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Science and Technology

Support Varieties for Finite Dimensional Algebras

Mads Hustad Sandøy

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Supervisor: Øyvind Solberg, MATH

Norwegian University of Science and Technology
Department of Mathematical Sciences

Problem description

The theory of support varieties for finite dimensional algebras has as its inspiration the corresponding theory for group algebras of finite groups, introduced by Jon F. Carlson in [11, 12]. The main construction in this theory is to associate to every finitely generated module over the group algebra a geometric object, an algebraic variety. The underlying geometric object in this case is the group cohomology ring of the group algebra, here employing the fact that every group algebra is a Hopf algebra. An arbitrary finite dimensional algebra does not necessarily possess a Hopf-structure, entailing that a direct translation of support varieties to this more general class of algebras is not possible. However, there does exist a related theory of support varieties for finite dimensional algebras using the Hochschild cohomology ring of a given algebra. This was introduced in [32] and further developed in [17], where one in [17] showed that much of the theory for group algebras can be generalised to finite dimensional algebras under some finite generation conditions called **(Fg)**. A consequence of this condition **(Fg)** is that the complexity of the algebra must be finite and that the algebra must be Gorenstein. In [34] the theory was further expanded to finite complexes over finite dimensional algebras that satisfy **(Fg)**.

Work with the thesis will involve understanding the theory for support varieties for finitely generated modules over group algebras of finite groups. Following this, one is to study the theory of support varieties for finite dimensional algebras over an algebraically closed field and write a presentation of it.

As previously mentioned, a consequence of the condition **(Fg)** is that an algebra satisfying it must be of finite complexity. Complexity is a measure of the polynomial growth of the projective modules in each degree of the minimal projective resolution of the simple modules of the algebra. The complexity of an algebra can be estimated by way of calculations, whereas directly verifying the condition **(Fg)** is quite complicated. Hence, the complexity of an algebra is the first invariant one checks to see if an algebra satisfies **(Fg)**. A trivial extension $T(\Lambda)$ of an algebra Λ is always a symmetric algebra, and, in particular, a Gorenstein algebra, implying hence that this is an interesting class to examine with regards to **(Fg)**. Here, M. Purin in [29] studied this beginning with some hereditary algebra H and forming the trivial extension $T(H)$. She has shown the following:

Theorem 0.0.1. *Let H be a hereditary algebra over a field. Then the following holds.*

(a) *If H is of finite representation type, the complexity of $T(H)$ is* 1.

(b) *If H is of tame representation type, the complexity of $T(H)$ is* 2.

(c) *If H is of wild representation type, then the complexity of $T(H)$ is infinite.*

One wishes to examine whether this result can be generalised. To do this one will calculate examples as well as make theoretical observations. One will especially study whether being of finite representation type is the vital assumption in statement (a).

Problembeskrivelse

Teorien for støttevarieteter for endeligdimensjonale algebraer har som en inspirasjonskilde den tilsvarende teorien for gruppealgebraer av endelige grupper introdusert av Jon F. Carlson i [11, 12]. Hovedkonstruksjonen i denne teorien er å assosiere til enhver endeliggenerert modul over gruppealgebraen et geometrisk objekt, en algebraisk varietet. Det underliggende geometriske objektet i dette tilfellet er gruppekohomologiringen til gruppealgebraen, der en bruker at enhver gruppealgebra er en Hopf algebra. En vilkårlig endeligdimensjonal algebra innehar ikke en Hopf-struktur, slik at en direkte oversettelse av støttevarieteter til denne mer generelle klassen av algebraer er ikke mulig. Men det finnes en tilsvarende teori av støttevarieteter for endeligdimensjonale algebraer ved å bruke Hochschild kohomologiringen til algebraen. Dette ble introdusert i [32] og videreført i [17], hvor en i [17] viste at mye av teorien for gruppealgebraer lar seg generalisere til endeligdimensjonale algebraer under noen endliggenererthets-betingelser kalt **(Fg)**. En konsekvens av denne betingelsen **(Fg)** er at kompleksiteten til algebraen må være endelig og at algebraen må være Gorenstein. I [34] ble teorien videre utvidet til endelige komplekser over endeligdimensjonale algebraer som tilfredsstiller **(Fg)**.

Oppgaven går ut på å sette seg inn i teorien for støttevarieteter for endeliggenererte moduler over gruppealgebraer av endelige grupper. Deretter skal en studere teorien for støttevarieteter for endeligdimensjonale algebraer over en algebraisk lukket kropp og skrive en presentasjon av dette.

Som tidligere nevnt er en konsekvens av betingelsen **(Fg)**, at algebraen har endelig kompleksitet. Kompleksitet er et mål for den polynomielle veksten til de projektive modulene i hver grad av den minimale projektive oppløsningen av de simple modulene over algebraen. Kompleksiteten til en algebra er det mulig å finne et estimat for ved beregninger, mens direkte å verifisere betingelsen **(Fg)** er svært komplisert. Derfor er kompleksiten til en algebra den første invarianten en sjekker for å se om en algebra tilfredsstiller **(Fg)**. Det er et åpent problem hvilke algebraer som tilfredsstiller **(Fg)**. En triviell ekstensjon $T(\Lambda)$ av en algebra Λ er alltid en symmetrisk algebra og spesielt en Gorenstein algebra, slik at dette er en interessant klasse å undersøke med hensyn til **(Fg)**. Her har M. Purin i [29] undersøkt dette når en starter med en hereditær algebra H og danner den trivielle ekstensjonen $T(H)$. Hun har vist følgende:

Theorem 0.0.2. *La H være en hereditær algebra over en kropp. Da holder følgende.*

(a) *Hvis H er av endelig representasjonstype, da er kompleksiteten til $T(H)$ lik* 1.

(b) *Hvis H er av tam representasjonstype, da er kompleksiteten til $T(H)$ lik* 2.

(c) *Hvis H er av vill representasjonstype, da er kompleksiteten til $T(H)$ uendelig.*

En vil undersøke om dette resultatet lar seg generalisere ved å beregne eksempler og samtidig gjøre teoretiske betraktninger. Spesielt vil en studere om det er det å være av endelig representasjonstype som er den vitale antakelsen ved utsagn (a).

Summary

We give a presentation of the theory of support varieties for finite dimensional algebras Λ using the Hochschild cohomology ring. Our presentation is especially focused on the finite generation hypotheses an algebra must satisfy to have an adequate theory of support varieties, as well as the consequences of these hypotheses for the complexity of the modules of such an algebra. To demonstrate that certain aspects of the corresponding theory for group algebras can be recovered, we show that by assuming the finite generation hypotheses we can prove that every closed homogeneous variety is the variety of some module. Following this, we investigate whether a result of Purin in [29] concerning the complexity of trivial extensions of hereditary algebras can be generalized: Firstly, using a result of Benson and some well-known results concerning radical square zero algebras, we give an example that shows that an algebra Λ can be of finite representation type while its trivial extension $T(\Lambda)$ has infinite complexity, hence showing that a straightforward generalization of Purin's result is not available. After this, we derive a weak bound on the length of the terms of the minimal $T(\Lambda)$-projective resolution of a Λ-module considered as a $T(\Lambda)$-module. Following this, we utilize the proof of a result of Guo et al. in [22] in giving a description of the syzygies and the minimal $T(\Lambda)$-projective resolution of a Λ-module considered as a $T(\Lambda)$-module. Using this and a result by Dichi and Sangare in [15], we are able to show that if a selfinjective algebra Λ satisfies the finite generation hypotheses (**Fg**) then the complexity of $T(\Lambda)$ is exactly one greater than that of Λ.

Oppsummering

Vi presenterer et utvalg av teorien om støttevarieter for endeligdimensjonelle algebraer Λ som benytter Hochschild kohomologiringen. Vår presentasjon omhandler spesielt endeliggenereringshets-betingelsene (**Fg**) en algebra må tilfredsstille for å ha en adekvat teori av støttevarieteter, samt konsekvensene av disse betingelsene for kompleksitetene til modulene over en slik algebra. For å demonstrere at enkelte aspekter av den korresponderende teorien for gruppealgebraer også gjelder for denne mer generelle teorien, viser vi at, såfremt man antar disse (**Fg**)-betingelsene, kan man bevise at enhver lukket homogen variete er varieteten til en modul. Etter dette studerer vi hvorvidt man kan generalisere et resultat av Purin i [29] som gjelder kompleksiteten av trivielle ekstensjoner av hereditære algebraer: Ved hjelp av et resultat av Benson og noen velkjente resultater som omhandler algebraer hvis radikaler kvadrert er null, gir vi et eksempel som demonsterer at en algebra Λ kan være av endelig representasjonstype samtidig som dens trivielle ekstensjon $T(\Lambda)$ har uendelig kompleksitet. Som følge av dette, ser vi at en direkte og rettfram generalisering av Purins resultat er ikke tilgjengelig. Etter dette, utleder vi en svak øvre skranke over lengdene av leddene av den minimale $T(\Lambda)$-projektive oppløsningen av en Λ-modul ansett som en $T(\Lambda)$-modul. Etter dette, benytter vi beviset av et resultat av Guo et al. i [22] til å gi en beskrivelse av syzygiene og den minimale $T(\Lambda)$-projektive resolusjonen av en Λ-modul ansett som en $T(\Lambda)$-modul. Ved å anvende denne beskrivelsen og et resultat av Dichi og Sangare i [15], er vi i stand til å vise at hvis en selvinjektiv algebra Λ tilfredsstiller endeliggenereringshypotesene (**Fg**), så er kompleksiteten til $T(\Lambda)$ nøyaktig én større enn kompleksiteten til Λ.

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Mads Hustad Sandøy
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Preface

This master's thesis constitutes most of the work of my final year as a master's student in Mathematical Sciences (Master's Programme) at Norges tekniske-naturvitenskapelige universitet, NTNU. Work on this text was done part time in the autumn of 2015, and semi-fulltime in the spring and summer of 2016. That this came to be so was due to me beginning my master's studies in the spring semester of 2015 while also choosing to include a couple of courses I had taken before that semester, hence resulting in the necessity of having to take a few additional courses in the spring semester of 2016, as well as the strange turn-in date of this work. However, nearly every major course of study I have undertaken was begun in a spring semester, and so these peculiar ways of doing things have, somehow, become something of a tradition for me.

We note that we assume the reader is familiar with material covered in courses such as MA3201 - Ringer og moduler, MA3202 - Galoisteori, MA3203 - Ringteori, and MA3204 - Homologisk algebra. While the second of these is really only used in one section contained in the second chapter, and is hence of not too great importance, the material from the others is used essentially throughout the text. [1] Additionally, in the third chapter of this thesis, we also assume some familiarity with affine algebraic varieties and some results concerning them, as well as some commutative algebra. Moreover, we note that in our work on this thesis, we have especially relied upon the following as references: [6] for material related to artin algebras; [3] for material on finite dimensional algebra over fields, quivers with relations, and related material; and [4] for material on commutative algebra.

We now briefly summarize the structure of the text:

Chapter 1 provides some necessary preliminaries. Among other things, we review some well-known material on different interpretations on the Ext-functor, but also show that the Hochschild cohomology ring of an algebra Λ over a commutative ring k is graded commutative if Λ is k-projective.

Chapter 2 develops the foundations of the theory of support varieties using the Hochschild cohomology ring. This chapter is based on [32].

[1] To put this another way, we assume the reader is familiar with, among other things, the terms and results in the following list, which, while perhaps long, is not meant to be exhaustive: rings, ideals, maximal ideals, prime ideals, modules, submodules, maximal submodules, simple modules, semisimple rings, semisimple modules, the First Isomorphism Theorem, the Correspondence Theorem, Noetherian, Artinian, the Wedderburn-Artin Theorem, algebraically closed fields, field extensions, separable field extensions, quivers, admissible relations, path algebras, quotients of path algebras, finite dimensional algebras over fields, artin algebras, the Jacobson radical of an algebra, projective modules, injective modules, projective covers, minimal projective resolutions, minimal injective resolutions, categories, functors, derived functors, pullbacks, pushouts, direct limits and colimits, Schanuel's Lemma, the Comparison Theorem, the Horseshoe Lemma, the Snake Lemma, and so on.

Chapter 3 investigates the consequences of the (**Fg**)-hypotheses, certain criteria an algebra must fulfil to have a useful theory of support varieties. With an eye to our goal, we especially concentrate on the effect assuming these hyotheses has for the complexity of the modules of an algebra. Additionally, to show how aspects of the theory of support varieties for group algebras can be recovered, we show that assuming an algebra satisfies the (**Fg**)-hypotheses allows one to show that every closed homogeneous variety is the variety of some module over that algebra. This chapter is based on [17], although some material from [7] is presented as well.

Chapter 4 attempts to investigate whether a result of Purin in [29] concerning the complexity of trivial extensions of hereditary algebras can be generalized. We begin the chapter by presenting some preliminary material analyzing the structure of $T(\Lambda)$-modules, and the relation between the projectives of Λ and those of $T(\Lambda)$. Following this, we give an example that shows that an algebra Λ can be of finite representation type while its trivial extension $T(\Lambda)$ has infinite complexity. To do this, we employ a result of Benson and some well-known results concerning radical square zero algebras. As a consequence, we see that Purin's result in [29] cannot be generalized in a straightforward fashion.

We continue by deriving a weak bound on the length of the terms of the minimal $T(\Lambda)$-projective resolution of a Λ-module considered as a $T(\Lambda)$-module. Inspired by this, we utilize the proof of a result of Guo et al. in [22] in giving a description of the syzygies and the minimal $T(\Lambda)$-projective resolution of a Λ-module considered as a $T(\Lambda)$-module. Finally, using this and a result by Dichi and Sangare in [15], we are able to show that if a selfinjective algebra Λ satisfies the finite generation hypotheses (**Fg**) then the complexity of $T(\Lambda)$ is exactly one greater than that of Λ.

Contents

Chapter 1

Preliminaries

This first chapter begins by briefly presenting some necessary preliminary results on different interpretations of certain derived functors: we show that for a ring Λ, the nth derived functors of $\mathrm{Hom}_\Lambda(M, N)$ for M and N Λ-modules, i.e. $\mathrm{Ext}^n_\Lambda(M, N)$, can be interpreted as sets of exact sequences with $n + 2$ consecutive non-zero terms for $n > 0$ under an equivalence relation and endowed with a certain group operation, the Baer sum. We further show that this interpretation readily yields a product structure on the direct product over $\mathbb{N}_0$ of these derived functors for $N = M$, i.e. $\bigoplus_{n\in\mathbb{N}_0} \mathrm{Ext}^n_\Lambda(M, M)$. The presentation here will be based mainly on that of [25]. Of course, while this material is well-known, it is nevertheless not covered in some of the more popular tracts on homological algebra, such as [31], and hence we have chosen to include it here.

In the second section of this chapter, we give the proof of a result which is foundational for the theory of support varieties we concern ourselves with in this text. As a consequence of this result, one can deduce that the Hochschild cohomology ring of an algebra Λ over a commutative ring k, i.e. $\mathrm{HH}^*(\Lambda)$, is a graded commutative ring provided Λ is k-projective.

1.1 Extensions

In this section we will lay the groundwork for the theory of support varieties that is to be presented in this text. Let Λ be a ring. To begin with, we will be reviewing some of the properties of

$$\mathrm{Ext}^*_\Lambda(M, M) = \bigoplus_{n\in\mathbb{N}_0} \mathrm{Ext}^n_\Lambda(M, M),$$

especially its representation as formal sums of equivalence classes of n-fold exact sequences, and its graded ring structure.

Definition 1.1.1. Let M and N be left Λ-modules. An **n-fold exact sequence** starting with N and ending with M is an exact sequence of left Λ-modules

$$0 \longrightarrow N \longrightarrow M^{n-1} \longrightarrow \cdots \longrightarrow M^0 \longrightarrow M \longrightarrow 0$$

beginning with N and ending with M, and with n terms in between.

We will need the following construction for short exact sequences to define the appropriate equivalence relation for n-fold exact sequences.

Construction. Let M, M', N and N' be left Λ-modules. Then, given some Λ-homomorphisms $f\colon N \to N'$ and $g\colon M' \to M$, and η a short exact sequence from N to M, define $f \cdot \eta$ to be the lower row of

$$\begin{tikzcd}
\eta\colon 0 \arrow[r] & N \arrow[r] \arrow[d, "f"'] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
f \cdot \eta\colon 0 \arrow[r] & N' \arrow[r] & E' \arrow[r] & M \arrow[r] & 0,
\end{tikzcd}$$

where the leftmost square is given by a pushout, and the lower right horizontal map is a cokernel; and $\eta \cdot g$ the upper row of

$$\begin{tikzcd}
\eta \cdot g\colon 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d, "g"] & 0 \\
\eta\colon 0 \arrow[r] & N \arrow[r] & E \arrow[r] & M \arrow[r] & 0,
\end{tikzcd}$$

where the rightmost square is given by a pullback, and the upper left horizontal map is a kernel.

Now, note that an m-fold exact sequence μ beginning in M and ending in L can be broken down into m 1-fold exact sequences, i.e. short exact sequences, as indicated in the following diagram:

$$\begin{tikzcd}
\mu\colon 0 \arrow[r] & M \arrow[r] & M^{m-1} \arrow[rr] \arrow[dr] & & M^{m-2} \arrow[rr] \arrow[dr] & & M^{m-3} \arrow[r] & \cdots \\
& & & K^{m-1} \arrow[ur] \arrow[dr] & & K^{m-2} \arrow[ur] \arrow[dr] & & \\
& & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

The short exact sequences involved in such a decomposition are unique up to isomorphism. We represent such decompositions as $\mu = M_n \cdot M_{n-1} \cdot \cdots \cdot M_1$, where M_n is the short exact sequence that begins in M, and M_1 is the one which ends in L.

Definition 1.1.2. Two n-fold exact sequences η and η' are **equivalent**, $\eta \cong \eta'$, if, when they are represented as products of short exact sequences, one can be obtained from the other in a finite number steps, wherein one in each step is permitted

1. to exchange a short exact sequence by one isomorphic to it;
2. to replace two adjacent short exact sequence of the form $(E \cdot f) \cdot E'$ with $E \cdot (f \cdot E')$;
3. or to replace two adjacent short exact sequence of the form $E \cdot (f \cdot E')$ with $(E \cdot f) \cdot E'$.

Now, although it is well-known that the collection of all equivalence classes of n-fold exact sequences beginning in N and ending in M is in bijection with $\mathrm{Ext}^n_\Lambda(M, N)$ under the appropriate assumptions, we shall, until we have shown this fact, let this collection be denoted by $\mathrm{ext}^n_\Lambda(M, N)$, for $n \geq 1$, and set $\mathrm{ext}^0_\Lambda(M, N) = \mathrm{Hom}_\Lambda(M, N)$. We also let the equivalence class of η be denoted by $\overline{\eta}$.

At this point, it should be remarked that there are some set theoretical difficulties with the presentation given here, namely that there is no guarantee that $\mathrm{ext}^n_\Lambda(M, N)$ is a set at all. These will be ignored. The reason for this is that, in practice, there will be placed sufficient restrictions on Λ ensuring that these troubles do not arise. To be more precise, the restrictions will be such that the category of finitely generated modules over Λ, i.e. $\mathrm{mod}\,\Lambda$, has enough projectives. Hence, it for instance suffices to assume that Λ is an artin algebra.

With this out of the way, we can define analogues of $f \cdot -$ and $- \cdot g$ for n-fold exact sequences: If $\eta = E_n \cdot E_{n-1} \cdots E_2 \cdot E_1$, then $f \cdot \eta = (f \cdot E_n) \cdot E_{n-1} \cdots E_2 \cdot E_1$; and similarly, $\eta \cdot g = E_n \cdot E_{n-1} \cdots E_2 \cdot (E_1 \cdot g)$. It is clear from the definition just given that $f \cdot (\eta \cdot g) = (f \cdot \eta) \cdot g$ holds.

We now look at what will form the multiplicative structure of $\mathrm{Ext}^*_\Lambda(M, M)$.

Definition 1.1.3. Let M, M' and N be left Λ-modules. If $\overline{\mu} \in \mathrm{ext}^m_\Lambda(L, M)$, $\overline{\nu} \in \mathrm{ext}^n_*(M', N)$ are represented by extensions

$$\mu\colon\ 0 \longrightarrow M \longrightarrow M^{m-1} \longrightarrow \cdots \longrightarrow M^0 \longrightarrow L \longrightarrow 0$$

and

$$\nu\colon\ 0 \longrightarrow N \longrightarrow N^{n-1} \longrightarrow \cdots \longrightarrow N^0 \longrightarrow M' \longrightarrow 0,$$

and provided $M = M'$, the **Yoneda splice** $\overline{\nu \cdot \mu} \in \mathrm{ext}^{m+n}_\Lambda(L, N)$ is defined and is given by the equivalence class of the following:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] & N^{n-1} \arrow[r] & \cdots \arrow[r] & N^0 \arrow[rr] \arrow[dr] & & M^{m-1} \arrow[r] & \cdots \arrow[r] & M^0 \arrow[r] & L \arrow[r] & 0 \\
& & & & & M \arrow[ur] \arrow[dr] & & & & & \\
& & & & 0 \arrow[ur] & & 0 & & & &
\end{tikzcd}$$

A more compact way to formulate this would be to say that the Yoneda splice of $\overline{\mu}$ and $\overline{\nu}$, provided it is defined, is $\overline{N_n \cdot N_{n-1} \cdot \cdots \cdot N_1 \cdot M_m \cdot M_{m-1} \cdot \cdots \cdot M_1}$, if $\mu = M_m \cdot M_{m-1} \cdot \cdots \cdot M_1$ and $\nu = N_{n-1} \cdot \cdots \cdot N_1$ are representatives of $\overline{\mu}$ and $\overline{\nu}$, respectively, and where the M_i and N_j for $1 \leq i \leq m$ and $1 \leq j \leq n$ correspond to the short exact sequences μ and ν are composed of, respectively.

From this point of view, it is rather immediate that this operation is well-defined: Let μ and μ' be extensions that begin with M and end in L, and ν and ν' be extensions that begin with N and end with M. Observe that if $\mu \cong \mu'$ and $\nu \cong \nu'$, then by (1.1.2), μ' can be obtained from μ in, say, k steps, and ν' can be obtained from ν in, say, k' steps. To obtain the splice of μ' and ν' from that of the splice of μ and ν, simply perform on it the k steps to obtain μ' from μ, and then perform the k' steps to obtain ν' from ν.

We now define what will induce the additive structure of $\operatorname{Ext}^*_\Lambda(M, M)$:

Definition 1.1.4. Let M, M', N and N' be left Λ-modules. If $\overline{\eta} \in \operatorname{ext}^n_*(M, N)$, $\overline{\nu} \in \operatorname{ext}^n_*(M', N')$ are represented by the n-fold exact sequence

$$\eta\colon\ 0 \longrightarrow N \longrightarrow E^{n-1} \longrightarrow \cdots \longrightarrow E^0 \longrightarrow M \longrightarrow 0$$

and

$$\nu\colon\ 0 \longrightarrow N' \longrightarrow N^{n-1} \longrightarrow \cdots \longrightarrow N^0 \longrightarrow M' \longrightarrow 0,$$

their **coproduct** $\overline{\eta} \oplus \overline{\nu} \in \operatorname{ext}^n_*(M \oplus M', N \oplus N')$ is given by the class of

$$\eta \oplus \nu\colon \quad \begin{array}{l} 0 \longrightarrow C \oplus C \longrightarrow E^{n-1} \oplus N^{n-1} \longrightarrow \cdots \\ \cdots \longrightarrow E^0 \oplus N^0 \longrightarrow A \oplus A \longrightarrow 0. \end{array}$$

For $\overline{\eta}$, $\overline{\nu} \in \operatorname{ext}^n_*(M, N)$, we then have the **Baer sum**

$$\overline{\eta} + \overline{\nu} = \nabla_N \cdot (\overline{\eta} \oplus \overline{\nu}) \cdot \Delta_M,$$

where $\nabla_N = \begin{pmatrix} 1 & 1 \end{pmatrix} : N \oplus N \to N$ and $\Delta_M = \begin{pmatrix} 1 \\ 1 \end{pmatrix} : M \to M \oplus M$.

We will not show here that these operations are well-defined, as we instead refer to the following proposition for this.

Proposition 1.1.5. *Let L, M, and N be left Λ-modules. Let $\overline{\nu}$, $\overline{\nu'} \in \operatorname{ext}^n_\Lambda(M,N)$, $\overline{\mu}, \overline{\mu'} \in \operatorname{ext}^m_\Lambda(L,M)$, and $\overline{\lambda} \in \operatorname{ext}^l(K,L)$. We have then that $\overline{\nu} + \overline{\nu'}$ and $\overline{\mu} \cdot \overline{\nu}$ are both well-defined, and that the distributive properties*

$$\overline{\mu}(\overline{\nu} + \overline{\nu'}) = \overline{\mu} \cdot \overline{\nu} + \overline{\mu} \cdot \overline{\nu'}$$

and

$$(\overline{\mu} + \overline{\mu'})\overline{\nu} = \overline{\mu} \cdot \overline{\nu} + \overline{\mu'} \cdot \overline{\nu'}$$

both hold.

In addition, one has the the associative property

$$\overline{\lambda} \cdot (\overline{\mu} \cdot \overline{\nu}) = (\overline{\lambda} \cdot \overline{\mu}) \cdot \overline{\nu}.$$

Moreover, the Baer sum operation makes $\operatorname{ext}^n_\Lambda(M,N)$ into an abelian group.

Proof. See [25, Theorem 5.3, Chapter 3]. □

Given the statement of the following corollary, it might be pertinent to recall the definition of a graded ring and some related notions: Following [4, Chapter 10], we define a **graded ring** to be a ring A together with a family $(A_n)_{n\geq 0}$ of subgroups of the additive group of A satisfying $A = \bigoplus_{n\in\mathbb{N}_0} A_n$ and $A_m A_n \subseteq A_{m+n}$ for all $m, n \geq 0$. If x is an element of A_n for some $n \geq 0$, then we say that x is **homogeneous** and of **degree** n. One can immediately observe some notable consequences, namely that A_0 becomes a subring of A and that each A_n is an A_0-module.

Corollary 1.1.6. *Let M be a left Λ-module. $\operatorname{ext}^*_\Lambda(M,M) = \bigoplus_{n\in\mathbb{N}_0} \operatorname{ext}^n_\Lambda(M,M)$ is then a graded ring.*

Proof. This is almost immediate by the preceding proposition: The additive structure is given by the direct sum structure of $\bigoplus_{n\in\mathbb{N}_0} \operatorname{ext}^n_\Lambda(M,M)$. As each n-fold exact sequence begins and ends with M, the Yoneda splice of two extensions is always defined, and can be extended as an operation linearly over formal sums of extensions of different length. Finally, we have that the identity element is

$$1_M \in \operatorname{ext}^0_\Lambda(M,M) = \operatorname{Hom}_\Lambda(M,M),$$

as, if $\mu \in \operatorname{ext}^*_\Lambda(M,M)$, $1_M \cdot \mu = \mu$ and $\mu \cdot 1_M = \mu$. □

We have need of the following notion in the next result: by a **morphism between n-fold exact sequences** η and η', $\Gamma\colon \eta \to \eta'$, we mean a collection of morphisms $(f, \dots, g)$ forming a commutative diagram of the following form:

$$\begin{tikzcd} \eta\colon 0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & M^{n-1} \arrow[r] \arrow[d] & M^{n-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & M^1 \arrow[r] \arrow[d] & M^0 \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, "g"] & 0 \\ \eta'\colon 0 \arrow[r] & M \arrow[r] & N^{n-1} \arrow[r] & N^{n-2} \arrow[r] & \cdots \arrow[r] & N^1 \arrow[r] & N^0 \arrow[r] & L \arrow[r] & 0 \end{tikzcd}$$

Proposition 1.1.7. *Let L and M be in* $\operatorname{mod}\Lambda$*, and let $P^* \to L$ be a projective resolution of L. Then there exists mutually inverse isomorphisms $\Phi\colon \operatorname{ext}^n_\Lambda(L, M) \to \operatorname{Ext}^n_\Lambda(L, M)$ and $\Psi\colon \operatorname{Ext}^n_\Lambda(L, M) \to \operatorname{ext}^n_\Lambda(L, M)$.*

Proof. Let $\overline{\mu} \in \operatorname{ext}^n_\Lambda(L, M)$. If μ is a representative of $\overline{\mu}$, and $P^* \to L$ is a projective resolution of L, then we derive an element of $\operatorname{Ext}^*_\Lambda(L, M)$ in the following fashion. Note that by our assumptions, we have a diagram of the form

$$\begin{tikzcd} \cdots \arrow[r] & P^{n+1} \arrow[r] & P^n \arrow[r] & P^{n-1} \arrow[r] & P^{n-2} \arrow[r] & \cdots \arrow[r] & P^1 \arrow[r] & P^0 \arrow[r] & L \arrow[r] \arrow[d, equal] & 0 \\ & 0 \arrow[r] & M \arrow[r] & M^{n-1} \arrow[r] & M^{n-2} \arrow[r] & \cdots \arrow[r] & M^1 \arrow[r] & M^0 \arrow[r] & L \arrow[r] & 0, \end{tikzcd}$$

where the lower row is μ. By the comparison theorem, this can be filled out to give

$$\begin{tikzcd} \cdots \arrow[r] & P^{n+1} \arrow[r] \arrow[d] & P^n \arrow[r] \arrow[d, "\iota_n"'] & P^{n-1} \arrow[r] \arrow[d, "\iota_{n-1}"'] & P^{n-2} \arrow[r] \arrow[d, "\iota_{n-2}"'] & \cdots \arrow[r] & P^1 \arrow[r] \arrow[d, "\iota_1"'] & P^0 \arrow[r] \arrow[d, "\iota_0"'] & L \arrow[r] \arrow[d, equal] & 0 \\ & 0 \arrow[r] & M \arrow[r] & M^{n-1} \arrow[r] & M^{n-2} \arrow[r] & \cdots \arrow[r] & M^1 \arrow[r] & M^0 \arrow[r] & L \arrow[r] & 0, \end{tikzcd}$$

i.e. we get ι, a lifting of 1_L.

As the leftmost square commutes, we have that ι_n is a cocycle, and hence its class lies in $\operatorname{Ext}^n_\Lambda(L, M)$. Define $\Phi\colon \operatorname{ext}^*_\Lambda(L, M) \to \operatorname{Ext}^*_\Lambda(L, M)$ by letting $\Phi(\overline{\mu})$ be the ι_n just obtained. This must be shown to be well-defined. Observe thus that if we had chosen some other lifting of 1_L, say ι', and had gotten some other $\iota'_n\colon P^n \to M$, then by the Comparison Theorem, these two liftings would be chain homotopic, implying that $\iota_n - \iota'_n = \partial_\mu s_n + s_{n-1}\partial_{P^*}$ would hold for $s_i\colon P^i \to M^{i+1}$. But $M^{n+1} = 0$, so that s_n is the zero-morphism. Hence, $\iota_n - \iota'_n = s_{n-1}\partial_{P^*}$ holds, i.e. ι and ι' are mapped to the same cocycle.

