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Nuclear Equation of State and Neutron Stars

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Preface

This master's thesis has been written as an integral part of the five-year Master's degree programme Applied Physics and Mathematics at the Norwegian University of Science and Technology (NTNU). It was written at the Department of Physics during the fall semester of 2016. It is a continuation of the specialization project [32] written during the spring semester of 2016.

Presumption on level of knowledge

The reader of this master's thesis is presumed to have knowledge of university level physics, including general relativity and quantum field theory.

Citing of references

References are cited in the following way: When used extensively throughout a section or chapter, the reference is cited at the begin of the respective section or chapter. When a particular result or information is used, the reference is cited in the main text.

Acknowledgements

I would like to thank my supervisor Professor Jens Oluf Andersen for the opportunity to work on such an interesting project and for his guidance and advice on both the specialization project and master's thesis. I would also like to thank my family and friends for their support during this time.

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Abstract

In this thesis, we study the mass-radius relation of neutron stars. This relation is computed using stellar structure equations and an equation of state. We use the Tolman-Oppenheimer-Volkoff (TOV) equation and the mass continuity equation as stellar structure equations. Using quantum field theory, we find the equation of state of an ideal gas of fermions. Interactions between the fermions are included using the σ-ω model, where we introduce mesons in a mean-field approximation. We also look at how the model can be extended by including scalar self-interactions and isospin force. Leptons are also included in order to impose electrical neutrality. We solve the equation of state for the σ-ω model numerically, and use it to solve the TOV and mass continuity equations for a given central pressure. Solving the equations for a range of central pressures gives us the mass-radius relation of neutron stars. It has a maximum mass $M_{\rm max} = 2.595 M_\odot$ for a radius $R_{\rm max} = 12.64$ km. This means that there is an upper bound on the mass of a neutron star, and beyond this limit, the neutron star is unstable and will continue to collapse into a black hole.

Sammendrag

I denne masteroppgaven ser vi på masse-radius-sammenhengen til nøytronstjerner. Masse-radius-sammenhengen finner vi ved hjelp av ligninger for stjernestrukturen og en tilstandsligning. Tolman-Oppenheimer-Volkoff (TOV)-ligningen og kontinuitetsligningen for masse brukes som ligninger for stjernestrukturen. Vi bruker kvantefeltteori til å utlede en tilstandsligning for en ideell gass av fermioner. For å inkludere vekselvirkninger mellom fermionene bruker vi σ-ω-modellen, hvor vi innfører mesoner i en middelfelttilnærming. Vi ser også på hvordan modellen kan bli utvidet ved å inkludere skalare selvvekselvirkninger og isospinnkraft. For å kunne kreve elektrisk ladningsnøytralitet innfører vi også leptoner. Vi finner tilstandsligningen for σ-ω-modellen numerisk og bruker den til å løse TOV-ligningen og kontinuitetsligningen for masse for et gitt sentraltrykk. Ved å løse ligningene for et spekter av sentraltrykk, finner vi sammenhengen mellom massen og radien til nøytronstjerner. Kurven har en maksimumsmasse $M_{\max} = 2.595 M_{\odot}$ for en radius på $R_{\max} = 12.64$ km. Dette betyr at det finnes en øvre grense på hvor stor masse en nøytronstjerne kan ha. Over denne grensen er nøytronstjernen ustabil og vil kollapse videre til et sort hull.

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Abbreviations, Notations and Conventions

- **Abbreviations** The following abbreviations are used

TOV equation	=	Tolman-Oppenheimer-Volkoff equation
EoS	=	Equation of state

- **Metric** For Minkowski space, the metric is $g_{\mu\nu} = \text{diag}\,(-1, 1, 1, 1)$.

- **Vectors** Spacial 3-vectors are written as $\mathbf{x} = (x_1, x_2, x_3)$. Spacetime 4-vectors are written as $x = x_\mu = (x_0, x_1, x_2, x_3) = (x_0, x_i) = (x_0, \mathbf{x})$.

- **Units** Natural units are used, except when working with the TOV equation, for which SI units are used.

- **Constants in natural units** We use the following constants in natural units

Reduced Planck's constant $\hbar$	=	1
Speed of light c	=	1
Neutron mass m	=	$939\ \text{MeV}$
Sigma meson mass m_σ	=	$550\ \text{MeV}$
Omega meson mass m_ω	=	$783\ \text{MeV}$

- **Constants in SI units** we use the following constants in SI units

Reduced Planck's constant $\hbar$	=	$1.054\,571\,8 \times 10^{-34}\ \text{m}^2\,\text{kg}\,\text{s}^{-1}$
Speed of light c	=	$299\,792\,458\ \text{m}\,\text{s}^{-1}$
Gravitational constant G	=	$6.674\,08 \times 10^{-20}\ \text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}$
Neutron mass m	=	$1.674\,927\,471 \times 10^{-27}\ \text{kg}$
Solar mass $M_\odot$	=	$1.9891 \times 10^{30}\ \text{kg}$

Chapter 1

Introduction

A neutron star is a high-density stellar object. It is the last stage of the evolution of a high-mass star and the remnant core that is left after the star has gone through a supernova explosion. Neutron stars are small, but have large masses. Typical values are a mass of one solar mass $M_\odot$ and radius of 10 km, corresponding to a density of order 10^{17} kgm^{-3}. This density is comparable to the density found in an atomic nucleus. Neutron stars are the second densest objects in the universe, their densities only surpassed by black holes. The extreme densities of neutron stars make them ideal objects for the study of extreme condition physics, such as general relativity and the interplay with nuclear and particle physics.

When studying neutron stars, we are interested in the relation between the mass and the radius of the star. To find this relation, we need equations for the stellar structure and an equation of state for nuclear matter. Assuming that the stellar object is static, spherically symmetric and consists of a perfect fluid, we can derive the Tolman-Oppenheimer-Volkoff (TOV) equation. This derivation was performed in the specialization project [32]. The TOV equation together with the mass continuity equation describe a relativistic stellar object and will be used as stellar structure equations. In the specialization project, the equation of state was found using a fully-degenerate, ideal Fermi gas. Solving the TOV equation with this equation of state, we found that there is an upper bound on the mass of about $M_{\max} = 0.77 M_\odot$. This upper bound on the mass is lower than several observed neutron star masses, which suggests that the model is too simplistic.

In the following master's thesis, we try to improve the model by changing the equation of state, while still using the TOV and mass continuity equations as stellar structure equations. The equation of state will be found using quantum field theory. We will treat the neutrons as Dirac fermions and find expressions for the pressure and energy density using a path integral formalism. Next, we will incorporate nuclear interactions in a mean-field approximation and find a new equation of state for this model, which we will use to find the mass-radius relationship of a neutron star.

Chapter 2

Thermodynamics and Statistical Mechanics

This chapter is based on references [1] and [20].

Thermodynamics describe the bulk properties of a system in equilibrium. Bulk properties are intensive and do not depend on the size of the system. Temperature, pressure and density are bulk properties of interest when working with neutron stars. The relation between the pressure and the density is called the equation of state (EoS) and is necessary to solve the equation for the stellar structure of the star.

2.1 Statistical ensembles

In thermodynamics, we want to describe systems consisting of very many particles. For these kind of systems, it is impossible to find an exact description using classical or quantum mechanics. Instead, we use statistical mechanics, where we look at the statistical behaviour of a large collection of identical systems, known as an ensemble. We are interested in the macroscopic properties of the ensemble, which are described by macroscopic variables such as temperature, entropy, pressure and energy. These variables are statistical averages over the ensemble.

In statistical mechanics, we are concerned with three different kinds of ensembles: the microcanonical ensemble, the canonical ensemble, and the grand canonical ensemble. Which of these ensembles one should use, depends on the system and its restrictions. The ensembles are also known by the names of which thermodynamical quantities that are conserved in the system.

The microcanonical ensemble describes an isolated system, with fixed energy E and a fixed number of particles N. The volume V of the system is also fixed. The microcanonical ensemble is therefore also known as the NVE ensemble.

The canonical ensemble describes a system in contact with a heat reservoir at temperature T. The system can exchange energy with its surroundings, so the energy is no longer conserved. The number of particles and the volume is still fixed. The canonical ensemble is also called the NVT ensemble. The Lagrange multiplier of the system is $\beta = T^{-1}$. Lagrange multipliers are used to fix the averages of quantities, and will for the canonical ensemble fix the mean energy.

Lastly, the grand canonical ensemble describes a system in contact with a heat reservoir at temperature T and a particle reservoir. The system can freely exchange energy and particles with the reservoirs, so neither the energy nor the number of particles are fixed. The energy E and number of particles N will therefore fluctuate when the system is in thermal equilibrium. The system has three fixed thermodynamical quantities: the temperature T, the volume V and the chemical potential μ. The grand canonical ensemble is therefore often called a μVT ensemble. The quantities $\beta = T^{-1}$ and μ are the Lagrange multipliers of the system: β fixes the mean energy and μ, assuming it is the chemical potential for the particles, fixes mean number of particles.

For the case of a neutron star, the system is best described by the grand canonical ensemble. In stars, particles are frequently created and destroyed, meaning we must allow for fluctuations in the number of particles. As stars emit light, it should also be possible for energy exchange with the surroundings. The volume and temperature of a star is often assumed to be constant, although these are simplifications to reality.

2.2 The partition function and thermodynamic observables

The grand canonical partition function Z, hereafter referred to as the partition function, is used to describe the grand canonical ensemble. It describes a system in thermodynamic equilibrium and it is the most fundamental quantity of the system, as every thermodynamical quantity can be derived from it. For a system with Hamiltonian $\hat{H}$ and a set of conserved quantities N_i, represented by the conserved number operators $\hat{N}_i$, the partition function is given by

$$Z = Z(V, T, \mu_i) = \mathrm{Tr}[e^{-\beta(\hat{H} - \mu_i \hat{N}_i)}] = \mathrm{Tr}\,\hat{\rho}, \tag{2.1}$$

where $\hat{\rho} = \exp\left[-\beta(\hat{H} - \mu_i \hat{N}_i)\right]$ is the statistical density matrix and μ_i is the chemical potential of the conserved quantity N_i. The operators $\hat{N}_i$ must commute with themselves and $\hat{H}$. The short-hand notation $\hat{H}' = \hat{H} - \mu_i \hat{N}_i$ is often used. The average of any observable represented by an operator $\hat{A}$, can be found from

$$A = \langle \hat{A} \rangle = \frac{\mathrm{Tr}\,\hat{A}\hat{\rho}}{\mathrm{Tr}\,\hat{\rho}}. \tag{2.2}$$

Using the partition function, the following expressions for some thermodynamical quantities are found; the pressure P is given by

$$P = \frac{\partial (T \ln Z)}{\partial V}, \tag{2.3}$$

the number of particles N_i by

$$N_i = \frac{\partial(T \ln Z)}{\partial \mu_i}, \tag{2.4}$$

the entropy S by

$$S = \frac{\partial(T \ln Z)}{\partial T}, \tag{2.5}$$

and the energy E by

$$E = -PV + TS + \mu_i N_i. \tag{2.6}$$

The pressure is also related to Helmholtz free energy F, which is given by

$$F = -T \ln Z, \tag{2.7}$$

such that

$$P = -\frac{\partial F}{\partial V}. \tag{2.8}$$

We will use these expressions to find the equation of state.

Chapter 3

Classical Mechanics, Quantum Mechanics and Quantum Field Theory

Before using quantum field theory to derive a new equation of state, we will look at some connections between classical mechanics, quantum mechanics, and quantum field theory. In particular, we will see how the quantities Lagrangian, Hamiltonian, and action are defined in classical mechanics, how a path-integral formalism is defined in quantum mechanics, and how these are extended to quantum field theory. In the last part of the chapter, we will look at a useful result called Noether's theorem and how we can work in imaginary time when using the path integral formalism in quantum field theory.

3.1 Lagrangian, Hamiltonian, and action in classical mechanics

This section is based on references [4] and [14].

In classical mechanics, the Lagrangian L is usually given as the difference between the kinetic energy T and the potential energy V

$$L = T - V, \tag{3.1}$$

and is a function of the generalized coordinates q, their derivatives $\dot{q}$, and the time t, such that $L = L(q, \dot{q}, t)$. We have suppressed the indices, such that $q_i = q$. The Lagrangian satisfies the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0, \tag{3.2}$$

which gives the equations of motion of the system. Sometimes, it is useful to go from the Lagrangian formulation, which depends on $(q, \dot{q}, t)$, to the Hamiltonian formulation, which depends on (q, p, t). Here, p is the conjugate momentum, given by

$$p = \frac{\partial L}{\partial \dot{q}}. \tag{3.3}$$

Changing variables in this way is called a Legendre transformation. For a system with Lagrangian L, the corresponding Hamiltonian H can be found through the Legendre transformation

$$H = H(q, p, t) = p\dot{q} - L(q, \dot{q}, t). \tag{3.4}$$

The differential of the Lagrangian is

$$dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \dot{q}} d\dot{q} + \frac{\partial L}{\partial t} dt = \dot{p} dq + p d\dot{q} + \frac{\partial L}{\partial t} dt, \tag{3.5}$$

where we have used (3.2) and (3.3). We can write the differential of the Hamiltonian in two ways: from the definition of a differential or by using (3.4). We then find that

$$dH = \frac{\partial H}{dq} dq + \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial t} dt = \dot{q} dp - \dot{p} dq - \frac{\partial L}{\partial t} dt. \tag{3.6}$$

Comparing the expressions of (3.6), we find the Hamiltonian equations

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}, \tag{3.7}$$

which, as the Euler-Lagrange equations, also give the equations of motion of the system. For a system with n degrees of freedom, the Euler-Lagrange equations give n second-order differential equations and determines the state of the system as a point in the n-dimensional configuration space, while the Hamiltonian equations give $2n$ first-order differential equations and determines the state of the system as a point in the $2n$-dimensional phase space.

For a simple system, such as a particle of mass m in a velocity-independent potential, the Hamiltonian is equal to the total energy E and can be written as

$$H = T + V = \frac{p^2}{2m} + V(q). \tag{3.8}$$

The corresponding Lagrangian of this system is

$$L = T - V = \frac{1}{2} m\dot{q}^2 - V(q), \tag{3.9}$$

which we see from the Legendre transformation (3.4).

In classical mechanics, the action S is defined as the time integral of the Lagrangian

$$S[q] = \int_a^b dt\, L(q, \dot{q}, t), \tag{3.10}$$

where a and b are two points in time. The action S is a functional of q as it maps q into real numbers. Using the variational principle on the action gives

$$\begin{aligned} 0 = \delta S[q] &= \int_a^b dt\, \delta L(q,\dot{q},t) = \int_a^b dt\, \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right) \\ &= \int_a^b dt \frac{\partial L}{dq}\delta q + \left[\frac{\partial L}{\partial \dot{q}}\delta q \right]_a^b - \int_a^b dt \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\delta q \\ &= \int_a^b dt\delta q \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right), \end{aligned} \tag{3.11}$$

where we have used that $\delta \dot{q} = \frac{d}{dt}\delta q$ and that the term $\frac{\partial L}{\partial \dot{q}}\delta q$ vanishes at the boundaries a and b. Since δq is arbitrary, the integrand must vanish, which leaves us with the the Euler-Lagrange equations (3.2).

3.2 Lagrangian, Hamiltonian and action in quantum mechanics and quantum field theory

This section is based on references [15] and [19].

We will now look at the how the quantities Lagrangian, Hamiltonian, and action from classical mechanics are extended to quantum mechanics and quantum field theory. In classical mechanics, we work both in the Lagrangian and Hamiltonian formalisms, depending on which is best suited for our problem. Problems in classical mechanics are usually concerned with finding the position as a function of time, $q(t)$. Quantum mechanics is most easily introduced with the Hamiltonian formalism and is usually concerned with finding the wave functions of the system. We promote the position and momentum variables to operators $\hat{q}$ and $\hat{p}$, which satisfy the canonical commutation relation $[\hat{q}, \hat{p}] = i$. The Hamiltonian also becomes an operator, $\hat{H}(\hat{q}, \hat{p}, t)$.

In Dirac notation, we represent the system with state vectors $|\Psi\rangle$, which form a complex Hilbert space. The state vectors can be used to find the wave function of a state. In the position basis, the wave function of a particle is $\Psi(q,t) = \langle q|\Psi(t)\rangle$, and its square $|\Psi(q,t)|^2$ gives the probability density of finding the particle in position q at time t. The operators act on the state vectors and return the eigenvalues if the state vectors are eigenvectors of the operators. The Hamiltonian is an energy operator, meaning that when acting on the state $|\Psi(t)\rangle$, its eigenvalue corresponds to the energy of the system, such that

$$\hat{H}\psi = E\psi. \tag{3.12}$$

The state vectors $|\Psi\rangle$ of the system satisfy the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle. \tag{3.13}$$

Various bases, such as the position basis $|q\rangle$ or the momentum basis $|p\rangle$, can be chosen for the state vectors. The Schrödinger equation can be solved in terms of a unitary time evolution operator $\hat{U}(t, t_0)$, defined by

$$|\Psi(t)\rangle = \hat{U}(t, t_0)\, |\Psi(t_0)\rangle\,. \tag{3.14}$$

Assuming the Hamiltonian $\hat{H}$ is time-independent, the time evolution operator becomes

$$\hat{U}(t, t_0) = e^{-i\hat{H}(t-t_0)}. \tag{3.15}$$

The time evolution operator also helps us to go from the Schrödinger picture, where the operators are constants and the time evolution is given by the change in the eigenvectors $|\Psi(t)\rangle$, to the Heisenberg picture, where the operators are time-dependent while the eigenvectors are constants in time. The operators $\hat{O}_S$ and $\hat{O}_H$, in the Schrödinger and the Heisenberg pictures respectively, are related through

$$\hat{O}_S = \hat{U}(t, t_0) O_H(t) \hat{U}^\dagger(t, t_0). \tag{3.16}$$

The connection between the states $|\Psi_S(t)\rangle$ in the Schrödinger picture and the states $|\Psi_H(t)\rangle$ in the Heisenberg picture is

$$|\Psi_S(t)\rangle = \hat{U}(t, t_0)\, |\Psi_H(t)\rangle\,. \tag{3.17}$$

When going to quantum field theory, we describe particles in terms of fields ϕ, instead of their positions and momenta. We can view quantum mechanics as $(1+0)$-dimensional quantum field theory, meaning that the position operator $\hat{q}$ at some point corresponds to the field operator $\hat{\phi}$. In the Heisenberg picture, which is most common to use in quantum field theory, we have

$$\hat{q}_H(t) \leftrightarrow \hat{\phi}_H(t, \mathbf{0}). \tag{3.18}$$

This means that a particle in quantum mechanics described by the mapping $t \to q(t)$, corresponds to a field in $(1+0)$-dimensions, described by the mapping $t \to \phi(t)$.

The Lagrangian in quantum field theory is a function of the fields ϕ, their time derivatives $\dot{\phi}$ and the time t, such that $L = L(\phi, \dot{\phi}, t)$. We often write the Lagrangian and the Hamiltonian as space integrals over their respective densities, that is

$$L(\phi, \dot{\phi}, t) = \int d^3x\, \mathcal{L}(\phi, \dot{\phi}, t), \tag{3.19}$$

$$H(\phi, \dot{\phi}, t) = \int d^3x\, \mathcal{H}(\phi, \dot{\phi}, t), \tag{3.20}$$

where $\mathcal{L}$ and $\mathcal{H}$ are the Lagrangian and the Hamiltonian density, respectively. The change from position to field means that both the action (3.10) and the Euler-Lagrange equations (3.2) can be written in terms of the fields. For the action, we find that

$$S[\phi] = \int_a^b dt\, L(\phi, \dot{\phi}, t) = \int_a^b dt \int d^3x\, \mathcal{L}(\phi, \dot{\phi}, t). \tag{3.21}$$

The Euler-Langrange in quantum field theory is usually written in terms of the Lagrangian density $\mathcal{L}$, such that

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \tag{3.22}$$

Similiarly to equation (3.4), we can also find the Hamiltonian density in terms of the fields from a Legendre transformation

$$\mathcal{H} = \pi \frac{\partial \mathcal{L}}{\partial t} - \mathcal{L}, \tag{3.23}$$

where

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \tag{3.24}$$

is the conjugate momentum to the field ϕ.

3.3 Path integral formalism in quantum mechanics and quantum field theory

This section is based on references [19], [23], and [28].

We want to work with quantum field theory in the path integral formalism. We can develop the formalism in quantum mechanics and extend it to quantum field theory by using the relation (3.18). Let us start by looking at the transition amplitude $F(q_f, t_f; q_i, t_i)$, that is the probability for a particle at position q_i at time t_i to be at position q_f at time t_f. Using the position basis $|q\rangle$ and the expression for the time evolution operator $\hat{U}(t_f, t_i)$ from equation (3.15), we find that the transition amplitude $F(q_f, t_f; q_i, t_i)$ can be written as

$$F(q_f, t_f; q_i, t_i) = \langle q_f| \, e^{-i\hat{H}(t_f - t_i)} \, |q_i\rangle \, . \tag{3.25}$$

We let $t = t_f - t_i$ and divide t into N infinitesimal intervals of size δt, so that

$$F(q_f, t_f; q_i, t_i) = \langle q_f| \, e^{-i\hat{H}\delta t} \cdots e^{-i\hat{H}\delta t} \, |q_i\rangle \, . \tag{3.26}$$

Next, we insert a complete set of position eigenstates $\int_{-\infty}^{\infty} dq_j \, |q_j\rangle \langle q_j| = 1$ between each exponential in (3.26), such that

$$F(q_f, t_f; q_i, t_i) = \int \prod_{j=1}^{N} dq_j \, \langle q_{j+1}| \, e^{-i\hat{H}\delta t} \, |q_j\rangle \, , \tag{3.27}$$

where the right-most and left-most states are $|q_1\rangle = |q_i\rangle$ and $\langle q_{N+1}| = \langle q_f|$, and we have suppressed the boundaries of the integrals. The exponentials of the transition amplitudes in (3.27) can be expanded since δt is small, giving

$$\begin{aligned} \langle q_{j+1}| \, e^{-i\hat{H}\delta t} \, |q_j\rangle &= \langle q_{j+1}| \left[1 - i\hat{H}\delta t - \frac{1}{2}\hat{H}^2 \delta t^2 + ... \right] |q_j\rangle \\ &= \langle q_{j+1}|q_j\rangle - i\delta t \, \langle q_{j+1}| \, \hat{H} \, |q_j\rangle + \mathcal{O}(\delta t^2). \end{aligned} \tag{3.28}$$

To evaluate the second term of (3.28), we assume that we have a Hamiltonian of the form $\hat{H} = \hat{p}^2/2m + V(\hat{q})$, as in (3.8). We insert a complete momentum basis $\int_{-\infty}^{\infty} \frac{dp_j}{2\pi} |p_j\rangle \langle p_j| = 1$ in front of the ket state and use that $\langle q_j|p_j\rangle = e^{ip_j q_j}$, such that

$$\begin{aligned} -i\delta t \langle q_{j+1}| \hat{H} |q_j\rangle &= -i\delta t \langle q_{j+1}| \left[\frac{\hat{p}^2}{2m} + V(\hat{q})\right] \int \frac{dp_j}{2\pi} |p_j\rangle \langle p_j|q_j\rangle \\ &= -i\delta t \int \frac{dp_j}{2\pi} \left[\frac{p_j^2}{2m} + V(q_{j+1})\right] e^{ip_j(q_{j+1}-q_j)} \\ &= -i\delta t \int \frac{dp_j}{2\pi} \left[\frac{p_j^2}{2m} + V(q_j)\right] e^{ip_j(q_{j+1}-q_j)}, \end{aligned} \tag{3.29}$$

where we used the approximation $V(q_{j+1}) \approx V(q_j)$ in the last step. This approximation is valid in the continuum limit $N \to \infty$. We can also rewrite the first term in (3.28) using the momentum basis

$$\langle q_{j+1}|q_j\rangle = \int \frac{p_j}{2\pi} e^{ip_j(q_{j+1}-q_j)}. \tag{3.30}$$

The transition amplitude between q_{j+1} and q_j therefore becomes

$$\begin{aligned} \langle q_{j+1}| e^{-i\hat{H}\delta t} |q_j\rangle &= \int \frac{p_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} \left[1 - i\delta t \hat{H}(q_j, p_j) + \mathcal{O}(\delta t^2)\right] \\ &= \int \frac{p_j}{2\pi} e^{ip_j(q_{j+1}-q_j)} e^{-i\delta t \hat{H}(q_j, p_j)} \\ &= \int \frac{p_j}{2\pi} e^{i\delta t\left[p_j \dot{q}_j - \hat{H}(q_j, p_j)\right]}, \end{aligned} \tag{3.31}$$

where we have dropped terms of order δt^2 and higher, and used that $(q_{j+1} - q_j)/\delta t \approx \dot{q}_j$, which is also valid in the continuum limit. Inserting (3.31) into (3.27), we find that the transition amplitude is

$$F(q_f, t_f; q_i, t_i) = \int \prod_{j=1}^{N} dq_j \frac{dp_j}{2\pi} \exp\left\{ i\delta t \sum_{j=1}^{N} \left[p_j \dot{q}_j - \hat{H}(q_j, p_j)\right] \right\}. \tag{3.32}$$

In the limit $N \to \infty$, $\delta t \to 0$, the exponential of (3.32) becomes an integral over time of the time-dependent functions for position $q(t)$ and momentum $p(t)$, and we denote the integral measure as $\mathcal{D}q\mathcal{D}p$. The transition amplitude in the continuum limit is therefore

$$F(q_f, t_f; q_i, t_i) = \int \mathcal{D}q\mathcal{D}p \exp\left\{ i \int_{t_i}^{t_f} dt \left[p(t)\dot{q}(t) - \hat{H}(q(t), p(t))\right] \right\}, \tag{3.33}$$

with the boundary conditions $q(t_i) = q_i$ and $q(t_f) = q_f$. Comparing the term in the square brackets of (3.33) with equations (3.4) and (3.10), we see that it is equal to the Lagrangian L of the system and the integral itself is the action S

$$F(q_f, t_f; q_i, t_i) = \int \mathcal{D}q\mathcal{D}p \exp\left[i \int_{t_i}^{t_f} dt\, L(q, \dot{q}, t) \right] = \int \mathcal{D}q\mathcal{D}p\, e^{iS[q]}. \tag{3.34}$$

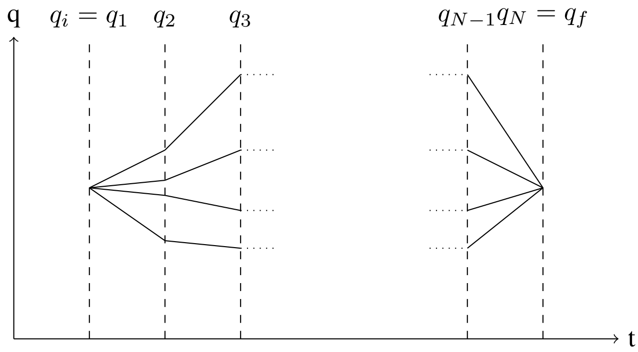

Figure 3.1: The concept of a path integral. We find the transition amplitude by integration over all possible paths between the initial point q_i and the final point q_f. The figure is inspired by Figure 2.1 in [19].

