the contrarian strategy. This is not the case for the low-volatility investment strategy. A low-volatility investor invests in stocks that has either low historic volatility or low forecasted volatility relative to their peers. A familiar axiom in financial theory states that high returns are associated high risk. Several studies exist that contradicts this and show that investors may not be rewarded for bearing systematic⁵ risk.

⁵Systematic risk is the same as market risk, which generally describe the degree to which securites co-move with the market (Berk and DeMarzo [12]).

Although low-volatility investing has risen in popularity following the financial crisis in 2007-2008, the concept is not new. Jensen et al. [61] test the CAPM model on NYSE stocks in the time period from 1931 to 1965 and find that the excess returns to these stocks are not strictly proportional to their betas, meaning that low beta stocks may perform better than high beta stocks. Since the beta captures the individual securities sensitivity to market risk, this means that low beta stocks can be seen as low-volatility stocks⁶. Haugen and Baker [49] further document that low beta stocks outperform market capitalization weighted portfolios. They use data with the 1000 largest US stocks in the time period from 1972 to 1989. They form low variance portfolios by each period choosing 100-150 stocks of the stocks with the lowest past volatility. Clarke et al. [29] use a optimization approach to construct minimum variance portfolio also with the 1000 largest stocks in the US CRSP database in the time period from January 1968 to December 2005. They find that these portfolios have 75% the realized risk of the market capitalization weighted index and comparable or higher returns, something which confirms some of the findings of Haugen and Baker [49].

Blitz and Van Vliet [15] investigates stocks in US, European and Japanese markets in the time period from December 1985 to January 2006. Their low-volatility model ranks stocks constituent of the FTSE World Developed index each month based on last three years volatility of monthly returns. Portfolios are formed of the top decile with equally weighting. They use monthly rebalancing and cite monthly returns. Their findings suggest that stocks with low historical volatility exhibit superior risk adjusted returns similar in size to the more familiar size and momentum factors. Pedersen and Lasse [83] reported similar findings with developed country data from the time period 1984 to 2009.

The explanations as to why low-volatility investing may be more profitable than holding the market are several, most of them behavioral. Baker et al. [9] claim that investors behave irrationally and use high-volatility stocks as lottery tickets. Investors does not mind a lower expected return, if they have a change to 'win the lottery' with high-risk securities. Investor overconfidence is another explanation. Investors tend to believe that their abilities to forecast the future are superior to those of others. Their view deviates more for high-risk securities and at the same time, it is easier to express a positive view of the future. The results is overpricing of high-volatility stocks, and higher returns to low volatile stocks. Another explanation along the same lines is given by Hsu et al. [54], who believe that analysts portray a too positive view of high-volatile stocks. This causes overpricing and lower returns relative to low volatile stocks.

There are mainly two ways of modelling low-volatility strategies found in the literature[11]. Those models that invests in some proportion of stocks ranked based on historical or forecasted volatility⁷, and those who use optimization techniques to find minimum variance portfolios. In this work, we create a model based on the former. Our model closely follows that of Blitz and Van Vliet [15] and will be introduced in Chapter 5. The SP500 volatility index is also modelled along these lines, only difference is that they weight the stocks in their portfolios based on the inverse of the historical volatility and the portfolio sizes are larger[11].

⁶For this reason low-volatility stocks are sometimes referred to as low beta stocks.

⁷Much like the momentum strategies previously introduced.

Chapter 3

Portfolio Theory in a Single Stage Setting

A portfolio is the mix of financial assets held by an investor. Portfolio optimization is the process of selecting, from a set of available instruments, the subset of those which in aggregate best achieve some objective under given constraints. This section presents an overview over portfolio theory in a single stage setting, which studies such allocation of wealth among a set of investment instruments. The objectives and constraints should properly reflect the investors' aims, preferences and attitude towards risk. Portfolio optimization provides a mathematical framework in which to encode these objects.

At the heart of portfolio optimization is the balancing of portfolio risk and reward. Section 3.1 introduces the concept of portfolio reward and is followed by a section on popular and commonly applied quantitative measures of risk. The foundations of modern portfolio theory were laid by Markowitz [77]. His static, one-stage approach is presented in Section 3.3 together with portfolio optimization models utilizing other, linear risk measures. Portfolio optimization in a two stage stochastic setting is introduced later in Chapter 4, and will be implemented algorithmically on portfolios of momentum and contrarian assets in the computational part of this thesis.

3.1 Portfolio Reward

One of the main characteristics that describe investor behavior is the greediness. Any rational investor wants to maximize the expected reward of her portfolio if other constraints are given. The portfolio reward can be seen as an aggregate of the rewards of the individual financial assets in the portfolio. Asset returns is a measure of such reward and represents the value development of the asset over some time-span. Another measure is the individual asset value¹. A distinction is made between portfolio value and return, despite the trivial relation between them, because of the inherent differences in statistical

¹The reason as to why we use value instead of the more commonly used price, is that our portfolio will contain complex assets consisting of several stocks with individual prices.

properties in their distributions. Whereas any statistical model used to describe asset values would have to yield non-negative values only, this is not the case for asset returns. Also, it is commonly accepted that returns follow a stationary process, i.e. they revolves around some mean with a constant variance [67, 90].

To describe returns, consider a universe \mathcal{I} of n financial instruments. With ω_i being the value of the investment in asset $i \in \mathcal{I}$ at time t, the individual asset return over a time-span from time t to time t+1 may be taken as

$$r_i = \frac{\omega_{it+1}}{\omega_{it}} - 1 \tag{3.1}$$

This measure is commonly noted simple return. Another measure of asset return, is the log return $\ln(\mathbf{r})_i$

$$ln(r)_{i} = ln(r_{i}+1) = ln(\omega_{it+1}) - ln(\omega_{it})$$
(3.2)

Log return corresponds to the continuously compounded rate of the simple returns. In this work we use simple return as measure of reward. From the relationships in equation 3.1 and 3.2, we note that optimizing with respect to simple return and log return is equivalent. However, since the log of a sum is not equal to the sum of logs, there is a difference in the calculation of the portfolio return that makes simple returns favorable in the sense that the portfolio expected return can be calculated simply as a linear combination of the returns of each constituent asset². Rudoy [90] argues that using log returns as a return measure is favorable since the multi-period return becomes additive and not multiplicative. I.e. for N periods the total return is given by

$$R_{tot} = \prod_{t=1}^{N} (1+r_{it}) - 1 \tag{3.3}$$

for simple returns and

$$ln(R)_{tot} = ln(1 + R_{tot}) = \sum_{t=1}^{N} (ln(\omega_{it+1}) - ln(\omega_t))$$
(3.4)

for log returns. In order to calculate total portfolio returns, however, Rudoy [90] assumes that the following Taylor series expansion holds

$$ln(1+r_{it}) \approx r_{it} \tag{3.5}$$

This approximation is only valid for returns close to zero, a common assumption for daily returns of stocks[3]. In this work, however, we optimize portfolios with monthly returns on complex assets and as such, a mean close to zero is not a valid assumption. For this reason, simple returns are chosen to represent portfolio reward.

At each point in time, the future return of each asset is considered a random variable \tilde{r}_i with expected value \bar{r} . In this thesis, as in any scenario based portfolio optimization scheme³, we assume that the random variable asset return \tilde{r}_i may be taken from plausible

²This is evident in equation 3.7

³Scenario optimization refers to optimization models that apply when the uncertain data are represented by a set of discrete scenarios[97]

sets, e.g. $\{r_i^l\}_{l\in\Omega}$, where $l\in\Omega$ are indexes of possible scenarios with probability of realization p^l . In such a setting, the expected return of asset i is given by

$$\bar{r}_i = \sum_{l \in \Omega} p^l r_i^l \tag{3.6}$$

The portfolio return is a function of the individual asset allocation x_i and individual asset return r_i^l and is then given by

$$R_p(x; r^l) = \sum_{i \in \mathcal{I}} x_i r_i^l \tag{3.7}$$

in each scenario l. The total expected portfolio reward can be found by

$$\bar{R}_p = \sum_{i \in \mathcal{I}} x_i \bar{r}_i \tag{3.8}$$

In Chapter 4, these measures are extended to a two stage setting in which we have different plausible sets for each stage, possibly depending on the previous set.

3.2 Quantitative Risk Measures and Coherency

From the equations in the previous section it follows that portfolio returns are uncertain. Risk can be understood as the variability of the portfolio returns due to market changes and uncertain, unforeseeable events. In this section we introduce means by which to quantity risk inherent in financial markets using appropriate risk metrics. Artzner et al. [6] argue that a good risk measure should display the property of coherency. Furthermore, in this work we need risk measures suited for an institutional investor. We do not want to put constraints the upside variability in our portfolio optimization models. After first presenting the variance risk metric introduced to portfolio management by Markowitz [77], we will present other more sensible risk metrics. Among which are the conditional value at risk and mean negative absolute deviation. One of which, the portfolio conditional value at risk, is coherent. Both of which put constraints on downside deviations when used in a portfolio optimization scheme and are suited for enterprise wide risk management[97].

3.2.1 Coherent Risk Measures

In Zenios [97] it is argued that risk measures that are to support decision making for financial institutions should display the property of coherency. The following definition gives an axiomatic characterization of the property[97, 6].

Definition 1. A coherent risk measure ρ is a function that assign numbers $\rho(\widetilde{X})$, $\rho(\widetilde{X})$ to two random variables \widetilde{X} and \widetilde{Y} such that for any pair \widetilde{X} , \widetilde{Y} , independent or not, and for each positive number $a, b \in \mathbb{R}$ the following relations hold.

- (1) Sub-Additivity: $\rho(\widetilde{X} + \widetilde{Y}) \leq \rho(\widetilde{X}) + \rho(\widetilde{Y})$ (2) Homogeneity: $\rho(a\widetilde{X}) = a\rho(\widetilde{X})$ (3) Monotonicity: $\rho(\widetilde{Y}) \leq \rho(\widetilde{X})$ if $\widetilde{X} \leq \widetilde{Y}$
- (4) **Risk-free condition**: $\rho(\widetilde{X} br_f) = \rho(\widetilde{X}) b$

The sub-additivity axiom in (1) states that the risk of two separate positions is less than or equal to the risk associated with a portfolio holding the two assets. This axiom ensures that the principle of diversification holds for the risk metric in question. Unless two assets in a portfolio is perfectly [positivly] correlated, there will be a diversification effect in holding both assets if the risk measure is coherent. Further, the homogeneity axiom in (2) states that if we increase our holding in one asset with a given factor, the risk increases with the same factor. This, together with sub-additivity, implies that the risk measure is convex [86]. I.e. from (1) and (2) it follows that

$$\rho(\alpha \widetilde{X} + (1 - \alpha)\widetilde{Y}) \le \alpha \rho(\widetilde{X}) + (1 - \alpha)\rho(\widetilde{Y})$$
(3.9)

In a portfolio optimization setting, axiom (3) states that if for all realizations, the return \widetilde{X} is worse than \widetilde{Y} , then the risk associated with holding a portfolio whose return distribution is given by \widetilde{Y} should be lower than holding a portfolio with return distribution \widetilde{X} . The last axiom, the risk free condition in (4), simply states that adding a risk free position b to portfolio should decrease the risk by the same amount, b.

One might argue that the coherency framework does not always ensure a risk metric that makes sense. Imagine, for instance, a portfolio of several different loans issued to the same customer. Clearly, the sub-additivity axiom shouldn't hold because there should be no clear diversification effect. On the contrary, the risk metric should then reflect that the risk of default increases as the number of loans to the same borrower increases. One could also question whether or not the risk free amount b in (4) would make a non diversified portfolio of say high yield bonds any less risky. These criticisms aside, the coherency framework is widely accepted to ensure a sensible risk metric in many cases[97].

3.2.2 Variance of Portfolio Returns

One of the most widely known measures of risk is the portfolio variance. Variance is a measure of the deviation of random values from the mean and was first introduced to portfolio management by Markowitz [77]. Given the covariance between each asset $i,j \in \mathcal{I}$, σ_{ij} , and the individual variance of each asset, σ_i^2 , we may define the variance of the portfolio as

$$\begin{aligned}
\sigma_p^2 &= \mathbb{E}((R_p - \mathbb{E}(R_p))^2) \\
&= \mathbb{E}(\sum_{i \in \mathcal{I}} x_i \widetilde{r}_i - \sum_{i \in \mathcal{I}} x_i \mathbb{E}(\widetilde{r}_i))^2) \\
&= \mathbb{E}(\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I} \mid i \neq j} x_i x_j (\widetilde{r}_i - \mathbb{E}(\widetilde{r}_i)) (\widetilde{r}_j - \mathbb{E}(\widetilde{r}_j))) \\
&= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} x_i x_j \sigma_{ij}
\end{aligned}$$
(3.10)

or in a discrete scenario setting with scenarios $l \in \Omega$

$$\sigma_p^2 = \frac{1}{1 - \sum_{l \in \Omega} (p^l)^2} \sum_{l \in \Omega} p^l (R_p(x; r_p^l) - \bar{R}_p(x; \bar{r}_p))^2$$

$$= \frac{1}{1 - \sum_{l \in \Omega} (p^l)^2} \sum_{l \in \Omega} p^l (\sum_{i \in \mathcal{I}} x_i r_i^l - \sum_{l \in \Omega} \sum_{i \in \mathcal{I}} p^l x_i r_i^l)^2$$
(3.11)

Despite its popularity among practitioners, the variance risk metric has some draw-downs. The most eminent of which is that it penalizes both upside deviations as well as downside deviations. Hence, large profits is considered equally risky as large losses. In this regard, it should be noted that Markowitz himself later proposed the use of portfolio semi-variance as a risk measure focusing on downside deviation[78]. Specifically, semivariance is the variance of the deviations below the mean. Neither semi-variance nor variance are coherent measures of risk and both require quadratic programming (QP) solvers when applied in a portfolio optimization model.

3.2.3 Value at Risk

Value at risk (V@R) is another risk metric widely used in the financial industry, that does not penalize upside variability. Loosely speaking, the portfolio V@R is a threshold for portfolio loss we can fairly certain that we do not exceed. More precisely stated; it's the loss ζ that will not be exceeded over some investment time-span with a probability level 1- β . Several other definitions can be found in the literature⁴. Following is a mathematical formulation⁵:

⁴For example, Gaivoronski and Pflug [40] defines the V@R as 'the largest under-performance relative to expected portfolio return that is possible in $1 - \beta$ cases of the outcomes.'

⁵Own formulation based on Zenios [97] and Rockafellar and Uryasev [86]

Understand by $L(x; \tilde{r})$ the portfolio loss function associated with wealth allocation x and random return realization \tilde{r} such that

$$L(x:\tilde{r}) = -R_p(x;\tilde{r}) \tag{3.12}$$

The probability that the loss function does not exceed some threshold value ζ is then given by

$$\Psi(x,\zeta) = \int_{L(x,\tilde{r}) \le \zeta} \rho(\tilde{r}) d\tilde{r}$$
(3.13)

where $\rho(\bullet)$ denotes the probability density function of \tilde{r} . With a confidence level 1- β we may now define the portfolio value at risk as

$$V@R(x,\zeta) = \min\{\zeta \in \mathbb{R} | \Psi(x,\zeta) \le \beta\}$$
(3.14)

In the case of a discrete scenario setting as previously introduced, the loss function takes the form

$$\Psi(x,\beta) = \sum_{l \in \Omega | L(x,p^l) \le \zeta} p^l$$
(3.15)

and the loss function is defined by

$$L(x:p^{l}) = -R_{p}(x;p^{l})$$
(3.16)

Portfolio returns are as defined in Section 3.1. The V@R risk measure is intuitive to understand and does not penalize upside variability. It is, however, not a coherent risk measure because it does only satisfy axioms (2), (3) and (4). V@R does not display the characteristic of sub-additivity[6]. This implies a risk measure that does not properly give diversification effects. Further, as sub-additivity is one of the preconditions for convexity, V@R may not be convex and can therefore be difficult to optimize due to being nonsmooth and possibly having multiple local minima[40]. For these reasons, V@R will not be applied in this thesis.

3.2.4 Conditional Value at Risk

Conditional Value at Risk (cV@R) is a coherent risk measure derived from the portfolio V@R (Artzner et al. [6]). Rockafellar and Uryasev [86] defines cV@R as the expected value of portfolio losses, conditioned on the losses being in excess of V@R. Following the notation in the previous section, cV@R may then be expressed mathematically as

$$cV@R(x,\beta) = \frac{1}{1-\beta} \int_{V@R(x,\tilde{r}) \le L(x,\tilde{r})} \rho(\tilde{r})L(x,\tilde{r})d\tilde{r}$$
(3.17)

Furthermore, suited for discrete stochastic programming with scenarios $l \in \Omega$ we have (Zenios [97])

$$V@R(x,\beta) = \mathbb{E}[L(x,p^{l})|\zeta \leq L(x,p^{l})]$$

$$= \frac{\sum_{\substack{l \in \Omega | L(x,p^{l}) > \zeta}} p^{l}L(x,\zeta)}{\sum_{\substack{l \in (x,p^{l}) > \zeta}} p^{l}}$$

$$= \frac{\sum_{\substack{l \in \Omega | L(x,p^{l}) > \zeta}} p^{l}L(x,\zeta)}{1-\beta} | \Psi(x,\zeta) = \beta$$
(3.18)

From this it follows that

$$V@R(x,\beta) \le cV@R(x,\beta) \quad \forall x,\beta \tag{3.19}$$

Where x is the allocation vector and β is the percentile parameter. In other words, cV@R is a more conservative risk metric than V@R. If the portfolio cV@R is restricted to some value ϖ , then it follows from equation 3.19 that the portfolio V@R is less than ϖ . Further, a portfolio optimization model utilizing cV@R as a risk metric actually considers the impact of losses *in excess* of V@R. This is obviously not the case for V@R. It is also of great practical significance that cV@R optimization models can be solved using linear programming software. In the following we derive how. The result is used in a portfolio optimization model with cV@R constraints in Section 3.3.3.

