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Multi-Factor Interest Rate Models and Portfolio Management within Life Insurance Companies in Low-Rate Environments

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Preface

This Master's thesis is the final submission of a five year Master's degree programme in Physics and Mathematics at the Norwegian University of Science and Technology, Department of Mathematical Sciences. The project falls under applied mathematics and finance and was supervised by Jacob Laading, Associate Professor at NTNU. I would like to thank my supervisor for the valuable information, guidance and constructive feedback provided to me throughout the work on this thesis.

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Abstract

In this Master's thesis we study the equity market and the two multi-factor interest rate models Heath-Jarrow-Morton model (HJM) and the LIBOR market model (LMM) on the Norwegian, European and US interest rate market. These models are used to analyze a floor and to investigate the management of paid-up policy portfolios kept by life insurance companies. The main concerns are the current low-rate environment experienced in the financial markets today, and their exposure to negative interest rates. The two interest rate models are calibrated to the market using volatility factors. The HJM uses principal component analysis to find the volatility factors, while the LMM uses Exponentially Weighted Moving Average. The paid-up policy portfolios are then analyzed using Value at Risk and Expected Shortfall. We find that the probability of negative rates is clearly present in the HJM-framework, while it is zero in LMM because of the log-normal assumption. Further, we also see that the probability of negative rates are larger in the European market, compared to the Norwegian and US market. This in turn leads to significantly higher floor prices in the European market. The prices calculated with HJM and LMM deviates the most near the current spot rate, with HJM always giving higher floor prices than LMM. In the end we see that the European paid-up policy portfolios give less gain to the insurance companies compared to the Norwegian and US portfolios. The simulation also shows that a higher gain demand requires a larger share in the equity markets. This result is in line with the general yield hunting strategies observed in the market today.

Sammendrag

I denne masteroppgaven studerer vi aksjemarkedet og de to flerfaktor-rentemodellene Heath-Jarrow-Morton modellen (HJM) og LIBOR markedsmodellen (LMM) på det norske, europeiske og amerikanske rentemarkedet. Disse modellene blir brukt til å analysere en nedre avkastningsgaranti (floor), og til å undersøke forvaltningen av fripoliseporteføljer tilbudt av livsforsikringsselskaper. Hovedfokuset til denne oppgaven er det lave rentemiljøet som oppleves i finansmarkedene idag, samt deres eksponering for negative renter. De to rentemodellene blir kalibrert til markedet ved hjelp av volatilitetsfaktorer. HJM bruker prinsipal komponent analyse til finne disse volatilitetsfaktorene, mens LMM bruker eksponentielt vektet gjennomsnitt (EWMA). Fripoliseporteføljene blir deretter analysert ved å bruke Value at Risk og Expected Shortfall. Sannsynligheten for negative renter er tydelig til stede i HJM, mens det er null sannsynlighet for negative renter i LMM på grunn av den lognormale antagelsen. Videre viser vi at sannsynligheten for negative renter er større i det europeiske markedet, sammenlignet med det norske og det amerikanske markedet. Dette i sin tur fører til betydelig høyere floorpriser i det europeiske markedet. Vi ser også at prisene beregnet med HJM og LMM avviker mest nær dagens spotkurs, med HJM som alltid gir høyere floorpriser enn LMM. Til slutt finner man ut at de europeiske fripoliseporteføljene gir mindre avkastning til forsikringsselskapene sammenlignet med de norske og amerikanske porteføljene. Simuleringene viser også at et større avkastningkrav fører til en høyere nødvendig andel i aksjemarkedet. Dette resultatet er i tråd med de generelle gevinst-jaktende strategiene observert i markedet i dag.

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Chapter 1

Introduction

As of April 2003, 80% of the top 500 companies in the world used interest rate derivatives to control their cash flows, while only 10% used stock options according to the International Swaps and Derivatives Association. In other words, understanding the future evolution of especially interest rates, but also stock prices, are crucial to maintain a good financial policy within a firm. However, modeling interest rates are much more complicated than modeling stock prices. Therefore, the mathematical models used to model rates are often more difficult to interpret, and we usually need higher order simulations to get feasible results.

Stock market indexes are highly diversified combinations of stocks. These indexes are mainly used to model stock markets as a whole. In other words, if an investor wants a portfolio of stocks that follows the market, he should buy a share of such an index. One of the most famous stock indexes are the S&P 500 (Standard & Poor 500 Index), which includes the 500 most traded publicly stocks in the US. This index may be a good indicator of how the economy in the US is performing all together. The behavior of these stock indexes and stock prices them self are far from predictable. Thus we need to model them in a stochastic way.

As stated earlier, the interest rate derivative market is the largest of the derivative markets in the world. The reason for this is that essentially all of the companies in the world have debt, hence they are also exposed to an interest rate. Therefore, interest rate derivatives are used to reduce risk and control cash flows. As with the stocks, we also need a fair pricing technique for the interest rate case. Thus we also model interest rates in stochastic way, but usually in a more complicated way.

Furthermore, from the dot-com bubble in 2001 until 2007 the financial market seemed stable with increasing rates. However, when the American housing bubble busted, the collapse of the Lehman Brothers in September 2008 almost took down the entire worlds financial system. After almost a decade, it is clear the crisis had several reasons. One of the biggest were maybe the financiers themselves. They thought they had eliminated risk by applying complicated and less intuitive models, when they actually just had lost track of it. The mortgage lending in America to subprime borrowers, i.e. borrowers with poor credit which struggled to repay their loans, was an example of this. Big banks thought they made these mortgages less risky by putting enough of them together in a pool, but this only works if they are not too correlated. The problem was that the financial institutions assumed that the economy in the different regions in the US would fall or rise independently [1]. This did obviously not hold and starting in 2006 America experienced a downturn in the house prices throughout the whole country.

The financial crisis made a big impact on the rate level in the economies, and in 2011 the rates in Europe had decreased drastically. This significant reduction in rate level forced the life insurance companies into new territories, which gives the motivation of this thesis. Namely to analyze the prices of rate instruments and the risk attached to the portfolios kept by the life insurance companies. This is particularly of interest during the low-rate environment we now see.

Life insurance companies are companies which offer both life and pension products. They usually have a long term commitment which says that the pension is to be paid out far out in the future, but also they have to pay out a yearly guaranteed return, which means they have a short term commitment as well. At 2010, this guaranteed return was maximum 2.75% in Norway, while older contracts used a return in between 3 and 4% [2]. This rate guarantee is a rate derivative which means it secures the downside without fixing any restrictions on an upper bound. This rate option has a premium which the customer has to pay, and the premium is usually calculated as the value of the option itself. However, it may be priced differently due to competition in the market or the credit rating of the customer. In this thesis we do not include these price modifications due to competition or credit rating.

In addition to the rate instruments, the life insurance companies also have several well-diversified portfolios of stocks. An example of such portfolios would be a share in the S&P 500 index, which is well diversified. However, in distressed times all uncorrelated stocks tends to fall in value, which means the correlation coefficient approaches 1. This means it would be dangerous to assume that these diversified portfolios always are low-risk. Therefore it has been issued new regulations regarding the amount of risk-less capital each insurance company needs, which has made it harder for some of the insurance companies to survive, for instance Silver [3].

Furthermore, a paid-up policy is an accumulated pension one has achieved from a previous employer. These paid-up policies can be placed in hands of a specific life insurance company, which is then obligated to manage these policies and pay out yearly returns to the customer. This accumulated pension may be placed in both risky and risk-less assets. The management of these paid-up policies are investigated in the end of this thesis for the Norwegian, European and US market. The gain from these paid-up policies are in general shared between the life insurance company and the customer. However, in this thesis we assume the life insurance company receives all of the profit above the yearly floor guarantee.

This thesis starts with an explanation of financial concepts and the fundamental theory behind. Especially we give a fair justification of the stock price model used in the thesis, as well as giving a short introduction to the different interest rates. Then we move on to derivatives and how to determine their values. In the end of Chapter 2 we argue why there can not exist any arbitrage opportunities in a financial market, and why it is important to price derivatives in a risk-neutral world. In Chapter 3 we move on to one of the most important partial differential equation used in the financial world, namely the Black Scholes equation. The key feature is that it can price any given option when the underlying asset follows a lognormal walk, assuming sufficient boundary and initial/final conditions. Further we list the assumptions used to derive the equation, as well as showing the solution of the Black Scholes equation for European options, also called the Black Scholes formula. Another perk with this formula is that it can be inverted to yield the implied volatility observed in a given market.