This type of integral is called a path integral, and the above result (3.34) is known as the phase-space path integral. The name path integral comes from the fact that we integrate over all possible paths from the initial point q_i to the final point q_f. A discretized version of the concept is shown in Figure 3.1.

When assuming a Hamiltonian of the form $\hat{H} = \hat{p}^2/2m + V(\hat{q})$, we can perform the momentum integrals in the the transition amplitude (3.32). Then the transition amplitude can be split into separate integrals over position and momentum, and the momentum integrals are Gaussian, which gives

$$F(q_f, t_f; q_i, t_i) = \int \prod_{j=1}^{N} dq_j \exp\left\{-i\delta t \sum_{j=1}^{N} V(q_j)\right\} \times \int \prod_{j=1}^{N} \frac{dp_j}{2\pi} \exp\left\{i\delta t \sum_{j=1}^{N}\left[p_j\dot{q}_j - \frac{p_j^2}{2m}\right]\right\}. \tag{3.35}$$

Using the formula for Gaussian integrals of this form from [19], we can calculate a momentum integral

$$\int \frac{dp_j}{2\pi} \exp\left[i\delta t\left(p_j\dot{q}_j - \frac{p_j^2}{2m}\right)\right] = \sqrt{\frac{m}{2\pi i\delta t}} \exp\left(\frac{i\delta t m\dot{q}_j^2}{2}\right). \tag{3.36}$$

We use this result in the transition amplitude (3.35) and find

$$F(q_f, t_f; q_i, t_i) = \left(\frac{m}{2\pi i\delta t}\right)^{N/2} \int \prod_{j=1}^{N} dq_j \exp\left\{i\delta t \sum_{j=1}^{N}\left[\frac{1}{2}m\dot{q_j}^2 - V(q_j)\right]\right\}. \tag{3.37}$$

The factor $\left(\frac{m}{2\pi i\delta t}\right)^{N/2}$ may be absorbed in $\mathcal{D}q$ as it does not depend on q. We recognize the term in the square brackets as the Lagrangian of the form given by (3.9). We can therefore write the transition amplitude as

$$F(q_f, t_f; q_i, t_i) = \int \mathcal{D}q \exp\left[i\delta t \sum_{j=1}^{N} L(q_j, \dot{q}_j, t)\right]. \tag{3.38}$$

In the continuum limit, the transition amplitude becomes

$$F(q_f, t_f; q_i, t_i) = \int \mathcal{D}q \exp\left[i \int_{t_i}^{t_f} L(q, \dot{q}, t)\right] = \int \mathcal{D}q e^{iS[q]}, \tag{3.39}$$

where we used that the temporal integral over the Lagrangian is equal to the action. Equation (3.39) is known as the configuration-space path integral.

Next, we see how the path integral formalism from quantum mechanics is extended to quantum field theory. Using the relation between the position operator in quantum mechanics and the field operator in quantum field theory (3.18) and the definition of conjugate momentum in quantum field theory (3.24), we can find an expression for the path integral in $(1+0)$-dimensional quantum field theory, which we denote by K. Using this expression in the quantum mechanical transition amplitude (3.33), we have

$$K = \int \mathcal{D}\pi\mathcal{D}\phi\, e^{iS[\phi]} = \int \mathcal{D}\pi\mathcal{D}\phi\, \exp\left[i \int d^4x\, \mathcal{L}(\phi, \partial_\mu\phi)\right]. \tag{3.40}$$

Again, if the Hamiltonian density can be assumed to have a simple form $\mathcal{H} = \frac{1}{2}\pi^2 + \mathcal{V}(\phi)$, the Gaussian integral over the conjugate momenta π can be performed to give

$$K = N \int \mathcal{D}\phi\, e^{iS[\phi]} = N \int \mathcal{D}\phi\, \exp\left[i \int d^4x\, \mathcal{L}(\phi, \partial_\mu\phi)\right]. \tag{3.41}$$

However, this step will not always be possible as many quantum fields have complicated Lagrangian densities, and we will see when working with fermionic fields that we must use the phase-space path integral (3.40).

3.4 Noether's theorem

This section is based on references [13], [19], and [31].

Noether's theorem [24] states that a conserved current is associated with each generator of a continuous symmetry of the Lagrangian of the system. The theorem is applicable to both classical and quantum field theory, even to the classical mechanics of a point particle. We will look at Noether's theorem applied to quantum fields described by a Lagrangian density $\mathcal{L}(\phi, \partial_\mu\phi)$. A infinitesimal change $\delta\phi$ in the field ϕ leads to a change $\delta\mathcal{L}$ in the Lagrangian density that is equal to

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\partial_\mu\phi. \tag{3.42}$$

Using the Euler-Lagrange equation (3.22) for the first term and the substitution $\delta\partial_\mu = \partial_\mu\delta$ for the second term, the change $\delta\mathcal{L}$ (3.42) becomes

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right) \delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \partial_\mu\delta\phi = \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \delta\phi \right). \tag{3.43}$$

If the Lagrangian density is invariant under the transformation $\phi \to \phi' = \phi + \delta\phi$, such that $\mathcal{L}'(\phi') = \mathcal{L}(\phi)$, we say that it has a continuous symmetry. When using $\delta\mathcal{L} = 0$ in equation (3.43), we see that the continuous symmetry implies the existence of a conserved current

$$\partial_\mu j^\mu = 0, \tag{3.44}$$

where

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \delta\phi. \tag{3.45}$$

The current j^μ is known as a Noether current. If the change $\delta\mathcal{L}$ is equal to a four divergence $\partial_\mu K^\mu$, the conserved current becomes

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} \delta\phi - K^\mu. \tag{3.46}$$

If we split the current into a temporal part j^0 and a spacial part $\mathbf{j}$, we recognize equation (3.44) as the continuity equation

$$0 = \partial_\mu j^\mu = \partial_0 j^0 + \boldsymbol{\nabla} \cdot \mathbf{j}, \tag{3.47}$$

We often interpret j^0 as a charge density and $\mathbf{j}$ as a current density.

Next, we take the space integral of the continuity equation (3.47) over the 3-volume V of the system

$$\int_V d^3x\, \partial_0 j^0 = -\int_V d^3x\, \boldsymbol{\nabla} \cdot \mathbf{j} = -\int_S \mathbf{j} \cdot d\mathbf{S}, \tag{3.48}$$

where we used Gauss' divergence theorem to write the volume integral of the divergence of the current as a surface integral over the current. Since the system is contained within the large volume V, there will be no current crossing the boundary defined by the surface S and hence the surface integral becomes zero. Using this in (3.48), we see that the volume integral over the time derivative of the current j^0 is zero, meaning that the volume integral of the current itself remains constant in time. We then have a conserved quantity Q defined by

$$Q = \int_V d^3x\, j^0. \tag{3.49}$$

Since j^0 is a charge density, Q represents the total conserved charge and is known as a Noether charge. Depending on the fields and symmetries we work with, the conserved charge Q has different physical meanings.

3.5 Imaginary time and periodicity of fields

This section is based on references [20] and [23].

It is common to work in imaginary time when using the path integral formalism in quantum field theory at finite temperature. The imaginary time variable τ is related to the real time variable through

$$t \to -i\tau. \tag{3.50}$$

The change in time variable corresponds to a Wick rotation from Minkowski to Euclidean space. We define the Euclidean action S_E as the integral over space and imaginary time of the Euclidean Lagrangian density $\mathcal{L}_E$, such that

$$S_E = \int d\tau \int d^3x \, \mathcal{L}_E. \tag{3.51}$$

We can then write the path integral in quantum field theory (3.40) in terms of the Euclidean action S_E, such that

$$K = \int \mathcal{D}\pi\mathcal{D}\phi \, e^{-S_E[\phi]} = \int \mathcal{D}\pi\mathcal{D}\phi \, \exp\left[-\int d\tau d^3x \, \mathcal{L}_E(\phi, \partial_\mu\phi)\right]. \tag{3.52}$$

Imaginary time in the path integral formalism is useful when working with the partition function. Using (2.2), we can find the average of the time-ordered product of two field operators

$$\begin{aligned}
\langle T\{\phi(\mathbf{x},t)\phi(\mathbf{y},0)\}\rangle &= \frac{1}{Z(\beta)} \operatorname{Tr}\left[e^{-\beta\hat{H}'}\phi(\mathbf{x},t)\phi(\mathbf{y},0)\right] \\
&= \frac{1}{Z(\beta)} \operatorname{Tr}\left[\phi(\mathbf{x},t)e^{-\beta\hat{H}'}e^{\beta\hat{H}'}\phi(\mathbf{y},0)e^{-\beta\hat{H}'}\right] \\
&= \frac{1}{Z(\beta)} \operatorname{Tr}\left[\phi(\mathbf{x},t)e^{-\beta\hat{H}'}e^{i(-i\beta\hat{H}')}\phi(\mathbf{y},0)e^{-i(-i\beta\hat{H}')}\right] \\
&= \frac{1}{Z(\beta)} \operatorname{Tr}\left[\phi(\mathbf{x},t)e^{-\beta\hat{H}'}\phi(\mathbf{y},-i\beta)\right] \\
&= \frac{1}{Z(\beta)} \operatorname{Tr}\left[e^{-\beta\hat{H}'}\phi(\mathbf{y},-i\beta)\phi(\mathbf{x},t)\right] \\
&= \langle T\{\phi(\mathbf{y},-i\beta)\phi(\mathbf{x},t)\}\rangle.
\end{aligned} \tag{3.53}$$

We used the cyclic property of trace in second and fifth step. For the third step we used the exponentials as time evolution operators acting on the field operator, as in (3.16). Going from real to imaginary time, we see that $t = -i\beta$ becomes $\tau = \beta$, such that inverse temperature β plays the role as imaginary time variable. Equation (3.53) can then be written as

$$\langle T\{\phi(\mathbf{x},\tau)\phi(\mathbf{y},0)\}\rangle = \langle T\{\phi(\mathbf{y},\beta)\phi(\mathbf{x},\tau)\}\rangle. \tag{3.54}$$

Equation (3.54) is known as the Kubo-Martin-Scwhinger (KMS) relation. It follows directly from (3.54) that

$$\phi(\mathbf{x}, 0) = \pm\phi(\mathbf{x}, \beta), \tag{3.55}$$

where the sign is determined by whether the fields commute (+) or anticommute (-). Commuting fields are bosonic, while anticommuting fields are fermionic.

It is useful to work in Fourier space. Because of the constraints on the periodicity and antiperiodicity of the fields given in (3.55), the Fourier transform of the field ϕ is a Fourier series rather than a Fourier integral. The Fourier transform is therefore

$$\phi(\mathbf{x}, t) = \sum_n \phi(\mathbf{x}, \omega_n) e^{i\omega_n \tau}. \tag{3.56}$$

The frequencies ω_n are known as Matsubara frequencies. They are discrete and given as

$$\omega_n = \frac{2\pi n}{\beta}, \quad n \in \mathbb{Z} \tag{3.57}$$

for bosonic fields, and

$$\omega_n = \frac{2\pi(n+1)}{\beta}, \quad n \in \mathbb{Z} \tag{3.58}$$

for fermionic fields.

Chapter 4

Thermal Field Theory

A neutron star can be modelled as an ideal Fermi gas. When the matter is assumed to be at zero temperature, the properties of the system can be found by using quantum statistical mechanics. This model was used in the preliminary specialization project [32]. At finite temperature, the gas is better described when quantum statistical mechanics is extended to finite-temperature field theory, using a path integral formalism. We will use the path integral formalism to find the partition function, from which all other thermodynamical quantities can be derived. The introduction of quantum field theory will also make it possible to include other contributions to the model, such as nuclear interactions. When taking the zero-temperature limit, we will see that in addition to our previous result for the partition function, we will end up with a divergent vacuum contribution, which is a result of using quantum field theory.

4.1 Fermionic oscillator

This section is based on reference [20].

Before deriving the path integral of the partition function for an ideal Fermi gas, it is useful to look at some of the properties of fermions and their quantum mechanical operators. The simplest kind of fermionic system is the fermionic oscillator, which can be seen as a non-interacting field in (1+0) dimension. Since fermions follow the Pauli exclusion principle, each single-particle mode can at most be occupied by one fermion. The system has therefore only two states; the ground state $|0\rangle$ and the one-particle state $|1\rangle$. The fermion creation operator $a^\dagger$ and the fermion annihilation operator a satisfy the anticommutation relations

$$\left\{a, a^\dagger\right\} = aa^\dagger + a^\dagger a = 1, \tag{4.1}$$

$$\{a, a\} = \left\{a^\dagger, a^\dagger\right\} = 0. \tag{4.2}$$

The anticommutation relations are a result of fermions following Fermi-Dirac statistics, and separates them from bosons, which follow Bose-Einstein statistics and therefore have operators that commute. From (4.2) we see that the squares of the operators are zero

$$aa = a^\dagger a^\dagger = 0. \tag{4.3}$$

We define the annihilation operator to act on the ground state in the following way

$$a\left|0\right\rangle = 0, \tag{4.4}$$

from which it follows that

$$a^\dagger \left|1\right\rangle = 0, \tag{4.5}$$

$$a^\dagger \left|0\right\rangle = \left|1\right\rangle, \tag{4.6}$$

$$a\left|1\right\rangle = \left|0\right\rangle. \tag{4.7}$$

The product of the two operators is the occupation number operator $\hat{N}$

$$\hat{N} = a^\dagger a, \quad \hat{N}\left|n\right\rangle = n\left|n\right\rangle, \tag{4.8}$$

where n is the eigenvalue of $\hat{N}$ and gives the number of fermions in the state. The eigenvalue n can therefore either be zero or one. The Hamiltonian $\hat{H}$ of the system can be written in terms of the occupation number operator, such that

$$\hat{H} = \omega\left(\hat{N} - \frac{1}{2}\right), \tag{4.9}$$

where $-\frac{1}{2}\omega$ and $\frac{1}{2}\omega$ are the energy of the states $\left|0\right\rangle$ and $\left|1\right\rangle$. The states form a complete set of eigenvectors for the Hamiltonian and spans a Hilbert space.

We find an expression for the partition function of a fermionic oscillator from equation (2.1), evaluating the trace of the complete set of eigenvectors of $\hat{H}$ such that

$$Z = \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})} = \sum_{n=0}^{1} \left\langle n\right| e^{-\beta(\omega-\mu)\hat{N}} e^{\frac{1}{2}\beta\omega} \left|n\right\rangle = \left(1 + e^{-\beta(\omega-\mu)}\right) e^{\frac{1}{2}\beta\omega}. \tag{4.10}$$

The factor $e^{\frac{1}{2}\beta\omega}$ corresponds to the zero-point energy and is often ignored. We can then use (2.4) to find the mean number of particles

$$N = \frac{1}{e^{\beta(\omega-\mu)} + 1}, \tag{4.11}$$

which is the Fermi-Dirac distribution.

We have looked at some properties of fermions, which we want to extend to field theory in order to describe a many-particle system of fermions. We therefore need to determine how to implement the analogue of the anticommutation relations for operators in quantum field theory. In order to do this, we introduce anticommuting variables called Grassmann variables.

4.2 Grassmann variables

This section is based on references [8] and [22].

The basic feature of Grassmann variables is that they anticommute. Therefore, they are also known as anticommuting numbers. Grassmann variables are especially useful when working with fermions. As seen from equation (4.9), we can find an expression for the Hamiltonian of the fermionic oscillator. However, if we want to find the corresponding Lagrangian, we run into problems if we try to use the usual expressions for coordinates and momenta. The reason is that the anticommutation property of the fermions has no classical analogue, and to find the Lagrangian we need to introduce anticommuting variables, that is, Grassmann variables.

In general, we have that the Grassmann variables η_i, $i = 1, 2, ..., N$, which act as generators for an N-dimensional Grassmann algebra G_N, obey the anticommutation relation

$$\{\eta_i, \eta_j\} = 0. \tag{4.12}$$

From this relation, we can find the properties of the Grassmann variables.

Let us first look at the properties of a complex two-dimensional Grassmann algebra, that we will later extend to N dimensions. The algebra can be generated from the complex Grassmann variables η and $\eta^\dagger$. The variables satisfy the anticommutation relations

$$\{\eta, \eta\} = \left\{\eta, \eta^\dagger\right\} = \left\{\eta^\dagger, \eta^\dagger\right\} = 0, \tag{4.13}$$

and they must be treated as independent variables. As seen from (4.13), the square of each of the variables must be zero

$$\eta^2 = \left(\eta^\dagger\right)^2 = 0. \tag{4.14}$$

The most general function $f(\eta, \eta^\dagger)$ of the two general variables can be written using Taylor expansions. Since the squares of the variables are zero, terms of higher order than $\mathcal{O}(\eta)$, $\mathcal{O}(\eta^\dagger)$, and $\mathcal{O}(\eta^\dagger\eta)$ vanish. The function $f(\eta, \eta^\dagger)$ is therefore

$$f\left(\eta, \eta^\dagger\right) = c_0 + c_1\eta + c_1'\eta^\dagger + c_{12}\eta^\dagger\eta, \tag{4.15}$$

where c_0, c_1, c_1', and c_{12} are the complex coefficients of the function. Integration of the Grassmann variables is defined as

$$\int d\eta = \int d\eta^\dagger = 0, \tag{4.16}$$

$$\int d\eta\eta = \int d\eta^\dagger\eta^\dagger = 1, \tag{4.17}$$

and when integrating over both variables, the ordering is chosen such that the inner integral is performed first:

$$\int d\eta^\dagger d\eta\, \eta\eta^\dagger = +1. \tag{4.18}$$

Integrating the function (4.15) then gives

$$\int d\eta^\dagger d\eta\, f\left(\eta, \eta^\dagger\right) = -c_{12}. \tag{4.19}$$

The Grassmann variables anticommute with the creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$ by assumption. Using this fact, we define a coherent fermionic state as

$$|\eta\rangle = e^{-\eta a^\dagger}|0\rangle = \left(1 - \eta a^\dagger\right)|0\rangle = |0\rangle + \eta\,|1\rangle\,, \tag{4.20}$$

and its bra conjugate as

$$\langle\eta| = \langle 0|\, e^{-a\eta^\dagger} = \langle 0|\left(1 - a\eta^\dagger\right) = \langle 0| - \langle 1|\,\eta^\dagger. \tag{4.21}$$

The states obey

$$a\,|\eta\rangle = \eta\,|0\rangle = \eta\,|\eta\rangle\,, \tag{4.22}$$

$$\langle\eta|\, a^\dagger = \langle 0|\,\eta^\dagger = \langle\eta|\,\eta^\dagger. \tag{4.23}$$

The inner products of the state $|\eta\rangle$ with the states $|0\rangle$ and $|1\rangle$ are

$$\langle\eta|0\rangle = \langle 0|\eta\rangle = 1, \quad \langle 1|\eta\rangle = \langle\eta|1\rangle^\dagger = -\eta, \tag{4.24}$$

while the inner product between two coherent fermion states is

$$\langle\eta'|\eta\rangle = \exp\left(\eta'^\dagger\eta\right) = 1 + \eta'^\dagger\eta. \tag{4.25}$$

The inner product between two coherent states can be seen as the transition amplitude between the two states. Similarly to equation (4.19), integrals over exponentials of Grassmann variables give

$$\int d\eta^\dagger d\eta\, e^{-\lambda\eta^\dagger\eta} = \lambda. \tag{4.26}$$

The analogue to the completeness relation can be found by considering the integral

$$\begin{aligned}\int d\eta^\dagger d\eta\, e^{-\eta^\dagger\eta}\,|\eta\rangle\langle\eta| &= \int d\eta^\dagger d\eta\, \left(-\eta^\dagger\eta\,|0\rangle\langle 0| + \eta\,|1\rangle\langle 1|\,\eta^\dagger\right) \\ &= |0\rangle\langle 0| + |1\rangle\langle 1| = 1,\end{aligned} \tag{4.27}$$

where equations (4.16) and (4.17) were used together with $a\,|0\rangle = 0$ and $\langle 0|\, a^\dagger = 0$. Another useful relation is the trace of an operator A. This can be found from the integral

$$\begin{aligned}\int d\eta^\dagger d\eta\, e^{-\eta^\dagger\eta}\,\langle -\eta|\, A\,|\eta\rangle &= \int d\eta^\dagger d\eta\, \left(-\eta^\dagger\eta\,\langle 0|\, A\,|0\rangle - \eta^\dagger\eta\,\langle 1|\, A\,|1\rangle\right) \\ &= \langle 0|\, A\,|0\rangle + \langle 1|\, A\,|1\rangle = \mathrm{Tr}[A],\end{aligned} \tag{4.28}$$

where we used (4.18).

The above relations can be generalized to a set of Grassmann variables η_i, $i = 1, ..N$, and their paired sets $\eta_i^\dagger$. This Grassmann algebra then satisfies the anticommutation relations

$$\{\eta_i, \eta_j\} = \left\{\eta_i, \eta_j^\dagger\right\} = \left\{\eta_i^\dagger, \eta_j^\dagger\right\} = 0. \tag{4.29}$$

A complete set of states is written as

$$\int d\eta_i^\dagger d\eta_i \, e^{-\eta_i^\dagger \eta_i} \left|\eta_i\right\rangle \left\langle \eta_i\right| = 1, \tag{4.30}$$

and the equivalent of equation (4.26) is

$$\int \prod_{i=1}^{N} d\eta_i^\dagger d\eta_i \, \exp\left(-\sum_{i=1}^{N} \eta_i^\dagger M_{ij} \eta_j\right) = \det M, \tag{4.31}$$

where M is an $N \times N$ matrix.

4.3 Partition function for anticommuting fields

This section is based on reference [22].

