NTNU
Norwegian University of
Science and Technology

Surfaces moving by Mean Curvature.

Fredrik Arbo Høeg

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Supervisor: Lars Peter Lindqvist, MATH

Norwegian University of Science and Technology
Department of Mathematical Sciences

Preface

I would like to thank my supervisor professor Peter Lindqvist for helping me in my work. Our discussions regarding the subject have been invaluable to me.

Abstract

We use a level-set method to describe surfaces moving by mean curvature. The interesting partial differential equation $u_t = |\nabla u| \mathrm{div} \left(\frac{\nabla u}{|\nabla u|} \right)$ arises. In this thesis, we prove uniqueness of solutions in the viscosity sense and singularities of the flow are taken into consideration. Our work is based on the demanding proof of Evans and Spruck, published in Journal of Differential geometry (1991).

Sammendrag

Vi beskriver overflater som beveger seg i forhold til den gjennomsnittlige kurvaturen med en metode som baserer seg på nivåflater. En interessant partiell differensialligning kan beskrive situasjonen, $u_t = |\nabla u| \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)$. I denne masteroppgaven beviser vi at denne ligningen har en unik viskositetsløsning, og vi tar hensyn til singulariteter. Arbeidet er basert på et krevende bevis av Evans og Spruck, gitt ut i Journal of Differential geometry (1991).

Contents

1 Introduction

Mean curvature flow is an example of a geometric flow of hypersurfaces. In this paper, we mainly study smooth surfaces in $\mathbb{R}^3$. We require that the surface moves with velocity equal to the mean curvature in the normal direction.

In the paper [B], Brakke studied motion of grain boundaries, in which he introduced motion by mean curvature for surfaces. There are other physical phenomena which can be explained by mean curvature flow. These include surface tension phenomena, horizons of black holes in general relativity, image processing and soap films stretched across a wire frame. Gage and Hamilton [GH] and Grayson [G1] showed that closed embedded curves in the plane remains embedded before they shrink to a point. Huisken [H] showed that convex surfaces in $\mathbb{R}^3$ remains convex until they shrink to a point under the mean curvature flow. In fact, Huisken and Ilmanen proved the Riemann Penrose inequality in [HI] studying the inverse mean curvature flow, where the velocity is equal to the reciprocal of the mean curvature. In these cases, a differential geometric approach to the problem has been used.

Here, we use a level-set method for the flow. The interesting mean curvature flow equation arises,

$$u_t = |\nabla u| \mathrm{div} \left(\frac{\nabla u}{|\nabla u|} \right).$$

The equation is not defined when $\nabla u = 0$. Introducing a notion of a weak solution, namely a *viscosity solution* turned out to be successful, see [ES]. Viscosity solutions were first introduced in [CL]. We intend to discuss the problem by studying this equation, and by gaining insight in the equation we derive some geometric properties of the flow. In particular, we prove *uniqueness* of solutions. When uniqueness is proved one can show several interesting properties of the flow, including that two surfaces initially disjoint remain disjoint under the flow. Our work regarding the level-set method is mainly based on the article by [ES].

There are two mathematical technicalities which arise in the proof of uniqueness of solutions. These are properties of semi-convex functions and inf- and sup convolutions. For semi-convex functions, the Alexandrov theorem is applied, which states that a convex function is twice differentiable almost everywhere. The inf- and sup convolutions are introduced to approximate the merely continuous function u. We base our discussion on these by using the celebrated Hopf-Lax formula, which solves the Hamilton-Jacobi partial differential equation [E].

In section 2 we give an introduction to the mean curvature flow in the plane, where calculations are easier. Then we derive the mean curvature for surfaces in $\mathbb{R}^3$ before introducing the level-set method. In section 3 we introduce viscosity solutions. Section 4 contains an introduction to inf- and sup convolutions. Further, we prove that we have uniqueness for classical solutions, provided $\nabla u \neq 0$. Finally, we give a proof of uniqueness of viscosity solutions. Having established uniqueness, we give some geometrical properties of the flow in section 5. Here, we will also make mention of the minimal surface equation, which turns out to be the elliptic counterpart of the mean curvature flow equation, just as the laplace equation is the elliptic counterpart to the heat equation.

2 The mean curvature flow equation

2.1 Curvature and the Curve-Shortening flow

2.1.1 Curvature

The concept of curvature can loosely be thought of as how much an object deviates from being flat. If the object is a curve, the curvature tells us how much the curve deviates from being a straight line. A curve C in $\mathbb{R}^3$ may be described as a smooth vector valued function of one parameter, $r(t) = (x(t), y(t), z(t))$ where $t \in I \subset \mathbb{R}$. For each t, $r(t)$ has a tangent vector, given by the derivative of r. The unit tangent vector T is defined by

$$T(t) = \frac{\dot{r}(t)}{|\dot{r}(t)|} \equiv \frac{1}{v}\dot{r}(t).$$

It will be useful to parametrize r so that $\dot{r}$ has length one. The arclength of C is given by $ds = vdt$. We see that

$$\left|\frac{dr}{ds}\right| = v\left|\frac{dt}{ds}\right| = 1$$

under this choice of the parameter s.

Definition 2.1. *The curvature of C, κ, is given by*

$$\kappa = \left|\frac{dT}{ds}\right| = |r''(s)|. \tag{1}$$

The signed curvature k is given by the same equation if the unit tangent vector rotates counterclockwise, and with a negative sign if the unit tangent vector rotates clockwise.

The next example shows that the curvature of a straight line is zero, which fits well with our intuition. Further, we calculate the curvature of a circle.

Example 2.2. ***(The circle and the straight line.)***
The circle in $\mathbb{R}^2$ of radius R can be parametrized by

$$r(t) = (R\cos\theta, R\sin\theta)$$

where $\theta \in [0, 2\pi]$. We have

$$\frac{dr}{d\theta} = (-R\sin\theta, R\cos\theta)$$

and $|\frac{dr}{d\theta}| = R$. Hence, by choosing $s = R\theta$ we have

$$T(s) = \left(-\sin\frac{s}{R}, \cos\frac{s}{R}\right).$$

The curvature is then

$$\kappa = \left|-\frac{1}{R}\left(\cos\frac{s}{R}, \sin\frac{s}{R}\right)\right| = \frac{1}{R}.$$

Consider now a straight line. Since T has constant components, equation (1) gives $\kappa = 0$. $\square$

2.1.2 Curve-shortening flow

Here, we give an introduction to the mean curvature flow in $\mathbb{R}^2$ based on the ideas of Gage and Hamilton [GH]. The flow in the plane is often referred to as the curve-shortening flow. As we will see, the flow has the property that the length of a curve decreases, and the area bounded by a closed curve decreases. We consider a vector

$$X : S^1 \times [0, T] \to \mathbb{R}^2$$

with the property that

$$\frac{\partial X}{\partial t} = kN$$

where N is the inward pointing unit normal vector of a curve parametrized by $X(u, t)$. We can define the parametrization in terms of the arclength s by

$$\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u}$$

where $v = \left|\frac{\partial X}{\partial u}\right|$. Using the Frenet equations

$$\frac{\partial T}{\partial u} = vkN, \quad \frac{\partial N}{\partial u} = -vkT,$$

we derive the evolution equation for the curvature and give some properties of the flow.

To find the change of the length of a curve

$$\frac{dL}{dt} = \int_{S^1} \frac{dv}{dt} du$$

we need the following lemma.

Lemma 2.3.

$$\frac{dv}{dt} = -k^2 v.$$

Proof. We calculate using the Frenet equations

$$\begin{aligned}\frac{dv^2}{dt} &= \frac{d}{dt}\left\langle \frac{\partial X}{\partial u}, \frac{\partial X}{\partial u}\right\rangle = 2\left\langle \frac{\partial}{\partial t}\frac{\partial X}{\partial u}, \frac{\partial X}{\partial u}\right\rangle = 2\left\langle \frac{\partial (kN)}{\partial u}, \frac{\partial X}{\partial u}\right\rangle \\ &= 2\left\langle \frac{\partial k}{\partial u}N - vk^2 T, \frac{\partial X}{\partial u}\right\rangle = -2vk^2\left\langle T, Tv\right\rangle = -2v^2k^2.\end{aligned}$$

□

Proposition 2.4. *The length of a curve under the curve-shortening flow decreases,*

$$\frac{dL}{dt} = -\int k^2 ds \leqslant 0.$$

Proof. By the previous lemma we find

$$\frac{dL}{dt} = \int_{S^1} \frac{dv}{dt} du = \int_{S^1} -k^2 v du = -\int k^2 ds.$$

□

We now compute the evolution equation for the curvature $k = k(s,t)$. However, as we will see in the next example, the operators $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ do not commute.

Example 2.5. ***(The Grim Reaper.)***
Consider a graph solution to the flow moving by translation,

$$F(x,t) = (x, t + y(x))$$

We calculate

$$F_s = \frac{1}{v}(1, y') \quad F_{st} = \frac{d\left(\frac{1}{v}\right)}{dt}(1, y') = k^2 X_s$$

while $F_{ts} = 0$. The solution to the curve-shortening flow is given by

$$\frac{\partial F}{\partial t} = (0,1) = kN. \tag{2}$$

For a graph we have

$$k = \frac{y''}{(1 + (y')^2)^{3/2}}, \quad N = \frac{(-y', 1)}{\sqrt{1 + (y')^2}}$$

Multiplying equation (2) by N gives $k = \frac{1}{\sqrt{1+(y')^2}}$, which can be rewritten into the differential equation

$$y''(x) = 1 + (y'(x))^2 .$$

This has a particular solution $y(x) = -\ln\cos(x)$, which is valid for $x \in (-\pi/2, \pi/2)$. The solution is often called the Grim Reaper, as seen in figure 1. $\square$

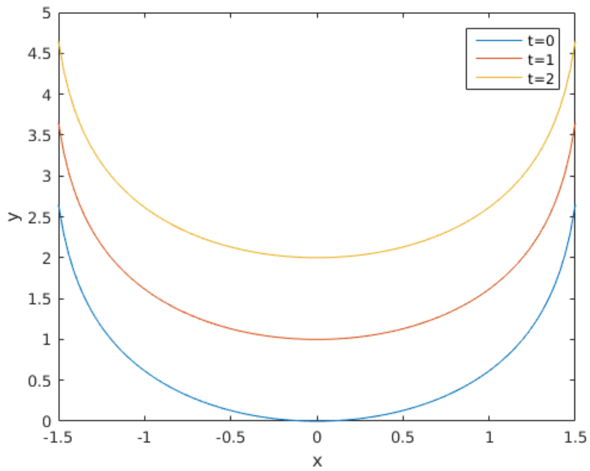

Figure 1: *A translating solution of the curve-shortening flow.*

Lemma 2.6. *The operators* $\frac{\partial}{\partial s}$ *and* $\frac{\partial}{\partial t}$ *are related in the following way*

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} = k^2 \frac{\partial}{\partial s} + \frac{\partial}{\partial s}\frac{\partial}{\partial t}.$$

Remark. We see that the lemma holds true for example 2.5.

Proof. We let the operators act on a vector F to get

$$\begin{aligned} F_{ts} &= \left(\frac{1}{v}F_u\right)_t = \frac{k^2 v}{v^2}F_u + \frac{1}{v}F_{tu} \\ &= k^2 F_s + \frac{1}{v}F_{ut} = k^2 F_{ss} + F_{st}. \end{aligned}$$

$\square$

We can now calculate, using the Frenet equations

$$\begin{aligned} k_t N + k N_t &= (kN)_t = X_{tss} = k^2 X_{ss} + X_{sts} \\ &= k^3 N + \left(k^2 X_s + X_{st}\right)_s = k^3 N + \left(k_s N + k N_s + k^2 T\right)_s \\ &= k^3 N + k_{ss} N - k k_s T \\ &= \left(k_{ss} + k^3\right) N - k k_s T. \end{aligned}$$

Since $\langle N, T \rangle = 0$ and they are both unit vectors, we get the following evolution equations

$$k_t = k_{ss} + k^3,$$

$$N_t = -k_s T.$$

The equation for curvature is of particular interest. From the equation we get following proposition, which also turns out to be true for higher dimensions for the mean curvature.

Proposition 2.7. *Suppose $\Omega \subset \mathbb{R}$ is a bounded domain and look at*

$$\begin{cases} k_t = k_{ss} + k^3, & (s,t) \in \Omega \times (0,T] \\ k(s,0) = k_0(s), & s \in \Omega \times \{t = 0\}, \end{cases}$$

where $k_0(s) > 0$. Then $k(s,t) > 0$ for all $(s,t) \in \Omega \times [0,T]$.

To prove this proposition, we need a version of the strong minimum principle for parabolic equations presented on p.169 in [PW].

Theorem 2.8. ***(The strong minimum principle.)***
Suppose that

$$k_t - k_{ss} \geqslant 0$$

for all $(s,t) \in E = \{(s,t) : s \in \Omega, t \leqslant t_1\}$ for some $t_1 > 0$. If $k \geqslant M$ in E and there is an s_1 so that $k(s_1, t_1) = M$, then $k \equiv M$ in E.

Proof. (Of proposition 2.7.) If k is not positive everywhere, we find the first point (t_1, s_1) so that $k(s_1, t_1) = 0$. By continuity, k is strictly positive up until this point, and so

$$k_t - k_{ss} = k^3 \geqslant 0$$

when $t \leqslant t_1$. Using theorem 2.8 we find that $k \equiv 0$ when $t \leqslant t_1$. This contradicts $k(s, 0) = k_0(s) > 0$.

□

Proposition 2.9. *Let C be a closed curve parametrized by $F(u, t)$. Then*

$$\frac{dA}{dt} = -2\pi,$$

where A is the area enclosed by C.

Proof. By Green's theorem in the plane,

$$2A = \int y dx + x dy = -\int \langle F, vN \rangle \, du$$

so that

$$\begin{aligned} 2\frac{dA}{dt} &= -\int_{S^1} \langle F_t, vN \rangle + \langle F, v_t N \rangle + \langle F, vN_t \rangle \, du \\ &= -\int_{S^1} kv + \left\langle F, -k^2 vN \right\rangle + \langle F, -k_u T \rangle \, du. \end{aligned}$$

The last term may be integrated by parts (the boundary term disappears) to get

$$\begin{aligned} \frac{dA}{dt} &= -\frac{1}{2}\int_{S^1} vk + \left\langle F, k^2 vN \right\rangle + k \left(v - \left\langle F, vk^2 N \right\rangle \right) du \\ &= -\int_{S^1} vk du = -\int k ds = -2\pi. \end{aligned}$$

The last equality follows from the definition of k. Since $k = T'(s)$ where T rotates counterclockwise, the integral around the closed curve is equal to 2π.

□

Example 2.10. ***(The circle under curve-shortening flow.)***
In example 2.2 we found $\kappa = \frac{1}{R}$ and hence can write $k = \frac{1}{R}$ with N pointing inwards. Since k is independent on where we are on the circle, the circle keeps its shape under the curve-shortening flow. With $k(s,t) = k(t) = \frac{1}{R(t)}$ we have

$$k_t = -\frac{1}{R^2}\dot{R} = k_{ss} + k^3 = \frac{1}{R^3}.$$

If $R(0) = R_0$ we get

$$R(t) = \sqrt{R_0^2 - 2t},$$

and we see that the circle shrinks to a point when $t = \frac{R_0^2}{2}$. We can also verify proposition 2.9 by noting that

$$A(t) = \pi R(t)^2 = \pi(R_0^2 - 2t),$$

so that $\frac{dA}{dt} = -2\pi$. □

2.2 Normal curvature

Suppose $S \subset \mathbb{R}^3$ is a surface and $p \in S$. Let $T_p(S)$ be the tangent space at p, i.e. the vector space of vectors tangent to S at p. If ν is the normal vector to S at p and $v \in T_p(S)$, the plane through p determined by v and ν intersects S in a curve. We call this curve r_v, see figure 2.

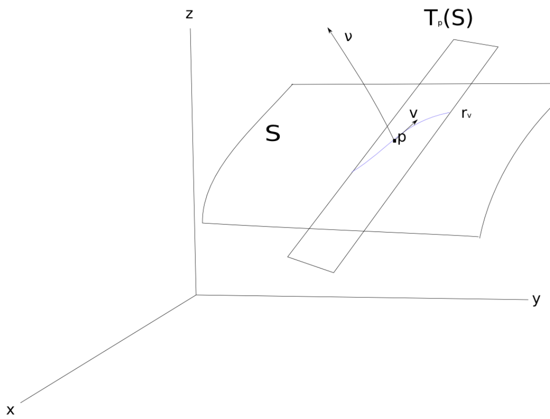

Figure 2: *The normal curvature at a point on a surface.*

Definition 2.11. *The normal curvature, $k_n(v)$ at a point $p \in S$ is given by the curvature of the curve parametrized by r_v.*

We denote $k_1 = \min_v k_n(v)$ and $k_2 = \max_v k_n(v)$. The mean curvature is defined to be the sum of the two.