Let now μ and μ' be such that there exists some morphism of n-fold exact sequences between them, say $f\colon \mu \to \mu'$, where this morphism is given by

$$f = (1_M, f^{n-1}, \dots, f^1, f^0, 1_L),$$

which is to say that f is some morphism of n-fold exact sequences that begins witht 1_M and ends in 1_L. It can then be seen that $f\iota$ is a lifting of 1_M over μ', so that the argument of the previous paragraph implies that μ and μ' map to the same element of $\mathrm{Ext}^n_\Lambda(L, M)$. As this is the case, it can be seen that the same holds for equivalent n-fold exact sequences. Indeed, this follows by two results of [25], the first of which is [25, Proposition 5.1, Chapter 3], which states the following: if there exists a morphism of extensions $\Gamma\colon \eta \to \eta'$ beginning with $f\colon M \to M'$ and ending with $g\colon L \to L'$, then $f \cdot \eta \cong \eta' \cdot g$ holds. The second is [25, Proposition 5.2, Chapter 3], which states: two n-fold exact extensions η and η' beginning at M and ending in L are equivalent if and only if there is an integer k such that there are $2k$ morphisms of n-fold exact sequences of the form

$$\eta = \eta_0 \longrightarrow \eta_1 \longleftarrow \eta_2 \longrightarrow \cdots \longleftarrow \eta_{2k-2} \longrightarrow \eta_{2k-1} \longleftarrow \eta_{2k} = \eta',$$

running alternately to the left and to the right, all of which begin with 1_M and end with 1_L. In other words, it suffices to repeat the argument just given $2k$ times to deduce our claim, namely that Φ maps equivalent n-fold exact extensions to the same cocycle in $\mathrm{Ext}^n_\Lambda(L, M)$.

We now show that Φ has an inverse $\Psi\colon\ \mathrm{Ext}^n_\Lambda(L, M) \to \mathrm{ext}^n_\Lambda(L, M)$. Observe that any cocycle $\sigma\colon P^n \to M$ vanishes on $\partial_{P^*}P^{n+1}$. Hence, it can be factored uniquely as a product through $\partial^*_P P^n$, as in the following commutative diagram:

$$\begin{tikzcd}
P^{n+1} \arrow[r] & P^n \arrow[d, "\partial'"] \arrow[dr, "\partial_{P^*}"] & & & & & & \\
0 \arrow[r] & \partial^*_P P^n \arrow[r, "j"] \arrow[d, "\sigma'"] & P^{n-1} \arrow[r] & \cdots \arrow[r] & P^1 \arrow[r] & P^0 \arrow[r] & L \arrow[r] & 0 \\
& M & & & & & &
\end{tikzcd}$$

If we let the middle row here be denoted by $\sigma(P^*, M)$, we can define our Ψ by setting $\Psi(\overline{\sigma'\partial'}) = \overline{(\sigma' \cdot \sigma(P^*, L)}$. By the distributive laws proven in (1.1.5), the right hand side is additive in σ', so that to show well-definedness of Ψ, it suffices to show that it vanishes whenever $\sigma'\partial'$ is a coboundary $\gamma\partial_{P^*}$, for $\gamma\colon P^{n-1} \to M$. But then one has that $\gamma\partial_{P^*} = \gamma j\partial'$. By associativity of the Yoneda splice, i.e. (1.1.5), we have that if we can show that $j \cdot \sigma(P^*, L) \cong 0$, then we would be done.

Examine thus the following diagram, in which the lower row is the leftmost short exact sequence of $j \cdot \sigma(P^*, M)$:

$$\begin{tikzcd}
0 \arrow[r] & \partial_P^* P^n \arrow[r, "j"] \arrow[d, "j"] & P^{n-1} \arrow[r] \arrow[d, "k'"] & \partial_P^* P^{n-1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P^{n-1} \arrow[r, "k"] & U \arrow[r] & \operatorname{Cok} k \arrow[r] & 0
\end{tikzcd}$$

It is known that any U, k and k' that satisfy such a diagram must be the pushout of $j\colon \partial_P^* P^n \to P^{n-1}$ with itself. However, $U = P^{n-1} \oplus \partial_P^* P^{n-1}$ with morphisms the inclusion of P^{n-1} and the projection onto $\partial_P^* P^{n-1}$ satisfies the diagram, entailing that any other solution must also be split. By the proof of [25, Theorem 5.3, Chapter 3], the additive identity element of $\operatorname{ext}^1_\Lambda(M, M)$ is the equivalence class of the split short exact sequences. This, of course, implies what we wished to show.

It remains to show that Ψ and Φ are each other's inverse, and that they are homomorphisms. With respect to the former, this follows by comparing the two diagrams we have produced. We see that $\Psi \circ \Phi = 1$ follows by a simple computation, followed by an application of the Comparison Theorem. For a statement of this result, see for instance [25, Theorem 6.1, Chapter 3]. To see that $\Phi \circ \Psi = 1$, examine the diagram below, where the lower row is the chosen representative of μ and the upper row is the representative μ that $\Phi \circ \Psi$ outputs.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & E \arrow[r] & \cdots \arrow[r] & P^1 \arrow[r] & P^0 \arrow[r] & L \arrow[r] & 0 \\
0 \arrow[r] & \partial_P^* P^n \arrow[r, "j"] \arrow[u, "\sigma'"] \arrow[d, "\sigma'"] & P^{n-1} \arrow[r] \arrow[u] \arrow[d] & \cdots \arrow[r] & P^1 \arrow[r] \arrow[u, equal] \arrow[d] & P^0 \arrow[r] \arrow[u, equal] \arrow[d] & L \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & M^{n-1} \arrow[r] & \cdots \arrow[r] & M^1 \arrow[r] & M^0 \arrow[r] & L \arrow[r] & 0
\end{tikzcd}$$

The result follows by yet again applying [25, Proposition 5.1, Chapter 3].

Finally, with respect to whether they are homomorphisms, we can observe that we have already established that Ψ is additive, which implies that Φ is as well, and we are done. $\square$

As promised, we henceforth use only the Ext notation. To finish out the section, we show a small, useful corollary.

Corollary 1.1.8. *If $\overline{\mu} \in \operatorname{Ext}^n_\Lambda(L, M)$ with $n \geq 1$, then there is a representative of $\overline{\mu}$ of the form*

$$0 \longrightarrow M \longrightarrow E \longrightarrow P^{n-2} \longrightarrow P^{n-3} \longrightarrow \cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow L \longrightarrow 0,$$

in which each P^i is projective.

Proof. If the notation is as above, let σ be a representative of $\Phi(\overline{\mu})$. Then $\sigma = \sigma'\partial'$, so that $\sigma' \cdot \sigma(P^*, L)$ is of the desired form. □

1.2 Graded commutativity of $\mathrm{HH}^*(\Lambda)$

Let Λ and Γ be algebras over a commutative ring k. We denote by Λ^e the enveloping algebra $\Lambda \otimes_k \Lambda^{\mathrm{op}}$. The goal will be to show that $\mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda)$ is graded commutative. For Λ k-projective, it is known that $\mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda)$ is isomorphic to the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$ of Λ. See [13, Proposition 4.3, Chapter IX] or [30]. As $\mathrm{HH}^*(\Lambda)$ is known to be graded commutative, these observations would suffice. However, we will be working under the weaker assumption that Λ is k-flat, and will instead prove directly that $\mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda)$ is graded commutative, hence also providing a proof for $\mathrm{HH}^*(\Lambda)$ being graded commutative for Λ k-projective.

To do this, we first show that there is a homomorphism of graded rings

$$\Phi_M \colon \mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda) \to \mathrm{Ext}^*_{\Lambda \otimes_k \Gamma^{\mathrm{op}}}(M, M),$$

given by $\Phi_M(\eta) = \eta \otimes_\Lambda M$, where M is some Λ-Λ-bimodule. Then, as this induces an $\mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda)$-module structure on $\mathrm{Ext}^*_{\Lambda \otimes_k \Gamma^{\mathrm{op}}}(M, N)$ for a pair of Λ-Γ-bimodules M and N, we prove that

$$\Phi_N(\eta)\theta = (-1)^{mn}\theta\Phi_M(\eta)$$

holds for η in $\mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda)$ and θ in $\mathrm{Ext}^*_{\Lambda \otimes_k \Gamma^{\mathrm{op}}}(M, N)$.

To begin with, we prove a minor technical result that will be needed in the sequel:

Proposition 1.2.1. *If Λ is flat over k, then $\Lambda^e = \Lambda \otimes_k \Lambda^{\mathrm{op}}$ is flat as a left and as a right Λ-module. Moreover, any projective Λ^e-module is flat as a left and as a right Λ-module.*

Proof. To see that the first statement must hold, let M be a right Λ-module. It follows then that

$$M \otimes_\Lambda \Lambda^e = M \otimes_\Lambda (\Lambda \otimes_k \Lambda^{\mathrm{op}}) \cong (M \otimes_\Lambda \Lambda) \otimes_k \Lambda^{\mathrm{op}} \cong M \otimes_k \Lambda^{\mathrm{op}}.$$

In other words, Λ-tensoring with Λ^e on the right is equivalent to k-tensoring with Λ^{op}, which is k-flat since Λ is k-flat and $\Lambda \cong_k \Lambda^{\mathrm{op}}$. The case for left Λ-modules is similar. Since any summand of a flat module is itself flat, the second statement follows. □

For this next proposition, we need to recall the following notion: we call a short exact sequence of left Λ-modules $0 \to A \to B \to C \to 0$ **pure exact** if we have exactness of

$$0 \to M \otimes_\Lambda A \to M \otimes_\Lambda B \to M \otimes_\Lambda C \to 0$$

for arbitrary right Λ-modules M. The definition for a short exact sequence of right Λ-modules is entirely analogous.

Proposition 1.2.2. *For a Λ-Γ-bimodule M, $\phi_M(\eta) = \eta \otimes_\Lambda M$ yields a homomorphism of graded rings $\phi_M \colon \operatorname{Ext}^*_{\Lambda^e}(\Lambda, \Lambda) \to \operatorname{Ext}^*_{\Lambda\otimes\Gamma^{\mathrm{op}}}(M, M)$.*

Proof. First of all, it must be shown that $\phi_M(\eta) \in \operatorname{Ext}^*_{\Lambda\otimes\Gamma^{\mathrm{op}}}(M, M)$ holds. Note that $\eta \otimes_\Lambda M$ is given by the n-fold exact sequence

$$0 \longrightarrow M \longrightarrow E \otimes M \longrightarrow P^{n-2} \otimes M \longrightarrow \cdots$$

$$\cdots \longrightarrow P^0 \otimes M \longrightarrow M \longrightarrow 0$$

if η is given by

$$0 \longrightarrow \Lambda \longrightarrow E \longrightarrow P^{n-2} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow \Lambda \longrightarrow 0,$$

where each of the P^i for $i \geq 0$ are Λ^e-projective. That we can choose such a representative of η follows by (1.1.8).

Observe that $\eta \otimes_\Lambda M$ is exact, since each of the short exact sequences η is composed of are themselves pure exact. Indeed, the rightmost sequence is pure exact by [31, Proposition 3.67, Chapter 3] since Λ is Λ-flat as it is clearly Λ-projective. Furthermore, since by (1.2.1), P^i for all i are Λ-flat, and flat modules are closed under kernels of epimorphisms (between flat modules), this argument can clearly be repeated for the next short exact sequence and so on.

What remains is to show that the homomorphism properties hold for ϕ_M. Note that if $\eta, \nu \in \operatorname{Ext}^n_{\Lambda^e}(\Lambda, \Lambda)$, then $\phi_M(\eta + \nu) = \phi_M(\eta) + \phi_M(\nu)$ holds if

$$\begin{aligned} \phi_M(\eta+\nu) &= \phi_M(\nabla_\Lambda \cdot (\eta \oplus \nu) \cdot \Delta_\Lambda) \\ &= \nabla_M \cdot \phi_M(\eta \oplus \nu) \cdot \Delta_M \\ &= \nabla_M \cdot (\phi_M(\eta) \oplus \phi_M(\nu)) \cdot \Delta_M. \end{aligned}$$

As it is clear that $\phi_M(\eta \oplus \nu) = \phi_M(\eta) \oplus \phi_M(\nu)$, what remains is to show that $\nabla_\Lambda \cdot -$ and $- \cdot \Delta_\Lambda$ commute with ϕ_M. Since $\phi_M(-) = - \otimes_\Lambda M$ is left adjoint, $f \cdot -$ is defined by way of a pushout, and left adjoint functors commute with colimits, one has that $\phi_M(f \cdot -) = f \otimes M \cdot \phi_M(-)$.

Let now A, B, C and D be Λ-Λ-bimodules, let M be a Λ-Γ-bimodule, and assume that C and D are Λ-flat. Examine the following diagram:

$$\begin{tikzcd}
(\mu \cdot g) \otimes_\Lambda M\colon 0 \arrow[r] & A \otimes M \arrow[r] \arrow[d, equal] & B' \otimes M \arrow[r] \arrow[d] & D \otimes M \arrow[r] \arrow[d, "g \otimes_\Lambda M"] & 0 \\
\mu \otimes_\Lambda M\colon 0 \arrow[r] & A \otimes M \arrow[r] & B \otimes M \arrow[r] & C \otimes M \arrow[r] & 0 \\
(\mu \otimes_\Lambda M) \cdot (g \otimes_\Lambda M)\colon 0 \arrow[r] & A \otimes M \arrow[r] \arrow[u, equal] & B'' \arrow[r] \arrow[u] & D \otimes M \arrow[r] \arrow[u, "g \otimes_\Lambda M"'] & 0
\end{tikzcd}$$

Here the upper sequence remains short exact after applying $- \otimes_\Lambda M$, since D flat entails that $\mu \cdot g$ is pure exact by [31, Proposition 3.67, Chapter 3]. As the upper leftmost vertical morphism is an isomorphism, this implies that the upper right square is a pulllback. Since pullbacks are unique up to isomorphism, it follows that $(\mu \cdot g) \otimes_\Lambda M = \mu \otimes_\Lambda M \cdot (g \otimes_\Lambda M)$. Hence, we have shown that the additive homomorphism property holds. As the multiplicative homomorphism property obviously holds, and since $\phi_M(1_\Lambda) = 1_M$ follows from $- \otimes_\Lambda M$ being a functor, we are done. $\square$

We are now finally in a position to prove the results which form the base upon which the theory of support varieties based on Hochschild cohomology is built.

Proposition 1.2.3. *Let Λ and Γ be two algebras over a commutative ring k. Assume that Λ is flat as a module over k. Let η be an element in $\mathrm{HH}^n(\Lambda)$, and let θ be an element in $\mathrm{Ext}^m_{\Lambda \otimes_k \Gamma^{\mathrm{op}}}(M, N)$ for two Λ-Γ-bimodules M and N. Then*

$$\Phi_N(\eta)\theta = (-1)^{mn} \theta \Phi_M(\eta).$$

Proof. Assume M and N to be Λ-Γ-bimodules, and let

$$\cdots \longrightarrow P^n \longrightarrow P^{n-1} \longrightarrow \cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow \Lambda \longrightarrow 0$$

be a Λ^e-projective resolution of Λ. As Λ is assumed to be k-flat, we have by (1.1.8) that any Λ^e-projective module is flat both as a left and as a right Λ-module. Moreover, since flat left and right Λ-modules are closed under kernels of epimorphisms, a syzygy of Λ as a Λ^e-module must be flat as both a left and as a right Λ-module. Let now η be given by

$$0 \longrightarrow \Lambda \longrightarrow E \longrightarrow P^{n-2} \longrightarrow \cdots \longrightarrow P^1 \longrightarrow P^0 \longrightarrow \Lambda \longrightarrow 0.$$

First we consider the case when $\theta \in \mathrm{Hom}_{\Lambda\otimes_k\Gamma^{\mathrm{op}}}(M,N)$ holds. It can be seen that $\mathrm{HH}^0(\Lambda) = \mathrm{Hom}_{\Lambda^e}(\Lambda,\Lambda) = Z(\Lambda)$ holds, where $Z(\Lambda)$ is the centre of Λ. Indeed, if $f \in \mathrm{Hom}_{\Lambda^e}(\Lambda,\Lambda)$, then we have

$$f(\lambda) = f((\lambda \otimes_k 1)\cdot 1) = f((1\otimes_k \lambda)\cdot 1).$$

But then we have that

$$f((\lambda \otimes_k 1)\cdot 1) = (\lambda\otimes_k 1)\cdot f(1) = \lambda\cdot f(1)$$

and

$$f((1 \otimes_k \lambda)\cdot 1) = (1\otimes_k \lambda)\cdot f(1) = f(1)\cdot \lambda.$$

Hence, it follows that $\lambda\cdot f(1) = f(1)\cdot\lambda$ holds for all $\lambda\in\Lambda$ and all $f \in \mathrm{Hom}_{\Lambda^e}(\Lambda,\Lambda)$, which, as every $f \in \mathrm{Hom}_{\Lambda^e}(\Lambda,\Lambda)$ is determined by its value at 1, is precisely equivalent to $\mathrm{Hom}_{\Lambda^e}(\Lambda,\Lambda) = Z(\Lambda)$. Clearly then the claim holds for $\eta \in \mathrm{HH}^0(\Lambda)$.

If we suppose that $\eta \in \mathrm{HH}^1(\Lambda)$, then it follows that there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \Lambda\otimes_\Lambda M \arrow[r] \arrow[d, "\Lambda\otimes\theta"] & E\otimes_\Lambda M \arrow[r] \arrow[d] & \Lambda\otimes_\Lambda M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Lambda\otimes_\Lambda N \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & \Lambda\otimes_\Lambda M \arrow[r] \arrow[d, "\Lambda\otimes\theta"] & 0 \\
0 \arrow[r] & \Lambda\otimes_\Lambda N \arrow[r] & E\otimes_\Lambda N \arrow[r] & \Lambda\otimes_\Lambda N \arrow[r] & 0
\end{tikzcd}$$

where the composition of the middle two horizontal morphisms is equal to $E\otimes\theta$. This then shows that $(\eta\otimes_\Lambda N)\theta = \theta(\eta\otimes_\Lambda M)$. Moreover, this argument can clearly be extended to any homogeneous element η in $\mathrm{HH}^*(\Lambda)$: to be precise, we do this by considering a representative of η and repeatedly employing the recently derived equation on a decomposition of that representative into short exact sequences of Λ^e-modules.

Let now $\theta\colon 0 \to N \to X \to M \to 0$ be an element of $\mathrm{Ext}^1_{\Lambda\otimes_k\Gamma^{\mathrm{op}}}(M,N)$. Since, as mentioned above, all the syzygies of Λ as a Λ^e-module are flat as right Λ-modules, we have the following exact and commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^i_{\Lambda^e}(\Lambda)\otimes_\Lambda N \arrow[r] \arrow[d] & P^{i-1}\otimes_\Lambda N \arrow[r] \arrow[d] & \Omega^{i-1}_{\Lambda^e}(\Lambda)\otimes_\Lambda N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^i_{\Lambda^e}(\Lambda)\otimes_\Lambda X \arrow[r] \arrow[d] & P^{i-1}\otimes_\Lambda X \arrow[r] \arrow[d] & \Omega^{i-1}_{\Lambda^e}(\Lambda)\otimes_\Lambda X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^i_{\Lambda^e}(\Lambda)\otimes_\Lambda M \arrow[r] \arrow[d] & P^{i-1}\otimes_\Lambda M \arrow[r] \arrow[d] & \Omega^{i-1}_{\Lambda^e}(\Lambda)\otimes_\Lambda M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

where $\Omega^0_{\Lambda^e}(\Lambda) = \Lambda$. To see this, note that while the commutativity is obvious, all of the rows are the images of pure exact sequences of right Λ-modules under tensoring on the right by some module, and thus remain exact. On the other hand, each of the columns is the image of some short exact sequence under tensoring on the left by some Λ-flat module, and are thus also exact.

Denote then the upper row of this diagram by σ_i, the right-most column by θ_i, the left-most column by θ_{i+1} and the lower row by σ'_i. It follows then by [25, 22, Lemma 3.2, Chapter VIII] that the equality $\sigma_i\theta_i = -\theta_{i+1}\sigma'_i$ holds for all $i \geq 1$. Since

$$\eta\otimes_\Lambda N = (0 \to N \to E\otimes_\Lambda N \to \Omega^{n-1}_{\Lambda^e}(\Lambda)\otimes_\Lambda N \to 0)\sigma_{n-1}\cdots\sigma_2\sigma_1$$

and

$$\eta\otimes_\Lambda M = (0 \to M \to E\otimes_\Lambda M \to \Omega^{n-1}_{\Lambda^e}(\Lambda)\otimes_\Lambda M \to 0)\sigma'_{n-1}\cdots\sigma'_2\sigma'_1,$$

so that by repeatedly applying the aforementioned equality, we find that

$$(\eta\otimes_\Lambda N)\theta = (-1)^n\theta(\eta\otimes_\Lambda M).$$

If $\theta \in \mathrm{Ext}^m_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}(M,N)$, we can view θ as a product of m short exact sequences. Hence, applying our newly derived equality m-times yields

$$(\eta\otimes_\Lambda N)\theta = (-1)^{mn}\theta(\eta\otimes_\Lambda M)$$

for $\eta \in \mathrm{HH}^n(\Lambda)$ and $\theta \in \mathrm{Ext}^m_{\Lambda\otimes_k\Lambda^{\mathrm{op}}}(M,N)$. $\square$

As previously mentioned, it is known that $\mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda)$ is isomorphic to the Hochschild cohomology ring of Λ when Λ is projective over k. As promised, we thus have result (b) of the following corollary, which follows easily by the preceding result by setting $\Gamma = \Lambda$.

Corollary 1.2.4. *Let Λ be an algebra over a commutative ring k, where Λ is flat as a module over k.*

(a) *The ring* $\mathrm{HH}^*(\Lambda) = \mathrm{Ext}^*_{\Lambda^e}(\Lambda, \Lambda)$ *is graded commutative.*

(b) *If Λ is projective as a module over k, then the Hochschild cohomology ring of Λ is graded commutative.*

Using part (a) of this corollary, we can see that any homogeneous element of $\mathrm{HH}^*(\Lambda)$ of odd degree is nilpotent with degree of nilpotency equal to 2 whenever the characteristic of k is different from 2.

The final result of this section is also an immediate consequence of (1.2.3).

Corollary 1.2.5. *Let Λ be an algebra over a commutative ring k, where Λ is flat as a module over k. Then the left and the right* $\mathrm{HH}^*(\Lambda)$*-module structures induced by Φ_N and Φ_M on* $\mathrm{Ext}^*_\Lambda(M, N)$ *are related as follows: if θ is an element in* $\mathrm{Ext}^m_\Lambda(M, N)$ *and η is in* $\mathrm{HH}^n(\Lambda)$*, then*

$$\Phi_N(\eta)\theta = (-1)^{mn}\theta\Phi_M(\eta).$$

Chapter 2

Support varieties

In the following chapter, building upon the basic results in the preceding chapter, we develop the foundations of a theory of support varieties using the Hochschild cohomology ring. We follow the presentation in [32]. In doing so, we gradually and successively assume increasingly restrictive hypotheses. While beginning by assuming more restrictive hypotheses might simplify some proofs, one would not quite so easily see which hypotheses could be dispensed with.

We note that this presentation is only a selection of the results in [32]: for instance, we have excluded some of the results concerning selfinjective algebras as well as the proof of the variety of a module being an invariant of the Auslander-Reiten component it belongs to. While these are natural and important developments, restrictions on the scope and length of this text prohibit their inclusion. Indeed, we make no use of them in our investigation of possible generalisations of the aforementioned results of Purin in [29].

2.1 Basic properties of support varieties

We have a few standing assumptions in this section. Namely, we assume that Λ is an algebra over a commutative ring k, and that Λ is flat as a module over k. Moreover, we will be working with $H = \bigoplus_{i\geq 0} H^i$, some graded subring of $\mathrm{HH}^*(\Lambda)$.

In this section, we define support varieties for pairs of modules (M, N) using the ring homomorphism

$$H \to \mathrm{HH}^*(\Lambda) \to \mathrm{Ext}^*_\Lambda(M, M).$$

Additionally, we derive some of the elementary properties of these varieties. To a given pair of Λ-modules (M, N) we can associate the left and the right annihilators of $\mathrm{Ext}^*_\Lambda(M, N)$ as a left H-module and as a right H-module, respectively, which we denote by $A^l_H(M, N)$ and $A^r_H(M, N)$, again respectively. Our first result of the section shows that these are graded ideals, and that, in fact, they are equal.

Lemma 2.1.1. *Let M and N be Λ-modules. The ideals $A^l_H(M,N)$ and $A^r_H(M,N)$ are equal and graded ideals in H. Moreover, if $\operatorname{Ext}^*_\Lambda(M,N)$ is non-zero, then $A^l_H(M,N) = A^r_H(M,N)$ is a proper ideal.*

Proof. Let x be an element of H. We have then that $x = \sum_{i=0}^N x_i$ for some x_i in H^i and some $N \in \mathbb{N}_0$. Assume now that $xm = 0$ for all $m \in \operatorname{Ext}^*_\Lambda(M,N)$. It follows then that $xm = 0$ holds for all homogeneous m, i.e. $m = m_j \in \operatorname{Ext}^j_\Lambda(M,N)$. Hence, we have that $xm = \sum_{i=0}^N x_i m_j = 0$ if and only if $x_i m_j = 0$ for $i = 0, 1, \ldots, N$. It follows that if $x \operatorname{Ext}^*_\Lambda(M,N) = (0)$, then all the homogeneous parts of x, i.e. the x_i such that $x = \sum_{i=0}^N x_i$, annihilate all the homogeneous elements in $\operatorname{Ext}^*_\Lambda(M,N)$ and thus also all of the elements of $\operatorname{Ext}^*_\Lambda(M,N)$. In other words, the left annihilator of $\operatorname{Ext}^*_\Lambda(M,N)$ is a graded right ideal. Since a similar argument entails that the right annihilator is a graded left ideal, it follows that (1.2.5) implies that the left and right annihilators of $\operatorname{Ext}^*_\Lambda(M,N)$ coincide, and thus that $A^l_H(M,N) = A^r_H(M,N)$ is a graded ideal in H.

If $\operatorname{Ext}^*_\Lambda(M,N)$ is non-zero, we have that there is some $i_0 \geq 0$ such that $\operatorname{Ext}^{i_0}_\Lambda(M,N) \neq (0)$, implying that the identity in H is not in $A^l_H(M,N)$, in which case it must be a proper ideal. □

Henceforth we let the graded ideal $A^l_H(M,N) = A^r_H(M,N)$ be denoted by $A_H(M,N)$.

An additional standing assumption for the remainder of the section is that H^0 is a local k-algebra. Given this assumption, the maximal ideal $\mathfrak{a} = \langle \operatorname{rad}(H^0), H^{\geq 1} \rangle$ will play a central role, namely by defining the modules with a trivial variety. For group rings, the modules with a trivial variety are precisely the projective modules. In our more general setting, we will later see that the class of modules which have a trivial variety will include those of finite injective or projective dimension, and those with no self extensions.

Moreover, one can see that H^0 is a local k-algebra if Λ is an indecomposable artin k-algebra, k is a commutative Artinian ring and $H = \operatorname{HH}^*(\Lambda)$. In fact, as seen before, since $H = \operatorname{HH}^*(\Lambda)$, it follows that $H^0 = Z(\Lambda)$, the centre of Λ. Moreover, for such a Λ, it is the case that H^0 has no non-trivial idempotents, as if it did, Λ would have a non-trivial central idempotent and would thus not be indecomposable. Finally, we can see that H^0 is an artin k-algebra by the same ring homomorphism that yields the k-algebra structure of Λ since it is clearly finitely generated as a k-module as it is a k-submodule of Λ. In sum, this thus implies that H^0 has only one indecomposable projective left module, namely itself as a left module, and its top is thus simple. Hence, $\operatorname{rad} H^0$ is maximal, i.e. H^0 is local.

This next lemma shows that the ideal $\mathfrak{a}$ contains the ideal $A_H(M,N)$ for any pair of modules (M,N) if $\operatorname{Ext}^*_\Lambda(M,N) \neq (0)$.

Lemma 2.1.2. *Let M and N be two Λ-modules.*

(a) *If* $\operatorname{Ext}^*_\Lambda(M,N)$ *is non-zero, then the ideal* $A_H(M,N)$ *is contained in the ideal*

$$\mathfrak{a} = \langle \operatorname{rad} H^0, H^{\geq 1}\rangle.$$

(b) *If* $\mathfrak{m}$ *is a non-zero maximal ideal in* H*, then* $\mathfrak{m}$ *contains the ideal generated by* $\operatorname{rad}(H^0)$*.*

Proof. (a) Let (M,N) be a pair of non-zero left Λ-modules with $\operatorname{Ext}^*_\Lambda(M,N)$ non-zero. Now, write $y = \sum_{i=0}^n y_i$ for $y \in A_H(M,N)$ and $y_i \in H^i$ for all i. Clearly it is sufficient to show that $y_0 \in \operatorname{rad}(H^0)$ holds. Note that as $A_H(M,N)$ is a graded ideal in H, it follows that $y_0 \in A_H(M,N)$.

We proceed by arguing reductio ad absurdum. Assume thus that y_0 is not an element of $\operatorname{rad}(H^0)$. As, by our standing assumption, H^0 is a local ring, y_0 must then be an invertible element. Otherwise, it would be contained in some maximal ideal, which would then have to be $\operatorname{rad}(H^0)$. However, if y_0 is invertible, the product of it and any non-zero element of $\operatorname{Ext}^*_\Lambda(M,N)$ must then also be non-zero. In other words, if y_0 were invertible, it could not annihilate $\operatorname{Ext}^*_\Lambda(M,N)$, and hence $y_0 \notin A_H(M,N)$ would have to hold, a contradiction. Thus the claim in (a) follows.

(b) Assume $\mathfrak{m}$ to be some maximal ideal in H. The ideal generated by $\operatorname{rad} H^0$ in H must equal $H \operatorname{rad} H^0 = \operatorname{rad} H^0 H$, where this last equality follows by (1.2.5) and the elements of $\operatorname{rad} H^0$ being homogeneous of degree 0. All ideals of H can clearly be considered H^0-submodules of H. Moreover, $\operatorname{rad} H^0 H$ must be contained in the radical of H as an H^0-module. See [24, Chapter 8, Proposition 24.4]. By the same result, it follows that $\operatorname{rad} H^0 H$ is a small submodule of H, which thus entails that $\mathfrak{m} + \operatorname{rad} H^0 H = \mathfrak{m}$, or in other words, $\mathfrak{m}$ contains $\operatorname{rad} H^0 H$. □

Let $\mathcal{N}_H$ denote the ideal in H generated by $\operatorname{rad}(H^0)$ and the homogeneous nilpotent elements of H, which is then clearly a graded ideal in H. As H is graded commutative and (2.1.2) entails that any maximal ideal in H contains $\operatorname{rad} H^0$, we can see that any maximal ideal contains $\mathcal{N}_H$. Indeed, if x is a homogeneous nilpotent element, $\langle x\rangle$ is a nilpotent ideal since H is graded commutative. Since it is a nilpotent ideal, and since a maximal ideal is a prime ideal, it follows that every maximal ideal contains $\langle x\rangle$ for every homogeneous nilpotent element. Hence, there is a 1-1 correspondence between the maximal ideals in H and the maximal ideals in $\overline{H} = H/\mathcal{N}_H$. For a given ideal I of $\overline{H}$, we let I' denote the inverse image of I in H. By way of these observations and (1.2.5), we can make the following definition.