Next, we use the Grassmann variables to find the path integral representation of the partition function for anticommuting fields. The derivation is similar to the derivation of the path integral formalism in quantum mechanics and quantum field theory performed in section 3.3. Starting from the definition of the partition function (2.1), the first step is to evaluate the trace in the partition function using equation (4.28). We split the exponent into N small terms, such that

$$Z = \mathrm{Tr}[e^{-\beta \bar{H}'}] = \int d\eta^\dagger d\eta \, e^{-\eta^\dagger \eta} \left\langle -\eta\right| e^{-\epsilon \hat{H}'} \cdots e^{-\epsilon \hat{H}'} \left|\eta\right\rangle, \tag{4.32}$$

where $\epsilon = \beta/N$. Inserting a complete set of states, given by equation (4.30), between each of the exponents in (4.32), gives terms of the form

$$\begin{aligned} e^{-\eta_{i+1}^\dagger \eta_{i+1}} \left\langle \eta_{i+1}\right| e^{-\epsilon \hat{H}'} \left|\eta_i\right\rangle &= e^{-\eta_{i+1}^\dagger \eta_{i+1}} e^{-\epsilon \hat{H}'\left(\eta_{i+1}^\dagger \eta_i\right)} \left\langle \eta_{i+1} | \eta_i\right\rangle \\ &= \exp\left\{-\epsilon\left[\eta_{i+1}^\dagger \frac{(\eta_{i+1} - \eta_i)}{\epsilon} + \hat{H}'\left(\eta_{i+1}^\dagger, \eta_i\right)\right]\right\}, \end{aligned} \tag{4.33}$$

where we used (4.25) for the inner product of two coherent fermion states. We define the right-most state as $|\eta\rangle = |\eta_1\rangle$. The left-most term will then take the form

$$e^{-\eta_1^\dagger \eta_1} \left\langle -\eta_1\right| e^{\epsilon \hat{H}'} \left|\eta_N\right\rangle = \exp\left\{\epsilon\left[-\eta_1^\dagger \frac{(-\eta_1 - \eta_N)}{\epsilon} + \hat{H}'\left(-\eta_1^\dagger, \eta_N\right)\right]\right\}, \tag{4.34}$$

which means that $\eta_{N+1} = -\eta_1$ and $\eta_{N+1}^\dagger = -\eta_1^\dagger$. The arguments of the exponentials can be recognized as a discrete approximation of the Euclidean action S_E

$$S_E = \epsilon \sum_{i=1}^{N} \left[\eta_{i+1}^\dagger \frac{\eta_{i+1} - \eta_i}{\epsilon} + \hat{H}'\left(\eta_{i+1}^\dagger, \eta_i\right)\right]. \tag{4.35}$$

The path integral representation of the partition function can then be written as

$$Z = \int \prod_{i=1}^{N} d\eta_i^\dagger d\eta_i \, \exp\left\{ -\epsilon \sum_{i=1}^{N} \left[\eta_{i+1}^\dagger \frac{\eta_{i+1} - \eta_i}{\epsilon} + \hat{H}' \left(\eta_{i+1}^\dagger, \eta_i \right) \right] \right\}$$
$$= \int \prod_{i=1}^{N} d\eta_i^\dagger d\eta_i \, e^{-S_E} \tag{4.36}$$

with the boundary conditions $\eta_{N+1} = -\eta_1$ and $\eta_{N+1}^\dagger = -\eta_1^\dagger$. In the continuum limit we let $N \to \infty$ and $\epsilon \to 0$, while $\beta = \epsilon N$ remains fixed. A collection of Grassmann variables, one at each point $\tau \in (0, \beta)$, represents a field $\eta(\tau)$. The variable τ represents imaginary time. The integration measure in the path integral representation is often written as

$$\lim_{N \to \infty} \int \prod_{i=1}^{N} d\eta_i^\dagger d\eta_i = \int \mathcal{D}\eta^\dagger (\tau) \, \mathcal{D}\eta (\tau) \, , \tag{4.37}$$

with the boundary conditions

$$\eta(\beta) = -\eta(0), \quad \eta^\dagger(\beta) = -\eta^\dagger(0). \tag{4.38}$$

In the continuum limit, the Euclidean action S_E becomes

$$S_E = \int_0^\beta \left[\eta^\dagger (\tau) \, \partial_\tau \eta (\tau) + \hat{H}' \left(\eta^\dagger (\tau) \, , \eta (\tau) \right) \right] d\tau. \tag{4.39}$$

The path integral representation of the partition function in the continuum limit is therefore

$$Z = \int \mathcal{D}\eta^\dagger (\tau) \, \mathcal{D}\eta (\tau) \, \exp\left\{ - \int_0^\beta \left[\eta^\dagger (\tau) \, \partial_\tau \eta (\tau) + \hat{H}' \left(\eta^\dagger (\tau) \, , \eta (\tau) \right) \right] d\tau \right\}$$
$$= \int \mathcal{D}\eta^\dagger (\tau) \, \mathcal{D}\eta (\tau) \, e^{-S_E}, \tag{4.40}$$

still with the boundary conditions given by (4.38). The Grassmann fields thereby obey antiperiodic boundary conditions over the Euclidean time interval $(0, \beta)$. To calculate the partition function in the path integral representation, further knowledge about the fermionic Hamiltonian is needed.

4.4 Dirac fermions

This section is based on references [9], [16], [20], [22], and [23].

We have established the properties and the path integral representation of the partition function of anticommuting fields. We next look at a certain type of anticommuting fields called Dirac fields. Dirac fields describe Dirac fermions, and we can use these Dirac fermions to model the Fermi gas of a neutron star. A Dirac fermion is a type of fermion which is not its own antiparticle. A fermion which is its own antiparticle, is called a Majorana fermion. All fermions in the Standard Model are Dirac fermions, except possibly neutrinos. Particles with half-integer spin are fermions. Dirac fermions have spin $\frac{1}{2}$, which is the case for neutrons.

We look at some properties of the Dirac fermions, which we will need in order to calculate the partition function. In particular we want to find expressions for the Lagrangian and Hamiltonian densities and the action of the Dirac fermions. The Lagrangian density of a Dirac fermion is

$$\mathcal{L}_M = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi, \tag{4.41}$$

where ψ and $\bar{\psi} = \psi^\dagger \gamma^0$ are the Dirac field and its adjoint, m is the mass of the Dirac fermion and γ^μ are the Dirac matrices. We used the Feynman slash notation $a\!\!\!/ = \gamma^\mu a_\mu = \gamma_\mu a^\mu$ in the second step. The subscript M is used to specify that the Lagrangian density $\mathcal{L}_M$ is for Minkowskian space. The Dirac matrices must obey the algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{4.42}$$

and are in standard representation given as

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}. \tag{4.43}$$

The matrices are 4×4, so the 0 denotes a 2×2 matrix of zeros, the I_2 denotes a 2×2 unit matrix, and $\boldsymbol{\sigma}$ is the vector consisting of the three Pauli matrices: $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$.

Using the Lagrangian density (4.41) in the Euler-Langrange equations for fields (3.22), we find the equations of motion for the field,

$$(i\gamma^\mu \partial_\mu - m)\psi = (i\partial\!\!\!/ - m)\psi = 0, \tag{4.44}$$

and its adjoint,

$$\bar{\psi}(i\gamma^\mu \partial_\mu - m) = \bar{\psi}(i\partial\!\!\!/ - m) = 0, \tag{4.45}$$

which are known as the Dirac equation and the adjoint Dirac equation, respectively [11]. The fields are four-component column vectors, called Dirac spinors. The spinor field ψ consists of the components ψ_α, where $\alpha = 1, 2, 3$ or 4. We treat ψ and $\bar{\psi}$ as independent variables. We can find plane wave solutions $\psi_\alpha = e^{ip\cdot x} u_\alpha(p)$ of the Dirac equation (4.44). The four-component bi-spinor $u(p)$ splits into two spinors $\tilde{u}(p)$ and $\tilde{v}(p)$, which represent the positive and negative energy solutions, respectively. As required, the Dirac field and its adjoint satisfy the anticommutation relations

$$\{\psi_\alpha(\mathbf{x}, t), \psi^\dagger_\beta(\mathbf{y}, t)\} = \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x} - \mathbf{y}), \tag{4.46}$$

$$\{\psi_\alpha(\mathbf{x}, t), \psi_\beta(\mathbf{y}, t)\} = \{\psi^\dagger_\alpha(\mathbf{x}, t), \psi^\dagger_\beta(\mathbf{y}, t)\} = 0. \tag{4.47}$$

We can then write the Fourier expansion of the Dirac field as

$$\psi(x) = \frac{1}{\sqrt{V}} \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left[b_s(\mathbf{p})\tilde{u}_s(\mathbf{p})e^{-ip\cdot x} + d^\dagger_s(\mathbf{p})\tilde{v}_s(\mathbf{p})e^{ip\cdot x}\right], \tag{4.48}$$

where V is the 3-volume of the system, s is the spin index, $E_p = \sqrt{\mathbf{p}^2 + m^2}$, $p{\cdot}x = E_p t - \mathbf{p}\cdot\mathbf{x}$, and $b_s(\mathbf{p})$ and $d_s^\dagger(\mathbf{p})$ are operators in field theory that satisfy the anticommutation relations

$$\{b_s(\mathbf{p}), b_r(\mathbf{p}')\} = \{d_s(\mathbf{p}), d_r(\mathbf{p}')\} = 0, \tag{4.49}$$

$$\{b_s(\mathbf{p}), b_r^\dagger(\mathbf{p})\} = \{d_s(\mathbf{p}), d_r^\dagger(\mathbf{p})\} = (2\pi)^3 2E_p \delta_{rs}\delta^{(3)}(\mathbf{p}-\mathbf{p}'). \tag{4.50}$$

Since we know the Lagrangian density (4.41), we can find the Hamiltonian density of the Dirac field through a Legendre transformation (3.23). Using the Lagrangian density (4.41) in (3.24), we find that the conjugate momentum is

$$\pi = \frac{\partial \mathcal{L}_M}{\partial\left(\partial\psi/\partial t\right)} = i\bar{\psi}\gamma^0 = i\psi^\dagger, \tag{4.51}$$

where we used $(\gamma^0)^2 = I_4$. The Hamiltonian density becomes

$$\begin{aligned}\mathcal{H} &= i\psi^\dagger\frac{\partial\psi}{\partial t} - \mathcal{L}_M = i\psi^\dagger\frac{\partial\psi}{\partial t} - i\bar{\psi}\gamma^0\frac{\partial\psi}{\partial t} - i\bar{\psi}\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\psi + m\bar{\psi}\psi \\ &= \bar{\psi}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\psi + m)\psi,\end{aligned} \tag{4.52}$$

where the differentiation was split into a temporal and a spacial part; $\partial_\mu = (\partial/\partial t, \boldsymbol{\nabla})$. The Hamiltonian is found by integrating the Hamiltonian density over space, which gives

$$\hat{H} = \int d^3x\,\bar{\psi}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\psi = \int d^3x\,\psi^\dagger(-i\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m\gamma^0)\psi. \tag{4.53}$$

Next, we want to include a chemical potential μ in the Hamiltonian, such that the full Hamiltonian is $\hat{H}' = \hat{H} + \mu\hat{N}$. The conserved Noether charge Q that corresponds to the conserved quantity $\hat{N}$, can be found using Noether's theorem from section 3.4. The Lagrangian density of the Dirac field is invariant under the global $U(1)$ phase transformation

$$\psi \to \psi' = e^{-i\alpha}\psi, \quad \bar{\psi} \to \bar{\psi}' = e^{i\alpha}\bar{\psi}. \tag{4.54}$$

An infinitesimal transformation leads to changes $\delta\psi$ and $\delta\bar{\psi}$ in the fields, given by

$$\psi' = (1 - i\alpha)\psi = \psi + \delta\psi, \quad \bar{\psi}' = (1 + i\alpha)\bar{\psi} = \bar{\psi} + \delta\bar{\psi}. \tag{4.55}$$

From equation (3.45), we find that

$$\alpha j^\mu = \frac{\partial\mathcal{L}_M}{\partial(\partial_\mu\psi)}\delta\psi + \frac{\partial\mathcal{L}_M}{\partial(\partial_\mu\bar{\psi})}\delta\bar{\psi} = \bar{\psi}i\gamma^\mu(-i\alpha\psi) + 0 = \alpha\bar{\psi}\gamma^\mu\psi. \tag{4.56}$$

We therefore find that the conserved Noether current of the Dirac Lagrangian is

$$j^\mu = \bar{\psi}\gamma^\mu\psi. \tag{4.57}$$

Using the conserved current (4.57) in (3.49), we find the conserved Noether charge

$$Q = \int d^3x\, j^0 = \int d^3x\,\bar{\psi}\gamma^0\psi = \int d^3x\,\psi^\dagger\psi. \tag{4.58}$$

The full Hamiltonian $\hat{H}'$ is thereby

$$\begin{aligned}\hat{H}' &= \int d^3x\, \psi^\dagger \left(-i\gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m\gamma^0 - \mu\right) \psi \\ &= \int d^3x\, \bar{\psi} \left(-i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m - \mu\gamma^0\right) \psi. \end{aligned} \tag{4.59}$$

Having found the full Hamiltonian (4.59) of the system, we can find an expression for the Euclidean action S_E. We insert (4.59) into (4.39) and find that the action is

$$S_E = \int_0^\beta d\tau \int d^3x\, \bar{\psi} \left(\gamma^0 \partial_\tau - i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m - \mu\gamma^0\right) \psi. \tag{4.60}$$

Using that the action S_E is related to the Lagrangian L_E and Lagrangian density $\mathcal{L}_E$ through

$$S_E = \int d\tau\, L_E = \int d\tau \int d^3x\, \mathcal{L}_E, \tag{4.61}$$

we find that the Euclidean Lagrangian density is

$$\mathcal{L}_E = \bar{\psi} \left(\gamma^0 \partial_\tau - i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} + m - \mu\gamma^0\right) \psi. \tag{4.62}$$

Comparing the Euclidean Lagrangian density $\mathcal{L}_E$ to the Minkowskian Lagrangian density $\mathcal{L}_M$ in (4.41), we see that $\mathcal{L}_E$ is the Wick rotation of $\mathcal{L}_M$, such that $\mathcal{L}_E(\tau) = -\mathcal{L}_M(t \to -i\tau)$. We note that $i\partial_t \to -\partial_\tau$. To simplify the expression of the Euclidean Lagrangian density (4.62), we can introduce the Euclidean Dirac matrices

$$\tilde{\gamma}_0 \equiv \gamma^0, \quad \tilde{\gamma}_k \equiv -i\gamma^k. \tag{4.63}$$

The Euclidean Dirac matrices satisfy

$$\{\tilde{\gamma}_\mu, \tilde{\gamma}_\nu\} = 2\delta_{\mu\nu}. \tag{4.64}$$

Also, we have that $\partial_t = \partial_\tau$. The Euclidean Lagrangian density can then be written as

$$\mathcal{L}_E = \bar{\psi} \left(\tilde{\gamma}_\mu \partial_\mu + m - \tilde{\gamma}_0 \mu\right) \psi, \tag{4.65}$$

where the repeated lower indices imply that we are using the Euclidean Dirac matrices.

4.5 Partition function for Dirac fermions

This section is based on references [20] and [21].

We have found the Hamiltonian density, the Lagrangian density, and the action of Dirac fermions, and we will now use them to calculate the partition function. Inserting the Dirac field ψ, its ajoint $\psi^\dagger$, and the expression for the Euclidean action S_E (4.60) into the path integral representation of the partition function of anticommuting fields (4.40), we find that the partition function becomes

$$\begin{aligned} Z &= \int_{\mathrm{BC}} \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\left[-\int_0^\beta d\tau \int d^3x \, \bar{\psi}\left(\gamma^0\partial_\tau - i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m - \mu\gamma^0\right)\psi\right] \\ &= \int_{\mathrm{BC}} \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\left[-\int_0^\beta d\tau \int d^3x \, \bar{\psi}\left(\tilde{\gamma}_\mu\partial_\mu + m - \tilde{\gamma}_0\mu\right)\psi\right], \end{aligned} \tag{4.66}$$

where BC are the antiperiodic boundary conditions

$$\psi(\mathbf{x},\beta) = -\psi(\mathbf{x},0), \quad \psi^\dagger(\mathbf{x},\beta) = -\psi^\dagger(\mathbf{x},0). \tag{4.67}$$

It is more convenient to work in Fourier space $(\mathbf{p},\omega_n)$ than in $(\mathbf{x},\tau)$-space. The Fourier transforms of the Dirac fields ψ and $\bar{\psi}$ are

$$\begin{aligned} \psi_\alpha(\mathbf{x},\tau) &= \frac{1}{\sqrt{V}}\sum_{n,\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x}+\omega_n\tau)}\tilde{\psi}_{\alpha;n}(\mathbf{p}), \\ \bar{\psi}_\alpha(\mathbf{x},\tau) &= \frac{1}{\sqrt{V}}\sum_{n,\mathbf{p}} e^{-i(\mathbf{p}\cdot\mathbf{x}+\omega_n\tau)}\tilde{\bar{\psi}}_{\alpha;n}(\mathbf{p}), \end{aligned} \tag{4.68}$$

where V is the volume of the system and ω_n are the fermionic Matsubara frequencies. From the antiperiodicity of the fields, we find that the fermionic Matsubara frequencies must satisfy the constraint given by (3.58). Using the Fourier transforms (4.68), the Euclidean action in (4.60) becomes

$$\begin{aligned} S_E &= \frac{1}{V}\int_0^\beta d\tau \int d^3x \sum_{n,n'}\sum_{\mathbf{p},\mathbf{q}} e^{i(\omega_n-\omega_{n'})\tau}e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\tilde{\bar{\psi}}\left(\gamma^0\partial_\tau - i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m - \mu\gamma^0\right)\tilde{\psi} \\ &= \beta\delta_{n,n'}\delta^{(3)}(\mathbf{p}-\mathbf{q})\sum_{n,n'}\sum_{\mathbf{p},\mathbf{q}}\tilde{\bar{\psi}}\left(i\gamma^0\omega_n + \boldsymbol{\gamma}\cdot\mathbf{p} + m - \mu\gamma^0\right)\tilde{\psi} \\ &= \sum_{n,\mathbf{p}}\tilde{\bar{\psi}}\left[\beta\left(i\gamma^0\omega_n + \boldsymbol{\gamma}\cdot\mathbf{p} + m - \mu\gamma^0\right)\right]\tilde{\psi}. \end{aligned} \tag{4.69}$$

In the second step, we used that the integral over the imaginary time variable τ gives a δ-function over the Matsubara indices n and n' and a factor β, while the integral over space gives a δ-function over the momenta $\mathbf{p}$ and $\mathbf{q}$ and a factor V, which cancels the factor $\frac{1}{V}$. Using (4.69), the partition function becomes

$$Z = \int \prod_{n,\mathbf{p}} \mathcal{D}\tilde{\bar{\psi}}\mathcal{D}\tilde{\psi}\exp\left\{-\sum_{n,\mathbf{p}}\tilde{\bar{\psi}}\left[\beta\left(i\gamma^0\omega_n + \boldsymbol{\gamma}\cdot\mathbf{p} + m - \mu\gamma^0\right)\right]\tilde{\psi}\right\}. \tag{4.70}$$

According to equation (4.31), this is equal to the determinant of the argument of the exponential,

$$Z = \det\left[\beta\left(i\gamma^0\omega_n + \boldsymbol{\gamma}\cdot\mathbf{p} + m - \mu\gamma^0\right)\right] = \det D, \tag{4.71}$$

where the determinant operation should be carried out over both the Dirac indices and Fourier space. The determinant over the Dirac indices is the determinant of a 4×4 matrix. Using the standard representation (4.43) for the gamma matrices, we can write D in matrix form

$$\begin{aligned} D &= \beta \left[\begin{pmatrix} i\omega_n & 0 \\ 0 & -i\omega_n \end{pmatrix} + \begin{pmatrix} 0 & \boldsymbol{\sigma} \cdot \mathbf{p} \\ -\boldsymbol{\sigma} \cdot \mathbf{p} & 0 \end{pmatrix} + \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} - \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix} \right] \\ &= \beta \begin{pmatrix} i\omega_n + m - \mu & \boldsymbol{\sigma} \cdot \mathbf{p} \\ -\boldsymbol{\sigma} \cdot \mathbf{p} & -i\omega_n + m + \mu \end{pmatrix}, \end{aligned} \tag{4.72}$$

where each element represents a 2×2 diagonal matrix. The scalar product of the vector of the three Pauli matrices and the momentum vector is

$$\boldsymbol{\sigma} \cdot \mathbf{p} = \sigma^1 p_1 + \sigma^2 p_2 + \sigma^3 p_3 = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix}. \tag{4.73}$$

Writing the matrix D out fully as a 4×4 matrix, we get

$$D = \beta \begin{pmatrix} i\omega_n + m - \mu & 0 & p_3 & p_1 - ip_2 \\ 0 & i\omega_n + m - \mu & p_1 + ip_2 & -p_3 \\ -p_3 & -(p_1 - ip_2) & -i\omega_n + m + \mu & 0 \\ -(p_1 + ip_2) & p_3 & 0 & -i\omega_n + m + \mu \end{pmatrix}. \tag{4.74}$$

To find the determinant of the matrix D, let us use a simpler notation by setting $a = i\omega_n + m - \mu$, $b = p_3$, $c = p_1 - ip_2$, $d = p_1 + ip_2$, and $e = -i\omega_n + m + \mu$. The matrix D is then

$$D = \beta \begin{pmatrix} a & 0 & b & c \\ 0 & a & d & -b \\ -b & -c & e & 0 \\ -d & b & 0 & e \end{pmatrix}. \tag{4.75}$$

The determinant can then be found the usual way, starting from 3×3 determinants and splitting into 2×2 determinants which can be calculated. We find that the determinant is

$$\begin{aligned} \det D &= \beta^4 \left[a \begin{vmatrix} a & d & -b \\ -c & e & 0 \\ b & 0 & e \end{vmatrix} + b \begin{vmatrix} 0 & a & -b \\ -b & -c & 0 \\ -d & b & e \end{vmatrix} - c \begin{vmatrix} 0 & a & d \\ -b & -c & e \\ -d & b & 0 \end{vmatrix} \right] \\ &= \beta^4 \left[a^2 \begin{vmatrix} e & 0 \\ 0 & e \end{vmatrix} - ad \begin{vmatrix} -c & 0 \\ b & e \end{vmatrix} - ab \begin{vmatrix} -c & e \\ b & 0 \end{vmatrix} - ab \begin{vmatrix} -b & 0 \\ -d & e \end{vmatrix} \right. \\ &\qquad \left. - b^2 \begin{vmatrix} -b & -c \\ -d & b \end{vmatrix} + ac \begin{vmatrix} -b & e \\ -d & 0 \end{vmatrix} - cd \begin{vmatrix} -b & -c \\ -d & b \end{vmatrix} \right] \\ &= \beta^4 \left(a^2 e^2 + 2aeb^2 + 2aecd + 2b^2 cd + c^2 d^2 + b^4 \right) \\ &= \beta^4 \left(ae + cd + b^2 \right)^2. \end{aligned} \tag{4.76}$$

Reinserting the expressions for a, b, c, d, and e into equation (4.76), the determinant becomes

$$\begin{aligned}\det D &= \beta^4 \left[(i\omega_n + m - \mu)(-i\omega_n + m + \mu) + (p_1 - ip_2)(p_1 + ip_2) + p_3^2\right]^2 \\ &= \beta^4 \left[m^2 - (i\omega_n - \mu)^2 + \mathbf{p}^2\right]^2 = \beta^4 \left[(i\omega_n - \mu)^2 - \omega^2\right]^2, \end{aligned} \tag{4.77}$$

where we used $\omega^2 = \mathbf{p}^2 + m^2$ and pulled out a -1 from the square brackets in the last step. For the determinant over Fourier space, we use the property that the logarithm of the determinant is equal to the trace of the logarithm [36]

$$\ln \det D = \operatorname{Tr} \ln D, \tag{4.78}$$

and evaluate the trace over the $(\mathbf{p}, \omega_n)$-space. Using equations (4.77) and (4.78), we find that the logarithm of the partition function is

$$\ln Z = \sum_{n,\mathbf{p}} \ln \left\{\beta^4 \left[(i\omega_n - \mu)^2 - \omega^2\right]^2\right\} = 2 \sum_{n,\mathbf{p}} \ln \left\{\beta^2 \left[(i\omega_n - \mu)^2 - \omega^2\right]\right\}, \tag{4.79}$$

where we pulled out a factor 2 from the logarithm in the second step.