Definition 2.12. *The mean curvature H at a point p on a surface S is given by*

$$H = k_1 + k_2.$$

Suppose $0 \in S$ and S is given by the surface $z = f(x,y)$. We suppose further that $f = f_x = f_y = f_{xy} = 0$ at $(0,0)$. Thus the tangent plane is spanned out by the vectors $(1,0,0)$ and $(0,1,0)$ and we can take $\nu = (0,0,1)$. A vector $v \in T_p(S)$ can be written $(s\cos\theta, s\sin\theta, 0)$ where $\theta \in [0,2\pi]$ and $s \in \mathbb{R}$. The curve r_v is an intersection of the surface $z = f(x,y)$ and the plane determined by ν and v. For $t \in \mathbb{R}$ and $\theta \in [0,2\pi]$ we can take

$$r_v(t) = (t\cos\theta, t\sin\theta, f(t\cos\theta, t\sin\theta)).$$

At the origin we have $|\dot{r}| = |(\cos\theta, \sin\theta, 0)| = 1$ so we easily calculate

$$\begin{aligned} k_n(v) = k_n(\theta) = |\ddot{r}(0)| &= \left|(0,0,\cos^2\theta f_{xx} + \sin^2\theta f_{yy})\right| \\ &= \cos^2\theta f_{xx}(0,0) + \sin^2\theta f_{yy}(0,0), \end{aligned}$$

at the origin. This is sometimes referred to as *Euler's formula.* We see that $H(0,0) = f_{xx}(0,0) + f_{yy}(0,0)$.

Example 2.13. ***(Curvatures at the origin for an elliptic paraboloid.)***

An elliptic paraboloid can be written

$$z = f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

Using Euler's formula we find $k_n = \frac{2}{a^2}\cos^2\theta + \frac{2}{b^2}\sin^2\theta$. Hence, at the origin,

$$H = \frac{2(a^2+b^2)}{a^2b^2}.$$

□

2.3 Mean curvature

In this section, we use the Einstein summation convention used in [F]. We describe a surface S in $\mathbb{R}^3$ as the image of the vector function

$$X(u^1,u^2) = X(x(u^1,u^2), y(u^1,u^2), z(u^1,u^2)).$$

In this section we assume that *the first and second partial derivatives of X exist and are continuous.*

Suppose

$$r(t) = X(u^1(t), u^2(t))$$

describe a curve C on the surface S. Then

$$\frac{dr}{dt} = \frac{du^1}{dt}\frac{\partial X}{\partial u^1} + \frac{du^2}{dt}\frac{\partial X}{\partial u^2} = u^{i'}\frac{\partial X}{\partial u^i}. \tag{3}$$

As in the planar case, we except a relation between curvature and the arclength s. We calculate

$$\begin{aligned}\left(\frac{ds}{dt}\right)^2 &= \left\langle \frac{dr}{dt}, \frac{dr}{dt}\right\rangle \\ &= \left(\frac{du^1}{dt}\right)^2 \left|\frac{\partial X}{\partial u^1}\right|^2 + 2\frac{du^1}{dt}\frac{du^2}{dt}\left\langle \frac{\partial X}{\partial u^1}, \frac{\partial X}{\partial u^2}\right\rangle + \left(\frac{du^2}{dt}\right)^2 \left|\frac{\partial X}{\partial u^2}\right|^2 \\ &\equiv g_{ij}u^{i'}u^{j'}\end{aligned}$$

or in differential form,

$$ds^2 = g_{ij}du^i du^j.$$

The metric

$$g_{ij} = \left\langle \frac{\partial X}{\partial u^i}, \frac{\partial X}{\partial u^j}\right\rangle$$

is called the *first fundamental form.*

Lemma 2.14. *Suppose the surface S is a regular surface, so that the unit normal at any point $P \in S$ satisfies*

$$\nu(P) = \frac{\frac{\partial X}{\partial u^1} \times \frac{\partial X}{\partial u^2}}{\left|\frac{\partial X}{\partial u^1} \times \frac{\partial X}{\partial u^2}\right|} \neq 0.$$

Then the matrix $G = (g_{ij})_{ij}$ is a positive matrix.[1]

[1] In fact, it is *strictly* positive, which means that $x^T G x \geqslant 0$ with equality only for $x = 0$.

Proof. Using the vector identity

$$\langle a \times b, c \times d \rangle = \langle a, c \rangle \langle b, d \rangle - \langle a, d \rangle \langle b, c \rangle$$

we get

$$\begin{aligned} 0 &\leqslant \left\langle \frac{\partial X}{\partial u^1} \times \frac{\partial X}{\partial u^2}, \frac{\partial X}{\partial u^1} \times \frac{\partial X}{\partial u^2} \right\rangle \\ &= g_{11} g_{22} - g_{12}^2. \end{aligned}$$

Since $g_{11} = \|\frac{\partial X}{\partial u^1}\|^2 \geqslant 0$ we have for all $x \in \mathbb{R}^2$

$$\begin{aligned} x^T G x &= x_1^2 g_{11} + 2 x_1 x_2 g_{12} + x_2^2 g_{22} \\ &= \frac{1}{g_{11}} \left(x_1 g_{11} + x_2 g_{12} \right)^2 + \frac{g_{11} g_{22} - g_{12}^2}{g_{11}} x_2^2 \geqslant 0. \end{aligned}$$

□

In view of this observation, the matrix G has an inverse. We denote it by $G^{-1} = (g^{ij})_{ij}$ so that

$$g_{ij} g^{ij} = \delta_{ij}.$$

In order to give a rigorous definition of the normal curvature at a point P on a surface S, we look at the normal component of $\frac{d^2 r}{ds^2}$. By equation (3), $\frac{dr}{ds} = u^{i'} \frac{\partial X}{\partial u^i}$ so that

$$\frac{d^2 r}{ds^2} = u^{i''} \frac{\partial X}{\partial u^i} + u^{i'} u^{j'} \frac{\partial^2 X}{\partial u^i \partial u^j}.$$

Taking the inner product with the unit normal ν gives

$$\left\langle \frac{d^2 r}{ds^2}, \nu \right\rangle = u^{i'} u^{j'} \left\langle \frac{\partial^2 X}{\partial u^i \partial u^j}, \nu \right\rangle \equiv L_{ij} u^{i'} u^{j'}.$$

Here,

$$L_{ij} = \left\langle \frac{\partial^2 X}{\partial u^i \partial u^j}, \nu \right\rangle$$

is called the *second fundamental form* on the surface.

Definition 2.15. *Let v be a unit vector tangent to the surface S at a point P, so that it can be written*

$$v = v^i \frac{\partial X}{\partial u^i}$$

The normal curvature of S at P in the direction of v is given by

$$k_n(v) = L_{ij} v^i v^j.$$

Remark. We see that this definition coincides with definition 2.11. If $r = r(s)$ is the curve created from intersecting S with the plane through p determined by v and ν we have for some s_0,

$$r(s_0) = p, \quad r'(s_0) = v, \quad r''(s_0) = \pm\nu.$$

Since

$$\frac{dr}{ds} = u^{i'} \frac{\partial X}{\partial u^i} = v = v^i \frac{\partial X}{\partial u^i}$$

we see that $v^i = u^{i'}$. Hence

$$k_n(v) = L_{ij} v^i v^j = L_{ij} u^{i'} u^{j'} = \left\langle \frac{d^2 r}{ds^2}, \nu \right\rangle = \pm \left| \frac{d^2 r}{ds^2} \right|,$$

the formula for curvature rediscovered.

We now give the formula for the mean curvature of a regular surface parametrized by X.

Lemma 2.16. *Suppose S is a regular surface. Then*

$$H = \operatorname{tr}(G^{-1} L) = g^{ij} L_{ij}.$$

Proof. We are trying to maximize and minimize

$$L_{ij} v^i v^j$$

with the restriction that $v = v^i \frac{\partial X}{\partial u^i}$ is of unit length. Note that

$$|v|^2 = g_{ij} v^i v^j.$$

Writing $x = (v^1, v^2)$ the problem is to find

$$\max_{x^T G x = 1} x^T L x, \quad \text{and} \quad \min_{x^T G x = 1} x^T L x.$$

Using corollary A.7 we get

$$H = k_1 + k_2 = \max_{x^T G x = 1} x^T L x + \min_{x^T G x = 1} x^T L x = \mathrm{tr}(G^{-1} L).$$

□

Example 2.17. ***(Mean curvature for a torus.)***

Consider a circle of radius $r < 1$ centered at $(1, 0)$ in the xz-plane. The circle may be parametrized by

$$\begin{aligned} x &= 1 + r\cos\theta \\ z &= r\sin\theta \end{aligned}$$

for $0 \leqslant \theta \leqslant 2\pi$. Revolving the circle about the z-axis gives us the following parametrization for the surface of a torus

$$X(\theta, \phi) = ((1 + r\cos\theta)\cos\phi, (1 + r\cos\theta)\sin\phi, r\sin\theta)$$

for $0 \leqslant \theta, \phi \leqslant 2\pi$. Upon differentiation we find

$$G^{-1} = \frac{1}{r^2 (1 + r\cos\theta)^2} \begin{bmatrix} r^2 & 0 \\ 0 & (1 + r\cos\theta)^2 \end{bmatrix}, \quad L = \begin{bmatrix} (1 + r\cos\theta)\cos\theta & 0 \\ 0 & r \end{bmatrix}.$$

Using $H = \mathrm{tr}(G^{-1}L)$ we find

$$H = \frac{1 + 2r\cos\theta}{r(1 + r\cos\theta)}.$$

□

For an explicit surface $z = f(x, y)$ we calculate

$$g^{ij} = \delta_{ij} - \frac{f_{x_i} f_{x_j}}{1 + |\nabla f|^2}, \qquad L_{ij} = \frac{1}{\sqrt{1 + |\nabla f|^2}} f_{x_i x_j}$$

so that

$$H(x, y) = g^{ij} L_{ij} = \frac{(1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy}}{(1 + f_x^2 + f_y^2)^{\frac{3}{2}}}. \tag{4}$$

Example 2.18. ***(Mean curvature for the elliptic paraboloid.)***

We check the calculations from example 2.13. An elliptic paraboloid may be written $z = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2$. From equation (4) we calculate

$$H(x,y) = \frac{\frac{2}{a^2}\left(1+4\frac{y^2}{b^4}\right) + \frac{2}{b^2}\left(1+4\frac{x^2}{a^4}\right)}{\left(1+4\frac{x^2}{a^4}+4\frac{y^2}{b^4}\right)^{\frac{3}{2}}}$$

and at the origin, $H = \frac{2(a^2+b^2)}{a^2b^2}$ as before. From the expression of $H(x,y)$ we see that $H(x,y)$ has a maximum at the origin. This fits well with our intuition. At the origin the paraboloid clearly deviates more than any other point from being a flat surface. See figure 3. $\square$

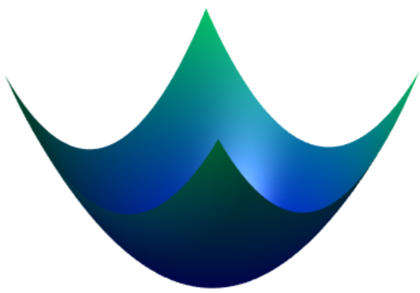

Figure 3: *The paraboloid given by $z = x^2 + y^2$.*

For the level-set method, the surface evolves accordingly to $u(x,y,z,t) = 0$, i.e. the surface is given implicitly. The next theorem shows how to calculate the mean curvature for implicit surfaces.

Theorem 2.19. *For a surface S given by $u(x,y,z) = 0$, the mean curvature is given by*

$$H(x,y,z) = -\mathrm{div}(\nu) \tag{5}$$

provided $\nabla u \neq 0$. Here, $\nu = \frac{\nabla u}{|\nabla u|}$ is the inward pointing unit normal vector.

Proof. First, assume $u_z \neq 0$. Then z can be written as a function of x and y, say $z = f(x, y)$. Since $u(x, y, z) = 0$, we get, by keeping x and y constant in turn,

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = 0$$
$$\left(\frac{\partial u}{\partial y}\right)_x = \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = 0.$$

Solving for the partial derivatives of $z = f(x, y)$ yields $f_x = -\frac{u_x}{u_z}$ and $f_y = -\frac{u_y}{u_z}$. Further, by again keeping y constant, we get

$$f_{xx} = \frac{2u_x u_z u_{xz} - u_x^2 u_{zz} - u_z^2 u_{xx}}{u_z^3}.$$

Similar calculations can be done to find expressions for f_{xy} and f_{yy}. Inserting the partial derivatives of f into equation (5) gives the same result as the calculation of $-\text{div}(\nu)$. If $f_z = 0$ at some point, we repeat the calculation by assuming either $f_x \neq 0$ or $f_y \neq 0$. Since $\nabla u \neq 0$ was assumed, the partial derivaties of f cannot all be zero at the same point.

□

2.4 The level set method

Let
$$\Gamma_t = \left\{(x, y, z) \in \mathbb{R}^3 : u(x, y, z, t) = 0\right\}.$$

Suppose first that, for all $t \geqslant 0$, $\Gamma_t \subset \Omega$ with $\nabla u \neq 0$ in Ω, where $\Omega \subset \mathbb{R}^3 \times [0, \infty)$. Then
$$\nu = \frac{\nabla u}{|\nabla u|}$$
chosen to be pointing inwards is a unit normal vector of Γ_t. Consequently, from theorem 5, we have $H = -\text{div}\left(\frac{\nabla u}{|\nabla u|}\right)$.

The idea is to follow the points on the surface $X = (x(t), y(t), z(t))$ as time passes. We define the motion by mean curvature as $\dot{X} = H\nu$. This means that the surface moves with velocity equal to the mean curvature in the normal direction. Since $u(X, t) = 0$ in Γ_t we have
$$\frac{d}{dt}u(X(t), t) = 0.$$

Hence, by the chain rule,

$$0 = \frac{\partial u}{\partial t} + \dot{X} \cdot \nabla u = \frac{\partial u}{\partial t} + H\nu \cdot \nabla u = \frac{\partial u}{\partial t} - |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

Now, suppose we are given an initial surface

$$\Gamma_0 = \left\{ x \in \mathbb{R}^3 : u(x) = g(x) = 0 \right\}.$$

and want to study how the surface evolves by mean curvature flow. That is, we ask the question, how does the following set behave

$$\Gamma_t = \left\{ x \in \mathbb{R}^3 : u(x,t) = 0 \right\}.$$

As we will see, this is equivalent to solving the problem

$$\begin{cases} u_t = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right), & (x,t) \in \mathbb{R}^n \times [0,\infty) \\ \\ u(x,0) = g(x), & (x,t) \in \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

The problem now is that the equation is not defined where $\nabla u = 0$. Further, we can not guarantee existence of a twice differentiable solution. We seek for a weak solution, namely a *viscosity solution* to overcome this difficulty.

Remark. Some C^2 solutions of the above equation are

$$|x|^2 + 4t, \quad e^{|x|^2+4t}, \quad e^{x_1}, \quad \cosh x_1, \quad \cosh\left(|x|^2 + 4t\right).$$

These are, however, not of particular interest, since their zero level sets are empty or trivial. We should however note that, if u solves the equation then it seems like $\phi(u)$, for a smooth function ϕ, also solves the equation. We will prove this assertion in section 5 in the viscosity sense, only requiring ϕ to be continuous.