In Rockafellar and Uryasev [86] it is proved that the following function is convex and continuously differentiable with respect to ζ

$$\Xi(\beta, x, \zeta) = \zeta + \frac{1}{1 - \beta} \int_{\widetilde{r} \in \mathbb{R}} [L(x, \widetilde{r}) - \zeta]^+ \rho(\widetilde{r}) d\widetilde{r}$$
(3.20)

With []⁺ denoting a function that takes the value $[L(x,\tilde{r})-\zeta]$ when $[L(x,\tilde{r})-\zeta] > 0$ and 0 otherwise. In addition it is derived that

$$\min_{x} cV@R(x,\beta) = \min_{x,\zeta} \Xi(\beta, x, \zeta)$$
(3.21)

Equation 3.21 implies that a mean-risk model minimizing $\Xi(\beta, x, \zeta)$ is equivalent to a mean-risk model minimizing cV@R(x, β). Since $\Xi(\beta, x, \zeta)$ is convex, this means that mean-cV@R optimization models can be solved with respect to a convex function. This makes cV@R considerably easier to handle than V@R, even though V@R is a part of the cV@R formulation [97]. In a discrete scenario setting, it is possible to approximate $\Xi(\beta, x, \zeta)$ in a number of ways. If the probabilities p^l of each scenario $l \in \Omega$ are of different magnitude⁶ the approximation may be given by

$$\Xi(\beta, x, \zeta) \approx \widetilde{\Xi}(\beta, x, \zeta) = \zeta + \frac{1}{1 - \beta} \sum_{l \in \Omega} p^l [L(x, p^l) - \zeta]^+$$
(3.22)

From this approximation is follows that if the loss function $L(x,p^l)$ is linear, then $\tilde{\Xi}(\beta, x, \zeta)$ is piece-wise linear. A mean-cV@R model may then be solved with a conventional LP solver[97]. Such a mean-cV@R optimization model is presented in Section 3.3.3.

⁶In Rockafellar and Uryasev [86] a formulation with equal probabilities is given

3.2.5 Mean Negative Absolute Deviation

A Mean Absolute Deviation (MAD) risk model is risk metric that penalizes deviations of the portfolio return from the mean. Provided that the reward function $R(x;\tilde{r})$ is linear, the penalty function takes the form of a linear function of the absolute value of the deviations.

$$MAD = \mathbb{E}[|R(x;\tilde{r}) - R(x;\bar{r})|]$$
(3.23)

Equation 3.23 describes a risk model that considers equally undesirable both downside and upside deviations in much the same way as portfolio variance. As mentioned, this poses a problem for an institutional investor as he is likely to consider upside deviations a good. An alternative to the MAD model that only penalizes downside deviations is the Mean Negative Absolute Deviation Model (MNAD). MNAD can be described by the following (Zenios [97]).

$$MNAD = \mathbb{E}[max[0, R(x; \bar{r}) - R(x; \tilde{r})]]$$
(3.24)

and with N scenarios $l \in \Omega$

$$MNAD = \frac{N}{N-1} \sum_{l \in \Omega} p^{l} max[0, \sum_{i=1}^{N} (\bar{r}_{i} - r_{i}^{l})x_{i}]$$
(3.25)

An advantage of a MNAD based portfolio optimization model is that is may be solved with a conventional LP solver. These solvers can solve large scale linear problems efficiently and are readily available. On the other hand, MNAD is not a coherent risk measure as it does not satisfy the risk-free (4) and monotonicity (3) axioms in Section 3.2.1. By implementing both cV@R and MNAD based portfolio optimization problems in this thesis, we get to compare what is a coherent risk measure and a non-coherent risk measure that both consider downside variability in a asset universe with momentum and contrarian investment algorithms.

3.3 Single Stage Portfolio Optimization

With the concepts of portfolio risk and portfolio reward introduced in the previous sections, we can now describe mathematical models of single stage portfolio optimization. The classical Mean-Variance model from the seminal work of Markowitz [77] is first introduced. Thereafter, we show how efficient portfolios formed with stocks from Oslo Stock Exchange, can be seen graphically with the efficient frontier. In the end of this section, portfolio optimization models with cV@R and MNAD constraints are presented.

3.3.1 Mean-Variance Optimization

Mean-Variance models addresses the problem of optimally selecting portfolios by allocating capital between a set of financial instruments with respect to reward versus risk in a single stage setting. The risk is measured by the portfolio variance and the reward is measured by the mean return of the portfolio. The goal is to maximize the portfolio mean given a constraint on the portfolio variance, or equivalently, minimize the risk with a constraint on the portfolio mean return.

To mathematically describe the problem, we assume the position of an investor that has access to a universe of \mathcal{I} assets. The random return of each asset i is denoted by \tilde{r}_i and is modeled on a probability space $(\Omega, \mathcal{F}(\Omega), \mathbb{P})$, where \mathbb{P} is the probability measure on the σ -algebra of events resolved at the end of the period, \mathcal{F} . Ω is the set of return scenarios l each with probability $p(r_i^l)$. The investor wants to optimally decide the proportional allocation x_i to the i'th assets such that

$$\sum_{i\in\mathcal{I}}x_i=1\tag{3.26}$$

Equation 3.26 is commonly denoted the *budget constraint*. Furthermore, we assume that short positions are disallowed, i.e.

$$0 \le x_i \tag{3.27}$$

Two equivalent mathematical formulations of the Mean-Variance portfolio problem are now given by (Zenios [97])

$$\begin{array}{ll} Maximize & R_p(x;\bar{r})\\ s.t. & \sigma^2(x) \le \beta\\ & \sum_{i\in\mathcal{I}} x_i = 1\\ & 0 \le x_i \quad \forall i\in\mathcal{I} \end{array}$$
(3.28)

$$\begin{array}{lll}
\text{Minimize} & \sigma^2(x) \\
\text{s.t.} & R(x; \bar{r}) \ge \mu \\
& \sum_{i \in \mathcal{I}} x_i = 1 \\
& 0 \le x_i \quad \forall i \in \mathcal{I}
\end{array}$$
(3.29)

Where β and μ denotes the upper and lower bound on risk and expected return, respectively. Since the variance is a quadratic function of the allocation vector x, neither of these models are linear. The second formulation, 3.29, involves a quadratic function with linear constraints only. This model can be solved using standard optimization software[97].

3.3.2 The Efficient Frontier

Solving the Mean-Variance models with given values for the parameters β and μ results in portfolios with maximum return given constraint on risk, or equivalently, minimum variance given lower bound on mean return. These portfolios are commonly known as *efficient portfolios*.

Definition 2. A portfolio is efficient if it has maximal expected return given an upper by ound on risk or, equivalently, it has minimal risk for a given expected return [12].

By changing the weights on each asset we obtain portfolios with different combinations of risk and reward. Further, by solving the Mean-Variance models with different values for the parameters β and μ we obtain different optimal portfolios. The [mean variance] *efficient frontier* is defined by the optimal portfolios x_{opt} for all parameters β and μ that yield feasible solutions for the portfolio problems given in 3.28 and 3.29, respectively. The efficient frontier can be represented graphically by plotting the set of optimal portfolios in a risk-reward space.

To illustrate this phenomenon, we randomly pick 8 stocks from the set of equities listed on Oslo Bors in the time period from 01.01.2012 to 31.08.2015. Their mean returns, volatilities (standard deviation) and total returns over the period are displayed in table 3.1. Next, we solve the Mean-Variance problem given in 3.29 with different values of the parameter μ . The implementation is done in Python with the Gurobi optimization library and Bokeh library for plotting. The resulting mean variance efficient frontier is displayed together with the individual stocks, in figure 3.1 as the blue upper half hyperbola in the mean-standard deviation space. The efficient frontier can also be found in other risk-reward spaces.

Randomly Chosen Equities on Oslo Stock Exchange				
Company	Ticker	Mean	St.Dev.	Tot.Return
Belships ASA	BELS	0.009%	2.40%	-16%
Seadrill Ltd	SDRL	-0.094%	2.26%	-68%
Eidesvik Offshore ASA	EIOF	-0.086%	1.85%	-62%
Lerøy Seafood Group ASA	LSG	0.165%	1.82%	300%
Havila Ariel ASA	HAVA	0.034%	1.10%	32%
Aktiv Kapital ASA	AIK	0.014%	0.67%	12%
DNB Bank ASA	DNB	0.010%	1.56%	127%
Hafslund ASA	HNA	0.030%	1.30%	21%

Table 3.1: Performance of Stocks on the Oslo Stock Exchange

Notes: this table shows the mean return, standard deviation and total period return for eight stocks listed on the Oslo Stock Exchange in the time period from 01.01.2012 to 30.08.2015. The stocks may not have been listed throughout the entire period.



Figure 3.1: The blue line displays the mean variance efficient frontier with stocks from Oslo Stock Exchange

3.3.3 Other Mean-Risk Models

In the following we present single stage portfolio optimization models based on the cV@R and MNAD risk measures introduced in Section 3.2. Both models will be formulated in a linear programming fashion making them solvable with conventional LP solvers. These models are extended to a two stage setting in the next chapter.

Mean-CV@R Optimization Model

In Section 3.2.4 we saw that minimizing the following function with respect to ζ and x is equivalent to minimizing the cV@R of the portfolio with respect to x.

$$\Xi(\beta, x, \zeta) = \zeta + \frac{1}{1 - \beta} \int_{\widetilde{r} \in \mathbb{R}} [L(x, \widetilde{r}) - \zeta]^+ \rho(\widetilde{r}) d\widetilde{r}$$

Furthermore, we showed that in a discrete scenarios setting $\Xi(\beta, \mathbf{x}, \zeta)$ can be approximated by $\widetilde{\Xi}(\beta, \mathbf{x}, \zeta)$.

$$\Xi(\beta, x, \zeta) \approx \widetilde{\Xi}(\beta, x, \zeta) = \zeta + \frac{1}{1 - \beta} \sum_{l \in \Omega} p^l [L(x, p^l) - \zeta]^+$$

If we introduce an auxiliary variable η^l that takes the value $[L(x,p^l)-\zeta]$ when $[L(x,p^l)-\zeta] > 0$ and 0 otherwise, the following linear programming model maximizes expected port-

folio return with a restriction on portfolio cV@R in a single stage setting.

$$\begin{aligned} Maximize \quad & \sum_{l \in \Omega} \sum_{i \in \mathcal{I}} p^{l} x_{i} r_{i}^{l} \\ s.t. \quad & \zeta + \frac{\sum_{l \in \Omega} p^{l} \eta^{l}}{1 - \beta} \leq \omega \\ & 0 \leq \eta^{l} \quad \forall l \in \Omega \\ & L(x; p^{l}) - \zeta \leq \eta^{l} \quad \forall l \in \Omega \\ & \sum_{i \in \mathcal{I}} x_{i} = 1 \\ & 0 \leq x_{i} \quad \forall i \in \mathcal{I} \end{aligned}$$
(3.30)

With notations as in the rest of this chapter. $\sum_{i \in \mathcal{I}} x_i = 1$ is the budget constraint and $x_i > 0$ denotes a short positions disallowed constraint as before. In model 3.30 the portfolio V@R is endogenously given. V@R is thus a variable of the model.

Mean-MNAD Optimization Model

In Section 3.2.5, the MNAD risk measure was described by the following equation in a discrete scenario setting.

$$MNAD = \frac{N}{N-1} \sum_{l \in \Omega} p^l max[0, \sum_{i=1}^{N} (\bar{r}_i - r_i^l)x_i]$$

Denote by v^l an auxiliary variable that takes the value $\left[\sum_{i=1}^{N} (\bar{r}_i - r_i^l)\right]$ when $\left[\sum_{i=1}^{N} (\bar{r}_i - r_i^l)\right] > 0$, and 0 otherwise. A single stage portfolio optimization model with MNAD constraints is then by given by

$$\begin{aligned} Maximize \quad & \sum_{l \in \Omega} \sum_{i \in \mathcal{I}} p^l x_i r_i^l \\ s.t. \quad & \sum_{l \in \Omega} p^l v^l \leq \gamma \\ & 0 \leq v^l \quad \forall l \in \Omega \\ & \sum_{i \in \mathcal{I}} x_i [\bar{r}_i - r_i^l] \leq \gamma \quad \forall l \in \Omega \\ & \sum_{i \in \mathcal{I}} x_i = 1 \\ & 0 \leq x_i \quad \forall i \in \mathcal{I} \end{aligned}$$
(3.31)

With notations as in the rest of this chapter. Again, $\sum_{i \in \mathcal{I}} x_i = 1$ is the budget constraint and $x_i > 0$ denotes a short positions disallowed constraint. The efficient frontier can easily be constructed in the mean-cV@R and mean-MNAD spaces by solving model 3.30 and 3.31 for different values of the parameters v and ω . The efficient frontier is an important concept that can be used for understanding theoretic risk and return behavior, this will be done in Chapter 7.

Chapter 4

Two-Stage Stochastic Portfolio Optimization

So far our analysis has been restricted by the assumption of a one-stage investment horizon. More realistically, however, one should consider the opportunity for the investor to rebalance her portfolio when random events realize. One interesting questions in this thesis is whether or not such consideration would yield any additional gains for an investor with access to a complex asset universe. A suitable mathematical framework for multistage portfolio optimization should not only allow the investor to anticipate future observations, but also allow adaptation to realized events by taking rebalancing decisions. Such a portfolio optimization model is part of a class of stochastic programming models known as *recourse models*[97].

In this work, we consider discrete scenario based optimization where the random variables \tilde{r} have a finite number of realizations for each stage. We can then visualize the possible sequence of events in a scenario tree structure. Formally, a scenario tree may be defined as a directed graph $\mathcal{G} = [\sum, \Lambda]$ where the nodes \sum represent a return realization at a given time, and the links Λ denote the set of possible sequential nodes. That is, \sum is a set of possible return realizations at time t such that $\sum[t] = \{N_v^t | v = 1, ..., \mathcal{N}_t\}$ where \mathcal{N}_t denotes the total number of realizations at any given time t. Λ is the set of pair of nodes that can feasibly follow each other $[N_{v(t)}^t, N_{v(t+1)}^{t+1}]$. $v(\cdot)$ indicates the dependence of the index v on t. In other words, $N_{v(t)}^t$ is a predecessor and $N_{v(t+1)}^{t+1}$ is a successor. Figure 4.1 depicts a scenario tree and for clarity we adapt the following definition from Gülten and Ruszczyński [47].

Definition 3. Scenario Tree

A scenario is a path from root node to leaf node A stage is the moment when a decision is taken A period is a time interval between two stages

Each node in the scenario tree is associated with a return realization \tilde{r}_n^t , which in our setting represents the end of period portfolio return. For this reason we need the under-



Figure 4.1: This figure displays the nodes in a scenario tree. Each node N_v^t is associated with a probability ρ and a return realization \tilde{r}_n^t .

lying stochastic process to be taken from a discrete distribution with a limited number of realizations. In Section 4.1, we introduce the concept of scenario tree generation. Furthermore, the moment-matching scenario generation heuristic applied in this thesis is thoroughly presented. Once the discrete distributed is presented in a scenario tree, we can solve stochastic optimization problems with appropriate programming techniques. Section 4.2 shows how the linear portfolio optimization models introduced in Chapter 3 can be extended to a two-stage setting.

4.1 Scenario Generation - Setting The Stage

In stochastic programming, scenario generation refers to the generation of a discrete approximation of the distribution of a random variable. In discrete scenario based portfolio optimization, we want to approximate the distribution of the constituent assets random returns and we want the end result to be in the form of a scenario tree. We understand that financial data alone may be insufficient in this regard as it is necessary to generate scenarios that may not have been present in the past, but that are possible according to statistics. Financial instruments can be risky and it is therefore important to model the possibility of occurrence of extreme events. Furthermore, the investor may have opinions of the future outcomes that deviate from those of the past. An adequate representation of the underlying random processes and their co-movement are of crucial importance in proper two-stage stochastic modelling.

In most practical financial applications, the distributions of the variables have to be approximated by discrete distributions with a limited number of outcomes [65]. Denote by \tilde{r} the true distribution of a random stochastic variable and by \check{r} an approximation. Furthermore, let \check{r} denote the multivariate approximation of the random process. For portfolio

optimization models, scenario approximation facilitates the following transformation

$$\max_{\boldsymbol{x}} \quad F(\boldsymbol{x}, \widetilde{\boldsymbol{r}}) \quad \longrightarrow \quad \max_{\boldsymbol{x}} \quad F(\boldsymbol{x}, \widetilde{\boldsymbol{r}}) \tag{4.1}$$

From 4.1 it is easy to understand that several academics claim that the scenario generation technique should be measured by its ability to facilitate an optimal approximate solution that is close to the true optimal solution, rather than by the closeness of \breve{r} to \tilde{r} (Hurt et al. [55], Kjetil Fagerholt).

In Kaut and Wallace [65] and Hurt et al. [55] a variety of scenario generation methods are reviewed and tested. *Conditional sampling*, a technique where samples are drawn from a stochastic process at each node in the scenario tree, is the most common. This method is straight forward if the underlying process is univariate. The sampling can be done either directly from a given distribution or from a return predicting model. For multivariate purposes we additionally need to model the correlation. This is explained in e.g. Loretan [75]. Similar to this method, it is also possible to generate scenarios by drawing samples from given specified marginal distributions with a given correlation matrix [76]. The problem with this method, of course, is that we may not know the given marginal distributions of the assets¹. Other methods exist, e.g. the *path based method* (Kaut [64]) or *optimal discretization* (Pflug [84]). These are somewhat different from the moment-matching scenario generation method implemented in this work.