Next, we want to manipulate equation (4.79) so that the sum over the Matsubara frequencies ω_n can be performed. First, we split the logarithm into two terms, such that

$$\ln Z = \sum_{n,\mathbf{p}} \Big\{2 \ln \left[\beta (i\omega_n - \mu + \omega)\right] + 2 \ln \left[\beta (i\omega_n - \mu - \omega)\right]\Big\}. \tag{4.80}$$

Splitting each of these terms again in two and using that the summation is over both negative and positive frequencies, meaning $\sum_n \omega_n = \sum_n (-\omega_n)$, we get

$$\begin{aligned}\ln Z = \sum_{n,\mathbf{p}} \Big\{ &\ln \left[\beta (i\omega_n - \mu + \omega)\right] + \ln \left[\beta (-i\omega_n - \mu + \omega)\right] \\ &+ \ln \left[\beta (i\omega_n - \mu - \omega)\right] + \ln \left[\beta (-i\omega_n - \mu - \omega)\right]\Big\}. \end{aligned} \tag{4.81}$$

Combining the first term with the second, and the third with the fourth, we find that the logarithm of the partition function is

$$\ln Z = \sum_{n,\mathbf{p}} \left\{\ln \left[\beta^2 \left(\omega_n^2 + (\omega - \mu)^2\right)\right] + \ln \left[\beta^2 \left(\omega_n^2 + (\omega + \mu)^2\right)\right]\right\}. \tag{4.82}$$

The sum over the Matsubara frequencies ω_n of the type

$$\sigma = \sum_{n=-\infty}^{\infty} \ln \left[\beta^2 \left(\omega_n^2 + (\omega \pm \mu)^2\right)\right], \tag{4.83}$$

is known as a Matsubara sum. This sum can be calculated using the residue theorem and contour integrals. We start by setting $a = \omega \pm \mu$ and differentiate the Matsubara sum

(4.83) with respect to a^2, such that

$$\begin{aligned}\frac{\partial \sigma}{\partial a^2} &= \sum_{n=-\infty}^{\infty} \frac{\partial}{\partial a^2} \ln\left[\beta^2\left(\omega_n^2 + a^2\right)\right] \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2 + a^2} = -\sum_{n=-\infty}^{\infty} \frac{1}{(i\omega_n)^2 - a^2}, \end{aligned} \tag{4.84}$$

where we used $\omega_n^2 = -(i\omega_n)^2$ in the last step. We recall Cauchy's residue theorem, which states that for a function $f(z)$ that is analytic inside and on a simple closed path C, except for finitely many singular points $z_1, z_2, ...z_k$ inside C, the integral of $f(z)$ over C is equal to $2\pi i$ times the sum of the residues of $f(z)$ at the poles $z_1, z_2, ..., z_k$, that is

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z). \tag{4.85}$$

A simple closed path is also called a contour, and the left-hand side of (4.85) is called a contour integral. The contour encircles the poles counterclockwise. For a simple pole z_0, the residue of the function $f(z)$ at the pole is defined as

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z - z_0) f(z). \tag{4.86}$$

Consider the following function in the complex plane

$$f(z) = \frac{g(z)}{z^2 - a^2}. \tag{4.87}$$

If we can find a function $g(z)$ that has poles in $z = i\omega_n$ inside the contour C and residue one at these poles, the sum in equation (4.84) is equal to the sum of the residues of the function $f(z)$, so

$$\sum_{n=-\infty}^{\infty} \frac{1}{(i\omega_n)^2 - a^2} = \sum \operatorname*{Res}_{z=i\omega_n} \frac{g(z)}{z^2 - a^2}. \tag{4.88}$$

One function that satisfies these requirements is $g(z) = \frac{\beta}{2} \tanh \frac{\beta z}{2}$. Using the residue theorem (4.85), we can write the sum (4.84) as a contour integral

$$\sum_{n=-\infty}^{\infty} \frac{1}{(i\omega_n)^2 - a^2} = \frac{\beta}{2} \sum \operatorname*{Res}_{z=i\omega_n} \frac{\tanh \frac{\beta z}{2}}{z^2 - a^2} = \frac{1}{2\pi i} \frac{\beta}{2} \oint_C \frac{\tanh \frac{\beta z}{2}}{z^2 - a^2} dz. \tag{4.89}$$

The function $g(z)$ has infinitely many poles along the imaginary axis and is bounded everywhere else. The contour in (4.89) encircles each of these poles, which is equal to a contour enclosing the imaginary axis. Since there are infinitely many poles, it will be impossible to calculate this integral. Instead, we can replace the contour enclosing the imaginary axis with two half-circle contours not enclosing the imaginary axis. We thereby

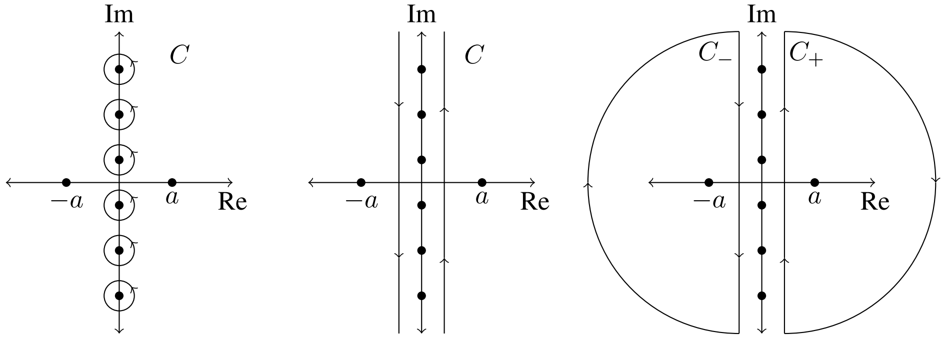

Figure 4.1: To calculate the contour integral (4.89), the contour C around the infinitely many poles $z = i\omega_n$ along the imaginary axis, is replaced with the contours C_+ and C_- enclosing the poles $z = \pm a$ on the real axis.

go from a contour C encircling the poles $z = i\omega_n$ to the contours C_+ and C_- enclosing the poles $z = \pm a$. These are the poles of the denominator $z^2 - a^2$ in the integral in equation (4.89). This manipulation of the contour is shown in Figure 4.1. We then have

$$\frac{\beta}{2}\oint_C \frac{\tanh\frac{\beta z}{2}}{z^2-a^2}dz = -\frac{\beta}{2}\oint_{C_-}\frac{\tanh\frac{\beta z}{2}}{z^2-a^2} - \frac{\beta}{2}\oint_{C_+}\frac{\tanh\frac{\beta z}{2}}{z^2-a^2}dz, \tag{4.90}$$

where the minus signs are due to the directions of the contours C_- and C_+ being clockwise, not counterclockwise. We can now reapply the residue theorem on right-hand side contour integrals in (4.90). We then find that the sum in equation (4.89) is equal to the residues of $f(z) = g(z)/(z^2 - a^2)$ at the poles $z = \pm a$

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{1}{(i\omega_n)^2-a^2} &= \frac{1}{2\pi i}\left(-\frac{\beta}{2}\oint_{C_-}\frac{\tanh\frac{\beta z}{2}}{z^2-a^2} - \frac{\beta}{2}\oint_{C_+}\frac{\tanh\frac{\beta z}{2}}{z^2-a^2}dz\right)\\ &= -\frac{\beta}{2}\sum\operatorname*{Res}_{z=\pm a}\frac{\tanh\frac{\beta z}{2}}{z^2-a^2},\end{aligned} \tag{4.91}$$

where the residues can be calculated using the definition (4.86). The residues are

$$\begin{aligned}\sum\operatorname*{Res}_{z=\pm a}\frac{\tanh\frac{\beta z}{2}}{z^2-a^2} &= \lim_{z\to a}(z-a)\frac{\tanh\frac{\beta z}{2}}{z^2-a^2} + \lim_{z\to -a}(z-(-a))\frac{\tanh\frac{\beta z}{2}}{z^2-a^2}\\ &= \frac{1}{2a}\tanh\frac{\beta a}{2} + \frac{1}{-2a}\tanh\frac{\beta(-a)}{2} = \frac{1}{a}\tanh\frac{\beta a}{2}.\end{aligned} \tag{4.92}$$

Inserting this result into (4.84), we find that the derivative of the Matsubara sum with respect to a^2 is

$$\frac{\partial\sigma}{\partial a^2} = \frac{\beta}{2a}\tanh\frac{\beta a}{2}. \tag{4.93}$$

Finally, we find the Matsubara sum by integrating (4.93), such that

$$\begin{aligned}\sigma &= \int \frac{\beta}{2a}\tanh\frac{\beta a}{2}da^2 = \int \frac{2}{u}du = 2\ln\left[\cosh\frac{\beta a}{2}\right] + C \\ &= \beta a + 2\ln\left[1+e^{-\beta a}\right] + C',\end{aligned} \tag{4.94}$$

where we used the substitution $u = \cosh\frac{\beta a}{2}$ in the second step. The constants C and C' do not depend on β and can in further calculations be omitted. We can now reinsert $a = \omega \pm \mu$ and use the result for the Matsubara sum (4.94) to find the logarithm of the partition function, which becomes

$$\ln Z = 2\sum_{\mathbf{p}}\left\{\beta\omega + \ln\left[1+e^{-\beta(\omega-\mu)}\right] + \ln\left[1+e^{-\beta(\omega+\mu)}\right]\right\}. \tag{4.95}$$

Finally, we can take the continuum limit $\sum_{\mathbf{p}} = V\int\frac{d^3p}{(2\pi)^3}$, which gives us our final expression for the logarithm of the partition function of Dirac fermions

$$\ln Z = 2V\int\frac{d^3p}{(2\pi)^3}\left\{\beta\omega + \ln\left[1+e^{-\beta(\omega-\mu)}\right] + \ln\left[1+e^{-\beta(\omega+\mu)}\right]\right\}. \tag{4.96}$$

The three terms in equation (4.96) represent the contributions from the vacuum, particles and antiparticles, respectively. The contributions for the particles and antiparticles to the logarithm of the partition function are the same that we find using a fully-degenerate ideal Fermi gas in quantum statistical mechanics. The vacuum contribution is however a result of using field theory. It appears even when there are no particles or antiparticles, and turns out to be divergent.

4.6 Free energy and pressure of an ideal Fermi gas

We have found the partition function (4.96) of an ideal Fermi gas consisting of Dirac fermions using a path integral formalism for anticommuting fields. Using the partition function, we can find expressions for the free energy and the pressure, which we need for the equation of state. Inserting the partition function (4.96) in equation (2.7), we find that the free energy is

$$F = -\frac{2V}{\beta}\int\frac{d^3p}{(2\pi)^3}\left[\beta\omega + \ln\left(1+e^{-\beta(\omega-\mu)}\right) + \ln\left(1+e^{-\beta(\omega+\mu)}\right)\right]. \tag{4.97}$$

The free energy density $\mathcal{F}$ is found by differentiating with respect to the volume, giving

$$\mathcal{F} = \frac{\partial F}{\partial V} = -\frac{2}{\beta}\int\frac{d^3p}{(2\pi)^3}\left[\beta\omega + \ln\left(1+e^{-\beta(\omega-\mu)}\right) + \ln\left(1+e^{-\beta(\omega+\mu)}\right)\right]. \tag{4.98}$$

Comparing this to equation (2.8), we find that the pressure is minus the free energy density, such that

$$P = -\mathcal{F} = \frac{2}{\beta}\int\frac{d^3p}{(2\pi)^3}\left[\beta\omega + \ln\left(1+e^{-\beta(\omega-\mu)}\right) + \ln\left(1+e^{-\beta(\omega+\mu)}\right)\right]. \tag{4.99}$$

As the partition function, the pressure also contain three terms, which represent contributions from the vacuum, particles and antiparticles.

Let us look at the pressure P in the zero-temperature limit. When $T \to 0$, $\beta \to \infty$. The logarithmic terms in (4.99) become

$$\lim_{\beta\to\infty} \frac{1}{\beta} \ln\left(1 + e^{-\beta(\omega\pm\mu)}\right) = \begin{cases} 0, & \omega \pm \mu > 0 \\ -(\omega \pm \mu), & \omega \pm \mu < 0 \end{cases}. \tag{4.100}$$

We assume that $\mu > 0$, such that for antiparticles $\omega + \mu > 0$. Then, the antiparticle contribution, which is the third term of equation (4.99), must vanish in the zero-temperature limit. The particle contribution is only non-zero when $\omega - \mu < 0$, which means we can write it in terms of a Heaviside step function $\theta(\omega, \mu)$. The pressure in the zero-temperature limit is therefore

$$P = 2 \int \frac{d^3p}{(2\pi)^3} \left[\omega + \theta\left(\mu - \omega\right)\left(\mu - \omega\right)\right]. \tag{4.101}$$

We see that the step function is zero when the energy ω is larger than the chemical potential μ. The energy that is equal to the chemical potential represents the highest occupied state, and we define it as the Fermi energy $\omega_F = \mu$. We can find the corresponding Fermi momentum p_F through

$$p_F = \sqrt{\omega_F^2 - m^2} = \sqrt{\mu^2 - m^2}, \tag{4.102}$$

where m is the mass of the Dirac fermion. Due to the step function, the integration over all momenta p is replaced with integration over momenta from zero up to the Fermi momentum p_F. Using the spherical symmetry of the integrand, we find that pressure becomes

$$P = \frac{1}{\pi^2} \int_0^{p_F} dp\, p^2 \left(\mu - \omega\right) + 2 \int \frac{d^3p}{\left(2\pi\right)^3} \omega = I_1 + I_2, \tag{4.103}$$

where we use I_1 and I_2 to denote each of the integrals. We can calculate the first integral I_1 by inserting $\mu = \sqrt{p_F^2 + m^2}$ and $\omega = \sqrt{m^2 + p^2}$. We then get

$$I_1 = \frac{1}{\pi^2} \int_0^{p_F} dp\, p^2 \left(\mu - \omega\right) = \frac{1}{\pi^2} \int_0^{p_F} dp\, p^2 \left[\sqrt{p_F^2 + m^2} - \sqrt{p^2 + m^2}\right]. \tag{4.104}$$

Using the substitution $x = p/m$, we find that

$$\begin{aligned} I_1 &= \frac{m^4}{\pi^2} \int_0^{x_F} dx\, x^2 \left(\sqrt{x_F^2 + 1} - \sqrt{x^2 + 1}\right) \\ &= \frac{m^4}{24\pi^2} \left[\left(2x_F^3 - 3x_F\right)\sqrt{1 + x_F^2} + 3\sinh^{-1}\left(x_F\right)\right], \end{aligned} \tag{4.105}$$

where $x_F = p_F/m$. We recognize that the expression for the particle contribution of the pressure is equal to the expression for the pressure found in the specialization project

[32]. When using field theory, we get an additional term due to the vacuum. The vacuum contribution, equal to the second integral I_2, is divergent, as seen from

$$I_2 = 2 \int \frac{d^3p}{(2\pi)^3} \sqrt{p^2 + m^2}. \tag{4.106}$$

To regulate it, one can use e.g. dimensional regularisation or UV-cutoff. In this thesis we will not focus on the vacuum contribution and will therefore omit this term from our calculations. A note on its effect can be found in the outlook section.

Chapter 5

Nuclear Field Theory

Next, we want to incorporate nuclear interactions between the Dirac fermions in the Fermi gas that we use to model our neutron star. The fermions in the Fermi gas will interact, and this interaction can be modelled by introducing mesons. The first model we look at is the σ-ω model, which is also known as the Walecka model [34]. Here, scalar mesons and vector mesons are introduced. The model can be used for both pure neutron matter and nuclear matter consisting of equal amounts of neutrons and protons, depending on the choice of how many baryons each momentum state can accommodate. We extend the model further by introducing scalar self-interactions and isospin force. The isospin force, which is introduced through a triplet of charged vector mesons, distinguishes neutrons and protons. This allows us to take into account that neutron stars will contain small amounts of protons in addition to neutrons, due to weak decay of the neutrons. Weak decay also produces electrons, which we together with muons add to our model so that we can impose that the system as a whole should be electrically neutral. We find the expressions for the pressure and energy density of the model and its extensions, which gives us a new equation of state.

5.1 The σ - ω model

This section is based on references [5], [13], and [27].

In the σ-ω model, nuclear interactions are introduced by using the fields of a scalar meson and a vector meson in addition to the Dirac field for nucleons. We will derive the σ-ω model for pure neutron matter. In section 4.4, we found the Lagrangian density of a free nucleon field

$$\mathcal{L}_{\text{nucleon}} = \bar{\psi}\left(i\gamma_\mu\partial^\mu - m\right)\psi. \tag{5.1}$$

The scalar meson is represented by a scalar field σ. Its free Lagrangian density is

$$\mathcal{L}_\sigma = \frac{1}{2}\left(\partial_\mu \sigma \partial^\mu \sigma - m_\sigma^2 \sigma^2\right), \tag{5.2}$$

where m_σ is the meson mass. A vector field ω_μ represents the vector meson of mass m_μ. The free Lagrangian of the vector meson is

$$\mathcal{L}_\omega = -\frac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{2}m_\omega^2 \omega_\mu \omega^\mu, \tag{5.3}$$

where we have defined

$$\omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu, \tag{5.4}$$

as in electromagnetic theory. In addition to these free fields, we want to include interactions between the nucleons and the meson fields. Since the Lagrangian density must be a Lorentz scalar, the scalar meson should be coupled to the scalar density $\bar{\psi}\psi$, while the vector meson should be coupled to the nucleon four-current $\bar{\psi}\gamma_\mu\psi$. We write the interaction terms of the Lagrangian density as

$$\mathcal{L}_{\text{int}} = g_\sigma \sigma \bar{\psi}\psi - g_\omega \omega^\mu \bar{\psi}\gamma_\mu \psi, \tag{5.5}$$

where g_σ and g_ω are the coupling constants between the meson fields and the nucleon fields. Finally, the total Lagrangian density is the sum of Lagrangian density of the free fields (5.1), (5.2), and (5.3), and their interactions (5.5)

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{\text{nucleon}} + \mathcal{L}_\sigma + \mathcal{L}_\omega + \mathcal{L}_{\text{int}} \\ &= \bar{\psi}\left[i\gamma_\mu\left(\partial^\mu + ig_\omega\omega^\mu\right) - \left(m - g_\sigma\sigma\right)\right]\psi \\ &\quad + \frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma - m_\sigma^2\sigma^2\right) - \frac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{2}m_\omega^2\omega_\mu\omega^\mu.\end{aligned} \tag{5.6}$$

Using the Lagrangian density (5.6) in the Euler-Lagrange equations (3.22) we find the equations of motion of the system. Inserting each of the fields σ, ω, and $\bar{\psi}$, we find the equations of motion

$$\left(\Box + m_\sigma^2\right)\sigma(x) = g_\sigma \bar{\psi}(x)\psi(x), \tag{5.7}$$

$$\left(\Box + m_\omega^2\right)\omega_\mu(x) - \partial_\mu\partial^\nu\omega_\nu(x) = g_\omega\bar{\psi}(x)\gamma_\mu\psi(x), \tag{5.8}$$

$$\Big\{\gamma_\mu\big[i\partial^\mu - g_\omega\omega^\mu(x)\big] - \big[m - g_\sigma\sigma(x)\big]\Big\}\psi(x) = 0, \tag{5.9}$$

where $\Box = \partial_\mu\partial^\mu$ is the d'Alembert operator and we have written the x-dependence explicitly. We recognize the similarity between the last equation (5.9) and the Dirac equation (4.44).

Our system consists of static, uniform matter in its ground state. We can therefore use a relativistic mean-field approximation to solve the equations of motions (5.7), (5.8), and (5.9). We replace the meson fields by their expectation values in this state. This removes the quantum fluctuations and the meson fields can therefore be treated as classical fields, instead of quantized. We write the replacements of the meson fields with their expectation values as $\sigma(x) \rightarrow \langle\sigma\rangle$ for the scalar meson and $\omega_\mu(x) \rightarrow \langle\omega_\mu\rangle$ for the vector meson. Also, since matter is static and uniform, the source currents $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\psi$ are independent of x, and they can be replaced by their ground-state expectation values $\langle\bar{\psi}\psi\rangle$ and $\langle\bar{\psi}\gamma^\mu\psi\rangle$. This means that also the expectation values of the meson fields are independent of x and all the derivatives in (5.7) and (5.8) vanish. The equation of motion for the expectation value of the σ meson field becomes

$$m_\sigma^2 \langle\sigma\rangle = g_\sigma \langle\bar{\psi}\psi\rangle . \tag{5.10}$$

We can split the equation of motion (5.8) for the expectation value of the ω meson field into a temporal and spacial part, such that

$$m_\omega^2 \langle\omega_0\rangle = g_\omega \langle\bar{\psi}\gamma_0\psi\rangle = g_\omega \langle\psi^\dagger\psi\rangle , \tag{5.11}$$

$$m_\omega^2 \langle\omega_i\rangle = g_\omega \langle\bar{\psi}\gamma_i\psi\rangle . \tag{5.12}$$

Inserting the expectation values of the meson fields in equation (5.9), we find

$$\Big[\gamma_\mu\big(i\partial^\mu - g_\omega \langle\omega^\mu\rangle\big) - \big(m - g_\sigma \langle\sigma\rangle\big)\Big]\psi(x) = 0. \tag{5.13}$$

We see that the terms in the square brackets do not depend on x, which means that we have translation invariance. We can rewrite (5.13) as an eigenvalue equation for momentum. Then the nucleon fields are momentum eigenstates, such that $\psi(x) = \psi(p)e^{-ip\cdot x}$, where $p = p_\mu = (p_0, \mathbf{p})$ is the four-momentum and $p \cdot x = p_\mu x^\mu$. We insert this into (5.13) and get

$$\Big[\gamma_\mu\big(p^\mu - g_\omega \langle\omega^\mu\rangle\big) - \big(m - g_\sigma \langle\sigma\rangle\big)\Big]\psi(p) = 0. \tag{5.14}$$

We define the expressions in the parentheses as

$$P^\mu = p^\mu - g_\omega \langle\omega^\mu\rangle , \tag{5.15}$$

$$m^* = m - g_\sigma \langle\sigma\rangle , \tag{5.16}$$

where m^* is known as the Dirac effective mass. Using these definitions in (5.14) and multiplying by $(\not{P} + m^*)$, leads to the eigenvalue equation

$$(P_\mu P^\mu - m^{*2})\psi(P) = 0, \tag{5.17}$$

which gives

$$P_0 = \sqrt{\mathbf{P}^2 + m^{*2}}. \tag{5.18}$$

We then have

$$p_0 = P_0 + g_\omega \langle\omega_0\rangle . \tag{5.19}$$

It is common to denote $P_0 = E(\mathbf{p})$ and $p_0 = e(\mathbf{p})$, such that the energy is

$$E(\mathbf{p}) = \sqrt{\left(\mathbf{p} - g_\omega \langle\omega_i\rangle\right)^2 + \left(m - g_\sigma \langle\sigma\rangle\right)^2}, \tag{5.20}$$

and the eigenvalues of the 3-momentum $\mathbf{p}$ for particles and antiparticles are

$$e(\mathbf{p}) = E(\mathbf{p}) + g_\omega \langle \omega_0 \rangle \,, \tag{5.21}$$
$$\bar{e}(\mathbf{p}) = E(\mathbf{p}) - g_\omega \langle \omega_0 \rangle \,. \tag{5.22}$$

Next, we want to find expressions for the source currents $\langle \bar{\psi}\psi \rangle$ and $\langle \bar{\psi}\gamma^\mu\psi \rangle$. We will use these to find the expectation values of the meson fields. The source currents can be found from the partition function of the model, which we derive in similar way as the partition function for Dirac fermions found in section 4.4. Since we are working with the expectation values of the meson fields, their derivatives vanish, and we are left with the following Lagrangian density

$$\begin{aligned}\mathcal{L}^{\text{mean}} = & \bar{\psi} \left[i\gamma_\mu \left(\partial^\mu + i g_\omega \langle \omega^\mu \rangle \right) - \left(m - g_\sigma \langle \sigma \rangle \right) \right] \psi \\ & - \frac{1}{2} m_\sigma^2 \langle \sigma \rangle^2 + \frac{1}{2} m_\omega^2 \langle \omega_\mu \rangle \langle \omega^\mu \rangle \,. \end{aligned} \tag{5.23}$$

We also include a chemical potential μ, which we from section 4.4 know should be coupled to the vector density $\bar{\psi}\gamma_0\psi$, such that

$$\begin{aligned}\mathcal{L}^{\text{mean}} = & \bar{\psi} \left[i\gamma_\mu \left(\partial^\mu + i g_\omega \langle \omega^\mu \rangle \right) - \left(m - g_\sigma \langle \sigma \rangle \right) + \gamma_0 \mu \right] \psi \\ & - \frac{1}{2} m_\sigma^2 \langle \sigma \rangle^2 + \frac{1}{2} m_\omega^2 \langle \omega_\mu \rangle \langle \omega^\mu \rangle \,. \end{aligned} \tag{5.24}$$

The above Lagrangian density (5.24) is in Minkowski space. When working with the partition function in the imaginary time formalism, we need the Euclidean Lagrangian density, which we find by doing the substitution $\mathcal{L}_E = -\mathcal{L}_M(t \to -i\tau)$, such that

$$\begin{aligned}\mathcal{L}_E^{\text{mean}} = & \bar{\psi} \left[\gamma_0 \partial^\tau - i\gamma_i \partial^i + g_\omega \gamma_\mu \langle \omega^\mu \rangle + \left(m - g_\sigma \langle \sigma \rangle \right) - \gamma_0 \mu \right] \psi \\ & + \frac{1}{2} m_\sigma^2 \langle \sigma \rangle^2 - \frac{1}{2} m_\omega^2 \langle \omega_\mu \rangle \langle \omega^\mu \rangle \,. \end{aligned} \tag{5.25}$$

Using (4.61) and the general expression for the path integral representation of the partition function of Grassmann fields (4.40), we find the partition function of the σ-ω model in the mean-field approximation

$$\begin{aligned} Z = & \int_{\text{BC}} \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp \Big\{ - \int_0^\beta d\tau \int d^3x \\ & \Big[\bar{\psi} \left[\gamma_0 \partial^\tau - i\gamma_i \partial^i + g_\omega \gamma_\mu \langle \omega^\mu \rangle + \left(m - g_\sigma \langle \sigma \rangle \right) - \gamma_0 \mu \right] \psi \\ & + \frac{1}{2} m_\sigma^2 \langle \sigma \rangle^2 - \frac{1}{2} m_\omega^2 \langle \omega_\mu \rangle \langle \omega^\mu \rangle \Big] \Big\}, \end{aligned} \tag{5.26}$$

where BC are the antiperiodic boundary conditions. The last two terms of the exponential are constants. We perform the integration over spacetime, which gives a factor β from integration over imaginary time and a factor V, equal to the volume of the system, from integration over space. The exponential can then be pulled outside the integral since the expectation values of the fields do not depend on $\bar{\psi}$ and ψ. We therefore get

$$
\begin{aligned}
Z = &\exp\left\{\beta V\left[-\frac{1}{2}m_\sigma^2\left\langle\sigma\right\rangle^2+\frac{1}{2}m_\omega^2\left\langle\omega_\mu\right\rangle\left\langle\omega^\mu\right\rangle\right]\right\} \\
&\times\int_{\mathrm{BC}}\mathcal{D}\bar{\psi}\mathcal{D}\psi\exp\Big\{-\int_0^\beta d\tau\int d^3x \\
&\bar{\psi}\left[\gamma_0\partial^\tau-i\gamma_i\partial^i+g_\omega\gamma_\mu\left\langle\omega^\mu\right\rangle+\left(m-g_\sigma\left\langle\sigma\right\rangle\right)-\gamma_0\mu\right]\psi\Big\}.
\end{aligned}
\tag{5.27}
$$