Example 2.20. ***(Mean curvature flow for the sphere, the plane, the cylinder and the torus.)***

A plane may be described as solutions to $g(x,y,z) = ax + by + cz - d = 0$ where $a, b, c, d \in \mathbb{R}$. We see that g satisfies the mean curvature flow equation, so we can take $u = g$. Hence

$$\Gamma_t = \{x : g(x) = 0\} = \Gamma_0,$$

so nothing happens to the plane under mean curvature flow. Consider now the initial surface $\Gamma_0 = \{x \in \mathbb{R}^3 : x^2 + y^2 + z^2 = R_0^2\}$, the sphere of radius R. Under mean curvature flow, the sphere's radius shrinks. Letting $R = R(t)$, we see that $\dot{R} = \frac{-2}{R}$, with $R(0) = R_0$. This can be seen by using the defining relation $\dot{X} = H\nu$. The solution to the differential equation is given by

$$R(t) = \sqrt{R_0^2 - 4t}.$$

We verify also that $u(x, y, z, t) = R(t)^2 - x^2 - y^2 - z^2$ satisfies the mean curvature flow equation. The sphere shrinks to a point in finite time, $t = \frac{R_0^2}{4}$. A similar calculation shows that a spherical cylinder shrinks to a line under mean curvature flow. For the torus, we calculated from example 2.17

$$H = \frac{1 + 2r\cos\theta}{r(1 + r\cos\theta)}.$$

We take $0 < r << 1$ (if r is close to 1, the evolution can be similar to that of a sphere, see [SS]). The expression does not depend on ϕ, the angle from revolving a circle about a line. The surface will remain a surface of revolution under the mean curvature flow, but the cross section will not remain a circle, since H varies with θ. As the evolution goes on, the cross section will shrink to a point, and hence the torus evolves under mean curvature flow until it becomes a circle. See figure 4, 5, 6 and 7 (we abbreviate MCF for mean curvature flow). $\square$

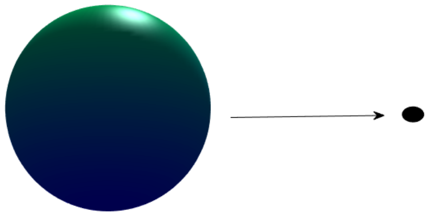

Figure 4: MCF for the sphere.

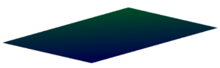

Figure 5: MCF for the plane.

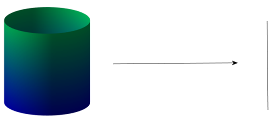

Figure 6: MCF for the cylinder

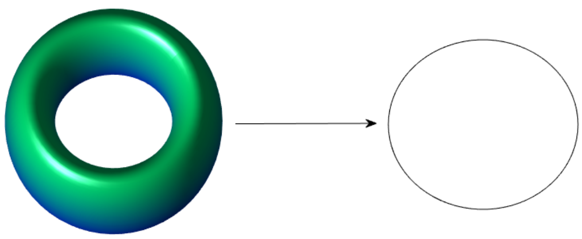

Figure 7: MCF for the torus.

3 Viscosity solutions

3.1 Introduction

When looking at the mean curvature flow equation

$$\begin{cases} u_t = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) & (x,t) \in \mathbb{R}^n \times [0,\infty) \\ u(x,0) = g(x) & (x,t) \in \mathbb{R}^n \times \{t=0\} \end{cases} \tag{6}$$

with only continuous $u = u(x,t)$ it is clear that u does not satisfy equation (6) in the classical sense. An often used technique to overcome this difficulty is to multiply the equation with a test function. With integration by parts one can pass the equation over to the test function in an integral form. However, with trial and error, one quickly realizes that the method does not work here. This is where the notion of *viscosity solutions* enters. In section 3.1 and 3.2, we assume that $\nabla u \neq 0$.

Definition 3.1. *A bounded and continuous function u is said to be a viscosity subsolution of equation* (6) *if for all $\phi \in C^2(\mathbb{R}^n \times [0,\infty])$,*

$$\phi_t \leqslant |\nabla \phi| \operatorname{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right)$$

at any point (x,t) where $u - \phi$ attains a local maximum. Similarly, u is a viscosity supersolution if

$$\phi_t \geqslant |\nabla \phi| \operatorname{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right)$$

at any point (x,t) where $u - \phi$ attains a local minimum. The function u is called a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution. In addition, it is required that $u(x,0) = g(x)$.

Remark.

a) We may assume that the local maximum is strict. To see this, replace $\phi(x,t)$ by $\phi(x,t) - |x - x_0|^4 - (t - t_0)^4$, where (x_0, t_0) is the point where $u - \phi$ has a local maximum. The same applies to the local minimum.

b) The equations $\phi_t = |\nabla\phi| \operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)$ and $-\phi_t = -|\nabla\phi| \operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)$ are not equivalent in the viscosity sense.

c) We can remove the restriction of continuity by requiring only upper and lower semi continuity for the viscosity sub- and supersolutions respectively. For discussion, see [CIL].

One of the first things one should check when making a definition is consistency, which is stated in the following lemma. We first note that the PDE in equation (6) can be rewritten to

$$u_t = \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|\nabla u|^2}\right) u_{x_i x_j} \tag{7}$$

where we sum over $1 \leqslant i, j \leqslant n$.

Lemma 3.2. ***(Consistency of viscosity solutions.)***
If $u \in C^2\big(\mathbb{R}^n \times [0,\infty)\big)$ is a classical solution of equation (7), then u is a viscosity solution. Further, if u is twice differentiable everywhere and u is a viscosity solution, then u is a classical solution.

Proof. Suppose first that u is a classical solution. Pick $\phi \in C^2$ and (x_0, t_0) so that $u - \phi$ has a local minimum point at (x_0, t_0). By the infinitesimal calculus, $\phi_t = u_t$, $\nabla\phi = \nabla u$ and

$$D^2(u - \phi) \geqslant 0$$

at the point (x_0, t_0). Hence, using equation (7)

$$\begin{aligned}
\phi_t = u_t = &\left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|\nabla u|^2}\right) u_{x_i x_j} \\
= &\left(\delta_{ij} - \frac{\phi_{x_i} \phi_{x_j}}{|\nabla \phi|^2}\right) \left(u_{x_i x_j} - \phi_{x_i x_j}\right) + \left(\delta_{ij} - \frac{\phi_{x_i} \phi_{x_j}}{|\nabla \phi|^2}\right) \phi_{x_i x_j} \\
\equiv &\operatorname{tr}(AD^2(u - \phi)) + \left(\delta_{ij} - \frac{\phi_{x_i} \phi_{x_j}}{|\nabla \phi|^2}\right) \phi_{x_i x_j}.
\end{aligned}$$

By proposition A.4 and example A.2 we get

$$\phi_t \geqslant \left(\delta_{ij} - \frac{\phi_{x_i} \phi_{x_j}}{|\nabla \phi|^2}\right) \phi_{x_i x_j}$$

at (x_0, t_0), which shows that u is a viscosity supersolution. A similar argument can be used to show that u is a viscosity subsolution.

If u is a C^2 viscosity solution, let $\phi = u$. Then $u - \phi$ has a maximum everywhere and since u is a viscosity subsolution,

$$u_t = \phi_t \leqslant |\nabla \phi| \operatorname{div}\left(\frac{\nabla \phi}{|\nabla \phi|}\right) = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right).$$

In addition, $u - \phi$ has a minumum everywhere and since u is a viscosity supersolution, $u_t \geqslant |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ at all points (x, t). This shows that u solves the equation in the classical sense. $\square$

Remark. In general, if a viscosity solution u is not twice differentiable everywhere, we cannot say that u is a classical solution. However, if u is a twice differentiable viscosity solution at a point (x, t), then u satisfies the equation in the classical sense at the point (x, t). A proof of this is given in [E] for first order equations and in [K] for general second order equations. The main idea behind the proof is that if u is twice differentiable at some point, there exists a $\phi \in C^2$ so that $u = \phi$ at this point as shown in figure 8.

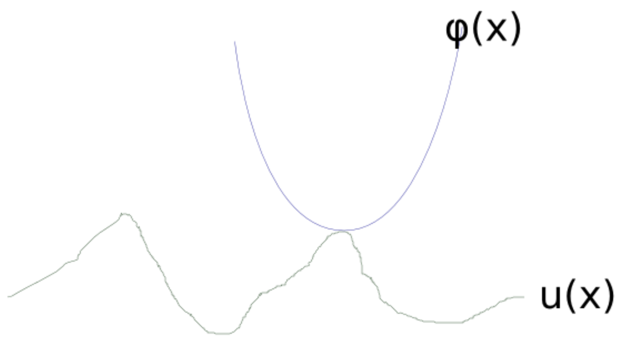

Figure 8: *Touching a C^2 function ϕ.*

3.2 The method of vanishing viscosity

The name *viscosity solution* as a notion of a weak solution has its origin from the method of vanishing viscosity. The idea is to add a viscosity term to a nonlinear partial differential equation. For the ongoing discussion to work for more general equations, suppose that

$$u_t^\epsilon + F(\nabla u^\epsilon, D^2 u^\epsilon) = \epsilon \Delta u^\epsilon,$$

where F is a continuous function. One hopes that, as $\epsilon \to 0$, the function $u = \lim_{\epsilon \to 0} u^\epsilon$ is a viscosity solution to the equation $u_t + F(\nabla u, D^2 u) = 0$. For the procedure to work, we need the following condition on F.

Definition 3.3. *If F satisfies*

$$F(p, X) \geqslant F(p, Y)$$

whenever $X \leqslant Y$, we say that F is degenerate elliptic.

Remark. For the mean curvature flow equation we have

$$F(p, X) = -\left(\delta_{ij} + \frac{p_i p_j}{|\nabla p|}\right) X_{ij}$$

and we see that F is degenerate elliptic.

Proposition 3.4. *Suppose $u^\epsilon \in C^2$ solves*

$$u_t^\epsilon + F(\nabla u^\epsilon, D^2 u^\epsilon) = \epsilon \sum_{i,j=1}^{n} a_{ij} u^\epsilon_{x_i x_j}, \tag{8}$$

where $A = (a_{ij})_{ij}$ satisfies

$$\xi^T A \xi \geqslant \theta |\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and some constant $\theta > 0$. Further, suppose that F is continuous and degenerate elliptic. If $u^\epsilon \to u$ uniformly on compact subsets of $\mathbb{R}^n \times \{t = 0\}$, then u is a viscosity solution of

$$u_t + F(\nabla u, D^2 u) = 0.$$

Proof. Here, it is only shown that u is a viscosity supersolution. Using similar arguments, one can show that u is a viscosity subsolution.

Suppose $u-\phi$ has a minimum at (x_0,t_0). Find (x_ϵ,t_ϵ) so that $u^\epsilon-\phi$ has a minimum at (x_ϵ,t_ϵ) and $(x_\epsilon,t_\epsilon)\to(x_0,t_0)$. These points exist (possibly taking some subsequence $(x_{\epsilon_j},t_{\epsilon_j})$) since $u^\epsilon\to u$ uniformly.

At (x_ϵ,t_ϵ) the first partial derivatives of ϕ and u^ϵ coincide, and

$$D^2\left(u^\epsilon-\phi\right)\geqslant 0,$$

from the infinitesimal calculus. Hence, at the point (x_ϵ,t_ϵ),

$$\begin{aligned}
\phi_t+F\left(\nabla\phi,D^2\phi\right)&=u^\epsilon_t+F\left(\nabla u^\epsilon,D^2\phi\right)\\
&\geqslant u^\epsilon_t+F\left(\nabla u^\epsilon,D^2u^\epsilon\right)\\
&=\epsilon a_{ij}u^\epsilon_{x_ix_j}\\
&=\epsilon a_{ij}\left(u^\epsilon_{x_ix_j}-\phi_{x_ix_j}\right)+\epsilon a_{ij}\phi_{x_ix_j}\\
&\geqslant \epsilon a_{ij}\phi_{x_ix_j}.
\end{aligned}$$

Passing to the limit $\epsilon\to 0$ using that F is continuous yields at (x_0,t_0)

$$\phi_t+F\left(\nabla\phi,D^2\phi\right)\geqslant 0$$

which shows that u is a viscosity supersolution.

□

This section ends with an example illustrating proposition 3.4.

Example 3.5. ***(The method of vanishing viscosity.)***
Consider the problem

$$\begin{cases} u^\epsilon_t+\frac{1}{2}(u^\epsilon_x)^2=\epsilon u^\epsilon_{xx}, & (x,t)\in\mathbb{R}\times[0,\infty)\\ u^\epsilon(x,0)=x^2, & (x,t)\in\mathbb{R}\times\{t=0\}\end{cases}$$

Here, $F(p,X)=F(p)$ so that F is automatically degenerate elliptic. For $\epsilon=0$, the solution is given by the Hopf-Lax formula which is discussed in section 4.2,

$$u(x,t)=\frac{x^2}{1+2t}.$$

Let

$$v(x,t) = e^{\frac{-u^\epsilon(x,t)}{2\epsilon}}.$$

Then $v(x,t)$ solves the heat equation with diffusion constant ϵ,

$$\begin{aligned} v_t &= \epsilon v_{xx} \\ v(x,0) &= e^{\frac{-x^2}{2\epsilon}}. \end{aligned}$$

The solution to the heat equation may be found using fourier analysis,

$$\begin{aligned} v(x,t) &= \frac{1}{\sqrt{4\pi\epsilon t}} \int_{-\infty}^{\infty} e^{\frac{-y^2}{2\epsilon}} e^{-\frac{(x-y)^2}{4\epsilon t}} dy \\ &= e^{\frac{-x^2}{2\epsilon(1-2t)}} \frac{1}{\sqrt{4\pi\epsilon t}} \int_{-\infty}^{\infty} e^{-\frac{1+2t}{4\epsilon t}(y-\frac{x}{1+2t})^2} dy \\ &= e^{\frac{-x^2}{2\epsilon(1-2t)}} \frac{1}{\sqrt{\pi(1+2t)}} \int_{-\infty}^{\infty} e^{-z^2} dz \\ &= \frac{1}{\sqrt{1+2t}} e^{-\frac{x^2}{2\epsilon(1+2t)}}. \end{aligned}$$

Here, we have completed the square and used the gaussian integral, $\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}$. We can now invert the formula for $v(x,t)$ to find a formula for $u^\epsilon(x,t)$. Note that $v(x,t) \geqslant 0$ for all x,t. For a strictly positive $v(x,0)$, this is always the case for the heat equation, since we integrate a positive function. This fact is crucial for the example, since we are working with logarithms. Hence, we get

$$\begin{aligned} u^\epsilon(x,t) &= -2\epsilon \ln(v(x,t)) \\ &= \epsilon \ln(1+2t) + \frac{x^2}{1+2t}. \end{aligned}$$

As $\epsilon \to 0$ we see that $u^\epsilon(x,t) \to \frac{x^2}{1+2t}$. Since u is a classical solution to the original equation, u is a viscosity solution by the consistency lemma 3.2. □

3.3 The problem with zero gradient.

The partial differential equation describing mean curvature flow only makes sense at points where $\nabla u \neq 0$. Thus, we need to somehow extend definition 3.1 to hold at points where $\nabla \phi(x_0, t_0) = 0$. Suppose $u \in C^2(\mathbb{R}^n \times [0,\infty))$ satisfies

$$u_t \leqslant \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|\nabla u|^2}\right) u_{x_i x_j}. \tag{9}$$

The idea is to look at the behavior of u close to the point (x_0, t_0) with $\nabla u(x_0, t_0) = 0$, where equation (9) is not defined.

If $\nabla u(x_0, t_0) = 0$, suppose there are points $(x_k, t_k) \to (x_0, t_0)$ so that $\nabla u(x_k, t_k) \neq 0$ for all $k \in \mathbb{N}$. Then, at (x_k, t_k),

$$u_t \leqslant (\delta_{ij} - \eta_i^k \eta_j^k) u_{x_i x_j} \tag{10}$$

for

$$\eta_i^k = \frac{u_{x_i}(x_k, t_k)}{|\nabla u(x_k, t_k)|}.$$

Since $|\eta_i^k| \leqslant 1$, $\{\eta_i^k\}_k$ is a bounded set of numbers, by the Bolzano- Weierstrass theorem (C.4) we can extract a convergent subsequence

$$\eta_i^{k_l} \to \eta_i$$

with $|\eta_i| \leqslant 1$. Passing to the limit $k_l \to \infty$ in equation (10) gives, at (x_0, t_0),

$$u_t \leqslant (\delta_{ij} - \eta_i \eta_j) u_{x_i x_j}$$

for some $\eta \in \mathbb{R}^n$ with $|\eta| \leqslant 1$.