In Chapter 4 we dive into the interest rate modeling world. First we list the general one-factor model before we deduce the bond pricing equation assuming a single-factor model for the stochastic interest rate. The biggest difference between this deduction and the deduction of the Black Scholes equation is that the bond pricing equation requires a hedge of one bond with another, opposite of the Black Scholes deduction which uses a hedge of the option itself combined with a fraction of the underlying asset. Further we list some common single factor models which gives a nice solution of the bond pricing equation. These models are either equilibrium models or no-arbitrage models, in other words if the parameters does not depend on time the model is an equilibrium model, while it is a no-arbitrage model if the parameters are time dependent. Chapter 4 also gives an introduction to the most interesting rate derivative for this thesis, namely the floor. It also gives several ways to price these options. One of these ways is the Blacks model, which is to interest rate derivatives like the Black Scholes formula is to stock derivatives.

In the next chapter we explain the two multi-factor interest rate models, namely HJM and LMM. The reason we use multi-factor instead of single-factor interest rate models is because the multi-factor models capture more of the dynamics in the yield curve. After deducing the HJM-model we give a description of the volatility factors used in the simulation. These factors are determined using PCA on historical data. The LIBOR market model also uses volatility factors to calibrate itself to the market, and in this thesis we use the covariance to calibrate the volatility factors estimated with exponentially weighted moving average. Both frameworks also give a nice way to price the floor.

Further, in Chapter 6 we give an introduction to risk measures and analysis. The two methods used in this thesis are the value at risk and expected shortfall. They are both easy to interpret and are consistent across different portfolio positions and risk factors. This means it is easy to compare the risk associated with different projects. The reason for performing risk management is to have an idea of the probability distribution the portfolio follows. This is the reason we explain the theory behind Monte Carlo simulation in Chapter 7. In the same chapter, we also go through how the gain of the paid-up policies are simulated. It is important to remember that calculation of a derivative's value is usually simulated in a risk-neutral world, namely under the probability measure $\mathbb{Q}$. The life insurance's floor prices are thus calculated in a risk-neutral world. However, when simulating the gain of the portfolios held by the life insurance companies, namely the returns above the yearly paid out floor guarantee, this requires simulation in real world. In other words, we simulate under the real probability measure $\mathbb{S}$.

The rest of the thesis is devoted to analysis of historical stock price and interest rate data from 2002-2015, the resulting simulation of future rates and the risk associated with the portfolios kept by the life insurance companies. First we look into the historical prices of several stocks and examines if their returns can be matched to a given probability distribution, namely the normal and the t-distribution. This gives an indication of whether the life insurance companies actually can assume that their stock shares are well diversified. Further we fit multi-factor interest rate models to the markets. The interest rate models are fitted in order to analyze the floor prices given by the insurance companies. We further simulate the LIBOR forward rates together with the stock indexes to investigate the positions needed for keeping the simulated mean gain of the paid-up policies held by the life insurance, positive. We also explore the risk attached to these positions.

Chapter 2

Financial Definitions

2.1 Financial Market

A financial market is a collection of particular markets that trade assets like stocks, bonds and derivative products. The prices of these traded assets are assumed to move randomly because of the efficient market hypothesis [4]. The hypothesis says that all previous information is already absorbed in the present price of the financial assets. In other words this means that markets with a legitimate price discovery mechanism collects all the information that the traders have. This results in that prices will only shift if new information comes to the table. However, new information tends to be random by nature, so prices will also have to change randomly. Moreover, it says it is impossible for investors to buy undervalued stocks or sell stocks for artificially high prices. This means that we cannot outperform the market and that buying into a market portfolio would be the optimal strategy. Note however, this does not mean we cannot earn more than the market, but it requires us to take on more risk. There has been several case studies to reject or accept this hypothesis. We can for instance look at the auto correlation in returns for a time period, study trading rules like momentum strategies(e.g. always buy if the price increases and always sell if the price is decreasing) and a vast majority of other tests. However, most of these tests conclude that the market is efficient.

2.2 Assets

A financial asset is a commitment that the holder of the asset will own future cash flows. The value of the asset are then determined by the expected present value of the cash flows. There exist several types of assets, but the most common are stocks, bonds, currencies and futures. We also have derivative assets like stock options or interest rate derivatives. In this thesis we will use the following model to analyze the underlying asset values and rates

$$dS(t) = u(t,S)dt + w(t,S)dX(t), \quad t > 0 \tag{2.2.1}$$

with $S(0) = S_0$. Here $S(t)$ is the value of the asset at time t and $u(t,S)$ and $w(t,S)$ are functions of time and asset value. $dX(t)$ is a random sample drawn from a probability distribution, also called a Wiener process. It contains the randomness of the asset prices and has the following properties

$$E[dX(t)] = 0 \tag{2.2.2}$$

$$Var[dX(t)] = dt. \tag{2.2.3}$$

Further, dt is called the drift term while $dX(t)$ is called the diffusion term.

2.3 Stocks

When companies need capital for a new project or a new product, they can sell partly ownership of themselves to investors to raise money, in other words they sell shares of the company. This means that the company is owned by it's stock holders, and the investors might earn money if the company increases its revenue. The money the investors receives from the stock is paid out as dividend per share. However, if the company collapse, the investors loose the money their shares are worth. Thus the values of these stocks reflects the future revenue and capital growth of the company.

The most basic way to model the stock prices is to use the equation (2.2.1), with $u(t,S) = \mu S(t)$ and $w(t,S) = \sigma S(t)$ with a geometric Brownian motion of the form

$$dS(t) = \mu S(t)dt + \sigma S(t)dX(t), \quad t > 0. \tag{2.3.1}$$

This results in a lognormal walk with μ as the mean, which measures the average rate of growth of the stocks prices, also known as the mean of the returns, where returns are defined as

$$r_i = \frac{S_i - S_{i-1}}{S_{i-1}}. \tag{2.3.2}$$

Here i can be an arbitrary index, but the most common is to use either daily, weekly or monthly increments. Further, σ is the volatility which gives the standard deviation of the returns. μ and σ are in this case constant, while in more complicated models they can also be a function of time and stock price. Equation (2.3.1) is a good model for the stock price because it gives an exponential fall or rise in prices which is consistent with the observed prices in the market. The prices can obviously not go below zero which makes the lognormal a good fit. Also it contains the random part which incorporates new information in the model. As stated above, the easiest way to model the stock is to use equation (2.3.1), however there also exist other types of models like the stochastic volatility model. In this model the volatility itself varies stochastically and is dependent on the current level of price of the stock, namely

$$dS(t) = \mu S(t)dt + \sqrt{\nu(t)}S(t)dX(t), \quad t > 0 \tag{2.3.3}$$

$$d\nu(t) = \alpha(\nu, t)dt + \beta(\nu, t)dY(t), \quad t > 0. \tag{2.3.4}$$

In this case μ is still constant, while the volatility function ν varies stochastically with a given drift $\alpha(\nu, t)$ and volatility $\beta(\nu, t)$, which again might depend on the current level of ν. Here $dY(t)$ is another Wiener process.

2.4 Bonds

A bond is a fixed-income security where one part lends a load of money to the other part, in return the lender gets his money back as fixed cash flows in the future. These cash flows might be spread equally out, may be paid back all in one at maturity or might be involved in other payback plans. Bonds are operated by banks, financial institutions and big companies to raise money.

The easiest bond to handle is a zero-coupon bond. This is a bond that pays no coupon but instead pays the whole returning sum at maturity. Some of these type of bonds are issued as zero-coupon bonds, while other are issued as regular bonds but then later have been stripped of their coupons by a financial institution. In this thesis we use the notation $Z(t,T)$ for the value of a zero-coupon bond at time t, that pays 1 (unit currency) at time T.