Note that all these presently available means by which to generate discrete scenarios from continuous distributions (or large discrete distributions) are heuristics. This means that, however small, there will always be an error $\epsilon > 0$ between the true optimal value of a given two-stage portfolio optimization model and the model obtained by approximating the scenario outcomes in a scenario tree structure[63]. In this work, we apply a heuristic for scenario generation introduced to stochastic programming by Høyland et al. [53]. This moment-matching scenario generation technique is presented in the following section and builds around the ability of the investor to specify her expectations of the future in terms of four moments of each asset marginal distribution and the correlation matrix between the assets. No assumptions on the particular marginal distributions are made. This way, the future expectations may in whole or in part be based on past historical data and experts may be consulted for their opinions. To our knowledge, this heuristic has never before been implemented with momentum and contrarian algorithms as assets in a portfolio optimization setting.

4.1.1 A Moment-Matching Scenario Generation Algorithm

In this section we introduce the moment-matching scenario generation algorithm by Høyland et al. [53]. The algorithm can be used to generate multistage scenario trees for a multivariate stochastic process and it is previously tested for financial application. The first part of this section is confined to the sub-problem of generating a limited number of scenarios for a multivariate stochastic process with given moments and correlations

¹This is certainly the case for the special portfolio optimization problem in this thesis where the assets are investment algorithms. The moment-matching scenario generation introduced in the next section is more suited for such a purpose.

in a single node of a scenario tree. The moments and correlations may be taken from historic data or appropriate forecasting and simulation models. These approximations may then in whole or in part be combined with expert knowledge and opinion. No assumption is thus made on the distribution of the underlying assets. This is indeed favorable for our situation in which limited or no research has been done on the return distribution of the underlying investment algorithms. In the second part we show how the moment-matching algorithm easily can be exploited to produce a two (or several) stage scenario tree. We draw upon theory introduced in Høyland et al. [53]. The algorithm and associated mathematics can be seen as a bit involved. For this reason we only introduce the main mathematical concepts used and assume that basic mathematics is familiar by the reader. The main mathematical concepts are a cubic transformation for moment-matching and a Cholesky transformation for correlation matching.

Let \tilde{r} denote the discrete multivariate distribution that is to be generated with four prescribed moments for each marginal and an associated given correlation matrix at each node. Then \tilde{r}_i describes a one dimensional random vector with one index for each possible scenario $s \in S$. Each scenario has an associated probability of occurrence $\rho(s)$. The prescribed four moments for each marginal $i \in \mathcal{I}$ dictates that

$$s_{1}^{i} = \mathbb{E}[\widetilde{r}_{i}]$$

$$s_{2}^{i} = \sqrt{\mathbb{E}[(\widetilde{r}_{i} - \mathbb{E}(\widetilde{r}_{i}))^{2}]}$$

$$s_{3}^{i} = \frac{\mathbb{E}[(\widetilde{r}_{i} - \mathbb{E}[\widetilde{r}_{i}])^{3}]}{(s_{2}^{i})^{3}}$$

$$s_{4}^{i} = \frac{\mathbb{E}[(\widetilde{r}_{i} - \mathbb{E}[\widetilde{r}_{i}])^{4}]}{(s_{2}^{i})^{4}}$$
(4.2)

To keep the formulas involved in the cubic transformation simple, we restrict ourselves to a process with zero means and variances equal to 1. To do this, we define a standardized distribution of \tilde{r} (\tilde{z}) by

$$\widetilde{z}_{i} = \frac{\widetilde{r}_{i} - \mathbb{E}[\widetilde{r}_{i}]}{s_{i}} \quad \forall i \in \mathcal{I}$$

$$(4.3)$$

From 4.3 and 4.2 it is possible to derive that the correlation matrix of \tilde{r} and \tilde{z} are the same. Further, it can easily be shown that

$$\mathbb{E}[\widetilde{z}_i] = 0 \quad \mathbb{E}[\widetilde{z}_i^2] = 1 \quad \mathbb{E}[\widetilde{z}_i^3] = s_3^i \quad \mathbb{E}[\widetilde{z}_i^4] = s_4^i \tag{4.4}$$

The process executed by the moment-matching algorithm is thus first to generate a distribution \tilde{z} with moments given by 4.4 and correct correlations and then to perform a linear transformation through 4.3 in the end.

4.1.2 Mathematical Transformations

There are two key mathematical transformations applied in the moment-matching algorithm. They are a cubic and a matrix transformation and will be presented in the following. **Cubic Transformation for Moment-Matching** A cubic transformation $\tilde{y}_i = a_i + b_i \tilde{z}_i^2 + b_i \tilde{z}_i^2 + d_i \tilde{z}_i^3$ is done for the purpose of creating a univariate random variable \tilde{y}_i with four prescribed moments, e.g. the moments in 4.4. For each variable $i \in \mathcal{I}$ the transformation consists in estimating the parameters a_i, b_i, c_i and d_i . To do this, we need to specify the first 12 moments of the \tilde{z}_i distribution. Thence we solve the following set of equations.

$$(1)\mathbb{E}[\tilde{y}_{i}] = \mathbb{E}[a_{i} + b_{i}\tilde{z}_{i} + c_{i}\tilde{z}_{i}^{2} + d_{i}\tilde{z}_{i}^{3}]$$

$$(2)\mathbb{E}[\tilde{y}_{i}^{2}] = \mathbb{E}[(a_{i} + b_{i}\tilde{z}_{i} + c_{i}\tilde{z}_{i}^{2} + d_{i}\tilde{z}_{i}^{3})^{2}]$$

$$(3)\mathbb{E}[\tilde{y}_{i}^{3}] = \mathbb{E}[(a_{i} + b_{i}\tilde{z}_{i} + c_{i}\tilde{z}_{i}^{2} + d_{i}\tilde{z}_{i}^{3})^{3}]$$

$$(4)\mathbb{E}[\tilde{y}_{i}^{4}] = \mathbb{E}[(a_{i} + b_{i}\tilde{z}_{i} + c_{i}\tilde{z}_{i}^{2} + d_{i}\tilde{z}_{i}^{3})^{4}]$$

$$(4.5)$$

This set of equations can be solved with readily available solvers in Python. This software solves the equations numerically, even though an analytical solution approach does in theory exist. The problem with this, of course, is that there might not be a feasible solution to the problem given by 4.5. The build-in software applies a least squares method that minimizes a sum of [square] deviations between the right-side and left-side of equations (1), (2), (3) and (4). This way we can obtain the parameters a_i, b_i, c_i and d_i that provide a solution that is [hopefully] close to feasible, and a variable \tilde{y}_i with the closest possible moments to the prescribed. In the implementation we assume that a feasible solution exist every time we run the simulation.

Matrix Transformation for Correct Correlations For the purpose of creating a multidimensional variable \tilde{r} with a given correlation matrix $\mathbf{R} = LL^T$, we apply a Cholesky based matrix transformation. In Høyland et al. [53] it is shown that if \tilde{r} is a n-dimensional random variable with zero means, variances equal to one and independent components \tilde{r}_i , then $\tilde{y} = L\tilde{r}$ is a n-dimensional vector with zero means, variances equal to 1 and a correlation matrix R. From this transformation, two important notes should be made;

- 1. The components of \tilde{r} need to be strictly independent for this transformation to work. This is important to note because it is impossible to programmatically create independent random components \tilde{r}_i of limited size. It is for this reason that the moment-matching heuristic cannot guarantee a perfect match².
- 2. While the transformation creates a variable \tilde{y} with first and second moments equal to those of \tilde{r} , higher marginal moments of \tilde{y} will not be the same. This is important to note because if we want to end up with correct prescribed moments for \tilde{y} , we need to start off with slightly different moments for \tilde{r} . In the following we show how.

Suppose we want to create a multidimensional random variable \tilde{y} with prescribed moments $\mathbb{E}[\tilde{y}]=0$, $\mathbb{E}[\tilde{y}^2]=1$, $\mathbb{E}[\tilde{y}^3]=s_3$ and $\mathbb{E}[\tilde{y}^4]=s_4$, and a correlation matrix $\mathbf{R}=LL^T$. If the components of \tilde{r} are independent, it is easy to proof that the relations between the

 $^{^{2}}$ It is also for this reason that if we increase the number of scenarios we want to create, the result often times get better.

third and fourth moments of \tilde{y} and \tilde{r} in the transformation $\tilde{y} = L\tilde{r}$ is given by

$$\mathbb{E}[\widetilde{y}_i^3] = \sum_{j=1}^i L_{ij}^3 \mathbb{E}[\widetilde{r}_j^3] \quad \forall i \in \mathcal{I}$$

$$\mathbb{E}[\widetilde{y}_i^4] = 3 + \sum_{j=1}^i L_{ij}^4 (\mathbb{E}[\widetilde{r}_j^4] - 3) \quad \forall i \in \mathcal{I}$$
(4.6)

With reference to note (2), we know that we will reach our objective if we start the transformation with the following higher marginal moments for \tilde{r} .

$$\mathbb{E}[\tilde{r}_{i}^{3}] = \frac{1}{L_{ii}^{3}} (\mathbb{E}[\tilde{y}_{i}^{3}] - \sum_{j=1}^{i-1} L_{ij}^{3} \mathbb{E}[\tilde{r}_{j}^{3}]) \quad \forall i \in \mathcal{I}$$

$$\mathbb{E}[\tilde{r}_{i}^{4}] = 3 + \frac{1}{L_{ii}^{4}} (\mathbb{E}[\tilde{y}_{i}^{4}] - 3 - \sum_{j=1}^{i-1} L_{ij}^{4} (\mathbb{E}[\tilde{r}_{j}^{4}] - 3)) \quad \forall i \in \mathcal{I}$$
(4.7)

Where 4.7 follows from 4.6. Some last remarks should also be made. First, for the equations 4.7 to yield the wanted results we need the correlation matrix R to be positivedefinite. For practical purposes (implementation), this imposes no restriction since the Cholesky decomposition doesn't work if the R isn't regular(which implies a semi-positive matrix). Second, if components of \tilde{r} are assumed both normally distributed and mutually independent such that the higher marginal moments of \tilde{r} are $\mathbb{E}[\tilde{r}^3]=0$ and $\mathbb{E}[\tilde{y}^4]=3$, then it follows from 4.7 that the transformation $\tilde{y} = L\tilde{r}$ preserves all four moments³.

4.1.3 The Moment-Matching Algorithm

With the key mathematical transformations introduced, we can now present the momentmatching algorithm. The idea behind the algorithm is the following: generate n independent discrete random variables with the desired marginal moments. This is done by the cubic transformation and the desired moments are zero means, variances equal to 1 and skewness and kurtosis given by 4.7. Thereafter we combine these variables to a n-dimensional multivariate random variable and perform the matrix transformation for desired correlation. In the end, the linear transformation in 4.3 is performed for each of the components of the random variable.

This procedure would create a perfect match if the components of the n-dimensional variable created were truly independent. As mentioned, however, it is impossible to create random vectors of finite length that are strictly independent using simulation. For this reason, Høyland et al. [53] exploits the properties inherent in the Cholesky transformation to decrease the dependence by ensuring uncorrelated variables instead. If \tilde{r} is a multidimensional random variable with correlation matrix $\mathbf{R} = LL^T$, then we know that $\tilde{r} = L^{-1}\tilde{r}$ is a random variable with the same dimension and a correlation matrix I.

³This means that by generating normally distributed variables, we don't have to apply the equations in 4.7. We can specify directly the prescribed standardized higher marginal moments . This is because linear transformations preserve normality [53].

By using this inverse transformation, we are able to lower the degree of dependence. The error distances between the obtained random variable and the prescribed with respect to target moments and correlations, are measured by the root mean square error⁴. This gives way for algorithm 1.

Alg	gorithm 1 Moment-Matching Scenario	Generation
1:	procedure Start Variable(S, I, M^{ta}	r) \triangleright Create Discrete Distribution
2:	for $i \in \mathcal{I}$ do	▷ Independent Generation
3:	$\widetilde{r}_i \leftarrow \mathcal{N}(0,1)$	\triangleright Draw random normal vector, length $ \mathcal{S} $
4:	for $j \in [1, 12]$	\triangleright Calculate 12 Moments
5:	$reve{M}[i,j] = \mathbb{E}[r_i^j]$	
6:	$\widetilde{r}_i = \mathbf{CubicTransformation}(\widetilde{r}_i,$	$\check{M}[i], M^{tar}[i]) $ \triangleright Match Moments
7:	$\mathbf{return}\;\widetilde{r}$	\triangleright Multidimentional Variable
8:		U U
9:	Generated is a multidimensional variable	le \widetilde{r} with moments (M) close to correct mo-
	ments (M^{tar}) and a correlation matrix H	that is close to I due to independent gener-
	ation.	
10:		
1:	procedure MAIN LOOP $(\tilde{r}, M^{tar}, R^{tar})$	▷ Main Algorithm Loop
2:	Set $p = 0$	
3:	Let $R^{tar} = LL^T$	\triangleright By Cholesky Decomposition
4:	while $\epsilon^{mom} > \epsilon^{mom}_{max} \& \epsilon^{cor} > \epsilon^{cor}_{max} de$)
5:	$R^p = \mathbf{CorrelationMatrix}(\widetilde{r}^p)$	
6:	$R^p = LL^T, \widetilde{r}^p = L^{-1}\widetilde{r}^p, \widetilde{y}^p =$	$= L \widetilde{r}^p$
7:	Now \widetilde{y}^p will have correct correlat	ions, wrong moments
8:	for $i \in \mathcal{I}$ do	
9:	for $\mathbf{j} \in [1, 12]$ do	\triangleright Calculate 12 Moments
10:	$M[\mathrm{i},\mathrm{j}]=\mathbb{E}[y_i^j]$	
11:	$\widetilde{y}_{i}^{p}= extbf{CubicTransformation}$	$(\widetilde{y}_i^p, \widetilde{M}[i], M^{tar}[i]) $ \triangleright Match Moments
12:	$\epsilon^{mom} = {f RootMeanSquareErro}$	$\mathbf{r}(\widetilde{y}^p, M^{tar})$ \triangleright Evaluate error
13:	$\epsilon^{cor} = \mathbf{RootMeanSquareError}$	$(\widetilde{y}^p, R^{tar})$
14:	$p \rightarrow p + 1$	
15:	Now \widetilde{y} has standardized moments a	nd correlations withing limits from the pre-
	scribed	
16:	Let $\widetilde{z}^i = s_2^i \widetilde{y} + s_1^i \forall i \in \mathcal{I}$	\triangleright Standard Linear Transformation
17:	return \widetilde{z}	

The biggest difference between the code implemented in Python and the presented pseudo-code, is that since we cannot be certain that the heuristic will provide a result that is within the prescribed error limits, we allow a predetermined number of iterations for both the *StartProcedure* and the *MainLoop*. Furthermore, we define a maximum

⁴The root mean square error is defined by: $RMSE = \sqrt{\frac{1}{N_{el}} \left[\sum_{k} (value_k - TARGET_k)\right]}$ where N_{el} is the number of elements in the sum[53]

Moment-Matching Scenario Generation			
	Definition		
I	Set of variables i		
S	Set of scenarios s		
M^{tar}, R^{tar}	Target Moments and Correlations. Ma-		
	trices $[\mathcal{I} x4]$		
s^1, s^2	Target 1^{st} and 2^{nd} moments. Vectors		
	$[\mathcal{I} \mathrm{x1}]$		
$\widetilde{r}, \widetilde{y}, \widetilde{z}$	Multidimensional Random Variables		
	$[\mathcal{I} \mathbf{x} \mathcal{S}]$		
$\widetilde{r}_i, \widetilde{y}_i, \widetilde{x}_i$	Random Vectors $[1x \mathcal{S}]$		
$\mathbb{E}[\widetilde{r}_i^j]$	Higher Marginal Moment when $\mathbb{E}[\widetilde{r}_i^j] = 0$		
	for $j = 1$		
ϵ_{max}^{cor} , ϵ_{max}^{mom}	Maximum allowed root mean square er-		
	ror for correlations and moments		

Table 4.1: Parameters, Variables and Sets

Table 4.2: Parameters, Variables and Sets

Scenario Tree Generation			
	Definition		
\mathcal{T}	Set of stages t		
\mathcal{N}^t	Set of nodes n in stage t		
\mathcal{U}_n^{t+1}	Subset of nodes in stage $(t+1)$ with same		
	parent N_n^t		
M^{tar}, R^{tar}	Target Conditional Moments and Corre-		
	lations. Matrices $[\mathcal{I} x4]$		
\widetilde{p}^t	Conditional probability matrix in stage t		
\widetilde{r}^t	Conditional return matrix in stage t		

allowed error for the Cubic Transformation and allow a certain amount of iterations for this as well.

Multistage Scenario Tree A multistage scenario tree can be built using the the moment-matching scenario generation heuristic. This is done by generating scenarios node for node. To create stochastic dependence between events in different stages of the tree, it is necessary to choose/specify/predict the target marginal moments and targets correlation matrices at each node in such a way that the specification in node t depends on the realization of events from parent node in node t-1 and possibly node t-2,t-3 and so on. If we are using historic data, this can be done simply by updating our sample. If we are using a stochastic model, e.g. a vector auto regression model, the model input should also include the past realization[s] of events. Obviously, expert knowledge and opinion cannot easily be considered in each node if there are several stages. A conceptual approach for creating scenario trees with the moments matching heuristic is provided in algorithm 2. The sets, parameters and variables are defined in table 4.1. We need the algorithm to return to us conditional returns and probabilities for each stage $t \in \mathcal{T}$. As each child node has one parent node, this will result in one two-dimensional return matrix \tilde{r}^t and one two-dimensional probability matrix \mathbb{P}^t for each stage. The exception is the root node. As a result, first stage (t=0) generate one-dimensional vectors \tilde{r}^1 and \tilde{p}^1 .