On the other hand, if we cannot find points $(x_k, t_k) \to (x_0, t_0)$ with $\nabla u(x_k, t_k) \neq 0$, there is a $\delta > 0$ so that

$$\nabla u = 0$$

when $|x - x_0|^2 + (t - t_0)^2 < \delta$. Fix $t = t_0$ and find $R > 0$ as large as possible so that $\nabla u = 0$ in $B_R(x_0)$. Since $u \in C^2$, $\nabla u = D^2 u = 0$ on $\partial B_R(x_0)$. However, there are points arbitrary close to $\partial B_R(x_0)$, say for example $y \in \mathbb{R}^n$, at which $\nabla u \neq 0$. Here, equation (9) holds. Hence, for $\xi \in \partial B_R(x_0)$,

$$\begin{aligned} u_t(\xi, t_0) \leftarrow u_t(y, t_0) &\leqslant \left(\delta_{ij} - \frac{u_{x_i}(y, t_0) u_{x_j}(y, t_0)}{|\nabla u(y, t_0)|^2} \right) u_{x_i x_j}(y, t_0) \\ &\leqslant \delta_{ij} u_{x_i x_j}(y, t_0) \\ &\to \delta_{ij} u_{x_i x_j}(\xi, t_0) = 0 \end{aligned}$$

upon passing to the limit $y \to \xi \in \partial B_R(x_0)$. Here we used the fact that $u \in C^2(\mathbb{R}^n \times [0, \infty))$. Since u does not vary with x in $B_R(x_0)$, we get

$$u_t(x, t_0) \leqslant 0$$

for all $x \in B_R(x_0)$ and in particular $u_t(x_0, t_0) \leqslant 0$. Hence, for any $\eta \in \mathbb{R}^n$, we have at (x_0, t_0),

$$u_t \leqslant (\delta_{ij} - \eta_i \eta_j) u_{x_i x_j}.$$

We are now able to give the complete definition of a viscosity solution of equation (6).

Definition 3.6. *Suppose u is a continuous and bounded function. We say that u is a viscosity subsolution of* (6) *if for all $\phi \in C^2$,*

$$\phi_t \leqslant \left(\delta_{ij} - \frac{\phi_{x_i} \phi_{x_j}}{|\nabla \phi|^2} \right) \phi_{x_i x_j} \tag{11}$$

at any point (x, t) where $u - \phi$ attains a local maximum, provided $\nabla \phi(x_0, t_0) \neq 0$. Further,

$$\phi_t \leqslant (\delta_{ij} - \eta_i \eta_j) \phi_{x_i x_j} \tag{12}$$

for some $\eta \in \mathbb{R}^n$ with $|\eta| \leqslant 1$ at any point (x, t) where $u - \phi$ attains a local maximum and $\nabla \phi(x_0, t_0) = 0$.
Similarly, u is a viscosity supersolution if the reversed inequality in equation (11) *holds where $u - \phi$ attains a local minimum and $\nabla \phi(x_0, t_0) \neq 0$. If $\nabla \phi(x_0, t_0) = 0$, the reversed inequality in equation* (12) *should hold.*

If u is both a viscosity sub- and supersolution, and $u(x, 0) = g(x)$, we say that u is a viscosity solution of (6).

3.4 Semi-Jets

3.4.1 An equivalent viscosity definition

We introduce an equivalent definition of viscosity solutions. First, we give the definition of the parabolic semi-jets of a function.

Definition 3.7. *Suppose u is bounded and continuous. If $(x_0, t_0) \in \mathbb{R}^n \times [0, \infty)$ and*

$$u(x,t) \leqslant u(x_0, t_0) + q(t - t_0) + p \cdot (x - x_0) + \frac{1}{2}(x - x_0)^T A (x - x_0) + o(|t - t_0| + |x - x_0|^2)$$

when $x \to x_0$, $t \to t_0$, we say that $(q, p, A) \in P^{2,+} u(x_0, t_0)$. Similarly, if

$$u(x,t) \geqslant u(x_0, t_0) + q(t - t_0) + p \cdot (x - x_0) + \frac{1}{2}(x - x_0)^T A (x - x_0) + o(|t - t_0| + |x - x_0|^2)$$

when $x \to x_0$, $t \to t_0$, we say that $(q, p, A) \in P^{2,-}u(x_0, t_0)$. In both cases, $p \in \mathbb{R}^n$, $q \in \mathbb{R}$ and A is a symmetric $n \times n$ matrix.

Proposition 3.8. *The following properties for the parabolic semi-jets holds.*

(i) If $u \in C^2(\mathbb{R}^n \times [0, \infty))$, then

$$P^{2,+}u(x_0, t_0) \cap P^{2,-}u(x_0, t_0) = \big(u_t(x_0, t_0), \nabla u(x_0, t_0), D^2 u(x_0, t_0)\big).$$

(ii)

$$P^{2,+}u(x_0, t_0) = -P^{2,-}(-u)(x_0, t_0).$$

Proof. (i) follows by expanding u in a Taylor series around (x_0, t_0). (ii) follows by a direct computation. $\square$

Definition 3.9. *A continuous and bounded function u is a viscosity subsolution at (x_0, t_0) of equation (6) if*

$$\begin{aligned} q &\leqslant \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) a_{ij}, \quad \text{if } p \neq 0,\\ q &\leqslant (\delta_{ij} - \eta_i \eta_j) a_{ij}, \quad \text{if } p = 0, \end{aligned}$$

for some $\eta \in \mathbb{R}^n$, provided $(q, p, A) \in P^{2,+}u(x_0, t_0)$ with $|\eta| \leqslant 1$.

A continuous and bounded function u is a viscosity supersolution at (x_0, t_0) of equation (6) if

$$\begin{aligned} q &\geqslant \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) a_{ij}, \quad \text{if } p \neq 0,\\ q &\geqslant (\delta_{ij} - \eta_i \eta_j) a_{ij}, \quad \text{if } p = 0, \end{aligned}$$

for some $\eta \in \mathbb{R}^n$, provided $(q, p, A) \in P^{2,-}u(x_0, t_0)$ with $|\eta| \leqslant 1$. Finally, u is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution. In addition, it is required that $u(x, 0) = g(x)$.

It is easy to see that definition 3.1 and 3.9 are equivalent. We show here the basic idea. Suppose that for $(x_0, t_0) \in \mathbb{R}^n \times [0, \infty)$ we have for $\phi \in C^2(\mathbb{R}^n \times [0, \infty))$

$$(u - \phi)(x_0, t_0) \geqslant (u - \phi)(x, t) \tag{13}$$

for all (x, t) close to (x_0, t_0). Expanding ϕ from equation (13) in a Taylor series around (x_0, t_0) gives

$$\begin{aligned} u(x,t) &\leqslant u(x_0,t_0) + \nabla\phi(x_0,t_0) \cdot (x - x_0) + \phi_t(x_0,t_0)(t - t_0) \\ &+ \frac{1}{2}(x - x_0)^T D^2\phi(x_0,t_0)(x - x_0) + o(|t - t_0| + |x - x_0|^2), \end{aligned}$$

which shows that $(\phi_t(x_0, t_0), \nabla\phi(x_0, t_0), D^2\phi(x_0, t_0)) \in P^{2,+}u(x_0, t_0)$. Similar reasoning holds when $(u - \phi)$ has a minimum at (x_0, t_0).

This observation gives us the following way to calculate the parabolic semi-jet of a function.

$$\begin{aligned} P^{2,+}u(x_0,t_0) = \Big\{ \big(\phi_t(x_0,t_0), \nabla\phi(x_0,t_0), D^2\phi(x_0,t_0)\big) &: \exists\phi \in C^2 \text{ such that} \\ &u - \phi \text{ has a maximum at } (x_0,t_0)\Big\}, \end{aligned}$$

$$\begin{aligned} P^{2,-}u(x_0,t_0) = \Big\{ \big(\phi_t(x_0,t_0), \nabla\phi(x_0,t_0), D^2\phi(x_0,t_0)\big) &: \exists\phi \in C^2 \text{ such that} \\ &u - \phi \text{ has a minimum at } (x_0,t_0)\Big\}. \end{aligned}$$

This follows from the remark under the consistency lemma 3.2. In the following example, we omit the t-variable for simplicity. A common notation is then to replace the parabolic semi-jet $P^{2,\pm}$ by the ordinary semi-jet $J^{2,\pm}$.

Example 3.10. ***(Calculations of semi-jets.)***
Suppose $u(x) = |x|$ for $x \in \mathbb{R}$. Since u is smooth at any point except $x = 0$, we have

$$J^{2,+}u(x) = J^{2,-}u(x) = (\{1\} \times 0) \cup (\{-1\} \times 0)$$

for all $x \in \mathbb{R}\backslash\{0\}$. At $x = 0$, note that $J^{2,+}u(0) = \varnothing$, since there can be no smooth ϕ so that $u - \phi$ has a maximum at $x = 0$.
For $J^{2,-}u(0)$, we look for $\phi \in C^2$ such that $u - \phi$ has a minimum at $x = 0$. For this, we may suppose $\phi(0) = u(0) = 0$. We note first that $|\phi'(0)| > 1$ is out of the question, see figure 9.

If $|\phi'(0)| = 1$ as in figure 10, we have $\phi''(0) < 0$. We must ensure $\phi < u$ for all $x \neq 0$ so we see that ϕ has negative curvature at the origin. Finally, if $|\phi'(0)| < 1$ we can allow positive curvature for ϕ, see figure 11. In total we have

$$J^{2,-}u(0) = (\{1\} \times (-\infty, 0]) \cup (\{-1\} \times (-\infty, 0]) \cup ((-1, 1) \times \mathbb{R})\,.$$

□

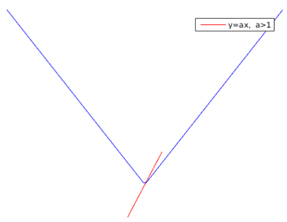

Figure 9: $|\phi'(0)| > 1$

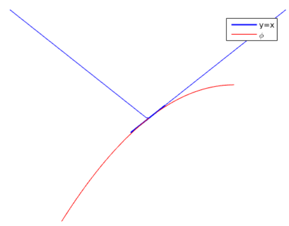

Figure 10: $|\phi'(0)| = 1$

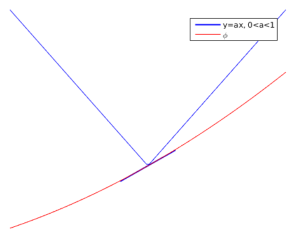

Figure 11: $|\phi'(0)| < 1$

3.4.2 A stability estimate

The closures of $P^{2,+}$ and $P^{2,-}$ are defined as follows. If $(q,p,A) \in \bar{P}^{2,+}u(x_0,t_0)$ there exists $(x_k,t_k) \to (x_0,t_0)$ and $(q^k,p^k,A^k) \to (q,p,A)$ so that

$$(q^k,p^k,A^k) \in P^{2,+}u(x_k,t_k)$$

for each $k \in \mathbb{N}$. We define $\bar{P}^{2,-}$ in a similar manner.

Lemma 3.11. ***Stability***.
Suppose u is a viscosity subsolution of equation (6) *and* $(q,p,A) \in \bar{P}^{2,+}u(x_0,t_0)$. *Then*

$$q \leqslant \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) a_{ij}, \quad \textit{if } p \neq 0,$$
$$q \leqslant (\delta_{ij} - \eta_i \eta_j) a_{ij}, \quad \textit{if } p = 0,$$

for some $\eta \in \mathbb{R}^n$ with $|\eta| \leqslant 1$. A similar result holds for viscosity supersolutions.

Proof. Since $(q,p,A) \in \bar{P}^{2,+}u(x_0,t_0)$ there are $(q^k,p^k,A^k) \to (q,p,A)$ so that

$$q^k \leqslant \left(\delta_{ij} - \gamma_i^k \gamma_j^k\right) a_{ij}^k$$

where

$$\gamma_i^k = \begin{cases} \frac{p_i^k}{|p^k|}, & p^k \neq 0 \\ \eta_i^k, & p^k = 0, |\eta| \leqslant 1 \end{cases}$$

We will use that if A, B are matrices and $A^k \to A, B^k \to B$ pointwise each entry then

$$\lim_{k\to\infty} \operatorname{tr}\left(A^k B^k\right) = \lim_{k\to\infty} \sum_{i,l} A_{il}^k B_{li}^k = \sum_{i,l} \left(\lim_{k\to\infty} A_{il}^k\right)\left(\lim_{k\to\infty} B_{li}^k\right) = \operatorname{tr}(AB).$$

First, if $p \neq 0$ then $p^k \neq 0$ for large enough k. Hence, $\gamma_i^k \to \frac{p_i}{|p|}$ and

$$q \leqslant \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) a_{ij}, \quad \text{if } p \neq 0.$$

Now, if $p = 0$ there is a subsequence

$$\gamma_i^{k_j} \to \eta_i$$

with $|\eta_i| \leqslant 1$. This follow from the Bolzano-Weierstrass theorem (C.4) since $|\gamma_i^k| \leqslant 1$ for all k. We pass to the limit $k_j \to \infty$ to get the result. The proof is similar for the viscosity supersolutions.

$\square$

4 Uniqueness of viscosity solutions

4.1 Uniqueness of C^2 solutions

We want to show that, given the initial function g, equation (6) has a unique solution. We first do it for a simplified case, when $u \in C^2(\mathbb{R}^n \times [0, \infty))$ and $\nabla u \neq 0$. Finally, we remove this restriction by assuming only that u is continuous and bounded. It is clear that for the latter part, we need to use the notion of viscosity solutions.

Theorem 4.1. *Suppose $u, v \in C^2(\mathbb{R}^n \times [0, \infty))$ with nonzero gradients and $u(x, 0) \leqslant v(x, 0)$. Finally assume $u \leqslant v$ when $|x| + t \geqslant R$ for some $R > 0$. Then*

$$u \leqslant v$$

in $\mathbb{R}^n \times [0, \infty)$.

Remark. The condition $u \leqslant v$ when $|x| + t \geqslant R$ will be explained later. An intuitive argument for this assumption is that we are looking at geometrical objects.

Proof. Let

$$\Omega = \{(x, t) \in \mathbb{R}^n \times [0, \infty) : |x| + t \leqslant R\}$$

and define

$$w(x, t) = u(x, t) - v(x, t) - \epsilon t,$$

for $\epsilon > 0$.

When $|x| + t = R$ we have

$$w(x, t) = u(x, t) - v(x, t) - \epsilon t \leqslant -\epsilon t \leqslant 0$$

and

$$w(x, 0) = u(x, 0) - v(x, 0) \leqslant 0.$$

Hence, $w \leqslant 0$ on $\partial\Omega$.

Suppose now that w has a local maximum point in Ω. At this point, $\nabla w = \nabla u - \nabla v = 0$ and $D^2 w = D^2(u-v) \leqslant 0$ from the infinitesimal calculus. Further, $w_t \geqslant 0$. This gives

$$\begin{aligned} 0 \leqslant w_t = u_t - v_t - \epsilon &= \left(\delta_{ij} - \frac{u_{x_i}u_{x_j}}{|\nabla u|^2}\right) u_{x_ix_j} - \left(\delta_{ij} - \frac{v_{x_i}v_{x_j}}{|\nabla v|^2}\right) v_{x_ix_j} - \epsilon \\ &= \left(\delta_{ij} - \frac{u_{x_i}u_{x_j}}{|\nabla u|^2}\right)(u_{x_ix_j} - v_{x_ix_j}) - \epsilon \leqslant -\epsilon, \end{aligned}$$

where the last inequality follows from proposition A.4. We have showed that w can not attain a local maximum in Ω. Thus

$$w(x,t) = u(x,t) - v(x,t) - \epsilon t \leqslant \max_{\partial\Omega} w = 0$$

in Ω. Passing to the limit $\epsilon \to 0$ gives $u \leqslant v$ in Ω.

□

4.2 Inf-and sup convolutions

We want to define an approximation to merely continuous functions. The approximated version should be twice differentiable almost everywhere, and coincide with the original function in some limit. We also want the approximation to preserve viscosity properties.

Definition 4.2. *Suppose u is continuous and bounded, say $-M \leqslant u \leqslant M$. For $\epsilon > 0$, we define*

$$\begin{aligned} u^\epsilon(x,t) &= \sup_{y\in\mathbb{R}^n, s\geqslant 0} \left\{ u(y,s) - \frac{1}{2\epsilon}\left(|x-y|^2 + (t-s)^2\right)\right\} \\ u_\epsilon(x,t) &= \inf_{y\in\mathbb{R}^n, s\geqslant 0} \left\{ u(y,s) + \frac{1}{2\epsilon}\left(|x-y|^2 + (t-s)^2\right)\right\}, \end{aligned}$$

to be the sup- and inf convolution of u, respectively.

Proposition 4.3. *The following properties holds for the inf- and sup convolutions:*

(i) $-(-u^\epsilon)(x,t) = u_\epsilon(x,t)$.

(ii) The supremum for u^ϵ and the infimum for u_ϵ are attained on a compact set.