2.5 Interest Rates

The interest rate are divided into either a discretely compounded rate or a continuously compunded rate. If we invest 1 at time t and has a discretely compounded rate, then the money is worth

$$1 \cdot \left(1 + \frac{r}{m}\right)^{m(T-t)}$$

at $T-t$ years later. Here m says how many interest payments there are per year. However, if r is continuously compounded, that is $m \to \infty$, we get

$$\left(\left(1 + \frac{r}{m}\right)^{m}\right)^{(T-t)} = \left(e^{mlog\left(1+\frac{r}{m}\right)}\right)^{(T-t)} \sim e^{r(T-t)}$$

for the value of the money at $T-t$ years. At least this is the result if we assume a fixed and constant rate. If the interest rate is not constant but a known function of time $r(t)$, then M, the value of the money, after $T-t$ years becomes

$$M = e^{-\int_t^T r(s)\,\mathrm{d}s}.$$

2.6 Yield Curve

The problem with having portfolios of derivative assets like stock options or interest rate derivatives is that normally we assume a deterministic or constant interest rate in the pricing models. However, for products with longer life span the problem with randomly fluctuating interest rates must be addressed. This is where the yield curve comes into the picture. It is a measure of future values of interest rates. The yield Y(t,T) is given by

$$Y(t,T) = -\frac{log(Z(t,T))}{(T-t)}, \tag{2.6.1}$$

and is derived from the following equation

$$Z(t,T) = e^{-Y(t,T)(T-t)}.$$

Here $Z(t,T)$ is the value of the zero-coupon bond, $Y(t,T)$ is the yield, $t(\leq T)$ is the time and T is the maturity time. The yield is simply the continuous compounded constant rate that your money is growing with if you pay $Z(t,T)$ at time t and receives 1 at maturity $t = T$.

2.7 The Continuous Forward Rate

The continuous forward rate is a rate we apply to a financial transaction in the future. It is the instantaneous continuously compounded rate, $f(t,T)$, we use when lending an amount at time t in the future with maturity at T. The relationship with a zero coupon bond is given by

$$Z(t,T) = e^{-\int_t^T f(t,s)\,\mathrm{d}s}. \tag{2.7.1}$$

Further, the spot rate $r(t)$ is related to the the continuous forward rate by

$$r(t) = f(t,t). \tag{2.7.2}$$

The forward rate $f(t,T)$ is a deterministic function for each t, which means the curve is known for all $t < T$.

2.8 The Simple Rate

A simple rate, $L(t,T)$, is the rate for an accrual time period of length $\delta = T - t$, with time measured in years. This means the interest earned in one time period is

$$\delta L(t,T).$$

We will see the importance of this simple rate in the LIBOR market model.

2.9 Derivatives

A derivative is a contract between two parties which has a price that is dependent upon one or several underlying assets. The price is determined by variations in the underlying asset and this asset might not be possible to trade with. These underlying assets might be stocks or simple spot rates, which is not possible to buy/sell. One of the simplest and most common option would be the European call option. This is a contract which says that the holder of the option at a determined time in the future may buy the prescribed underlying stock, for a previously determined price called the strike price. The word may means it is a right, not a necessity. This option gives the holder a payoff

$$max(S(T) - E, 0)$$

at a time T, where $S(T)$ is the price of the underlying stock and E is the fixed, constant strike price. We may also have an option with the right to sell the asset, this is called a put and its value is

$$max(E - S(T), 0).$$

Further, we also have the interest rate swap, which is a contract between two parties to give each other interest rate payments on a certain amount of money for a prescribed period of time. One example of this can be that party 1 pays a fixed interest rate r on an amount Z to party 2, while party 2 pays interest rate back to party 1 on a floating interest rate r^*, on the same amount. This can be seen as a bond with $(r^* - r)Z$ coupon payments. In addition there exist floors and caps which are features that bound the size of the interest rate. For instance, a floor is a bond with a fluctuating interest rate, but the interest rate cannot go under a fixed prescribed level. The same goes for a cap, however in this case the interest rate cannot exceed a specific value. We can have several options on these bond characteristics, like swaptions, captions and floortions. For example, we may have the option to buy a swap for an amount E at time $T < T_S$. This option will have the value

$$V(r, T) = max(V_S(r, T) - E, 0).$$

Her $V_S(r, T)$ is the value of the swap, T_S is the expiring time of the swap and E is the strike price. This is equivalent for captions and floortions, aswell as there exists other types of options.

2.10 Hedging

Hedging is an investment strategy where the risk of the portfolio is reduced by taking advantage of correlations between assets and the movement of option prices. This can be done by creating a portfolio with both assets and options, such that when small unpredictable changes in the assets does not cause an unpredictable move in the value of the portfolio. The most common hedging is delta hedging, but there also exists other type of hedging.

2.11 Diversification

The diversification effect is one of the few "free lunches" in the financial world. It is based upon the fact that combining several assets into one portfolio, the respective covariances between the assets will decrease the total variance of the whole portfolio. In other words, the risk can be reduced by combining several assets into one portfolio, as long as the correlations between the assets are not close to one.

2.12 Arbitrage

Given a financial market, an arbitrage opportunity is a way to possibly earn money without any risk. This means we make an investment and it is guaranteed that we receive back the initial paid amount, in addition there exist a probability that we receive even more. Arbitrage opportunities are not common and if they do occur, they will vanish very quickly because the market will adjust. Therefore models in financial markets do not include arbitrage opportunities.

2.13 Risk-Neutral Valuation

The risk-neutral valuation comes from the fact that the expected payoff of any given risky asset can be discounted with the risk-free rate to find the fair value of the asset. This discounting need to be used with risk-neutral probabilities, or in other words the prices need to be modeled in a risk-neutral world. In the stock option world this emerges when μ does not occur in the Black Scholes equation, which is derived in Chapter 3.1. Even though the variance of the derivative's price affects the value of an option, the option's value does not depend on the underlying's rate of growth. This means that the risk preferences of investors does not matter because all the risk built in to the option can be hedged away. In other words if a portfolio can be created with a derivative product and the underlying asset such that the the random part can be removed, then the derivative product can be evaluated as if all the random walks used are risk-neutral. This is done by replacing the drift term in the stochastic differential equation with the risk-free rate r wherever it appears. However, it is important to remember that the probability density using r as drift is valid under risk-neutral valuation, and can not be used to show the distribution of prices in the real world. Further, it is also important to valuate interest rate options in the risk-neutral world. This risk-neutral feature occurs when the bond pricing equation is deduced, namely that we end up with a risk-neutral drift of $u - \lambda w$, showed later in the thesis.

Chapter 3

Black-Scholes

3.1 Derivation of Black-Scholes

In order to derive the simplest form of the Black-Scholes formula we need to make the following assumptions

- The asset price follows the lognormal random walk given in equation (2.2.1) with $u(t, S) = \mu S$ and $w(t, S) = \sigma S$ explained in Chapter 2.3.
- μ and σ are given functions of time over the whole life time of the option.
- There are no transaction costs.
- The underlying assets pay no dividents during the lifetime of the option
- There are no opportunities for arbitrage.
- We assume that trading can be done continuously and that the assets can be divided into any given fraction.
- Short selling is possible.

Given that there exists an option (we could also have a portfolio of options) with value $V(t, S)$ and that the underlying assets follow the model stated in the assumptions, then using Ito's lemma in the appendix we write

$$dV = \sigma S \frac{\partial V}{\partial S} dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt. \qquad (3.1.1)$$

Here we require that V has atleast one t derivative aswell as two S derivatives. Further we construct a portfolio of one option and a -δ of the underlying asset.

Here δ can be any real number. The value of this portfolio is

$$\Pi = V - \delta S, \tag{3.1.2}$$

while the change in the portfolio is equal to

$$d\Pi = dV - \delta dS. \tag{3.1.3}$$

Here δ is held fixed during the time step.

Combining the model that the asset is following, with equation (3.1.1) and (3.1.3) we obtain the random walk

$$d\Pi = \sigma S \left(\frac{\partial V}{\partial S} - \delta \right) dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu S \right) dt.$$

In order to remove the random component from this equation we choose $\delta = \frac{\partial V}{\partial S}$.

This results in a deterministic increment for the portfolio

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$

Relying on the arbitrage free assumption we have that the return on an amount Π invested in assets without any risks would give a growth of $r\Pi dt$ during a time dt. This means we have

$$r\Pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt. \tag{3.1.4}$$

Again, if we combine equation (3.1.2) and (3.1.4) with δ, and divide by dt we get

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \tag{3.1.5}$$

Thus we have derived the Black-Scholes partial differential equation. There exist exact, explicit solutions for both the European call and put option [5]. However, for more complicated derivatives we need to solve it with numerical methods. Note that $V(S,t)$ can be any function of S and t, which means equation (3.1.5) holds for every derivative which value depends only on S and t. One interesting observation is that equation (3.1.5) does not include the growth parameter μ, which means the value of an option is priced independently of the growth rate of the underlying asset. This means that two parties that are disagreeing on the correct value of μ yet still agree on the correct value of the option.