Alg	gorithm 2 Scenario Tree Structure	
1:	procedure Make $TREE(\mathcal{T})$	> Make tree with $ \mathcal{T} $ stages
2:	Define \mathcal{N}	$\triangleright \mathcal{N}^0 = 1$
3:	Set $\tilde{p}^0 = 1$	
4:	$\mathbf{for} \ \mathbf{t} \in \mathcal{T} \ \mathbf{do}$	
5:	$\mathbf{for}\;\mathrm{n}\in\mathcal{N}^t\;\mathbf{do}$	\triangleright Node by node
6:	Specify conditional properties M^{tar} , R^{tar}	
7:	Generate Scenarios \tilde{r}_n^{t+1} with given M^{tar} , R^{ta}	r
8:	Assign corresponding child nodes \mathcal{U}_n^{t+1}	
9:	Assign conditional probabilities \widetilde{p}_n^{t+1} to \mathcal{U}_n^{t+1}	
10:	Stack vectors $\widetilde{r}_n^{t+1}, \widetilde{p}_n^{t+1} \to \widetilde{r}^{t+1}, \widetilde{p}^{t+1}$	\triangleright Create Matrices
11:	$\mathbf{if} \mathrm{t} = \mathcal{T} $ -1 \mathbf{then}	
12:	Stop	
13:	else	
14:	$t \rightarrow t + 1$	
15:	$\mathbf{return} \widetilde{r}^t, \widetilde{p}^t orall t \in \mathcal{T}$	\triangleright Returns and Probabilities

4.2 Two-Stage Mean-Risk Models

When the scenario tree describing the underlying stochastic process is generated, it is possible to solve multistage portfolio optimization problems with appropriate programming. The purpose of this section is to extend the linear portfolio optimization models introduced in Chapter 3 to a two-stage setting in which a portfolio rebalance decision can be made. First a formal description of the two-stage problem domain is given together with an introduction of appropriate constraints. Thereafter, we formulate two-stage cV@R and MNAD portfolio optimization models which are evaluated in the computational part of this thesis.

In two-stage portfolio optimization we are faced with the problem of allocating wealth among a set of investment instruments over two consecutive time periods. There exists an option of rebalancing after returns from the first time period have realized. Formally, we define a set of stages $\mathcal{T} = \{1, 2\}$. Initial allocation happens in stage 1, and rebalancing takes place in stage 2. As with one-stage programming, we assume that the returns are modeled on some complete probability space $(\Omega, \mathbb{F}, \mathbb{P})$. We cannot see into the future, however, we must therefore apply a filtration on the space for multistage modeling. Then \mathbb{F} is the σ -algebra of events that has resolved at time t, and \mathbb{P} is the associated probability measure. A filtration $\{\mathbb{F}_t\}_{t\geq 0}$ is applied such that $\mathbb{F}_{t=2} = \mathbb{F}[63]$. Note that the rebalancing decision in stage 2 takes place after an event in \mathbb{F}_1 has been observed. A scenario tree⁵ can visualise the possible sequence of events $\tilde{r} = [\tilde{r}^1, \tilde{r}^2]$. With a formal description of the underlying stochastic process, we can now turn to multistage constraints. For readability we adapt the notations in table 4.3 in the following.

Inventory and Cashflow Constraints In the first stage (t=0) the investor must allocate wealth among a set of instruments given an initial composition of the portfolio ω_0^i , knowing with full certainty the prices of the assets. An *inventory constraint* is then given by

$$x_1^i = \omega_0^i + b_1^i - s_1^i \tag{4.8}$$

Where b_1^i and s_1^i denote the amounts bought and sold and x_1^i is the first stage allocation in asset i. Equation 4.8 must hold $\forall i \in \mathcal{I}$. When purchasing and selling assets, there must also be a constraint ensuring feasible cash flows. Here, we could allow the investor to draw funding from some initial outstanding cash should there be a deficit. Another possibility is to ensure self-financing. This is done through the following self-financing *cashflow constraint*.

$$\sum_{i\in\mathcal{I}} x_1^i = \sum_{i\in\mathcal{I}} \omega_0^i \tag{4.9}$$

Equation 4.9 also simply states that the investor cannot allocate more wealth than she initially has.

Time-stage Constraints and Transaction Costs As discussed in Chapter 2, there will always be some sort of penalizing transaction cost associated with a given trading

⁵Depicted in figure 4.1

Indices

- i : assets
- first stage event realization j :
- second stage event realization dependent on jk:

$\underline{\mathbf{Sets}}$

 \mathcal{S} : scenarios: $\mathcal{S} \in \{(\mathbf{j},\mathbf{k}) | \mathbf{j} \in \{1,...,J\}, k \in \{1,...,K_j\}\}$

: assets: $\mathcal{I} \in \{1, ..., I\}$ \mathcal{I}

Parameters

		<u>r arameters</u>
p_1^j	:	probability of first stage event j
p_2^{jk}	:	probability of second stage event k given j
r_{1}^{ij}	:	return on asset i in event j
r_2^{ijk}	:	return on asset i in event k given event j
α	:	confidence level for V@R and $cV@R$
β	:	maximal allowable cV@R
ω_0	:	initial wealth
ω_0^i	:	initial wealth/position in asset i
ϵ_i	:	proportional transaction cost asset i

First Stage Variables

x_1^i	:	first stage wealth allocation to asset i
b_1^i	:	first stage amount bought of asset i
s_1^i	:	first stage amount sold of asset i
$\omega_1^{i,j}$:	wealth in asset i given first stage realization of event j

Second Stage Variables

:	second stage wealth allocation to asset i given event j
:	second stage amount bought of asset i
:	second stage amount sold of asset i
:	wealth in asset i in scenario k, j
:	total horizon V@R
:	total horizon return in scenario k, j
:	auxiliary variable for cV@R formulation
:	auxiliary variable for MNAD formulation
:	maximum allowable deviation
	· · · · · · · · · · · · · · · · · · ·

policy for financial instruments. The most obvious is the brokerage fee. There also exists costs related to the bid/ask spread and the fact that prices of financial instruments change when purchases/sales are made (price impact). See e.g. Wagner and Edwards [96]. Modelling of transaction costs can be done in several ways[97]. In the following we consider proportional transaction costs, where costs of trading are set to a fixed proportion (ϵ) of the traded amount.

In addition to trading amounts and transaction costs, first stage return realizations r_1^{ij} also impacts the end of first period (t=1) wealth ω_1^{ij} . A time-stage constraint is then given by

$$\omega_1^{ij} = (1 + r_1^{ij})x_1^i - (b_1^i + s_1^i)\epsilon_i \tag{4.10}$$

Equation 4.10 must hold for all assets $i \in \mathcal{I}$ and for all first stage scenarios $j \in \mathcal{S}$.

Rebalancing and Second Stage Constraints In the second stage (t=1), the investor has the opportunity to rebalance her portfolio. The rebalancing constraint is quite similar to the inventory constraint in 4.8 and is given by

$$x_2^{ij} = \omega_1^{ij} + b_2^{ij} - s_2^{ij} \tag{4.11}$$

Equation 4.11 must hold $\forall i \in \mathcal{I}$ and $\forall j \in \mathcal{S}$. Furthermore, a time stage constraint similar to 4.10 and cash flow balance similar to 4.9 should also be satisfied. These will be fully stated in the next section when the two-stage portfolio optimization models are formulated.

4.2.1 Two-Stage CV@R Optimization Model

In this section we formulate a linear two-stage cV@R portfolio optimization model with the constraints from the previous section. This model is one of two mean-risk models implemented with momentum and contrarian investment algorithms as assets in the computational part of this thesis. We are interested in maximizing the end of horizon expected portfolio return R_{tot} with a restriction on the end of horizon cV@R ζ . The one-stage model was presented together with the cV@R risk measure in Chapter 3. The model formulated is a two-stage stochastic optimization model with recourse and transaction costs. Note that by forcing the amounts bought and sold in the second stage, b_2^{ij} and s_2^{ij} , equal to zero, we may obtain a one-stage (static) model instead. The notation is explained in table 4.3.

$$\max z = \sum_{k,j\in\mathcal{S}} p_1^j p_2^{jk} R_{tot}^{jk}$$
(4.12)

Subject to:

$$x_1^i = \omega_0^i + b_1^i - s_1^i \quad \forall i \in \mathcal{I}$$

$$(4.13)$$

$$\sum_{i\in\mathcal{I}} x_1^i = \sum_{i\in\mathcal{I}} \omega_0^i \tag{4.14}$$

$$\omega_1^{ij} = (1 + r_1^{ij})x_1^i - (b_1^i + s_1^i)\epsilon_i \quad \forall i \in \mathcal{I} \quad \forall j \in \mathcal{S}$$

$$(4.15)$$

$$x_1^i \ge 0, \, \omega_1^{ij} \ge 0 \tag{4.16}$$

$$x_2^{ij} = \omega_1^{ij} + b_2^{ij} - s_2^{ij} \quad \forall i \in \mathcal{I} \quad \forall j \in \mathcal{S}$$

$$(4.17)$$

$$\sum_{i \in \mathcal{S}} x_2^{ij} = \sum_{i \in \mathcal{S}} \omega_1^{ij} \quad \forall j \in \mathcal{S}$$
(4.18)

$$\omega_2^{ijk} = (1 + r_2^{ijk})x_2^{ij} - (b_2^{ij} + s_2^{ij})\epsilon_i \quad \forall i \in \mathcal{I} \quad \forall j, k \in \mathcal{S}$$

$$(4.19)$$

$$\omega_2^{jk} = \sum_{i \in \mathcal{I}} \omega_2^{ijk} \quad \forall j, k \in \mathcal{S}$$
(4.20)

$$\zeta + \frac{1}{1 - \alpha} \sum_{j,k \in \mathcal{S}} p_1^j p_2^{jk} \eta^{jk} \le \beta$$
(4.21)

$$\sum_{i \in \mathcal{I}} \omega_0^i - \omega_2^{jk} - \zeta \le \eta^{jk} \quad \forall j, k \in \mathcal{S}$$
(4.22)

$$0 \le \eta^{jk} \quad \forall j, k \in \mathcal{S} \tag{4.23}$$

$$R_{tot}^{jk} = \frac{\omega_2^{jk}}{\omega_0} - 1 \quad \forall j, k \in \mathcal{S}$$

$$(4.24)$$

$$x_2^{ij} \ge 0, \, \omega_2^{ijk} \ge 0 \tag{4.25}$$

4.2.2 Two-Stage MNAD Optimization Model

In this section we formulate a linear two-stage mean negative absolute deviation (MNAD) portfolio optimization model. This model is one of two mean-risk models implemented with momentum and contrarian investment algorithms as assets in the computational part of this thesis. We are interested in maximizing the end of horizon expected portfolio return R_{tot} with a restriction on the end of horizon MNAD. The one-stage model was presented together with the MNAD risk measure in Chapter 3. The model formulated is a two-stage stochastic optimization model with recourse and transaction costs. Note that by forcing the amounts bought and sold in the second stage, b_2^{ij} and s_2^{ij} , equal to zero, we may obtain a one-stage (static) model instead. The notation is explained in table 4.3.

$$\max z = \sum_{k,j \in S} p_1^j p_2^{jk} R_{tot}^{jk}$$
(4.26)

Subject to:

$$x_1^i = \omega_0^i + b_1^i - s_1^i \quad \forall i \in \mathcal{I}$$

$$(4.27)$$

$$\sum_{i\in\mathcal{S}} x_1^i = \sum_{i\in\mathcal{S}} \omega_0^i \tag{4.28}$$

$$\omega_1^{ij} = (1 + r_1^{ij})x_1^i - (b_1^i + s_1^i)\epsilon_i \quad \forall i \in \mathcal{I} \quad \forall j \in \mathcal{S}$$

$$(4.29)$$

$$x_1^i \ge 0, \, \omega_1^{ij} \ge 0 \tag{4.30}$$

$$x_2^{ij} = \omega_1^{ij} + b_2^{ij} - s_2^{ij} \quad \forall i \in \mathcal{I} \quad \forall j \in \mathcal{S}$$

$$(4.31)$$

$$\sum_{i \in \mathcal{S}} x_2^{ij} = \sum_{i \in \mathcal{S}} \omega_1^{ij} \quad \forall j \in \mathcal{S}$$
(4.32)

$$\omega_2^{ijk} = (1 + r_2^{ijk})x_2^{ij} - (b_2^{ij} + s_2^{ij})\epsilon_i \quad \forall i \in \mathcal{I} \quad \forall j, k \in \mathcal{S}$$

$$(4.33)$$

$$\omega_2^{jk} = \sum_{i \in \mathcal{I}} \omega_2^{ijk} \, \mathbf{do} \quad \forall j, k \in \mathcal{S}$$
(4.34)

$$R_{tot}^{jk} = \frac{\omega_2^{jk}}{\omega_0} - 1 \quad \forall j, k \in \mathcal{S}$$

$$(4.35)$$

$$\sum_{j,k\in\mathcal{S}} p_1^j p_2^{jk} v_2^{jk} \le \gamma \tag{4.36}$$

$$\sum_{j,k\in\mathcal{S}} p_1^j p_2^{jk} R_{tot}^{jk} - R_{tot}^{jk} \le v_2^{jk} \quad \forall j,k\in\mathcal{S}$$

$$(4.37)$$

$$v_2^{jk} \le 0 \quad \forall j,k \in \mathcal{S} \tag{4.38}$$

$$x_2^{ij} \ge 0, \, \omega_2^{ijk} \ge 0 \tag{4.39}$$

Chapter 5

Data and Model Design

This chapter presents the data and models applied when back-testing factor investment strategies on US and Norwegian equity markets in the time period from January 2000 to December 2015. Section 5.1 details the data samples used. Section 5.2 presents the factor models applied, and the framework applied in algorithmically performing stochastic portfolio optimization with factor model investment strategies as assets, is described in section 5.3.

All tests were run on a 64-bit Windows 7 PC with 3.40 GHz Intel Core i7-3770 CPUs (4 cores) and 16 GB RAM.

5.1 Samples

The samples used in this work comprise stocks from Norwegian and US equity markets. The analysis in Norway is conducted on daily price data of all stocks traded on the Oslo Stock Exchange (OSE) in the time period from 03.01.2000 to 31.12.2015. A total of 4175 data points is considered. Although the effect of survivorship bias¹ on factor strategies remain relatively unexplored, including all stocks eliminates the problem, should it exist. Norwegian market return data, gathered from Thomas Reuters database, is defined as the periodic return of an equally weighted Oslo Stock Exchange index with monthly rebalancing. The cumulative returns to this index is depicted in figure 5.4.

In the US equity market, the back-tests are conducted on all US stocks being or having been constituent of the MSCI ACWI (world index) in the time period from 03.01.2000 to 31.12.2015. A total of 4175 data points is considered, gathered from Thomas Routers Database. MSCI ACWI captures large and mid cap representation. Small cap stocks are thus excluded from the study². US market return data, gathered from Thomas Reuters

¹Could cause the results of studies to skew higher because only companies which were successful enough to survive until the end of the period are included

²This may not be optimal seeing as several US studies report the momentum effect to be especially strong among small cap firms.

database, is defined as the periodic return of an equally weighted MSCI US Index with monthly rebalancing. The cumulative returns to this index is depicted in figure 5.4.

US and Norway Equity Markets			
Data Set	Oslo SE	MSCI US	
Start Date	03.01.2000	03.01.2000	
End Date	31.12.2015	31.12.2015	
Data Points	4175	4175	
Buy/Sell	Adjusted Close	Adjusted Close	
# of Equities	501	1049	

Table 5.1: Data Sets

Notes: this table gives an overview over the different data sets used for simulation in this thesis. Only past price data is considered.

Other Considerations

As seen in Chapter 2, momentum strategies implemented in the literature rely heavily on short selling stocks that are under performing as to create zero-cost portfolios³. However, the investor faces short-selling restriction when dealing stocks. Not all stocks are listed for short-selling, and the transaction costs are considerably higher. The majority of the literature assumes this obstacle to be nonexistent and calculates returns to investment strategies which may be of great value in theory, but costly or impossible to implement in practice. We deal with short-selling in two ways;

- 1. We calculate returns when short selling is not permitted. This entails calculating returns to strategies only holding the winner portfolios. These long-only strategies are subjects of particular interest in this thesis (see Chapter 1).
- 2. For comparison with existing literature and to obtain results that may be of academic value, we also include results for investment strategies that allow for shortselling in the initial analysis. For simplicity all stocks are included in these calculations.

Missing observations have been padded with the preceding stock price for up to a limit of 20 trading days. Accordingly, stocks that de-list in the holding period are sold at the last days adjusted close price and held as cash with no additional return. The data sets will introduce new stocks as they are introduced in the time period. As new entries are prone to momentum effects, these are important to include in a momentum study⁴. Implicitly in this approach is the assumption that transactions are always executable. In reality however, there may be limited supply and demand of equities.

The returns are measured using daily adjusted close prices. The assumption that underpin these calculations is that an investor could rank and trade stocks based on the closing

³See e.g. [59, 60, 21, 46, 45, 87, 20, 8, 28]

⁴Vegard Egeland, Fronteer Solutions



Figure 5.1: Cumulative returns; equally weighted indexes for the Norwegian and US stock market samples in the time period from 03.01.2000 to 31.12.2015

price. This is a simplification, of course. One should never devise an investment strategy solely on the adjusted $close[19]^5$. We further assume 20 trading days per month.

5.2 Factor Model Design

In this section we present the factor model strategies back-tested on the data samples. As mentioned, for academic purposes and completeness, contrarian and minimum volatility factor models will be implemented in addition to momentum models. In the second part of this thesis, the factor model investment strategies will constitute an investment universe of complex assets, available for algorithmic portfolio optimization. For a thorough discussion and explanation of the different strategies and their rationale, we refer the reader to Chapter 2.