(iii) $u_\epsilon \leqslant u \leqslant u^\epsilon$.

(iv) $-M \leqslant u_\epsilon, u^\epsilon \leqslant M$.

(*v*) *If we have found points* $(y,s) \in \mathbb{R}^n \times [0,\infty)$ *so that*

$$u^\epsilon(x,t) = u(y,s) - \frac{1}{2\epsilon}\left(|x-y|^2 + (t-s)^2\right)$$

then $|x-y|, |t-s| \leqslant 2\sqrt{M\epsilon}$. *A similar assertion holds for* u^ϵ. *From this we see that definition 4.2 is only valid for* $t \geqslant 2\sqrt{M\epsilon}$.

Proof. The calculation

$$\begin{aligned} -(-u^\epsilon)(x,t) &= -\sup_{y\in\mathbb{R}^n, s\geqslant 0}\left\{-\left(u(y,s) + \frac{1}{2\epsilon}\left(|x-y|^2 + (t-s)^2\right)\right)\right\} \\ &= \inf_{y\in\mathbb{R}^n, s\geqslant 0}\left\{u(y,s) + \frac{1}{2\epsilon}\left(|x-y|^2 + (t-s)^2\right)\right\} \end{aligned}$$

shows that (i) holds true.

To prove (ii) we only prove the assertion for u^ϵ. Property (i) tells us that it is enough to work with one of the convolutions. Define $f(y,s;x,t) = u(y) - \frac{1}{2\epsilon}\left(|x-y|^2 + (t-s)^2\right)$, where (x,t) are kept fixed. We calculate

$$f(y,s;x,t) \geqslant \min_{\mathbb{R}^n\times[0,\infty)} u - \frac{1}{2\epsilon}\left(|x-y|^2 + (t-s)^2\right) \to -\infty$$

when $|y|, s \to \infty$. It then follows that since f is continuous, it must attains its maximum on a compact set, which shows that the supremum in the definition of u^ϵ is a maximum.

For (iii) we simply take $y = x$ and $s = t$ in the definition.

We show (iv) for the sup convolution. The first inequality, $u^\epsilon \geqslant -M$ follows from (iii). Fix $(x,t) \in \mathbb{R}^n \times [0,\infty)$. Then

$$u(y,s) - \frac{1}{2\epsilon}\left(|x-y|^2 + (t-s)^2\right) \leqslant u(y,s)$$

for all $(y,s) \in \mathbb{R}^n \times [0,\infty)$. Hence,

$$u^\epsilon(x,t) \leqslant \sup_{y\in\mathbb{R}^n, s\geqslant 0} u(y,s) = M.$$

We prove the assertion (v) for the sup-convolution. If

$$u^\epsilon(x,t) = u(y,s) - \frac{1}{2\epsilon}\left(|x-y|^2 + (t-s)^2\right)$$

we see from (iv) that

$$\begin{aligned} |x-y|^2+(t-s)^2 &= 2\epsilon\left(u^\epsilon(x,t)-u(y,s)\right) \\ &\leqslant 4\epsilon M. \end{aligned}$$

□

Example 4.4. ***(The Hopf-Lax formula.)***
The solution of the Hamilton-Jacobi equation

$$u_t+H(\nabla u)=0$$

where H is convex is given by

$$u(x,t)=\inf_{y\in\mathbb{R}^n}\left\{u(y,0)+tL\left(\frac{x-y}{t}\right)\right\}$$

where L is the Lagrangian of H. We refer to [E] for formal treatment of the Hopf-Lax formula. For $H(p)=\frac{1}{2}p^2$, it turns out that $L=H$ and the solution is then

$$u(x,t)=\inf_{y\in\mathbb{R}^n}\left\{u(y,0)+\frac{1}{2t}|x-y|^2\right\}$$

and we see the clear relation with the inf-convolution of u given in definition 4.2. Taking $u(x,0)=|x|^2$, we obtain the solution

$$u(x,t)=\frac{|x|^2}{1+2t},$$

in correspondence with example 3.5, the method of vanishing viscosity. □

Lemma 4.5. *The functions u_ϵ and u^ϵ are locally Lipschitz continuous.*

Proof. We prove the result for u_ϵ which is enough by proposition 4.3 (i). Find (y,s) so that

$$u_\epsilon(x,t)=u(y,s)+\frac{1}{2\epsilon}\left(|x-y|^2+(t-s)^2\right).$$

Then we have

$$\begin{aligned} &u_\epsilon(\hat{x},t)-u_\epsilon(x,t)= \\ &\inf_{z\in\mathbb{R}^n,\tau\geqslant 0}\left\{u(z,\tau)+\frac{1}{2\epsilon}\left(|\hat{x}-z|^2+(t-\tau)^2\right)\right\}-u(y,s)-\frac{1}{2\epsilon}\left(|x-y|^2+(t-s)^2\right). \end{aligned}$$

Choose now $z = y$ and $\tau = s$ so that

$$u_\epsilon(\hat{x}, t) - u_\epsilon(x, t) \leqslant \frac{1}{2\epsilon}\left(|y - \hat{x}|^2 - |y - x|^2\right) = \frac{1}{2\epsilon}\left(|\hat{x} - x|^2 + 2\left\langle \hat{x} - x, x - y\right\rangle\right).$$

Note that $|x - y| \leqslant 2\sqrt{M\epsilon}$ from proposition 4.3 (v). If $\Omega \subset \mathbb{R}^n \times [0, \infty)$ is compact, the Cauchy-Schwarz' inequality yields for $x, \hat{x} \in \Omega$

$$u_\epsilon(\hat{x}, t) - u_\epsilon(x, t) \leqslant \frac{1}{\epsilon}\left(\max_{x \in \Omega} x + \sqrt{M\epsilon}\right)|\hat{x} - x| \equiv C|\hat{x} - x|.$$

Interchanging the role of x and $\hat{x}$, we see that u_ϵ is Lipschitz continuous in the space variable. A similar proof shows that u is Lipschitz in the time variable.

□

We want to show that $u^\epsilon, u_\epsilon \to u$ uniformly on compact sets. We derive this by applying *Dini's theorem.*

Theorem 4.6. ***(Dini's theorem.)***
Suppose $\{f_n\}_{n=1}^\infty$ is a monotone sequence of continuous functions defined on a compact set $\Omega \subset \mathbb{R}^n \times [0, \infty)$. Further, suppose that $f_n \to f$ pointwise, with f being continuous. Then the convergence is uniform in Ω.

Proof. We suppose without loss of generality $f = 0$ and $\{f_n\}_{n=1}^\infty$ is nonincreasing. Set

$$A_n = \{x \in \Omega : f_n(x) < \epsilon\}.$$

Since $\{f_n\}_{n=1}^\infty$ is nonincreasing, we have

$$A_1 \subset A_2 \subset A_3 \subset ...$$

and further

$$\Omega = \bigcup_{n=1}^{\infty} A_n.$$

The last statement follows from the pointwise convergence: if $x \in \Omega$, there is an n so that $f_n(x) < \epsilon$ which shows that x is in the countable union. By the Heine-Borel theorem (C.2) there is an $N \in \mathbb{N}$ so that

$$\Omega = \bigcup_{n=1}^{N} A_n = A_N.$$

Hence, given $\epsilon > 0$, there is an $N > 0$ so that $f_n(x) < \epsilon$ for all $x \in \Omega$, which shows that $f_n \to 0$ uniformly in Ω.

□

Proposition 4.7. *The inf- and sup convolutions satisfy*

$$u_\epsilon, u^\epsilon \to u$$

uniformly on compact subsets of $\mathbb{R}^n \times [0, \infty)$.

Proof. We prove the assertion for u^ϵ. By proposition 4.3 (ii) we can write

$$u^\epsilon(x,t) = \sup_{(y,s)\in\Omega} \left\{ u(y,s) - \frac{1}{2\epsilon}\left(|x-y|^2 + (t-s)^2\right) \right\}$$

where Ω is a compact subset of $\mathbb{R}^n \times [0, \infty)$.

Let $(x,t) \in \Omega$ and find $y_\epsilon, s_\epsilon \in \Omega$ so that

$$u^\epsilon(x,t) = u(y_\epsilon, s_\epsilon) - \frac{1}{2\epsilon}\left(|x-y_\epsilon|^2 + (t-s_\epsilon)^2\right).$$

From proposition 4.3 (v), $y_\epsilon \to x$ and $s_\epsilon \to t$ when $\epsilon \to 0$. Continuity of u then gives

$$u(x,t) = \lim_{\epsilon\to 0} u(y_\epsilon, s_\epsilon) \geqslant \lim_{\epsilon \to 0} u^\epsilon(x,t).$$

By proposition 4.3 (iii), $u^\epsilon \geqslant u$ for any $\epsilon > 0$, and we see that $u^\epsilon(x)$ decreases pointwise to $u(x)$. Since the limit function u is continuous, we can apply theorem 4.6 to see that $u^\epsilon \to u$ uniformly on Ω. □

We see that the functions u^ϵ and u_ϵ satisfy many of the desired properties that we searched for. However, we aim for them to be twice differentiable in some sense, and we want to know under what conditions they are viscosity solutions of the mean curvature flow equation.

Lemma 4.8. *The sup convolution u^ϵ is semi-convex in space with semi-convexity constant $C = \frac{1}{\epsilon}$. Similarly, u_ϵ is semi-concave in space with the semi-concavity constant $\frac{1}{\epsilon}$.*

Proof. We use definition B.1 to show that u^ϵ is semi-convex. That is, we want to show that, for any $x, h \in \mathbb{R}^n$,

$$u^\epsilon(x+h,t) - 2u^\epsilon(x,t) + u^\epsilon(x-h,t) \geqslant -\frac{1}{\epsilon}|h|^2.$$

First, find $(y, s) \in \mathbb{R}^n \times [0, \infty)$ so that

$$u^\epsilon(x,t) = u(y,s) - \frac{1}{2\epsilon}\left(|x-y|^2 + (t-s)^2\right).$$

From the definition we can take

$$\begin{aligned} u^\epsilon(x+h,t) &= \sup_{z\in\mathbb{R}^n, \tau>0}\left\{u(z,\tau) - \frac{1}{2\epsilon}\left(|x-z+h|^2 + (t-\tau)^2\right)\right\} \\ &\geqslant u(y,s) - \frac{1}{2\epsilon}\left(|x-y+h|^2 + (t-s)^2\right) \end{aligned}$$

and similarly

$$u^\epsilon(x-h,t) \geqslant u(y,s) - \frac{1}{2\epsilon}\left(|x-y-h|^2 + (t-s)^2\right).$$

Using these estimates, we calculate

$$\begin{aligned} &u^\epsilon(x+h,t) - 2u^\epsilon(x,t) + u^\epsilon(x-h,t) \\ &\geqslant -\frac{1}{2\epsilon}\left(|x-y+h|^2 - 2|x-y|^2 + |x-y-h|^2\right) = -\frac{1}{\epsilon}|h|^2, \end{aligned}$$

which shows that u^ϵ is semi-convex in space with semi-convexity constant $\frac{1}{\epsilon}$.

□

The next theorem shows that if u is a viscosity solution of

$$\begin{cases} u_t = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) & (x,t) \in \mathbb{R}^n \times [0,\infty) \\ u(x,0) = g(x) & (x,t) \in \mathbb{R}^n \times \{t=0\} \end{cases} \tag{14}$$

then u^ϵ is a viscosity subsolution and u_ϵ is a viscosity supersolution of the same equation.

Theorem 4.9. *Suppose u is a viscosity solution of equation* (14)*. Then u^ϵ (u_ϵ) is a viscosity subsolution (supersolution) on $\mathbb{R}^n \times (2\sqrt{M\epsilon}, \infty)$.*

Remark. Recall that definition 4.2 is only valid for $t \geqslant 2\sqrt{M\epsilon}$. We actually need a strict inequality to preserve the viscosity property, which is seen in the proof below.

Proof. We show that the theorem holds for viscosity supersolutions. Suppose $u_\epsilon - \phi$ has a local minimum for $\phi \in C^2$ at the point (x_0, t_0) where $t_0 > 2\sqrt{M\epsilon}$. Find (y_0, s_0) close to (x_0, t_0) so that

$$u_\epsilon(x_0,t_0) = u(y_0,s_0) + \frac{1}{2\epsilon}\left(|x_0-y_0|^2 + (t_0-s_0)^2\right).$$

Further, from the definition of u_ϵ, pick (y,s) close to (x,t) so that

$$u_\epsilon(x,t) \geqslant u(y,s) + \frac{1}{2\epsilon}\left(|x-y|^2 + (t-s)^2\right).$$

Since $u_\epsilon - \phi$ has a local minimum at (x_0,t_0) we have $(u_\epsilon-\phi)(x_0,t_0) \leqslant (u_\epsilon-\phi)(x,t)$ for all (x,t) close to (x_0,t_0). This gives

$$\begin{aligned}&u(y_0,s_0) - \phi(x_0,t_0) + \frac{1}{2\epsilon}\left(|x_0-y_0|^2 + (t-s)^2\right)\\ &\leqslant u(y,s) - \phi(x,t) + \frac{1}{2\epsilon}\left(|x-y|^2 + (t-s)^2\right).\end{aligned}$$

Now, pick $x = y + (x_0 - y_0)$, $t = s + (t_0 - s_0)$ to find

$$(u - \hat{\phi})(y_0,s_0) \leqslant (u - \hat{\phi})(y,s)$$

where $\hat{\phi}(y,s) = \phi(y + x_0 - y_0, s + t_0 - s_0)$. Since this holds for all (y,s) close to (y_0,s_0) and the partial derivatives of $\hat{\phi}$ at (y_0,s_0) coincide with the partial derivatives of ϕ at (x_0,t_0) the result follows, as u is a viscosity solution.

□

4.3 Uniqueness of viscosity solutions

For convenience, we divide the proof into different parts. The first part introduces a technique called "doubling the number of variables" and uses the inf-and sup convolutions of a continuous and bounded function. We then show that the viscosity property is preserved under the convolution. Finally, we complete the proof of the theorem and compare our method to the well known Ishii's lemma. The statement of the theorem is similar to the one given in section 4.1.

Theorem 4.10. *Suppose u is a viscosity subsolution and v is a viscosity supersolution of equation (6) with $u \leqslant v$ when $t = 0$. Assume further that u and v are constant and $u \leqslant v$ when $|x| + t \geqslant R$ for some $R > 0$.*
Then

$$u \leqslant v$$

in $\mathbb{R}^n \times [0,\infty)$.

Proof.

- **Doubling the number of variables.**

As in the proof of the uniqueness of C^2 solutions, we define

$$\Omega = \{(x,t) \in \mathbb{R}^n \times [0,\infty) : |x| + t \leqslant R\}$$

and we want to show that $u \leqslant v$ in Ω. Now suppose

$$\max_{\Omega} (u - v) \equiv \theta > 0.$$

For small $\alpha > 0$ and $(x,s) \in \Omega$ we have $\max_{\Omega} (u - v - \alpha s) \geqslant \frac{\theta}{2}$. We now introduce the inf-and sup convolution of u. Since $u^{\epsilon} \geqslant u$ and $v_{\epsilon} \leqslant v$ we have

$$\max_{\Omega} (u^{\epsilon} - v_{\epsilon} - \alpha s) \geqslant \frac{\theta}{2} > 0.$$

Given $\delta > 0$, let

$$\Phi(x,y,t,s) = u^{\epsilon}(x,t) - v_{\epsilon}(y,s) - \alpha s - \frac{1}{\delta}\left(|x-y|^4 + (t-s)^2\right) \tag{15}$$

where $x, y \in \mathbb{R}^n$, $t, s > 0$. The idea is that this function has a maximum, attained on a compact set. To see this note that

$$\sup_{(x,y,t,s) \in \mathbb{R}^{2n} \times [0,\infty)^2} \Phi \geqslant \frac{\theta}{2} > 0$$

by equation (15). To see why this is a maximum, and that this is attained in some compact set, note that for large $|x|, |y|, |s|$ and t we have $u^{\epsilon} = u$, $v_{\epsilon} = v$ and $u^{\epsilon} < v_{\epsilon}$ by assumption. This gives

$$\Phi \to -\infty$$

when $|x|, |y|, s, t \to \infty$. Hence, the supremum is attained in some large ball. Since Φ is continuous, it attains its maximum on a compact set.