3.2 Solution of Black-Scholes for European options

When r and σ are constant, there exist analytical solutions for both European call and put option as stated earlier. The solution for a European call is

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2). \tag{3.2.1}$$

Here $N(x)$ is the standard normal cumulative distribution function

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\, \mathrm{d}y.$$

Further

$$d_1 = \frac{log(\frac{S}{E}) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{log(\frac{S}{E}) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

To find the value of European put we may use the put-call parity. The parity says that if we have a portfolio Π which consist of longing one asset, longing one put and shorting one call

$$\Pi = S + P - C, \tag{3.2.2}$$

then the payoff for (3.2.2) at expiry is

$$S + max(E - S, 0) - max(S - E, 0).$$

This can also be written as

$$S + (E - S) = E \quad S \leq E,$$

$$S - (S - E) = E \quad E \leq S.$$

This means that the payoff at expiry is always E. Again using the no arbitrage assumption we understand that the price of the portfolio in (3.2.2) at time t should equal the discounted final value of the portfolio, namely

$$S + P - C = Ee^{-r(T-t)}. \tag{3.2.3}$$

This is the put-call parity. Further, using equation (3.2.1) with (3.2.3) we arrive at

$$P(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1) \tag{3.2.4}$$

for the value of a European put.

Chapter 4

Interest Rate Models and the Bond Price Equation

So far, the interest rate r has been assumed constant in the Black-Scholes analysis. This might be a good approximation for options over a shorter time period. However, for a longer period of time rates have a significant random fluctuation. This means that for options and other derivatives it would be more useful to include a stochastic interest rate model. Further, the interest rate for the shortest possible time to make a deposit is usually called the spot rate. This spot rate is common to be modeled as in equation (2.2.1), with $r = S$, namely

$$dr(t) = u(r,t)dt + w(r,t)dX(t), \quad t > 0. \tag{4.0.1}$$

When interest rates are stochastic as in equation (4.0.1), a bond has the price of the form $V(r,t;T)$. Pricing these bonds are harder than pricing options because there are no underlying assets we can hedge with. Rates are the obvious underlying asset, but it is not possible to buy nor sell a rate. However, to hedge a portfolio constructed of only bonds we need to hedge one bond with another bond of a different maturity. This means bond 1 have maturity T_1 and price $V(r,t;T_1)$ and equivalent for bond 2. Further, we construct the portfolio

$$\Pi = V_1 - \delta V_2.$$

Again using Ito's lemma we obtain

$$d\Pi = \frac{\partial V_1}{\partial t}dt + \frac{\partial V_1}{\partial r}dr + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2}dt - \delta\left(\frac{\partial V_2}{\partial t}dt + \frac{\partial V_2}{\partial r}dr + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}dt\right)$$

Further, to remove the randomness terms we choose

$$\delta = \frac{\partial V_1}{\partial r}\frac{\partial r}{\partial V_2}.$$

With this choice of δ we get

$$d\Pi = \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - \frac{\partial V_1}{\partial r}\frac{\partial r}{\partial V_2}\left(\frac{\partial V_2}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}\right)\right)dt.$$

Again, using the no arbitrage assumption in the Black-Scholes analysis we have that

$$\Pi = r\Pi dt = r\left(V_1 - \left(\frac{\partial V_1}{\partial r}\frac{\partial r}{\partial V_2}\right)V_2\right)dt. \tag{4.0.2}$$

Here the risk free rate r is the spot rate. Further, collecting all terms of V_1 on the left side and all terms of V_2 on the right side we obtain

$$\frac{\frac{\partial V_1}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - rV_1}{\frac{\partial V_1}{\partial r}} = \frac{\frac{\partial V_2}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2} - rV_2}{\frac{\partial V_2}{\partial r}}. \tag{4.0.3}$$

The left side of equation (4.0.3) only contains T_1 parts but no T_2 parts, while this is equivalent for T_2 on the right side. This can only happen if both sides are independent of the maturity date. This means we may drop the subscript of V and we get

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} - rV}{\frac{\partial V}{\partial r}} = a(r,t) \tag{4.0.4}$$

The most convenient way to write the right hand side of equation (4.0.4) is

$$a(r,t) = w(r,t)\lambda(r,t) - u(r,t).$$

This results in the following bond price equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0 \tag{4.0.5}$$

For equation (4.0.5) to have a unique solution we need one final condition as well as two boundary conditions. The final condition would in this case be the payoff at maturity, so for a zero-coupon bond we have

$$V(r, T; T) = 1.$$

If we have a coupon paying bond and assumes that an amount of $K(r, t)dt$ is received during a time dt then the new bond pricing equation will be

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV + K(r, T) = 0.$$

However, when the coupons are paid discretely we obtain the jump condition

$$V(r, t_c^-; T) = V(r, t_c^+; T) + K(r, t_c).$$

Here the coupon $K(r, t_c)$ is paid at time t_c.

4.1 Market Price of Risk and Risk Neutrality

When we model something stochastic which can not be traded, we get too few equations compared to the number of unknowns. We solve this by defining the market price of risk. This can be seen if we hold one bond with maturity T, then the change in the value of the bond in time dt is

$$dV = w\frac{\partial V}{\partial r}dX + \left(\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r}\right)dt. \tag{4.1.1}$$

Combining equation (4.0.5) with (4.1.1) we obtain

$$dV - rVdt = w\frac{\partial V}{\partial r}(dX + \lambda dt).$$

Here it can be seen that the term on the right contains both a random and a deterministic part. We understand the deterministic part as the excess return over the risk free rate for accepting a specific level of risk. Since the portfolio have accepted an extra amount of risk then we earn an extra $\lambda\ dt$ per unit of extra risk, dX. This is the reason why the function λ is called the market price of risk. Further, in the bond pricing equation (4.0.5) we have time decay, diffusion, drift and discounting parts respectively. We interpret a solution of this equation as the expected value of all future cash flows. So, consider a payoff at time T, then the present value of this agreement would be

$$E\left[e^{\int_{-t}^{T} r(\tau)\,\mathrm{d}\tau}\text{Payoff}\right]$$

This expectation is with respect to the risk neutral variable and not with respect to the real random variable. The difference occurs because the drift term in the bond pricing equation (4.0.5) is not the drift of the real spot rate u, but the drift of the so-called risk neutral spot rate. The risk neutral spot rate has a drift of $u - \lambda w$. This means that when pricing derivatives it is important to use the risk neutral spot rate which satisfies

$$dr = (u - \lambda w)dt + wdX. \tag{4.1.2}$$

4.2 Tractable Interest Rate Models

History shows that the coefficients in equation (4.1.2) need to be more complicated than the coefficients in the equity random walk, in order to grasp the dynamics of the real spot rate. However, making the coefficients too advanced will in turn make it hard to find solutions of the bond pricing equation. Therefore, in this chapter we look at coefficients of the form

$$u(r,t) - \lambda(r,t)w(r,t) = \eta(t) - \gamma(t)r \tag{4.2.1}$$

and

$$w(r,t) = \sqrt{\alpha(t)r + \beta(t)}. \tag{4.2.2}$$

Fixing the coefficients in this way, with a few restrictions, we make sure that r in the random walk (4.0.1) has the following properties

- The interest rates can be held positive. The spot rate can be bounded below by $-\frac{\beta}{\alpha}$ if $\alpha(t) > 0$ and $\beta(t) \leq 0$. If $\alpha(t) = 0$ then $\beta(t) \geq 0$. Here r still can go to ∞ but with probability equal to zero.

- The interest rates have mean reversion. For large r the risk neutral interest rate will start decreasing to its mean, while when the rate is very small it will increase. This can be seen from the drift term.

We also want the model to never reach its lower bound, this is satisfied if

$$\eta(t) \geq -\frac{\beta(t)\gamma(t)}{\alpha(t)} + \frac{\alpha(t)}{2}. \tag{4.2.3}$$

See [6] for proof. In addition, we need to impose two boundary conditions for a zero coupon bond, namely

$$V(r,t;T) \to 0 \quad as \quad r \to \infty,$$

and that V remains finite on $r = -\frac{\beta}{\alpha}$. This last condition is only achieved if equation (4.2.3) is valid. See [6] again for proof.