Factor Model Implementation

Implementation of the factor models is done in Python using object oriented programming. The simulation is done by each month ranking stocks based on the past performance criteria. We use overlapping portfolios. The investor holds several momentum portfolios each month⁶. This is consistent with the majority of the literature and increases the power of the empirical tests. Thus in any month, we initiate positions in the winner portfolios, and close out positions in old portfolios. We rebalance monthly to maintain equal weights on the momentum portfolios held and on the equities in each portfolio. With monthly equally weighted rebalancing we obtain a time series of monthly

 $^{{}^{5}}$ In initial testing, we are interested in whether or not the strategy generates abnormal returns. If so, we can apply the models with more realistic constraints(Vegard Egeland, Fronteer Solutions).

⁶Except for the case when the holding period is equal to the time period between rebalancing, then the investor only holds one momentum portfolio

returns for the investment strategies implemented. Figure 5.2 illustrates the algorithmic approach used in the back-tests performed in this work.



Figure 5.2: This figure displays a flow-diagram describing the general approach for backtesting factor model strategies.

Investment Strategies and Parameters

Individual Stock Price Momentum The individual stock price momentum factor strategy implemented in this thesis ranks stocks each month t based on their compounded return over the formation period, J.

$$C_{JT} = \prod_{j=t-J}^{t-1} (1+r_{ij}) = (1+r_{ij-1})(1+r_{ij-2})...(1+r_{ij-J})$$

Equally weighted portfolios are constructed based on these rankings. We use portfolios sizes of 10% to form portfolios. The results are obtained with a skipping period S = 0. When S=0, the stocks are bought and sold immediately after ranking at the closing
price. A skipping period of S=1 was also implemented, but results didn't deviate much⁷. Formation period J and holding period K range from 1 to 12 months. Returns to the strategy are calculated with monthly rebalancing, daily prices and equal weights assigned to each portfolio in holding. In addition to calculating returns to the strategy where no short sales are allowed, we construct zero cost portfolios, as in the literature⁴ where

$$r_{tot,m} = \frac{1}{MN} \sum_{j=1}^{M} \left(\sum_{i=1}^{N} \prod_{k=t-T}^{T} \frac{P_{it}}{P_{it-1}} - 1\right)_{W} - \sum_{j=1}^{M} \left(\sum_{i=1}^{N} \prod_{k=t-T}^{T} \frac{P_{it}}{P_{it-1}} - 1\right)_{L}\right)$$

where:

 P_t = End of day adjusted close price for an equity P_{t-1} = Adjusted close the day before

 $r_{tot,m}$ = The strategy monthly return

M = Number of portfolios in holding each month (both winner and loser portfolios)

N = Number of stocks in each portfolio

T = Number of trading days in a month

W,L = Subscripts of the winner and loser portfolio, respectively

The volatility-scaled momentum factor, inspired by the MSCI momentum index[11], is implemented with ranking criterion

$$C_{MSCI} = \frac{\bar{r}_T}{\sigma_T}$$

where:

 σ_T = The volatility of the daily returns over the formation period. \bar{r}_T = The average daily return over the formation period.

52weekhigh & Cross-sectional Momentum At the end of each month stocks are ranked based on their nearness to the 52-week high price for the 52weekhigh strategy, and the average daily deviation from the cross-sectional mean for for the cross-sectional momentum model.

$$C_{GH} = \frac{P_{it}}{HIGH_{it}}$$
$$C_{DE} = \frac{1}{TJ} \sum_{t=1}^{TJ} [r_{it} - \frac{1}{I} \sum_{i=1}^{I} r_{it}]$$

where:

 $HIGH_{it}$ = Highest equity price of the past 252 trading days J = Number of formation months T = Number of trading days per month I = Number of active assets of the period r_{it} = Daily return on asset i

⁷Commonly found in the momentum literature is that profits are somewhat higher for a skipping period of S=1. This was not the case in our back-tests.

From the highest ratio to the lowest ratio. Equally weighted portfolios are then formed. The top performers constitute the winner portfolio and the bottom performers constitute the loser portfolio. As with the JT model, the strategies are implemented with different model parameters. Portfolio size is 10%. The monthly returns are also calculated in a similar fashion to the individual stock price momentum, with monthly rebalancing to maintain equal weights on the portfolios held.

When using these ranking criteria with data samples containing 'dead'⁸ and newly introduced stocks, some caution must be made. If the stocks are recently listed, naturally the closeness to the 52-week high will be high. Similarly, 'dead' stocks will display low variance, but obviously yield no return in the future. For these reasons we only consider stocks that have been active for six months with at least 5 observations the last 15 trading days, when ranking based on these these criteria.

Contrarian Strategy The contrarian strategy implemented, uses the opposite investment logic of the JT model. The stocks are ranked based on the same performance measure, but equally weighted portfolios are formed where the top % constitute the 'loser' portfolio and the bottom % of the ranked stocks the 'winner' portfolio. We initiate a long position in the stocks constituent of the 'loser' portfolio. As return reversal commonly has been found in the short term, we rebalance the portfolios every two weeks, and use two weeks holding and formation periods⁹. With these adjustments, the returns are calculated in a similar fashion to the individual stock price momentum.

Low-volatility Investing In this work we try to capture low-volatility stocks by ranking based on historical variance. Each third month, stocks are ranked based on the last twelve months historical variance of daily returns.

$$C_{Vol} = \sigma_{it}^2 \tag{5.1}$$

From the lowest ratio to the highest ratio. An equally weighted portfolios is then formed. The top 10% constitute the low variance portfolio. This model in similar to that of the S&P500 volatility index[24], only with a smaller portfolio size and different weighting of the constituents. With regards to 'dead' and newly introduced stocks, the same procedure as with the 52weekhigh is applied to the historic data.

Performance Evaluation To evaluate the performance of the different strategies, we apply several descriptive statistics. Commonly sited in the literature is the average monthly return and the variance of the monthly returns. In addition, we calculate a risk-reward given by

$$\mathcal{R} = \frac{\bar{r}}{\sigma} \tag{5.2}$$

⁸Stocks that have been delisted

⁹We could have implemented with even shorter rebalancing periods and holding and formation periods, however, this would increase the volume traded and make the assumption of no trading cost more questionable (Vegard Egeland, Fronteer Solutions)

This could be seen as a Sortina-ratio with a zero target return. In the second part of the computational of this thesis we also apply the value at risk and conditional value at risk. These measures were introduced in Chapter 3.

$$V@R(x,\zeta) = min\{\zeta \in \mathbb{R} | \Psi(x,\zeta) \le \beta\}$$
$$CV@R(x,\beta) = \mathbb{E}[L(x,p^l)|\zeta \le L(x,p^l)]$$

The value at risk is the loss ζ that will not be exceeded over some investment time-span with a probability level 1- β . The conditional value at risk is the expected loss, given that the loss is in excess of ζ . These are then calculated with historical data at a 95% confidence level. Further, a stocks' beta generally describe the degree to which securites co-move with the market[12]. In this work we calculate strategy betas with respect to the equally weighted market indexes described in 5.1.

$$\beta = \frac{\sigma_{im}}{\sigma_m^2} \tag{5.3}$$

where:

 σ_{im} : Covariance between market and security σ_m^2 : Market variance

Moreover, the significance of the returns obtained are examined using a t-statistic. We perform a one-sided test whereby the statistic is given by

$$t = \frac{\bar{r}}{\frac{\sigma}{\sqrt{n}}} \tag{5.4}$$

where n is the number of observations. We thus test whether the sample mean is significantly different from zero.

5.3 Algorithmic Portfolio Optimization Framework

In Chapter 3 and Chapter 4 we saw how stochastic portfolio optimization problems could be formulated linearly with different risk measures. In this section we present the algorithmic system applied in this thesis, which involves monthly stochastic portfolio optimization for a portfolio of factor model investment strategies. The purpose of this model is to investigate the possibility for a factor investor of dynamically changing the momentum factor model they invest on basis of. First the framework is presented together with a flow diagram depicting the overall back-testing approach. Next, we describe the particular methods used for forecasting future returns. The stochastic portfolio optimization models applied are described to a thorough extent in Chapter 4.

Model Implementation

The system initiates each month by retrieving historic return data to each of the factor model algorithms available in the investment universe. Based on these returns, we next generate a scenario tree representing possible future returns for each of the constituent algorithms, with associated correlation matrices. This is done with moment-matching scenario generation¹⁰. Thence we solve a stochastic optimization¹¹ problem to decide this months buying and selling amounts. The particular portfolio optimization models are presented in Section 4.2. Finally, we rebalance our portfolio according to these weights and evaluate this months wealth. The algorithm is illustrated in the figure 5.3.

When using this framework with associated portfolio optimization problems for backtesting, there are several assumptions underpinning the calculations. In addition to the assumptions made with the factor models, we assume that the investor may allocate proportions of wealth to each factor. It is academically interesting to find if a strategy with restrictions on expected risk could yield favorable risk/reward performance out-ofsample. Realistically, however, it is easier for a practitioner to trade based on signals from a single factor¹². To facilitate switching behavior consistent with such investing, we may assume no transactions costs related to changing factor and no risk restrictions. From theory we know that this will yield a model that each month chooses a single constituent based solely on expected return.

The framework, including portfolio optimization models and the scenario generation algorithm, is implemented using Python and object oriented programming. The optimization problems are solved using Gurobi, a complete mathematical programming solver with a Python interface.

5.3.1 Forecasting and Scenario Tree Generation

As limited work has previously been conducted on forecasting returns of factor model algorithms, different approaches to forecasting for scenario tree generation are performed in this thesis. Note that, in both cases, the very scenario three generation hinges on the moment-matching heuristic previously introduced. It is the forecasting approach that differs. With different approaches we may perform a comparison. The methods are as follows.

- 1. Future means, standards deviations, skewness, kurtosis and correlation matrix for the algorithm assets are computed using historical returns.
- 2. Future means, standards deviations, skewness, kurtosis for the algorithm assets are computed from a discrete distribution generated with auto-regression and Monte Carlo simulation with moment-matching for the error terms. The correlation matrix is computed using historical return data.

Rolling samples of different sizes are used. While the first approach is rather intuitive¹³, the second approach to forecasting is partly our own proposition, and will be explained to a further extent in the following.

 $^{^{10}\}mathrm{Algorithm}$ presented in Chapter 4.

 $^{^{11}\}mathrm{Both}$ one-stage and two-stage models are implemented in this work

¹²Vegard Egeland, Fronteer

¹³This method was applied in a one-stage setting for stock investments in the Taiwanese stock market in Chen and Yang [23].



Figure 5.3: This figure displays a flow-diagram describing the algorithmic portfolio optimization framework applied with factor model assets

Auto-Regression Simulation with Moment-Matching

A common way in econometrics to model the economy is by means of an auto-regression (AR) model. With this model we assume that the evolution of a process may be predicted by some linear combination of its past evolution.

$$r_{it+1} = \alpha + \beta_1 r_{it} + \beta_2 r_{it-1} + \dots + \beta_p r_{i-c} + \epsilon_{it+1}$$

$$\mathbb{E}(\epsilon) = 0 \qquad (5.5)$$

$$\mathbb{E}(\epsilon_t \epsilon_{t-s}) = 0 \quad \forall s \neq 0$$

An auto-regression model with p lags, AR(p) can be described by 5.5. The error term ϵ_{it+1} is assumed to have a zero mean and no correlation across time. The model in this work is fitted using historic data with linear regression and the method of ordinary least squares. A discrete distribution of possible future return realizations r_{it+1} may then be simulated by randomly generating realizations of the error term ϵ_{it+1} . This method of

simulation is noted Monte Carlo simulation[89]. A common assumption is that the error term is normally distributed[3].

$$\epsilon_{t+1} \sim \mathcal{N}(0, \sigma)$$

In which case the simulation of ϵ_{t+1} can be done by drawing samples from a normal distribution. In this work we make no such assumption. Instead, we use historical data to calculate the mean, standard deviation, skewness and kurtosis of the error term. Next, we use the cubic transformation introduced in Section 4.1 to generate realizations of the error term with matching moments. This way, no assumption on the particular marginal distributions of the error terms are made, only that it may be described by four historical moments. The discrete distribution of next period return for each variable is generated with 10 000 realizations of the error term ϵ_{it} . It is with the error term distribution and equation 5.5 that we obtain a discrete distribution for each asset possible future return. From the resulting forecasted distribution of r_{it+1} , the first four moments are calculated. To simulate dependence between the assets, the historical correlation matrix is used in this method of forecasting also¹⁴.

The reason we do not assume the error term to be normally distributed, is that since the transformation from $\epsilon_{it+1} \rightarrow r_{it+1}$ is linear, this would result in a normal distribution for possible realizations of r_{it+1}^{15} . If we could assume normality for the returns of our complex assets, the moment-matching scenario heuristic would be of limited use for this approach to forecasting. We would simply draw from a multivariate normal distribution instead. Our method of Monte Carlo forecasting is thus consistent with the momentmatching scenario generation heuristic by Høyland et al. [53] introduced in Chapter 4.

Scenario Tree Generation

With both the first and second method of forecasting, we obtain four moments for each asset future distribution, and a correlation matrix describing the linear dependence between the assets. These are inputted into the momentum matching scenario heuristic introduced in 4.1 to create the first stage scenarios of a scenario tree. For each first stage node, the method of forecasting is repeated as to create the second stage. The procedure for creating a scenario three is also thoroughly presented in 4.1. We generate 75 realizations in each stage for a total of 5625 possible scenarios in each tree. This generation is time-consuming. In addition, the purpose of this thesis is to observe behavior of different momentum factor strategies with an emphasis on recent years. For these reasons, the portfolio optimization framework is only be implemented in the time-period following 2005.

¹⁴Unlike predictability in stock returns, which challenges the efficient market hypothesis, there is some academic acceptance on the possibility of predicting covariance matrices with historic data[80, 39, 3].

¹⁵Since linear transformations preserve normality. See Section 4.1



(b) Generated Discrete Distribution

0.0

0.1

0.2

0.3

20

10

0 -0.3

-0.2

-0.1

Figure 5.4: Moment-matching; from 100 observations of historical returns to discrete distribution of 1000 scenarios via moment-matching scenario generation. This illustration is with returns from the individual stock price momentum strategy.

Chapter 6

Computational Part 1: Factor Model Returns

This chapter reports results to four momentum strategies, one contrarian strategy and one low-volatility investment strategy back-tested on US and Norwegian equity samples from the time period between January 2000 to December 2015. The purpose is to find which momentum strategies that yield the highest performance and to assess whether some formulations are superior in different periods of time. The contrarian and lowvolatility strategies are, similar to the momentum strategy, based on past price history and included in this work for completeness and to assess whether such strategies could offer additional value to the momentum investor.

Section 6.1 gives an overview over initial results relevant for comparison with existing momentum literature. In Section 6.2 and Section 6.3 we examine more thoroughly results to long-only investment models. Section 6.2 evaluates the effect of different holding and formation periods and Section 6.3 examines performance of selected factor strategies over different subperiods of time. Since a factor investor may be interested in finding whether the underlying anomaly driving stock returns persists, it is of special interest whether the factor strategies have performed well in recent years. In the next chapter, several of the factor models presented will constitute an investment universe available for algorithmic portfolio optimization.

6.1 Factor Model Returns Across Two Markets

This section presents results to factor models across the Norwegian and US equity markets that are comparable to those commonly found in the academic momentum literature.

Table 6.1 reports average monthly returns of winner, loser and zero-cost investment strategies in the time period from 03.01.2000 to 31.12.2015. Standard deviations of the zerocosts¹ are also displayed. Panel A documents results in Norway. Panel B displays results

 $^{^1{\}rm The\ zero-cost}$ (winner - loser) strategy is commonly noted as the 'momentum' strategy in the literature.

from US data. The momentum strategies are implemented with a 6-month holding period and a 6-month formation period, consistent with most of the literature. The contrarian is implemented with a 2-week holding period and 2-week formation period with biweekly rebalancing. The winner portfolios represent the momentum strategies without shortsales².

From table 6.1 we observe that the individual stock price momentum, the volatility-scaled momentum and cross-sectional momentum strategies are all found to be profitable and the returns to the zero-cost investment strategies are statistically significant, in the Norwegian market. The average monthly returns to these strategies are found in the range from 1.67% for the cross-sectional strategy to 1.96% for the individual stock price momentum strategy. The zero-cost 52weekhigh is negative, which is interesting to note because both the winner-portfolios and loser portfolios are profitable. This means that long-only investing based both on the 52-week low and the 52-week high could be profitable strategies. The size of the momentum returns are somewhat higher than what have been found in the Norwegian market in previous studies. Rouwenhorst [87] and Chui et al. [28] both documented average monthly returns of around 1%. This could mean that momentum investing has gotten more profitable in recent years, since these studies report from the time-period up until 2003. This is somewhat surprising since financial theory dictates that anomalies should erode once they get exposed.

The difference between the zero-cost and the long-only for the individual stock price momentum is found to be significantly lower than in the work performed by Israel and Moskowitz [57], who considered long-only investing in the US market. They found the long-side to account for around 50% of the momentum profits. In this work, we find that most of the value added is from the long-side. These findings imply that the performance of the past worst performing firms are better in the Norwegian market than in US market. The average return to the zero-cost is largely driven by the returns to the winner portfolios. Therefore, focusing our analysis on the winner strategy is not only more realistic since not all stocks are listed for short-sales ('approved securities'). Considering short-selling related cost, this could also be a more profitable strategy.

We furthermore observe that the long-only contrarian strategy with 2-weeks holding and 2-weeks formation yields the highest average monthly returns altogether. It is somewhat perplexing that this strategy with the opposite investment logic of the momentum strategy, yields positive returns at the same time. However, this strategy would entail biweekly rebalancing, which gives us reason to question whether it would have been the most profitable if we were to consider transaction costs. Nonetheless, the contrarian strategy may not have been the main focus of this thesis, but these results suggest that the contrarian strategy should be further researched. Moreover, the poor results to the low-volatility strategy is opposing to what we have seen in recent literature for other markets, but in line with the familiar axiom that states that high returns are associated with high risk. We note that a strategy with long positions in the stocks with highest historical volatility,

²While this strategy may not be directly comparable with the momentum strategy commonly found in the literature, which rely heavily on short-selling the under-performers, the exclusion of short-selling makes it more realistic when additional short-selling transaction costs is not considered and the data-sets consist of securities not approved for short-selling.