We now wish to apply theorem 4.9 which tells us that if $u = u(x,t)$ is a viscosity subsolution, then $u^{\epsilon}(x,t)$ is a viscosity subsolution for $t > 2\sqrt{M\epsilon}$, where M is the maximum of u. A similar result holds for viscosity supersolutions. Denote (x_0, y_0, t_0, s_0) by the maximum point of Φ.

- **The viscosity property persists under the convolutions.**

We show that the maximum point of Φ satisfies $t_0, s_0 > C\sqrt{\epsilon}$, where $C = 2\sqrt{M}$. It then follows from theorem 4.9 that u^ϵ and v_ϵ are viscosity sub- and supersolutions close to the maximum point, respectively.

Suppose for contradiction that $t_0 \leqslant C\sqrt{\epsilon}$. Then

$$u^\epsilon(x_0,t_0) - v_\epsilon(y_0,s_0) \longrightarrow u(x_0,0) - v(y_0,s_0)$$

as $\epsilon \to 0$ in view of proposition 4.7 and the continuity of u. By the definition of Φ in equation (15) we have

$$|x-y|^4 + (t-s)^2 \leqslant \delta \left(\max_{\mathbb{R}^{2n}\times[0,\infty)^2} (u^\epsilon - v_\epsilon - \alpha s) + \max_{\mathbb{R}^{2n}\times[0,\infty)^2} \Phi \right).$$

so that

$$|x_0 - y_0| \leqslant \tilde{C}\delta^{\frac{1}{4}}, \quad |t_0 - s_0| \leqslant \tilde{C}\delta^{\frac{1}{2}}$$

for a constant $\tilde{C} > 0$. Since u and v are continuous,

$$u^\epsilon(x_0,t_0) - v_\epsilon(y_0,s_0) \longrightarrow u(x_0,0) - v(x_0,0)$$

as $\epsilon, \delta \to 0$. Recall that $u \leqslant v$ when $t = 0$ by assumption. Hence, passing to the limit $\epsilon, \delta \to 0$ we get

$$\begin{aligned} \frac{\theta}{2} &\leqslant \Phi(x_0,y_0,t_0,s_0) \leqslant u^\epsilon(x_0,t_0) - v_\epsilon(y_0,s_0) \\ &\longrightarrow u(x_0,0) - v(x_0,0) \leqslant 0. \end{aligned}$$

This contradicts $\theta > 0$ so we must have $t_0 > C\sqrt{\epsilon}$.

- **Semi-convex functions and matrices.**

We see that Φ is semi-convex, since $u^\epsilon - v_\epsilon$ is semi-convex. As Φ has a maximum at (x_0,y_0,t_0,s_0) we have from corollary B.5 that there exists $\xi_k = (x_k,y_k,t_k,s_k) \to (x_0,y_0,t_0,s_0)$ so that

$$\begin{aligned} &\Phi, u^\epsilon, v_\epsilon \text{ are twice differentiable in the sense of Alexandrov at } \xi_k \\ &\nabla_{x,y,t,s}\Phi(\xi_k) \to 0 \\ &D^2_{x,y}\Phi(\xi_k) \leqslant \frac{1}{k} I_{2n}. \end{aligned}$$

We now find points (x_k, y_k, t_k, s_k) for which

$$\Phi(x, y, t, s) = u^\epsilon(x, t) - v_\epsilon(y, s) - \alpha s - \frac{1}{\delta}\left(|x-y|^4 + (t-s)^2\right)$$

is twice differentiable. For convenience, write

$$\begin{aligned} p^k &= \nabla u^\epsilon(x_k, t_k), & \bar{p}^k &= \nabla v_\epsilon(y_k, s_k), \\ q^k &= u^\epsilon_t(x_k, t_k), & \bar{q}^k &= v_{\epsilon,t}(y_k, s_k), \\ X^k &= D^2 u^\epsilon(x_k, t_k), & Y^k &= D^2 v_\epsilon(y_k, s_k). \end{aligned}$$

Differentiating, we see that

$$\begin{aligned} p^k, \bar{p}^k &\to \frac{4}{\delta}|x_0 - y_0|^2(x_0 - y_0) \equiv p \\ q^k - \bar{q}^k &\to q - \bar{q} = \alpha. \end{aligned}$$

We may assume that these limits exists, in the view that the inf- and sup convolutions are Lipshitz continuous, see lemma 4.5. For the second derivatives, note that

$$D^2_{x,y}\Phi(x_k, y_k, t_k, s_k) = \begin{bmatrix} X^k & 0 \\ 0 & -Y^k \end{bmatrix} - \frac{12}{\delta}|x_k - y_k|^2 \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} \leqslant \frac{1}{k} I_{2n}.$$

In the view that $u^\epsilon - v_\epsilon$ is semi-convex, we also have for any $\epsilon > 0$

$$\begin{bmatrix} X^k & 0 \\ 0 & -Y^k \end{bmatrix} \geqslant -\frac{1}{\epsilon} I_{2n}.$$

In view of these two inequalities, we see that there exists a limit $X^k \to X$ and $Y^k \to Y$. Further, letting $k \to \infty$ in the first equation, we see that

$$X \leqslant Y$$

since the matrix

$$\begin{bmatrix} I & -I \\ -I & I \end{bmatrix}$$

annihilates vectors in $\mathbb{R}^{2n}$. We now plan to use the viscosity property of u^ϵ and v_ϵ. The problem arrives when $x_0 = y_0$, because then $p = 0$. However, in view of the matrix inequalities we see that

$$-\frac{1}{\epsilon} I_{2n} \leqslant \begin{bmatrix} X^k & 0 \\ 0 & -Y^k \end{bmatrix} \leqslant \frac{1}{k} I_{2n} + \frac{12}{\delta}|x_k - y_k|^2 \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + \epsilon \frac{2 \cdot 12^2}{\delta^2}|x_k - y_k|^4 \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}.$$

The last term is added by noting that for a symmetric matrix A, we have $A^2 \geqslant 0$. Choose now

$$\epsilon = \epsilon(k) = \frac{\delta}{24|x_k - y_k|^2}.$$

This yields

$$-\frac{24}{\delta}|x_k - y_k|^2 I_{2n} \leqslant \begin{bmatrix} X^k & 0 \\ 0 & -Y^k \end{bmatrix} \leqslant \frac{1}{k} I_{2n} + \frac{24}{\delta}|x_k - y_k|^2 \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}$$

and we see that $X = Y = 0$ if $x_0 = y_0$.

Now, using that u^ϵ and v_ϵ are viscosity sub- and supersolutions close to the maximum point, we have

$$\begin{aligned} &(q, p, X) \in \bar{P}^{2,+} u^\epsilon(x_0, t_0), \\ &(\bar{q}, \bar{p}, Y) \in \bar{P}^{2,-} v_\epsilon(y_0, s_0). \end{aligned}$$

Using the stability result for viscosity solutions, lemma 3.11, we get

$$\begin{aligned} &q \leqslant \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) X_{ij} \quad \text{if } x_0 \neq y_0 \\ &q \leqslant 0 \quad \text{if } x_0 = y_0 \end{aligned}$$

and

$$\begin{aligned} &\bar{q} \geqslant \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) Y_{ij} \quad \text{if } x_0 \neq y_0 \\ &\bar{q} \geqslant 0 \quad \text{if } x_0 = y_0. \end{aligned}$$

Since $q - \bar{q} = \alpha$ we get the contradiction

$$\alpha \leqslant 0$$

regardless of whether $x_0 = y_0$ or not. We conclude that $u \leqslant v$ in Ω.

□

Remark. We have actually shown, in doing this proof, a version of Ishii's lemma, or the theorem of sums, which is one of the main tools when studying viscosity solutions. It was first proven in [CIL], and the elliptic version stated in [K] is given here: Suppose u and v are continuous in $\bar{\Omega}$. For $\phi \in C^2\left(\bar{\Omega} \times \bar{\Omega}\right)$ suppose $(x_0, y_0) \in \bar{\Omega} \times \bar{\Omega}$ is a maximum for the function

$$u(x) + v(y) - \phi(x, y).$$

Then, for each $\mu > 1$ there are symmetric matrices X, Y such that

$$(D_x\phi(x_0, y_0), X) \in \bar{J}^{2,+}u(x_0), \quad (D_y(\phi(x_0, y_0), Y) \in \bar{J}^{2,+}v(y_0)$$

and

$$-\left(\mu + \|D^2\phi(x_0, y_0)\|\right)\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \leqslant \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \leqslant D^2\phi(x_0, t_0) + \frac{1}{\mu}\left(D^2\phi(x_0, t_0)\right)^2.$$

5 Geometric properties of the mean curvature flow

5.1 Mean curvature flow for compact sets

In this section, we present some geometric properties of the mean curvature flow. Recall that we are given

$$\Gamma_0 = \{x \in \mathbb{R}^n : g(x) = 0\}$$

for some continuous function $g : \mathbb{R}^n \to \mathbb{R}$. We also assume that g is constant when $|x| \geqslant r$ for some $r > 0$. We then look for

$$\Gamma_t = \{x \in \mathbb{R}^n : u(x,t) = 0\}$$

where u is the unique weak solution to the mean curvature flow equation,

$$\begin{cases} u_t = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) & (x,t) \in \mathbb{R}^n \times [0,\infty) \\ u(x,0) = g(x) & (x,t) \in \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{16}$$

Lemma 5.1. *Suppose u is a viscosity solution of the mean curvature flow equation* (16)*. Further, suppose f is a continuous function. Then*

$$v = f(u)$$

is also a viscosity solution of equation (16)*.*

Proof. We only prove that f is a viscosity subsolution under the assumption that f is strictly increasing.
Suppose first that f is smooth, so that it has an inverse $h = f^{-1}$ and $f' > 0$. Choose $\phi \in C^2$ so that

$$0 = (v - \phi)(x_0, t_0) \geqslant (v - \phi)(x,t)$$

for all (x,t) close to (x_0, t_0). We rearrange the equation and compose both sides with h to get

$$(u - h(\phi)(x_0, t_0) \geqslant (u - h(\phi)(x,t)$$

for all (x,t) close to (x_0, t_0). Using the fact that u is a viscosity solution,

$$
\begin{aligned}
h'\phi_t &\leqslant \left(\delta_{ij} - \frac{\phi_{x_i}\phi_{x_j}}{|\nabla\phi|^2}\right)\left(h''\phi_{x_i}\phi_{x_j} + h'\phi_{x_ix_j}\right), \quad \text{if } \nabla\phi \neq 0 \\
h'\phi_t &\leqslant \left(\delta_{ij} - \eta_i\eta_j\right) h'\phi_{x_ix_j}, \quad \nabla\phi = 0, \quad |\eta| = 1.
\end{aligned}
$$

The term involving $h''(\phi)$ is zero, since if $p \neq 0$ we have

$$
\begin{aligned}
&\sum_{i,j=1}^{n} \left(\delta_{ij} - \frac{p_ip_j}{|p|^2}\right) p_ip_j \\
&= \frac{1}{|p|^2}\left(|p|^4 - \left(\sum_{i=1}^{n} p_i^2\right)^2\right) = 0.
\end{aligned}
$$

Hence, in both cases we can divide by $h' > 0$ which shows that $v = f(u)$ is a viscosity subsolution.

For continuous and strictly increasing functions f, we find a smooth sequence of functions $\{f_k\}_{k=1}^{\infty}$ with $f_k' > 0$ for all k and $f_k \to f$. A simple application of the stability lemma 3.11 shows that f is a viscosity subsolution. □

We mention two methods to show existence of solutions of the mean curvature flow equation. One is given by the Perron method, see for example [CIL]. One can also look at an approximation of the original equation. The idea is that the theory of *uniformly* elliptic partial differential equations gives existence of smooth solutions [LSU]. Considering

$$
u_t = \left((1+\theta)\delta_{ij} - \frac{u_{x_i}u_{x_j}}{\epsilon^2 + |\nabla u|^2}\right) u_{x_ix_j} \equiv \mathrm{tr}(A_{\theta,\epsilon}D^2u)
$$

before letting $\theta, \epsilon \to 0$, we can show that

$$
x^T A_{\theta,\epsilon} x \geqslant \theta |x|^2,
$$

for all $x \in \mathbb{R}^n$. Thus $A_{\theta,\epsilon}$ satisfies the uniform ellipticity condition, and the equation is uniformly elliptic.

Theorem 5.2. *Suppose $g : \mathbb{R}^n \to \mathbb{R}$ is continuous. Further, assume that g is constant when $|x| \geqslant r$ for some $r > 0$. Then there exists a continuous viscosity solution u of equation* (16) *so that u is constant when $|x| + t \geqslant R$ for some $R > 0$.*

In order to derive geometric properties of the flow, given only the initial surface Γ_0, we need to show that the subsequent surfaces $\{\Gamma_t\}_{t\geqslant 0}$ are well defined.

Theorem 5.3. *Γ_t, for $t \geqslant 0$ is well defined. More precisely, suppose that u and v are viscosity solutions of the mean curvature flow equation, with $u(x,0) = g(x)$, $v(x,0) = f(x)$. If*

$$\Gamma_0 = \{x : g(x) = 0\} = \{x : f(x) = 0\}$$

then

$$\Gamma_t = \{x : u(x,t) = 0\} = \{x : v(x,t) = 0\}.$$

Remark. This shows that the flow is independent of the choice of our initial function, as long as our choice agrees on Γ_0.

Proof. Suppose first that $f, g \geqslant 0$. Find a continuous function Φ so that

$$\begin{aligned}
&\Phi(g(x)) \geqslant f(x), \quad x \in \mathbb{R}^n \\
&\Phi(0) = 0 \\
&\Phi > 0, \quad x \in \mathbb{R}^n \backslash \{0\}.
\end{aligned}$$

By lemma 5.1 the function $\Phi(u)$ solves the same equation with $\Phi(u) = \Phi(g)$ at $t = 0$. By the comparison principle, theorem 4.10, we have

$$\Phi(u) \geqslant v \geqslant 0$$

on $\mathbb{R}^n \times [0, \infty)$. We see that if $u(x,t) = 0$ then $v(x,t) = 0$. Repeating the procedure, but now choosing Φ so that

$$\Phi(f(x)) \geqslant g, \quad x \in \mathbb{R}^n$$

which shows that $v(x,t) = 0$ implies $u(x,t) = 0$.

For general $g : \mathbb{R}^n \to \mathbb{R}$ note that $|u|$ is a viscosity solution with $|u| = |g|$ at $t = 0$. Since

$$\Gamma_0 = \{x : g(x) = 0\} = \{x : |g(x)| = 0\}$$

we may consider the positive function $|g|$ in the proof. The same argument holds for f.

□

We want to use the comparison principle for solutions of the mean curvature flow equation to show a similar type of statement for the geometric flow. The next proposition tells us that, if a compact set sits in a bigger compact set, and the two sets evolve by mean curvature flow, the smaller set will always lie inside the bigger set. In addition, the distance between initially disjoint surfaces increases under the flow. We put the proof from [ES] with some modifications. In both cases we use the *distance* function to describe the initial function g. It is defined by

$$\operatorname{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\}.$$

Proposition 5.4. *Suppose* Γ_0, Λ_0 *are nonempty compact subsets of* $\mathbb{R}^n$.

(i) $\Gamma_0 \subset \Lambda_0$ *implies*

$$\Gamma_t \subset \Lambda_t$$

for all times $t \geqslant 0$.

(ii) If Γ_0 *and* Λ_0 *are disjoint, then*

$$\operatorname{dist}(\Gamma_0, \Lambda_0) \leqslant \operatorname{dist}(\Gamma_t, \Lambda_t)$$

up until a time T *where either* $\Gamma_T = \varnothing$ *or* $\Lambda_t = \varnothing$.

Proof. For (i), let $u(x, 0) = g(x)$, $v(x, 0) = f(x)$ where u, v are viscosity solutions of equation (16). Here, the zero-level sets of $u(\cdot, t)$ correspond to Γ_t and the zero-level sets of $v(\cdot, t)$ correspond to Λ_t. By theorem 5.3 we can take

$$g(x) = \operatorname{dist}(x, \Gamma_0) \geqslant 0,$$
$$f(x) = \operatorname{dist}(x, \Lambda_0) \geqslant 0$$

since we notice that Γ_0 and Λ_0 are really the sets where g and f are zero, respectively. Since $\Gamma_0 \subset \Lambda_0$ we have $g(x) \geqslant f(x)$ for all $x \in \mathbb{R}^n$. By the comparison principle, theorem 4.10,

$$u(x, t) \geqslant v(x, t) \geqslant 0$$

for all $(x, t) \in \mathbb{R}^n \times [0, \infty)$. Hence, if $x \in \Gamma_t$ we see that $v(x, t) = 0$ which implies $\Gamma_t \subset \Lambda_t$.