Further, looking at a zero coupon bond $Z(r,t;T)$ with the coefficients given in equation (4.2.1) and (4.2.2), the solution of the bond pricing equation (4.0.5) takes the simple form

$$Z(r,t;T) = e^{A(t;T)-rB(t;T)} \tag{4.2.4}$$

To find the form of A and B we insert (4.2.4) into (4.0.5). This leads to

$$\frac{\partial A}{\partial t} - r\frac{B}{\partial t} + \frac{1}{2}w^2B^2 - (u - \lambda w)\, B - r = 0. \tag{4.2.5}$$

Taking the double derivative of this with respect to r and divide by B gives

$$\frac{1}{2}B\frac{\partial^2}{\partial r^2}(w^2) - \frac{\partial^2}{\partial r^2}\,(u - \lambda w) = 0.$$

Since only B is a function of T this means

$$\frac{\partial^2}{\partial r^2}(w^2) = 0$$

and

$$\frac{\partial^2}{\partial r^2}(u - \lambda w) = 0.$$

From this equations (4.2.1) and (4.2.2) are derived. Further, substitution of (4.2.1) and (4.2.2) into equation (4.2.5) leads to

$$\frac{\partial A}{\partial t} = \eta(t)B - \frac{1}{2}\beta(t)B^2 \tag{4.2.6}$$

$$\frac{\partial B}{\partial t} = \frac{1}{2}\alpha(t)B^2 + \gamma(t)B - 1. \tag{4.2.7}$$

We also need $A(T;T) = B(T;T) = 0$ to satisfy the final condition $Z(r,t;T) = 1$. Usually when α, β, γ and η are time dependent we can not integrate explicitly the two equations (4.2.6) and (4.2.7). However, assuming constant parameters we obtain a nice solution of the form

$$B(t;T) = \frac{2\left(e^{\Phi_1(T-t)} - 1\right)}{(\gamma + \Phi_1)\left(e^{\Phi_1(T-t)} - 1\right) + 2\Phi_1},$$

and

$$A(t;T) = \frac{2}{\alpha}\left(a\Phi_2\text{log}(a - B) + b(\Phi_2 + \frac{1}{2}\beta)\text{log}(\frac{B+b}{b}) - \frac{1}{2}B\beta - a\phi_2\text{log}a\right)$$

where

$$\phi_1 = \sqrt{\gamma^2 + 2\alpha},$$

$$\phi_2 = \frac{\eta - \frac{a\beta}{2}}{a+b}$$

and

$$b, a = \frac{\pm\gamma + \sqrt{\gamma^2 + 2\alpha}}{\alpha}.$$

Further, it is obvious in the constant parameter case that A and B only depends on a single variable $\tau = T - t$. This means the yield curve can easily be estimated from equation (2.6.1). In this case

$$Y \to \frac{2}{(\gamma + \Phi_1)^2} \left(\eta(\gamma + \Phi_1) - \beta\right) \tag{4.2.8}$$

as $\tau \to \infty$. So the model leads to a fixed long term interest rate when using constant parameters.

The Vasicek model is one of the most common risk neutral interest rate models. It uses equations (4.2.1) and (4.2.2) with $\alpha = 0$, $\beta > 0$ and the other parameters with no time dependence

$$dr = (\eta - \gamma r)dt + \beta^{\frac{1}{2}} dX. \tag{4.2.9}$$

Some of the flaws with the model is that it is hard to get humped yield curves and also that interest rates can go negative. However, one of the perks is the mean reverting to a constant level feature.

In the Cox, Ingersoll and Ross model β is zero and there is still no time dependence in the rest of the parameters

$$dr = (\eta - \gamma r)dt + \sqrt{\alpha r} dX. \tag{4.2.10}$$

This model is also mean reverting and it also includes an additional feature, namely that the spot rate stays positive for $\eta > \frac{\alpha}{2}$. Also, the value of $Z(r, t; T)$ is again given by equation (4.2.4) with A and B given in equations (4.2.1) and (4.2.2) with $\beta = 0$.

4.3 Bond Option and Interest Rate Derivatives

Much of the theory analyzing equity options can be used to evaluate bond options and interst rate derivatives. The bond option is much like an equity option except that the underlying asset is a bond. Also in this case there exist both European and American versions. Consider a zero coupon bond $Z(r, t; T_B)$ satisfying equation (4.0.5) and a call option on this bond with exercise price E and expiry date $T \leq T_B$. The value $V(r, T)$ of the call option also depends on r, so it also satisfies equation (4.0.5), however the final value of the option is now

$$V(r,T) = \max\left(Z(r,t;T_B) - E, 0\right).$$

Further, a cap contract typically has payments at times t_i, each period, of a floating interest on a principal. Hence the cash flow is of the form

$$Principal \cdot m \cdot \max(r_L - r_c, 0),$$

while for a floor the cash flow is

$$Principal \cdot m \cdot \max(r_f - r_L, 0).$$

Here m is the tenor of interest payments, for instance 0.5 if payments are semi-annual, r_c, r_f are the fixed cap and floor rates and r_L is the floating rate. r_L might be three-month LIBOR if payments are made quarterly and this rate that is to be paid at t_i is set at t_{i-1}. Each of the cash flows $V(r,T)$ is called a caplet/floorlet, which means if we assume that $r_L \approx r$(spot rate), these "lets" can be priced with equation (4.0.5) with

$$V(r,T) = Principal \cdot m \cdot \max(r - r_c, 0)$$

for a caplet and

$$V(r,T) = Principal \cdot m \cdot \max(r_f - r)$$

for a floorlet. Further, we see that this is equivalent to a call option on the floating rate r for the caplet, and a put for the floorlet.

4.4 Black's Model

Another way to price the "lets" is to use the Black Scholes formula since it simple to use. This is done by modeling a caplet/floorlet as a call/put on a lognormally distributed interest rate. This means the model takes in the strike price r_c/r_f, annualized volatility σ of the interest rate, the time to the cash flow $t_i - t$ and two interest rates. One of them is used as a "stock price" and is the current forward rate between t_{i-1} and t_i, while the other is the yield on a bond having maturity

$T = t_i$. The latter one is used for discounting to present. From the lecture notes in [7] we have that the value V of a floorlet is

$$V = Principal \cdot m \cdot e^{-r^*(t_i - t)} \left(-F(t, t_{i-1}, t_i) N(-d_1^{'}) + r_f N(-d_2^{'}) \right), \qquad (4.4.1)$$

with

$$d_1^{'} = \frac{\log(\frac{F}{r_f}) + \frac{1}{2}\sigma^2 t_{i-1}}{\sigma\sqrt{t_{i-1}}} \quad and \quad d_2^{'} = d_1^{'} - \sigma\sqrt{t_{i-1}}.$$

Here $F(t, t_{i-1}, t_i)$ is the forward rate and r^* is the yield to maturity for a maturity of $t_i - t$. Note that this model is not limited to one-factor models. Further, there exist an one-to-one link between the floorlet price and the volatility. Therefore, option prices are sometimes listed by giving the implied volatility. Namely the unique volatility which gives the observed market price. An important problem with the Black's model however, is that the implied volatility does not exist for negative rates. This is the reason why LMM-rates are used in the Black's pricing model, and not HJM-rates because they may very vell become negative.

4.5 Floating Rate Approximation

For short maturities we may solve the zero coupon bond pricing equation in (4.0.5) with a taylor series expansion. This is done by substituting

$$Z(r, t; T) = 1 + a(t)(T - t) + \frac{1}{2}b(r)(T - t)^2 + ...$$

into equation (4.0.5). Doing this we find

$$a(r) = -r \quad \text{and} \quad b(r) = r^2 - (u - \lambda w).$$

From these results we then find the yield curve for small maturities

$$-\frac{\log Z}{T - t} \sim r + \frac{1}{2}(u - \lambda w)(T - t) + ... \quad \text{as} \quad t \to T \qquad (4.5.1)$$

This means that we have another approximation than $r_l \approx r$(spotrate), namely

$$r_l \approx r + \frac{m}{2}(u - \lambda w).$$

Here m is the maturity measured in years, for instance $m = \frac{1}{4}$ for a three-month rate.