Definition 2.1.3. Let M and N be any pair of Λ-modules. Define the support variety $V_H(M,N)$ in $\overline{H}$ associated with the pair (M,N) by

$$V_H(M,N) = \{\mathfrak{m} \in \operatorname{MaxSpec} \overline{H} \mid A_H(M,N) \subseteq \mathfrak{m}'\}$$

whenever $\operatorname{Ext}^*_\Lambda(M,N)$ is non-zero. If $\operatorname{Ext}^*_\Lambda(M,N)$ is zero, then let $V_H(M,N)=\emptyset$.

It follows from the fact that $\operatorname{HH}^*(\Lambda)$ is graded commutative that the even part of $\operatorname{HH}^*(\Lambda)$ is a commutative ring, and that $\operatorname{HH}^*(\Lambda)$ is a commutative ring whenever the characteristic of k is 2. Moreover, for k of characteristic different from 2, $\overline{H}$ can identified with the quotient of the even part of H over the ideal generated by the homogeneous nilpotent elements of even degree and $\operatorname{rad}(H^0)$. In other words, the algebra $\overline{H}$ is always commutative with no non-zero nilpotent elements and is an algebra over the commutative ring $H^0/\operatorname{rad}(H^0)$.

We call a variety of a pair of modules (M,N) trivial if $V_H(M,N)=\{\mathfrak{m}_{\text{gr}}\}$, where $\mathfrak{m}_{\text{gr}}$ is the ideal $\langle\mathcal{N}_H,H^{\geq 1}\rangle/\mathcal{N}_H$ of $\overline{H}$, which is in fact maximal. Note that the claim in the latter clause holds since any ideal of $\overline{H}$ containing $\mathfrak{m}_{\text{gr}}$ would have to correspond to an ideal of H containing $\langle\mathcal{N}_H,H^{\geq 1}\rangle$. Since $\mathcal{N}_H$ is generated in particular by $\operatorname{rad}(H^0)$, we have that H modulo $\langle\mathcal{N}_H,H^{\geq 1}\rangle$ is isomorphic to the top of H^0, which is simple as H^0 is local.

Now, since $\mathcal{N}_H$ is the ideal generated by $\operatorname{rad}(H^0)$ and all of the homogeneous nilpotent elements of H, it follows that $\mathfrak{m}_{\text{gr}}$ is equal to $\langle\operatorname{rad}(H^0),H^{\geq 1}\rangle/\mathcal{N}_H$, which is to say that $\mathfrak{m}_{\text{gr}}=\mathfrak{a}/\mathcal{N}_H$.

Having stated all this, we are ready to state and prove the next result: as it is desirable to classify the class consisting of pairs of modules which have a trivial variety, the following is a partial result in that direction.

Proposition 2.1.4. *Suppose one of the following conditions holds:*

(i) *M is a Λ-module with* $\operatorname{Ext}^i_\Lambda(M,M)=(0)$ *for* $i\gg 0$,

(ii) *N is a Λ-module with* $\operatorname{Ext}^i_\Lambda(N,N)=(0)$ *for* $i\gg 0$, *or*

(iii) $\operatorname{Ext}^i_\Lambda(M,N)=(0)$ *for* $i\gg 0$ *with* $\operatorname{Ext}^*_\Lambda(M,N)$ *non-zero.*

Then $V_H(M,N)=\{\mathfrak{m}_{\text{gr}}\}$.

Proof. Clearly we can assume that $\operatorname{Ext}^*_\Lambda(M,N)$ is non-zero and one of the following hold:

(i) M is a Λ-module with $\operatorname{Ext}^i_\Lambda(M,M)=(0)$ for $i\geq n$,

(ii) N is a Λ-module with $\operatorname{Ext}^i_\Lambda(N,N)=(0)$ for $i\geq n$, or

(iii) $\operatorname{Ext}^i_\Lambda(M,N)=(0)$ for $i\geq n$.

In each of these cases, we can see that $H^{\geq n+1}$ is contained in $A_H(M,N)$, since $H^{n+1}\cdot\operatorname{Ext}^*_\Lambda(M,N)$ consists of elements which are sums of homogeneous elements of degree greater than n and is thus equal to (0). As a consequence, $(H^i)^{n+1}$

is contained $A_H(M,N)$ for all $i = 1, 2, \ldots, n$. Hence, it follows then that any maximal ideal $\mathfrak{m}'$ of H satisfying $(H^i)^{n+1} \subseteq A_H(M,N) \subseteq \mathfrak{m}'$ must then also contain $H^{\geq 1}$ as any maximal ideal is also a prime ideal. On the other hand, (2.1.2) (b) ensures that any non-zero maximal ideal in H contains the ideal generated by $\operatorname{rad}(H^0)$. But then it follows that any maximal ideal $\mathfrak{m} \in V_H(M,N)$ must contain $\mathfrak{m}_{\mathrm{gr}} = \langle \operatorname{rad}(H^0), H^{\geq 1}\rangle/\mathcal{N}_H$, that is, we must have $\mathfrak{m} = \mathfrak{m}_{\mathrm{gr}}$, which is equivalent to what was to be shown. □

It follows immediately from the preceding result that the variety of any pair of modules (M,N) with $\operatorname{Ext}^*_\Lambda(M,N)$ non-zero where M or N has finite projective or injective dimension must be trivial.

We end this section on the following three results. They describe the relationship between the varieties of modules occurring in an exact sequence.

Lemma 2.1.5. *Let $\eta\colon 0 \to X_1 \to X_2 \to X_3 \to 0$ be an exact sequence of Λ-modules, and let M and N be two Λ-modules. Then the following statements hold:*

(a) $V_H(M, X_{i_1}) \subseteq V_H(M, X_{i_2}) \cup V_H(M, X_{i_3})$ *whenever* $\{i_1, i_2, i_3\} = \{1, 2, 3\}$*;*

(b) $V_H(X_{i_1}, N) \subseteq V_H(X_{i_2}, N) \cup V_H(X_{i_3}, N)$ *whenever* $\{i_1, i_2, i_3\} = \{1, 2, 3\}$.

Proof. (a) It follows by the Long Exact Sequence of Homology that the short exact sequence η induces an exact sequence of right H-modules

$$\operatorname{Ext}^*_\Lambda(M, X_1) \to \operatorname{Ext}^*_\Lambda(M, X_2) \to \operatorname{Ext}^*_\Lambda(M, X_3).$$

Indeed, to see this, it suffices to show that there is a commutative diagram

$$\begin{tikzcd} \operatorname{Ext}^n_\Lambda(M,X) \arrow[r, "-\cdot h_m"] \arrow[d, "f_*"] & \operatorname{Ext}^{n+m}_\Lambda(M,X) \arrow[d, "f_*"] \\ \operatorname{Ext}^n_\Lambda(M,Y) \arrow[r, "-\cdot h_m"] & \operatorname{Ext}^{n+m}_\Lambda(M,Y) \end{tikzcd}$$

where f_* corresponds to a homomorphism $f\colon X \to Y$ acting on the left on elements, and where the horizontal maps correspond to an $h_m \in \mathrm{HH}^m(\Lambda)$ acting on the right on elements. However, that this diagram commutes is then immediate from the associative property of the Yoneda splice, as recorded in (1.1.5).

It follows then that $A_H(M, X_2) \supseteq A_H(M, X_1)A_H(M, X_3)$. Indeed, one can see that for $x \in \operatorname{Ext}^*_\Lambda(M, X_2)$, we have that the image of an element of $x \cdot A_H(M, X_3)$ in $\operatorname{Ext}^*_\Lambda(M, X_3)$ is zero. But then every element of $x \cdot A_H(M, X_3)$ is the image of something in $\operatorname{Ext}^*_\Lambda(M, X_1)$, which then entails $x \cdot A_H(M, X_1)A_H(M, X_3)$ is zero in $\operatorname{Ext}^*_\Lambda(M, X_2)$.

Suppose that $A_H(M, X_2) \subseteq \mathfrak{m}'$ for some $\mathfrak{m} \in \operatorname{MaxSpec} \overline{H}$. Since $\mathfrak{m}$ is a prime ideal, we have that $A_H(M, X_3) \subseteq \mathfrak{m}'$ or $A_H(M, X_1) \subseteq \mathfrak{m}'$ and thus

$$V_H(M, X_2) \subseteq V_H(M, X_1) \cup V_H(M, X_3).$$

It also follows by the Long Exact Sequence of Homology that η induces the exact sequences of right H-modules

$$\operatorname{Ext}^*_\Lambda(M, X_2) \to \operatorname{Ext}^*_\Lambda(M, X_3) \to \operatorname{Ext}^{*+1}_\Lambda(M, X_1),$$

$$\operatorname{Ext}^*_\Lambda(M, X_3) \to \operatorname{Ext}^*_\Lambda(M, X_1) \to \operatorname{Ext}^{*+1}_\Lambda(M, X_2),$$

and

$$0 \to \operatorname{Hom}_\Lambda(M, X_1) \to \operatorname{Hom}_\Lambda(M, X_2).$$

By arguments entirely analogous to those just given, it follows that both

$$A_H(M, X_1) \supseteq A_H(M, X_2) A_H(M, X_3)$$

and

$$A_H(M, X_3) \supseteq A_H(M, X_1) A_H(M, X_2)$$

hold, so that

$$V_H(M, X_1) \subseteq V_H(M, X_2) \cup V_H(M, X_3)$$

and

$$V_H(M, X_3) \subseteq V_H(M, X_1) \cup V_H(M, X_2)$$

also both hold, and we have shown the claim in (a).

The proof of (b) is similar to that of (a), and is thus not included. $\square$

Proposition 2.1.6. *Let $\eta\colon 0 \to X' \to E \to X \to 0$ be an exact sequence of Λ-modules where $V_H(E, N) = V_H(M, E) = \{\mathfrak{m}_{\mathrm{gr}}\}$ for any two Λ-modules M and N. Then the following assertions hold:*

(a) $V_H(M, X) \cup \{\mathfrak{m}_{\mathrm{gr}}\} = V_H(M, X') \cup \{\mathfrak{m}_{\mathrm{gr}}\}$ *for any left Λ-module M, where $\mathfrak{m}_{\mathrm{gr}}$ is in one of the varieties $V_H(M, X)$ or $V_H(M, X')$;*

(b) $V_H(X, N) \cup \{\mathfrak{m}_{\mathrm{gr}}\} = V_H(X', N) \cup \{\mathfrak{m}_{\mathrm{gr}}\}$ *for any left Λ-module N, where $\mathfrak{m}_{\mathrm{gr}}$ is in one of the varieties $V_H(X, N)$ or $V_H(X', N)$.*

Proof. (a) From (2.1.5) we can deduce that

$$V_H(M, X) \subseteq V_H(M, X') \cup V_H(M, E)$$

and

$$V_H(M, X') \subseteq V_H(M, X) \cup V_H(M, E),$$

and hence surely

$$V_H(M,X) \cup V_H(M,E) \subseteq V_H(M,X') \cup V_H(M,E)$$

and

$$V_H(M,X') \cup V_H(M,E) \subseteq V_H(M,X) \cup V_H(M,E),$$

from which the first part of the claim in (a) easily follows as $V_H(M,E) = \{\mathfrak{m}_{\mathrm{gr}}\}$ holds by hypothesis. Since

$$V_H(M,E) \subseteq V_H(M,X) \cup V_H(M,X'),$$

also follows by (2.1.5), the second part also holds, and we have proven the claim in (a).

As the proof of (b) is analogous to that of (a), we leave it to the reader. □

This final result is an easy consequence of (2.1.5).

Proposition 2.1.7. *Let $M = \bigoplus_{i=1}^r M_i$ and $N = \bigoplus_{i=1}^r N_i$. Then*

$$V_H(M,N) = \bigcup_{i,j=1}^{r,s} V_H(M_i, M_j).$$

Proof. Since we have that $\mathrm{Ext}^l_\Lambda(M,N) = \bigoplus_{i,j=1}^{r,s} \mathrm{Ext}^l_\Lambda(M_i,N_j)$, the claim follows from (2.1.5). □

2.2 Support varieties for artin algebras

In this section, we have as our standing assumptions that Λ is an artin algebra over a commutative Artinian ring k, and that Λ is flat as a k-module. Since k is Artinian and hence also Noetherian and Λ is finitely generated as a k-module, the latter assumption is actually equivalent to requiring Λ to be projective as a k-module. Finally, we let H be a graded subalgebra of $\mathrm{HH}^*(\Lambda)$, and we assume throughout that H^0 is a local k-algebra. At this point, it also follows immediately that H^0 is Artinian.

In the following we introduce the support variety $V_H(M)$ of a module $M \in \operatorname{mod}\Lambda$ as $V_H(M, \Lambda/\operatorname{rad}\Lambda)$ after having shown the equalities $V_H(M,\Lambda/\operatorname{rad}\Lambda) = V_H(M,M) = V_H(M,\Lambda/\operatorname{rad}\Lambda)$. Assuming Λ to be an artin algebra ensures that any finitely generated module of Λ has a finite filtration in finitely generated semisimple modules. An implication of this is that the variety of a finitely generated Λ-module is contained in the variety of $\Lambda/\operatorname{rad}\Lambda$. Support varieties are defined for left and right modules in analogous ways, and both are contained in $\operatorname{MaxSpec}\overline{H}$.

We begin by showing a result needed for deriving the equalities which motivate the aforementioned definition of the variety of a module.

Proposition 2.2.1. *Let M and N be in* $\operatorname{mod}\Lambda$. *The following assertions hold:*

(a) $V_H(M,N) \subseteq V_H(M,\Lambda/\operatorname{rad}\Lambda) \cap V_H(\Lambda/\operatorname{rad}\Lambda, N) \subseteq V_H(\Lambda/\operatorname{rad},\Lambda/\operatorname{rad})$;

(b) *if* $\operatorname{Ext}^*_\Lambda(M,N)$ *is non-zero, then* $\{\mathfrak{m}_{\mathrm{gr}}\} \subseteq V_H(M,N) \subseteq V_H(\Lambda/\operatorname{rad},\Lambda/\operatorname{rad})$;

(c) $V_H(M,N) \subseteq V_H(M,M) \cap V_H(N,N) \supseteq V_H(N,M)$.

Proof. (a) By (2.1.7), it follows that we have for any simple Λ-module S that $V_H(M,S) \subseteq V_H(M,\Lambda/\operatorname{rad}\Lambda)$, as, since Λ is an artin algebra, any simple Λ-module is the top of an indecomposable projective Λ-module, and is thus a summand of $\Lambda/\operatorname{rad}\Lambda$.

Moreover, we have for all L in $\operatorname{mod}\Lambda$ short exact sequences

$$0 \to \operatorname{rad}^{i+1} L \to \operatorname{rad}^i L \to \operatorname{rad}^i L/\operatorname{rad}^{i+1} L \to 0$$

for all $i = 0,1,\dots,l$, where l is the radical length of L, and where as a further consequence of Λ being an artin algebra, $\operatorname{rad}^i L/\operatorname{rad}^{i+1} L$ is semisimple. Hence, by repeated applications of (2.1.5)(a), we can deduce that

$$V_H(M,N) \subseteq \bigcup_{i=0}^{l} V_H(M, \operatorname{rad}^i\Lambda/\operatorname{rad}^{i+1}\Lambda).$$

Furthermore, by repeated applications of (2.1.7) we infer that

$$V_H(M, \operatorname{rad}^i\Lambda/\operatorname{rad}^{i+1}\Lambda) \subseteq V_H(M,\Lambda/\operatorname{rad}\Lambda)$$

must hold for all $i = 0,1,\dots,l$.

Repeating the argument with M and N interchanged yields that

$$V_H(M,N) \subseteq V_H(\Lambda/\operatorname{rad}\Lambda, N).$$

Moreover, using the first inclusion with $M = \Lambda/\operatorname{rad}\Lambda$, we find that

$$V_H(\Lambda/\operatorname{rad}\Lambda, N) \subseteq V_H(\Lambda/\operatorname{rad}\Lambda, \Lambda/\operatorname{rad}\Lambda),$$

and by combining all of these observations we derive the claim claim in (a).

(b) This follows by (a) and (2.1.2).

(c) Since

$$A_H(X,X) \subseteq A_H(M,N)$$

and

$$A_H(X,X) \subseteq A_H(N,M)$$

both hold for $X = M$ or $X = N$, the claim follows. □

The following result is now but an easy consequence of the preceding one.

Proposition 2.2.2. *The equalities*

$$V_H(M, \Lambda/\operatorname{rad}\Lambda) = V_H(\Lambda/\operatorname{rad}\Lambda, M) = V_H(M, M)$$

hold for any module M in $\operatorname{mod}\Lambda$.

Proof. It follows from (2.2.1)(a) and (c) that we have

$$V_H(M, \Lambda/\operatorname{rad}\Lambda) \subseteq V_H(M, M) \subseteq V_H(M, \Lambda/\operatorname{rad}\Lambda),$$

and hence $V_H(M, \Lambda/\operatorname{rad}\Lambda) = V_H(M, M)$. Similarly, we find that

$$V_H(\Lambda/\operatorname{rad}\Lambda, M) = V_H(M, M),$$

and hence we are done. □

As promised, we are now ready to define the support variety of a module in $\operatorname{mod}\Lambda$.

Definition 2.2.3. The **support variety** $V_H(M)$ of a module M in $\operatorname{mod}\Lambda$ with respect to H is given by $V_H(M) = V_H(M, \Lambda/\operatorname{rad}(\Lambda))$.

As for the variety of a pair of modules (M, N), we say that a module M has a trivial variety if $V_H(M) = \{\mathfrak{m}_{\mathrm{gr}}\}$. The final result of this section illustrates the continued importance of the ideal $\mathfrak{m}_{\mathrm{gr}}$ in this current context.

Proposition 2.2.4. (a) *We have $\{\mathfrak{m}_{\mathrm{gr}}\} \subseteq V_H(M) \subseteq V_H(\Lambda/\operatorname{rad}\Lambda)$ for all non-zero modules M in* $\operatorname{mod}\Lambda$.

(b) *If $V_H(\Lambda/\operatorname{rad}\Lambda) = \{\mathfrak{m}_{\mathrm{gr}}\}$, then $V_H(M) = \{\mathfrak{m}_{\mathrm{gr}}\}$ for all non-zero modules M in* $\operatorname{mod}\Lambda$.

(c) *If $\operatorname{Ext}^i_\Lambda(M, M) = (0)$ for $i \gg 0$, or the projective or the injective dimension of M is finite for a non-zero module M, then $V_H(M) = \{\mathfrak{m}_{\mathrm{gr}}\}$.*

(d) *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence in* $\operatorname{mod}\Lambda$, *then*

$$V_H(M_{i_1}) \subseteq V_H(M_{i_2}) \cup V_H(M_{i_3})$$

whenever $\{i_1, i_2, i_3\} = \{1, 2, 3\}$.

(e) *If $0 \to M' \to E \to M \to 0$ is an exact sequence with non-zero end terms in* $\operatorname{mod}\Lambda$ *where $V_H(E)$ is trivial, then $V_H(M) = V_H(M')$. In particular, for any non-zero module M in* $\operatorname{mod}\Lambda$ *the equality $V_H(M) = V_H(\Omega^n_\Lambda(M))$ holds for all integers n such that $\Omega^n_\Lambda(M) \neq (0)$.*

(f) *If* $M = \bigoplus_{i=1}^n M_i$, *then* $V_H(M) = \bigcup_{i=1}^n V_H(M_i)$.

Proof. (a) Note that $\operatorname{Ext}^*_\Lambda(M, \Lambda/\operatorname{rad}\Lambda)$ is non-zero for a non-zero module M. In fact, there is clearly some non-zero map $f\colon M \to \Lambda/\operatorname{rad}\Lambda$ that factors as

$$f\colon M \to \operatorname{top} M \to \Lambda/\operatorname{rad}\Lambda,$$

since $\operatorname{top} M$ is semisimple. This then implies that $\operatorname{Hom}_\Lambda(M, \Lambda/\operatorname{rad}\Lambda) \neq (0)$. Hence, the result follows from (2.2.1)(b).

(b) This follows from (a).

Parts (c), (d), (e), and (f) follow from (2.1.4), (2.1.5), (2.1.6) and (2.1.7). □

2.3 Support varieties for finite dimensional algebras

Our standing assumptions for this section are that Λ is a finite dimensional algebra over a field k. As k is a field, all modules over k are projective, and so we need not assume anything corresponding to our previous assumptions of Λ being k-flat. Finally, as before, we let H be a graded subalgebra of $\operatorname{HH}^*(\Lambda)$, and we assume throughout that H^0 is a local k-algebra.

For Λ a finite dimensional algebra over a field k, we show that the varieties of M and $D(M)$ are equal, where D denotes standard duality $\operatorname{Hom}_k(-, k)$ from $\operatorname{mod}\Lambda$ to $\operatorname{mod}\Lambda^{\mathrm{op}}$.

Let $P^* \to \Lambda$ be a minimal Λ^e-projective resolution of Λ. That is to say, $P^* \to \Lambda$ is of the form

$$\cdots \longrightarrow P^n \xrightarrow{\partial^n} P^{n-1} \xrightarrow{\partial^{n-1}} \cdots \longrightarrow P^1 \xrightarrow{\partial^1} P^0 \xrightarrow{\partial^0} \Lambda \longrightarrow 0.$$

Given our assumptions, $P^* \otimes_\Lambda M$ is a projective resolution of M for any Λ-module M. Indeed, observe that

$$(\Lambda \otimes_k \Lambda^{\mathrm{op}}) \otimes_\Lambda M \cong \Lambda \otimes_k (\Lambda^{\mathrm{op}} \otimes_\Lambda M) \cong \Lambda \otimes_k M \cong \Lambda^{\dim M},$$

so that if P^n is a Λ^e-projective, $P^n \otimes_\Lambda M$ is a summand of some number of copies of Λ, and is hence Λ-projective.

The varieties of the modules M and $D(M)$ are defined by, respectively, the kernel of the map

$$H \xrightarrow{-\otimes_\Lambda M} \operatorname{Ext}^*_\Lambda(M, M)$$

and the kernel of the map

$$H \xrightarrow{D(M)\otimes_\Lambda -} \operatorname{Ext}^*_{\Lambda^{\mathrm{op}}}(D(M), D(M))$$

implying that both are contained in MaxSpec $\overline{H}$, and hence they can be compared. This next proposition does just that and shows that they are actually equal.

Proposition 2.3.1. *Let Λ be a finite-dimensional algebra over a field k. For any module M in* $\operatorname{mod}\Lambda$ *the varieties $V_H(M)$ and $V_H(D(M))$ are equal.*

Proof. Let $P^* \to \Lambda$ of the form

$$\cdots \longrightarrow P^n \xrightarrow{\partial^n} P^{n-1} \longrightarrow \cdots \longrightarrow P^0 \longrightarrow \Lambda \longrightarrow 0$$

be a minimal Λ^e-projective resolution of Λ. Let η be in H^n. By (1.1.7), we can represent this element by a map $\eta\colon P^n \to \Lambda$. Assume now that $\eta \in A_H(M,M)$ holds. That is to say, $\eta \otimes_\Lambda M\colon P^n \otimes_\Lambda M \to \Lambda \otimes_\Lambda M$ factors through

$$P^n \otimes_\Lambda M \xrightarrow{\partial^n \otimes_\Lambda M} P^{n-1} \otimes_\Lambda M.$$

Of course, the goal is to show that $\eta \in A_H(D(M), D(M))$ holds.

Now, note that we have that $D(M) \otimes_\Lambda X$ and $D(\operatorname{Hom}_\Lambda(X, M))$ are isomorphic as right Λ-modules for an arbitrary Λ-Λ-bimodule X. To see this, note that we have that

$$\begin{aligned} D(D(M) \otimes_\Lambda X) &= \operatorname{Hom}_k(D(M) \otimes_\Lambda X, k) \\ &\cong \operatorname{Hom}_\Lambda(D(M), D(X)) \\ &\cong \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(X, M), \end{aligned}$$

where the equality holds by definition, the first left Λ-isomorphism holds by Hom-$\otimes$-adjunction while the second left Λ-isomorphism holds by $D(-)$ being a duality. The desired isomorphism of right Λ-modules then follows by applying $D(-)$ to both sides and observing that $D(-)$ is a duality and that right Λ-modules correspond precisely to left Λ^{op}-modules.

Putting this observation to use, we find that the map

$$\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(D(M) \otimes_\Lambda P^{n-1}, D(M) \otimes_\Lambda \Lambda) \to \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(D(M) \otimes_\Lambda P^n, D(M) \otimes_\Lambda \Lambda)$$

is isomorphic to

$$\begin{aligned} &\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(D(\operatorname{Hom}_\Lambda(P^{n-1}, M)), D(\operatorname{Hom}_\Lambda(\Lambda, M))) \\ &\qquad\qquad \to \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(D(\operatorname{Hom}_\Lambda(P^n, M)), D(\operatorname{Hom}_\Lambda(\Lambda, M)). \end{aligned}$$

Since $D(-)$ is a duality, we find that this is isomorphic to

$$\operatorname{Hom}_\Lambda(\operatorname{Hom}_\Lambda(\Lambda, M), \operatorname{Hom}_\Lambda(P^{n-1}, M)) \to \operatorname{Hom}_\Lambda(\operatorname{Hom}_\Lambda(\Lambda, M), \operatorname{Hom}_\Lambda(P^n, M)).$$

Using Hom-$\otimes$-adjunction once again, we find that this in turn is isomorphic to

$$\mathrm{Hom}_\Lambda(P^{n-1} \otimes_\Lambda \mathrm{Hom}_\Lambda(\Lambda, M), M) \to \mathrm{Hom}_\Lambda(P^n \otimes_\Lambda \mathrm{Hom}_\Lambda(\Lambda, M), M),$$

which is isomorphic to

$$\mathrm{Hom}_\Lambda(P^{n-1} \otimes_\Lambda M, M) \to \mathrm{Hom}_\Lambda(P^n \otimes_\Lambda M, M)$$

by the well-known isomorphism $\mathrm{Hom}_\Lambda(\Lambda, X) \cong X$ for arbitrary Λ-modules X.

It can be seen that these isomorphisms bijectively map the homomorphism

$$D(M) \otimes_\Lambda \eta \colon D(M) \otimes_\Lambda P^n \to D(M) \otimes_\Lambda \Lambda$$

to the homomorphism

$$\eta \otimes_\Lambda M \colon P^n \otimes_\Lambda M \to \Lambda \otimes_\Lambda M.$$

And since the latter factors through $\partial^n \otimes_\Lambda M \colon P^n \otimes_\Lambda M \to P^{n-1} \otimes_\Lambda M$, the former factors through $D(M) \otimes_\Lambda \partial^n \colon D(M) \otimes_\Lambda P^n \to D(M) \otimes_\Lambda P^{n-1}$. It follows thus that $A_H(M, M)$ is contained in $A_H(D(M), D(M))$. Repeating the argument while interchanging M and $D(M)$ yields the reverse inclusion. In other words, we have that $A_H(M, M) = A_H(D(M), D(M))$, and thus also that $V_H(M) = V_H(D(M))$. □

2.4 $\mathrm{HH}^*(\Lambda)$ for Λ over an algebraically closed field

The goal of this section will be to investigate more closely the structure of $\mathrm{HH}^*(\Lambda)$ for when Λ is a finite dimensional algebra over an algebraically closed field. While this assumption is not strictly necessary, as will become clear below, it is sufficient to have a good characterisation of the elementary structure of $\Lambda^e = \Lambda \otimes_k \Lambda^{\mathrm{op}}$. We also assume all of the standing assumptions from the previous section.

To begin with, we work to identify the radical of Λ^e. With respect to that, we need the following result, which we cite without proof:

Proposition 2.4.1. *Let Λ and Γ be finite-dimensional semisimple algebras over a field k such that at least one of them has as its simple components matrix rings over separable extensions over k. Then $\Lambda \otimes_k \Gamma$ is semisimple.*

Proof. See [23, Chapter 2, Corollary 2.37]. Note that our statement of the result is obtained by including Knapp's definition of a separable semisimple algebra, which can be found between Proposition 2.33 and Proposition 2.33' of [23, Chapter 2]. □

Using this, it is rather immediate that $\Lambda/\operatorname{rad}\Lambda \otimes_k \Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}}$ is semisimple. Indeed, since $\Lambda/\operatorname{rad}\Lambda$ is a semisimple algebra over a field, we know by the Wedderburn-Artin Theorem that $\Lambda/\operatorname{rad}\Lambda$ is a direct sum of full matrix rings over finite dimensional division rings over k. However, for k algebraically closed, any finite dimensional division ring D over k is in fact equal to k: Let x be in D. Then since D is finite dimensional over k, there is some n such that $\{1, x, \ldots, x^n\}$ is linearly dependent. Hence, x is a root of some polynomial with coefficients in k, and thus $x \in k$ must hold since k is algebraically closed. Moreover, since the minimal polynomial of an element of k over k itself is a linear polynomial, it has distinct roots, implying that k is a separable extension over itself.

Employing what we have just shown, we identify the radical of Λ^e by a simple diagram chase.

Proposition 2.4.2. *Let Λ be a finite dimensional algebra over an algebraically closed field. Then the radical of Λ^e is equal to $\Lambda \otimes_k \operatorname{rad}\Lambda^{\mathrm{op}} + \operatorname{rad}\Lambda \otimes_k \Lambda^{\mathrm{op}}$. Moreover, the top of Λ^e is $\Lambda/\operatorname{rad}\Lambda \otimes_k \Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}}$.*

Proof. Since k-tensoring is exact, we have the following commutative diagram with rows and columns exact:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \Lambda \otimes_k \operatorname{rad}\Lambda^{\mathrm{op}} \arrow[r] \arrow[d] & \Lambda/\operatorname{rad}\Lambda \otimes_k \operatorname{rad}\Lambda^{\mathrm{op}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{rad}\Lambda \otimes_k \Lambda^{\mathrm{op}} \arrow[r] \arrow[d] & \Lambda \otimes_k \Lambda^{\mathrm{op}} \arrow[r] \arrow[d] & \Lambda/\operatorname{rad}\Lambda \otimes_k \Lambda^{\mathrm{op}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{rad}\Lambda \otimes_k \Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}} \arrow[r] \arrow[d] & \Lambda \otimes_k \Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}} \arrow[r] \arrow[d] & \Lambda/\operatorname{rad}\Lambda \otimes_k \Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

If an element x in $\Lambda \otimes_k \Lambda^{\mathrm{op}}$ maps to some x_1 in $\Lambda \otimes_k \Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}}$ which then maps to zero in $\Lambda/\operatorname{rad}\Lambda \otimes_k \Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}}$, this latter element x_1 must be the image of some x_2 in $\operatorname{rad}\Lambda \otimes_k \Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}}$. By surjectivity, this x_2 is the image of some x_3 in $\operatorname{rad}\Lambda \otimes_k \Lambda^{\mathrm{op}}$. Identifying x_3 with its image in $\Lambda \otimes_k \Lambda^{\mathrm{op}}$, we find that the difference $x - x_3$ maps to zero in $\Lambda \otimes_k \Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}}$, and is thus the image of some x_4 in $\Lambda \otimes_k \operatorname{rad}\Lambda^{\mathrm{op}}$.