For the second statement, we can take the initial function g with a zero level-set corresponding to Γ_0. We also want it to be related to the set Λ_0 and the distance between Γ_0 and Λ_0.

We find a Lipschitz continuous function $g : \mathbb{R}^n \to \mathbb{R}$ so that

- $\Gamma_0 = \{x : g(x) = 0\}, \Lambda_0 = \{x : g(x) = 1\}$
- $g(x) = 2$ for $x \in A = \{x : |x| \geqslant r$ for large $r > 0\}$
- $$|g(x) - g(y)| \leqslant \frac{1}{\operatorname{dist}(\Gamma_0, \Lambda_0)}|x - y|$$

  for all $x \neq y$.

Such a function g exists, for example let

$$g(x) = \min_{y \in \Gamma_0 \cup \Lambda_0 \cup A} \left\{ g(y) + \frac{1}{\operatorname{dist}(\Gamma_0, \Lambda_0)}|x - y| \right\}.$$

Considering different possibilities for x, we see that Γ_0 and Λ_0 are the sets where $g = 0$ and $g = 1$ respectively.

Again, we apply theorem 5.3. Let u be the solution of the mean curvature flow equation with $u(x, 0) = g(x)$. Then

$$\Gamma_t = \{x : u(x, t) = 0\}, \Lambda_t = \{x : u(x, t) = 1\}.$$

From a contraction property given in theorem 3.3 in [ES] which follows from the comparison principle, we can extract that

$$|u(x, t) - u(y, t)| \leqslant \frac{1}{\operatorname{dist}(\Gamma_0, \Lambda_0)}|x - y|$$

for all $x \neq y$.
Before we reach the critical time T, we can find points $x \in \Gamma_t$, $z = \Lambda_t$ so that

$$|x - z| = \operatorname{dist}(\Gamma_t, \Lambda_t).$$

Then, since $u = 0$ on Γ_t and $u = 1$ on Λ_t we have

$$1 = u(x, t) - u(z, t) \leqslant \frac{1}{\operatorname{dist}(\Gamma_0, \Lambda_0)}|x - z| = \frac{\operatorname{dist}(\Gamma_t, \Lambda_t)}{\operatorname{dist}(\Gamma_0, \Lambda_0)}.$$

$\square$

In the paper [G1], Grayson showed that, for $n = 2$, any embedded curve converges to a convex curve under the mean curvature flow before contracting to a point. This, along with the previous proposition leave us with a peculiar example.

Example 5.5. ***(Why the snake does not move by mean curvature.)***
In example 2.10 we calculated that, for $n = 2$, the unit circle shrinks into a point after $t = \frac{1}{2}$ seconds. By the previous observation, if we take a closed spiral-shaped curve inside the unit circle, it will unwrap itself in less than half a second, become convex and shrink to a point. See figure 12. $\square$

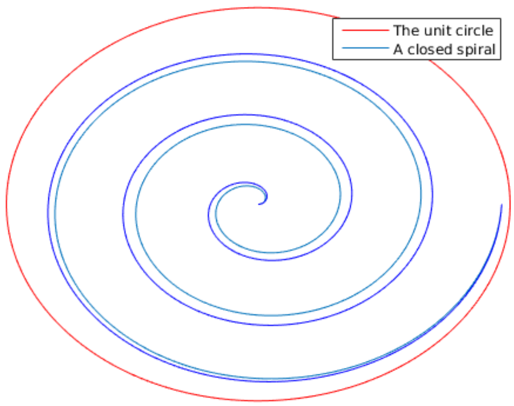

Figure 12: *The snake unwraps itself under mean curvature flow.*

Example 5.5 could not happen in higher dimensions than $n = 2$. In the paper [G2], Grayson created an example, for which a smooth initial surface changed its topology under the mean curvature flow. Figure 13 shows a dumbbell, the initial surface. Under the mean curvature flow, provided the cylinder separating the two bells is long and narrow enough, the surface will develop a singularity. Here, the two bells are separated, creating two convex bodies. By the work of [H], the two convex bodies will then shrink to a point.

We remark that the geometric applications only work for codimension one. Thus, we can not use these results for curves in $\mathbb{R}^3$.

Example 5.6. ***(Codimensions.)***
Let

$$\Gamma_0 = \{(x, y, z) : x^2 + y^2 = 9, z = 0\}, \quad \Lambda_0 = \{(x, y, z) : (x - 2)^2 + z^2 = 9, y = 0\}$$

describe circles in $\mathbb{R}^3$. In example 2.10 we calculated the evolution of circles under mean curvature flow,

$$\Gamma_t = \{(x, y, z) : x^2 + y^2 = 9 - 2t, z = 0\}, \quad \Lambda_t = \{(x, y, z) : (x - 2)^2 + z^2 = 9 - 2t, y = 0\}.$$

Figure 13: *Under the mean curvature flow, the surface will evolve smoothly until a time when the two ends pinch of the cylinder.*

At time $t = 4$ we see that $(1, 0, 0) \in \Gamma_t \cap \Lambda_t$, but $\Gamma_0 \cap \Lambda_0$ is empty. This clearly violates proposition 5.1 (ii).

On the other hand, taking

$$\Gamma_0 = \{y^2 + z^2 = 4,\ -1 \leqslant x \leqslant 1\}$$
$$\Lambda_0 = \{(x-4)^2 + y^2 + z^2 = 4\},$$

to be a cylinder and a sphere respectively, we have

$$\Gamma_t = \{y^2 + z^2 = 4 - 2t,\ -1 \leqslant x \leqslant 1\}$$
$$\Lambda_t = \{(x-4)^2 + y^2 + z^2 = 4 - 4t\}.$$

The closest point from the cylinder to the sphere is clearly given at $x = 1$ for the cylinder. Further, by symmetry, the distance does not depend on where on the circle $y^2 + z^2 = 4 - 2t$ we are. A calculation shows that

$$\mathrm{dist}(\Gamma_t, \Lambda_t)^2 = 17 - 6t - 4\sqrt{(13-2t)(1-t)}$$

for $0 \leqslant t \leqslant 1$ (at time $t = 1$ the sphere has contracted to a point). We see that the distance is increasing with time, in correspondence with proposition 5.1 (ii). $\square$

5.2 Minimal surfaces and decrease in surface area

A minimal surface is a surface that minimizes its area. If we consider all smooth surfaces $z = u(x, y)$ in a bounded domain Ω, we want to minimize

$$A(u) = \iint_{\Omega} \sqrt{1 + |\nabla u|^2} dxdy. \tag{17}$$

The following theorem states the relationship between minimal surfaces and mean curvature.

Theorem 5.7. *Suppose that $u \in C^2(\bar{\Omega})$ minimizes the integral in equation* (17) *among all similar functions with the same boundary values. Then $H = 0$ for the surface $z = u(x, y)$. Hence, a minimal surface is a surface with zero mean curvature.*

Proof. Let $\eta \in C_0^2(\bar{\Omega})$ with $\eta = 0$ on $\partial\Omega$. By assumption, $A(u + \epsilon\eta)$ has a minimum at $\epsilon = 0$ so that

$$\frac{d}{d\epsilon} A(u + \epsilon\eta)\Big|_{\epsilon=0} = 0.$$

We calculate

$$\frac{d}{d\epsilon} A(u + \epsilon\eta)\Big|_{\epsilon=0} = \iint_{\Omega} \frac{\langle \nabla u, \nabla \eta \rangle}{\sqrt{1 + |\nabla u|^2}} dxdy.$$

Using Green's identity and $\eta = 0$ on $\partial\Omega$ gives

$$\iint_{\Omega} \eta \nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) dxdy = 0.$$

Since this holds for all $\eta \in C_0^2(\bar{\Omega})$ the variational lemma (C.1) yields

$$u_{xx}(1 + u_y^2) - 2u_x u_y u_{xy} + u_{yy}(1 + u_x^2) = 0.$$

This is often called the *minimal surface equation.* Comparing with the expression for H in equation (4), the minimal surface has zero mean curvature. $\square$

We note that minimal surfaces under mean curvature flow satisfy $\dot{X} = H\nu = 0$, so that nothing happens to minimal surfaces under mean curvature flow. As we have already seen, the plane satisfies $H = 0$, so the plane is a minimal surface. We give here some other examples.

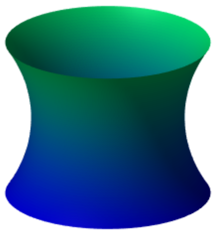

Figure 14: The cateniod.

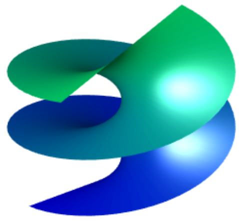

Figure 15: The helicoid.

Example 5.8. ***(Minimal surfaces.)***
The catenoid is defined by

$$x = a\cosh(v/a)\cos u,$$
$$y = a\cosh(v/a)\sin u,$$
$$z = v,$$

where a is a non-zero real constant, $v \in \mathbb{R}$ and $u \in [-\pi, \pi]$. The catenoid can be written explicitly as

$$z = a\cosh^{-1}\left(\frac{\sqrt{x^2+y^2}}{a}\right)$$

and we see that the catenoid is a minimal surface. The helicoid is given by

$$r(u, v) = (u\cos v, u\sin v, cv),$$

where $a \leqslant v \leqslant b$ and $u \in [-\pi, \pi]$ for some constants a, b and c. It is also a minimal surface. Further, it is a *ruled* surface, it can be written

$$r(u, v) = b(v) + u\gamma(v),$$

where b is the base curve and γ is the director curve. In our case, we see that γ describes the unit circle, $\gamma(v) = (\cos v, \sin v, 0)$. See figure 14 and 15. $\square$

In the plane, we found that the area of a closed curve is decreasing with a rate of 2π under mean curvature flow. Here, we show a similar result for the surface area of a graph solution of the mean curvature flow equation, $z = u(x, y)$. We mention that this result holds true for a parametric surface moving under mean curvature flow, see [H]. As when experimenting with soap films, we keep the boundary fixed for all times. We note that in this case,

$$\Gamma_t = \left\{(x, y, z) \in \mathbb{R}^3 : z = u(x, y, t)\right\}$$

and the mean curvature flow equation reduces to

$$u_t = -\sqrt{1 + |\nabla u|^2}\; H = \sqrt{1 + |\nabla u|^2}\; \mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right).$$

Proposition 5.9. *Suppose $u \in C^2$. Let $\Omega \subset \mathbb{R}^n \times [0, \infty)$ and consider*

$$\begin{cases} u_t = \sqrt{1 + |\nabla u|^2}\; \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right), \quad (x, t) \in \Omega \\ u(x, t) = f(x), \quad (x, t) \in \partial\Omega \\ u(x, 0) = g(x). \end{cases}$$

Then the surface area

$$A(t) = \int_\Omega \sqrt{1 + |\nabla u|^2} dx dy$$

is decreasing with time.

Proof. We calculate, using Green's theorem

$$\begin{aligned} \frac{dA}{dt} &= \int_\Omega \frac{1}{\sqrt{1 + |\nabla u|^2}} \langle \nabla u, \nabla u_t \rangle \, dx dy \\ &= \oint_{\partial\Omega} \frac{u_t}{\sqrt{1 + |\nabla u|^2}} \langle \nabla u, \nu \rangle \, dS - \int_\Omega u_t \mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) dx dy. \end{aligned}$$

The boundary term vanishes, since u is kept fixed in time on $\partial\Omega$. We use the evolution equation to manipulate the last term,

$$\frac{dA}{dt} = -\int_\Omega H^2 \sqrt{1 + |\nabla u|^2} dx dy \leqslant 0.$$

$\square$

6 Concluding remarks and further work.

Most of the work in this thesis has been made in connection with viscosity solutions and uniqueness of solutions. By investigating properties of semi-convex functions and inf- and sup convolutions, we were also able to see the connection between the mean curvature flow equation and other second order partial differential equations, where uniqueness is often showed by an application of the *Ishii lemma*.

Expanding the ideas from the previous section, one can find the minimal surfaces corresponding to a given boundary. Simply putting any surface touching the boundary and letting it flow by mean curvature flow, yields a minimal surface once the evolution stops. This theorem is summarized in theorem 2.1 in Huisken [H2].

In further work, it would be interesting to include the Gaussian curvature and other geometric quantities, controlling moving surfaces. Here, interesting equations and problems appear and for non-local equations involving the fractional Laplacian this seems to be *terra incognita*.

Appendices

A Matrices

From elementary linear algebra, a real matrix $A = (A_{ij})_{ij}$ is symmetric if $A_{ij} = A_{ji}$. A symmetric matrix can be diagonalized by using an orthogonal matrix S consisting of the eigenvectors of A. If Λ is the matrix consisting of the the eigenvalues of A on its diagonal, and zero otherwise, we can write

$$A = S^T \Lambda S.$$

That S is orthogonal means that $S^T = S^{-1}$.

Remark. If $f : \mathbb{R}^n \to \mathbb{R}$ is in $C^2(\mathbb{R}^n)$ then $D^2 f$ is symmetric.

For example, when $n = 2$, $D^2 f$ is the matrix given by

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

Since $f \in C^2$, $f_{xy} = f_{yx}$ so $D^2 f$ is symmetric.

Definition A.1. *For a symmetric matrix A, if*

$$\xi^T A \xi \geqslant 0,$$

for all $\xi \in \mathbb{R}^n$, we say that $A \geqslant 0$ or A is positive.

Example A.2. The identity matrix is positive. If $x = (x_1, x_2, ..., x_n)$ then

$$x^T I x = \sum_{i=1}^{n} x_i^2 \geqslant 0.$$

The matrix $A = (a_{ij})_{ij}$ given by

$$a_{ij} = \delta_{ij} - \frac{p_i p_j}{|p|^2}$$

is a positive matrix.

$$x^T A x = \sum_{i,j} x_i \left(\delta_{ij} - \frac{p_i p_j}{|p|^2} \right) x_j$$

and so, by the Cauchy-Schwarz' inequality

$$\begin{aligned}
|p|^2 x^T A x &= \sum_{i,j} x_i \delta_{ij} |p|^2 x_j - \sum_{i,j} x_i p_i p_j x_j \\
&= |x|^2 |p|^2 - \left(\sum_i x_i p_i \right) \left(\sum_j x_j p_j \right) \\
&= |x|^2 |p|^2 - \left(\sum_i x_i p_i \right)^2 \\
&\geqslant |x|^2 |p|^2 - \left(\sum_i x_i^2 \right) \left(\sum_i p_i^2 \right) = 0.
\end{aligned}$$

□

From the definition, one can extract the following property of a positive matrix.

Lemma A.3. *If $A \geqslant 0$, the eigenvalues of A are non-negative.*

Proof. Write $A = S^T \Lambda S$. Then

$$0 \leqslant \xi^T A \xi = \xi^T S^T \Lambda S \xi = (S\xi)^T \Lambda (S\xi) = x^T \Lambda x,$$

where $x = S\xi$. Further,

$$x^T \Lambda x = \sum_{k=1}^{n} x_k (\Lambda x)_k = \sum_{k=1}^{n} \sum_{l=1}^{n} x_k x_l \Lambda_{kl}.$$

Since Λ is a diagonal matrix, the only terms that are nonzero are the terms where $k = l$. Hence,

$$0 \leqslant x^T \Lambda x = \sum_{k=1}^{n} \lambda_k x_k^2.$$

This holds for all $x \in \mathbb{R}^n$ (ξ was arbritrary), so $\lambda_k \geqslant 0$ for $k = 1, .., n$. □

Since all the eigenvalues of a positive matrix are positive, define the matrix

$$\begin{matrix} \sqrt{\lambda_1} & 0 & \dots & 0 \\ 0 & \sqrt{\lambda_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{\lambda_n} \end{matrix}$$

to be $\Lambda^{\frac{1}{2}}$. Indeed, $\Lambda^{\frac{1}{2}}\Lambda^{\frac{1}{2}} = \Lambda$.

Proposition A.4. *Suppose $A \geqslant 0$ and $B \geqslant 0$ are matrices. Then* $\operatorname{tr}(AB) \geqslant 0$.

In proposition A.4, the trace of a matrix A, denoted $\operatorname{tr}(A)$ is given by

$$\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

To prove proposition A.4, we need the following lemma.