In order to perform the integral in (5.27) we go to Fourier space using the Fourier transforms of the nucleon fields, given by equation (4.68). The argument of the exponential in (5.27) then becomes

$$
\begin{aligned}
&\frac{1}{V}\int_0^\beta d\tau\int d^3x\sum_{n,n'}\sum_{\mathbf{p},\mathbf{q}}e^{i(\omega_n-\omega_{n'})\tau}e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}} \\
&\times\tilde{\bar{\psi}}\left[\gamma_0\partial^\tau-i\gamma_i\partial^i+g_\omega\gamma_\mu\left\langle\omega^\mu\right\rangle+\left(m-g_\sigma\left\langle\sigma\right\rangle\right)-\gamma_0\mu\right]\tilde{\psi} \\
=&\sum_{n,\mathbf{p}}\tilde{\bar{\psi}}\beta\left[i\gamma_0\omega_n+\boldsymbol{\gamma}\cdot\mathbf{p}+g_\omega\gamma_\mu\left\langle\omega^\mu\right\rangle+\left(m-g_\sigma\left\langle\sigma\right\rangle\right)-\gamma_0\mu\right]\tilde{\psi},
\end{aligned}
\tag{5.28}
$$

and the integral in (5.27) becomes

$$
\begin{aligned}
&\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\exp\Big\{-\int_0^\beta d\tau\int d^3x \\
&\bar{\psi}\left[\gamma_0\partial^\tau-i\gamma_i\partial^i+g_\omega\gamma_\mu\left\langle\omega^\mu\right\rangle+\left(m-g_\sigma\left\langle\sigma\right\rangle\right)-\gamma_0\mu\right]\psi\Big\} \\
=&\int\prod_{n,\mathbf{p}}\mathcal{D}\tilde{\bar{\psi}}\mathcal{D}\tilde{\psi}\exp\Big\{-\sum_{n,\mathbf{p}} \\
&\tilde{\bar{\psi}}\beta\left[i\gamma_0\omega_n+\boldsymbol{\gamma}\cdot\mathbf{p}+g_\omega\gamma_\mu\left\langle\omega^\mu\right\rangle+\left(m-g_\sigma\left\langle\sigma\right\rangle\right)-\gamma_0\mu\right]\tilde{\psi}\Big\}.
\end{aligned}
\tag{5.29}
$$

The integral (5.29) now has the form of equation (4.31) and we need only find the determinant of the argument of its exponential. Written out in matrix form, we see that the argument M is equal to the matrix (4.75) with

$$
\begin{aligned}
a =&i\omega_n+g_\omega\left\langle\omega_0\right\rangle+m-g_\sigma\left\langle\sigma\right\rangle-\mu, \\
b =&p_3+g_\omega\left\langle\omega_3\right\rangle, \\
c =&p_1-ip_2+g_\omega\left\langle\omega_1\right\rangle-ig_\omega\left\langle\omega_2\right\rangle, \\
d =&p_1+ip_2+g_\omega\left\langle\omega_1\right\rangle+ig_\omega\left\langle\omega_2\right\rangle, \\
e =&-i\omega_n-g_\omega\left\langle\omega_0\right\rangle+m-g_\sigma\left\langle\sigma\right\rangle+\mu.
\end{aligned}
\tag{5.30}
$$

Using equation (4.76), we find the determinant over the Dirac indices

$$\begin{aligned}\det M &= \beta^4 \Big[\left(i\omega_n + g_\omega \langle\omega_0\rangle + m - g_\sigma \langle\sigma\rangle - \mu\right)\left(-i\omega_n - g_\omega \langle\omega_0\rangle + m - g_\sigma \langle\sigma\rangle + \mu\right) \\ &\qquad + \left(p_1 - ip_2 + g_\omega \langle\omega_1\rangle - ig_\omega \langle\omega_2\rangle\right)\left(p_1 + ip_2 + g_\omega \langle\omega_1\rangle + ig_\omega \langle\omega_2\rangle\right) \\ &\qquad + \left(p_3 + g_\omega \langle\omega_3\rangle\right)^2 \Big]^2 \\ &= \beta^4 \Big[(m - g_\sigma \langle\sigma\rangle)^2 - (i\omega_n + g_\omega \langle\omega_0\rangle - \mu)^2 + (\mathbf{p} + g_\omega \langle\omega_i\rangle)^2 \Big]^2 \\ &= \beta^4 \Big[(i\omega_n + g_\omega \langle\omega_0\rangle - \mu)^2 - E(\mathbf{p})^2 \Big]^2, \end{aligned} \tag{5.31}$$

where we used (5.20) in the last step. We should also take the determinant over momentum space. Using equation (4.78) to do this, we find that the logarithm of the integral (5.29) is

$$\begin{aligned}\ln\det M &= \sum_{n,\mathbf{p}} \ln\left\{ \beta^4 \left[(i\omega_n + g_\omega \langle\omega_0\rangle - \mu)^2 - E(\mathbf{p})^2 \right]^2 \right\} \\ &= \sum_{n,\mathbf{p}} \Big\{ \ln\left[\beta^2 \left(\omega_n^2 + (E(\mathbf{p}) + g_\omega \langle\omega_0\rangle - \mu)^2 \right) \right] \\ &\qquad + \ln\left[\beta^2 \left(\omega_n^2 + (E(\mathbf{p}) - g_\omega \langle\omega_0\rangle + \mu)^2 \right) \right] \Big\}, \end{aligned} \tag{5.32}$$

where we repeated the steps (4.80) – (4.82) to obtain to the last expression. We recognize it as a Matsubara sum on the form (4.83), and use the result found in equation (4.94) with $a = E(\mathbf{p}) \pm g_\omega \langle\omega_0\rangle \mp \mu$. Again omitting the terms that do not depend on β, we are left with

$$\begin{aligned}\ln\det M &= 2\sum_{\mathbf{p}} \Big\{ \beta E(\mathbf{p}) + \ln\left[1 + e^{-\beta(E(\mathbf{p}) + g_\omega\langle\omega_0\rangle - \mu)} \right] \\ &\qquad + \ln\left[1 + e^{-\beta(E(\mathbf{p}) - g_\omega\langle\omega_0\rangle + \mu)} \right] \Big\} \\ &= 2\sum_{\mathbf{p}} \Big\{ \beta E(\mathbf{p}) + \ln\left[1 + e^{-\beta(e(\mathbf{p}) - \mu)} \right] + \ln\left[1 + e^{-\beta(\bar{e}(\mathbf{p}) + \mu)} \right] \Big\}, \end{aligned} \tag{5.33}$$

where we used (5.21) and (5.22) for the 3-momentum eigenvalues of particles and antiparticles. In the continuum limit $\sum_{\mathbf{p}} = V \int \frac{d^3p}{(2\pi)^3}$, we find that

$$\begin{aligned}\ln\det M = 2V \int \frac{d^3p}{(2\pi)^3} \Big\{ &\beta E(\mathbf{p}) \\ &+ \ln\left[1 + e^{-\beta(e(\mathbf{p}) - \mu)} \right] + \ln\left[1 + e^{-\beta(\bar{e}(\mathbf{p}) + \mu)} \right] \Big\}. \end{aligned} \tag{5.34}$$

Using this result, we find the logarithm of the partition function in (5.27) becomes

$$\begin{aligned}\ln Z = 2V &\int \frac{d^3p}{(2\pi)^3} \left\{ \beta E(\mathbf{p}) + \ln\left[1 + e^{-\beta(e(\mathbf{p}) - \mu)} \right] + \ln\left[1 + e^{-\beta(\bar{e}(\mathbf{p}) + \mu)} \right] \right\} \\ &+ \beta V \left[-\frac{1}{2} m_\sigma^2 \langle\sigma\rangle^2 + \frac{1}{2} m_\omega \langle\omega_\mu\rangle \langle\omega^\mu\rangle \right]. \end{aligned} \tag{5.35}$$

We are interested in the zero temperature limit of the system, i. e. $\beta \to \infty$. This yields

$$\frac{1}{\beta}\ln\left[1+e^{-\beta(e(\mathbf{p})-\mu)}\right] = \begin{cases} 0, & (e(\mathbf{p})-\mu) > 0 \\ [\mu - e(\mathbf{p})], & (e(\mathbf{p})-\mu) < 0 \end{cases} \tag{5.36}$$

$$\frac{1}{\beta}\ln\left[1+e^{-\beta(\bar{e}(\mathbf{p})+\mu)}\right] = \begin{cases} 0, & (\bar{e}(\mathbf{p})+\mu) > 0 \\ -[\bar{e}(\mathbf{p}) + \mu], & (\bar{e}(\mathbf{p})+\mu) < 0. \end{cases} \tag{5.37}$$

Again we assume $\mu > 0$, such that $\bar{e}(\mathbf{p}) + \mu$ is always larger than zero, the antiparticle contribution vanishes and we are left with

$$\begin{aligned} \frac{1}{\beta}\ln Z = 2V \int \frac{d^3p}{(2\pi)^3}\Big[E(\mathbf{p}) + (\mu - e(\mathbf{p}))\,\theta\,(\mu - e(\mathbf{p}))\Big] \\ + V\left[\frac{1}{2}m_\sigma^2 \langle\sigma\rangle^2 - \frac{1}{2}m_\omega^2 \langle\omega_\mu\rangle\langle\omega^\mu\rangle\right]. \end{aligned} \tag{5.38}$$

The first term of (5.38) involving $E(\mathbf{p})$ is divergent and we will for now omit it in our derivations. We next want to use equations (5.27), (5.29), and (5.38) in

$$\frac{\partial}{\partial\zeta}\frac{1}{\beta}\ln Z = \frac{1}{\beta}\frac{1}{Z}\frac{\partial Z}{\partial\zeta}, \tag{5.39}$$

where ζ is a suitable parameter. If we make the correct choices for ζ, we see that the right-hand side of (5.39) can give us the expectation values of the source currents $\langle\bar{\psi}\psi\rangle$ and $\langle\bar{\psi}\gamma^\mu\psi\rangle$. With $\zeta = \mu$, we find

$$\frac{1}{Z}\frac{\partial Z}{\partial\mu} = \beta V \sum_n \int \frac{d^3p}{(2\pi)^3}\tilde{\bar{\psi}}\gamma_0\tilde{\psi} = \beta V \langle\bar{\psi}\gamma_0\psi\rangle = \beta V \langle\psi^\dagger\psi\rangle, \tag{5.40}$$

$$\frac{\partial}{\partial\mu}\frac{1}{\beta}\ln Z = 2V\int\frac{d^3p}{(2\pi)^3}\theta(\mu - e(\mathbf{p})). \tag{5.41}$$

Inserting (5.40) and (5.41) into (5.39), we find that the expectation value of the zeroth component of the nucleon four-current $\langle\psi^\dagger\psi\rangle$, which is equal to the the nucleon density, is

$$\langle\psi^\dagger\psi\rangle = 2\int\frac{d^3p}{(2\pi)^3}\theta(\mu - e(\mathbf{p})) = \frac{1}{\pi^2}\int_0^{p_F} p^2 dp = \frac{p_F^3}{3\pi^2}. \tag{5.42}$$

Next with $\zeta = p^i$, we get

$$\frac{1}{Z}\frac{\partial Z}{\partial p^i} = -\beta V \sum_n \int \frac{d^3p}{(2\pi)^3}\tilde{\bar{\psi}}\gamma_i\tilde{\psi} = -\beta V \langle\bar{\psi}\gamma_i\psi\rangle, \tag{5.43}$$

$$\frac{\partial}{\partial p^i}\frac{1}{\beta}\ln Z = -2V\int\frac{d^3p}{(2\pi)^3}\frac{\partial e(\mathbf{p})}{\partial p^i}\theta(\mu - e(\mathbf{p})). \tag{5.44}$$

Since we have assumed a uniform system at rest, the expectation value of the nucleon current must be zero

$$\langle\bar{\psi}\gamma_i\psi\rangle = 0, \tag{5.45}$$

which means that the integral (5.44) must be zero. From the equation of motion (5.12), we see that this leads to the spacial components of the ω-meson field being zero; $\langle\omega_i\rangle = 0$. Finally, with $\zeta = m$, we find

$$\frac{1}{Z}\frac{\partial Z}{\partial m} = -\beta V \sum_n \int \frac{d^3p}{(2\pi)^3}\tilde{\bar{\psi}}\tilde{\psi} = -\beta V \left\langle \bar{\psi}\psi \right\rangle, \tag{5.46}$$

$$\frac{\partial}{\partial m}\frac{1}{\beta}\ln Z = -2V\int \frac{d^3p}{(2\pi)^3}\frac{\partial e(\mathbf{p})}{\partial m}\theta(\mu - e(\mathbf{p})). \tag{5.47}$$

Using the expression for $e(\mathbf{p})$ (5.21) and $\langle\omega_i\rangle = 0$, we find the expectation value of the scalar density

$$\langle\bar{\psi}\psi\rangle = \frac{1}{\pi^2}\int_0^{p_F} p^2 dp \frac{m - g_\sigma\langle\sigma\rangle}{\sqrt{p^2 + (m - g_\sigma\langle\sigma\rangle)^2}}. \tag{5.48}$$

We have found expressions for the expectation values of the source currents $\langle\psi^\dagger\psi\rangle$, $\langle\bar{\psi}\gamma^i\psi\rangle$, and $\langle\bar{\psi}\psi\rangle$, which we insert in the equations of motion (5.10), (5.11), and (5.12) to find expressions for the expectation values of the meson fields. This yields

$$\langle\sigma\rangle = \frac{g_\sigma}{m_\sigma^2}\frac{1}{\pi^2}\int_0^{p_F} p^2 dp \frac{m - g_\sigma\langle\sigma\rangle}{\sqrt{p^2 + (m - g_\sigma\langle\sigma\rangle)^2}}, \tag{5.49}$$

$$\langle\omega_0\rangle = \frac{g_\omega}{m_\omega^2}\frac{p_F^3}{3\pi^2}, \tag{5.50}$$

$$\langle\omega_i\rangle = 0. \tag{5.51}$$

We note that the $\langle\sigma\rangle$ field is determined by an integral equation depending on itself, which can be solved for a given p_F.

Now that we have found the expectation values of the meson fields, we can use the partition function to find the pressure of the system. Inserting (5.38) in equation (2.3), we find

$$P = 2\int \frac{d^3p}{(2\pi)^3}\Big[E(\mathbf{p}) + (\mu - e(\mathbf{p}))\,\theta\,(\mu - e(\mathbf{p}))\Big] - \frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_\mu\rangle\langle\omega^\mu\rangle. \tag{5.52}$$

Comparing to the pressure of an ideal Fermi gas (4.101), we see that including nuclear interactions shifts the energy of the fermions and adds contributions from the meson fields. We can also find the pressure by using the energy-momentum tensor. In general, the energy-momentum tensor is given as

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\partial^\nu\phi_i - n^{\mu\nu}\mathcal{L}, \tag{5.53}$$

where ϕ_i are the fields of the theory. With the Lagrangian density given by (5.23), only the ψ-field will contribute to the first term of (5.53). Using that $i\partial_\nu\psi = p_\nu\psi$, we find that the first term becomes $\langle\bar{\psi}\gamma^\mu p^\nu\psi\rangle$. For the second term, we use the Minkowski metric

$n^{\mu\nu} = g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Using the equation of motion for the Dirac field (5.13), we see that the first term of the Lagrangian density is zero. We also know that the spacial part of the ω-meson field is zero. We are then left with just the contributions from the $\langle\sigma\rangle$ and $\langle\omega_0\rangle$ fields. This means that the energy-momentum tensor is

$$T^{\mu\nu} = \langle\bar{\psi}\gamma^\mu p^\nu \psi\rangle - g^{\mu\nu}\left(-\frac{1}{2}m_\sigma^2 \langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2 \langle\omega_0\rangle^2\right). \tag{5.54}$$

If we work in the rest frame of the matter and assume that it is a perfect fluid, the energy-momentum tensor is diagonal and equal to

$$T^{\mu\nu} = \begin{pmatrix} \mathcal{E} & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}, \tag{5.55}$$

where $\mathcal{E}$ is the energy density and P is the pressure. Combining (5.54) and (5.55), we find expressions for the energy density and pressure;

$$\mathcal{E} = \langle\bar{\psi}\gamma^0 p^0 \psi\rangle + \frac{1}{2}m_\sigma^2 \langle\sigma\rangle^2 - \frac{1}{2}m_\omega^2 \langle\omega_0\rangle^2, \tag{5.56}$$

$$P = \frac{1}{3}\langle\bar{\psi}\gamma^i p^i \psi\rangle - \frac{1}{2}m_\sigma^2 \langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2 \langle\omega_0\rangle^2, \tag{5.57}$$

where the factor $1/3$ is due to that the pressure occurs three times in (5.55). We find the expectation values in (5.56) and (5.57) by using the derivations of the source currents in equations (5.42), (5.43), and (5.44). We then find

$$\begin{aligned} \langle\bar{\psi}\gamma^0 p^0 \psi\rangle &= \frac{1}{\pi^2}\int_0^{p_F} dp\, p^2 e(p) \\ &= m_\omega^2 \langle\omega_0\rangle^2 + \frac{1}{\pi^2}\int_0^{p_F} dp\, p^2 \sqrt{p^2 + (m - g_\sigma \langle\sigma\rangle)^2}, \end{aligned} \tag{5.58}$$

where we used (5.11) and (5.21), and

$$\begin{aligned} \langle\bar{\psi}\gamma^i p^i \psi\rangle &= \frac{1}{\pi^2}\int_0^{p_F} p^2 dp \frac{\partial e(p)}{\partial p_i} p_i \\ &= \frac{1}{\pi^2}\int_0^{p_F} dp\, \frac{p^4}{\sqrt{p^2 + (m - g_\sigma \langle\sigma\rangle)^2}}. \end{aligned} \tag{5.59}$$

Using equations (5.58) and (5.59) in equations (5.56) and (5.57), the energy density and pressure become

$$\mathcal{E} = \frac{1}{\pi^2}\int_0^{p_F} dp\, p^2 \sqrt{p^2 + (m - g_\sigma \langle\sigma\rangle)^2} + \frac{1}{2}m_\sigma^2 \langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2 \langle\omega_0\rangle^2, \tag{5.60}$$

$$P = \frac{1}{3}\frac{1}{\pi^2}\int_0^{p_F} dp\, \frac{p^4}{\sqrt{p^2 + (m - g_\sigma \langle\sigma\rangle)^2}} - \frac{1}{2}m_\sigma^2 \langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2 \langle\omega_0\rangle^2. \tag{5.61}$$

With expressions for both the energy density and the pressure we can find the equation of state that gives the relation between the two. We also notice that the expressions are similar to the ones for a free fermion gas, but in addition contains finite contributions from the expectation values of the meson fields. The expressions are for pure neutron matter, but differs only from the expressions for symmetric nuclear matter by a factor 2.

5.2 Scalar self-interactions

This section is based on reference [13].

We next extend the σ-ω model by including scalar self-interactions for the scalar meson field σ. This model is sometimes called the nonlinear σ-ω model. Their contribution to the Lagrangian density are cubic and quartic

$$\mathcal{L}_{I,\sigma} = -\frac{1}{3}bm(g_\sigma\sigma)^3 - \frac{1}{4}c(g_\sigma\sigma)^4, \tag{5.62}$$

where b and c are dimensionless constants. The full Lagrangian density of the system is then

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_{\text{nucleon}} + \mathcal{L}_\sigma + \mathcal{L}_\omega + \mathcal{L}_{\text{int}} + \mathcal{L}_{I,\sigma} \\ &= \bar{\psi}\left[i\gamma_\mu\left(\partial^\mu + ig_\omega\omega^\mu\right) - (m - g_\sigma\sigma)\right]\psi + \frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma - m_\sigma^2\sigma^2\right) \\ &\quad - \frac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{2}m_\omega^2\omega_\mu\omega^\mu - \frac{1}{3}bm(g_\sigma\sigma)^3 - \frac{1}{4}c(g_\sigma\sigma)^4.\end{aligned} \tag{5.63}$$

We repeat our derivations from section 5.1 to find expressions for the expectation values of the meson fields, the energy density and the pressure in the mean-field approximation. The equation of motion for the expectation value of the σ-meson field becomes

$$\begin{aligned}\langle\sigma\rangle = \frac{g_\sigma}{m_\sigma^2}\Bigg[&\frac{1}{\pi^2}\int_0^{p_F} p^2 dp \frac{m - g_\sigma\langle\sigma\rangle}{\sqrt{p^2 + (m - g_\sigma\langle\sigma\rangle)^2}} \\ &- bmg_\sigma^2\langle\sigma\rangle^2 - cg_\sigma^3\langle\sigma\rangle^3\Bigg],\end{aligned} \tag{5.64}$$

while the equations of motion for the temporal and spacial parts of the expectation value of the ω-meson field remain the same

$$\langle\omega_0\rangle = \frac{g_\omega}{m_\omega^2}\frac{p_F^3}{3\pi^2}, \tag{5.65}$$

$$\langle\omega_i\rangle = 0. \tag{5.66}$$

The energy density and pressure become

$$\mathcal{E} = \frac{1}{\pi^2}\int_0^{p_F} dp\, p^2 \sqrt{p^2 + (m - g_\sigma \langle\sigma\rangle)^2} + \frac{1}{2}m_\sigma^2 \langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2 \langle\omega_0\rangle^2 + \frac{1}{3}bmg_\sigma^3 \langle\sigma\rangle^3 + \frac{1}{4}cg_\sigma^4 \langle\sigma\rangle^4, \quad (5.67)$$

$$P = \frac{1}{3}\frac{1}{\pi^2}\int_0^{p_F} dp\, \frac{p^4}{\sqrt{p^2 + (m - g_\sigma \langle\sigma\rangle)^2}} - \frac{1}{2}m_\sigma^2 \langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2 \langle\omega_0\rangle^2 - \frac{1}{3}bmg_\sigma^3 \langle\sigma\rangle^3 - \frac{1}{4}cg_\sigma^4 \langle\sigma\rangle^4, \quad (5.68)$$

from which we see that the scalar self-interactions lead to additional finite terms to the energy density and pressure.

5.3 Isospin force

This section is based on references [13] and [27].

In the σ-ω model we considered pure neutron matter. It can also be used for symmetric nuclear matter, meaning that we assume the masses and chemical potentials of the neutrons and protons to be identical. Inclusion of isospin force allows for antisymmetric nuclear matter, which makes it possible to distinguish neutrons and protons. In particular, since the chemical potentials can be different, we can have different proton and neutron densities. This is usually the case for a neutron star, which mainly consists of neutrons and only a small fraction of protons. This model is sometimes called the σ-ω-ρ model.

We introduce the isospin force through the ρ meson, which is a triplet of mesons ρ^0, $\rho^\pm$ with the same mass and charges $(0, \pm 1)$. We describe the mesons by a charged vector field $\boldsymbol{\rho}^\mu$, whose isospin components are

$$\boldsymbol{\rho}^\mu = (\rho_1^\mu, \rho_2^\mu, \rho_3^\mu). \quad (5.69)$$

The Lagrangian density of the free ρ meson is

$$\mathcal{L}_\rho = \frac{1}{4}\boldsymbol{\rho}_{\mu\nu} \cdot \boldsymbol{\rho}^{\mu\nu} + \frac{1}{2}m_\rho^2 \boldsymbol{\rho}_\mu \cdot \boldsymbol{\rho}^\mu, \quad (5.70)$$

where m_ρ is the mass of the ρ mesons and we define

$$\boldsymbol{\rho}_{\mu\nu} = \partial_\mu \boldsymbol{\rho}_\nu - \partial_\nu \boldsymbol{\rho}_\mu, \quad (5.71)$$

as for the ω vector meson. We can form a real and two complex fields from the three isospin components. We construct the two complex fields as

$$\rho_\pm^\mu = \frac{1}{\sqrt{2}}(\rho_1^\mu \pm i\rho_2^\mu), \quad (5.72)$$

which means that they are each others complex conjugates. These can be seen as combinations of the raising and lowering operators for charged ρ mesons. The ρ meson has an interaction term in the Lagrangian density

$$\mathcal{L}_{\text{isospin}} = -g_\rho \boldsymbol{\rho}_\nu \cdot \mathbf{I}^\nu, \tag{5.73}$$

where g_ρ is a coupling constant and $\mathbf{I}^\nu$ is the total conserved isospin current, which contains contributions from both the nucleon and the ρ fields. We find these conserved isospin current using Noether's theorem from 3.4.