What we have just shown is equivalent to $\Lambda \otimes_k \operatorname{rad}\Lambda^{\mathrm{op}} + \operatorname{rad}\Lambda \otimes_k \Lambda^{\mathrm{op}} \subseteq \operatorname{rad}\Lambda^e$. Since it follows by simple computations that $\Lambda \otimes_k \operatorname{rad}\Lambda^{\mathrm{op}} + \operatorname{rad}\Lambda \otimes_k \Lambda^{\mathrm{op}}$ is a nilpotent ideal of Λ^e, we are done. □

Note that if λ is an element of Λ, we write λ' whenever we consider it as an element of Λ^{op}. Additionally, let $M_n(k)$ denote the full ring of $n \times n$ matrices over k. The next proposition yields a description of a complete set of primitive idempotents of Λ^e in terms of a corresponding set for Λ. Of course, this allows us to give a description of the indecomposable projectives of Λ^e as well.

Proposition 2.4.3. *Let Λ be a finite dimensional algebra over an algebraically closed field. Assume that $\{e_j\}_{1\leq j\leq n}$ is a complete set of primitive idempotents of Λ. Then the following hold:*

(i) $\{e_i \otimes_k e'_j\}_{1\leq i,j\leq n}$ is a complete set of primitive idempotents of Λ^e.

(ii) The indecomposable projective modules of Λ^e are of the form $\Lambda e_i \otimes_k e_j\Lambda$.

(iii) The simple modules of Λ^e are of the form $S(i) \otimes_k S'(j)$, where $S(i)$ is the simple Λ-module corresponding to e_i, whereas $S'(j)$ is the simple Λ^{op}-module corresponding to e'_j.

Proof. (i) By employing the Wedderburn-Artin Theorem, observing that the simple components of $\Lambda/\operatorname{rad}\Lambda$ and $\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}}$ are central simple, and employing [23, Chapter 2, Corollary 2.36], we see that the simple components of $\Lambda/\operatorname{rad}\Lambda \otimes_k \Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}}$ are of the form $M_l(k) \otimes_k M_m(k)$, where $M_l(k)$ and $M_m(k)$ are simple components of $\Lambda/\operatorname{rad}\Lambda$ and $\Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}}$, respectively. Moreover, it follows that $M_l(k) \otimes_k M_m(k) \cong M_{lm}(k)$ must hold.

Consider now subsets of $\{e_j\}_{1\leq j\leq n}$ of the form $\{e_{j_i}\}_{1\leq i\leq l}$ and $\{e_{j_i}\}_{1\leq i\leq m}$ such that images of the former under the standard projection correspond to a complete set of primitive idempotents of $M_l(k)$, and the latter such that the images of $\{e'_{j_i}\}_{1\leq i\leq m}$ under the standard projection corresponds to a complete set of primitive idempotents of $M_m(k)$. Note also that since $\{e_j\}_{1\leq j\leq n}$ is assumed to be a complete set of idempotents of Λ, the elements are also orthogonal. It is then easily seen that $\{e_{j_i} \otimes_k e'_{j_{i'}}\}_{1\leq i\leq l, 1\leq i'\leq m}$ is a complete set of orthogonal idempotents, and thus corresponds to a decomposition of $M_{lm}(k)$ into a direct sum of lm non-zero projective $M_{lm}(k)$-modules. By counting and comparing with the number of indecomposable projective modules of $M_{lm}(k)$, we see that these projective $M_{lm}(k)$-modules must be indecomposable by applying the Krull-Schmidt Theorem. Hence, the elements of $\{e_{j_i} \otimes_k e'_{j_{i'}}\}_{1\leq i\leq l, 1\leq i'\leq m}$ must be primitive. Repeating this for each simple component of $\Lambda/\operatorname{rad}\Lambda \otimes_k \Lambda^{\mathrm{op}}/\operatorname{rad}\Lambda^{\mathrm{op}}$, we arrive at a complete set of primitive idempotents that lifts to $\{e_i \otimes_k e'_j\}_{1\leq i,j\leq n}$, and we have shown the claim in (i).

(ii) This follows immediately from (i).

(iii) From (2.4.2), we know that that $\Lambda e_i \otimes_k e_j \operatorname{rad}\Lambda^{\mathrm{op}} + \operatorname{rad}\Lambda e_i \otimes_k e_j \Lambda^{\mathrm{op}}$ is the radical of $\Lambda e_i \otimes_k e_j \Lambda^{\mathrm{op}}$. Examine then the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & \Lambda e_i \otimes_k e_j \operatorname{rad}\Lambda^{\mathrm{op}} \arrow[r] \arrow[d] & S(i) \otimes_k e_j \operatorname{rad}\Lambda^{\mathrm{op}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{rad}\Lambda e_i \otimes_k e_j\Lambda^{\mathrm{op}} \arrow[r] \arrow[d] & \Lambda e_i \otimes_k e_j\Lambda^{\mathrm{op}} \arrow[r] \arrow[d] & S(i) \otimes_k e_j\Lambda^{\mathrm{op}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{rad}\Lambda e_i \otimes_k S'(j) \arrow[r] \arrow[d] & \Lambda e_i \otimes_k S'(j) \arrow[r] \arrow[d] & S(i) \otimes_k S'(j) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

□

We note that this proof would over all have been somewhat easier if we assumed we were working over a basic finite dimensional algebra over an algebraically closed field. Indeed, in that case, the simples of Λ and Λ^{op} would have dimension 1, in which case $S(i) \otimes_k S'(j)$ for $1 \leq i, j \leq n$ would also have dimension 1. At that point we could immediately pass to the commutative diagram in (iii) to show that $\{e_i \otimes_k e'_j\}_{1 \leq i,j \leq n}$ is a complete set of primitive idempotents, as primitive idempotents correspond exactly to indecomposable projective modules which correspond exactly to the projective covers of the simple modules. Incidentally, we note that this argument also shows that if Λ is basic, then Λ^e is as well.

2.5 The annihilator of $\operatorname{Ext}^*_\Lambda(M,M)$

Given our definition of varieties, the annihilator of a module M, i.e. $A_H(M,M)$, plays a central role. This annihilator is given by the kernel of the ring homomorphism consisting of the composition of the inclusion of H into $\mathrm{HH}^*(\Lambda)$ and the ring homomorphism induced by the functor $- \otimes_\Lambda M$:

$$H \to \mathrm{HH}^*(\Lambda) \to \operatorname{Ext}^*_\Lambda(M,M).$$

Our goal in this section will simply be to investigate these structures more closely, and to derive some technical results which we need in the sequel. To be precise, they will see heavy use when we show that any closed variety occurs as the variety of some module. Let thus

$$\cdots \to P^n \to P^{n-1} \to \cdots \to P^1 \to P^0 \to \Lambda \to 0$$

be a minimal projective resolution of Λ as a Λ^e-module. Note that we keep all of the standing assumptions from the previous section.

Definition 2.5.1. Given a homogeneous element η in $\mathrm{HH}^*(\Lambda)$ of degree n, represented by a map $\eta\colon \Omega^n_{\Lambda^e}(\Lambda) \to \Lambda$, we define the Λ^e-module M_η by the following pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^n_{\Lambda^e}(\Lambda) \arrow[r, "\iota^n"] \arrow[d, "\eta"] & P^{n-1} \arrow[r] \arrow[d] & \Omega^{n-1}_{\Lambda^e}(\Lambda) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Lambda \arrow[r, "\alpha_n"] & M_\eta \arrow[r, "\beta_n"] & \Omega^{n-1}_{\Lambda^e}(\Lambda) \arrow[r] & 0
\end{tikzcd}$$

where we denote by $\mathcal{E}_\eta$ the bottom row short exact sequence.

We note that the isomorphism class of the module M_η is independent of the representation of η as a map $\Omega^n_{\Lambda^e}(\Lambda) \to \Lambda$. Indeed, to see this, observe that if η' is another representative of of the same element of $\mathrm{HH}^n(\Lambda)$, then by definition it follows that there is some $\epsilon\colon P^{n-1} \to \Lambda$ such that if $\iota^n\colon \Omega^n_{\Lambda^e}(\Lambda) \to P^{n-1}$ is the inclusion, then $\eta - \eta' = \epsilon\iota$. Moreover, observe also that by the definition of M_η, there exists an exact sequence of the form

$$0 \longrightarrow \Omega^n_{\Lambda^e}(\Lambda) \xrightarrow{\begin{pmatrix} \iota^n \\ -\eta \end{pmatrix}} P^{n-1} \oplus \Lambda \xrightarrow{\begin{pmatrix} \gamma^n & \alpha^n \end{pmatrix}} M_\eta \longrightarrow 0$$

where $\gamma^n\colon P^{n-1} \to M_\eta$ is the middle vertical map in the commutative diagram defining M_η. Since this also holds for $M_{\eta'}$, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^n_{\Lambda^e}(\Lambda) \arrow[r, "\begin{pmatrix} \iota^n \\ -\eta \end{pmatrix}"] \arrow[d, equal] & P^{n-1} \oplus \Lambda \arrow[r] \arrow[d, "\begin{pmatrix} 1 & 0 \\ \epsilon & 1 \end{pmatrix}"] & M_\eta \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \Omega^n_{\Lambda^e}(\Lambda) \arrow[r, "\begin{pmatrix} \iota^n \\ -\eta' \end{pmatrix}"] & P^{n-1} \oplus \Lambda \arrow[r] & M_{\eta'} \arrow[r] & 0
\end{tikzcd}$$

Note that by elementary homological algebra, we have an induced map $M_\eta \to M_{\eta'}$. Moreover, since the middle vertical map is clearly an isomorphism, by the Five Lemma it follows that this induced map must be an isomorphism as well.

With this preliminary discussion out of the way, we present this section's first result:

Proposition 2.5.2. *Let η be a homogeneous element of degree n in* $\mathrm{HH}^*(\Lambda)$, *and let M be in* $\operatorname{mod}\Lambda$. *Then the following are equivalent.*

(i) η *is in* $A_{\mathrm{HH}^*(\Lambda)}(M,M)$.

(ii) $\mathcal{E}_\eta \otimes_\Lambda M$ *is a split short exact sequence.*

(iii) $M_\eta \otimes_\Lambda M \cong M \oplus \Omega^{n-1}_\Lambda(M) \oplus Q$ *for some projective* Λ*-module* Q.

Proof. We assume the notation used in the discussion preceding this result. Our strategy in showing these equivalences is straightforward, and we begin by showing that (i) implies (ii). Note that after applying $-\otimes_\Lambda M$ to the diagram defining M_η, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^n_{\Lambda^e}(\Lambda)\otimes_\Lambda M \arrow[r, "\iota^n\otimes_\Lambda M"] \arrow[d, "\eta\otimes_\Lambda M"] & P^{n-1}\otimes_\Lambda M \arrow[r] \arrow[d] & \Omega^{n-1}_{\Lambda^e}(\Lambda)\otimes_\Lambda M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Lambda\otimes_\Lambda M \arrow[r] & M_\eta\otimes_\Lambda M \arrow[r] & \Omega^{n-1}_{\Lambda^e}(\Lambda)\otimes_\Lambda M \arrow[r] & 0
\end{tikzcd}$$

where the rows remain exact, as they were split exact when considered as short exact sequences of right Λ-modules. Since $\eta \in A_{\mathrm{HH}^n(\Lambda}(M,M)$ is assumed to hold, it follows that there is some $\epsilon\colon P^{n-1}\otimes_\Lambda M \to \Lambda\otimes_\Lambda M$ such that $\eta\otimes_\Lambda M = \epsilon(\iota^n\otimes_\Lambda M)$. This implies the desired result.

(ii) implies (i). If $\mathcal{E}_\eta \otimes_\Lambda M$ is split exact, there is some $\zeta\colon M_\eta\otimes_\Lambda M \to \Lambda\otimes_\Lambda M$ satisfying $\zeta(\alpha^n \otimes_\Lambda M) = 1_{\Lambda\otimes_\Lambda M}$. By the commutativity of the left square in the above diagram, it follows that $\eta\otimes_\Lambda M$ factors as $\zeta(\gamma^n\otimes_\Lambda M)(\iota^n\otimes_\Lambda M)$, which establishes the claim.

(ii) implies (iii). This follows since $P^*\otimes_\Lambda M$ is a Λ-projective resolution of M, whereupon we may apply Schanuel's Lemma to find $\Omega^n_{\Lambda^e}(\Lambda)\otimes_\Lambda M \cong \Omega^n_\Lambda(M)\oplus Q$ for some projective Λ-module Q.

(iii) implies (ii). To begin with, we observe that we can, without loss of generality, assume that M has no projective summands. Indeed, adding or removing a projective summand does not alter the annihilator of a module, and hence has no bearing on the conclusion; but also, since the syzygies of a module are insensitive to such a summand, it has no effect on the hypothesis.

Now, note that we are in the situation where $\mathcal{E}_\eta\otimes_\Lambda M$ is isomorphic to

$$0 \to M \to M\oplus\Omega^{n-1}_\Lambda(M)\oplus Q \to \Omega^{n-1}_\Lambda(M)\oplus Q' \to 0,$$

since

$$M_\eta\otimes_\Lambda M \cong M\oplus\Omega^{n-1}(M)\oplus Q$$

holds by hypothesis, while

$$\Omega^{n-1}_{\Lambda^e}(\Lambda)\otimes_\Lambda M \cong \Omega^{n-1}_\Lambda(M)\oplus Q'$$

for some projective Q' follows by Schanuel's Lemma and the fact that if P^* is Λ^e-projective presentation of Λ then $P^* \otimes_\Lambda M$ is a Λ-projective presentation of M.

Now, we note that if Q'' is a summand of $\Omega_\Lambda^{n-1}(M)$, then we can split it off the $\Omega_\Lambda^{n-1}(M)$-summand in both the middle and the rightmost terms. We denote the result of removing all projective summands in this manner by $\Omega_\Lambda^{n-1}(M)'$. Hence, we pass to the situation

$$0 \to M \to M \oplus \Omega_\Lambda^{n-1}(M)' \oplus Q \to \Omega_\Lambda^{n-1}(M)' \oplus Q' \to 0.$$

Here we see that since neither M nor $\Omega_\Lambda^{n-1}(M)'$ have projective summands, Q' must be a summand of Q. Yet, by the additivity of dimension, $Q' \cong Q$. Hence, we can deduce by a familiar result of elementary homological algebra that the sequence must be split: namely, if $0 \to A \to B \to C \to 0$ is a short exact sequence with $B \cong A \oplus C$, then that sequence is split. $\square$

Let $\{\eta_1, \ldots, \eta_t\}$ be a finite set of homogeneous elements of $\mathrm{HH}^*(\Lambda)$. We define $\alpha_{\eta_1,\ldots,\eta_t}$ to be the map

$$\Lambda \otimes_\Lambda \cdots \otimes_\Lambda \Lambda \xrightarrow{\alpha_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda \alpha_{\eta_t}} M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t}.$$

It can be seen that this map is a monomorphism. To see this, note that

$$\alpha_{\eta_1} \otimes_\Lambda \alpha_{\eta_2} \otimes_\Lambda \cdots \otimes_\Lambda \alpha_{\eta_t} =$$
$$(\alpha_{\eta_1} \otimes_\Lambda 1_\Lambda \otimes_\Lambda \cdots \otimes_\Lambda 1_\Lambda) \cdot (1_{M_{\eta_1}} \otimes_\Lambda \alpha_{\eta_2} \otimes_\Lambda 1_\Lambda \otimes_\Lambda \cdots \otimes_\Lambda 1_\Lambda \cdots (1_{\eta_1} \otimes_\Lambda 1_{M_{\eta_2}} \otimes_\Lambda \cdots \otimes_\Lambda \alpha_{\eta_t})$$

where each factor on the right hand side is a monomorphism since each is the result of repeatedly tensoring monomorphisms on the left or on the right by right or left Λ-projective modules.

Hence, it induces an exact sequence

$$0 \longrightarrow \Lambda \xrightarrow{\alpha_{\eta_1,\ldots,\eta_t}} M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \longrightarrow X_{\eta_1,\ldots,\eta_t} \longrightarrow 0$$

which we denote by $\mathcal{E}_{\eta_1,\ldots,\eta_t}$. We now use this construction to give some sufficient conditions for when the ideal generated by homogeneous elements $\{\eta_1, \ldots, \eta_t\}$ is in the annihilator $A_{\mathrm{HH}^*(\Lambda)}(M, M)$.

Theorem 2.5.3. *Let $\{\eta_1, \ldots, \eta_t\}$ be a finite set of homogeneous elements in $\mathrm{HH}^*(\Lambda)$, and let M be in* $\operatorname{mod} \Lambda$.

(a) *The following are equivalent.*

   (i) The ideal generated by $\{\eta_1, \ldots, \eta_t\}$ is contained in $A_{\mathrm{HH}^*(\Lambda)}(M, M)$.

(ii) $\mathcal{E}_{\eta_i} \otimes_\Lambda M$ *is a split short exact sequence for all* $i = 1, 2, \dots, t$.

(iii) $\mathcal{E}_{\eta_1,\dots,\eta_t} \otimes_\Lambda M$ *is a split short exact sequence.*

(b) *If a finite set* $\{\eta_1, \dots, \eta_t\}$ *of homogeneous elements in* $\mathrm{HH}^*(\Lambda)$ *is in* $A_{\mathrm{HH}^*(\Lambda)}(M,M)$, *then* M *is a direct summand of* $M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \otimes_\Lambda M$.

Proof. (a) It follows by (2.5.2) that (i) and (ii) are equivalent. What remains is thus to prove the equivalence of (ii) and (iii). We begin with showing that (iii) implies (ii). Since $\alpha_{\eta_1,\dots,\eta_t}$ can be viewed as the composition of $\alpha_{\eta_2,\dots,\eta_t}$ and α_{η_1}, any splitting of $\alpha_{\eta_1,\dots,\eta_t}$ induces a splitting of α_{η_1}. As we can similarly for all $1 \le i \le t$ factor $\alpha_{\eta_1,\dots,\eta_t}$ into the map obtained by replacing α_{η_i} by 1_Λ in the definition of $\alpha_{\eta_1,\dots,\eta_t}$ and α_{η_i} itself, the result follows. To see that (ii) implies (iii), note that our factorisation of $\alpha_{\eta_1,\dots,\eta_t}$ in our remarks preceding this proposition implies that if each $\mathcal{E}_{\eta_i}$ is a split exact monomorphism, then $\mathcal{E}_{\eta_1,\dots,\eta_t}$ is a composition of split exact monomorphisms and is itself thus also a split exact monomorphism. The claim in (a) is thus shown.

(b) follows immediately from the claim in (a). □

The following is a refinement of (2.5.2) for the case of Λ selfinjective. Recall that we denote the projectively stable category by $\underline{\mathrm{mod}}\Lambda$.

Proposition 2.5.4. *Let* Λ *be a selfinjective algebra. Let* η *be a homogeneous element of* $\mathrm{HH}^*(\Lambda)$ *of degree* n, *and let* M *be in* $\mathrm{mod}\,\Lambda$. *Then* η *is in* $A_{\mathrm{HH}^*(\Lambda)}(M,M)$ *if and only if* $\eta \otimes_\Lambda 1_M \colon \Omega^n_{\Lambda^e}(\Lambda) \otimes_\Lambda M \to \Lambda \otimes_\Lambda M$ *is zero in* $\underline{\mathrm{mod}}\Lambda$.

We finish the section with the following result:

Lemma 2.5.5. *Let* $\eta_1, \eta_2, \dots, \eta_t$ *be homogeneous elements in* $A_{\mathrm{HH}^*(\Lambda)}(M)$. *Then*

$$M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \otimes_\Lambda M$$

is in $\mathrm{add}\{\Omega^i_\Lambda(M)\}_{i=0}^N \cup \mathrm{add}\,\Lambda$ *for some integer* N.

Proof. For $t = 1$, this is just the claim in (2.5.2). Suppose then that $t > 1$, and let $\widetilde{M} = M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_{t-1}} \otimes_\Lambda M$. By (2.5.2) we have that $M_{\eta_t} \otimes_\Lambda M \cong M \oplus \Omega^{\deg \eta_t - 1}_\Lambda(M) \oplus Q$ for Q some Λ-projective module. After tensoring this isomorphism on the left with $M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_{t-1}}$ we have

$$M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \otimes_\Lambda M \cong \widetilde{M} \oplus M_{\eta_1} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_{t-1}} \otimes_\Lambda \Omega^{\deg \eta_t - 1}_\Lambda(M) \oplus Q'$$

where Q' is some Λ-projective module. Here the middle summand on the right hand side can be identified under an isomorphism with $\Omega^{\deg \eta_t - 1}_\Lambda(\widetilde{M})$ modulo projectives. To see this, note that we arrive at a projective resolution of $\widetilde{M}$ by iteratively tensoring a projective resolution of M with $M_{\eta_{t-1}}, \dots, M_{\eta_1}$ since these modules are all right Λ-projective, and then we need only apply Schanuel's Lemma to the appropriate kernel. Hence, the claim follows by induction. □

Chapter 3

Finite generation hypotheses

Having laid a foundation, we now build upon it. We follow more or less the presentation in [17] and [7]. This has, however, required some reorganization. To be precise, we have tried to include some results of Chapter 4 of [7] that [17] calls upon, entailing that they must be discussed prior to the results of the latter.

Moreover, we note that yet again we have made a selection of the results of the article we base the chapter on: indeed, we have tried to keep in mind certain restrictions on scope and length, but also more importantly to the problem which we are to investigate. As such, we have given greater weight to thoroughly presenting the proofs concerning the finite generation hypotheses, (**Fg1**) and (**Fg2**), and their relation to the complexity of an algebra satisfying them. To give an indication of to what extent results similar to those in the group ring case can be recovered, we also show that every closed homogeneous variety is the variety of some module, provided one assumes these hypotheses.

However, as a consequence, we have for instance included none of the applications of the theory. Among other things, we have thus excluded a proof of a variation on Webb's Theorem giving a classification of the stable components of the Auslander-Reiten quiver of a selfinjective algebra satisfying the finite generation hypotheses.

Finally, we mention that we henceforth assume the reader is familiar with some standard results on affine algebraic varieties, equivalent to, for instance, the first three or four pages of Chapter 5.4 of [7]. Among other things, we thus assume the reader is familiar with the Weak Nullstellensatz and Hilbert's Nullstellensatz. Although we do not make a great deal of use of these, they nevertheless inform the development of the theory, and are another reason for assuming that our algebras are finite dimensional over an algebraically closed field.

3.1 Finite generation hypotheses (Fg1) and (Fg2)

Group rings of finite groups and finite dimensional cocommutative Hopf algebras are classes of algebras which have a corresponding theory of support varieties. For both of these theories, it is a crucial property of them that $\mathrm{Ext}^*_\Lambda(M,N)$ is finitely generated as a module over the Noetherian ring defining the varieties. In our case, this is a graded subalgebra H of the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$.

It is the immediate goal of this section to derive some particular consequences of assuming the finite generation of $\mathrm{Ext}^*_\Lambda(M,N)$ over a commutative graded Noetherian subalgebra H of $\mathrm{HH}^*(\Lambda)$. We show among other things that if these hypotheses hold, Λ must be a Gorenstein algebra, which is to say that the injective dimensions of Λ as a left and as a right Λ-module are both finite.

We introduce now the first of the two finite generation hypotheses.

Definition 3.1.1. Let Λ be a finite dimensional algebra over a field k, and let H be a fixed graded subalgebra of $\mathrm{HH}^*(\Lambda)$. We say that the pair (Λ, H) satisfy (**Fg1**) if the following hold:

(i) H is a commutative Noetherian ring.

(ii) $H^0 = \mathrm{HH}^0(\Lambda) = Z(\Lambda)$.

We will be assuming for the rest of this chapter that our finite dimensional algebras Λ and our fixed graded subalgebra H of $\mathrm{HH}^*(\Lambda)$ satisfy this assumption unless it is stated otherwise. It can be seen that the assumption of commutativity in (i) is not too cumbersome, as we can simply pass to the even part of a given H, which will then be commutative graded subalgebra of $\mathrm{HH}^*(\Lambda)$. Moreover, at this point it might be pertinent to state the other standing assumptions for this section, namely that an algebra Λ is assumed to be a finite dimensional algebra over a field k. Moreover, unless otherwise stated, this field k is assumed to be algebraically closed. As regards this latter assumption, we note that it is made necessary by, among other things, some subtleties relating to the structure of the enveloping algebra Λ^e of Λ, as was expanded upon in a earlier section.

Recall that for a commutative ring B one calls B a **finitely generated** A-algebra provided there is a finite set of elements $\{x_0, \ldots, x_n\}$ of B such that every element of B is expressible as a polynomial in the elements $\{x_0, \ldots, x_n\}$ with coefficients in A, or, in other words, there is an A-algebra homomorphism from a polynomial ring $A[t_0, \ldots, t_n]$ onto B [4, p. 30, Chapter 2]. From this latter perspective, it follows almost immediately that B is Noetherian if A is Noetherian, as can be seen by applying Hilbert's Basis Theorem and the fact that the image of a Noetherian ring under a ring homorphism is Noetherian. In particular, since a field k is Noetherian, a finitely generated commutative algebra over k is Noetherian as well.

The following is an immediate, elementary implication of our first finite generation hypothesis.

Proposition 3.1.2. *Let Λ be a finite dimensional algebra over a field k, and let the pair (Λ, H) satisfy* (**Fg1**)*. Then H is a finitely generated commutative graded algebra over k.*

Proof. Indeed, by [4, Proposition 10.7, Chapter 10], it follows that for (Λ, H) satisfying (**Fg1**), H^0 is Noetherian and H is finitely generated as an H^0-algebra. Since by (**Fg1**) we have $H^0 = Z(\Lambda)$, which is a finite dimensional k-algebra since Λ is a finite dimensional k-algebra, we are done. □

To properly begin analysing the consequences of assuming these hypotheses, we present the definition of the complexity of a module and then, after that, the definition of the dimension of a variety. With that goal in mind, recall that we denote the length of a module M by $l(M)$, and that if $P^* \to M$ of the form

$$\cdots \to P^n \to P^{n-1} \to \cdots \to P^0 \to M \to 0,$$

is a minimal projective resolution of M, then the i**th Betti number** of a M, β^i, is the number of indecomposable projective summands of P^i. Notice that since the tops of indecomposable projective modules are simple, we have that $\beta^i = l(\operatorname{top} \Omega^i(M))$.

Definition 3.1.3. If M is a Λ-module, and its minimal projective resolution is given by $P^* \to M$, then we define the **complexity of a module** M **over** Λ to be

$$\operatorname{cx}_\Lambda M = \inf\{n \in \mathbb{N}_0 \mid \exists \alpha \in \mathbb{R} \text{ such that } \beta^i \leq \alpha i^{n-1} \text{ for all } i \geq 0\}.$$

Moreover, the **complexity of an algebra** Λ, denoted by $\operatorname{cx} \Lambda$, is defined to be the maximum of the complexities of the simple modules of Λ.

There are a few well-known observations one can make with respect to the first part of this definition: namely, one can easily see that a module having complexity 0 is equivalent to it having finite projective dimension, and that it having complexity 1 is equivalent to the terms in its projective resolution being of bounded length.

Moreover, the latter part of this definition is perhaps in need of some justification. It can be seen that for M in $\operatorname{mod} \Lambda$, one has that $\operatorname{cx} \Lambda \geq \operatorname{cx}_\Lambda M$ holds. This follows from the fact that M in $\operatorname{mod} \Lambda$ can be filtered in semisimple modules, and an elementary result concerning the complexity of Λ-modules: Let L, M, N be in $\operatorname{mod} \Lambda$. Then given a short exact sequence $0 \to L \to M \to N \to 0$, the complexity of each term of this sequence is at most the maximum of the complexity of the other two terms.

Since we make no use of this result other than in justifying our definition, we only sketch its proof: Firstly, we can observe almost immediately from the definition of the complexity of a module that a module and its syzygies have the same complexity. Moreover, our desired result can be seen to hold for the middle term of the above short exact sequence by applying the Horseshoe Lemma. However, we can apply the Snake Lemma to an appropriately formed diagram derived from this short exact sequence to arrive at the short exact sequence

$$0 \to \Omega^1_\Lambda(M) \to \Omega^1_\Lambda(L) \oplus Q \to L \to 0,$$

wherein Q is some projective Λ-module. We can yet again apply the Horseshoe Lemma. Since this can be repeated, one would be done.

Our definition for the dimension of a variety will depend on the notion of the **Krull dimension** of a commutative Noetherian ring. Hence, recall that we say that a commutative Noetherian ring A has **Krull dimension** n if there is a chain of prime ideals of length n in A

$$\mathfrak{p}_\mathfrak{n} \supseteq \mathfrak{p}_{\mathfrak{n}-1} \supseteq \cdots \supseteq \mathfrak{p}_\mathfrak{o}$$

but none of length $n+1$. If there are arbitrarily long chains of prime ideals in A, we say that A has infinite Krull dimension.

Although this is a standard definition, we note that we have an alternative definition available to us since the rings $H \subseteq \mathrm{HH}^*(\Lambda)$ are finitely generated algebras over k by (3.1.2). For the sake of stating this alternative definition, we recall the statement of the Noether Normalization Lemma:

Lemma 3.1.4. *(Noether Normalisation Lemma). Suppose that k is a field and A is a finitely generated commutative algebra over k. Then there exist elements $y_1, \dots, y_n \in A$ generating a polynomial subalgebra $k[y_1, \dots, y_n] \subseteq A$ over which A is finitely generated as a module.*

If A is graded then $y_1, \dots, y_n$ may be chosen to be homogeneous elements.

Proof. See [7, Lemma 5.4.5, Chapter 5]. □

It follows by the discussion on [7, p. 166, Chapter 5] that the Krull dimension of a commutative ring A being n is equivalent to the number of generators in the preceding proposition being equal to n.

Having recalled all of this, we are now almost ready to give our definition. In the following we use a slightly different definition of a variety, namely we write $V_H(M) = \mathrm{MaxSpec}(H/A_H(M))$. This is easily seen to be equivalent to the one we gave before.

Definition 3.1.5. If the pair (Λ, H) satisfies (**Fg1**) and M is a Λ-module, we define the **dimension** of its associated variety $V_H(M) = \mathrm{MaxSpec}(H/A_H(M))$ to be the number n in the Noether Normalisation Lemma for $A = H/A_H(M)$.

What we have done, of course, is to define the dimension of the variety of M to be the Krull dimension of the associated commutative ring $H/A_H(M)$.

Remark 3.1.6. We note that if a module M has a trivial variety, then by definition $V_H(M) = \operatorname{MaxSpec} H/A_H(M,M) = \{\mathfrak{m}_{\mathrm{gr}} + A_H(M,M)\}$. [1] However, by Hilbert's Nullstellensatz [7, Chapter 5.4, Theorem 5.4.2], we find that $\sqrt{A_H(M,M)} = \mathfrak{m}_{\mathrm{gr}}$, where we have for some ideal I of a commutative Noetherian ring R

$$\sqrt{I} = \{r \in R \mid \text{ for some } j > 0, r^j \in I\}.$$

Moreover, by the same result, we have that $V_H(M) = V(\mathfrak{m}_{\mathrm{gr}}) = \operatorname{MaxSpec} H/\mathfrak{m}_{\mathrm{gr}}$. Since $H/\mathfrak{m}_{\mathrm{gr}}$ is a field, it is clear that $\dim V_H(M) = 0$.