Lemma A.5. *If $A \geqslant 0$, then* $\operatorname{tr}(DAD^T) \geqslant 0$ *for all $n \times n$ matrices D.*

Proof. Look at

$$\begin{aligned} (DAD^T)_{ii} &= \sum_{k=1}^{n} (DA)_{ik}(D^T)_{ki} \\ &= \sum_{k=1}^{n} (DA)_{ik} D_{ik} \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} D_{il} A_{lk} D_{ik}. \end{aligned}$$

This seems fairly similar to the product $x^T Ax = \sum_{k=1}^{n} \sum_{l=1}^{n} x_l A_{lk} x_k$. Letting D^i be the vector containing the *ith* row of D, $D^i = (d_{i1}, d_{i2}, ..., d_{in})$, we see that

$$(DAD^T)_{ii} = (D^i)^T AD^i \geqslant 0$$

since $A \geqslant 0$. Hence, all the diagonal elements of DAD^T are positive, so the trace is positive. □

Proof. (Of proposition A.4)
Note first that for any $n \times n$ matrices A and B we have

$$\operatorname{tr}(AB) = \sum_{k=1}^{n} (AB)_{kk} = \sum_{k=1}^{n} \sum_{l=1}^{n} A_{lk} B_{kl} = \sum_{l=1}^{n} \sum_{k=1}^{n} B_{kl} A_{lk} = \sum_{l=1}^{n} (BA)_{ll} = \operatorname{tr}(BA).$$

Writing $A = S^T\Lambda^{\frac{1}{2}}\Lambda^{\frac{1}{2}}S$ gives

$$\begin{aligned}\mathrm{tr}(AB) &= \mathrm{tr}(S^T\Lambda^{\frac{1}{2}}\Lambda^{\frac{1}{2}}SB)\\ &= \mathrm{tr}\big((\Lambda^{\frac{1}{2}}S)^T B(\Lambda^{\frac{1}{2}}S)\big).\end{aligned}$$

With $C = \Lambda^{\frac{1}{2}}S$ we see that

$$\mathrm{tr}(AB) = \mathrm{tr}(C^T BC) = tr(CBC^T) \geqslant 0,$$

by lemma A.5 since B is positive. □

We use the square root matrix $\Lambda^{\frac{1}{2}}$ to optimize a quadratic form $x^T Ax$. If $B \geqslant 0$ we can define $B^{\frac{1}{2}} = S^T\Lambda^{\frac{1}{2}}S$, so that $B^{\frac{1}{2}}B^{\frac{1}{2}} = B$.

Proposition A.6. *Suppose $B \geqslant 0$ and A is a symmetric matrix. Then*

$$\max_{x^T Bx=1} x^T Ax = \max_i \lambda_i$$

where $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of $B^{-1}A$. Similarly, the minimum of the above expression is given by the minimum eigenvalue of $B^{-1}A$.

Proof. (For maximum)
Let $y = B^{\frac{1}{2}}x$ so that $y^T y = 1$ under the restriction $x^T Bx = 1$. This gives

$$\max_{x^T Bx=1} x^T Ax = \max_{y^T y=1}\left\{y^T B^{-\frac{1}{2}}AB^{-\frac{1}{2}}y\right\} = \max_{y^T y=1}\left\{y^T S^T\Lambda Sy\right\}.$$

Here, we have written

$$B^{-\frac{1}{2}}AB^{-\frac{1}{2}} = S^T\Lambda S$$

since $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ is a symmetric matrix. Now, defining $z = Sy$ gives $z^T z = 1$ and further

$$\max_{x^T Bx=1} x^T Ax = \max_{z^T z=1} z^T\Lambda z = \sum_{i=1}^n \lambda_i z_i^2 \leqslant \max_i \lambda_i.$$

Noting that

$$B^{-\frac{1}{2}}AB^{-\frac{1}{2}} = \lambda x$$

implies

$$B^{-1}Ay = \lambda y$$

for $y = B^{-\frac{1}{2}}x$ shows that the matrices $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ and $B^{-1}A$ share the same eigenvalues.

$\square$

Corollary A.7. *Under the same assumptions, if* $n = 2$ *we have*

$$\max_{x^T Bx=1} x^T Ax + \min_{x^T Bx=1} x^T Ax = \mathrm{tr}(B^{-1}A).$$

Proof. Since $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ is symmetric, write

$$B^{-\frac{1}{2}}AB^{-\frac{1}{2}} = S^T \Lambda S.$$

The eigenvalues λ_1, λ_2 of Λ satisfies

$$\max_{x^T Bx=1} x^T Ax + \min_{x^T Bx=1} x^T Ax = \lambda_1 + \lambda_2$$

by the previous proposition. Using tr(AB) =tr(BA) we get

$$\lambda_1 + \lambda_2 = \mathrm{tr}(S^T \Lambda S) = \mathrm{tr}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) = \mathrm{tr}(B^{-1}A).$$

$\square$

B Semi-convex functions

We suppose in this section that Ω is a bounded domain, and $\Omega \subset \mathbb{R}^n$.

Definition B.1. *A function $f : \Omega \to \mathbb{R}$ is said to be semi-convex with semi-convexity constant $C > 0$ if*

$$f(x+h) - 2f(x) + f(x-h) \geqslant -C|h|^2$$

for all $x, h \in \Omega$.

A perhaps more common way to define semi-convex functions is that $f(x) + \frac{C}{2}|x|^2$ is convex. As we will use both definitions in the rest of the appendix, we show that the two are equivalent.

Proposition B.2. *If f is continuous then f satisfies the condition given in definition B.1 if and only if $f(x) + \frac{C}{2}|x|^2$ is convex.*

Proof. Suppose first $f(x) + \frac{C}{2}|x|^2$ is convex, so that

$$f\left(\frac{x+y}{2}\right) + \frac{C}{2}\left|\frac{x+y}{2}\right|^2 \leqslant \frac{1}{2}\left(f(x) + f(y)\right) + \frac{C}{4}\left(|x|^2 + |y|^2\right). \tag{18}$$

Let $x = z + h$ and $y = z - h$. Then $\frac{x+y}{2} = z$ and so

$$f(z+h) - 2f(z) + f(z-h) \geqslant C|z|^2 - \frac{C}{2}\left(|z+h|^2 + |z-h|^2\right) = -C|h|^2.$$

For the other direction, it is easy to verify that equation (18) is valid when f satisfies the condition in definition B.1. Since f is continuous, $f(x) + \frac{C}{2}|x|^2$ is convex.

□

We mention two theorems about semi-convex functions. The first one says that a semi-convex function is twice differentiable almost everywhere in the sense of Alexandrov.

Theorem B.3. *(The Alexandrov theorem)*
If $f : \Omega \to \mathbb{R}$ is semi-convex, then, for almost every $x \in \Omega$ there is a $p \in \mathbb{R}^n$ and a symmetric matrix X so that, as $h \to 0$

$$f(x) = f(a) + \langle p, x-h \rangle + \frac{1}{2}(x-h)^T X (x-h) + o(|h|^2).$$

Lemma B.4. *(Jensen's lemma)*
Let $\phi : \Omega \to \mathbb{R}$ be semi-convex and let x_0 be a strict local maximum point of ϕ. Set $\phi_p(x) = \phi(x) + \langle x, p\rangle$ for $x, p \in \Omega$. Then, for $r, \delta > 0$, the set

$$K = \left\{x \in \bar{B}_r(x_0) :\ \exists p \in B_\delta \text{ such that } \phi_p \text{ has a local max at } x\right\}$$

has positive measure.

A proof of the Alexandrov theorem can be found in [EG]. We give here a proof of Jensen's lemma, which states that, if we perturb a semi-convex function which achieves a local maximum, we can get functions with local maximum close to the original maximum point. In a given ball around the original point, the set of maximum points for the perturbed function has positive measure. It should be clear why we need ϕ to be semi-convex and not convex, since a convex function cannot achieve a strict local maximum (recall that $D^2 f \geqslant 0$ for convex functions f).

Proof. (Of lemma B.4.)
We suppose $x_0 = 0$, otherwise we can consider $\tilde{\phi}_p(x) = \phi_p(x + x_0)$.
Assume first $\phi \in C^2(\Omega)$. We choose $r > 0$ so that ϕ has a unique maximum in $\bar{B}_r$. Find $\epsilon > 0$ so that

$$\phi(0) > \sup_{x \in \partial B_r} \{\phi(x)\} + \epsilon \equiv M + \epsilon.$$

The supremum is clearly attained, since ϕ is continuous. Further, we choose $\delta > 0$ so that

$$\delta \leqslant \frac{\epsilon}{2r}.$$

Then, for $x \in \partial B_r$ and $p \in \bar{B}_\delta$ the Cauchy-Schwarz' inequality gives

$$\phi_p(x) = \phi(x) + \langle p, x\rangle \leqslant M + |p||x| \leqslant M + \frac{\epsilon}{2}.$$

At $x = 0$ we see that

$$\phi_p(0) = \phi(0) > M + \epsilon > \sup_{x \in \partial B_r} \{\phi_p(x)\}.$$

Hence, there is a point $x \in \bar{B}_r \backslash \partial B_r$, $x \neq 0$, for which $\phi_p(x) > M$, so any maximum point of ϕ_p lies in the interior of $\bar{B}_r$. Since this holds for all $p \in \bar{B}_\delta$, and for local maximum $x \in B_r$ we have

$$\nabla \phi_p(x) = \nabla \phi(x) + p = 0,$$

we see that $B_\delta \subset \nabla\phi(K)$. Since ϕ is semi-convex, $\phi(x) + \frac{\lambda}{2}|x|^2$ is convex so that $-\lambda I \leqslant D^2\phi \leqslant 0$ in K. This gives

$$|B_\delta| \leqslant |\nabla\phi(K)| = \int_{\nabla\phi(K)} dx \leqslant \int_K \left|\mathrm{Det}\left(D^2\phi(x)\right)\right| dx \leqslant |\lambda|^n |K|.$$

The second inequality uses the change-of-variables formula from [EG]. We have shown that K has positive measure, provided $\phi \in C^2(\Omega)$.

For the general case, replace ϕ by $\phi^\epsilon = \phi * \rho_\epsilon$, where ρ_ϵ is the standard mollifier. Set

$$K_{1/l} = \{x \in \bar{B}_r : \ \exists p \in B_\delta \text{ such that } \phi_p^{1/l} \text{ has a local maximum at } x\}.$$

We now show that

$$\bigcap_{m=1}^{\infty} \bigcup_{l=m}^{\infty} K_{1/l} \subset K.$$

If $x \in \bigcap_{m=1}^{\infty} \bigcup_{l=m}^{\infty} K_{1/l}$ then $x \in K_{1/m}$ for infinitely many m. Thus, for infinitely many m, there exists $p_m \in B_\delta$ such that $\phi_{p_m}^{1/m}$ has a maximum at x. In particular, we can find a subsequence $\{m_k\}_{k=1}^{\infty}$ with the properties that

$$p_{m_k} \to p, \quad p \in B_\delta$$

$$\phi_{p_{m_k}}^{1/m_k} \text{ has a max at } x \text{ for all } k$$

$$\phi_{p_{m_k}}^{1/m_k} \to \phi_p \text{ locally uniformly.}$$

Using these properties, we see that $x \in K$.

Now let $A_m = \bigcup_{l=m}^{\infty} K_{1/l}$. We see that

$$A_1 \supset A_2 \supset A_3 \supset \ldots$$

and $|A_1| \leqslant |B_r| < \infty$. From chapter 3 in [MW] we get

$$\left|\bigcap_{m=1}^{\infty} A_m\right| = \lim_{m\to\infty} |A_m|.$$

Since $A_m = \bigcup_{l=m}^{\infty} K_{1/l} \supset K_{1/m}$ we have

$$|K| \geqslant \left| \bigcap_{m=1}^{\infty} \bigcup_{l=m}^{\infty} K_{1/l} \right| \geqslant \lim_{m\to\infty} |K_{1/m}| \geqslant 0.$$

The last inequality follows from the fact that $K_{1/m}$ has positive measure, independent on m by the previous case.

$\square$

From Jensen's lemma and the Alexandrov theorem we get the following corollary.

Corollary B.5. *Suppose $f : \Omega \to \mathbb{R}^n$ is semi-convex and f has a local maximum at x_0. Then there are $x_k \to x_0$ so that*

$$\nabla f(x_k) \to 0$$
$$D^2 f(x_k) \leqslant \frac{1}{k} I \to 0$$

when $k \to \infty$.

Proof. Set

$$\tilde{f}(x) = f(x) - \frac{1}{2k}|x - x_0|^2.$$

Then $\tilde{f}$ is also semi-convex and it has a *strict* maximum at x_0. By Jensen's lemma and the Alexandrov theorem, there are

$$x_k \in B_{\frac{1}{k}}(x_0)$$
$$p_k \in B_{\frac{1}{k}}(0)$$

so that

$$\tilde{f}_p(x) = \tilde{f}(x) - \langle p_k, x \rangle$$

has a local maximum at x_k with f twice differentiable at x_k. By the infinitesimal calculus we have

$$\begin{aligned} \nabla f(x_k) &= \nabla \tilde{f}(x_k) + \frac{1}{k}(x_k - x_0) = \nabla f_p(x_k) + p_k + \frac{1}{k}(x_k - x_0) \\ &= p_k + \frac{1}{k}(x_k - x_0) \to 0 \end{aligned}$$

as $k \to \infty$. Further,

$$D^2 f(x_k) = D^2 \tilde{f}(x_k) + \frac{1}{k} I = D^2 \tilde{f}_p(x_k) + \frac{1}{k} I \leqslant \frac{1}{k} I.$$

$\square$

C Some useful results from real analysis

C.1 The variational lemma

Lemma C.1. ***(The variational lemma.)***
Let Ω be a bounded domain in $\mathbb{R}^n$ and suppose F is continuous in Ω. Further, suppose

$$\int_\Omega F(x)\phi(x)dx = 0$$

for all test functions $\phi \in C_0^\infty(\Omega)$. Then

$$F \equiv 0.$$

Proof. Fix $x_0 \in \Omega$ and choose

$$\phi(x) = \rho_\epsilon(x - x_0)$$

where $0 < \epsilon <$dist$(x_0, \partial\Omega)$ and note that $\phi \in C_0^\infty(\Omega)$. Here, ρ_ϵ is the standard mollifier in $\mathbb{R}^n$. The mollification of F at x_0 is then

$$F_\epsilon(x_0) = \int_\Omega F(x)\rho_\epsilon(x - x_0)dx = 0$$

by assumption. We also have

$$F(x_0) = \lim_{\epsilon \to 0} F_\epsilon(x_0) = 0.$$

Since x_0 was arbitrary, $F = 0$ in Ω. □

C.2 Results from real analysis

Theorem C.2. ***(The Heine-Borel theorem.)***
A set $E \subset R^n$ is compact if and only if any collection O of open sets that covers E contains a finite subcover that also covers E.

A proof of theorem C.2 can be found in [MW] chapter 11.

Lemma C.3. ***(Monotone convergence theorem for sequences**.)*
Suppose the sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is bounded for each n. Further, suppose $\{a_n\}_{n=1}^{\infty}$ is a monotone sequence. Then there exists a limit $L = \lim_{n\to\infty} a_n$.

Proof. We suppose without loss of generality that $\{a_n\}$ is increasing. Set $L = \sup_n a_n$. By definition of L, for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ so that

$$L - \epsilon < a_N.$$

We estimate, for $n \geqslant N$

$$L - \epsilon < a_N \leqslant a_n \leqslant L + \epsilon.$$

Put differently, for any $\epsilon > 0$ we have

$$|a_n - L| \leqslant \epsilon$$

for all $n \geqslant N$. □

Theorem C.4. ***(The Bolzano-Weierstrass theorem.)***
Suppose the sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is bounded for each n. Then there exists a convergent subsequence $\{a_{n_k}\}_{k=1}^{\infty}$.

Proof. Our plan is to construct a monotone subsequence of $\{a_n\}$, which will also be a bounded sequence. The result then follows from lemma C.3.

Suppose first that the set

$$S_N = \{a_n : n > N\}$$

obtains its maximum for all $N \in \mathbb{N}$. Choose $n_1 < n_2 < n_3 < ...$ and set

$$a_{n_1} = \max_{n>1} a_n$$
$$a_{n_k} = \max_{n>n_{k-1}} a_n.$$

We see that the sequence $\{a_{n_k}\}$ is decreasing. If S_N fails to reach its maximum, put

$$a_{n_1} = a_{N+1}.$$

Then, since S_N has no maximum, we can find $n_2 > n_1$ so that $a_{n_2} > a_{n_1}$. Continuing this process gives an increasing sequence $\{a_{n_k}\}$. □

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