If we use the free Lagrangian density $\mathcal{L}_\rho$ of the ρ meson to find the isospin current, we will find that it contains a derivative. This derivative will in itself also make a contribution to the current. We should therefore add an extra term

$$\mathcal{L}_{\text{int},\rho} = -g_\rho (\boldsymbol{\rho}^\nu \times \boldsymbol{\rho}^\mu) \cdot \boldsymbol{\rho}_{\nu\mu} \tag{5.74}$$

to the Lagrangian density of the ρ meson when finding the isospin current. To find the isospin current of the ρ meson, we use that the Lagrangian density is invariant under a rotation about the 3-axis in isospin space, which can be written as

$$\boldsymbol{\rho}_\mu \to \boldsymbol{\rho}_\mu + \boldsymbol{\Lambda} \times \boldsymbol{\rho}_\mu, \tag{5.75}$$

where $\boldsymbol{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3)$ is an infinitesimal vector in isospin space. Using Noether's theorem we find that the conserved current, which is the conserved isospin current for the ρ meson, is

$$\mathbf{I}^\nu_\rho = \boldsymbol{\rho}_\mu \times \frac{\delta(\mathcal{L}_\rho + \mathcal{L}_{\text{int},\rho})}{\delta(\partial_\nu \boldsymbol{\rho}_\mu)} = \boldsymbol{\rho}_\mu \times \boldsymbol{\rho}^{\nu\mu} - 2g_\rho \boldsymbol{\rho}_\mu \times (\boldsymbol{\rho}^\nu \times \boldsymbol{\rho}^\mu). \tag{5.76}$$

We find the isospin current of the nucleons by considering the continuous phase transformation

$$\psi \to \psi' = e^{-i\boldsymbol{\tau}\cdot\boldsymbol{\Lambda}/2}\psi, \tag{5.77}$$

where $\boldsymbol{\tau}$ is the vector of the three Pauli isospin matrices. An infinite transformation leads to a change $\delta\psi$ in the field given by

$$\psi' = (1 - \frac{i}{2}\Lambda \cdot \boldsymbol{\tau})\psi = \psi + \delta\psi. \tag{5.78}$$

We use with Noether's theorem and insert in (3.45)

$$\boldsymbol{\Lambda} \cdot \mathbf{j}^\nu = \frac{\delta \mathcal{L}_{\text{nucleon}}}{\delta \partial_\nu \psi}\delta\psi = \bar{\psi} i\gamma^\nu (-\frac{i}{2}\Lambda \cdot \boldsymbol{\tau}\psi) = \frac{1}{2}\bar{\psi}\gamma^\nu \boldsymbol{\Lambda} \cdot \boldsymbol{\tau}\psi, \tag{5.79}$$

which means that the conserved Noether current, which is the conserved isospin current for the nucleons, is

$$\mathbf{I}^\nu_{\text{nucleon}} = \mathbf{j}^\nu = \frac{1}{2}\bar{\psi}\gamma^\nu \boldsymbol{\tau}\psi. \tag{5.80}$$

The total conserved isospin current is thereby

$$\mathbf{I}^{\nu} = \frac{1}{2}\bar{\psi}\gamma^{\nu}\boldsymbol{\tau}\psi + \boldsymbol{\rho}_{\mu} \times \boldsymbol{\rho}^{\nu\mu} + 2g_{\rho}(\boldsymbol{\rho}^{\nu} \times \boldsymbol{\rho}^{\mu}) \times \boldsymbol{\rho}_{\mu}, \tag{5.81}$$

containing both terms for the nucleons and the ρ meson.

Again, we want to find the expressions for the expectation values of the meson fields, the energy density and the pressure in a mean-field approximation. In the mean-field approximation we assume that we are in the ground state of the system and that there are no fluctuations. Then, the expectation values of the charged ρ mesons vanish, which we see from (5.72), and we are left with the third component ρ_3^{μ} of the $\boldsymbol{\rho}^{\mu}$ field, which corresponds to the neutral ρ meson. As with the ω meson, the spacial part vanishes since the system is static and uniform. We are left with just the temporal part of the neutral meson field ρ_3^0, and only the nucleon contribution to the isospin current (5.81) will remain. The full Lagrangian density of the system is therefore

$$\begin{aligned}\mathcal{L} = \bar{\psi}&\left[i\gamma_{\mu}\left(\partial^{\mu} + ig_{\omega}\omega^{\mu}\right) - (m - g_{\sigma}\sigma) - \frac{1}{2}g_{\rho}\gamma_{\mu}\boldsymbol{\tau}\cdot\boldsymbol{\rho}^{\mu}\right]\psi \\ &+ \frac{1}{2}\left(\partial_{\mu}\sigma\partial^{\mu}\sigma - m_{\sigma}^2\sigma^2\right) - \frac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{2}m_{\omega}^2\omega_{\mu}\omega^{\mu} \\ &- \frac{1}{4}\boldsymbol{\rho}_{\mu\nu}\cdot\boldsymbol{\rho}^{\mu\nu} + \frac{1}{2}m_{\rho}^2\boldsymbol{\rho}_{\mu}\cdot\boldsymbol{\rho}^{\mu} - \frac{1}{3}bm(g_{\sigma}\sigma)^3 - \frac{1}{4}c(g_{\sigma}\sigma)^4.\end{aligned} \tag{5.82}$$

We repeat our derivations from section 5.1, remembering that we now instead of the Fermi momentum p_F, have two separate Fermi momenta p_n and p_p for the neutrons and the protons, respectively. The equation of motion for the expectation value of the σ field becomes

$$\begin{aligned}\langle\sigma\rangle = \frac{g_{\sigma}}{m_{\sigma}^2}&\left[\frac{1}{\pi^2}\int_0^{p_n} p^2 dp \frac{m - g_{\sigma}\langle\sigma\rangle}{\sqrt{p^2 + (m - g_{\sigma}\langle\sigma\rangle)^2}} + \frac{1}{\pi^2}\int_0^{p_p} p^2 dp \frac{m - g_{\sigma}\langle\sigma\rangle}{\sqrt{p^2 + (m - g_{\sigma}\langle\sigma\rangle)^2}}\right. \\ &\left. - bm(g_{\sigma}\langle\sigma\rangle)^2 - c(g_{\sigma}\langle\sigma\rangle)^3\right].\end{aligned} \tag{5.83}$$

While the equations of motion for the temporal and spacial parts of the expectation value of the ω-meson field become

$$\langle\omega_0\rangle = \frac{g_{\omega}}{m_{\omega}^2}\frac{1}{3\pi^2}\left(p_n^3 + p_p^3\right) = \frac{g_{\omega}}{m_{\omega}^2}\left(\rho_n + \rho_p\right), \tag{5.84}$$

$$\langle\omega_i\rangle = 0, \tag{5.85}$$

where we have used that the proton and neutron densities are

$$\rho_n = \frac{p_n^3}{3\pi^2}, \quad \rho_p = \frac{p_p^3}{3\pi^2}. \tag{5.86}$$

We also find equations of motion for the ρ meson

$$\langle\rho_3^0\rangle = \frac{1}{2}\frac{g_{\rho}}{m_{\rho}^2}\langle\bar{\psi}\gamma^0\tau_3\psi\rangle = \frac{1}{2}\frac{g_{\rho}}{m_{\rho}^2}(\rho_p - \rho_n), \tag{5.87}$$

$$\langle\rho_3^i\rangle = 0, \tag{5.88}$$

where we used that the nucleon field can be written as a bispinor for the fields of the proton and the neutron. The Dirac equation becomes

$$\left[\gamma_\mu\left(p^\mu - g_\omega\left\langle\omega^\mu\right\rangle - \frac{1}{2}g_\rho\tau_3\left\langle\rho_3^\mu\right\rangle\right) - \left(m - g_\sigma\left\langle\sigma\right\rangle\right)\right]\psi(p) = 0, \tag{5.89}$$

and its eigenvalues become

$$e_{I_3}(p) = g_\omega\left\langle\omega^0\right\rangle + g_\rho\left\langle\rho_3^0\right\rangle I_3 + E(p), \tag{5.90}$$

$$E(p) = \sqrt{p^2 + (m - g_\sigma\sigma)^2}, \tag{5.91}$$

where I_3 is the isospin, which is $\frac{1}{2}$ for protons and $-\frac{1}{2}$ for neutrons. We notice that the eigenvalues are shifted up or down depending on whether they are for protons or neutrons. Finally, we find the expressions for the energy density and the pressure

$$\begin{aligned}\mathcal{E} &= \frac{1}{\pi^2}\int_0^{p_n} dp\, p^2\sqrt{p^2 + (m - g_\sigma\left\langle\sigma\right\rangle)^2} + \frac{1}{\pi^2}\int_0^{p_p} dp\, p^2\sqrt{p^2 + (m - g_\sigma\left\langle\sigma\right\rangle)^2} \\ &+ \frac{1}{2}m_\sigma^2\left\langle\sigma\right\rangle^2 + \frac{1}{2}m_\omega^2\left\langle\omega_0\right\rangle^2 + \frac{1}{2}m_\rho^2\left\langle\rho_3^0\right\rangle^2 + \frac{1}{3}bmg_\sigma^3\left\langle\sigma\right\rangle^3 + \frac{1}{4}cg_\sigma^4\left\langle\sigma\right\rangle^4, \end{aligned} \tag{5.92}$$

$$\begin{aligned}P &= \frac{1}{3}\frac{1}{\pi^2}\int_0^{p_F} dp\,\frac{p^4}{\sqrt{p^2 + (m - g_\sigma\left\langle\sigma\right\rangle)^2}} + \frac{1}{3}\frac{1}{\pi^2}\int_0^{p_F} dp\,\frac{p^4}{\sqrt{p^2 + (m - g_\sigma\left\langle\sigma\right\rangle)^2}} \\ &- \frac{1}{2}m_\sigma^2\left\langle\sigma\right\rangle^2 + \frac{1}{2}m_\omega^2\left\langle\omega_0\right\rangle^2 + \frac{1}{2}m_\rho^2\left\langle\rho_3^0\right\rangle^2 - \frac{1}{3}bmg_\sigma^3\left\langle\sigma\right\rangle^3 - \frac{1}{4}cg_\sigma^4\left\langle\sigma\right\rangle^4. \end{aligned} \tag{5.93}$$

5.4 Electrons and muons

This section is based on reference [13].

In order for the neutron star as a whole to be electrically neutral, we need negative charges to balance the positive charges from the protons. We therefore introduce leptons, more specifically electrons e^- and muons μ^-, to the model. Since leptons are fermions, we can describe them with Dirac fields. Their Lagrangian density can therefore be written as

$$\mathcal{L}_{\mu,e} = \sum_{\lambda=\mu,e}\bar{\psi}_\lambda(i\gamma_\mu\partial^\mu - m_\lambda)\psi_\lambda, \tag{5.94}$$

where m_μ and m_e are the muon and electron masses. The introduction of muons and electrons lead to two new Fermi momenta: p_e and p_μ. The chemical potentials of the electrons μ_e and muons μ_μ should be equal to assure that the reaction $e^- \to \mu^- + \nu_e + \bar{\nu}_\mu$ is in equilibrium, such that

$$\sqrt{m_e^2 + p_e^2} = \mu_e = \mu_\mu = \sqrt{m_\mu^2 + p_\mu^2}. \tag{5.95}$$

Adding the Lagrangian density of electrons and muons (5.94) to the σ-ω-ρ model in the mean-field approximation leaves the expressions for the expectation values of the meson fields unchanged, so we can still use equations (5.83), (5.84), (5.85), (5.87), and (5.88). The energy density and pressure of this model become

$$\begin{aligned}
\mathcal{E} &= \frac{1}{\pi^2}\int_0^{p_n} dp\, p^2\sqrt{p^2+(m-g_\sigma\langle\sigma\rangle)^2} + \frac{1}{\pi^2}\int_0^{p_p} dp\, p^2\sqrt{p^2+(m-g_\sigma\langle\sigma\rangle)^2} \\
&+ \frac{1}{\pi^2}\int_0^{p_e} dp\, p^2\sqrt{p^2+m_e^2} + \frac{1}{\pi^2}\int_0^{p_\mu} dp\, p^2\sqrt{p^2+m_\mu^2} \\
&+ \frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 + \frac{1}{2}m_\rho^2\langle\rho_3^0\rangle^2 + \frac{1}{3}bm(g_\sigma\langle\sigma\rangle)^3 + \frac{1}{4}c(g_\sigma\langle\sigma\rangle)^4, \quad (5.96) \\
P &= \frac{1}{3}\frac{1}{\pi^2}\int_0^{p_F} dp\,\frac{p^4}{\sqrt{p^2+(m-g_\sigma\langle\sigma\rangle)^2}} + \frac{1}{3}\frac{1}{\pi^2}\int_0^{p_F} dp\,\frac{p^4}{\sqrt{p^2+(m-g_\sigma\langle\sigma\rangle)^2}} \\
&+ \frac{1}{3}\frac{1}{\pi^2}\int_0^{p_e} dp\,\frac{p^4}{\sqrt{p^2+m_e^2}} + \frac{1}{3}\frac{1}{\pi^2}\int_0^{p_\mu} dp\,\frac{p^4}{\sqrt{p^2+m_\mu^2}} \\
&- \frac{1}{2}m_\sigma^2\langle\sigma\rangle^2 + \frac{1}{2}m_\omega^2\langle\omega_0\rangle^2 + \frac{1}{2}m_\rho^2\langle\rho_3^0\rangle^2 - \frac{1}{3}bm(g_\sigma\langle\sigma\rangle)^3 - \frac{1}{4}c(g_\sigma\langle\sigma\rangle)^4. \quad (5.97)
\end{aligned}$$

To be able to solve this system, we must impose several restrictions. Firstly, we require electrical neutrality, which means that the proton density should be equal to the sum of the electron density ρ_e and muon density ρ_μ

$$\rho_p = \rho_e + \rho_\mu. \tag{5.98}$$

The electron density and muon density are given by

$$\rho_e = \frac{p_e^3}{3\pi^2}, \quad \rho_\mu = \frac{p_\mu^3}{3\pi^2}, \tag{5.99}$$

so the charge neutrality can also be written in terms of the Fermi momenta

$$p_p^3 = p_e^3 + p_\mu^3. \tag{5.100}$$

We also require baryon charge to be conserved, which puts a constraint on the chemical potentials

$$\mu_p = \mu_n - \mu_e. \tag{5.101}$$

This constraint is also required for the reaction $n \to p + e^- + \bar{\nu}_e$ to be in equilibrium. With these constraints, it is possible to determine the density fraction and Fermi momenta of each type of particle for a given density of the system. We can then calculate the energy density and pressure for a range of densities to find the equation of state.

Chapter 6

Numerical Solutions

In chapter 5 we found expressions for the pressure and energy density for a Fermi gas consisting of interacting Dirac fermions, which we now will calculate numerically to find the equation of state. We next use the equation of state to solve the Tolman-Oppenheimer-Volkoff equation and the mass continuity equation, which are derived in Appendix A. Finally, solving the equations for a range of central pressures, gives the mass-radius relation of neutron stars. All the numerical solutions in this thesis are performed with the mathematical computation program *Mathematica*. The code can be found in Appendix B.

6.1 Note on units

When working with quantum field theory and elementary particles, it is most common to use natural units, where $c = \hbar = 1$. The particles we are interested in, such as neutrons, protons and mesons, then have mass of order 10^2-10^3 MeV. However, when working with the TOV equation to find the mass of a neutron star, natural units are impractical, as the mass is on the order of $1M_\odot$ and the radius is on the order of 10 km. We will therefore, after having solved the system in terms of dimensionless variables, restore our results to SI units.

6.2 Dimensionless stellar structure equations

In order to solve the equations for stellar structure numerically, we write them in terms of dimensionless variables. We scale the mass M by the solar mass $M_\odot$ and the pressure P and the energy density $\mathcal{E}$ by $\mathcal{E}_0$, which we define as

$$\mathcal{E}_0 = \frac{m^4 c^5}{3\pi^2 \hbar^3}, \tag{6.1}$$

where m is the neutron mass. This constant shows up when working with ideal Fermi gases and is therefore a natural choice. The dimensionless variables are then

$$\bar{M} = \frac{M}{M_\odot}, \quad \bar{P} = \frac{P}{\mathcal{E}_0}, \quad \bar{\mathcal{E}} = \frac{\mathcal{E}}{\mathcal{E}_0}. \tag{6.2}$$

Inserting (6.2) into the TOV equation (A.38), we get

$$\frac{d\bar{P}}{dr} = -\frac{\left[\bar{\mathcal{E}}(r) + \bar{P}(r)\right]\left[\bar{M}(r) + \frac{4\pi\mathcal{E}_0}{M_\odot c^2} r^3 \bar{P}(r)\right]}{\frac{c^2}{GM_\odot} r^2 - 2\bar{M}(r) r}, \tag{6.3}$$

while the mass continuity equation (A.26) becomes

$$\frac{d\bar{M}}{dr} = \frac{4\pi\mathcal{E}_0}{M_\odot c^2} r^3 \bar{\mathcal{E}}(r). \tag{6.4}$$

We define two constants

$$\alpha = \frac{4\pi\mathcal{E}_0}{M_\odot c^2}, \quad R_0 = \frac{c^2}{GM_\odot}, \tag{6.5}$$

such that equations (6.3) and (6.4) can be written as

$$\frac{d\bar{P}}{dr} = -\frac{\left[\bar{\mathcal{E}}(r) + \bar{P}(r)\right]\left[\bar{M}(r) + \alpha r^3 \bar{P}(r)\right]}{R_0 r^2 - 2\bar{M}(r) r}, \tag{6.6}$$

$$\frac{d\bar{M}}{dr} = \alpha r^3 \bar{\mathcal{E}}(r). \tag{6.7}$$

6.3 Equation of state for σ - ω model

We calculate the equation of state for the σ-ω model with the energy density and pressure given by equations (5.60) and (5.61). The energy density and pressure are functions of the Fermi momentum p_F and the expectation values of the two meson fields $\langle\sigma\rangle$ and $\langle\omega_0\rangle$, which also are functions of the Fermi momentum and given by equations (5.49) and (5.50). The system can therefore be solved for a given p_F, first finding the meson fields, then inserting the meson fields into the expressions for the energy density and pressure. Repeating the procedure for a suitable range of Fermi momenta, relating the corresponding pressures and energy densities, gives the equation of state.

To solve the system numerically, we also write the equation of state in terms of dimensionless variables. We scale the Fermi momentum by the neutron mass and the meson fields by the neutron mass and their respective coupling constants, such that

$$\bar{p}_F = \frac{p_F}{m}, \quad \bar{\sigma} = \frac{g_\sigma \langle\sigma\rangle}{m}, \quad \bar{\omega}_0 = \frac{g_\omega \langle\omega_0\rangle}{m}. \tag{6.8}$$

The coupling constants are parameters of the theory and should be chosen so that the saturation density and binding per energy is in accord with empirical values. In our computation we will use the coupling constants from [7], given as

$$g_\sigma = 9.569, \quad g_\omega = 11.665. \tag{6.9}$$

Using the dimensionless variables (6.8), we can write the two equations for the meson fields as

$$\bar{\sigma} = \frac{g_\sigma^2}{m_\sigma^2}\frac{m^2}{\pi^2}\int_0^{\bar{p}_F} \bar{p}^2 d\bar{p}\frac{1-\bar{\sigma}}{\sqrt{\bar{p}^2+(1-\bar{\sigma})^2}}, \tag{6.10}$$

$$\bar{\omega}_0 = \frac{g_\omega^2}{m_\omega^2}\frac{m^2}{3\pi^2}\bar{p}_F^3. \tag{6.11}$$

As for the TOV equation, we scale the energy density and pressure by $\mathcal{E}_0$, which in natural units become

$$\mathcal{E}_0 = \frac{m^4}{3\pi^3}. \tag{6.12}$$

The dimensionless expressions for the energy density and pressure then become

$$\bar{\mathcal{E}} = \frac{1}{\mathcal{E}_0}\left(\frac{m^4}{\pi^2}\int_0^{\bar{p}_F} d\bar{p}\,\bar{p}^2\sqrt{\bar{p}^2+(1-\bar{\sigma})^2} + \frac{1}{2}\frac{m_\sigma^2 m^2}{g_\sigma^2}\bar{\sigma}^2 + \frac{1}{2}\frac{m_\omega^2 m^2}{g_\omega}\bar{\omega}_0^2\right), \tag{6.13}$$

$$\bar{P} = \frac{1}{\mathcal{E}_0}\left(\frac{m^4}{3\pi^2}\int_0^{\bar{p}_F} d\bar{p}\,\frac{\bar{p}^4}{\sqrt{\bar{p}^2+(1-\bar{\sigma})^2}} - \frac{1}{2}\frac{m_\sigma^2 m^2}{g_\sigma^2}\langle\sigma\rangle^2 + \frac{1}{2}\frac{m_\omega^2 m^2}{g_\omega^2}\bar{\omega}_0^2\right), \tag{6.14}$$

which we solve numerically to find the equation of state.

We first plot the pressure as a function of the Fermi momentum in Figure 6.1. As seen from the figure, the pressure is negative for Fermi momenta in the range $0.2 < \bar{p}_F < 0.3$. These negative pressures mean that also the equation of state will have negative values, as seen in Figure 6.2. To resolve this problem, we replace the low values of the equation of state for the σ-ω model with the equation of state for a free Fermi gas, described by the only first terms of (6.13) and (6.14). The two solutions intersect at $\bar{P}_i = 0.00131$, so we make a new equation of state using the free Fermi gas for pressures lower than $\bar{P}_i$, and the σ-ω model for pressures higher than $\bar{P}_i$. The new equation of state in shown in Figure 6.3.

6.4 Mass-radius relation

Using the equation of state for the σ-ω model shown in Figure 6.3, we can solve the TOV equation (6.6) and the mass continuity equation (6.7) for a given dimensionless central pressure $\bar{P}_0$. Figures 6.4 and 6.5 show the dimensionless mass $\bar{M}(r) = M(r)/M_\odot$ and dimensionless pressure $\bar{P}(r) = P(r)/\mathcal{E}_0$ as functions of the radial distance for $\bar{P}_0 = 5$. The mass and radius of this neutron star are $M_{\text{star}} = 1.886 M_\odot$ and $R_{\text{star}} = 9.749$ km. We next solve the TOV equation and mass continuity equation for a range of dimensionless central pressures. We can then plot the neutron star radii and neutron star masses as functions of $\bar{P}_0$, as seen in Figures 6.6 and 6.7. Finally we find the relation between the mass and the radius of a neutron star in Figure 6.8. The curve has a maximum mass $M_{\max} = 2.595 M_\odot$ at radius $R_{\max} = 12.64$ km.

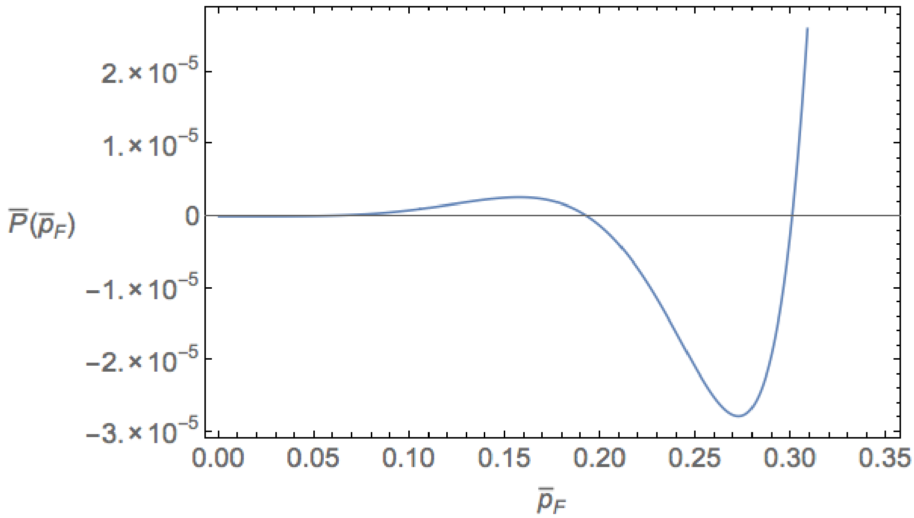

Figure 6.1: Dimensionless pressure $\bar{P}$ as a function of dimensionless Fermi momentum $\bar{p}_F$ for the σ-ω model.