Our goal at this point is to show that

(i) if $\operatorname{Ext}^*_\Lambda(M, \Lambda/\operatorname{rad}\Lambda)$ is a finitely generated H-module, then

$$\dim V_H(M) = \operatorname{cx}_\Lambda(M) < \infty;$$

(ii) if $\operatorname{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, M)$ is a finitely generated H-module, then

$$\dim V_H(M) = \operatorname{cx}_{\Lambda^{\mathrm{op}}}(D(M)) < \infty.$$

However, before we can do this, we need to present some auxiliary results and definitions. The presentation of the following is based on that of [7, Chapter 5.3], as given the assumption of (**Fg1**) the proof of the analogous result in the group ring case essentially carries over.

Hence, recall that we say that a graded vector space V over a field k is of **finite type** provided $V = \bigoplus_{r\geq 0} V_r$ with each V_r a finite dimensional vector space over k. Given this, we have that the **Poincaré series** of V is defined to be

$$p(V,t) = \sum_{r\geq 0} t^r \dim_k V_r$$

as a formal power series in the indeterminate t.

The following proposition, which we include albeit without proof, describes the form of the Poincaré series of a finitely generated module V over a finitely generated commutative graded ring of finite type.

[1] Note that we are abusing notation a bit here, as in this case $\mathfrak{m}_{\mathrm{gr}}$ is used to denote the ideal $\mathfrak{a}$ defined in the preceding chapter. This abuse, however, is in line with the notation used in [17], which, as mentioned, this chapter is based on.

Proposition 3.1.7. *Suppose that A is a commutative graded ring of finite type over k, finitely generated over A^0 by homogeneous elements $x_1, \ldots, x_s$ in degrees $k_1, \ldots, k_s$. Suppose V is a finitely generated graded A-module (i.e. we have $A_i \cdot A_j \subseteq A_{i+j}$ and $A_i \cdot V_j \subseteq V_{i+j}$). Then the Poincaré series $p(V,t)$ is of the form*

$$\frac{f(t)}{\prod_{j=1}^{s}(1-t^{k_j})}$$

where $f(t)$ is a polynomial in t with integer coefficients.

Proof. See [7, Proposition 5.3.1, Chapter 5]. □

The next result shows that the complexity of the graded components of a graded module is determined by the order of the pole of its Poincaré series at $t = 1$.

Proposition 3.1.8. *Suppose*

$$p(t) = \frac{f(t)}{\prod_{j=1}^{s}(1-t^{k_j})} = \sum_{r \in \mathbb{N}_0} a_r t^r$$

where $f(t)$ is a polynomial with integer coefficients and the a_r are non-negative integers. Let b be the order of the pole of $p(t)$ at $t = 1$. Then

(i) there exists a constant $\alpha > 0$ such that $a_i \leq \alpha \cdot i^{b-1}$ for $i > 0$, but

(ii) there does not exist a constant $\alpha > 0$ such that $a_i \leq \alpha \cdot i^{b-2}$ for $i > 0$.

Proof. The hypotheses and the conclusion of the proposition remain unchanged if we replace $p(t)$ by $p'(t) = p(t) \cdot (1 + t + \cdots + t^{k_j - 1})$. Indeed, the order of the pole of $p(t)$ remains the same. Moreover, the points (i) and (ii) are unaffected. We show this for (i). To see that this claim holds for (i), note that if $p'(t) = \sum_{r \in \mathbb{N}_0} a'_r t^r$, then $a'_r = \sum_{i=0}^{k_j - 1} a_{r-i}$, if we agree that $a_s = 0$ for $s < 0$. Thus, if $a'_i \leq \alpha' i^{b-1}$ holds for $i > 0$, then $a_i \leq a'_i \leq \alpha' i^{b-1}$ holds for $i > 0$. Hence, without loss of generality, we may assume that $p(t) = f(t)/(1-t)^b$ with $f(1) \neq 0$.

Assume now that $f(t) = \phi_m t^m + \cdots + \phi_0$. Recall that

$$\frac{1}{1-t} = 1 + t + t^2 \cdots$$

which upon taking the derivative b times and dividing by $(b-1)!$ yields

$$\frac{1}{(1-t)^b} = \sum_{i \in \mathbb{N}_0} \binom{b+i-1}{b-1} t^i.$$

Hence, it follows that

$$p(t) = f(t) \cdot \sum_{i \in \mathbb{N}_0} \binom{b+i-1}{b-1} t^i,$$

and thus

$$a_i = \phi_0 \binom{b+i-1}{b-1} + \phi_1 \binom{b+i-2}{b-1} + \cdots + \phi_m \binom{b+i-m-1}{b-1}$$

holds for all $i \geq m$, (where m depends on $f(t)$).

Since $f(1) \neq 0$, we have that $\phi_0 + \phi_1 + \cdots + \phi_m \neq 0$ holds, implying that a_i is a polynomial in i of degree b. In fact, this can be seen by collecting the terms on the right side of this equation to yield

$$a_i = \frac{f(1)}{(b-1)!} i^{b-1} + \cdots ,$$

where we display only the highest degree term. □

We immediately note the following corollary of the proof of this proposition.

Corollary 3.1.9. *Suppose*

$$p(t) = \frac{f(t)}{(1-t)^s} = \sum_{r \in \mathbb{N}_0} a_r t^r$$

where $f(t)$ is a polynomial with integer coefficients and the a_r are non-negative integers. Let b be the order of the pole of $p(t)$ at $t = 1$. Assume $f(t) = \sum_{i=0}^{N} \phi_i t^i$. Then a_r for $r \geq N$ is given by a polynomial in r of degree $b - 1$ with rational coefficients.

Proof. Repeat the part of the proof of the preceding proposition that follows the first paragraph. □

We note that by (3.1.7), the hypotheses of this corollary are satisfied if $p(t)$ is the Poincaré series of a finitely generated graded module V of finite type over some ring A which is finitely generated over A^0 by elements $x_0, \ldots, x_s$ of A of degree 1.

For a graded vector space V with a Poincaré series $p(V, T)$ of the form detailed in the hypothesis of the preceding proposition, we write $\gamma(V)$ for the order of the pole of $p(V, t)$ at $t = 1$. By the same proposition, $\gamma(V)$ is a measure of the polynomial growth rate of the graded components V_r of $V = \bigoplus_{r \in \mathbb{N}_0} V_r$. Clearly, for a minimal projective resolution $P^* \to M$ of M such that $p(\bigoplus_{r \in \mathbb{N}_0} P^r, t)$ is of the given form, it follows that $\operatorname{cx}_\Lambda M = \gamma(\bigoplus_{r \in \mathbb{N}_0} P^r)$.

Note that if $0 \to V'' \to V' \to V \to 0$ is a short exact sequence of finitely generated graded modules over a commutative graded ring A which is finitely generated and of finite type over k, then (3.1.7) entails that $\gamma(V'')$, $\gamma(V')$ and $\gamma(V)$ are all defined. In this case, it can be seen that $\gamma(V')$ is equal to the maximum of $\gamma(V'')$ and $\gamma(V)$. Indeed, this follows from the additivity of dimension over short exact sequences and V'',V' and V being of finite type.

This additive property of $\gamma(-)$ can be seen to imply that $\gamma(V) \leq \gamma(A)$ holds for all finitely generated graded modules V over A: in fact, this follows easily by noting that every finitely generated graded module over A is the quotient of $\bigoplus_{i=0}^n A$ for some n and by some graded ideal I of A, that $\gamma(\bigoplus_{i=0}^n A) = \gamma(A)$, and that either $\gamma(A) = \gamma(V)$, or $\gamma(A) = \gamma(I)$, in which case $\gamma(A) > \gamma(V)$.

We need the following remark in the coming proposition:

Remark 3.1.10. In the same vein as in the preceding paragraph we can note that if A is a finitely generated graded algebra over k of finite type, then by the Noether Normalisation Theorem we have a polynomial subalgebra $k[y_1, \ldots, y_n]$ for homogeneous elements $y_1, \ldots, y_n$ in A of positive degree such that

$$\gamma(A) = \gamma(k[y_1, \ldots, y_n]) = n.$$

Lemma 3.1.11. *Let Λ be a basic finite dimensional algebra over a field. Assume $P^* \to M$ of the form*

$$\cdots \to P^n \to P^{n-1} \to \cdots \to P^0 \to M \to 0,$$

to be a minimal projective resolution of M in $\operatorname{mod}\Lambda$. Then the following hold.

(i) The multiplicity of the indecomposable projective $P(i)$ corresponding to a simple $S(i)$ as a summand of P^r is equal to

$$\dim_k \operatorname{Hom}_\Lambda(P^r, S(i)).$$

(ii)

$$\dim_k P^r = \sum_i \dim_k P(i) \cdot \dim_k \operatorname{Ext}^r_\Lambda(M, S(i)).$$

Proof. Indeed, this follows by the well-known k-linear isomorphism

$$\operatorname{Hom}_\Lambda(P(j), S(i)) \cong \operatorname{Hom}_\Lambda(\Lambda e_j, S(i)) \cong e_j S(i),$$

where e_j is the primitive idempotent corresponding to $P(j)$, and the fact that $e_j S(i) \ncong (0)$ if and only if $i = j$. This takes care of the claim in (i).

Since $P^* \to M$ is a minimal projective resolution, it follows that every homomorphism $\eta\colon P^r \to S(i)$ is a cocycle. Indeed, this follows from the fact that

$$\operatorname{Im} \partial^{r+1} \subseteq \operatorname{rad} P^r \subseteq \operatorname{Ker} \eta,$$

where the first inclusion follows by the definition of a minimal projective resolution and the second by the properties simple modules and Jacobson radicals: Namely, since η has a simple module as its codomain, it is either the zero map or an epimorphism. If it is the zero map, then $\operatorname{rad} P^r \subseteq \operatorname{Ker} \eta = P^r$ obviously holds. On the other hand, if η is an epimorphism, then since its image is simple, its kernel $\operatorname{Ker} \eta$ must be maximal. Hence, in this case, by the definition of the radical of a module, one must also have $\operatorname{rad} P^r \subseteq \operatorname{Ker} \eta$.

One can also note that $\operatorname{Im} \partial^{r+1} \subseteq \operatorname{Ker} \eta$ for $\eta\colon P^r \to S(i)$ a homomorphism also implies that there are no non-zero coboundaries, so that all of this in sum implies that

$$\operatorname{Hom}_\Lambda(P^r, S(i)) \cong \operatorname{Ext}^r_\Lambda(M, S(i)).$$

As a consequence of this, we have that

$$\dim_k P^r = \sum_i \dim_k P(i) \cdot \dim_k \operatorname{Ext}^r_\Lambda(M, S(i)).$$

□

If $\operatorname{Ext}^*_\Lambda(M, S)$ is finitely generated as an H-module, we have by (3.1.7) that the Poincaré series

$$p(\operatorname{Ext}^*_\Lambda(M, S), t)$$

is of the form $\frac{f(t)}{\prod_{j=1}^{s}(1-t^{k_j})}$, in which $f(t)$ is a polynomial with integer coefficients and the k_j are the degrees of the generators of H. Given this, it is easily seen that the graded vector space $\bigoplus_{r\in\mathbb{N}_0} P^r$ has a Poincaré series of such a form as well.

We are nearly ready to prove the desired result, and only need to present a few elementary observations before we begin: note that if N is another finitely generated Λ-module, then we have that $\operatorname{Ext}^r_\Lambda(M, N)$ is a quotient of a submodule of $\operatorname{Hom}_\Lambda(P^r, N)$, which is itself a subspace of $\operatorname{Hom}_k(P^r, N)$. Hence it follows that

$$\dim_k \operatorname{Ext}^r_\Lambda(M, N) \leq \operatorname{Hom}_k(P^r, N) \leq \dim_k P^r \cdot \dim_k N,$$

where the last inequality follows by basic linear algebra. In other words, we have that

$$\gamma(\operatorname{Ext}^*_\Lambda(M, N)) \leq \gamma(\bigoplus_{r\geq 0} P^r) = \operatorname{cx}_\Lambda M.$$

Thus, having finished our preparations, we give the promised result:

Proposition 3.1.12. *Let M be in* $\operatorname{mod}\Lambda$.

(i) If $\operatorname{Ext}^*_\Lambda(M, \Lambda/\operatorname{rad}\Lambda)$ *is a finitely generated H-module, then*

$$\dim V_H(M) = \operatorname{cx}_\Lambda(M) < \infty.$$

(ii) If $\operatorname{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, M)$ *is a finitely generated H-module, then*

$$\dim V_H(M) = \operatorname{cx}_{\Lambda^{\mathrm{op}}}(D(M)) < \infty.$$

Proof. (i) Assume that $\operatorname{Ext}^*_\Lambda(M, \Lambda/\operatorname{rad}\Lambda)$ is finitely generated as an H-module. If N is of Loewy length 0, then it follows by the assumption and the additivity of $\operatorname{Ext}^n_\Lambda(M, -)$ for $n \in \mathbb{N}_0$ that $\operatorname{Ext}^*_\Lambda(M, N)$ is finitely generated as an H-module.

Suppose then that $\operatorname{Ext}^*_\Lambda(M, N)$ is finitely generated as an H-module for N in $\operatorname{mod}\Lambda$ of Loewy length less than or equal to n. Let then N' be of Loewy length $n+1$. Observe that we have

$$0 \longrightarrow \operatorname{rad} N' \xrightarrow{i} N' \xrightarrow{p} \operatorname{top} N' \longrightarrow 0,$$

and hence we also have an exact sequence of H-modules

$$\operatorname{Ext}^*_\Lambda(M, \operatorname{rad} N') \to \operatorname{Ext}^*_\Lambda(M, N') \to \operatorname{Ext}^*_\Lambda(M, \operatorname{top} N'),$$

of which we can note that its end terms are finitely generated over H by the induction hypothesis. Moreover, we can see that the middle term must be finitely generated over H as we can canonically factor both the leftmost map and the rightmost map onto their respective images to arrive at an exact sequence

$$0 \to \operatorname{Im} i_* \to \operatorname{Ext}^*_\Lambda(M, N') \to \operatorname{Im} p_* \to 0.$$

The leftmost term of this sequence is a finitely generated H-module since it is the image of a finitely generated H-module; whereas the rightmost is an an H-submodule of $\operatorname{Ext}^*_\Lambda(M, \operatorname{top} N')$. This latter H-module is then itself finitely generated since $\operatorname{Ext}^*_\Lambda(M, \operatorname{top} N')$ is finitely generated and Noetherian. In other words, the inductions step follows, and we have that if $\operatorname{Ext}^*_\Lambda(M, \Lambda/\operatorname{rad}\Lambda)$ is a finitely generated H-module, then $\operatorname{Ext}^*_\Lambda(M, N)$ is as well for all N in $\operatorname{mod}\Lambda$.

As such, it can be seen that our observations preceding this proposition imply that we have

$$\begin{aligned}\max_i \gamma(\operatorname{Ext}^*_\Lambda(M, S(i))) \leq \gamma(\operatorname{Ext}^*_\Lambda(M, M)) \leq \gamma(\bigoplus_{r \in \mathbb{N}_0} P^r) = \operatorname{cx}_\Lambda(M) \\ \leq \max_i \gamma(\operatorname{Ext}^*_\Lambda(M, S(i))).\end{aligned}$$

In other words, if this holds, it follows that

$$\gamma(\operatorname{Ext}^*_\Lambda(M,M)) = \gamma(H/A_H(M,M)) = \operatorname{cx}_\Lambda(M).$$

Hence, $\dim V_H(M) = \operatorname{cx}_\Lambda M$. Moreover, since (Λ, H) is assumed to satisfy (**Fg1**), we could see by (3.1.10) that $\dim V_H(M) = n$ for some n and we would be done.

Note that the first inequality follows from the fact that the H-module structure of $\operatorname{Ext}^*_\Lambda(M, S(i))$ factors through $\operatorname{Ext}^*_\Lambda(M,M)$, thus entailing that $\operatorname{Ext}^*_\Lambda(M, S(i))$ is a finitely generated $\operatorname{Ext}^*_\Lambda(M,M)$-module. Hence, there is some H-epimorphism of the form

$$\bigoplus_{i=0}^{n} \operatorname{Ext}^*_\Lambda(M,M) \to \operatorname{Ext}^*_\Lambda(M, S(i))$$

with kernel a finitely generated graded $\operatorname{Ext}^*_\Lambda(M,M)$-module and hence also a finitely generated graded H-module, where the finite generation of the kernel follows from $\operatorname{Ext}^*_\Lambda(M,M)$ being Noetherian, which itself follows from $\operatorname{Ext}^*_\Lambda(M,M)$ being a finitely generated H-module and (**Fg1**) entailing that H is Noetherian. Thus, by the previously noted additive property of $\gamma(-)$, it follows that

$$\gamma(\operatorname{Ext}^*_\Lambda(M, S(i)) \leq \gamma(\operatorname{Ext}^*_\Lambda(M,M))$$

since $\gamma(\bigoplus_{i=0}^n \operatorname{Ext}^*_\Lambda(M,M)) = \gamma(\operatorname{Ext}^*_\Lambda(M,M))$ clearly holds.

The final inequality follows by the formula for $\dim_k P^r$ that we established earlier, namely

$$\dim_k P^r = \sum_i \dim_k P(i) \cdot \dim_k \operatorname{Ext}^r_\Lambda(M, S(i)).$$

(ii) Since (2.3.1) shows that $V_H(M) = V_H(D(M))$, the claim follows from (i). $\square$

The corresponding theories of support varieties of group rings and complete intersections depend crucially on the finite generation of the extension groups $\operatorname{Ext}^*(M,N)$ as modules over the relevant ring of cohomological operators for modules M and N. We, however, begin by analysing finite generation of $\operatorname{Ext}^*_\Lambda(M,N)$ as H-modules for certain particular pairs in $\operatorname{mod}\Lambda$.

Proposition 3.1.13. *Let M be in* $\operatorname{mod}\Lambda$.

(a) *Suppose that* $\operatorname{Ext}^*_\Lambda(M, \Lambda/\operatorname{rad}\Lambda)$ *is finitely generated as an H-module. If the variety of M is trivial, then the projective dimension of M is finite.*

(b) *Suppose that* $\operatorname{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, M)$ *is finitely generated as an H-module. If the variety of M is trivial, then the injective dimension of M is finite.*

(c) *Suppose that* $\operatorname{Ext}^*_\Lambda(D(\Lambda^{\mathrm{op}}), \Lambda/\operatorname{rad}\Lambda)$ *and* $\operatorname{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, \Lambda)$ *are finitely generated as H-modules. Then Λ is a Gorenstein algebra.*

Proof. (a) By the remark (3.1.6), a module M with trivial variety must satisfy $\dim V_H(M) = 0$, so that by (3.1.12) it follows that $\operatorname{cx}_\Lambda(M) = 0$. By the remarks following the definition of the complexity of a module, this is equivalent to the projective dimension of M being finite.

The claims in (b) and (c) follow from the claim in (a). □

Finitely generated modules over finite dimensional algebras can all be finitely filtered by semisimple modules, as can be seen by considering the ascending or descending Loewy series of a module. This next result uses this fact to show that the finite generation of $\operatorname{Ext}^*_\Lambda(M, N)$ for arbitrary pairs of modules M and N of $\operatorname{mod}\Lambda$ over a subalgebra H in $\operatorname{HH}^*(\Lambda)$ is equivalent to the finite generation of $\operatorname{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, \Lambda/\operatorname{rad}\Lambda)$.

Proposition 3.1.14. *The following are equivalent.*

(i) $\operatorname{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, \Lambda/\operatorname{rad}\Lambda)$ *is finitely generated as an H-module.*

(ii) $\operatorname{Ext}^*_\Lambda(M, N)$ *is finitely generated as an H-module for all pairs of Λ-modules M and N in* $\operatorname{mod}\Lambda$.

(iii) $\operatorname{HH}^*(\Lambda, B)$ *is finitely generated as an H-module for all B in* $\operatorname{mod}\Lambda^e$.

Proof. (iii) implies (ii): since we have that $\operatorname{Ext}^n_\Lambda(M, N) \cong \operatorname{HH}^n_\Lambda(\Lambda, \operatorname{Hom}_k(M, N))$ as groups by [13, Corollary 4.4, Chapter 9], we can see that there is induced an H-isomorphism $\operatorname{Ext}^*_\Lambda(M, N) \cong \operatorname{HH}^*_\Lambda(\Lambda, \operatorname{Hom}_k(M, N))$, and this implication is established.

(ii) implies (i): this is obvious.

(i) implies (iii): Clearly $\operatorname{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, \Lambda/\operatorname{rad}\Lambda)$ being finitely generated as an H-module is equivalent to the same holding for $\operatorname{HH}^*_\Lambda(\Lambda, \operatorname{Hom}_k(\Lambda/\operatorname{rad}\Lambda, \Lambda/\operatorname{rad}\Lambda))$. By (2.4.3) and the isomorphism $M \otimes_k N \cong N \otimes_k M$ holding for all k-modules M and N, we know that both the left and the right simple Λ^e-modules are of the form $S \otimes_k T$ for a left simple Λ-module S and a right simple Λ-module T. This implies that a left simple Λ^e-module is of the form $D(S \otimes_k T)$ for a left simple Λ-module S and a right simple Λ-module T. By the computation

$$D(S \otimes_k T) = \operatorname{Hom}_k(S \otimes_k T, k) \cong \operatorname{Hom}_k(S, D(T)) \cong \operatorname{Hom}_k(S, T'),$$

where T' is a simple left Λ-module satisfying $T = D(T')$, it thus follows that every left simple Λ^e-module is of the form $\operatorname{Hom}_k(S, T)$ for some S and T, both left simple Λ-modules.

Using this, the fact that $\mathrm{HH}^*_\Lambda(\Lambda, \mathrm{Hom}_k(\Lambda/\operatorname{rad}\Lambda, \Lambda/\operatorname{rad}\Lambda))$ is finitely generated as an H-module, and the fact that every Λ^e-module is filtered in simple Λ^e-modules, we see that $\mathrm{HH}^*_\Lambda(\Lambda, B)$ is finitely generated as an H-module for all B in $\operatorname{mod}\Lambda^e$, and we are thus done. □

This result provides motivation for the second finite generation assumption:

Definition 3.1.15. Let Λ be a finite dimensional algebra over a field k, and let H be a fixed graded subalgebra of $\mathrm{HH}^*(\Lambda)$. We say that the pair (Λ, H) satisfy (**Fg2**) if $\mathrm{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, \Lambda/\operatorname{rad}\Lambda)$ is a finitely generated H-module.

We note that if we assume both (**Fg1**) and (**Fg2**), then by the preceding proposition, $\mathrm{HH}^*(\Lambda)$ is a finitely generated H-module, and is thus finitely generated as a k-algebra. By nearly the same argument, $\mathrm{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, \Lambda/\operatorname{rad}\Lambda)$ is a finitely generated k-algebra as well.

The following theorem summarises some of the results and the observations we have made in this section:

Theorem 3.1.16. *Suppose that a pair (Λ, H) satisfy* (**Fg1**) *and* (**Fg2**).

(a) *The algebra Λ is Gorenstein.*

(b) *The following are equivalent for a module M in* $\operatorname{mod}\Lambda$.

  (i) The variety of M is trivial.

  (ii) The projective dimension of M is finite.

  (iii) The injective dimension of M is finite.

(c) $\dim V_H(M) = \operatorname{cx}_\Lambda(M)$ *for any module M in* $\operatorname{mod}\Lambda$.

3.2 Modules with given varieties

In the following section we derive some results utilizing the finite generation hypotheses discussed above. As such, we continue to hold the standing assumptions of the preceding section while additionally assuming that the pair (Λ, H) satisfies (**Fg1**) and (**Fg2**).

The variety of a module is generally a closed homogeneous variety. Our goal in the following is to show that any closed homogeneous variety occurs as the variety of some module. To do this, we pick up the thread of Section 2.5, which concerned itself with the annihilator of a Λ-module M, introduced the construction yielding the bimodules M_η, and gave several criteria relating these two notions. As we noted also there, the modules we construct to reach our current goal are not necessarily indecomposable.

We begin by examining the variety of $M_\eta \otimes_\Lambda M$ for η some homogeneous element η of positive degree in H. With respect to that, we let L_η denote the module $M_\eta \otimes_\Lambda \Lambda/\operatorname{rad}\Lambda$. The following result gives, among other things, some relations between the varieties of $M_\eta \otimes_\Lambda M$ and L_η. We also note that this result, as the lone exception in this section, does not make use of our finite generation assumptions.

Proposition 3.2.1. *Let η be a homogeneous element of positive degree in H, and let M be in* $\operatorname{mod}\Lambda$.

(a) $V_H(M_\eta \otimes M) \subseteq V_H(L_\eta) \cap V_H(M)$.

(b) *The element η^2 is in $A_H(M_\eta \otimes_\Lambda M, \Lambda/\operatorname{rad}\Lambda)$. In particular, $V_H(L_\eta)$ is contained in $V_H(\langle\eta\rangle)$, and consequently $V_H(M_\eta \otimes_\Lambda M)$ is contained in $V_H(\langle\eta\rangle) \cap V_H(M)$.*

(c) *Let $\{\eta_1, \dots, \eta_t\}$ be homogeneous elements in $\mathrm{HH}^*(\Lambda)$. Then $V_H(M_\eta \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \otimes_\Lambda M)$ is contained in $V_H(\langle\eta_1, \dots, \eta_t\rangle) \cap V_H(M)$.*

Proof. (a) First of all, we note that the sequence

$$\mathcal{E}_\eta \otimes_\Lambda M \colon 0 \longrightarrow M \longrightarrow M_\eta \otimes_\Lambda M \longrightarrow \Omega^{n-1}_{\Lambda^e} \otimes_\Lambda M \longrightarrow 0$$

is exact. Indeed, this follows since $\mathcal{E}_\eta$ splits as a sequence of right Λ-modules. By (2.2.4), we have that $V_H(M_\eta \otimes_\Lambda M) \subseteq V_H(M)$, as $\Omega^{n-1}_{\Lambda^e}(\Lambda) \otimes_\Lambda M \cong \Omega^{n-1}_\Lambda(M) \oplus F$ for some Λ-projective module F and since varieties are invariant under taking syzygies. In fact, the latter statement is (2.1.4)(d), while the former follows by recalling Schanuel's Lemma and the fact that $\Omega^{n-1}_{\Lambda^e}(\Lambda) \otimes_\Lambda M$ is a kernel of a projective resolution of M.

Now, recalling that M has a filtration in semisimple modules, one sees that $V_H(M_\eta \otimes_\Lambda M)$ is contained in $V_H(L_\eta)$ by repeatedly applying (2.2.4). Thus we have shown the claim in (a).

(b) By once again observing that $\Omega^{n-1}_{\Lambda^e}(\Lambda) \otimes_\Lambda M \cong \Omega^{n-1}_\Lambda(M) \oplus F$ and employing Dimension Shift, we see that

$$\operatorname{Ext}^i_\Lambda(\Omega^{n-1}_{\Lambda^e}(\Lambda) \otimes_\Lambda M, \Lambda/\operatorname{rad}\Lambda) \cong \operatorname{Ext}^{i+n-1}_\Lambda(M, \Lambda/\operatorname{rad}\Lambda)$$

must hold for $i \geq 1$.

Hence, it follows that the short exact sequence

$$\mathcal{E}_\eta \otimes_\Lambda M \colon 0 \longrightarrow M \longrightarrow M_\eta \otimes_\Lambda M \longrightarrow \Omega^{n-1}_{\Lambda^e} \otimes_\Lambda M \longrightarrow 0$$

yields the two exact sequences

$$\cdots \to \operatorname{Ext}^i_\Lambda(M, \Lambda/\operatorname{rad}\Lambda) \xrightarrow{\cdot\eta} \operatorname{Ext}^{i+n}_\Lambda(M, \Lambda/\operatorname{rad}\Lambda) \to \operatorname{Ext}^{i+1}_\Lambda(M_\eta \otimes_\Lambda M, \Lambda/\operatorname{rad}\Lambda)$$

$$\longrightarrow \operatorname{Ext}^{i+1}_\Lambda(M, \Lambda/\operatorname{rad}\Lambda) \xrightarrow{\cdot\eta} \operatorname{Ext}^{i+n+1}_\Lambda(M, \Lambda/\operatorname{rad}\Lambda) \longrightarrow \cdots$$

and

$$0 \longrightarrow \operatorname{Hom}_\Lambda(\Omega^{n-1}_{\Lambda^e}(\Lambda) \otimes_\Lambda M, \Lambda/\operatorname{rad}\Lambda) \longrightarrow \operatorname{Hom}_\Lambda(M_\eta \otimes_\Lambda M, \Lambda/\operatorname{rad}\Lambda) \longrightarrow$$

$$\operatorname{Hom}_\Lambda(M, \Lambda/\operatorname{rad}\Lambda) \xrightarrow{\cdot\eta} \operatorname{Ext}^n_\Lambda(M, \Lambda/\operatorname{rad}\Lambda).$$

From these, we can immediately derive the following short exact sequences

$$0 \longrightarrow \operatorname{Hom}_\Lambda(\Omega^{n-1}_{\Lambda^e}(\Lambda) \otimes_\Lambda M, \Lambda/\operatorname{rad}\Lambda) \longrightarrow \operatorname{Hom}_\Lambda(M_\eta \otimes_\Lambda M, \Lambda/\operatorname{rad}\Lambda) \xrightarrow{\nu_0}$$

$$\operatorname{Ker}(\cdot\eta)_{|\operatorname{Hom}_\Lambda(M,\Lambda/\operatorname{rad}\Lambda)} \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Ext}^{*+n-1}_\Lambda(M, \Lambda/\operatorname{rad}\Lambda)/(\eta \operatorname{Ext}^{*-1}_\Lambda(M, \Lambda/\operatorname{rad}\Lambda)) \xrightarrow{\mu_*}$$

$$\operatorname{Ext}^*_\Lambda(M_\eta \otimes_\Lambda M, \Lambda/\operatorname{rad}\Lambda) \xrightarrow{\nu_*} \operatorname{Ker}(\cdot\eta)_{|\operatorname{Ext}^*_\Lambda(M,\Lambda/\operatorname{rad}\Lambda)} \longrightarrow 0$$

where the index $*$ is allowed to vary over natural numbers greater than or equal to 1. Let θ be an element of $\operatorname{Ext}^i_\Lambda(M_\eta \otimes_\Lambda M, \Lambda/\operatorname{rad}\Lambda)$. Then $\nu_{i+n}(\eta\theta) = \eta\nu_i(\theta) = 0$, and thus $\eta\theta$ is in $\operatorname{Ker}\nu_{n+i}$, which equals $\operatorname{Im}\mu_{n+i}$. As η annihilates $\operatorname{Im}\mu_{n+i}$, it follows that $\eta^2\theta = 0$, which implies what was to be shown.

(c) One can see that this follows rather immediately from the claim in (b). $\square$

We note that Λ is Gorenstein, and, as such, its injective dimensions as a left and as a right over itself are finite and, in fact, equal, for instance to n. We let ${}^\perp\Lambda$ denote the full subcategory $\{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(X, \Lambda) = (0) \text{ for all } i > 0\}$ of $\operatorname{mod}\Lambda$. As varieties are preserved under the taking of syzygies, it follows that all the different varieties of modules occur for a module in ${}^\perp\Lambda$.