Table 6.1: Returns to Factor Strategies in Two Markets

Fallel A: FTOILTS III NOI way				
	Winner	Loser	Win-Lose	St. Dev
Individual Stock Price Momentum	1.71%	-0.25%	1.96%	7.68%
(t-stat)	3.43^{***}	-0.34	3.63^{***}	-
52weekhigh Momentum	1.12%	1.44%	-0.32%	5.04%
(t-stat)	3.79^{***}	3.21***	-0.88	-
Volatility-Scaled Momentum	1.75%	-0.20%	1.95%	5.93%
(t-stat)	4.11***	-0.33	4.56^{***}	-
Cross-Sectional Momentum	1.51%	-0.15%	1.67%	6.63%
(t-stat)	3.20^{***}	-0.26	3.59^{***}	-
Contrarian	1.86%	0.27%	1.59%	7.62%
(t-stat)	2.98^{***}	1.13	2.49^{***}	-
Low Volatility	0.77%	1.30%	-0.53%	7.98%
(t-stat)	2.92***	1.86^{**}	-0.93	-
Panel B: Profits in US				
	Winner	Loser	Win-Lose	St. Dev.
Individual Stock Price Momentum	0.92%	0.93%	-0.01%	7.58%
(t-stat)	2.35^{**}	1.46^{*}	-0.22	-
52weekhigh Momentum	0.41%	0.93%	-0.52%	8.63%
(t-stat)	2.06^{**}	1.46^{*}	-0.82	-
Volatility-Scaled Momentum	0.98%	0.66%	0.32%	5.82%
(t-stat)	2.69^{***}	1.06	0.77	-
Cross-Sectional Momentum	0.89%	0.91%	-0.02%	6.34%
(t-stat)	2.16^{**}	1.43^{*}	-0.09	-
Contrarian	0.98%	0.57%	0.31%	4.67%
(t-stat)	1.86^{**}	1.32*	1.34^{*}	-
Low Volatility	0.71%	0.99%	-0.28%	9.31%
(t-stat)	3.28^{***}	1.32	-0.42	-

Notes: *** Significant at the 1% level; ** Significant at the 5% level; * Significant at the 10% level. This table reports results to strategies back-tested on Norwegian and US data in the time period from 03.01.2000 to 31.12.2015. Holding periods are 6-months for the momentum strategies and 2-weeks for the contrarian. For the individual stock price momentum the formation period is 6-months. The formation period is by definition 12-months (52 weeks) for the 52weekhigh. The formation period is 2-weeks for the contrarian. Portfolio sizes are 10%. The zero-cost 'win - lose' invests each month in the top 10% performers and short-sells the bottom 10% performers. would have yielded returns of 1.3%. These strategies, and their appropriateness as hedging instruments, are examined more closely over different subperiods of time in Section 6.3.

The factor model strategies were also back-tested on US data. While the zero-costs yield negative returns for all strategies but the contrarian, we note that a strategy longing only the winners (or losers) yields statistically significant positive returns and the highest returns are found for the contrarian strategy in this regard. We also see that the magnitude of the US returns are significantly lower than the results in the Norwegian market³. These findings are in line with the works of Hwang and Rubesam [56] and Lesmond et al. [73], claiming that the momentum effect has disappeared in US markets. We should note, however, that our US universe comprises mid cap and large cap stocks. The momentum effect is suggested by the literature to be stronger in small cap stocks, this may explain some of these findings.

It is interesting to see whether the strategies 'beat the market'. The average monthly return to an equally weighted Norwegian market index was 0.74% over the period. The average monthly return to an equally weighted index of the US stocks was 1.01%. Indeed, all factor models except the low-volatility strategy seem to be more profitable than the index in Norway with higher average monthly returns over the period. This is not the case for the US sample, where none yields above-market returns. The efficient market hypothesis states that it is impossible to systematically beat the market and that all available information should fully be reflected in current prices. It is then especially remarkable that a strategy based on a readily available piece of information, the 52-week high equity price, is found to generate above market returns. Nearly every financial newspaper states the 52-week high on a daily basis.

However, we know that there are several explanations appertaining to the sources of momentum and contrarian profits, and that not all of them violate the efficient market hypothesis. No transaction costs and other simplifying assumptions made in this work, as common in the literature, could explain the abnormal returns⁴ found. Albeit, if the momentum effect appertains to market inefficiencies⁵, one explanation for my findings could indeed be that the larger US market is more efficient than the smaller Norwegian market. In the next section, we examine more thoroughly the long-only strategies with respect to different holding and formation periods.

6.2 Momentum Model Specification

A factor investor may specify her momentum factor in a number of other ways. This section documents results to momentum models with long positions only for a wide range of holding and formation periods. By examining results for different holding and formation periods, we may find which specification of the momentum factor models that

³This makes them somewhat less interesting to analyse, which is why the rest of this work emphasises the Norwegian market.

⁴See Chapter 5 for an overview over the assumptions made.

⁵Something which may not be the case, see Section 2.2

yields the highest performance. Results from US data are included for completeness, despite that neither of the factor models were found to generate above-market returns on the US equity sample examined.

Table 6.2 and 6.3 document average monthly returns to four long only momentum factor strategies for a range of holding and formation periods in the time period from 03.01.2000 to 31.12.2015. Standard deviations, t-statistics and a ratio of mean return returns to standard deviation are also displayed. Portfolio sizes are 10%. Complete sensitivity plots for momentum models yielding above market returns are included in figures⁶ 6.1, 6.2 and 6.3.

From table 6.2, we first observe that short holding periods seems to be considerably more profitable for all momentum models. Figures 6.1, 6.2 and 6.3 confirms this for a range of formation periods, the surfaces are skewed. The plots also expose that longer formation periods are associated with higher reward. These findings are in line with the majority of the momentum literature claiming that momentum profits are more profound for short holding periods and long formation periods. A familiar tendency may furthermore be observed. Higher profits seems to be associated with higher risk for most of the specifications. Existing literature places little emphasis on this aspect of the momentum strategies. An investor may not only care for high returns he may also have a preference for low risk. Regardless of whether we value reward or reward/risk, the factor model formulations with short holding period and long formation periods are found to be most attractive, and the results to these strategies are found statistically significant in the Norwegian market.

Comparing the performance of the different specifications, we notice that the 12-2 formulation of the individual stock price momentum generates the highest returns, and that the volatility-scaled strategy performs best in terms of risk/reward. This strategy is less volatile over the period. Fronteer Solutions⁷ invests partly based on a 9-3 individual stock price momentum strategy. It is therefore interesting to observe that the 12-2 seems to dominate the 9-3 both in terms of average return and in terms of risk/reward for all momentum formulations tested on the Norwegian market. Nonetheless, it is difficult to conclude based on average results from a sixteen year long time period. Recent performance may be more predicative of the future.

Table 6.3 shows that the tendency for short holding periods and long formation periods to yield higher returns also can be found on the US data sample. The strategies, absent transaction cost, does not seem to yield abnormal returns for any of the model specifications back-tested. This could imply a more efficient US market, but note again that our US sample only comprises large and medium size companies. This may not be optimal for momentum investing, as the momentum effect previously has been found more profound in small cap equities.

⁶Information for the 52weekhigh is present in tables. This strategy has a 12 month formation period by definition.

⁷Fronteer Solutions is a financial practitioner that has supported this work, see Chapter 1.



Figure 6.1: This figure displays the average monthly return as a function of holding and formation period for the individual stock price momentum strategy on the Norwegian market.



Figure 6.2: This figure displays the average monthly return as a function of holding and formation period for the volatility-scaled momentum strategy on the Norwegian market.

Panel A: Ind	ividual Stoc	ck Price	Moment	um								
Formation(K)	Holding(J)	2	3	4	6	8	10					
9		1.99%	1.94%	1.85%	1.79%	1.73%	1.62%					
(t-stat)		3.87***	3.79^{***}	3.66^{***}	3.57^{***}	3.50^{***}	3.30***					
σ		7.06%	7.03%	6.96%	6.89%	6.78%	6.75%					
${\cal R}$		0.28	0.27	0.27	0.26	0.25	0.24					
12		2.29%	2.23%	2.15%	1.95%	1.73%	1.61%					
(t-stat)		4.46^{***}	4.24^{***}	4.09^{***}	3.75^{***}	3.38 ***	3.18***					
σ		7.26%	7.20%	7.18%	7.11%	7.00%	6.88%					
\mathcal{R}		0.32	0.31	0.30	0.27	0.25	0.233					
Panel B: 52weekhigh Momentum												
Formation(K)	Holding(J)	2	3	4	6	8	10					
12		1.39%	1.31%	1.26%	1.12%	1.11%	1.11%					
(t-stat)		4.20^{***}	4.52^{***}	4.28^{***}	4.09***	3.92^{***}	3.85^{***}					
σ		4.8%	4.38%	4.19%	4.09%	4.00%	3.99%					
\mathcal{R}		0.31	0.33	0.31	0.30	0.28	0.28					
Panel C: Cross-Sectional Momentum												
Formation(K)	Holding(J)	2	3	4	6	8	10					
9		1.74%	1.67%	1.62%	1.58%	1.51%	1.45%					
(t-stat)		3.70^{***}	3.68^{***}	3.61^{***}	3.56^{***}	3.49^{***}	3.44***					
σ		6.46%	6.25%	6.16%	6.10%	5.93%	5.77%					
\mathcal{R}		0.27	0.27	0.26	0.26	0.25	0.25					
12		1.91%	1.78%	1.72%	1.61%	1.49%	1.39%					
(t-stat)		4.22***	4.12***	4.02***	3.82***	3.62 ***	3.46***					
σ		6.16%	5.91%	5.83%	5.73%	5.62%	5.46%					
\mathcal{R}		0.31	0.30	0.30	0.28	0.27	0.25					
Panel D: Vol	atility-Scale	ed Mome	ntum									
Formation(K)	Holding(J)	2	3	4	6	8	10					
9		1.89%	1.83%	1.81%	1.77%	1.68%	1.57%					
(t-stat)		4.15^{***}	4.05^{***}	3.94^{***}	3.88^{***}	3.75^{***}	3.55^{***}					
σ		0.63%	6.31%	6.32%	7.26%	6.16%	6.10%					
${\cal R}$		0.30	0.29	0.29	0.28	0.27	0.26					
12		2.24%	2.14%	2.05%	1.82%	1.63%	1.54%					
(t-stat)		4.69^{***}	4.50^{***}	4.29***	3.85^{***}	3.53 ***	3.37***					
σ		6.53%	6.49%	6.5%	6.43%	6.31%	6.22%					
\mathcal{R}		0.34	0.33	0.31	0.28	0.26	0.25					

Table 6.2: Norway: Performance of Momentum Factor Models

Notes: *** Significant at the 1% level; ** Significant at the 5% level. This table reports average monthly returns, standard deviations and a ratio of mean returns to standard deviations, for momentum investment strategies in Norway in the time period from 03.01.2000 to 01.01.2016 for different holding and formation periods. Portfolio sizes are 10% with monthly rebalancing. Skipping period is 0.

Panel A: Ind	ividual Stoc	k Price	Momen	ntum							
Formation(K)	Holding(J)	2	3	4	6	8	10				
9		0.83%	0.83%	0.86%	0.81%	0.80%	0.79%				
(t-stat)		2.12**	2.18**	2.21**	2.05^{**}	2.04**	2.02***				
σ		5.39%	5.42%	5.45%	5.41%	5.41%	5.39%				
${\cal R}$		0.15	0.16	0.16	0.15	0.15	0.15				
12		0.86%	0.82%	0.81%	0.77%	0.77%	0.80%				
(t-stat)		2.17^{**}	2.05^{**}	1.98^{**}	1.89^{**}	3.92**	1.99^{**}				
σ		5.43%	5.47%	5.56%	5.56%	5.52%	5.49%				
\mathcal{R}		0.16	0.15	0.14	0.14	0.15	0.15				
Panel B: 52weekhigh Momentum											
Formation(K)	Holding(J)	2	3	4	6	8	10				
12		0.63%	0.65%	0.63%	0.61%	0.62%	0.68%				
(t-stat)		2.21**	2.29**	2.18^{**}	2.06^{**}	2.08^{**}	2.27^{**}				
σ		3.88%	3.86%	3.97%	4.04%	4.09%	4.08%				
\mathcal{R}		0.16	0.17	0.16	0.15	0.15	0.16				
Panel C: Cross-Sectional Momentum											
Formation(K)	Holding(J)	2	3	4	6	8	10				
9		0.81%	0.81%	0.81%	0.76%	0.70%	0.71%				
(t-stat)		2.05^{**}	2.09^{**}	2.08^{**}	1.93**	1.84^{**}	1.84^{**}				
σ		5.41%	5.31%	5.34%	5.35%	5.28%	5.26%				
${\cal R}$		0.15	0.15	0.15	0.14	0.13	0.13				
12		0.75%	0.71%	0.67%	0.61%	0.59%	0.64%				
(t-stat)		1.89^{**}	1.81^{**}	1.69^{**}	1.54	1.52	1.60				
σ		5.34%	5.36%	5.41%	5.36%	5.33%	5.32%				
\mathcal{R}		0.14	0.13	0.12	0.11	0.11	0.12				
Panel D: Vol	atility-Scale	d Mom	entum								
Formation(K)	Holding(J)	2	3	4	6	8	10				
9		0.79%	0.84%	0.83%	0.82%	0.80%	0.71%				
(t-stat)		2.14**	2.26^{**}	2.22**	2.13**	2.16^{**}	2.23**				
σ		5.05%	5.09%	5.12%	5.24%	5.06%	5.02%				
${\mathcal R}$		0.16	0.16	0.16	0.16	0.16	0.15				
12		0.87%	0.85%	0.83%	0.82%	0.76%	0.79%				
(t-stat)		2.30^{**}	2.22**	2.15^{**}	2.13**	2.29^{*}	2.38^{*}				
σ		5.18%	5.21%	5.27%	5.24%	5.16%	5.13%				
\mathcal{R}		0.17	0.16	0.16	0.16	0.17	0.17				

Table 6.3: US: Performance of Momentum Factor Models

Notes: *** Significant at the 1% level; ** Significant at the 5% level; *Significant at the 10% level. This table reports average monthly returns, standard deviations and a ratio of mean returns to standard deviations, for investment strategies in US in the time period from 03.01.2000 to 01.01.2016 for different holding and formation periods. Portfolio sizes are 10% with monthly rebalancing. Skipping period is 0.



Figure 6.3: This figure displays the average monthly return as a function of holding and formation period for the cross-sectional momentum strategy on the Norwegian market.

6.3 Subperiod Performance

A central issue for a factor investor is whether the underlying anomaly driving stock returns persists. If markets are well-developed, one would expect any near-arbitrage opportunity to erode by the acts of arbitrageurs. Furthermore, it is of interest whether some strategies are more profitable than others during different periods of time. This section investigates subperiod performance of momentum models that was found to be most promising in the last section⁸. To evaluate if there could be value to dynamically change factor strategy, the contrarian and low-volatility strategies are also included. Since the factor strategies yielded modest returns in the US market for a wide range of holding and formation periods, this section focuses on strategies implemented in the Norwegian market. It is the factor models examined in this section that will constitute a portfolio for algorithmic portfolio optimization in Chapter 7.

Table 6.4 documents statistics to different long-only factor models for back-tests performed over different subperiods. Mean monthly returns and risk-reward ratios given by the mean returns divided by the standard deviations are displayed together with the strategy beta. The momentum strategies are formed with 12-month formation and 2months holding, the contrarian strategy with 2-weeks holding and two-weeks formation and the low-volatility strategy with 12-months formation and 3-months holding. Portfolio sizes are 10%. Year-by-year average monthly return plots are depicted in figure 6.4. The results to an equally weighted Norwegian market index are included for comparison.

From table 6.4 and figure 6.4 it is interesting to observe that all strategies yields positive, above-market returns over the last three years. We notice that the cross-sectional strategy has generated positive returns the last seven years and that the volatility-scaled strategy has outperformed the market index consistently since 2003. Furthermore, the 12-2 individual stock price momentum strategy seems to have performed best in terms of average returns in recent years, particularly in 2015 with a monthly average of around $4\%^9$. In terms of risk-reward, however, we see that the best performance is yielded by the 12-2 volatility-scaled. This means that the returns to the 12-2 volatility-scaled are less volatile since the returns to this strategy are lower. A lower volatility could to some degree be expected since the stocks are ranked partly based on the inverse of the volatility, and historical volatility is known to have some predictive power[3, 70, 80, 39].

We notice that the size of the returns are smaller in recent years compared to the time period prior to the financial crises. We know that there was a consistent upwards trend in Norwegian stocks prior to the financial crisis and it has also been suggested by Cooper et al. [32] that momentum profits tend to be higher in bull periods. The returns to the market index has declined in much the same way, so this could be an explanation. Nevertheless, a tendency for falling profits could be a sign that the momentum anomaly is starting to erode.

As for the strategies that are not based on the momentum anomaly, we first observe

 $^{^{8}}$ A comparison is also made with the 9-month formation – 3-month holding momentum formulation that Fronteer Solution partly bases their investing on. See Chapter 1.

⁹Note that this is absent trading costs, see Section 5.1 for assumptions and model specifications and Section 7.3 for a related discussion.



Figure 6.4: Year-by-year average monthly returns to factor strategies and an equally weighted Norwegian market index. Portfolio sizes are 10%.