Chapter 5

Multi-Factor Interest Rate Models

5.1 The Heath, Jarrow and Morton Framework

In stead of modeling the spot rate with the previous models stated, the HJM models the forward rate curve. It is based on an implicit yield curve fitting, which means it builds a model for the whole forward rate curve from the forward rates currently available today. From equation (2.7.1) we have

$$f(t,T) = -\frac{\partial}{\partial T}\log Z(t;T). \tag{5.1.1}$$

Further, the forward rate curve is modeled by

$$df(t,T) = \mu(t,T)dt + \boldsymbol{\sigma}(t,T)^T \boldsymbol{dW}(t). \tag{5.1.2}$$

The process $\boldsymbol{dW}(t)$ is a d-dimensional standard Brownian motion. To make the discounted zero coupon bond prices positive martingales as in the book [8] we need it on the form

$$\frac{dZ(t,T)}{Z(t,T)} = r(t,T)dt + \boldsymbol{\nu}(t,T)^T \boldsymbol{dW}(t).$$

Using Ito's formula on equations (5.1.1) and (5.1.2) and interchanging the order of differentiation we obtain

$$df(t,T) = \frac{\partial}{\partial T}\left(\frac{1}{2}\nu^T(t,T)\nu(t,T) - r(t,T)\right)dt - \frac{\partial}{\partial T}\boldsymbol{\nu}(t,T)^T\boldsymbol{dW}(t).$$

This means

$$\boldsymbol{\sigma}(t,T) = -\frac{\partial}{\partial T}\boldsymbol{\nu}(t,T),$$

while the drift term is seen to be

$$\mu(t,T) = \boldsymbol{\sigma}(t,T)^T\int_t^T \boldsymbol{\sigma}(t,u)\,\mathrm{d}u. \tag{5.1.3}$$

Equation (5.1.3) is known as the no-arbitrage condition in the HJM framework. Further, equation (5.1.2) can now be written

$$df(t,T) = \left(\boldsymbol{\sigma}(t,T)^T\int_t^T \boldsymbol{\sigma}(t,u)\,\mathrm{d}u\right)dt + \boldsymbol{\sigma}(t,T)^T\boldsymbol{dW}(t) \tag{5.1.4}$$

with $f(0,T) = f^*(0,T)$ where $f^*(0,T)$ is the observed forward rates at today's time.

5.1.1 Discretization of HJM model

It is in general hard to represent the full continuously forward rate curve in (5.1.4), except for a few special choices of σ. This means we rather fix the same time grid $0 = t_0 < t_1 < t_2 < ... < t_{M-1} < t_M$ both for t and T. In other words, we are modeling the forward rate curved for the same set of times. Using the same set of time grid for both variables simplifies the notation significantly with little loss of generality. Further, letting $\hat{f}(t_i,t_j)$ be the discretized forward rate at time t_i with maturity t_j, the discretized version of the zero coupon bond price is

$$\hat{Z}(t_i,t_j) = e^{\sum_{l=i}^{j-1}\hat{f}(t_i,t_j)[t_{l+1}-t_l]}$$

Creating the time grid obviously introduces a discretization error. To minimize it, we choose the initial values of the discretized zero coupon bonds to match the true values of the bond prizes for all maturities on the time grid, i.e.

$$\hat{Z}(0, t_j) = Z(0, t_j) \quad j = 0, 1, ..., M.$$

Further, from equation (5.1.1) we have the following condition for the initial forward rates

$$\hat{f}(0, t_j) = \frac{1}{t_{j+1} - t_j} \log\left(\frac{Z(0, t_j)}{Z(0, t_{j+1})}\right). \tag{5.1.5}$$

After the initialization the simulated forward rate curve is computed for $i = 1, ..., M$,

$$\hat{f}(t_i, t_j) = \hat{f}(t_{i-1}, t_j) + \hat{\mu}(t_{i-1}, t_j)\left[t_i - t_{i-1}\right] + \sqrt{t_i - t_{i-1}}\boldsymbol{\sigma}(t_{i-1}, t_j)^T \boldsymbol{Z}_i, \quad j = i, ..., M. \tag{5.1.6}$$

Here $\boldsymbol{Z}_i$'s are d-dimensional independent $N(0, I)$ random vectors, the drift term is

$$\hat{\mu}(t_{i-1}, t_j) = \sum_{k=1}^{d} \hat{\mu}_k(t_{i-1}, t_j),$$

where

$$\hat{\mu}_k(t_{i-1}, t_j)\left[t_{j+1} - t_j\right] = \frac{1}{2}\left(\sum_{l=i}^{j} \hat{\sigma}_k(t_{i-1}, t_l)\left[t_{l+1} - t_l\right]\right)^2 - \frac{1}{2}\left(\sum_{l=i}^{j-1} \hat{\sigma}_k(t_{i-1}, t_l)\left[t_{l+1} - t_l\right]\right)^2. \tag{5.1.7}$$

Equation (5.1.7) is the discrete analogue to the no-arbitrage condition for a multi-factor HJM model. Having obtained an expression for the simulated forward rate curve the only things remaining are finding the initial forward rates and determining the volatility structure. However, fixing a daily grid means we do not have bonds maturing at each grid point. The solution is to interpolate between the known maturity points which gives you the initial forward rates for all the grid points. In this thesis we use linear interpolation. Further, in this chapter we also use the Musiela parametrization [9]

$$\sigma(t, T) = \sigma(T - t) = \sigma(\tau).$$

Namely that the volatility in the forward rate curve is only dependent on the time to maturity T. This means we use the change in historical forward rates for several maturities, to estimate the volatility in the forward rate curve. This is done by first finding the daily changes in each time series and then applying principal component analysis.

5.1.2 Principal Component Analysis in the HJM Framework

In this chapter we give a brief explanation of the PCA, see [13] for a more detailed explanation. We start with the daily historical changes in the forward rate, for d different maturities. The PCA will then convert a set of possibly correlated variables into linearly uncorrelated variables by using an orthogonal transformation on the historical changes in the forward rate. The first principal component will explain the largest variance in the data by the definition of the transformation, and then the second principal component will include the second largest variance under the constraint that it must be orthogonal to the previous component. In other words it is uncorrelated with the previous principal component. First we find the covariance matrix $\mathbf{\Sigma}$, which has dimension $d \times d$, where $\mathbf{\Sigma}_{ij}$ gives the covariance between movement of the i'th and j'th forward rate. Then $\mathbf{\Sigma}$ is decomposed into

$$\mathbf{\Sigma} = \boldsymbol{V}\mathbf{\Lambda}\boldsymbol{V}^{\mathbf{-1}},$$

where $\boldsymbol{V}$ is a matrix whose columns gives the eigenvectors of $\mathbf{\Sigma}$, while $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\mathbf{\Sigma}$. Further, the first eigenvector will give the most important move in the forward rate curve, while the i'th entry gives the movement of the i'th maturity. Also the j'th column of $\boldsymbol{V}$ will give the j'th principal component. Using k of these principal components will give a

$$\frac{\sum_{j=1}^{k} \lambda_j}{\sum_{j=1}^{d} \lambda_j} \cdot 100\%$$

explanation of the variability in the data. All together this gives the volatility factors in equation (5.1.6)

$$\sigma_j(\tau_i) = \sqrt{\lambda_j}\boldsymbol{V}_{ij} \tag{5.1.8}$$

Again, we need to interpolate (5.1.8) to get a an expression for the volatility factors over the whole time grid. In this case we also uses linear interpolation. Further, we usually assume that three factors can explain the most of the variability of the American yield curve. For instance, in Chapman and Pearson [14] it is shown that the three first principal components explains 99% of the variability in the American yield curve, namely the first component gives a parallel shift, the second a twisting and the third gives a bending of the yield curve. However, it is not so clear for forward rates and the intuitive economical meaning behind each of the components.

5.1.3 Pricing Derivatives Under the HJM Framework

From the relationship in equation (2.7.2) we obtain a HJM-model for the spot rate by evaluating the forward rate at $t = T$

$$r(t) = f(t, t).$$

This means we simulate the spot rate using equation (5.1.6) on our time grid. Furthermore, given a derivative we set the payment dates of this derivative to be annually. This makes it trivial to apply Monte Carlo simulation to price the derivative. For a life insurance company the floorlet would be the relevant derivative, and in a floorlet the floating rate is generally based on discrete compounding. Assuming the floorlets considered in this thesis coincides with each simulation interval, the discretely compounded forward rate, $\hat{F}$ for the interval $t \in [t_i, t_{i+1}]$ satisfies

$$\frac{1}{1 + \hat{F}(t_i)[t_{i+1} - t_i]} = e^{-\hat{f}(t_i,t_i)[t_{i+1}-t_i]},$$

i.e.

$$\hat{F}(t_i) = \frac{1}{t_{i+1} - t_i} \left(e^{\hat{f}(t_i,t_i)[t_{i+1}-t_i]} - 1 \right).$$

Fixing $h_{i+1} = t_{i+1} - t_i$ and the floor rate to E, the discounted price, P, of each simulated floorlet at time t_{i+1} is

$$P = e^{-\sum_{l=1}^{i} \hat{f}(t_{l-1},t_{l-1})[h_l]} \left(E - \hat{F}(t_i) \right)^{+}.$$

The total price of the floor is then the sum of the discounted price for each of the floorlets. This means we simulate many trajectories of the forward rate, then calculate the discounted cash flows of each floorlet and then calculate the average price of the derivative.