We note that by the fact that Λ is a cotilting module, there is for a module M in ${}^\perp\Lambda$ a complete resolution of the form

$$\mathbb{P}^*\colon \cdots \longrightarrow P^2 \xrightarrow{\partial^2} P^1 \xrightarrow{\partial^1} P^0 \xrightarrow{\partial^0} P^{-1} \xrightarrow{\partial^{-1}} P^{-2} \longrightarrow \cdots$$

satisfying $\operatorname{Im} d^0 \cong M$. See [5] for details. We define $\widehat{\operatorname{Ext}}^i_\Lambda(M,N)$ as the homology of $\operatorname{Hom}_\Lambda(\mathbb{P}^*,N)$ at stage i. In other words, $\widehat{\operatorname{Ext}}^i_\Lambda(M,N) = \ker d^{i+1}_* / \operatorname{Im} d^i_*$ for i in $\mathbb{Z}$.

We call an H-module X an $\mathfrak{m}_{\mathrm{gr}}$-torsion module provided each x in X is annihilated by some power of $\mathfrak{m}_{\mathrm{gr}}$, where in the current setting we abuse our notation a bit by letting $\mathfrak{m}_{\mathrm{gr}}$ denote the ideal of H generated by $\mathcal{N}_H$ and $\langle H^{i\geq 1}\rangle$, which is then the unique maximal graded ideal of H.

Remark 3.2.2. It can be seen that we have an exact sequence

$$0 \longrightarrow \mathcal{P}(M,N) \longrightarrow \operatorname{Ext}^*_\Lambda(M,N) \longrightarrow \widehat{\operatorname{Ext}}^*_\Lambda(M,N) \longrightarrow \widehat{\operatorname{Ext}}^-_\Lambda(M,N) \longrightarrow 0$$

of H-modules, in which the end terms are $\mathfrak{m}_{\mathrm{gr}}$-torsion modules. Note that by $\mathcal{P}(M,N)$ one means the set of all homomorphisms from M to N that factor through a projective module, whereas $\widehat{\operatorname{Ext}}^-_\Lambda(M,N) \cong \bigoplus_{i\leq -1} \widehat{\operatorname{Ext}}^i_\Lambda(M,N)$.

To show that this holds, we begin by giving the H-module structure of the endterms: $\mathcal{P}(M,N)$ is an H-module under the action whereby it is annihilated by $\bigoplus_{i\geq 1} H^i$ and acted upon by precomposition by a homomorphism under $H^0 \to \operatorname{End}_\Lambda(M)$ corresponding to the restriction of the familiar ring homomorphism $\Phi_M\colon H \to \operatorname{Ext}^*_\Lambda(M,M)$. Indeed, if $f \in \mathcal{P}(M,N)$ and $g \in \Phi_M(H^0) \subseteq \operatorname{End}_\Lambda(M)$, then $fg \in \mathcal{P}(M,N)$, showing that closure holds. It is similarly rather straightforward to show that the other module axioms hold. Clearly, $\mathcal{P}(M,N)$ is $\mathfrak{m}_{\mathrm{gr}}$-torsion under this action.

When it comes to $\widehat{\operatorname{Ext}}^*_\Lambda(M,N)$ and $\widehat{\operatorname{Ext}}^-_\Lambda(M,N)$, things are somewhat more involved: Let x be an element of $\widehat{\operatorname{Ext}}^i_\Lambda(M,N)$ for i in $\mathbb{Z}$, and also let η be an element of $\mathrm{HH}^n(\Lambda)$. We choose now some non-negative integer m satisfying $i+2m>0$. We now consider x to be an element of $\widehat{\operatorname{Ext}}^{i+2m}_\Lambda(\Omega^{-2m}_\Lambda(M),N)$, which can be seen to equal $\operatorname{Ext}^{i+2m}_\Lambda(\Omega^{-2m}_\Lambda(M),N)$. Hence, we can employ the ordinary H-module structure of extension groups to define the H-module structure of $\widehat{\operatorname{Ext}}^*_\Lambda(M,N)$. This structure can be seen to be well-defined.

As the H-module structure of $\widehat{\operatorname{Ext}}^-_\Lambda(L,M)$ is obtained from it being a quotient of $\widehat{\operatorname{Ext}}^*_\Lambda(L,M)$ by $\operatorname{Ext}^*_\Lambda(L,M)$, it is clear that it must be $\mathfrak{m}_{\mathrm{gr}}$-torsion.

We also note that the second morphism is injective except in the 0-degree, where kernel is of course $\operatorname{Im} d^0_*$. Now, a morphism in $\operatorname{Im} d^0_*$ can be described by the following commutative diagram:

$$\begin{tikzcd}
P^0 \arrow[rr, "d^0"] \arrow[dr, two heads, "\pi^0"'] & & P^{-1} \arrow[d] \\
& M \arrow[ur, tail] & N
\end{tikzcd}$$

Since π^0 is an epimorphism, an element of $\operatorname{Im} d_*^0$ corresponds to an element of $\mathcal{P}(M,N)$. On the other hand, an element of $\mathcal{P}(M,N)$ can be seen to correspond to an element of $\operatorname{Im} d_*^0$.

We use this remark in an essential fashion in the following lemma.

Lemma 3.2.3. *For any maximal ideal $\mathfrak{p}$ in $\operatorname{MaxSpec} H$ with $\mathfrak{p} \neq \mathfrak{m}_{\mathrm{gr}}$,*

$$\operatorname{Ext}_\Lambda^*(M,N)_{\mathfrak{p}} \cong \widehat{\operatorname{Ext}}_\Lambda^*(M,N)_{\mathfrak{p}}$$

for all modules M in ${}^\perp\Lambda$ and N in $\operatorname{mod}\Lambda$.

Proof. Since $\mathfrak{m}_{\mathrm{gr}}$ is the maximal ideal generated by $\mathcal{N}_H$ and $\langle H^{\leq 1}\rangle$, it follows that if $\mathfrak{p} \neq \mathfrak{m}_{\mathrm{gr}}$, then $(H - \mathfrak{p}) \cap \mathfrak{m}_{\mathrm{gr}} \neq (0)$. Hence, the localization of any $\mathfrak{m}_{\mathrm{gr}}$-torsion module at such a $\mathfrak{p}$ vanishes. As localization of modules at a prime ideal is an exact functor, the result thus follows by considering the exact sequence we derived in remark (3.2.2). □

We immediately put this to use in proving that $V_H(M_\eta \otimes_\Lambda M) = V_H(\langle\eta\rangle) \cap V_H(M)$ holds for arbitrary homogeneous elements η of H of positive degree and for all M in $\operatorname{mod}\Lambda$.

Proposition 3.2.4. *Let η be a homogeneous element of positive degree in H. Then*

$$V_H(M_\eta \otimes_\Lambda M) = V_H(\langle\eta\rangle) \cap V_H(M).$$

In particular, $V_H(L_\eta) = V_H(\langle\eta\rangle)$.

Proof. By (3.2.1), it follows that if $V_H(\langle\eta\rangle) \cap V_H(M)$ is trivial, then we have nothing to prove. As such, we assume that $V_H(\langle\eta\rangle) \cap V_H(M)$ is non-trivial. Let η be of degree $n > 0$.

It is clear that for Λ a Gorenstein algebra, the syzygies $\Omega_\Lambda^m(X)$ for X in $\operatorname{mod}\Lambda$ are in ${}^\perp\Lambda$ for m greater than or equal to the injective dimension of Λ. As the varieties of $M_\eta \otimes_\Lambda M$ and $M_\eta \otimes_\Lambda \Omega_\Lambda^m(M)$ are equal, given that the latter is a syzygy of the former, we can without loss of generality assume that M is in ${}^\perp\Lambda$, in which case $M_\eta \otimes_\Lambda M$ is an element of ${}^\perp\Lambda$ as well.

As we know, we can associate to an element η the exact sequence $\mathcal{E}_\eta$

$$0 \longrightarrow \Lambda \longrightarrow M_\eta \longrightarrow \Omega_{\Lambda^e}^{n-1}(\Lambda) \longrightarrow 0$$

of Λ^e-modules. Similarly to how we did in the first proposition of this section, by tensoring with M and applying $\operatorname{Hom}_\Lambda(-, \Lambda/\operatorname{rad}\Lambda)$ we can derive a long exact sequence of the form

$$\cdots \longrightarrow \widehat{\mathrm{Ext}}_{\Lambda}^{i}(M, \Lambda/\operatorname{rad}\Lambda) \xrightarrow{\cdot\eta} \widehat{\mathrm{Ext}}_{\Lambda}^{i+n+1}(M, \Lambda/\operatorname{rad}\Lambda) \longrightarrow$$

$$\widehat{\mathrm{Ext}}_{\Lambda}^{i+1}(M_\eta \otimes_\Lambda M, \Lambda/\operatorname{rad}\Lambda) \longrightarrow \widehat{\mathrm{Ext}}_{\Lambda}^{i+1}(M, \Lambda/\operatorname{rad}\Lambda)$$

$$\xrightarrow{\cdot\eta} \widehat{\mathrm{Ext}}_{\Lambda}^{i+n+1}(M, \Lambda/\operatorname{rad}\Lambda) \longrightarrow \cdots$$

which itself yields the short exact sequence

$$\widehat{\theta}\colon 0 \longrightarrow \widehat{\mathrm{Ext}}_{\Lambda}^{*+n-1}(M, \Lambda/\operatorname{rad}\Lambda)/(\eta\widehat{\mathrm{Ext}}_{\Lambda}^{*-1}(M, \Lambda/\operatorname{rad}\Lambda) \xrightarrow{\mu_*}$$

$$\widehat{\mathrm{Ext}}_{\Lambda}^{*}(M_\eta \otimes_\Lambda M, \Lambda/\operatorname{rad}\Lambda) \xrightarrow{\nu_*} \operatorname{Ker}(\cdot\eta)|_{\widehat{\mathrm{Ext}}_{\Lambda}^{*}(M,\Lambda/\operatorname{rad}\Lambda)} \longrightarrow 0$$

where the index $*$ is allowed to vary over natural number greater than or equal to 1. We note here that the fact that the connecting homomorphism of this long exact sequence is given by $\cdot\eta$, follows from considering [25, Theorem 9.1, Chapter 3]: This result implies that prior to identifying $\widehat{\mathrm{Ext}}_{\Lambda}^{i+1}(\Omega_{\Lambda^e}^{n-1}(\Lambda) \otimes_\Lambda M, \Lambda/\operatorname{rad}\Lambda)$ with $\widehat{\mathrm{Ext}}_{\Lambda}^{i+n+1}(M, \Lambda/\operatorname{rad}\Lambda)$, the connecting homomorphism is given by $\cdot\mathcal{E}_\eta$. Under the isomorphism given by (1.1.7), it is clear that $\mathcal{E}_\eta$ corresponds to the first short exact sequence from the left that η is composed of when considered as an extension. This thus implies that the aforementioned identification maps $\cdot\mathcal{E}_\eta$ to $\cdot\eta$.

We consider now a maximal ideal $\mathfrak{p} \neq \mathfrak{m}_{\mathrm{gr}}$ containing $\langle \eta, A_H(M, \Lambda/\operatorname{rad}\Lambda)\rangle$. Assume that $\mathfrak{p}$ does not contain $A_H(M_\eta \otimes_\Lambda M, \Lambda/\operatorname{rad}\Lambda)$. This implies of course that $\mathrm{Ext}_{\Lambda}^{*}(M_\eta \otimes_\Lambda M, \Lambda/\operatorname{rad}\Lambda)_{\mathfrak{p}} = (0)$, so that by (3.2.3) we have that the localization $\widehat{\mathrm{Ext}}_{\Lambda}^{*}(M_\eta \otimes_\Lambda M, \Lambda/\operatorname{rad}\Lambda)_{\mathfrak{p}} = (0)$. By examining the exact sequence $\widehat{\theta}$ we then deduce that $\widehat{\mathrm{Ext}}_{\Lambda}^{*}(M, \Lambda/\operatorname{rad}\Lambda)_{\mathfrak{p}} \cong \eta\widehat{\mathrm{Ext}}_{\Lambda}^{*}(M, \Lambda/\operatorname{rad}\Lambda)_{\mathfrak{p}}$. Once again employing (3.2.3), we find $\widehat{\mathrm{Ext}}_{\Lambda}^{*}(M, \Lambda/\operatorname{rad}\Lambda)_{\mathfrak{p}} \cong \mathrm{Ext}_{\Lambda}^{*}(M, \Lambda/\operatorname{rad}\Lambda)_{\mathfrak{p}}$.

Hence, since we are assuming that (Λ, H) satisfy (**Fg1**) and (**Fg2**), it follows that $\widehat{\mathrm{Ext}}_{\Lambda}^{*}(M, \Lambda/\operatorname{rad}\Lambda)_{\mathfrak{p}}$ is a finitely generated $H_{\mathfrak{p}}$-module. By assumption, η is in $\mathfrak{p}$, and hence also in $\mathfrak{p}H_{\mathfrak{p}}$. Given the general properties of localization at a prime ideal, we know that $\mathfrak{p}H_{\mathfrak{p}}$ is the unique maximal ideal of $H_{\mathfrak{p}}$. As a consequence, we may employ the Nakayama Lemma to deduce $\widehat{\mathrm{Ext}}_{\Lambda}^{*}(M, \Lambda/\operatorname{rad}\Lambda)_{\mathfrak{p}} = (0)$, and hence also $\mathrm{Ext}_{\Lambda}^{*}(M, \Lambda/\operatorname{rad}\Lambda)_{\mathfrak{p}} = (0)$.

Now, since $\mathrm{Ext}_{\Lambda}^{*}(M, \Lambda/\operatorname{rad}\Lambda)$ is a finitely generated H-module, the annihilator

$$A_H(M, \Lambda/\operatorname{rad}\Lambda)$$

cannot be contained in $\mathfrak{p}$, a contradiction given our constraints on $\mathfrak{p}$. Thus we must have that $\mathfrak{p}$ does contain $A_H(M_\eta \otimes_\Lambda M, \Lambda/\operatorname{rad}\Lambda)$. As the reverse inclusion is proved in (3.2.1), we have shown the first claim.

The second claim of the proposition follows from the first by the fact that

$$V \subseteq V_H(\Lambda/\operatorname{rad}\Lambda)$$

holds for arbitrary varieties V, and the definition of L_η. □

It should be noted that this provides an alternative proof for the corresponding result in the theory of support varieties for group rings. Indeed, the usual proof uses rank varieties, of which no corresponding notion exists in our more general setting, and reduction to abelian subgroups.

The result just given easily implies the stated goal of this section: namely, that any closed homogeneous variety occurs as a variety of a module.

Theorem 3.2.5. *Let* $\mathfrak{a}$ *be any homogeneous ideal in* H. *Then there exists a module* M *in* $\operatorname{mod}\Lambda$ *such that* $V_H(M) = V_H(\mathfrak{a})$.

Proof. Suppose $\mathfrak{a}$ is some homogeneous ideal of H: i.e. $\mathfrak{a} = \langle \eta_1, \eta_2, \ldots, \eta_t \rangle$ for some homogeneous elements $\{\eta_1, \eta_2, \ldots, \eta_t\}$. By repeatedly applying (3.2.4), we see that

$$V_H(M_{\eta_1} \otimes_\Lambda M_{\eta_2} \otimes_\Lambda \cdots \otimes_\Lambda M_{\eta_t} \otimes_\Lambda \Lambda/\operatorname{rad}\Lambda) = V_H(\langle \eta_1, \eta_2, \ldots, \eta_t \rangle),$$

and thus we are done. □

Chapter 4

Complexity and trivial extensions

A trivial extension $T(\Lambda)$ of an algebra Λ is always a symmetric algebra, and, in particular, a Gorenstein algebra, implying hence that this is an interesting class to examine with regards to **(Fg)**. M. Purin in [29] has shown using results of H. Tachikawa that beginning with some hereditary algebra H and forming the trivial extension $T(H)$ one can demonstrate the following:

Theorem 4.0.1. *Let H be a hereditary algebra over a field. Then the following holds.*

(a) *If H is of finite representation type, the complexity of $T(H)$ is* 1.

(b) *If H is of tame representation type, the complexity of $T(H)$ is* 2.

(c) *If H is of wild representation type, then the complexity of $T(H)$ is infinite.*

In this chapter we wish to examine whether the result in (a) can be generalised. To do this, we compute examples and make theoretical observations. It turns out that a straightforward generalisation is unavailable. In particular, in a later section we examine an example that shows that there are algebras of finite representation type whose trivial extensions have infinite complexity. Having said that, we also show that there are some weak bounds on the complexity of the trivial extension of a general finite dimensional algebra. Moreover, we obtain particularly good results in the case of selfinjective finite dimensional algebras.

4.1 Trivial extensions

We begin by studying the relationship between Λ-modules and $T(\Lambda)$-modules, and describe the projectives of $T(\Lambda)$ using those of Λ. For the following section, our standing assumptions are that, unless otherwise specified, Λ is a connected [1] finite dimensional algebra over a field; and likewise, unless otherwise specified, all modules are assumed to be finitely generated left modules, regardless of whichever ring they are modules over.

Definition 4.1.1. Let Λ be a finite dimensional algebra over a field k. We denote then by $T(\Lambda)$ the **trivial extension** of Λ, namely $T(\Lambda) = \Lambda \ltimes \mathrm{D}(\Lambda)$, where $\mathrm{D}(\Lambda) = \mathrm{Hom}_k(\Lambda, k)$. The additive structure of $T(\Lambda)$ is given by the direct sum $\Lambda \oplus \mathrm{D}(\Lambda)$, i.e. as ordered pairs of elements (λ, f) with componentwise addition, $\lambda \in \Lambda$ and $f \in D(\Lambda)$, whereas the multiplication is given by

$$(\lambda, f)(\lambda', f') = (\lambda\lambda', \lambda f' + f\lambda')$$

for $\lambda, \lambda' \in \Lambda$ and $f, f' \in D(\Lambda)$.

Note that we have a ring homomorphism $\alpha\colon \Lambda \to T(\Lambda)$ given by $\lambda \longmapsto (\lambda, 0)$. This entails that every $T(\Lambda)$-module is a Λ-module. Moreover, if $\phi\colon k \to \Lambda$ is the morphism giving Λ its k-algebra structure, it can be seen that $\alpha\phi$ gives $T(\Lambda)$ a k-algebra structure. Indeed, if $a \in k$ and $\lambda \in \Lambda$, then we find that

$$(\phi(a), 0)(\lambda, 0) = (\phi(a)\lambda, 0) = (\lambda\phi(a), 0) = (\lambda, 0)(\phi(a), 0).$$

Likewise, for $f \in D(\Lambda)$, we find that $(\phi(a), 0)(0, f) = (0, \phi(a)f)$, where we have that

$$(\phi(a)f)(\lambda) = f(\lambda\phi(a)) = f(\phi(a)\lambda) = (f\phi(a))(\lambda).$$

Together, this implies that $\mathrm{Im}(\alpha\phi) \subseteq \mathrm{Z}(T(\Lambda))$, which is what was to be shown.

Remark 4.1.2. We note that this definition works equally well when substituting some general Λ-bimodule M for $D(\Lambda)$. Indeed, one then has the trivial extension of Λ with respect to M, for which one writes $\Lambda \ltimes M$. Moreover, in actual fact, nearly all of the results in this chapter can just as easily be stated and proven for this case, although one may need to make further assumptions with respect to the choice of M. However, restrictions on the scope and length of this text have kept us from employing this greater generality: we are interested in investigating when those trivial extensions we know are Gorenstein have finite complexity, so as to determine cases where these algebras may satisfy the finite generation hypotheses (**Fg1**) and (**Fg2**).

[1] Note that the change in terminology from "indecomposable" to "connected" is motivated by the fact that the latter is both more suggestive and conducive to the intuition in the case of finite dimensional algebras over fields.

We begin now to investigate under which conditions a Λ-module is a $T(\Lambda)$-module. If we view a $T(\Lambda)$-module M as a Λ-module, we can see that $\beta\colon D(\Lambda) \to T(\Lambda)$ given by $f \longmapsto (0, f)$ for $f \in D(\Lambda)$, induces a Λ-biadditive map $D(\Lambda) \times M \to M$ of the form $(f, m) \longmapsto (0, f) \cdot m$, for $m \in M$. Hence, by the universal property of tensor products, there exists for each $T(\Lambda)$-module M a Λ-homomorphism $\psi\colon D(\Lambda) \otimes_\Lambda M \to M$ that is unique up to isomorphism, mapping $f \otimes m$ to $(0, f) \cdot m$.

Observe that this homomorphism satisfies $\psi \circ (1_{D(\Lambda)} \otimes_\Lambda \psi) = 0$. In the following proposition, we shall see that, in fact, there is a sense in which a converse of this holds.

Proposition 4.1.3. *Let M be a Λ-module, and let also $\psi\colon D(\Lambda) \otimes_\Lambda M \to M$ be a Λ-homomorphism. Define a map $\mu\colon T(\Lambda) \times M \to M$ by*

$$(\lambda, f) \cdot m = \lambda m + \psi(f \otimes_\Lambda m),$$

where $m \in M$, $\lambda \in \Lambda$ and $f \in D(\Lambda)$. If $\psi(1_{D(\Lambda)} \otimes_\Lambda \psi) = 0$, we have that μ is a left action of $T(\Lambda)$ on M, i.e. M is a left $T(\Lambda)$-module.

Proof. It is clear that the distributive relations hold. In addition, we have that $(1, 0) \cdot m = 1 \cdot m + \psi(0 \otimes m) = m + 0 = m$. Thus, what remains to be shown is that the associative relation holds. We check this by simple computation:

$$\begin{aligned}
&(\lambda', f')((\lambda, f) \cdot m) \\
&= (\lambda', f')(\lambda m + \psi(f \otimes m)) \\
&= \lambda'(\lambda m) + \lambda'\psi(f \otimes m) + \psi(f' \otimes (\lambda m)) + \psi(f' \otimes (\psi(f \otimes m)) \\
&= \lambda'(\lambda m) + \lambda'\psi(f \otimes m) + \psi(f' \otimes (\lambda m)) + \psi(1 \otimes \psi)(f' \otimes (f \otimes m)) \\
&= \lambda'(\lambda m) + \lambda'\psi(f \otimes m) + \psi(f' \otimes (\lambda m)) + 0 \\
&= \lambda'(\lambda m) + \lambda'\psi(f \otimes m) + \psi(f' \otimes (\lambda m)) \\
&= (\lambda'\lambda, \lambda' f + f'\lambda) \cdot m \\
&= ((\lambda', f')(\lambda, f)) \cdot m
\end{aligned}$$

□

At this point, we want to recall some facts about the projectives of $T(\Lambda)$. First of all, by [21, Corollary 1.6, Chapter 1], we have that if P is Λ-projective, then $T(\Lambda) \otimes_\Lambda P$ is $T(\Lambda)$-projective. Furthermore, if M is indecomposable, $T(\Lambda) \otimes_\Lambda M$ can be shown to be indecomposable as well, and hence the indecomposable projectives of Λ map under $T(\Lambda) \otimes_\Lambda -$ to the indecomposable projectives of $T(\Lambda)$. Indeed, this follows by the properties of the functors introduced on pages 7 to 9 of [21]. In particular, we need the following three facts: Firstly, note that the composition of $T(\Lambda) \otimes_\Lambda -$ followed by the functor induced by $-/(D(\Lambda) \cdot -)$ is naturally isomorphic to the identity functor on Λ. Moreover, the functor induced by $-/(D(\Lambda) \cdot -)$ is additive, and the only $T(\Lambda)$-module it maps to the zero module is the zero module itself. Combining these, it is clear that if $T(\Lambda) \otimes_\Lambda P$ is decomposable, then P is as well, and we are done.

We can use this to construct the projective cover of a Λ-module as a $T(\Lambda)$-module. First, though, we need to prove these next well-known facts about trivial extensions.

Proposition 4.1.4. *Let Λ be a finite dimensional algebra. If $T(\Lambda)$ is its trivial extension, then the following holds.*

(a) $\operatorname{rad} T(\Lambda) = \operatorname{rad} \Lambda \oplus D(\Lambda)$.

(b) $\operatorname{rad}^2 T(\Lambda) = \operatorname{rad}^2 \Lambda \oplus (\operatorname{rad} \Lambda \cdot D(\Lambda) + D(\Lambda) \cdot \operatorname{rad} \Lambda)$

(c) $\operatorname{rad} T(\Lambda)/ \operatorname{rad}^2 T(\Lambda)$ *and* $\operatorname{rad} \Lambda/ \operatorname{rad}^2 \Lambda \oplus (D(\Lambda)/(\operatorname{rad} \Lambda \cdot D(\Lambda) + D(\Lambda) \cdot \operatorname{rad} \Lambda)))$ *are isomorphic vector spaces.*

Proof. To see that (a) holds, note that $(\operatorname{rad} \Lambda, D(\Lambda))$ is a nilpotent ideal of $T(\Lambda)$, and that $T(\Lambda)/(\operatorname{rad} \Lambda, D(\Lambda)) \cong \Lambda/ \operatorname{rad} \Lambda$, since $\Phi \colon \Lambda \to T(\Lambda)/(\operatorname{rad} \Lambda, D(\Lambda))$ given by $\lambda \mapsto (\lambda, 0) + (\operatorname{rad} \Lambda, D(\Lambda))$ is a surjective ring homomorphism with kernel $\operatorname{rad} \Lambda$, and so the result follows by [3, Corollary 1.4(c), Chapter 1].

We then have that (b) follows from (a) and a quick computation, whereas (c) follows from (a) and (b) and the First Isomorphism Theorem for vector spaces. $\square$

We are now ready to give the promised result:

Proposition 4.1.5. *Let M be a Λ-module. Let $\pi_\Lambda \colon P_\Lambda(M) \to M$ be the Λ-projective cover of M. We have then that the $T(\Lambda)$-projective cover of M is*

$$\pi_{T(\Lambda)} \colon T(\Lambda) \otimes_\Lambda P_\Lambda(M) \to M,$$

where $\pi_{T(\Lambda)}$ as a Λ-homomorphism is given for $p \in P_\Lambda(M), (\lambda, f) \in T(\Lambda)$ by

$$(\lambda, f) \otimes_\Lambda p \mapsto \pi_\Lambda(\lambda p).$$

Moreover, the kernel of this morphism, $\Omega^1_{T(\Lambda)}(M)$, can be identified with $\Omega^1_\Lambda(M) \oplus \nu P_\Lambda(M)$ as a Λ-module with the $T(\Lambda)$-action defined by

$$(\lambda, f) \cdot (p, g \otimes_\Lambda p') = (\lambda p, f \otimes_\Lambda p)$$

for $(\lambda, f) \in T(\Lambda)$ and $p, p' \in P_\Lambda(M)$.

Proof. First of all, we note that $(\lambda, f) \otimes_\Lambda p \mapsto \pi_\Lambda(\lambda p)$ can easily be seen to be a well-defined Λ-homomorphism since $((\lambda, f), p) \mapsto \pi_\Lambda(\lambda p)$ is a Λ-biadditive map. As we have that

$$(\mu, g)(\lambda, f) \otimes_\Lambda p = (\mu\lambda, \mu f + g\lambda) \otimes_\Lambda p \mapsto \pi_\Lambda(\mu\lambda p) = \pi_\Lambda((\mu, g)\lambda p) = (\mu, g)\pi_\Lambda(\lambda p)$$

holds, it is also a $T(\Lambda)$-homomorphism. Since π_Λ is an epimorphism, $\pi_{T(\Lambda)}$ is as well.

Furthermore, we can see that $T(\Lambda) \otimes_\Lambda P_\Lambda(M) \cong P_\Lambda(M) \oplus \nu P_\Lambda(M)$, since $D(\Lambda) \otimes_\Lambda -$ and ν are functorially isomorphic. Hence, the kernel of π_Λ can then be identified with the submodule $\Omega^1_\Lambda(M) \oplus \nu P_\Lambda(M)$ with the stated $T(\Lambda)$-action. Moreover, by (4.1.4), we see that

$$\operatorname{rad}(T(\Lambda) \otimes_\Lambda P_\Lambda(M)) = \operatorname{rad} T(\Lambda) \otimes_\Lambda P_\Lambda(M) = \operatorname{rad} P_\Lambda(M) \oplus \nu P_\Lambda(M),$$

and thus that

$$\Omega^1_\Lambda(M) \oplus \nu P_\Lambda(M) \subseteq \operatorname{rad}(T(\Lambda) \otimes_\Lambda P_\Lambda(M)).$$

Consequently, $\pi_{T(\Lambda)}$ is minimal, and we are done. □

We note that this entails that $\Omega^1_{T(\Lambda)}(P) \cong \nu P$ when P is a projective Λ-module. This fact will be of crucial importance in a later section.

4.2 Finite representation type and infinite complexity

In this section we examine an example demonstrating that statement (a) of Purin's theorem cannot be fully generalised. That is to say, to even assure that the complexity of the trivial extension of an algebra is finite one needs something more than it merely being of finite representation type. A straightforward approach seems difficult: it turns out that explicitly computing the minimal projective resolution of a module with complexity greater than 2 is difficult. Fortunately, there are some theoretical results that allow us to bypass this obstacle. Thus, pursuant to this task, we have need of a result of Benson's, namely [8, Theorem 1.1]. For the convenience of the reader, we include an excerpt of this result below. We note that we have suppressed some parts which are irrelevant to our current endeavour. Note that the matrix $E = E(\Lambda)$, which we call the adjacency matrix of Λ, is given by $E_{i,j} = \dim_k \operatorname{Ext}^1_\Lambda(S(i), S(j))$.

Theorem 4.2.1. *Let Λ be a weakly symmetric algebra over an algebraic closed field k satisfying $\operatorname{rad}^3 \Lambda = (0)$. The matrix E is symmetric, so its eigenvalues are real. The eigenvalue λ with largest absolute value is positive, and is a simple root of the characteristic polynomial of E.*

1. *If $\lambda > 2$ then the minimal projective resolution of each finitely generated non-projective Λ-module has exponential growth.*

2. *If $\lambda = 2$ then the dimensions of the modules in the minimal projective resolution of each finitely generated Λ-module are either bounded or grow linearly.*

3. *If $\lambda < 2$ then the dimensions of the module in the minimal projective resolution of each finitely generated non-projective Λ-module are bounded.*

Note that by (4.1.4), if $\operatorname{rad}^2 \Lambda = (0)$ then $\operatorname{rad}^3 T(\Lambda) = (0) \oplus \operatorname{rad} \Lambda D(\Lambda) \operatorname{rad} \Lambda$ by a simple computation. However, the Λ-Λ-bimodule structure of $D(\Lambda)$ entails that

$$\operatorname{rad} \Lambda D(\Lambda) \operatorname{rad} \Lambda = (0)$$

if $\operatorname{rad}^2 \Lambda = (0)$. Our plan is to use this in combination with the above and some results on the stable equivalence of algebras satisfying $\operatorname{rad}^2 \Lambda = (0)$ with hereditary algebras.