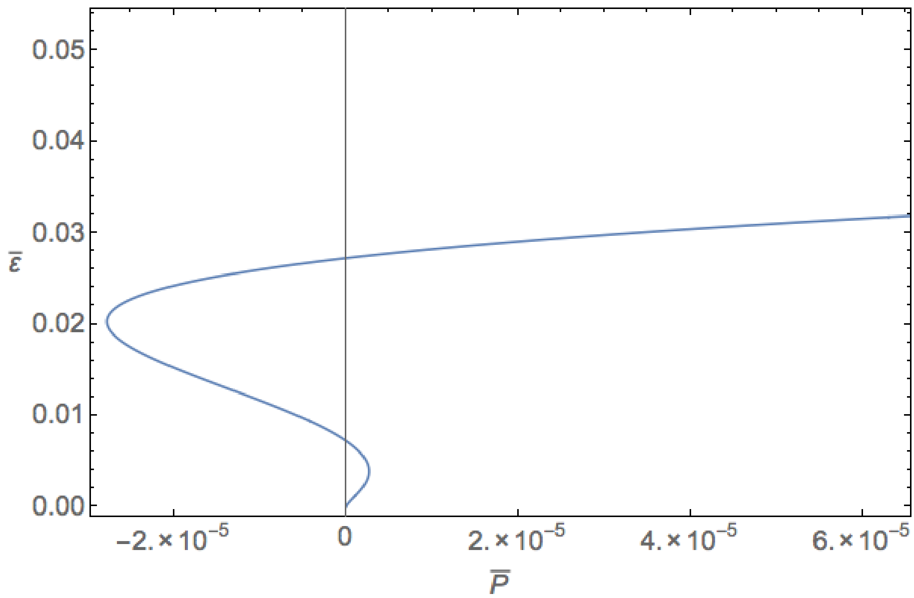

Figure 6.2: Dimensionless equation of state for the σ-ω model.

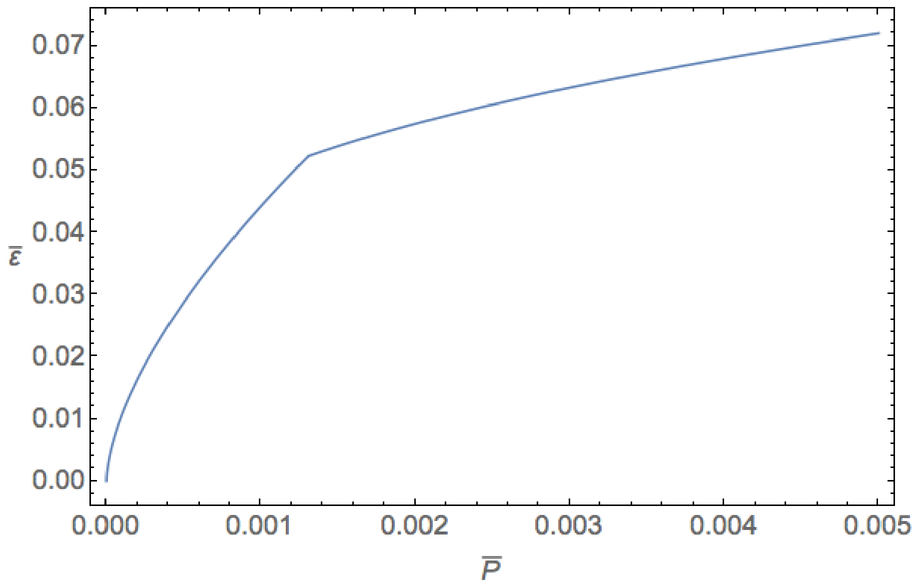

Figure 6.3: Combined dimensionless equation of state for a free Fermi gas and the σ-ω model.

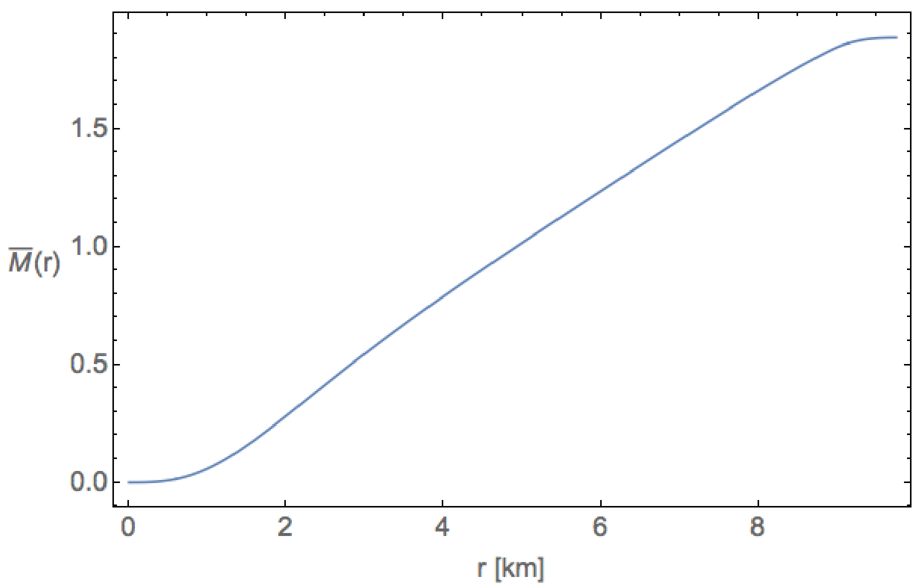

Figure 6.4: Dimensionless mass $\bar{M}(r)$ as a function of radial distance r in km for a neutron star with dimensionless central pressure $\bar{P}_0 = 5$.

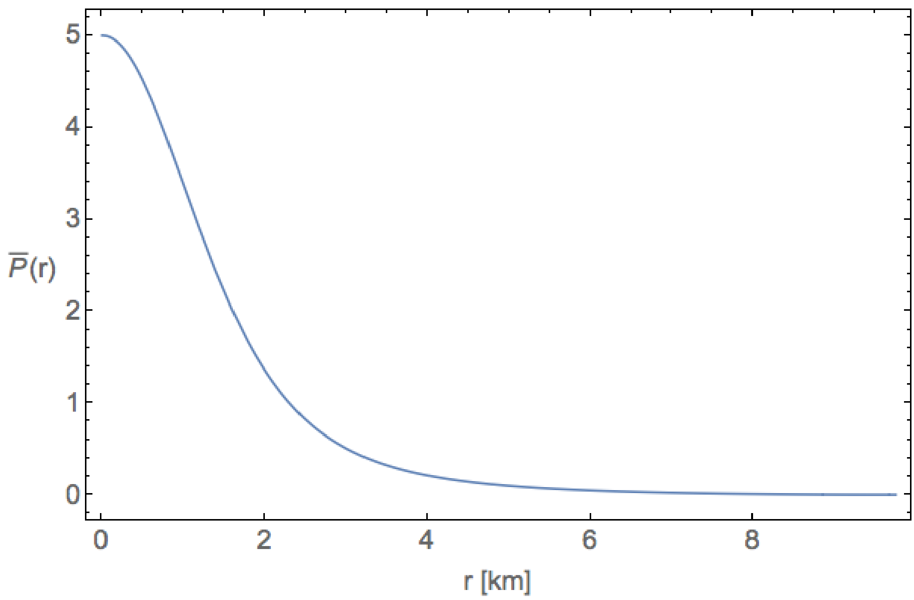

Figure 6.5: Dimensionless pressure $\bar{P}(r)$ as a function of radial distance r in km for a neutron star with dimensionless central pressure $\bar{P}_0 = 5$.

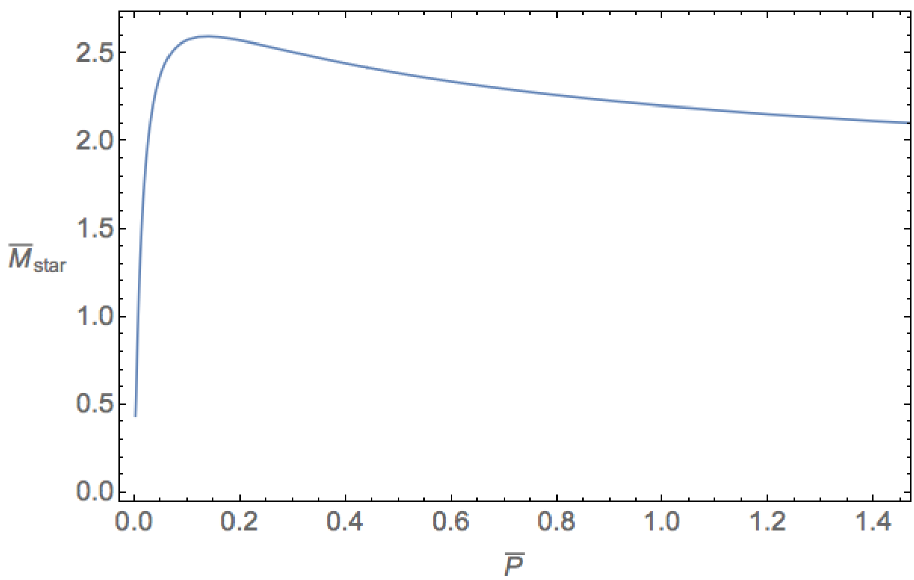

Figure 6.6: Dimensionless neutron star mass as a function of dimensionless central pressure $\bar{P}_0$.

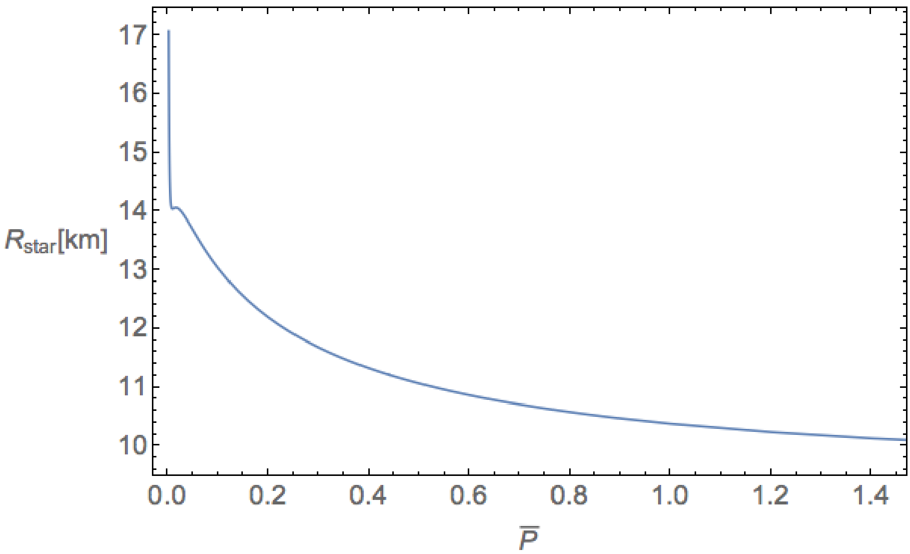

Figure 6.7: Neutron star radius in km as a function of dimensionless central pressure $\bar{P}_0$.

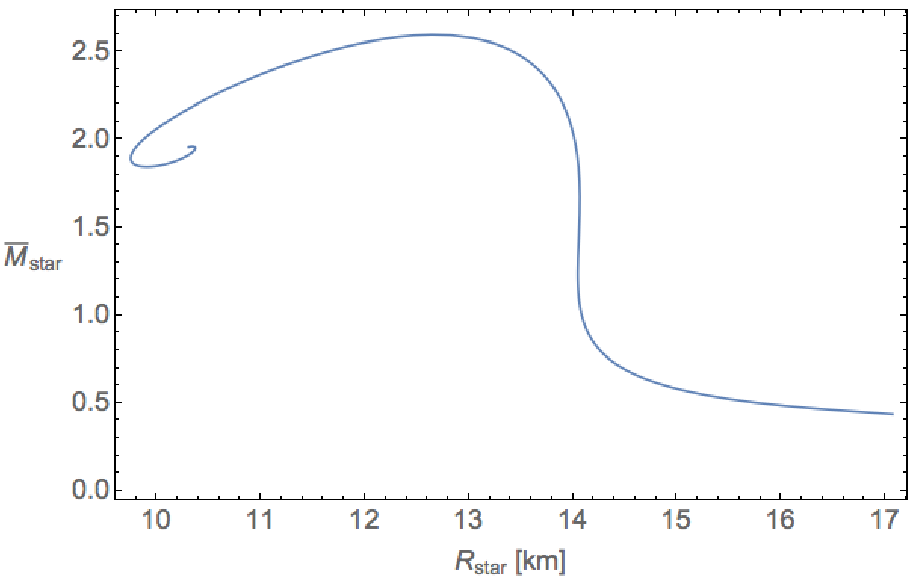

Figure 6.8: The relation between the mass and radius of neutron stars. The mass is scaled by solar masses $M_{\odot}$ and the radius in km.

Chapter 7

Upper Bound On Neutron Star Mass and Stability

7.1 Upper bound neutron star mass

As seen from Figure 6.8, the mass-radius relation of neutron stars has a maximum mass $M_{\max} = 2.595 M_{\odot}$ for a radius $R_{\max} = 12.64$ km. This maximum means that there is an upper bound on the mass of a neutron star. The same situation arises for a white dwarf, where the upper bound on the mass is $1.39 M_{\odot}$ and is called the Chandrasekhar limit [6]. The maximum mass for a neutron star happens when the central pressure is $P_0 = 7.684 \cdot 10^{34}$ kgm^{-1}s^{-2} and central energy density is $\mathcal{E}_0 = 1.573 \cdot 10^{35}$ kgm^{-1}s^{-2}, which corresponds to a central density of $\rho_0 = 1.750 \cdot 10^{18}$ kgm^{-3}. The core of a neutron star with this mass is almost ten times as dense as the nucleus of an atom.

The upper bound on the mass of a neutron star is a result of relativistic gravity and low temperatures. Relativistic causality requires that the speed of sound does not exceed the speed of light [30]. The speed of sound is given by $c_s^2 = \frac{dP}{d\rho}$, meaning that the requirement can be written as $\frac{dP}{d\rho} \leq c^2$ or $\frac{dP}{d\mathcal{E}} \leq 1$. For low temperatures, the pressure mainly depends on the density, while its temperature dependence is negligible. An increase in density will lead to an increase in gravitational attraction. To balance this, the pressure must increase equivalently. However, there is a limit to how much the pressure can increase and beyond this the pressure will no longer be able to balance the gravitational attraction, leading to gravitational collapse. There must therefore be an upper bound on how large a mass a neutron star can have, which corresponds to the limiting case where the pressure still manages to balance the gravitational attraction.

As expected, the inclusion of nuclear interactions in the equation of state allows for larger neutron star masses than the equation of state of an ideal Fermi gas, which gives a maximum mass of $0.77 M_{\odot}$ [32]. This result is in better agreement with observations, where neutron stars with masses up to $2 M_{\odot}$ have been found [2][10]. However, the σ-ω model suffers problems with negative pressures, as seen in Figure 6.1. It has been argued that this region can be seen as a liquid-gas phase transition, similar to the Van der Waals equation of state [20]. Using a Maxwell construction to avoid the negative values, the authors of [7] find the maximum mass of a neutron star to be $2.57M_{\odot}$. Our result is in close agreement with this, although the equations of state differ for low pressures. Another problem is that the parameters of the σ-ω model, such as the compression modulus and the Dirac effective mass, are in poor agreement with empirical values [13]. Further extensions of the model, such as described in sections 5.2 - 5.4, is therefore necessary. These extensions will lead to lower maximum masses, such as $2.15M_{\odot}$ [12] and $2.02M_{\odot}$ [18].

7.2 Stability of solution

The neutron stars with radii shorter than R_{max}, are unstable. We see in Figure 6.8 that the solutions to the left of the maximum mass make up a spiral shape. The spiral converges to the point where $M_{\mathrm{spiral}} = 1.956M_{\odot}$ and $R_{\mathrm{spiral}} = 10.30$ km. This happens for large values of the dimensionless central pressure $\bar{P}_0$, as seen from Figures 6.6 and 6.7, where the mass and radius approach M_{spiral} and R_{spiral} for large $\bar{P}_0$. Since the solutions are unstable, they are easily affected by small fluctuations, which will lead the gravitational attraction to exceed the pressure, allowing further gravitational collapse to continue, thereby turning the neutron star into a black hole.

Chapter 8

Conclusion and Outlook

8.1 Conclusion

In this master's thesis we have studied the mass-radius relation of neutron stars. To describe the stellar structure of a neutron star, we have used the Tolman-Oppenheimer-Volkoff equation together with the mass continuity equation. In order to solve the TOV and mass continuity equations, we need an equation of state. We first derived expressions for the energy density and pressure of an ideal Fermi gas using quantum field theory. Next, interactions between the fermions were included using the σ-ω model, where mesons were introduced in a mean-field approximation. We also looked at extensions of this model. In the non-linear σ-ω model, scalar self-interactions for the scalar meson were included. In the σ-ω-ρ model, the isospin force was introduced using a ρ meson, which allow us to distinguish between protons and neutrons. In order to impose electrical neutrality, leptons were also included in the model.

For our numerical computations, we used the expressions for the pressure and energy density of the σ-ω model to find an equation of state. Using this equation of state, we solved TOV equation and mass continuity equation numerically for a given central pressure. By iterating over a range of central pressures, the relation between the mass and radius of neutron star was found. This curve has a maximum mass of $M_{\max} = 2.595 M_{\odot}$, for a radius of $R_{\max} = 12.64\,\mathrm{km}$. We argued that this maximum means that there is an upper bound on the mass of a neutron star, and that solutions beyond this point are unstable, meaning that these neutron stars will collapse further into black holes.

8.2 Outlook

The present model can be extended in several directions to better describe a neutron star and its mass-radius relation.

As described in section 5.2, we can extend the σ-ω model into the nonlinear σ-ω model by including scalar self-interactions for the scalar meson. Protons, electrons and muons should also be included in the model, as described in sections 5.3 and 5.4. The neutrons in a neutron star can undergo weak decay, where a neutron n decays to a proton p, an electron e^- and an anti-neutrino $\bar{\nu}_e$, via $n \to p + e^- + \bar{\nu}_e$. There will arise an equilibrium where the rate of weak decay is balanced by rate of neutron capture, such that the neutron star also contains small amounts of protons and electrons.

For high enough densities, we should also allow for the occurrence of hyperons [13]. The six hyperons Σ^0, Σ^+, Σ^-, Ξ^0, Ξ^- and Λ are baryons with spin $\frac{1}{2}$ and make up the baryon octet together with the neutron and proton. They are composed of the three quarks up, down, and strange. They all have masses larger than that of the neutron and the proton, and it is when the Fermi energy of the system exceeds these masses that these particles appear. To include hyperons in the model, we can use the Lagrangian density

$$
\begin{aligned}
\mathcal{L} = \sum_B \bar{\psi}_B \left[i\gamma_\mu \left(\partial^\mu + i g_{\omega B}\omega^\mu\right) - (m_B - g_{\sigma B}\sigma) - \frac{1}{2} g_{\rho B}\gamma_\mu \boldsymbol{\tau} \cdot \boldsymbol{\rho}^\mu \right] \psi_B \\
+ \frac{1}{2}\left(\partial_\mu \sigma \partial^\mu \sigma - m_\sigma^2 \sigma^2\right) - \frac{1}{4}\omega_{\mu\nu}\omega^{\mu\nu} + \frac{1}{2} m_\omega^2 \omega_\mu \omega^\mu - \frac{1}{4} c (g_\sigma \sigma)^4 \\
- \frac{1}{4}\boldsymbol{\rho}_{\mu\nu} \cdot \boldsymbol{\rho}^{\mu\nu} + \frac{1}{2} m_\rho^2 \boldsymbol{\rho}_\mu \cdot \boldsymbol{\rho}^\mu - \frac{1}{3} b m (g_\sigma \sigma)^3 + \sum_{\lambda = \mu^-, e^-} \bar{\psi}_\lambda (i\gamma_\mu \partial^\mu - m_\lambda)\psi_\lambda,
\end{aligned}
$$

where B denotes the baryon species, $g_{\sigma B}$, $g_{\omega B}$ and $g_{\rho B}$ are the hyperon coupling constants and the sum should be taken over all the states in the baryon octet. The hyperon coupling constants have not been determined experimentally. They have great impact on the system, and [18] finds that the maximum mass varies from $1.44 M_\odot$ to $2.02 M_\odot$ when using different values for the coupling constants.

In our computations we have omitted the vacuum contribution. The term is divergent, but can be calculated using e.g. dimensional regularisation. It is a result of using quantum field theory, and replaces the notion of empty space with that of a vacuum state, which corresponds to the ground state of a collection of quantum fields. The quantum fields fluctuate, even when there are no particles or radiation. These zero-point fluctuations lead to a vacuum energy density ρ_{vac}, which is believed to contribute to the cosmological constant Λ from Einstein's field equations

$$
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu},
$$

thereby connecting quantum field theory and general relativity. However, the observed value of the cosmological constant is very small, while the theoretical estimates for various contributions to the vacuum energy density are much larger. This discrepancy is known as the cosmological constant problem and is one of the fundamental problems in modern physics [26].

Another assumption to consider is whether the equation of state is valid for all densities. In our computations we have assumed that the same equation of state can be applied for the whole neutron star. This however is too simplistic. As the density increases from the surface inwards to the core, the character of the matter changes considerably. The low-density surface of a neutron star is usually made up of iron ^{56}Fe nuclei arranged in a lattice. As the density increases, the electrons become free and subsequently fully relativistic. The lowest energy state of matter will gradually change from a lattice of iron ^{56}Fe nuclei to lattices of nuclei progressively richer in neutrons. At the neutron drip point $\rho \approx 4.3 \cdot 10^{-11}\ \mathrm{gcm}^{-3}$, the continuum neutron state begin to populate, and as the density increases, the matter consists of nuclei in a lattice with a gas of free neutrons and electrons penetrating it. At even higher densities, the nuclei will no longer be present, and the matter is mainly a mix of neutrons, protons and electrons. Finally, at the core the density is so high that hyperons may appear. Each of these regions should be represented by its own equation of state, as described in [3], and the equation of state for the whole neutron star should be put together from these.

We next consider the zero-temperature assumption. Neutron stars are the remnants after supernova explosions and are believed to have temperatures over 10^{11} K when formed. The temperature decreases rapidly through neutrino emission and will within a day drop to 10^{9}-10^{10} K. When the core reaches 10^{8} K, the cooling mechanism is dominated by photon emission. The surface temperature is usually two orders of magnitude smaller [29]. When working with non-zero temperature in the nonlinear σ-ω model it is possible to avoid the problem with negative pressures [20].

It is unlikely that neutron stars are static, although we have made this assumption. Most stellar objects rotate, and when a massive star collapses during a supernova explosion, the conservation of angular momentum will cause the resulting neutron star to rotate rapidly. The neutron star called the Crab pulsar rotates with a frequency of 30 rotations per second [13]. With a non-static neutron star, we can no longer use the diagonal metric. Instead we can replace it with a perturbation of the diagonal matrix on the form

$$\begin{aligned} ds^2 = -A(r)\left[1 + 2\left(h_0 + h_2 P_2\right)\right](cdt)^2 + B(r)\left[1 + 2\frac{m_0 + m_2 P_2}{r - 2GM/c^2}\right] dr^2 \\ + r^2\left[1 + 2\left(v_2 - h_2\right) P_2\right]\left[d\theta^2 + \sin^2\theta\left(d\phi - \omega dt\right)^2\right], \end{aligned}$$

where P_2 is a Legendre polynomial, ω is the angular velocity of the local inertial frame, and h_0, h_2, m_0, m_2 and v_2 are functions of r and proportional to square of the angular velocity of the star. This is called the Hartle-Thorne metric and can be used to describe slowly rotating neutron stars [17].

Many neutron stars are surrounded by strong magnetic fields. The direction of the magnetic field breaks the spherical symmetry of the system, meaning we can no longer use the TOV equation as stellar structure equation. Its magnitude will increase with density, which mean the equation of state will be changed.

As we have seen, describing the stellar structure of a neutron star is not a simple task. Considerations must be done to decide in what manner the neutron star should be described and which assumptions should be applied.

Appendices

Appendix A

The Tolman-Oppenheimer-Volkoff Equation

The following derivation of the Tolman-Oppenheimer-Volkoff (TOV) equation was performed as part of the specialization project [32]. The derivations are done in SI units, as this will be used in the numerical computations so that the neutron star mass can be given in solar masses $M_\odot$ and the radius in km.

The section is based on references [35] and [37] .

The Tolman-Oppenheimer-Volkoff equation describes the interior of a relativistic stellar object, such as a neutron star. The TOV equation is the relativistic equivalent of the equation of hydrostatic equilibrium for nonrelativistic stellar objects. It can be derived from Einstein's field equations,

$$R^{\mu\nu} = \frac{8\pi G}{c^4}(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T), \tag{A.1}$$

where $R^{\mu\nu}$ is the Ricci tensor, $T^{\mu\nu}$ is the energy-momentum tensor and $g^{\mu\nu}$ is the metric. The following conditions are assumed:

- **Static, spherically symmetric interior** The interior is assumed to be static and spherically symmetric, which can be described by the diagonal metric

$$ds^2 = -A(r)(cdt)^2 + B(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{A.2}$$

  where $A(r)$ and $B(r)$ are functions of r to be determined.

- **Perfect fluid** The interior is assumed to consist of a perfect fluid with energy-momentum tensor $T^{\mu\nu}$ given as

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right)U^\mu U^\nu + Pg^{\mu\nu} = \frac{1}{c^2}\left(\mathcal{E} + P\right)U^\mu U^\nu + Pg^{\mu\nu}, \tag{A.3}$$

where U^μ, ρ, $\mathcal{E}$ and P are the 4-velocity, density, energy density and pressure of the fluid, respectively.