Panel A	: Individual	Stock Prices	Momentum	l								
Strategy	(2005-2006)	(2007-2008)	(2009-2010)	(2011-2012)	(2013-2014)	(2015)						
12-2	4.84%	-0.5%	1.76%	0.33%	2.84%	4.32%						
${\mathcal R}$	0.57	-0.06	0.26	0.06	0.57	0.79						
β	1.67	0.95	0.86	0.79	0.84	0.37						
9-3	5.04%	-0.59%	1.36%	-0.20%	2.59%	3.93%						
${\cal R}$	0.57	-0.07	0.20	-0.04	0.51	0.67						
β	1.77	0.89	0.90	0.87	0.91	0.51						
Panel B: 52weekhigh Momentum												
Strategy	(2005-2006)	(2007-2008)	(2009-2010)	(2011-2012)	(2013-2014)	(2015)						
12-2	4.14%	-1.24%	3.46%	0.07%	2.12%	1.95%						
${\cal R}$	0.84	-0.25	0.47	0.02	0.67	0.67						
eta	0.89	0.55	0.15	0.62	0.52	0.26						
Panel C	: Cross-Secti	ional Momen	itum									
Strategy	(2005-2006)	(2007-2008)	(2009-2010)	(2011-2012)	(2013-2014)	(2015)						
12-2	4.28%	-0.03%	2.59%	0.45%	2.04%	2.35%						
${\cal R}$	0.55	0.00	0.36	0.12	0.33	0.52						
β	1.54	0.55	0.30	0.52	1.09	0.40						
Panel D	: Volatility-S	Scaled Mome	entum									
Strategy	(2005-2006)	(2007-2008)	(2009-2010)	(2011-2012)	(2013-2014)	(2015)						
12-2	4.76%	-0.03%	3.14%	0.49%	2.69%	3.18%						
${\cal R}$	0.66	-0.05	0.45	0.10	0.62	0.54						
eta	1.42	0.74	0.43	0.52	0.66	0.66						
Panel E:	Contrarian											
Strategy	(2005-2006)	(2007-2008)	(2009-2010)	(2011-2012)	(2013-2014)	(2015)						
2-2	3.37%	-1.46%	4.11%	-0.01%	2.94%	2.69%						
${\cal R}$	0.70	-0.15	0.47	-0.03	0.36	0.20						
β	1.04	0.98	1.19	1.24	1.63	1.94						
Panel F:	Low-Volati	lity										
Strategy	(2005-2006)	(2007-2008)	(2009-2010)	(2011-2012)	(2013-2014)	(2015)						
12-3	2.20%	-1.15%	1.58%	-0.01%	0.03%	1.39%						
${\mathcal R}$	0.85	-0.22	0.43	-0.03	0.09	0.49						
β	0.50	0.60	0.50	0.54	0.66	0.54						

 Table 6.4:
 Subperiod Performance Norway

This table reports average monthly returns, a ratio of mean returns to standard deviation and the beta to factor model investment strategies on Norway equity data. Portfolio sizes are 10%. Skipping period is 0. Strategies are long-only.

that the returns to the contrarian are rather high in several subperiods. This means that, if there exist some predictability in the relative performance between a momentum and a contrarian strategy, a contrarian strategy could definitely offer some value for a momentum investor. However, with low risk-rewards, we then know that the contrarian returns are volatile. Further, high strategy betas suggests that there is a significant degree of systematic risk associated with contrarian investing and could explain some of the abnormal profits to such a strategy. The low-volatility strategy, on the other hand, is confirmed to perform poorly over a full range of subperiods¹⁰. The strategy only outperforms the index in one of thirteen years. This performance is, as mentioned, somewhat opposing to what we have seen in the literature, but in line with the familiar axiom that states that high returns are associated with high risk. The initial hypothesis motivating some of the work done in this thesis was, based on existing literature, that a low-volatility strategy could be utilized as some hedging instrument during times of financial turmoil. In other words, if an investor held a portfolio of momentum algorithm assets together with a low-volatility strategy, the low-volatility strategy would be weighted heavy during times when the market was down. Clearly, the ranking based low-volatility strategy back-tested in this work, is not suitable for such a purpose. From figure 6.4 we can see that the poor returns to the low-volatility strategy is, on a average monthly annual basis, rather correlated with both the market and the other momentum strategies¹¹.

If we compare the subperiod performance of the 9-month formation and 3-month holding individual stock price momentum strategy to the 12-2 formulation, we find the 12-2 formulation to be superior both in terms of risk and reward. The 12-2 formulation has yielded higher average monthly return for the past six years, and higher risk-reward for each of the subperiods displayed in table 6.4. Furthermore, the betas to the 9-3 formulation is higher indicating that this specification of the individual stock price momentum strategy holds a higher degree of systematic risk. These findings are in line with existing literature and suggests that Fronteer Solution could benefit from a different specification of their momentum factor. As we have seen, if they want a more steady momentum factor, they could also consider scaling by the historical volatility.

We further observe that, although other formulations of the momentum factor has performed worse than the individual stock price momentum strategy in recent years, the 52weekhigh, volatility-scaled momentum and cross-sectional momentum stratety all performed considerably better than during and after the financial crises in 2007-2008. This means that the momentum investor indeed could have earned higher profits by changing momentum strategy in these years¹². Albeit, it is then relevant to note from the shapes of the plot in figure 6.4, an otherwise relatively high degree of co-movement between the momentum strategies in terms of annual monthly average returns. This could be expected since these strategies try to exploit the same underlying anomaly and consequently, often constitute the same stocks. Accordingly, we could also question how suitable such strategies are for algorithmic portfolio optimization alone, since the possibility of diversifying

¹⁰Several rebalancing frequencies, holding and formation periods were tested with similar results

 $^{^{11}}$ For these reasons, the algorithmic portfolio optimization was performed with portfolios of contrarian and momentum algorithm assets only.

¹²If this could have been foreseen. In Chapter 7 we examine whether stochastic programming could have forecasted and capitalized on such time-variability among the momentum strategies.

risk is small, and the task of predicting future relative performance is more difficult. In this regard, including a contrarian strategy in the portfolio of assets might have positive effects because it doesn't co-move with the other assets to the same degree. This is something we get back to in Chapter 7.

By examining the average return plots in figure 6.4 and the statistics in table 6.4, other important aspects can be noted. The strategies and their betas are time-varying. All factor strategies, in particular the momentum strategies, seem to co-move with the market to a significant degree. If market returns are high, momentum profits tends to be high, and vice versa. As each momentum portfolio is equally weighted with up to 10% of the stocks in the market, this could to some degree be expected, and it is a common finding in the literature. Part of the momentum literature claims that such market dependence is related to momentum strategies bearing a significant amount of systematic risk, and that this partly is the source of the momentum profits. The betas calculated with respect to the equally weighted market index suggest that bearing of systematic risk fails to explain the momentum abnormal returns found in this work. The momentum strategies have relatively high betas, but not above one. This implies that the strategies bear less nondiversifyable risk than the market, with higher returns. If market frictions and the other assumptions underlying these investment models cannot explain the factor abnormal returns, our findings strongly challenge the hypothesis of an efficient market.

Moreover, the market dependence has an important implication for a factor investor. If the momentum strategy, regardless of the formulation, returns negative or modest returns during times in which the market is down, one should consider other investment strategies or holding a 'risk-free' instrument during these 'down-periods'. The factor investor could rely more heavily on other factors instead. To facilitate this kind of investing, one would have to carefully monitor the market. If an appropriate metric for describing market movements is chosen, and there exist some predictability in the relationship between momentum returns and market-movements, this could readily be incorporated in an automated trading system. Market states and momentum investing in this respect, is an interesting area for future research.

Chapter 7

Computational Part 2: Algorithmic Portfolio Optimization

The previous chapter documented results to several factor model strategies back-tested on samples from the US and Norwegian equity market. Different formulations of the momentum and contrarian factors were found to generate abnormal returns on the Norwegian stock market. This chapter concerns the performance of a stochastic portfolio optimization framework applied algorithmically on an investment universe consisting of the factor strategies found successful in the last chapter¹. The purpose is to assess the possibility to dynamically change the factor strategy with which we invest, and thereby try to exploit time-variability in momentum strategy returns by use of stochastic programming.

In Section 7.1 we examine results to an algorithmic portfolio optimization strategy maximizing expected returns. Further, in Section 7.2, we take the position of a factor investor with funds to allocate portion wise between different factor strategies and detail results to such a dynamic allocation strategy for different preferences on risk-level. Finally, Section 7.3 provides some comments on the results regarding validity and underlying assumptions.

7.1 Maximizing Expected Return

From theory, we know that if we apply a mean-risk model with no transaction costs and no restriction on expected risk, then this will yield a model that each month chooses a single constituent based solely on expected return. This section presents results to such a return maximizing model applied algorithmically on an investment universe of factor model strategies. Maximizing expected return facilitates a switching behavior easily implementable for a practitioner who relies on 'buy', 'hold' or 'sell' signals from single factors².

Table 7.2 reports descriptive statistics of the resulting time series of returns for the

¹While this does introduce some bias with respect to the obtained results, we are interested in any additional gains from a stochastic programming approach.

²This is the case for Fronteer solutions, see the introduction in Chapter 1



(a) Investment strategies - algorithm assets



(b) Algorithmic portfolio optimization strategies with algorithm assets

Figure 7.1: Cumulative returns; a comparison of different portfolio optimization strategies together with the algorithm assets.

Asset	μ	σ	${\cal R}$	cV@R	MNAD
Individual Stock Price	1.94%	7.04%	0.27	-15.7%	2.62%
Volatility-Scaled	2.12%	6.30%	0.34	-12.8%	2.26%
Contrarian	1.97%	8.84%	0.22	-18.9%	3.32%
52weekhigh	1.65%	5.23%	0.32	-10.3%	1.78%
Cross-Sectional	1.79%	6.13%	0.29	-11.4%	2.20%
Buy-and-Hold	1.95%	5.89%	0.33	-12.8%	2.22%

Table 7.1: Investment Strategies - Algorithm Assets

Table 7.2: Portfolio Optimization - Maximizing Expected Return

Momentum & Contrarian Portfolio

Momentum Only Portfolio

Model	$\mid \mu$	σ	\mathcal{R}	cV@R	Model	$\mid \mu$	σ	\mathcal{R}	cV@R
AR(1)	2.00%	7.44%	0.27	-15.4%	AR(1)	1.82%	6.63%	0.27	-14.6%
AR(2)	1.83%	7.39%	0.25	-16.5%	AR(2)	1.76%	6.72%	0.26	-13.9%
12mth	1.47%	8.23%	0.18	-19.3%	12mth	1.84%	6.92%	0.27	-14.9%
24mth	1.51%	7.91%	0.19%	-17.7%	24mth	1.96%	6.89%	0.28	-14.6%

Table 7.3: Momentum & Contrarian - Two-Asset Strategy

Model - $AR(1)$	$\mid \mu$	σ	\mathcal{R}	cVaR
Individual Stock Price & Contrarian	2.41%	8.49%	0.28	-17.1%
Volatility-Scaled & Contrarian	2.29%	7.36%	0.31	-14.3%
Cross-Sectional & Contrarian	2.08%	7.39%	0.28	-14.1%
52weekhigh & Contrarian	1.93%	7.18%	0.27	-13.76%

Tables 7.1 to 7.3 displays descriptive statistics for out-of-sample tests with both the constituent strategies, and the portfolio optimization performed algorithmically with investment strategies as complex assets.

portfolio optimization scheme applied with different forecasting methods in the time period from January 2005 to December 2015. Auto-regression forecasting with one and two lags is reported together with results from forecasting with historical returns based on a rolling sample of one and two years. Results are displayed for a portfolio of momentum strategies only, where the strategies are constructed with 2-months holding and 12-months formation. Furthermore, results for a portfolio of assets that contains both momentum strategies and a contrarian strategy are documented. The cumulative returns to the individual strategies are plotted in figure 7.1 and statistics are provided in table 7.1.

Based on a finding that the inclusion of a contrarian strategy tended to increased portfolio return³, we tested a two-asset strategy where the optimization algorithm chooses each month between one contrarian and one single momentum algorithm. Descriptive statistics to such strategies are reported in table 7.3. The cumulative returns to a wide variety

 $^{^{3}\}mathrm{Compared}$ to a portfolio of only momentum strategies, see Section 7.2.

of strategies are depicted in figure 7.1. In these plots the forecasting is done with autoregression with one lag and Monte Carlo simulation with moment-matching for the error terms⁴.

From figure 7.1 and table 7.1, we first observe that the highest performance of the constituent algorithms was yielded by the volatility-scaled momentum strategy over the time period examined. The risk-rewards of this strategy are superior. The cumulative return plot confirms the tendency of co-movement between the assets, something that will affect the ability to diversify risk. The algorithms yield poor performance during the financial crisis in 2007-2008, a region in which the assets co-move to an especially high degree. This is in accordance with the familiar financial axiom stating that the only thing that rises in a falling market is the correlation, implying that the correlations between the past best performers were high during the financial crisis.

From table 7.2 we observe that algorithmic portfolio optimization with momentum strategies only did not yield superior results with any of the forecasting methodologies tested. This could to a certain degree be expected since the correlations between the assets are high. The strategies are trying to exploit the same underlying anomaly and consequently, often hold the same stocks. This makes it even more difficult to predict relative future performance based on historic data as the assets perform fairly equally in the same periods of time. If the portfolio consisted of assets that performed consistently different over longer periods of time, one could expect the algorithm to be able to choose more wisely since it forecasts based on recent returns. This is not the case with strategies trying to exploit the same underlying anomaly. The momentum investor would be better off with a buy-and-hold of the constituents or, better yet, holding the volatility-scaled strategy alone.

From table 7.2, we also observe that algorithmic optimization with momentum and contrarian assets manage to beat a buy-and-hold of the portfolio in terms of mean return, for optimization with auto-regression forecasting. However, this is only barely and the algorithm falls short of the volatility-scaled momentum strategy in terms of both risk and reward. Considering the constituent assets, it is not surprising that the algorithm performs poorly during the financial crisis. Accordingly, it is reasonable to suggest that a significant performance increase could have been made, if the portfolio of assets also constituted a hedging instrument of some sort⁵, possibly a 'risk-free' free option. This is another way of stating what we found in Chapter 6, that a momentum investor might benefit from periodically changing strategy or holding a 'risk-free' instrument because the returns are time-varying and not always positive. Especially not in times when the market is 'down'⁶.

In terms of total cumulative and average returns, table 7.3 and figure 7.1 report the best performance to be yielded by a strategy that switches entirely between a contrarian and an individual stock price momentum strategy. This is somewhat surprising since these

⁴See Chapter 5 for a description of this methodology.

⁵This was initially our intention with the low-volatility strategy, but this strategy was found to offer very poor performance throughout the entire period.

⁶Poor performance to momentum strategies in times of financial turmoil can also be found in Cooper et al. [32] and Hwang and Rubesam [56] in US markets.

two strategies are not the most profitable in isolation. We observe that the individual stock price momentum strategy generated an average monthly return of 1.94% and a cumulative return of 2.7, and that the contrarian generated an average monthly return of 1.94% and a cumulative return of 2.8. Remarkably, by combining the two strategies we generate an average monthly return of 2.41% and a total cumulative return of 3.4. Similar results can also be found for other single momentum strategies when combined with a contrarian strategy, indicating that the combination of one single momentum strategy and a contrarian strategy generates some additional value. This could have practical implications for a momentum investor because it suggests that periodically investing with a contrarian strategy could generate additional returns, or in the case of the 52weekhigh, a strategy with higher risk/reward performance. Based on this, it seems as if the algorithm is better at forecasting future relative performance when there are only two assets in the mix with a lesser tendency to co-move. Finally, it is relevant to note that the two-asset strategies performs especially well in recent years, which should further motivate a current momentum investor to consider this kind of investing.

Widely discussed in financial literature is whether auto-regression based forecasting is superior to forecasting with historical data alone. In this work, the auto-regression with one lag gave the best overall results for different parameters with both portfolios. The difference was especially high for the momentum and contrarian portfolio where the use of historical returns yielded sluggish performance. The highest returns were also, as mentioned, found for the two-asset strategies for auto-regression with one lag. On the other hand, the highest returns for the momentum portfolio was based on historical forecasting with two years of data, and the difference between one and two lags of regression was marginal. Based on our findings, it is difficult to argue any forecasting method tested in this work to be superior. However, since the one lag auto-regression simulation yielded relatively good results for both portfolios across different parameters, the risk-reward analysis in the next section is based on this forecasting method. If we are to model dependency between the first and second stage in a scenario tree, mean-reversion forecasting⁷ is also preferred.

7.2 Risk-Reward Optimization

At the heart of portfolio optimization is the balancing of risk and reward. If we assume the position of an investor with the ability to allocate wealth portion-wise between factor model strategies, it is interesting to see if stochastic portfolio optimization could yield an overall strategy that performs with respect to both the risk and reward preferences of the investor. This section reports results to algorithmic portfolio optimization with constraints on expected portfolio risk. Both the conditional value at risk and the mean negative absolute deviation introduced in Chapter 3, are considered. The forecasting methodology applied in this section is the auto-regression based with one lag.