Recall then that if Λ is a basic and connected finite dimensional algebra over an algebraically closed field and $Q = (Q_0, Q_1)$ is its ordinary quiver, we can associate to it the separated quiver Q^s: if $Q_0 = \{1, \ldots, n\}$ then $Q_0^s = \{1, \ldots, n, 1', \ldots, n'\}$; and if $\alpha\colon i \to j \in Q_1$ then there is a corresponding $\alpha\colon i \to j' \in Q_1$. We now recall the following result [6, Theorem 2.6, Chapter 10], whose statement we adapt somewhat:

Theorem 4.2.2. *Let Λ be a basic and connected finite dimensional algebra Λ over an algebraically closed field satisfying* $\operatorname{rad}^2 \Lambda = (0)$. *Then Λ is of finite representation type if and only if the separated quiver for Λ is a finite disjoint union of Dynkin quivers.*

We are now ready to give the promised example:

Example 4.2.3. Let Λ be the algebra given by the quotient of the path algebra of the quiver

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[loop right] \end{tikzcd}$$

over the arrow ideal squared. That is to say, Λ satisfies $\operatorname{rad}^2 \Lambda = (0)$. Clearly we can see that the separated quiver of Λ is given by

$$\begin{tikzcd} 1 \arrow[dr] & 2 \arrow[d] \\ 1' & 2' \end{tikzcd}$$

which is the union of the Dynkin quivers A_1 and A_3. In other words, by (4.2.2), Λ is of finite representation type.

By [20, Proposition 2.2], it follows that the ordinary quiver of $T(\Lambda)$ has the same number of vertices as Λ. Moreover, by (4.1.4), we have that the arrows from vertices i to j of $T(\Lambda)$ are in correspondence with a k-basis of

$$e_i \operatorname{rad} \Lambda / \operatorname{rad}^2 \Lambda e_j \oplus e_i(D(\Lambda)/(\operatorname{rad} \Lambda \cdot D(\Lambda) + D(\Lambda) \cdot \operatorname{rad} \Lambda))e_j.$$

Since $\operatorname{rad}^2 \Lambda = (0)$, the second summand corresponds to elements of $D(\Lambda)$ which do not vanish on elements of Λ corresponding to arrows of the quiver of Λ. Moreover, an element of

$$e_i(D(\Lambda)/(\operatorname{rad} \Lambda \cdot D(\Lambda) + D(\Lambda) \cdot \operatorname{rad} \Lambda))e_j$$

is of the form $[\lambda \mapsto f(e_j \cdot \lambda \cdot e_i)]$. In sum, the arrows of the quiver of $T(\Lambda)$ from i to j correspond to arrows of the quiver of Λ from both i to j and j to i.

As a consequence, the ordinary quiver of $T(\Lambda)$ is given below:

$$\begin{tikzcd} 1 \arrow[r, shift left] & 2 \arrow[l, shift left] \arrow[loop above] \arrow[loop below] \end{tikzcd}$$

Thus $E(T(\Lambda))$ is given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix},$$

whose characteristic polynomial is thus $\lambda^2 - 2\lambda - 1$. One can then see that the maximal eigenvalue of $E(T(\Lambda))$ is $1 + \sqrt{2} > 2$, and, as a consequence of (4.2.1), the complexity of $T(\Lambda)$ is infinite.

4.3 Complexity of general trivial extensions

In this section we investigate the complexity of trivial extensions of arbitrary, general algebras. After recalling the appropriate definitions, we show that there exists an upper bound on the length of the terms of the minimal projective resolution of a Λ-module considered as a $T(\Lambda)$-module. It turns out that, in general, this bound seems to be of little use. However, we briefly investigate some specific cases which then lead to the subject matter of the next section, wherein we also obtain more substantial results.

We already gave the definition of the complexity of a Λ-module in (3.1.3). Nevertheless, we repeat it here for the convenience of the reader: With regard to that, recall that we denote the length of a module M by $l(M)$, and that if $P^* \to M$ of the form

$$\cdots \to P^n \to P^{n-1} \to \cdots \to P^0 \to M \to 0,$$

is a minimal projective resolution of M, then the ith Betti number of a M, β^i, is the number of indecomposable projective summands of P^i. Recall also that since the tops of indecomposable projective modules are simple, we have that $\beta^i = l(\operatorname{top}\Omega^i(M))$

Recall that if M is a Λ-module, and its minimal projective resolution is given by $P^* \to M$, then we define the complexity of M over Λ to be

$$\operatorname{cx}_\Lambda M = \inf\{n \in \mathbb{N}_0 \mid \exists \alpha \in \mathbb{R} \text{ such that } \beta^i \leq \alpha i^{n-1} \text{ for all } i \geq 0\}.$$

Moreover, the complexity of Λ, $\operatorname{cx}\Lambda$, is defined to be the maximum of the complexities of the simple modules of Λ.

We turn now to deriving the promised upper bound. To provide a satisfactory setting, we begin by defining some auxiliary notions. Our upper bound will be given by the sum of the "lengths" of the terms of iterates of an endomorphism of the free abelian group generated by the isomorphism classes of indecomposable modules of Λ. By analogy with the Grothendieck group of Λ, i.e. $K_0(\Lambda)$, we denote this group by $L_0(\Lambda)$. It is clear that we can construct $L_0(\Lambda)$ in the following manner: Let $\mathcal{F}$ be the free abelian group generated by all finitely generated Λ-modules, and let $\mathcal{F}'$ be the subgroup of $\mathcal{F}$ generated by all expressions $[M \oplus N] - [M] - [N]$, where $[M \oplus N]$, $[M]$ and $[N]$ are the isomorphism classes of respectively $M \oplus N$, M, and N in $\operatorname{mod}\Lambda$. We see then that $L_0(\Lambda) \cong \mathcal{F}/\mathcal{F}'$.

To see that this is well defined, we must show that the collection of all indecomposable modules is indeed a set. Hence, we note that an indecomposable Λ-module is, like every other finitely generated Λ-module, the quotient of $\bigoplus_{i=0}^n \Lambda$ for some $n \in \mathbb{N}_0$ and by some submodule. Hence, we can identify the class of all indecomposable modules with some subset of the infinite Cartesian product $\prod_{n\in\mathbb{N}_0}(\bigoplus_{i=0}^n \Lambda \times \mathcal{P}(\bigoplus_{i=0}^n \Lambda))$, where $\mathcal{P}(\bigoplus_{i=0}^n \Lambda)$ is the power set of $\bigoplus_{i=0}^n \Lambda$. Thus we are done.

The endomorphism we wish to investigate is the following: Let $[M] \in L_0(\Lambda)$ be the isomorphism class of M in $\operatorname{mod}\Lambda$. We let $S\colon L_0(\Lambda) \to L_0(\Lambda)$ be given by $[M] \mapsto [\Omega_\Lambda(M)] + [\nu P(M)]$. That this is an endomorphism follows by the fact that the projective cover of a direct sum is the direct sum of the projective covers of the summands. See for instance [6, Proposition 4.3, Chapter 1].

We are now almost ready to give the upper bound. We note that the length of a module $l(-)$ induces an homomorphism from $L_0(M)$ to the positive integers. Moreover, we agree to let S^n for $n \in \mathbb{N}_0$ denote the nth iterate of our endomorphism S, where S^0 is just the identity endomorphism $[M] \mapsto [M]$.

Finally, we note that, as previously remarked, any $T(\Lambda)$-module can be regarded as a Λ-module by way of the ring homomorphism mapping Λ injectively into $T(\Lambda)$. We denote that we are regarding a $T(\Lambda)$-module N as a Λ-module by writing ${}_\Lambda N$. One can now observe that if N is some $T(\Lambda)$-module, then

$$l_{T(\Lambda)}(N) \leq l_\Lambda({}_\Lambda N),$$

as any composition series of N over $T(\Lambda)$ can be regarded as a chain of submodules over Λ, and thus also completed to a composition series over Λ. To avoid some of the clutter of having too many subscripts, we henceforth let $l_\Lambda(-) = l(-)$.

Proposition 4.3.1. *Let M be a Λ-module. Then $l_{T(\Lambda)}(\Omega^n_{T(\Lambda)}(M)) \leq l(S^n([M]))$ holds for all $n \in \mathbb{N}_0$.*

Proof. We prove this by induction. Since

$$l_{T(\Lambda)}(\Omega^0_{T(\Lambda)}(M)) = l_{T(\Lambda)}(M) \leq l(M) = l(S^0([M]))$$

holds by our remarks immediately preceding this proposition, the base step follows.

Assume that the result holds for $n = k-1$. We observe now that we have a short split exact sequence of Λ-modules

$$0 \longrightarrow \nu P(M) \xrightarrow{\ i\ } \Omega^1_\Lambda(M) \oplus \nu P(M) \xrightarrow{\ p\ } \Omega^1_\Lambda(M) \longrightarrow 0,$$

where the maps are just the canonical inclusion and projection.

It can be seen that this is in fact a short exact sequence of $T(\Lambda)$-modules. Indeed, we endow the end terms with the trivial $D(\Lambda)$-action and the middle term with the $T(\Lambda)$-structure detailed in (4.1.5). The middle term can then be identified with $\Omega^1_{T(\Lambda)}(M)$.

By the Horseshoe Lemma, we see that

$$\begin{aligned} l_{T(\Lambda)}(\Omega^k_{T(\Lambda)}(M)) &= l_{T(\Lambda)}(\Omega^{k-1}_{T(\Lambda)}(\Omega^1_{T(\Lambda)}(M))) \\ &\leq l_{T(\Lambda)}(\Omega^{k-1}_{T(\Lambda)}(\nu P(M)) + l_{T(\Lambda)}(\Omega^{k-1}_{T(\Lambda)}(\Omega^1_\Lambda(M)). \end{aligned}$$

Hence, applying our induction hypothesis twice, we find

$$\begin{aligned} &l_{T(\Lambda)}(\Omega^{k-1}_{T(\Lambda)}(\nu P(M)) + l_{T(\Lambda)}(\Omega^{k-1}_{T(\Lambda)}(\Omega^1_\Lambda(M)) \\ &\qquad\qquad \leq l(S^{k-1}([\nu P(M)])) + l(S^{k-1}([\Omega^1_\Lambda(M)]) = l(S^k([M])), \end{aligned}$$

where the equality follows by the definition of S. Hence, the induction step follows and we are done. □

Remark 4.3.2. We note that if we were working over a basic algebra Λ over an algebraically closed field k, we could have replaced the notion of length in this proposition with that of the dimension vector of a module: namely, recall that if n is the number of non-isomorphic simple modules of Λ, then the dimension vector of a Λ-module M is defined to be the vector in $\mathbb{Z}^s$ given by

$$\mathbf{dim}\, M = \begin{bmatrix} \dim_k e_1 M \\ \vdots \\ \dim_k e_n M \end{bmatrix},$$

where $\{e_i\}_{i=1}^n$ is the complete set of primitive idempotents corresponding to the vertices Q_0 of the ordinary quiver of Λ. See [3, Chapter 3.3] for details.

Although we only sketch the argument, we can see that, given such assumptions, the dimension vector of a Λ-module would coincide with that of it as a $T(\Lambda)$-module. Indeed, this follows by the fact that the set of vertices of Λ and $T(\Lambda)$ are the same, and that any representation of a Λ-module can be regarded as a representation of a $T(\Lambda)$-module by setting the k-linear maps corresponding to arrows induced by $D(\Lambda)$ equal to the zero map. See for instance [20] for details on how to construct the ordinary quiver of $T(\Lambda)$ based on the bound quiver of Λ. Moreover, the given proof would almost carry over with only the indicated substitutions, as the crucial property of length that we utilise is its additivity as a function, a property which $\mathbf{dim} -$ also has. In a similar fashion, we can also see that $\mathbf{dim} -$ induces a homomorphism from $L_0(\Lambda)$ to $\mathbb{Z}^n$.

Example 4.3.3. We take another look at the algebra from Example 4.2.3. That is to say, let Λ be the algebra given by the quotient of the path algebra of the quiver

$$1 \longrightarrow 2 \circlearrowleft$$

over the arrow ideal squared.

We wish to examine our recently derived inequality for the simple modules in the the case $n = 2$. Upon computing, we find that

$$\mathbf{dim}\, \Omega^2_{T(\Lambda)}(S(1)) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

On the other hand, we see that

$$\begin{aligned} \mathbf{dim}\, S^2([S(1)]) &= \mathbf{dim}\, \Omega^2_\Lambda(S(1)) + \mathbf{dim}\, \Omega^1_\Lambda(\nu P(S(1))) \\ &\quad + \mathbf{dim}\, \nu P(\Omega^1_\Lambda(S(1))) + \mathbf{dim}\, \nu P(\nu P(S(1))) \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}. \end{aligned}$$

Likewise, we find

$$\mathbf{dim}\,\Omega^2_{T(\Lambda)}(S(2)) = \begin{bmatrix} 3 \\ 6 \end{bmatrix},$$

whereas

$$\begin{aligned}\mathbf{dim}\,S^2([S(2)]) &= \mathbf{dim}\,\Omega^2_\Lambda(S(2)) + \mathbf{dim}\,\Omega^1_\Lambda(\nu P(S(2))) \\ &\quad + \mathbf{dim}\,\nu P(\Omega^1_\Lambda(S(2))) + \mathbf{dim}\,\nu P(\nu P(S(2))) \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.\end{aligned}$$

Hence, it would seem as if this inequality is also in some sense the best one possible in the general case: i.e. we should expect that no improvements can be made without adding further assumptions on our algebra Λ or restricting our choice of modules further.

What we have derived is of course not directly a bound on the $T(\Lambda)$-complexity of a Λ-module. However, if β^i is the ith Betti number of a module M and P^i is the ith term of a minimal projective resolution of the same module, then it is easy to see that

$$\beta^i \leq l(P^i) = l(\Omega^{i+1}(M)) + l(\Omega^i(M)).$$

Let us now recall that the complexity of a sequence of natural numbers $\{s_n\}_{n\in\mathbb{N}_0}$, which we denote by the symbol $\mathrm{cx}\{s_n\}_{n\in\mathbb{N}_0}$, is given, if it exists, by the least $b \in \mathbb{N}_0$ such that there is some $\alpha \in \mathbb{R}$ so that

$$s_i \leq \alpha \cdot i^{b-1}$$

holds for all $i \gg 0$. If there is no such b, we say that the complexity is infinite and we write $\{s_n\}_{n\in\mathbb{N}_0} = \infty$. Given this and the preceding remarks, we see that we have the following corollary

Corollary 4.3.4. *Let M be a Λ-module. Then*

$$\mathrm{cx}_{T(\Lambda)}(M) \leq \mathrm{cx}\{l(S^{n+1}([M])) + l(S^n([M]))\}_{n\in\mathbb{N}_0}.$$

Generally, if there are no restriction on Λ, then there is no guarantee that any of the terms of $S^n([M])$ vanish. Hence, as there are $2^{n+1} + 2^n$ terms, in the worst case

$$\mathrm{cx}\{l(S^{n+1}([M])) + l(S^n([M]))\}_{n\in\mathbb{N}_0} = \infty.$$

Having said that, we can perhaps allow ourselves to speculate a bit: We see that if $S(i)$ is the ith simple module of Λ, the growth of the numbers of terms of $S^n([S(i)])$ depends on whether or not various syzygies vanish. This follows since the projective cover of a module always exists and is a non-zero module provided the module itself is non-zero. As such, since none of the simple modules are the zero module, we can see that the best case occurs when $S(i)$ is projective and all the terms of the form $\nu P(\cdots)$ are projective as well. Clearly, given such assumptions, $S^n([S(i)])$ consists of only one term for $n \geq 1$. Of course, this occurs for Λ semisimple.

However, this is too restrictive. If we are willing to drop the assumption that $S(i)$ is projective, we see that we nevertheless are left with a possibly nice bound: Let P^j be the ith term of the minimal projective resolution of $S(i)$. Then

$$S^n([S(i)]) = [\Omega^n(S(i))] + [\nu P^{n-1}] + [\nu^2 P^{n-2}] + \cdots + [\nu^n P^0].$$

Of course, this state of affairs is obtained if Λ is assumed to be selfinjective. In the next section, we show that we in fact have that

$$\Omega^n_{T(\Lambda)}(S(i)) = \Omega^n(S(i)) \oplus \nu P^{n-1} \oplus \nu^2 P^{n-2} \oplus \cdots \oplus \nu^n P^0$$

for Λ selfinjective.

If we cast our net even more widely, then we can see from the form of the bound that we are interested in algebras Λ such that all of its injective modules have finite projective dimension. Since a Gorenstein algebra has finite right injective dimension, it is clear that its minimal injective cogenerator has finite left projective dimension. By (3.1.13), an algebra which satisfies the finite generation conditions (**Fg1**) and (**Fg2**) must be Gorenstein. As such, we may expect that whether or not an algebra is Gorenstein or satisfies the finite generation conditions is somehow connected to whether or not its trivial extension has finite complexity. We will return to this point in a later section wherein we will see that there is some evidence for there being such a connection for certain classes of algebras, namely Nakayama algebras and truncated path algebras.

4.4 Trivial extensions of selfinjective algebras

In this section we will be restricting our investigation to trivial extensions of selfinjective algebras. To begin with, restricting to this case is sensible since M. Purin's results in [29], also cover iterated tilted algebras Λ from hereditary algebras H. This follows from her having shown that complexity is preserved under stable equivalence of selfinjective algebras. As such, since iterated tilted algebras have finite global dimension, our present case is disjoint from that of Purin's.

In fact, assuming Λ to be selfinjective allows us to compute explicitly the minimal $T(\Lambda)$-projective resolution of a $T(\Lambda)$-module M with trivial $D(\Lambda)$-action, i.e. $D(\Lambda) \cdot M = (0)$, which is to say, a Λ-module. We will see that the relevant terms, syzygies and $T(\Lambda)$-homomorphisms will be constructed out of those in the minimal Λ-projective resolution. A more general version of the result we give is to be found in Lemma 3.2 of Guo et al. in [22]. While the hypotheses assume that the algebra in question must be Koszul, that assumption is not utilized in the relevant parts of that result.

This allows us, among other things, to show that the complexity of a simple Λ-module considered as a $T(\Lambda)$-module is at most 1 greater than its complexity as a Λ-module, and hence, that the complexity of $T(\Lambda)$ is also at most 1 greater than Λ, provided that Λ has finite complexity. Moreover, in the following section, we show that in certain circumstances, the complexity of the trivial extension of a selfinjective algebra Λ is exactly one greater than that of Λ itself.

The proof of our main result will be rather straightforward and elementary. Before we can begin the proof proper, we need to recall some relevant facts, and give some preliminary definitions and results.

Firstly, as Λ is selfinjective, $T(\Lambda)$ will be a Λ-projective module, and hence, $T(\Lambda)\otimes_\Lambda -$ will be an exact functor. Moreover, the Nakayama functor $\nu(-)$, usually defined as $D\operatorname{Hom}_\Lambda(-,\Lambda)$, and which in general is functorially isomorphic with $D(\Lambda)\otimes_\Lambda -$, becomes an exact autoequivalence of $\operatorname{mod}\Lambda$, and preserves length, minimal projective resolutions, and projective covers. In addition, it commutes with syzygies. In case an algebra is symmetric, then we have by definition that Λ is isomorphic to $D(\Lambda)$ as a bimodule, and hence $\nu-$ is functorially isomorphic to the identity functor.

The following is the definition of a structure that will turn out to be the $T(\Lambda)$-syzygies of Λ-modules. We first give their Λ-structure, and then prove that they also have a well-defined $T(\Lambda)$-structure. As a matter of convention, we identify $\nu-$ with $D(\Lambda)\otimes_\Lambda -$, as this makes our formulas and expressions a great deal cleaner. Moreover, as we at no point make use of the standard definition of $\nu-$, there is little chance of confusion arising.

Definition 4.4.1. Assume Λ to be a finite dimensional algebra over a field. Let M be a Λ module, and let $\pi^*_\Lambda\colon P^* \to M$ be its minimal Λ-projective resolution. We define for $n \geq 1$ the n**th induced module** $\mathrm{I}_n(M)$ to be given as a Λ module by the direct sum

$$\Omega^n_\Lambda(M) \oplus \nu P^{n-1} \oplus \nu^2 P^{n-2} \oplus \nu^3 P^{n-3} \oplus \cdots \oplus \nu^n P^0.$$

Proposition 4.4.2. *Assume Λ to be a finite dimensional algebra. The nth induced module $\mathrm{I}_n(M)$ is a $T(\Lambda)$ module under the $T(\Lambda)$ action defined by the underlying*

Λ-action and the $D(\Lambda)$-action defined for $p = (p_n, p_{n-1}, p_{n-2}, \dots, p_0) \in \mathrm{I}_n(M)$ by

$$f \cdot (p_n, p_{n-1}, p_{n-2}, \dots, p_0) =$$
$$(0, f \otimes_\Lambda \iota_\Lambda^{n-1}(p_n), (-1)^1 f \otimes_\Lambda \nu(\pi^{n-1})(p_{n-1}), \dots, (-1)^{n-1} f \otimes_\Lambda \nu^{n-1}(\pi^1)(p_1)),$$

where $\iota_\Lambda^{n-1} \colon \Omega_\Lambda^n(M) \to P^{n-1}$ is the inclusion of the nth Λ-syzygy into the $(n-1)$th term in the minimal projective resolution of M as a Λ-module.

Proof. It suffices to show that $g \cdot (f \cdot p) = 0$ for f and g in $D(\Lambda)$ and p in $\mathrm{I}_n(M)$. Indeed, if this holds, then it is easy to show that induced modules satisfy the associative $T(\Lambda)$-module axiom, whereas the other axioms are in this case trivial to verify to begin with.

By the proposed definition, this is

$$(0, 0, (-1)^1 g \otimes_\Lambda \nu^1(\pi^{n-1})(f \otimes_\Lambda \iota_\Lambda^{n-1}(p_n)),$$
$$(-1)^2 g \otimes_\Lambda \nu^2(\pi^{n-2})((-1)^1 f \otimes_\Lambda \nu^1(\pi^{n-1})(p_{n-1})), \dots),$$

which is equal to

$$(0, 0, (-1)^1 g \otimes_\Lambda 0, (-1)^2 g \otimes_\Lambda \nu^2(\pi^{n-2})((-1)^1(\nu^2(\pi^{n-1}))(f \otimes_\Lambda p_{n-1})), \dots),$$

which, by the the fact that $\pi^{i-1}\pi^i = 0$, is equal to

$$(0, 0, 0, (-1)^2 g \otimes_\Lambda 0, \dots),$$

or simply $(0, 0, 0, ...)$, and hence we are done. $\square$

Observe that this construction is also valid in the non-selfinjective case. For the sake of completeness, we also note that there is a more general construction that encompasses the one just given. Namely, for any bounded complex of Λ-modules, it is clear that we can, up to choices of sign, construct a module in this manner. Moreover, there is a dual construction involving the quasi-inverse of the Nakayama functor, wherein the Λ-module structure is given as

$$\nu^{-n} M^0 \oplus \nu^{-(n-1)} M^1 \oplus \cdots \oplus \nu^{-1} M^1 \oplus M^0$$

for a bounded complex of Λ-modules M^*, and where the $D(\Lambda)$-action is induced by evaluating an element of $\nu^{-i} M^{n-i}$ at the given element of $D(\Lambda)$ and then mapping under an appropriate homomorphism into $\nu^{-(i-1)} M^{n-(i-1)}$. This also lends some legitimacy to identifying $\nu-$ with $D(\Lambda) \otimes_\Lambda -$ in the present case, as it is in this form that $\nu-$ also gives rise to $T(\Lambda)$-modules. Finally, it can be seen that these constructions generalise the functors T and H of [21] in the case of the trivial extension of a ring.

Having that said, we are now ready to give the proof proper.

Proposition 4.4.3. *Assume Λ to be a finite dimensional selfinjective algebra over a field. Let M be a Λ-module, and let $\pi_\Lambda^*\colon P^* \to M$ be its Λ-minimal projective resolution. Then the projective cover of the nth induced module of M is given as a $T(\Lambda)$-module*

$$T(\Lambda) \otimes_\Lambda P^n \oplus T(\Lambda) \otimes_\Lambda \nu P^{n-1} \oplus \cdots T(\Lambda) \otimes_\Lambda \nu^n P^0,$$

which as a Λ-module is isomorphic to

$$P^n \oplus \nu P^n \oplus \nu P^{n-1} \oplus \nu^2 P^{n-1} \oplus \cdots \oplus \nu^n P^0 \oplus \nu^{n+1} P^0$$

The $T(\Lambda)$-epimorphism onto the nth induced module of M has the following form as a Λ-homomorphism for $(p_i, p_i') \in T(\Lambda) \otimes_\Lambda \nu^{n-i} P^i \cong \nu^{n-i} P^i \oplus \nu^{n+1-i} P^i$, $0 \le i \le n$:

$$\pi^n_{T(\Lambda)}\colon ((p_n, p_n'), (p_{n-1}, p_{n-1}'), (p_{n-2}, p_{n-2}'), \ldots, (p_1, p_1'), (p_0, p_0'))$$
$$\mapsto$$
$$(\pi^n_\Lambda(p_n), \nu(\pi^n_\Lambda)(p_n') + p_{n-1}, (-1)^1\nu^2(\pi^{n-1}_\Lambda)(p_{n-1}') + p_{n-2}, \ldots, (-1)^{(n-1)}\nu^n(\pi^1_\Lambda(p_1')) + p_0).$$

Moreover, if $\iota^n_\Lambda\colon \Omega^{n+1}_\Lambda(M) \to P^n$ is the inclusion of the $(n+1)$th Λ-syzygy into the nth term in the minimal projective resolution of M as a Λ-module, the kernel of the projective cover of the nth induced module is the $(n+1)$th induced module of M, and the kernel monomorphism is given as a Λ-homomorphism for $(p_{n+1}, p_n, p_{n-1}, \ldots, p_0) \in \mathrm{I}_{n+1}(M)$ by

$$\iota^n_{T(\Lambda)}\colon (p_{n+1}, p_n, p_{n-1}, p_{n-2}, \ldots, p_1, p_0) \in \mathrm{I}_{n+1}(M)$$
$$\mapsto$$
$$((\iota^n_\Lambda(p_{n+1}), p_n), ((-1)^1\nu(\pi^n_\Lambda)(p_n), p_{n-1}), ((-1)^2\nu^2(\pi^{n-1}_\Lambda)(p_{n-1}), p_{n-2}), \ldots, ((-1)^n\nu^n(\pi^1_\Lambda)(p_1), p_0)).$$

Proof. By examining the coordinates, it is clear that $\pi^n_{T(\Lambda)}$ is an epimorphism and that $\iota^n_{T(\Lambda)}$ is a monomorphism. Furthermore, again from examining the coordinates, it is also clear that $\iota^n_{T(\Lambda)}$ is the kernel of $\pi^n_{T(\Lambda)}$. Indeed, this is obvious for both the leftmost and the rightmost coordinates, while for the other coordinates we have for $0 \le i \le n-1$ that $(-1)^i\nu^i(\pi^{n-i}_\Lambda)(p_{n-i}') + p_{n-(i+1)} = 0$ holds if and only if $p_{n-(i+1)} = (-1)^{i+1}\nu^i(\pi^{n-i}_\Lambda)(p_{n-i}')$ holds.

Hence, what remains is to show that these two Λ-homomorphisms are also $T(\Lambda)$-homomorphisms, and that $\pi^n_{T(\Lambda)}$ is essential. If we begin with the latter, we can see that

$$\operatorname{rad} T(\Lambda) \cdot (T(\Lambda) \otimes_\Lambda P^n \oplus T(\Lambda) \otimes_\Lambda \nu P^{n-1} \oplus \cdots T(\Lambda) \otimes_\Lambda \nu^n P^0)$$

is really

$$\operatorname{rad} T(\Lambda) \otimes_\Lambda P^n \oplus \operatorname{rad} T(\Lambda) \otimes_\Lambda \nu P^{n-1} \oplus \cdots \operatorname{rad} T(\Lambda) \otimes_\Lambda \nu^n P^0,$$

or

$$(\operatorname{rad} P^n \oplus \nu P^n) \oplus (\operatorname{rad} \nu^2 P^{n-1} \oplus \nu^3 P^{n-1}) \oplus \cdots \oplus (\operatorname{rad} \nu^n P^0 \oplus \nu^{n+1} P^0).$$

At this point, we can observe that the component morphisms of $\iota^n_{T(\Lambda)}$ either identify a Λ-summand $\nu^i P^{n-i}$ with the corresponding one in the codomain, in which case it is the rightmost one of a pair, or maps it into the leftmost element of a pair via $(-1)^{i+1}\nu^i(\pi^{n-i}_\Lambda)$, in which case its image is contained in $\operatorname{rad} \nu^i P^{n-i}$ since π^{n-i} is epimorphic onto $\Omega^{n-i}_\Lambda(M)$ by construction and ν preserves minimal projective resolutions. It follows that $\pi^n_{T(\Lambda)}$ is essential.

Only one point remains now, namely to show that $\pi^n_{T(\Lambda)}$ and $\iota^n_{T(\Lambda)}$ are homomorphisms also over $T(\Lambda)$. Towards this, it suffices to show that they commute with the action of $D(\Lambda)$. As such, we compute:

$$\begin{aligned}
&f \cdot ((p_n, p'_n), (p_{n-1}, p'_{n-1}), (p_{n-2}, p'_{n-2}), \ldots, (p_1, p'_1), (p_0, p'_0)) \\
&= ((0, f \otimes_\Lambda p_n), (0, f \otimes_\Lambda p_{n-1}), (0, f \otimes_\Lambda p_{n-2}), \ldots, (0, f \otimes_\Lambda p_1), (0, f \otimes_\Lambda p_0))
\end{aligned}$$

$$\mapsto$$

$$\begin{aligned}
&(0, \nu(\pi^n_\Lambda)(f \otimes_\Lambda p_n) + 0, (-1)^1 \nu^2(\pi^{n-1}_\Lambda)(f \otimes_\Lambda p_{n-1}) + 0, \\
&\qquad \ldots, (-1)^{(n-1)} \nu^n(\pi^1_\Lambda)(f \otimes_\Lambda p_1) + 0)
\end{aligned}$$

$$\begin{aligned}
&= (0, f \otimes_\Lambda \pi^n_\Lambda(p_n), (-1)^1 (f \otimes_\Lambda \nu(\pi^{n-1}_\Lambda)(p_{n-1})), \\
&\qquad \ldots, (-1)^{(n-1)} (f \otimes_\Lambda \nu^{n-1}(\pi^1_\Lambda)(p_1)))
\end{aligned}$$

$$\begin{aligned}
&= f \cdot (\pi^n_\Lambda(p_n), \nu\pi^n_\Lambda(p'_n) + p_{n-1}, (-1)^1 \nu^2 \pi^{n-1}_\Lambda(p'_{n-1}) + p_{n-2}, \\
&\qquad \ldots, (-1)^{(n-1)} \nu^n \pi^1_\Lambda(p'_1) + p_0).
\end{aligned}$$

Hence, the stated result holds for $\pi^n_{T(\Lambda)}$. Finally, we compute just once more to show that $\iota^n_{T(\Lambda)}$ also commutes with the $D(\Lambda)$-action.

$$\begin{aligned}
&f \cdot (p_{n+1}, p_n, p_{n-1}, \ldots, p_1, p_0)\\
&\quad = (0, f \otimes_\Lambda \iota^n_\Lambda(p_{n+1}), (-1)^1 f \otimes_\Lambda \nu(\pi^n_\Lambda)(p_n), (-1)^2 f \otimes_\Lambda \nu^2(\pi^{n-1}_\Lambda)(p_{n-1}),\\
&\qquad\qquad \ldots, (-1)^n f \otimes_\Lambda \nu^n(\pi^1_\Lambda)(p_1))
\end{aligned}$$

$$\mapsto$$

$$\begin{aligned}
&((0, f \otimes_\Lambda \iota^n_\Lambda(p_{n+1})), ((-1)^1 \nu(\pi^n_\Lambda)(f \otimes_\Lambda \iota^n_\Lambda(p_{n+1})), (-1)^1 f \otimes_\Lambda \nu(\pi^n_\Lambda)(p_n)),\\
&\qquad \ldots, (\ldots, (-1)^n f \otimes_\Lambda \nu^n(\pi^1_\Lambda)(p_1)))
\end{aligned}$$

$$\begin{aligned}
&= ((0, f \otimes_\Lambda \iota^n_\Lambda(p_{n+1})), ((-1)^1 f \otimes_\Lambda \pi^n_\Lambda(\iota^n_\Lambda(p_{n+1})), (-1)^1 f \otimes_\Lambda \nu(\pi^n_\Lambda)(p_n)),\\
&\qquad \ldots, (\ldots, (-1)^n f \otimes_\Lambda \nu^n(\pi^1_\Lambda)(p_1)))
\end{aligned}$$

$$\begin{aligned}
&= ((0, f \otimes_\Lambda \iota^n_\Lambda(p_{n+1})), (0, (-1)^1 f \otimes_\Lambda \nu(\pi^n_\Lambda)(p_n)),\\
&\qquad \ldots, (\ldots, (-1)^n f \otimes_\Lambda \nu^n(\pi^1_\Lambda)(p_1)))
\end{aligned}$$

$$\begin{aligned}
&= f \cdot ((\iota^n_\Lambda(p_{n+1}), p_n), ((-1)^1 \nu(\pi^n_\Lambda)(p_n), p_{n-1}), ((-1)^2 \nu^2(\pi^{n-1}_\Lambda)(p_{n-1}), p_{n-2}),\\
&\qquad \ldots, (\ldots, ((-1)^n \nu^n(\pi^1_\Lambda)(p_1), p_0))
\end{aligned}$$

□

This immediately yields the following result.