5.2 LIBOR Market Model

The model which is now presented is closely related to the HJM-framework in the sense that they both explain the arbitrage-free dynamics of the interest rate through the development of forward rates. However, the HJM-model were based on continuously compounded forward rates, which is unobservable abstract rates invented by mathematicians to ease the calculations. On the contrary, the LMM-models are based on simple rates explained in Chapter 2.8. LIBOR stands for London Inter-Bank Offered Rate and is updated daily as the average of several rates offered by different banks in London. Further, define

$$\delta_i = T_{i+1} - T_i, \quad i = 0, ..., M$$

with a finite set of maturity dates

$$0 = T_0 < T_1 < ... < T_M < T_{M+1}.$$

Then, for each maturity date T_n, $Z_n(t)$ gives the zero-coupon bond price of a bond maturing at T_n at a time $t \in [0, T_n]$. Similarly, $L_n(t)$ denotes the LIBOR forward rate at time $t \in [0, T_n]$ for the period $[T_n, T_{n+1}]$. Given these definitions it is trivially to find the definition of the forward LIBOR rate

$$L_n(t) = \frac{Z_n(t) - Z_{n+1}(t)}{\delta Z_{n+1}(t)}, \quad 0 \leq t \leq T_n, \quad n = 0, 1, ..., M. \tag{5.2.1}$$

We solve equation (5.2.1) to find the bond price

$$Z_n(T_i) = \prod_{j=i}^{n-1} \frac{1}{1 + \delta_j L_j(T_i)}, \quad n = i + 1, ..., M + 1.$$

However, obviously this has a fault in the sense that it fails to give the properl dicount factor for intervals shorter than the maturity periods. This means if we try to find the discount factor of a bond $Z_n(t)$ for some $n > i+1$, when $T_i < t < T_{i+1}$, the factor

$$Z_n(T_i) = \prod_{j=i+1}^{n-1} \frac{1}{1+\delta_j L_j(t)}$$

discounts the bond's payment at T_n back to T_{i+1}, but it does not include the discount factor from T_{i+1} to t. We solve this by defining a function $\eta : [0, T_{m+1}) \to 1, ..., M+1$ by defining $\eta(t)$ to be the unique integer which does not violate

$$T_{\eta(t)-1} \leq t \leq T_{\eta(t)}.$$

This means $\eta(t)$ gives the index of the next tenor date at time t. Further, this gives the correct bond price

$$Z_n(t) = Z_{\eta(t)}(t) \prod_{j=\eta(t)}^{n-1} \frac{1}{1+\delta_j L_j(t)}, \quad 0 \leq t < T_n.$$

Further, under the spot measure [8] the evolution of the LIBOR forward rates follows a system of SDEs of the form

$$\frac{dL_n(t)}{L_n(t)} = \mu_n(t)dt + \boldsymbol{\sigma}_n(t)^T \mathbf{dW(t)}, \quad 0 \leq t \leq T_n, \quad n = 1, ..., M. \tag{5.2.2}$$

Here $\mathbf{dW}$ is a d-dimensional standard Brownian motion, while μ_n and σ_n may depend on both the current LIBOR rates aswell as the time t. Just like in the HJM-framework, the LIBOR market model also has a no-arbitrage drift condition

$$\mu_n(t) = \sum_{j=\eta(t)}^{n} \frac{\delta_j L_j(t) \boldsymbol{\sigma}_n(t)^T \boldsymbol{\sigma}_j(t)}{1+\delta_j L_j(t)}. \tag{5.2.3}$$

Combining this with equation (5.2.2) we obtain

$$\frac{dL_n(t)}{L_n(t)} = \sum_{j=\eta(t)}^{n} \frac{\delta_j L_j(t) \boldsymbol{\sigma}_n(t)^T \boldsymbol{\sigma}_j(t)}{1+\delta_j L_j(t)} dt + \boldsymbol{\sigma}_n(t)^T \mathbf{dW}(t), \quad 0 \leq t \leq T_n, \quad n = 1, ..., M. \tag{5.2.4}$$

Here we observe that $L_n(t)$ is lognormal distributed and it's drift is again determined by the volatility factors just like in the HJM framework.

5.2.1 Simulation of LMM

Just as in the HJM case we fix a time grid $0 = t_0 \leq t_1 \leq t_2 \leq, ..., \leq t_M$ to simulate the LIBOR market model at, while choosing the tenor dates T_1, T_2,..., T_M to fully coincide with the time grid. Further, we apply Euler scheme log$L_n(t)$ in equation (5.2.4) under the spot measure again given in [16], which gives

$$\hat{L}_n(t_{i+1}) = \hat{L}_n(t_i)\text{exp}\left(\left(\mu_n\left(\hat{L}_n, t_i\right)t_i - \frac{1}{2}\|\sigma_n(t_i)\|^2\right)(t_{i+1} - t_i) + \sqrt{t_{i+1} - t_i}\sigma_n(t_i)^T Z_{i+1}\right). \tag{5.2.5}$$

Here μ_n is given in equation (5.2.3) and Z_{i+1} is an independent $N(0, 1)$ random variable. We further assume we have today's yield curve which enables us to initialize the forward LIBOR rates with

$$\hat{L}_n(0) = \frac{Z_n(0) - Z_{n+1}(0)}{\delta_n Z_{n+1}(0)}, \quad n = 1, ..., M. \tag{5.2.6}$$

5.2.2 Pricing Derivatives under LMM

Again, the pricing of derivatives looks a lot like the pricing in the HJM-framework. We simulate $\hat{L}_1$, $\hat{L}_2$,...,$\hat{L}_M$ and assuming we have caplets/floorlets at each tenor the properly discounted price of each payment would be

$$P = \left(E - \hat{L}_{n-1}(t_{n-1})\right)^+ \prod_{j=0}^{n-1} \frac{1}{1 + \delta_j \hat{L}_j(T_j)}.$$

This means the simulated price of the derivative is the sum of the discounted cash flows from all the lets combined. Further, we simulate the price for many paths and taking the average to obtain an estimate for the price at time 0.

5.3 Volatility Structure in LMM

5.3.1 Implied Volatility

The most common way to define the volatility structure in LMM is to find the implied volatility from Black's formula. This automatically calibrates the LIBOR market model to the observed prices of the rate options in the market. Further, if we take the forward rate in equation (4.4.1) as a LIBOR forward rate, $L_n(t)$, then we find the unique implied volatility from inverting equation (4.4.1), assuming we have the price for the interval $[T_n, T_{n+1]}$. Further imposing the constraint

$$\frac{1}{T_n}\int_0^{T_n} \|\sigma_n(t)\|^2 dt = v_n^2$$

on the deterministic R^d-valued functions σ_j's calibrates this model properly. Another way to fix the volatility structure is to apply the same trick as in the HJM-setting, which is to use the Musiela parametrization and set the volatility dependent only on the time to maturity. This volatility could simply be defined in the following linear model

$$\sigma_n(t_i) = 0.15 + 0.0025(n-i), \quad i = 0, ..., n-1, \quad n = 1, ..., M, \tag{5.3.1}$$

which is defined in the paper [15]. This model turned out to be consistent with the implied volatilities observed in the U.S. dollar term structure at 1997.