⁷Auto-regression models are, under certain conditions, mean-reverting.

cV@R	μ	σ	\mathcal{R}	cV@R	•	cV@R	μ	σ	\mathcal{R}	cV@R
11%	2.05%	6.93%	0.30	-13.2%		11%	1.93%	6.65%	0.29	-14.2%
12%	2.14%	7.14%	0.30	-14.2%		12%	2.04%	6.67%	0.31	-14.0%
$14\% \\ 16\%$	2.17% 2.07%	7.46% 7.70%	$0.29 \\ 0.27$	-14.8% -15.8%		$14\% \\ 16\%$	2.08% 1.98%	6.91% 7.17%	$\begin{array}{c} 0.30\\ 0.28\end{array}$	-14.0% -15.0%

Table 7.4: Momentum & Contrarian Portfolio - cV@R Constraints

 Table 7.5: Momentum Strategies Only Portfolio - cV@R Constraints

Two-stage Optimization

Two-stage Optimization

One-stage Optimization

One-stage Optimization

cV@R	$\mid \mu$	σ	${\cal R}$	cV@R	cV@R	$\mid \mu$	σ	\mathcal{R}	cV@R
11%	1.95%	6.43%	0.30	-12.6%	11%	1.87%	7.32%	0.26	-13.0%
12%	1.94%	6.56%	0.30	-13.3%	12%	1.92%	7.28%	0.26	-14.5%
14%	1.88%	6.76%	0.28	-14.5%	14%	1.82%	6.62%	0.27	-13.8%
16%	1.82%	6.63%	0.27	-14.6%	16%	1.82%	6.67%	0.27	-14.3%

Table 7.6: Momentum & Contrarian Portfolio - MNAD Constraints

	\cap \cdot \cdot \cdot \cdot
L'WO-STAGE	()ntimization
I WO SUAGO	Opumization

One-stage Optimization

MNAD	$\mid \mu$	σ	\mathcal{R}	MNAD	-	MNAD	μ	σ	\mathcal{R}	MNAD
2.2% 2.5% 2.0%	2.02% 2.08%	5.88% 6.31%	0.34 0.33	2.16% 2.26% 2.47%	_	2.2% 2.5% 2.0%	1.99% 2.09%	5.92% 6.33%	0.33 0.33	2.25% 2.38%
$3.0\% \\ 3.6\%$	2.09% 2.07%	6.62% 7.17%	$0.32 \\ 0.29$	2.47% 2.70%		3.0% 3.6%	2.16% 2.13%	6.61% 7.10%	$\begin{array}{c} 0.33 \\ 0.30 \end{array}$	2.58% 2.64%

Table 7.7: Momentum Strategies Only Portfolio - MNAD Constraints

MNAD $\mid \mu$	σ	\mathcal{R}	MNAD	_	MNAD	$\mid \mu$	σ	\mathcal{R}	MNAD
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	6.11% 6.28% 6.33% 6.33%	$0.32 \\ 0.29 \\ 0.30 \\ 0.28$	$2.18\% \\ 2.24\% \\ 2.29\% \\ 2.35\%$		$2.2\% \\ 2.5\% \\ 3.0\% \\ 3.6\%$	$ 1.84\% \\ 1.85\% \\ 1.85\% \\ 1.76\% $	6.36% 5.94% 6.15% 6.26%	0.29 0.31 0.30 0.28	2.30% 2.20% 2.26% 2.29%

Two-stage Optimization

One-stage Optimization

Notes: tables 7.4 to 7.7 displays out-of-sample performance of a stochastic portfolio optimization algorithm applied on portfolios consisting of factor investment strategies, as a function of constraint on expected risk. The conditional value at risk is at a 95% level.

Table 7.4 to 7.7 document results to an algorithmic portfolio optimization strategy performed on portfolios of factor strategy assets with constraints on expected risk. In table 7.5 and 7.7 the investment universe constitutes momentum strategies only. In table 7.4 and 7.6 the portfolio also include a contrarian investment strategy. In table 7.4 and 7.5 the constraints on expected risk are made with respect to conditional value at risk and in table 7.6 and 7.7 the constraints on expected risk are made with respect to the mean negative absolute deviation. One-stage and two-stage optimization results are displayed in both cases.

From table 7.4 to 7.8, we observe that constraining portfolio risk tends to decrease the realized risk for portfolios with both risk measures. The realized risk is, however, not always within the prescribed limit. This could to some degree be expected since the problem is stochastic and we are allocating wealth based on forecasts. With the mean negative absolute deviation we are able to construct a strategy that yields a better realized relationship between reward and MNAD than any one asset alone, but only in one case. With a MNAD constraint of 2.2% for the portfolio of assets containing both momentum and a contrarian asset, the optimization strategy beats the superior volatility-scaled momentum strategy. For any practical purposes, of course, holding the volatility-scaled momentum alone would be more reasonable. Nevertheless, by constraining MNAD to 2.2% we create a strategy with lower realized risk than the volatility-scaled, with high performance in comparison to the other low risk strategies in the asset universe. Furthermore, several of the MNAD constrained strategies outperform both the buy-and-hold benchmark and the equally weighted market index.

With the conditional value at risk, on the other hand, we are unable to outperform the volatility-scaled in terms of risk/reward in any case, and furthermore unable to generate a strategy with lower realized cV@R. The reader must appreciate that in this stochastic optimization framework, we try to model and constrain tail-risk each month based on historical data that may be without extreme occurrences. However sophisticated the statistical means of scenario generation, to forecast extreme occurrences is difficult exactly because they may not have been present in the past⁸. Without the generation of statistical distributions with tail properties representative of the future, it is difficult to capture and constrain expected tail-risk. Nevertheless, with constraints on cV@R, we are able to constrain the conditional value at risk to a certain extent, and generate a strategy that outperforms other constituent assets than the volatility-scaled strategy, and outperforms the buy-and-hold benchmark both in terms of mean return and mean-cV@R.

Interestingly, constraining the mean negative absolute deviations also significantly constrains the portfolio variance. Mean negative absolute deviation attempts to constrain downside deviations from the mean. We know from theory, however, that there is a strict relationship between the mean absolute deviation and mean negative absolute deviation. It is easy to derive that MNAD = 0.5^*MAD^9 . In other words, by constraining the mean negative absolute deviation, we are in fact also constraining upside deviations from the mean. Hence, with both the MNAD and the portfolio variance constraining dispersion

⁸This is why the use of scenario generation by means of statistical methods is so important. See Chapter 4 for a discussion on the topic.

⁹This equivalence holds only when the deviations are measured against the mean[97].

about the mean, it is not very surprising that the constraints on mean absolute negative deviations also constrain the portfolio variance to a significant degree. Notice that the same does not hold for constraining the portfolio cV@R, a risk metric that captures expected tail-risk and not dispersion about the mean.

A somewhat counter-intuitive relationship between risk and reward can be observed for both portfolios. Contrary to the familiar financial axiom that states that high returns are associated with high risk, this does not seem to be the case for realized risk and return. Constraining risk increases portfolio return in several cases. To understand this, we must consider both the constituent assets and the optimization algorithm. The volatility-scaled momentum strategy is a low-risk, high-reward asset compared to the other algorithms. Constraints on risk makes the algorithm weight more heavily this asset. This is probably because it offers superior risk-reward performance. When the optimization is done with less constraint on portfolio risk, the algorithm favours all the high reward assets. The other high-reward algorithms have higher dispersion about the mean and will periodically yield higher expected return than the volatility-scaled strategy, but at the cost of higher risk. Also, we must understand that because we cannot fully forecast the future, there will be a difference between expectations and realization of returns. The optimization problem is stochastic, which may also explain some of these findings. The algorithm seems better at forecasting and constraining risk than optimizing with respect to expected portfolio return.

From theory, we know that absent transaction costs, the only way for a two-stage optimization approach to outperform a one-stage approach, is if there is dependency in the tree. In other words, if there are no transaction costs and statistical independence between the return distribution in the first and second stage, then there should be no advantage to a two-stage approach. In a mean-risk space, the theoretical efficient frontier for the one-stage approach should be to the right and below the efficient frontier of a two-stage approach. Of course, given the inherent stochasticity, this relationship is not guaranteed to hold for the actual realized efficient frontier. In this work, we attempt to model a mean-reverting dependency in our scenario trees. Closely examining the tables in 7.4 to 7.7, we may observe a slight tendency for the two-stage optimization to be better than the one stage in terms of realized risk-reward¹⁰. Hence, for the factor investor there is some incentive to consider a multistage approach in this setting if she constraints the portfolio expected risk. However, the performance increase with two-stage optimization is marginal and not consistent across the whole range of parameters tested. The algorithmic framework seems somewhat better at constraining dispersion about the mean than expected tail-risk. As mentioned, modelling and capturing tail-risk is particularly difficult.

Moreover, while providing a useful means by which to generate discrete multivariate distributions without making strict assumptions on the marginals, the moment-matching algorithm also has some limitations that could have affected our results. First, there are no realistic conditions under which the heuristic will create a perfect match between the target moments and correlations, and the moments and correlations of the generated discrete distribution. Second, to model dependency between the assets, the heuristic uses

 $^{^{10}\}mathrm{In}$ the sense of risk we are constraining with respect to

historical correlation. From theory, we know that correlation is a linear and symmetric measure of association and is not flexible enough to capture non-linear or asymmetric dependencies¹¹. This gives reason to suggest that we could more successfully have captured and constrained tail-loss if we had applied another method¹².

Including a contrarian strategy in the mix of available assets seems to have two general effects on portfolio performance. First, the realized risk measured by different risk metrics gets higher. Second, the realized portfolio reward gets higher. In one sense, one could expect the portfolio risk to increase as we are adding a relatively risky asset to the portfolio. On the other hand, if the asset added to the universe is not perfectly [positively] correlated with the other assets, portfolio theory dictates that there should be some diversification. We were unable to generate strategies with lower realized risk than the less risky algorithm asset. We must again consider that there is a difference between realized and expected characteristics in a stochastic setting. The algorithm is making choices based on monthly forecasts; it has no deterministic information of the future. We note, however, that the ratio of risk to reward generally gets higher with the inclusion of the contrarian asset. Especially with the mean negative absolute deviation constraints. This could be seen as in line with portfolio theory, lower risk for the same reward.

These results, together with the findings with the two-asset momentum-contrarian strategies in the last section, could imply that there exists some predictability in the relative performance between the contrarian and the momentum strategy. That they perform well in different periods, and that this is possible to forecast to some extent. An implication by this is then that an investor could benefit from periodically changing between contrarian and momentum investing, exploiting both return reversal effects and continuation patterns in equity prices.

7.3 Comments on the Results

Despite showing promising results in the Norwegian market, we realize that the models have their limitations. The investment strategies are modelled in close accordance with models from existing literature. These models are designed to mimic investor behavior and are not real implementable trading systems[19]. In Chapter 5, we introduced several of the underpinning assumptions. With equally weighted portfolios, we assume that the investor could buy fractions of stocks. With returns being measured using daily adjusted close prices, we assume that an investor could rank and trade stocks based on the closing price, and that transactions are always executable. In reality, there may be limited supply and demand of stocks and prices vary during the trading day with a spread between the bid- and asking-price. There is a cost associated with the bid-ask spread, the lack of liquidity, and the impact that a trade may have on the price when it raises the demand. In addition, the models assume that the investor always has capacity to invest,

¹¹Financial instruments are typically more dependent on each other when markets are falling.

¹²Copulas, for instance, are great tools to capture asymmetric tail-dependencies[3]. Nonetheless, the moment-matching heuristic was assessed to be more than sufficient for this experimental work.

which might involve borrowing of funds even for long-position stocks, and we know that borrowing is associated with additional costs.

Furthermore, transaction costs were not considered¹³. On the one hand, monthly rebalancing typically makes transaction costs less significant¹⁴ and this work centered on the relative performance of different factor strategies. On the other, there is reason to suggest that transactions costs, together with the other underlying assumptions, would have altered the size of the reported returns. Our findings expose monthly average returns up to 2.4%. With monthly compounding this work suggests yearly returns of over 30% to selected, simple investment strategies based on past price history only. If the underlying assumptions and bearing of high systematic risk do not explain the momentum and contrarian abnormal returns, this work suggests near-arbitrage opportunities in the Norwegian stock market that strongly challenge the hypothesis of an efficient market.

The optimization framework applied is subject to the same underlying assumptions as those of the constituent factor models. Except that, the framework would demand even higher capacity and liquidity¹⁵ since we would change weights on entire portfolios of equities.

Lastly, in finance, model risk can be seen as risk of losses generated by the usage of financial models. A particular type of model risk is associated with the engineering and implementation of the models[97]. In performing back-tests and creating a stochastic optimization framework, it is important to note that programming errors could be made and that the data-sets used may be erroneous. This could also lead to misleading and wrongful results. We note also that Chapter 6 exposed that back-testing models can be sensitive to model specifications. This is important to note for a practitioner, as she is likely to be especially interested in generating profits in the future. Not in the past.

¹³In Section 2.1, we saw that some academics claim transaction costs to greatly affect the momentum premium, while others disagree. Nevertheless, most studies are conducted with the assumption of no trading costs.

¹⁴Vegard Egeland Fronteer Solutions

¹⁵These are important aspects also in relation to scalability of our models.

Chapter 8

Concluding Remarks

The following two sections will present the conclusion of this thesis and propose recommendations for future research.

8.1 Conclusion

The first objective of this thesis has been to evaluate the relative performance of different momentum investment strategies on the Norwegian and US equity markets. We evaluated the performance of four long-only momentum strategies and compared to those of a contrarian and a low-volatility investment strategy. Out-of-sample tests were conducted on US and Norwegian equity samples from the time period between January 2000 to December 2015. From this, we found the momentum and contrarian strategies to yield statistically significant abnormal returns on the Norwegian stock market only. If factor returns appertain to market inefficiencies, a widely discussed topic among financial academics¹, these findings imply the US market to be more efficient than the smaller Norwegian market.

The highest performance in terms of average monthly returns over the sample period was yielded by the individual stock price momentum strategy proposed by Jegadeesh and Titman [59], which has also been the most profitable strategy in recent years. The best risk-reward performance was found for a volatility-scaled momentum strategy. In line with existing literature, we found momentum profits to be more profound for short holding periods and long formation periods. Further, the returns to the strategies were found time-varying and not always positive, suggesting that a momentum investor could benefit from periodically changing strategy or investing in a risk-free instrument.

The second objective of this work has been to add to the theoretical momentum literature by modelling a situation in which the momentum investor can dynamically change investment strategy. We built a stochastic portfolio optimization framework with momentmatching scenario generation, and applied it out-of-sample on portfolios of momentum

¹See Section 2.2 for different explanations appertaining to the sources of momentum profits.

and contrarian strategies in the Norwegian stock market. Algorithmic portfolio optimization failed to yield superior performance with a portfolio of momentum algorithm assets only, which we believe to be due to high correlations between constituent strategies attempting to capitalize on the same anomaly. However, we found a significant performance increase in adding a contrarian investment strategy to the mix of available assets. By allowing the investor to allocate wealth portion-wise between these algorithms, we generated strategies with higher risk-reward performance than a buy-and-hold benchmark of the portfolio for different preferences on expected risk. With our approach, we managed to more adequately constrain portfolio mean negative absolute deviation than expected tail-loss, and a two-stage optimization approach was found marginally beneficial in this setting when portfolio expected risk was constrained.

Finally, we found superior return performance by forcing the investor to each month choose between a contrarian and a single momentum strategy based on expected return. In particular, we found a contrarian strategy in combination with an individual stock price momentum strategy to yield the highest cumulative and average return among the strategies tested. These findings from stochastic programming suggest that an investor could benefit from periodically changing between contrarian and momentum investing, exploiting both return reversal effects and continuation patterns in equity prices.

8.2 Future Work

The recommendations for future research are categorized into three different fields of interest.

Investable Momentum Models

Throughout this thesis, we have emphasized several assumptions underlying the models applied both in this work as well as in existing literature². Limited investability, as implied by the many underlying model assumptions, might offer an explanation as to why academic literature still reports abnormal returns to a strategy first documented over 20 years ago. If markets are well-developed, one would expect such near-arbitrage opportunities to erode. Therefore, an implementable and realistic momentum model should be further explored.

Some work in the direction of an implementable momentum model has occasionally been documented. This thesis and Israel and Moskowitz [57] consider long-only investing. An alternative to trading on the adjusted close can be found in Demir et al. [35], where they use volume-weighted average prices, or in Lesmond et al. [73], where they ad bid-ask spreads as additional transaction costs. Korajczyk and Sadka [69] furthermore modelled extensive transactions costs in their back-tests. An interesting next step would be to document the results to a momentum trading system that trades in worldwide markets

²See Chapter 5 and discussion in Section 7.3, in particular.

in real time. Alternatively, one could construct an actual trading system and simulate a market with conditions as close as possible to reality.

Market States and Factor Investing

Several financial researchers claim that momentum abnormal returns are due to bearing of high systematic risk. A consequence of having a high beta is that the strategy will yield large losses when markets are down. Historically, this has shown to be true[4], and it was also found to be the case for the strategies tested in this work. The market dependence has an important implication for a factor investor. If the momentum strategy returns negative or modest returns during times in which the market is down, one should consider other investment strategies or holding a 'risk-free' instrument during these 'downperiods'. The interplay between market states and momentum profits is an interesting area for future research. If predictability between market movements and momentum returns can be found, this could both be of great value for a practitioner and add to the academic debate on the hypothesis of market efficiency.

Stochastic Optimization & Algorithm Portfolios

A particular emphasis is in this work put on momentum strategy formulations. However, the optimization framework and approach could readily be applied in analysis of other factor strategies. This would be a natural extension of the work as we saw promising results to the adding of a contrarian strategy to the portfolio of momentum investment strategies. This work was limited to strategies based on historic price data only, mainly because of availability. However, as seen in the literature review provided, several other documented factor anomalies are associated with company fundamentals such as value, size and dividend yield. Furthermore, as we found the returns to the strategies to be time-varying and not always positive, it would be interesting to include also a 'risk-free' instrument in the portfolio.

Moreover, as the world of trading is becoming increasingly computerized and complex, portfolio optimization with a portfolio of other trading strategies is not difficult to imagine³, and may also be a very interesting topic for future research in its own right. In any case, if stochastic programming is to be applied to a portfolio of investment algorithms, it is of high importance that further research is done on the distributions and predictability of model returns. As emphasized throughout this text, favorable results to stochastic optimization hinges on proper forecasting of future distributions.

 $^{^{3}}$ Even for private investors, the idea of an algorithm portfolio is not far-fetched. A startup company called Huddlestock[1], for instance, evolves their business around the idea of letting private investors choose between different algorithms in which to invest.

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