Corollary 4.4.4. *Assume Λ to be a finite dimensional selfinjective algebra over a field. Let M be a Λ-module. Then we have that the nth $T(\Lambda)$-syzygy of M, $\Omega^n_{T(\Lambda)}(M)$, is isomorphic to the nth induced module of M, $\mathrm{I}_n(M)$.*

Proof. This follows by induction. By (4.1.5), the first $T(\Lambda)$-syzygy of M is isomorphic to the first induced module of M, and so the base step holds. The previous result shows that the induction step also holds, and we are done. □

Remark 4.4.5. Although we do not discuss this in depth, we note that this and the preceding result can probably easily be generalised to selfinjective artin algebras. Indeed, if Λ is instead assumed to be a selfinjective artin algebra over a commutative Artinian ring R, we believe the proofs should go through verbatim with the understanding that the duality $D(-)$ is interpreted everywhere as $\operatorname{Hom}_R(-, J)$, where J is the injective envelope of $R/\operatorname{rad} R$, i.e. it is the minimal injective cogenerator of R. See [6] for an introduction to the representation theory of artin algebras, and specifically Chapter 4 of that work for details on selfinjective artin algebras.

Moreover, these results should, as previously mentioned, also generalise to trivial extensions of Λ with respect to bimodules M where M is projective both as a left and as a right Λ-module. That is to say, in the definition we gave of the trivial extension of an algebra Λ, we replace $D(\Lambda)$ by a Λ^e-module M. In point of fact, the crucial property of the selfinjective case that we exploit is that $\nu(-) \cong D(\Lambda) \otimes_\Lambda -$ maps projective modules to projective modules. By familiar arguments using the Hom-$\otimes$-adjunction, we see that if M is a bimodule Λ-projective on both sides, then $M \otimes_\Lambda -$ maps projective Λ-modules to projective Λ-modules. Additionally, as already mentioned, the results in the first and the third section of this chapter should hold for trivial extensions with respect to bimodules.

For the sake of clarity, it may have been preferable to have written the presentation to take account of this. However, as stated before, keeping in mind the restrictions on the scope of the text, we have chosen not to. Additionally, the problem description calls for investigating the case of trivial extensions of algebras Λ with respect to the minimal injective cogenerator of those Λ. This is not unjustified, as it is immediately clear that trivial extensions are Gorenstein, and our stated goal of investigating whether or not they have finite complexity is to be a first step in investigating whether or not they can satisfy the finite generation hypotheses.

We turn now to demonstrating the general upper bound promised in the introduction of this section.

Corollary 4.4.6. *Assume Λ to be a finite dimensional selfinjective algebra over a field. If Λ has finite complexity, then $\operatorname{cx}\Lambda \leq \operatorname{cx} T(\Lambda) \leq \operatorname{cx}\Lambda + 1$ holds. If $\operatorname{cx}\Lambda = \infty$, then $\operatorname{cx} T(\Lambda) = \infty$ as well.*

Proof. Let $T = \Lambda/\operatorname{rad}\Lambda$. Note that T is both the top of Λ and $T(\Lambda)$, as can be seen to follow by (4.1.4). Let now β^i_Λ and $\beta^i_{T(\Lambda)}$ be the ith Betti numbers of T over Λ and $T(\Lambda)$ respectively. Then by the form of the minimal projective resolution of T we see that $\beta^i_\Lambda \leq \beta^i_{T(\Lambda)} = \sum_{r=0}^{i} \beta^r_\Lambda$ holds for all $i \geq 1$. Hence, if Λ has infinite complexity, $T(\Lambda)$ has as well: indeed, observe that for all b in $\mathbb{N}_0$ and all α in $\mathbb{R}$ there is some $j \geq 1$ such that $\alpha j^b < \beta^j_\Lambda \leq \beta^j_{T(\Lambda)}$. Moreover, if Λ has finite complexity we know that for all α in $\mathbb{R}$ there is some $j \geq 1$ such that $\alpha j^{\operatorname{cx}_\Lambda T - 2} < \beta^j_\Lambda \leq \beta^j_{T(\Lambda)}$, and so both our second claim and the first half of our first claim have been established.

For the second half of our first claim, we recall Faulhaber's formula:

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j}B_j n^{p+1-j}$$

Here, B_j is the jth Bernoulli number. If $\alpha \in \mathbb{R}$ is such that $\beta_\Lambda^i \leq \alpha i^{\mathrm{cx}_\Lambda T-1}$ for all $i \geq N$ for some sufficiently large N, then we can see that also for all $i \geq N$

$$\beta_{T(\Lambda)}^i = \sum_{r=0}^{i}\beta_\Lambda^r \leq \sum_{r=0}^{N}\beta_\Lambda^r + \frac{\alpha}{p+1}\sum_{j=0}^{\mathrm{cx}_\Lambda T-1}(-1)^j\binom{\mathrm{cx}_\Lambda T}{j}B_j i^{\mathrm{cx}_\Lambda T-j}.$$

Now, upon choosing

$$\alpha' = \sum_{r=0}^{N}\beta_\Lambda^r + \frac{\alpha}{p+1}\sum_{j=0}^{\mathrm{cx}_\Lambda T-1}\binom{\mathrm{cx}_\Lambda T}{j}|B_j|,$$

we see that $\beta_{T(\Lambda)}^i \leq \alpha' i^{cx_\Lambda T}$ holds for all $i \geq N$. Hence, the other half of our claim follows. $\square$

4.5 (Fg) and complexity revisited

Although it might seem a priori reasonable to expect the complexity of the trivial extension of a selfinjective algebra Λ to be precisely 1 greater than that of Λ itself, demonstrating this has proven difficult. Moreover, it has likewise proven difficult to find a counterexample. We note here some possible explanations for this, as well as motivating the particular approach we take in this section: Firstly, we can observe that by the formula in (4.4.3), if a Λ-module has complexity 1 over Λ, it must have complexity 2 over $T(\Lambda)$. Though this might seem innocuous, it has the effect of excluding the case for which this might otherwise have been a tractable problem. Indeed, computing enough of a minimal projective resolution to find a pattern so that one can deduce a general formula for it or otherwise deduce its complexity seems to be difficult in the case of complexity 2 or higher.

Moreover, we recall that, as previously mentioned, something more can be said given further assumptions: In fact, using their Corollary 3.2, Guo et al. in [22] have shown that the complexity of a selfinjective Koszul algebra increases by one when taking its trivial extension. Additionally, our desired result seems to hold if the lengths of the terms of the minimal projective resolutions of modules are given by polynomials.

Of course, a state of affairs such as in the latter is something we have already seen suggestions of in our earlier results. Namely, by combining (3.1.10), (3.1.11) and (3.1.9), we see that the following holds: Let (Λ, H) satisfy (**Fg1**) and (**Fg2**), and $P^* \to M$ be the minimal projective resolution of some M in $\operatorname{mod}\Lambda$. Then for $i \gg 0$ one has $\dim P^i = p(i)$ where $p(x) \in \mathbb{Q}[x]$, provided the generators of H all have degree 1. Upon observing this, it is clear that we can employ Faulhaber's formula to deduce that by taking the trivial extension the complexity for such an algebra Λ increases by exactly 1.

However, such assumptions on the generators of H are unfortunately too restrictive. Indeed, it is known by a result of A. Dugas in [16] that the selfinjective finite dimensional algebras over algebraically closed fields of finite representation type satisfy our finite generation hypotheses. Yet experimenting with representatives of the derived equivalence classes of these shows that not all of them have resolutions described by polynomials. See the appendix of [2] for quivers with relations for representatives of each of these derived equivalence classes. Furthermore, this is perhaps not surprising: clearly, the dimensions of the terms in the resolution of a periodic module cannot be described by a polynomial.

Moreover, it might be instructive to examine the case of the selfinjective Nakayama algebras. Indeed, the dimensions of the terms in the resolutions of modules over these can be seen to be described by polynomials. Nevertheless, by examining the structure of the Hochschild cohomology rings of this class of algebras as they are detailed in [18], one can see that this does not necessarily follow by the argument we have outlined. As it so happens, it is not clear whether or not there is some subalgebra H generated by its degree 1 part so that (Λ, H) satisfies our hypotheses. In fact, over fields of characteristic different from 2, then it is clear that no such subalgebra can exist, as in that case odd degree elements square to zero.

We note also that certain special biserial algebras whose Hochschild cohomology is described in Snashall and Taillefer's [33] also seem to exhibit this behaviour: i.e. by this we mean to say that while the dimensions of the terms of the minimal projective resolutions of these algebras are described by polynomials, it is not clear whether there is some H generated by degree 1 elements. See the end of that paper for a delineation of which of these algebras satisfy the finite generation hypotheses.

Fortunately, it turns out that by modifying our approach somewhat, more can still be said. In the following, we show that the complexity increases by one when taking the trivial extension for pairs (Λ, H) satisfying (**Fg1**) and (**Fg2**) and for which Λ is selfinjective. To do this, we must introduce the notion of a quasi-polynomial function:

Definition 4.5.1. Let k and d in $\mathbb{Z}$ satisfy $k \geq 1$ and $d \geq -1$. A function $\phi\colon \mathbb{Z} \to \mathbb{Z}$ is called a **quasi-polynomial function** of **period** k and **degree** d if there exists a necessarily unique sequence $F = (F_0, F_1, \ldots, F_{k-1})$ of polynomials

$F_j \in \mathbb{Q}[x]$ such that $j = 0, 1, \ldots, k-1$,

(i) $\phi(n) = F_j(n)$ for all $n \gg 0$ with $n \equiv j \mod k$ and

(ii) $\max_{0 \le j \le k-1} \deg F_j = d$.

Moreover, we say that F is the **quasi-polynomial associated with** ϕ and that d is the degree of F. In case all the polynomials F_j have the same leading coefficient, we call that the leading coefficient of F.

We remark that what we have referred to as the Poincaré series of a graded vector space V of finite type is also commonly called its Hilbert series. Now, for the sake of balance and our current purposes, we will henceforth use the latter term. Moreover, from this alternate nomenclature there is derived another term which we now find ourselves in need of: Namely, for n in $\mathbb{N}_0$ and M a finitely generated module over a commutative Noetherian graded ring $A = \bigoplus_{n \in \mathbb{N}_0} A_n$ where A_0 is Artinian, let $H(M, n)$ be such that

$$p(M, t) = \sum_{n \in \mathbb{N}_0} H(M, n) t^n.$$

Then $H(M, n)$ is called the **Hilbert function** of M. Building on this, we let the **cumulative Hilbert function** of M be given by

$$H^*(M, n) = \sum_{j=0}^{n} H(M, j).$$

We can now give the statement of the following result of Dichi and Sangare from [15], although we do not give its proof. Note that its formulation has only been amended in a few, trivial ways to align it more with the terminology we have used earlier in the text.

Theorem 4.5.2. *Let $A = \bigoplus_{n \in \mathbb{N}_0} A_n$ be a graded Noetherian ring of finite Krull dimension. Assume that $A = A_0[x_1, \ldots, x_r]$, where each x_i is homogeneous of degree $k_i \ge 1$ and that A_0 is an Artinian ring. Let $M = \bigoplus_{n \in \mathbb{N}_0} M_n$ be a finitely generated graded A-module and let $d = \dim M$. If $l = \operatorname{lcm}(k_1, k_2, \ldots, k_r)$ then:*

(1) The Hilbert function $H(M, -)$ of M is a quasi-polynomial function of period l and degree $d-1$. More precisely, there exists a sequence $F = (F_0, F_1, \ldots, F_{l-1})$ of polynomials $F_j \in \mathbb{Q}[x]$ such that

(i) $H(M, n) = F_j(n)$ for all $n \gg 0$ with $n \equiv j \mod l$;

(ii) $\max_{0 \le j \le l-1} \deg F_j = d - 1 = \deg F_0$.

(2) *The cumulative Hilbert function $H^*(M,-)$ of M is a quasi-polynomial function of period l and degree d. More precisely there exists a sequence $G = (G_0, G_1, \ldots, G_{l-1})$ of polynomials $G_j \in \mathbb{Q}[x]$ such that*

   (i) *$H^*(M,n) = G_j(n)$ for all $n \gg 0$ with $n \equiv j \mod l$.*

   (ii) *All the polynomials G_j have the same degree equal to d and the same leading coefficients.*

From this it is immediately clear that the Hilbert function describing the Betti numbers, the lengths, or the dimensions of the terms of the minimal projective resolution of a module over an algebra Λ satisfying the finite generation hypotheses is in fact a quasi-polynomial function. In relation to this, we note that Bergh has in [9, Theorem 3.1] shown a stronger and far more general result implying also this, formulated in and making use of the setting of the bounded derived category of a ring or an algebra Λ, i.e. $\mathcal{D}^b(\Lambda)$. In fact, if we were to use a generalization of Faulhaber's formula such as [9, Lemma 2.3], we could give a closed form formula for the cumulative Hilbert function showing that it too was a quasi-polynomial function, essentially replicating the second part of Dichi and Sangare's result. Hence, we could make do with Bergh's result. However, the result we cite above is simply too on the nose for our purposes to not use it.

We now state and prove the following result:

Theorem 4.5.3. *Assume Λ to be a selfinjective finite dimensional algebra. Let (Λ, H) be a pair satisfying* (**Fg1**) *and* (**Fg2**). *Then* $\operatorname{cx} T(\Lambda) = \operatorname{cx} \Lambda + 1$.

Proof. Recall that for a pair (Λ, H) satisfying our finite generation hypotheses, we have by (3.1.14) and (3.1.11) that we can associate to the minimal Λ-projective resolution $P^* \to M$ of an M in $\operatorname{mod} \Lambda$ a Hilbert series

$$\begin{aligned} p(\bigoplus_{n\in\mathbb{N}_0} P^n, t) &= \sum_{n\in\mathbb{N}_0} l(P^n)t^n \\ &= \sum_{n\in\mathbb{N}_0} H(\bigoplus_{m\in\mathbb{N}_0} P^m, n)t^n \\ &= \sum_{n\in\mathbb{N}_0} \sum_i l(P(i)) \cdot H(\operatorname{Ext}^*_\Lambda(M, S(i)), n)t^n, \end{aligned}$$

where $\operatorname{Ext}^*_\Lambda(M, S(i))$ is finitely generated as an H-module for all M in $\operatorname{mod} \Lambda$ and all i in I, an indexing set of the simple modules of Λ. By (4.4.3), if $Q^* \to M$ is the minimal $T(\Lambda)$-projective resolution of M, then we can see that

$$l(Q^n) = 2 \cdot H^*(\bigoplus_{m\in\mathbb{N}_0} P^m, n).$$

Hence, for $n \gg 0$, we can conclude by the aforementioned result that $l(Q^n)$ is given by a quasi-polynomial of some degree d, where $d-1$ is the degree of the quasi-polynomial $H(\bigoplus_{m\in\mathbb{N}_0} P^m, n)$. Namely, we can use the result since by assuming (**Fg1**) we know that H is commutative Noetherian, and H^0 is Artinian. Moreover, note that H must have finite Krull dimension: To see this, note that this is implicit in our discussion surrounding our definition of the dimension of a variety and the discussion in [7, Chapter 5.4]. Alternatively, by (3.1.2), we can employ the Noether Normalisation Lemma and [10, Corollary A.8, Appendix] to deduce that H must be of finite Krull dimension.

We now show that if M is $T = \Lambda/\operatorname{rad}\Lambda$, then $d = \operatorname{cx}\Lambda$ and that this entails that $\operatorname{cx} T(\Lambda) = \operatorname{cx}\Lambda + 1$. Observe that if $P^* \to M$ is a minimal projective resolution of M a Λ-module, then $\operatorname{cx}\{l(P^n)\}_{n\in\mathbb{N}_0} = \operatorname{cx}_\Lambda M$ holds for arbitrary artin algebras Λ by elementary considerations: i.e. we see that $\beta^i \leq l(P^i) \leq l(\Lambda)\beta^i$. Thus to establish $d = \operatorname{cx}\Lambda$, it suffices to show that

$$\operatorname{cx}\{l(P^n)\}_{n\in\mathbb{N}_0} = \operatorname{cx}\{H(\bigoplus_{m\in\mathbb{N}_0} P^m, n)\}_{n\in\mathbb{N}_0} = d.$$

Let $1 \leq j \leq l$ for l the period of $H(\bigoplus_{m\in\mathbb{N}_0} P^m, n)$ and $F_j(x) \in \mathbb{Q}[x]$ be such that $H(\bigoplus_{m\in\mathbb{N}_0} P^m, n) = F_j(n)$ for $n \equiv j \mod l$ and $\deg F_j(x) = d-1$. Now, clearly $\operatorname{cx}\Lambda = \operatorname{cx}\{H(\bigoplus_{m\in\mathbb{N}_0} P^m, n)\}_{n\in\mathbb{N}_0} \geq d$ holds. Indeed, if we assume otherwise, then given some $\alpha \in \mathbb{R}$, there is some $i \gg 0$ satisfying $i \equiv j \mod l$ such that

$$\alpha i^{\operatorname{cx}\Lambda - 1} < H(\bigoplus_{m\in\mathbb{N}_0} P^m, i) = F_j(i).$$

To be specific, we can take $i = |\lceil a_{j,d-1}^{-1}\rceil|(\alpha + \gamma + \sum_{r=0}^{d-1} |\lceil a_{j,r}\rceil|)$ where $a_{j,r}$ for $1 \leq r \leq d-1$ are the coefficients of F_j and $\gamma \in \mathbb{N}_0$ is chosen so that i is both sufficiently large and so that $i \equiv j \mod l$. In this case, provided $d > \operatorname{cx}\Lambda$, the highest degree term of F_j clearly dominates both $\alpha i^{\operatorname{cx}\Lambda-1}$ and the other terms of F_j, and we have a contradiction.

On the other hand, we can show directly that there is some choice of $\alpha \in \mathbb{R}$ such that $\alpha i^{d-1} \geq H(\bigoplus_{m\in\mathbb{N}_0} P^m, i)$ for $i \gg 0$. In fact, we take $\alpha = \sum_{r=0}^{d-1} |\max_{0\leq j\leq l} a_{j,r}|$, and hence $d \geq \operatorname{cx}\Lambda$, and thus $d = \operatorname{cx}\Lambda$.

Since

$$\begin{aligned}
\operatorname{cx} T(\Lambda) &= \operatorname{cx}_{T(\Lambda)} T \\
&= \operatorname{cx}\{l(Q^n)\}_{n\in\mathbb{N}_0} \\
&= \operatorname{cx}\{H^*(\bigoplus_{m\in\mathbb{N}_0} P^m, n)\}_{n\in\mathbb{N}_0},
\end{aligned}$$

and we can repeat the arguments we gave just now almost verbatim to show that

$$\operatorname{cx}\{H^*(\bigoplus_{m\in\mathbb{N}_0} P^m, n)\}_{n\in\mathbb{N}_0} = d + 1 = \operatorname{cx}\Lambda + 1,$$

we are in fact done. □

4.6 Periodicity of modules and trivial extensions

In this section we define what it means for a module to be periodic, and briefly investigate what passing to the trivial extension entails for the periodic modules of a selfinjective algebra Λ. Moreover, we derive some additional results related to periodicity following from those of the previous section. We note that our standing assumptions still hold.

We begin by giving the appropriate definition for periodicity in the selfinjective setting.

Definition 4.6.1. Let M be a Λ-module. We say that M is Ω_Λ**-periodic** if there is some $m \geq 1$ such that $\Omega_\Lambda^m(M) \cong M$.

As long as context prevents confusion from arising, we will henceforth simply be calling such modules periodic, or periodic over some given algebra.

The following result shows that the periodic modules of Λ embed under $T(\Lambda)\otimes_\Lambda -$ as periodic modules of $T(\Lambda)$.

Proposition 4.6.2. *Let M be a Λ-module, and let $P^* \to M$ be a minimal projective resolution of M over Λ. Then $T(\Lambda)\otimes_\Lambda P^* \to T(\Lambda)\otimes_\Lambda M$ is a minimal projective resolution of $T(\Lambda)\otimes_\Lambda M$ over $T(\Lambda)$.*

Proof. To begin with, note that since Λ is selfinjective, $D(\Lambda)$ is Λ-projective, and hence also $T(\Lambda)$ is Λ-projective, thus entailing that $T(\Lambda)\otimes_\Lambda -$ is exact. It follows that $T(\Lambda)\otimes_\Lambda P^* \to T(\Lambda)\otimes_\Lambda M$ is a projective resolution. To see that it is minimal, note that it is composed of short exact sequence of the form

$$0 \longrightarrow \Omega_\Lambda^{n+1}(M) \oplus \nu\Omega_\Lambda^{n+1}(M) \xrightarrow{T(\Lambda)\otimes_\Lambda i^n} P^n \oplus \nu P^n \longrightarrow \Omega_\Lambda^n(M) \oplus \nu\Omega_\Lambda^n(M) \longrightarrow 0,$$

where as before we are interpreting $\nu-$ as $D(\Lambda)\otimes_\Lambda -$. Now, note that

$$\operatorname{rad}_{T(\Lambda)}(P^n \oplus \nu P^n) = \operatorname{rad}_\Lambda P^n \oplus \nu P^n$$

holds by (4.1.4) and a quick computation. Since $\nu-$ preserves minimal projective resolutions, and since $T(\Lambda)\otimes_\Lambda i^n$ can be seen to simply be an inclusion, the desired result follows. □

Corollary 4.6.3. *If M is a periodic Λ-module, $T(\Lambda) \otimes_\Lambda M$ is a periodic $T(\Lambda)$-module.*

Proof. This follows immediately from the preceding proposition. □

Hence, the trivial extension $T(\Lambda)$ has at least as many periodic modules as Λ itself. Moreover, certain Λ-modules become periodic when considered as $T(\Lambda)$-modules, as we show in the next result:

Proposition 4.6.4. *Let P be a projective Λ-module. Then P is a periodic $T(\Lambda)$-module.*

Proof. This follows from an earlier observation, namely that the first $T(\Lambda)$-syzygy of a Λ-projective P is νP, which itself is a projective Λ-module. Since $\nu-$ for selfinjective algebras is an autoequivalence of $\mathcal{P}(\Lambda)$, it follows that the $\nu-$ order of P is finite, i.e. that there is some i such that $\nu^i P \cong P$. Of course, this entails that its $T(\Lambda)$-minimal projective resolution is periodic. □

Finally, we note that by the results of the previous section, it is clear that none of the simple modules of the trivial extension of a non-semisimple selfinjective finite dimensional algebra are periodic.

4.7 An open question

In the case of selfinjective algebras Λ, we have seen that there are restrictions on the complexity of their trivial extensions $T(\Lambda)$. Moreover, as we have stated several times before, by Purin's results in [29], we know that there are also strong restrictions on the complexities of the trivial extensions of hereditary algebras and also iterated tilted algebras. On the one hand, one may then ask whether one can say something for either more general classes of algebras, or, alternatively, other disjoint classes of algebras; on the other hand, as we also have remarked before, we are interested in when the trivial extensions of algebras Λ may possibly satisfy the finite generation hypotheses necessary for the theory of support varieties presented in [32] and [17]. As regards the latter, the scope of this text prohibits even an elementary discussion, and we thus content ourselves with discussing only the former. Finally, we make note of the fact that in this section we allow ourselves to be a bit more sloppy than we have been elsewhere in this text: we assume the reader is familiar with certain common terminology, and only sketch certain arguments.

With this in mind, we note with respect to the first of these questions that it may seem as if there is a connection between whether or not a non-selfinjective algebra Λ satisfies the finite generation hypotheses (**Fg1**) and (**Fg2**) and whether its trivial extension $T(\Lambda)$ has infinite complexity: First of all, the example we gave in Section 4.2 is in fact indicative of a more general pattern. That is to say, basic and connected radical square zero algebras that are not an oriented cycle, which is to say they are not Nakayama algebras, seem to have infinite complexity if their ordinary quivers contain an oriented cycle. In working with this thesis, using among other things the strategy in Section 4.2, this has been verifies for tens of examples, albeit all with less than ten simple modules.

Now, by Cibil's work in [14], we know that a radical square zero algebra that is not a Nakayama algebra satisfies the finite generation conditions if and only if its quiver does not contain an oriented cycle. In particular, [14, Corollary 3.2] states that if a quiver Q is not itself an oriented cycle, then, if I is the arrow ideal of kQ, kQ/I^2 has finite dimensional Hochschild cohomology if and only if Q contains no oriented cycle. Moreover, [14, Lemma 4.1] states that if a Q is not an oriented cycle, then the product of positive degree elements of the Hochschild cohomology of kQ/I^2 is zero. Hence, we see that it is clear that if Q properly contains an oriented cycle, $\mathrm{HH}^*(kQ/I^2)$ cannot be a finitely generated algebra over the zeroth degree part of that algebra, and thus, by recalling [4, Proposition 10.7], we deduce that $\mathrm{HH}^*(kQ/I^2)$ cannot be Noetherian. Yet, it is known by [34, Proposition 5.7] that $\mathrm{HH}^*(\Lambda)$ must be Noetherian if Λ is to satisfy the finite generation conditions, thus entailing that one of the claimed implications hold. The other follows by observing that if Q contains no oriented cycle, $\Lambda = kQ/I^2$ must have finite global dimension and so $\mathrm{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, \Lambda/\operatorname{rad}\Lambda)$ is finite dimensional and thus also finitely generated over any ring it is a module over. Thus, since by [34, Proposition 5.7] it is sufficient that $\mathrm{HH}^*(\Lambda)$ be Noetherian and to finitely generate $\mathrm{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, \Lambda/\operatorname{rad}\Lambda)$ for Λ to satisfy the finite generation hypotheses, we can see that the other implication must hold.

In other settings one does not have Benson's result in [8] to rely on, and so, as previously remarked, it is difficult to determine what the complexity of a given module is, and thus also what the complexity of an algebra is, at least in situations of interest, where the complexity is greater than zero. However, employing the useful Quivers and Path Algebras package for GAP [35], we have found some indications that this pattern also occurs in more general situations.

We note, though, that there is an important caveat: While we have both used the program's function to estimate complexities and its function to compute minimal projective resolutions, our evidence consists of observing that the former either returns high estimates or fails to finish computing (possibly in a reasonable amount of time) for far, far lower values of $n \in \mathbb{N}_0$ than it usually does, whereas the latter indicates that the relevant resolutions grow exceedingly quickly. Of course, while the dimensions of the terms in the minimal projective resolutions seem to grow exponentially fast, it is not inconceivable that they may halt or slow in their growth eventually, and hence all of this is not inconsistent with the complexities simply being very large or otherwise difficult to compute.

Having stated this, we enumerate some cases we have investigated with QPA and where we know whether or not an algebra satisfies (**Fg1**) and (**Fg2**): By the results of Nagase in [26], we know that a Nakayama algebra satisfies the finite generation conditions if and only if it is Gorenstein. Yet, in this case, going by our investigations in QPA of these algebras (over $\mathbb{Q}$), we have that whether or not a Nakayama algebra is Gorenstein seemingly correlates with whether or not that algebra's trivial extension has finite or infinite complexity.

Moreover, we have investigated the case of truncated quiver algebras. That is to say, for a path algebra kQ of a quiver Q, if I is its radical, i.e. its arrow ideal, then a truncated quiver algebra over Q is a bound quiver algebra of the form kQ/I^n for $n \geq 2$. Of course, for $n = 2$ we recognize these as being radical square zero algebras. Now, using the main results of [36] and [1], one can mimic the argument we gave above for radical square zero algebras to show that truncated quiver algebras kQ/I^n for $n \geq 3$ behave in a somewhat similar fashion with respect to the finite generation criteria: to be more precise, while the main result of [36] states that $\mathrm{HH}^*(kQ/I^n)$ for $n \geq 2$ is finite dimensional if and only if Q contains no oriented cycle, the central results in [1] state that the product in $\bigoplus_{i\geq 1} \mathrm{HH}^i(kQ/I^n)$ is zero in the case when either Q contains no oriented cycle [1, Theorem 8.1], or when it contains no sinks or sources but is not an oriented cycle [1, Theorem 8.6]. Yet again, it seems as if our computations in QPA suggest that in the case where the algebras do not satisfy the relevant hypotheses, one finds that their trivial extensions have infinite complexity.

Finally, we mention that in [27] and [28], Obara has investigated the Hochschild cohomology rings of certain quiver algebras defined by two oriented cycles meeting in a point and being bound by a "quantum-like" relation. Essentially, these can be viewed as generalizations of the much studied quantum complete intersections. However, according to Obara, unlike quantum complete intersection, these are not in general selfinjective. Furthermore, Obara has shown in [28, Theorem 4.3] that these algebras satisfy the finite generation criteria if and only if the parameter in the quantum-like relation is a root of unity. See [19, Proposition 9.1] for the corresponding result for a restricted case of quantum complete intersections. In this case we have made only limited computations in the most simple cases, as made necessary by the size of these algebras, given that they invariably are of double digit or greater dimension. Nevertheless we have found that the pattern continues to hold.

Given all of this, we thus ask: if an algebra Λ does not satisfy (**Fg1**) and (**Fg2**), must its trivial extension $T(\Lambda)$ have infinite complexity? A stronger formulation still consistent with the cases we have discussed would be the following:

if an algebra Λ is not Gorenstein, must its trivial extension $T(\Lambda)$ have infinite complexity?

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