In the last step of (A.3), the relation between the density and energy density

$$\mathcal{E} = \rho c^2, \tag{A.4}$$

was used. Inserting equation (A.3) into (A.1) gives

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left[\left(\rho + \frac{P}{c^2}\right)U_\mu U_\nu + \frac{1}{2}\left(\rho c^2 - P\right) g_{\mu\nu}\right], \tag{A.5}$$

where the normalization condition $g_{\mu\nu}U^\mu U^\nu = -c^2$ and $g_{\mu\nu}g^{\mu\nu} = 4$ were used. The Ricci tensor $R_{\mu\nu}$ is found from contracting the Riemann curvature tensor $R^\sigma_{\mu\lambda\nu}$ with an equal lower and upper index, such that

$$R_{\mu\nu} = R^\sigma_{\mu\sigma\nu} = \left(\partial_\sigma \Gamma^\sigma_{\mu\nu} + \Gamma^\sigma_{\kappa\sigma}\Gamma^\kappa_{\mu\nu}\right) - \left(\partial_\nu \Gamma^\sigma_{\mu\sigma} + \Gamma^\sigma_{\kappa\nu}\Gamma^\kappa_{\mu\sigma}\right). \tag{A.6}$$

The Christoffel symbols $\Gamma^\rho_{\mu\nu}$ can be found from the metric $g_{\mu\nu}$ using

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}), \tag{A.7}$$

giving a total of 4^3 Christoffel symbols to be calculated. The number can be reduced by symmetry considerations. From the diagonal metric (A.2), one finds that the only nonzero components of $g_{\mu\nu}$ are the ones with identical indices: g_{tt}, g_{rr}, $g_{\theta\theta}$ and $g_{\phi\phi}$. Therefore, the Christoffel symbols must have at least two identical indices to be nonzero. This condition reduces the number of Christoffel symbols by 24. Also, the Christoffel symbols are symmetric in the lower indices, reducing the number further by 12. Finally, the condition that the stellar interior is static, means that the metric should be invariant under the transformation $t \to -t$. Any Christoffel symbols with an odd number of index t, of which there are seven, must therefore be zero. This leaves us with 21 Christoffel symbols to calculate.

An alternative way of finding the Christoffel symbols is interpreting the geodesic equations,

$$\frac{d^2X^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\lambda}\frac{dX^\nu}{d\sigma}\frac{dX^\lambda}{d\sigma} = 0, \tag{A.8}$$

as the Euler-Langrange equations,

$$\frac{d}{d\sigma}\frac{\partial L}{\partial \dot{X}^\mu} - \frac{\partial L}{\partial X^\mu} = 0. \tag{A.9}$$

The Lagrangian is given as

$$L = \sqrt{-g_{\alpha\beta}\frac{\partial X^\alpha}{\partial\sigma}\frac{\partial X^\beta}{\partial\sigma}}. \tag{A.10}$$

For the metric given by the diagonal metric (A.2) and the proper time τ chosen as the parameter, the Lagrangian (A.10) becomes

$$L = \sqrt{A(r)\left(\frac{cdt}{d\tau}\right)^2 - B(r)\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2}. \qquad \text{(A.11)}$$

Inserting the Lagrangian (A.11) into the Euler-Lagrange equations (A.9) gives

$$\frac{cd^2t}{d\tau^2} + \frac{A'(r)}{A(r)}\frac{dr}{d\tau}\frac{cdt}{d\tau} = 0, \qquad \text{(A.12)}$$

$$\frac{d^2r}{d\tau^2} + \frac{A'(r)}{2B(r)}\left(\frac{cdt}{d\tau}\right)^2 + \frac{B'(r)}{2B(r)}\left(\frac{dr}{d\tau}\right)^2 - \frac{r}{B(r)}\left(\frac{d\theta}{d\tau}\right)^2 - \frac{r\sin^2\theta}{B(r)}\left(\frac{d\phi}{d\tau}\right)^2 = 0, \qquad \text{(A.13)}$$

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau} - \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0, \qquad \text{(A.14)}$$

$$\frac{d^2\phi}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau} + \frac{2\cos\theta}{\sin\theta}\frac{d\theta}{d\tau}\frac{d\phi}{d\tau} = 0. \qquad \text{(A.15)}$$

Comparing equations (A.12)-(A.15) to the geodesic equations (A.8) and exploiting that the Christoffel symbols are symmetric in the lower indices, give nine nonzero Christoffel symbols, as shown in Table A.1.

Table A.1: Christoffel symbols for the diagonal metric $ds^2 = -A(r)(cdt)^2 + B(r)dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$.

Γ^t_{tr}	Γ^r_{tt}	Γ^r_{rr}	$\Gamma^r_{\theta\theta}$	$\Gamma^r_{\phi\phi}$	$\Gamma^\theta_{r\theta}$	$\Gamma^\theta_{\phi\phi}$	$\Gamma^\phi_{r\phi}$	$\Gamma^\phi_{\theta\phi}$
$\frac{A'(r)}{2A(r)}$	$\frac{A'(r)}{2B(r)}$	$\frac{B'(r)}{2B(r)}$	$-\frac{r}{B(r)}$	$-\frac{r\sin^2\theta}{B(r)}$	$\frac{1}{r}$	$-\sin\theta\cos\theta$	$\frac{1}{r}$	$\cot\theta$

Under the given conditions, the Ricci tensor must be diagonal. The condition of a static interior is equal to no fluid flow, which means that the spacial part of the local 4-velocity of the fluid should be zero, $U^i = 0$, while $U^t \neq 0$. The spherical symmetry makes the metric $g_{\mu\nu}$ diagonal. Then, the energy-momentum tensor of a perfect fluid, given by equation (A.3), must also be diagonal. Inserting a diagonal $T_{\mu\nu}$ and $g_{\mu\nu}$ into (A.1) leaves only R_{tt},

R_{rr}, $R_{\theta\theta}$ and $R_{\phi\phi}$ nonzero. Using the results for the Christoffel symbols from Table A.1 in equation (A.6) gives

$$R_{tt} = \frac{A''(r)}{2B(r)} + \frac{A'(r)}{rB(r)} - \frac{A'(r)}{4B(r)}\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right), \tag{A.16}$$

$$R_{rr} = -\frac{A''(r)}{2A(r)} + \frac{B'(r)}{rB(r)} + \frac{A'(r)}{4A(r)}\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right), \tag{A.17}$$

$$R_{\theta\theta} = 1 - \frac{1}{B(r)} - \frac{r}{2B(r)}\left(\frac{A'(r)}{A(r)} - \frac{B'(r)}{B(r)}\right), \tag{A.18}$$

$$R_{\phi\phi} = \left[1 - \frac{1}{B} - \frac{r}{2B(r)}\left(\frac{A'(r)}{A(r)} - \frac{B'(r)}{B(r)}\right)\right]\sin^2\theta. \tag{A.19}$$

We note that $R_{\phi\phi} = R_{\theta\theta}\sin^2\theta$. The Ricci tensor can also be expressed by the right-hand side of equation (A.5). Using that the only nonzero component of the 4-velocity is U^t together with the normalization condition $g_{\mu\nu}U^\mu U^\nu = -c^2$, gives that $U_t = -cA(t)^{1/2}$. Inserting into equation (A.5) leads to

$$R_{tt} = \frac{8\pi G}{c^4}\frac{1}{2}(\rho c^2 + 3P)A(r), \tag{A.20}$$

$$R_{rr} = \frac{8\pi G}{c^4}\frac{1}{2}(\rho c^2 - P)B(r), \tag{A.21}$$

$$R_{\theta\theta} = \frac{8\pi G}{c^4}\frac{1}{2}(\rho c^2 - P)r^2, \tag{A.22}$$

$$R_{\phi\phi} = \frac{8\pi G}{c^4}\frac{1}{2}(\rho c^2 - P)r^2\sin^2\theta. \tag{A.23}$$

Equations (A.16)-(A.19), or the equivalent equations (A.20)-(A.23), constitute a set of three coupled differential equations (one equation is redundant since $R_{\phi\phi} = R_{\theta\theta}\sin^2\theta$).

To find the TOV equation, the functions $A(r)$ and $B(r)$ should be found. Using the combination $\frac{R_{tt}}{A(r)} + \frac{R_{rr}}{B(r)} + \frac{2R_{\theta\theta}}{r^2}$ leads to

$$\frac{1}{B(r)} + r\left(\frac{1}{B(r)}\right)' = 1 - \frac{8\pi G\rho r^2}{c^2}. \tag{A.24}$$

Similarly to the Schwarzschild solution, one can choose

$$\frac{1}{B(r)} = 1 - \frac{2GM(r)}{c^2 r}, \tag{A.25}$$

where $M(r)$ is a mass function that describes the mass contained within a radius r. Inserting the solution (A.25) into equation (A.24) gives

$$\frac{dM}{dr} = 4\pi r^2 \rho(r), \tag{A.26}$$

which is equal to the mass continuity equation from Newtonian theory. An expression for $A(r)$ can be found by using the combination $\frac{R_{tt}}{A(r)} + \frac{R_{rr}}{B(r)}$, leading to

$$\frac{A'(r)}{A(r)} = \frac{2GM(r)}{c^2 r^2} \frac{\left(1 + \frac{4\pi r^3 P(r)}{M(r)c^2}\right)}{\left(1 - \frac{2GM(r)}{c^2 r}\right)}. \tag{A.27}$$

In the Newtonian limit, the diagonal metric becomes $ds^2 = -(cdt)^2 + dr^2 + r^2 d\theta + r^2 \sin\theta d\phi^2$, meaning that $A(\infty) = 1$. Using that $\frac{A'(r)}{A(r)} = \frac{d\ln A(r)}{dr}$ and integrating equation (A.27) gives

$$\int_{A(r)}^{A(\infty)} d\ln A(r) = \frac{2G}{c^2} \int_r^\infty \frac{M(r)\left(1 + \frac{4\pi r^3 P(r)}{M(r)c^2}\right)}{r^2\left(1 - \frac{2GM(r)}{c^2 r}\right)} dr. \tag{A.28}$$

Choosing $r \geq R$ outside the star means that the pressure is zero, $P(r \geq R) = 0$, and the mass function remains constant, $M(r \geq R) = M_{\text{star}}$. Inserting this into (A.28) one finds that

$$A(r \geq R) = 1 - \frac{2GM_{\text{star}}}{c^2 r}. \tag{A.29}$$

To arrive at the TOV equation, the Bianchi identity,

$$D_\nu R_{\rho\mu\sigma\lambda} + D_\sigma R_{\rho\mu\lambda\nu} + D_\lambda R_{\rho\mu\nu\sigma} = 0, \tag{A.30}$$

is needed. Contracting (A.30) repeatedly with the metric $g_{\mu\nu}$, it can be written as

$$D^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = 0, \tag{A.31}$$

where D^μ is the covariant divergence of a tensor. Einstein's field equations (A.1) can be rewritten as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{A.32}$$

leading to

$$D^\mu T_{\mu\nu} = 0. \tag{A.33}$$

Inserting the expression for the energy-momentum tensor from equation (A.3) and using that $D_\mu g^{\mu\nu} = 0$ by definition, gives

$$D^\mu T_{\mu\nu} = g^{\mu\nu}\partial_\mu P + \frac{1}{\sqrt{-g}}\partial_\mu\{\sqrt{-g}\left(\rho + \frac{P}{c^2}\right) U^\mu U^\nu\} + \Gamma^\nu_{\mu\lambda}\left(\rho + \frac{P}{c^2}\right) U^\mu U^\lambda, \tag{A.34}$$

where $-g = det\,(g_{\mu\nu}) = A\,(r)\,B\,(r)\,r^4 \sin^2\theta$. Again exploiting that the only nonzero component of U^μ is U^t and that the object is static, simplifies (A.34) to

$$D^\mu T_{\mu\nu} = g^{\mu\nu}\partial_\mu P + \frac{\Gamma^\nu_{tt}}{A(r)}\left(\rho c^2 + P\right). \tag{A.35}$$

Because of the spherical symmetry, the pressure is assumed to depend only on r, such that $P = P\,(r)$. The only nonzero metric tensor component and Christoffel symbol are thus $g^{rr} = \frac{1}{B(r)}$ and $\Gamma^r_{tt} = \frac{A'(r)}{2B(r)}$, which inserted into (A.35) gives

$$\frac{A'(r)}{A(r)} = \frac{-2P'}{\rho c^2 + P}. \tag{A.36}$$

Using equations (A.25) and (A.36) in the expression for $R_{\theta\theta}$, given by equations (A.18) and (A.22), lead to

$$\frac{dP}{dr} = -\frac{GM\,(r)\,\rho\,(r)}{r^2}\left(1 + \frac{P\,(r)}{c^2\rho\,(r)}\right)\left(1 + \frac{4\pi r^3 P\,(r)}{c^2 M\,(r)}\right)\left(1 - \frac{2GM\,(r)}{c^2 r}\right)^{-1}, \tag{A.37}$$

which is known as the Tolman-Oppenheimer-Volkoff equation [25][33]. The TOV equation can also be written in terms of the energy density $\mathcal{E}$, using equation (A.4), such that

$$\frac{dP}{dr} = -\frac{GM\,(r)\,\mathcal{E}\,(r)}{c^2 r^2}\left(1 + \frac{P\,(r)}{\mathcal{E}\,(r)}\right)\left(1 + \frac{4\pi r^3 P\,(r)}{c^2 M\,(r)}\right)\left(1 - \frac{2GM\,(r)}{c^2 r}\right)^{-1}. \tag{A.38}$$

The TOV equation is the relativistic equivalent of the equation of hydrostatic equilibrium and should approach the Newtonian solution in the nonrelativistic limit. The corrections in the two first parentheses of (A.37) arise from the special relativistic correction of the energy, $\varepsilon = \sqrt{p^2c^2 + m^2c^4}$, while the correction in the third term is connected to the metric of general relativity. All three terms lead to increase in the gravitational attraction. In the Newtonian limit $c^2 \to \infty$, the three parentheses approach unity and the expression becomes

$$\frac{dP}{dr} \to -\frac{GM\,(r)\,\rho\,(r)}{r^2}, \tag{A.39}$$

as expected.

Appendix B

Mathematica Code

The following *Mathematica* code is used for numerical computations. The equation of state is found by combining the solution for a free Fermi gas with the solution for the σ-ω model at their intersection. The equation of state is used to solve the TOV equation and mass continuity equation. The solutions are performed for a range of central pressures, which gives the mass-radius relation.

```
(*MASS RADIUS RELATION FOR NEUTRON STAR*)
(*Program for solving TOV equation
  with a combined EoS for ideal Fermi gas and  sigma-
 omega model to find the mass-radius relation for a neutron star*)

(*CONSTANTS*)

(*For TOV equation, in SI units*)

(*Plank's reduced constant*)
hbar := 1.0545718 * 10^(-34) (*kg*m^2/s*)
(*Speed of light*)
c := 299 792 458 (*m/s*)
(*Solar mass*)
Msol := 1.9891 * 10^(30) (*kg*)
(*Gravitational constant*)
G := 6.67408 * 10^(-11) (*m^3/(kg*s^2)*)
(*Neutron mass, in kg*)
m := 1.674927471 * 10^(-27) (*kg*)
(*Scale for energy density and pressure*)
E0 := (m^4 * c^5) / (3 * Pi^2 * hbar^3)
(*Constants used in TOV equation*)
R0 := (c^2) / (G * Msol)
alpha := (4 * Pi * E0) / (Msol * c^2)

(*For sigma-omega model, in nautral units*)

(*Masses*)
mnucleon := 939(*MeV*)
msigma := 550 (*MeV*)
momega := 783 (*MeV*)
(*Coupling constants*)
gsigma := Sqrt[266.9] * 550 / 939
gomega := Sqrt[195.7] * 783 / 939

(*Scaling for EoS*)
E1 = (mnucleon^4) / (3 * Pi^2)

(*Degeneracy factor for pure neutron matter*)
g := 2

(*Maximum Fermi momentum and for loop step *)
pfmax := 3500 / mnucleon
pfstep := 1 / mnucleon


(*IDEAL FERMI GAS*)

(*Energy density as a function of Fermi momentum*)
energydensityIFG[pf_] :=
  1/E1 * (g * mnucleon^4) / (2 * Pi^2) * NIntegrate[Sqrt[p^2 + 1] * p^2, {p, 0, pf}]
(*Pressure as a function of Fermi momentum*)
```

```
pressureIFG[pf_] := 1/E1 * (g * mnucleon^4)/(3 * 2 * Pi^2) * NIntegrate[p^4/Sqrt[p^2 + 1], {p, 0, pf}]

(*Solution of pressure and energy density for a range of Fermi momenta,
combined to give an equation of state. *)
PvsEIFG = {};

For[pf = 0, pf < pfmax, pf += pfstep,
 PvsEIFG = Append[PvsEIFG, {pressureIFG[pf], energydensityIFG[pf]}];
]
(*Equation of state for ideal fermi gas*)
EoSIFG = Interpolation[PvsEIFG]

(*SIGMA-OMEGA MODEL*)

(*Integral equation for sigma meson field*)
sigmaintegral[sigma_?NumericQ, pf_] :=
 (g * gsigma^2 * mnucleon^2)/(2 * msigma^2 * Pi^2) NIntegrate[(p^2 * (1 - sigma))/Sqrt[p^2 + (1 - sigma)^2], {p, 0, pf}]
(*Omega meson field*)
omegafield[pf_] := (gomega^2 * mnucleon^2)/(momega^2) * (g * pf^3)/(6 * Pi^2)
(*Energy density as a function of sigma field,
omega field and Fermi momentum*)
energydensitySOM[sigma_, omega_, pf_] :=
 1/E1 (1/2 * (msigma^2 * mnucleon^2)/(gsigma^2) * sigma^2 + 1/2 * (momega^2 * mnucleon^2)/(gomega^2) * omega^2 +
    (g * mnucleon^4)/(2 * Pi^2) * NIntegrate[Sqrt[p^2 + (1 - sigma)^2] * p^2, {p, 0, pf}])

(*Pressure as a function of sigma field, omega field and Fermi momentum*)
pressureSOM[sigma_, omega_, pf_] :=
 1/E1 (-1/2 * (msigma^2 * mnucleon^2)/(gsigma^2) * sigma^2 + 1/2 * (momega^2 * mnucleon^2)/(gomega^2) * omega^2 +
    (g * mnucleon^4)/(6 * Pi^2) * NIntegrate[p^4/Sqrt[p^2 + (1 - sigma)^2], {p, 0, pf}])

(*Solution of pressure and energy density for a range of Fermi momenta,
combined to give an equation of state. *)
PvsESOM = {};
PvspfSOM = {};
(*Guess for value of sigma field(will be updated in for loop)*)
sigmaguess = 0
For[pf = 0, pf ≤ pfmax, pf += pfstep,
 sigmasolution =
  FindRoot[sigmaintegral[sigma, pf] == sigma, {sigma, sigmaguess}];
 sigma1 = sigma /. sigmasolution;
 omega1 = omegafield[pf];
 pressure = pressureSOM[sigma1, omega1, pf];
 energydensity = energydensitySOM[sigma1, omega1, pf];
 PvsESOM = Append[PvsESOM, {pressure, energydensity}];
 PvspfSOM = Append[PvspfSOM, {pf, pressure}];
 sigmaguess = sigma1;
]
```

```
ListPlot[{Take[PvsEIFG, 400], Take[PvsESOM, 400]}]

ListPlot[Take[PvspfSOM, 330], Frame → True,
 FrameLabel → {Style["pF/m", 10], Style["P (pF)/εo", 10]},
 FrameStyle → Medium, RotateLabel → False]

ListPlot[Take[PvsESOM, 350], Frame → True,
 FrameLabel → {Style["P/εo", 10], Style["ε/εo", 10]},
 FrameStyle → Medium, RotateLabel → False ]

(*Equation of state for sigma-omega model*)
EoSSOM = Interpolation[Take[PvsESOM, -3240]]

(*EQUATION OF STATE*)
(*Intersection between the equations of state*)
intersectionEoSsolution = FindRoot[EoSIFG[x] == EoSSOM[x], {x, 0.002}]
intersection = x /. intersectionEoSsolution
(*Combine ideal Fermi gas for low pressures and sigma
 omega model for high pressure to give new equation of state *)
EoS[P_] := Piecewise[
  {{EoSIFG[P], P < intersection}, {EoSSOM[P], P > intersection}}]

Plot[EoS[P], {P, 0, 0.005}, Frame → True,
 FrameLabel → {Style["P/εo", 10], Style["ε/εo", 10]},
 FrameStyle → Medium, RotateLabel → False]

(*TOV EQUATION*)

(*Function that solves the TOV equation for a given central
 pressure P0. Returns the radius and mass of the neutron star,
and the pressure and mass as functions of radial distance. *)

NeutronStar[P0_] :=
  Module[{Epsilon, Start, End, StellarStructure, InitialConditions,
    FinalConditions, Rstar, Mstar, Equations, Solution},

   Epsilon = 1 * 10^(-10);
   Start = 1 * 10^(-10);
   End = 10^6;

   StellarStructure = {
     M'[r] == alpha * r^2 * EoS[P[r]],
     P'[r] == -((EoS[P[r]] + P[r]) * (M[r] + alpha * P[r] * r^3))/(R0 * r^2 - 2 * M[r] * r)
    };

   InitialConditions = {
     M[Start] == 0,
     P[Start] == P0
    };

   FinalConditions = {
     WhenEvent[P[r] ≤ Epsilon,
```

```
      {"StopIntegration",
       Rstar = r,
       Mstar = M[r]}
     ]
    };

   Equations = Join[StellarStructure, InitialConditions, FinalConditions];

   Solution = Evaluate[NDSolve[Equations, {M, P},
      {r, Start, End}, AccuracyGoal → 20, PrecisionGoal → 10]];

   Return[{Rstar, Mstar, Solution}];

  ];

(*Example of solution for P0=5*)
{rstar, mstar, sstar} = NeutronStar[5];
Plot[M[r] /. sstar, {r, 0, rstar}, Frame → True,
 FrameLabel → {Style["r [m]", 10], Style["M (r)/M☉", 10]},
 FrameStyle → Medium, RotateLabel → False]
Plot[P[r] /. sstar, {r, 0, rstar}, Frame → True,
 FrameLabel → {Style["r [m]", 10], Style["P (r)/ε₀", 10]},
 FrameStyle → Medium, RotateLabel → False]
rstar
mstar


(*MASS-RADIUS RELATION*)
(*Solves NeutronStar for a range of central pressures P0,
to find the mass-radius relation of a neutron star*)

(* Arrays to store results*)
PressureVsMass = {};
PressureVsRadius = {};
RadiusVsMass = {};

(*Make range of P0*)
P0 = {}
For[ n = -3, n ≤ 2, n++,
 For[i = 1, i ≤ 9, i += 0.2,
  P0 = Append[P0, i * 10^n]
 ]
]

(*Find solutions and store in arrays*)
For[i = 1, i ≤ Length[P0], i++,
 {Rstar, Mstar, Solution} = NeutronStar[P0[[i]]];
 PressureVsMass = Append[PressureVsMass, {P0[[i]], Mstar}];
 PressureVsRadius = Append[PressureVsRadius, {P0[[i]], Rstar / 10^3}];
 RadiusVsMass = Append[RadiusVsMass, {Rstar / 10^3, Mstar}];
 ]

ListPlot[Take[PressureVsMass, 150], Frame → True,
```

```
 FrameLabel → {Style["P0/ε0", 10], Style["Mstar/M⊙", 10]},
 FrameStyle → Medium, RotateLabel → False]
ListPlot[Take[PressureVsRadius, 150], Frame → True,
 FrameLabel → {Style["P0/ε0", 10], Style["Rstar[km]", 10]},
 FrameStyle → Medium, RotateLabel → False]
ListPlot[RadiusVsMass, Frame → True,
 FrameLabel → {Style["Rstar [km]", 10], Style["Mstar/M⊙", 10]},
 FrameStyle → Medium, RotateLabel → False]

(*Maximum mass and corresponding radius*)
Mmax = Max[RadiusVsMass[[All, 2]]]
Maxposition = Position[RadiusVsMass, Mmax]
Rmax = RadiusVsMass[[Maxposition[[1, 1]], 1]]


(* Max mass and pressure of Mstar vs P0 curve*)
Mmax2 = Max[PressureVsMass[[All, 2]]]
MaxpositionP0 = Position[PressureVsMass, Mmax2]
P0ForM = PressureVsMass[[MaxpositionP0[[1, 1]], 1]]

(*Minimum radius and pressure of Rstar vs P0 curve*)
Rmin2 = Min[PressureVsRadius[[All, 2]]]
MinpositionP0 = Position[PressureVsRadius, Rmin2]
P0ForR = PressureVsRadius[[MinpositionP0[[1, 1]], 1]]

(*Asymptotes*)
Masymptote = Last[PressureVsMass]
Rasymptote = Last[PressureVsRadius]
MvsRasymptote = Last[RadiusVsMass]

(*Central pressure, energy density and density for *)
PatMax = P0ForM * E0
EatMax = EoS[P0ForM] * E0
rhoatMax = EoS[P0ForM] * E0 / c^2
```

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