5.3.2 Volatility Structure with Diagonalization

We may also create a volatility structure based on the historical LIBOR forward rates. It is obvious that the LIBOR rate $L_n(t)$ does not vary after maturity T_i, which means it's volatility has expired. This means we need a volatility structure of the form

Table 5.1: Volatility Structure

Maturity	t_0	t_1	t_2	$\dots$	t_{m-1}
T_1	$\sigma_{1,0}$	expired	expired	$\dots$	expired
T_2	$\sigma_{2,0}$	$\sigma_{2,1}$	expired	$\dots$	$\vdots$
T_3	$\sigma_{3,0}$	$\sigma_{3,1}$	$\sigma_{3,2}$	$\dots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$
T_i	$\sigma_{i,0}$	$\sigma_{i,1}$	$\sigma_{i,2}$	$\dots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	expired
T_M	$\sigma_{M,0}$	$\sigma_{M,1}$	$\sigma_{M,2}$	$\dots$	$\sigma_{M,M-1}$

Here $\sigma_{n,i}$ denotes the volatility for the LIBOR rate $L_n(t)$ where $t \in (t_i, t_{i+1}]$. In order to find such a matrix from the historical data we first find the covariance matrix $\mathbf{C}$ for the daily changes in the forward LIBOR rates for n maturities. However, we need to turn $\mathbf{C}$ into a lower tridiagonal matrix $\hat{\mathbf{C}}$, which is done by fixing the elements above the diagonal to zero. However, the matrix is of order percent squared, which means we diagonalize $\hat{\mathbf{C}}$

$$\hat{\mathbf{C}} = \mathbf{V\Lambda V^{-1}},$$

and define

$$\sigma_n(t_i) = \mathbf{Y_{n,i}}$$

where Y is defined as

$$\mathbf{Y} = \mathbf{V\Lambda}^{0.5}\mathbf{V^{-1}}.$$

Further, note that $\mathbf{Y}$ is square root of $\hat{\mathbf{C}}$ because

$$\mathbf{YY} = \mathbf{V\Lambda}^{0.5}\mathbf{V^{-1}V\Lambda}^{0.5}\mathbf{V^{-1}} = \mathbf{V\Lambda V^{-1}} = \hat{\mathbf{C}}.$$

5.3.3 Rolling Volatility

The rolling volatility are based on moving averages of an observed time series with either equally weighted or different weighted moving averages. These are useful for detecting periods of instability or unnatural behavior, but may also produce misleading results if they are used for short-term forecasting. This is especially valid for equally weighted averages because they are sensitive to extreme values. In this thesis rolling volatility will be used in the LMM-simulation.

The most easily computed rolling volatility is the simple moving average (SMA), which means the moving average is equally weighted. Since we apply an Euler scheme to $\log L_n$, we assume we have N daily log returns of the n'th forward rate $l_{n,i} = \log \frac{L_n(t_{i+1})}{L_n(t_i)}$ for the last K trading days. Depending on the rolling window, N, the annualized covariance matrix valid for today is found by the unbiased estimator

$$\hat{\Sigma}_{i,j} = 252 \frac{1}{N-1} \sum_{k=0}^{N-1} \left(l_{i,K-k} - \bar{l}_i \right) \left(l_{j,K-k} - \bar{l}_j \right).$$

Here $\bar{l}_i$ is the average log return of the i'th forward rate over the last K days

$$\bar{l}_i = \frac{1}{N} \sum_{k=0}^{N-1} l_{i,K-k}.$$

Hence the volatility $\sigma_i(t_j)$ for the LIBOR rate $L_n(t)$ is given by

$$\sigma_i(t_j) = \Sigma_{i,j} \quad i, j = 1, ..., M,$$

where M is the total number of LIBOR forward rates simulated. This method indeed finds the historic volatility, but since the real market volatility is not constant over time we do not know whether it is a good estimator for today's volatility. In this case we have a conflict between making K as large as possible to decrease the estimator's variance, and minimizing K in order to capture the latest volatility. This means there are better methods than SMA, one of them are exponentially weighted moving average (EMWA). EMWA weights the moving average by giving more importance to the data points in the near past compared to the older data points. This method also has a nice recursive updating formula, namely that today's EMWA is a function of yesterdays EWMA. Assuming we are at the N'th trading day, EMWA uses all of the N log returns but with decaying weights. This gives the annualized covariance matrix valid for today

$$\hat{\Sigma}_{i,j}^{N} = 252 \frac{1-\lambda}{1-\lambda^N} \sum_{k=0}^{N-1} \lambda^k \left(l_{i,N-k} - \bar{l}_{i,N} \right) \left(l_{j,N-k} - \bar{l}_{j,N} \right),$$

where $0 < \lambda < 1$ is the rate of decay and $\bar{l}_{i,N} = \frac{1}{N}\sum_{k=1}^{N} l_{i,k}$. We also see that the weighting is adequate in the sense that it sums to one,

$$\frac{1-\lambda}{1-\lambda^N}\sum_{k=0}^{N-1}\lambda^k = \frac{1-\lambda}{1-\lambda^N}\frac{1-\lambda^N}{1-\lambda} = 1.$$

Also we have,

$$\text{EMWA}(N) = \frac{1-\lambda}{1-\lambda^N}\sum_{k=0}^{N-1}\lambda^k Y_{N-k},$$

$$= \frac{1-\lambda}{1-\lambda^N}Y_n + \lambda\frac{1-\lambda}{1-\lambda^N}\sum_{k=1}^{N-1}\lambda^{k-1}Y_{N-k},$$

$$= \frac{1-\lambda}{1-\lambda^N}Y_n + \lambda\frac{1-\lambda^{n-1}}{1-\lambda^N}\frac{1-\lambda}{1-\lambda^{N-1}}\sum_{k=0}^{N-2}\lambda^k Y_{N-k-1},$$

$$= \frac{1-\lambda}{1-\lambda^N}Y_n + \lambda\frac{1-\lambda^{N-1}}{1-\lambda^N}\text{EMWA}(n-1).$$

This means we update the covariance matrix recursively

$$\hat{\Sigma}_{i,j} = 252\frac{1-\lambda}{1-\lambda^N}\left(l_{i,N} - \bar{l}_{i,N}\right)\left(l_{j,N} - \bar{l}_{j,N}\right) + \lambda\frac{1-\lambda^{N-1}}{1-\lambda^N}\hat{\Sigma}_{i,j}^{N-1}.$$

With these results we have the volatility $\sigma_i(t_j)$ for the LIBOR rate $L_n(t)$ is given by

$$\sigma_i(t_j) = \Sigma_{i,j} \quad i,j = 1,...,M.$$

In this thesis we use $\lambda = 0.94$ as proposed in [10].

Chapter 6

Risk Measure

The financial market contains several degrees of risk and with new financial innovations coming up every day, it makes risk management more important than ever. There are four main types of risk, namely

- Market Risk - Market risk is due to possibly changes in price of an asset.
- Credit Risk - Credit risk is the possibility that the counterpart does not meet contractual commitments, for example that the interest of a bond is not paid in time.
- Liquidity Risk - Liquidity risk comes from the fact that there may incur extra cost of liquidation a position because buyers are hard to find.
- Operational Risk - Operational risk is due to problems like flaws in management, frauds and human errors.

In this thesis we focus only on market risk analysis.

6.1 Coherent Measures

Market risk are one of the most important risks financial institutions need to consider. This risk might come from complicated portfolios and therefore they are in need of intuitive and effective ways of measuring risk. Earlier it was normal to measure risk with for instance duration analysis, but it turned out primitive and of only limited applicability. Therefore we need risk measures with certain well-behaved properties. In [11] they present a class of coherent measures, which all have these well-behaved properties.

If we let X, $Y \in S$ with S being a set of stochastic processes, for instance a set of returns from a portfolio, then a coherent risk measure $\rho : S \to \mathbb{R}$ satisfies the following

- **Translation Invariant** - $\forall X \in S$, $\forall \gamma \in \mathbb{R}, \quad \rho(X + \gamma) = \rho(X) - \gamma$. This means adding a or subtracting a risk-free quantity to a portfolio changes the risk measure by that exact amount.
- **Subadditive** - $\forall X, Y \in S, \quad \rho(X+y) \leq \rho(X)+\rho(Y)$. The risk of combining portfolios can never exceed the sum of the respective portfolios risks.
- **Positive Homogenous** - $\forall X \in S$, $\forall \nu \geq 0, \quad \rho(\nu X) = \nu\rho(X)$. Scaling the portfolio changes the risk by the same amount.
- **Monotonicity** - $\forall X, Y \in S$ such that $X \geq Y, \quad \rho(X) \leq \rho(Y)$. This means if returns from one portfolio always are greater than for another portfolio, then the risk associated with the first portfolio are always less than for the other one.

6.2 Value at Risk

Value at risk (VaR) is one of the most common risk estimation of portfolios. The reason for this is that it can be used on all kind of risks and assets, also on complex portfolios. VaR takes in two parameters, namely the time horizon T and the confidence level $1-\alpha$. By applying these parameters VaR gives us a bound such that the loss over the time period T is less than the VaR bound with probability $1 - \alpha$.

Letting $\Gamma(t)$ be the portfolio value at time t, $\mathcal{L} = \Gamma(t) - \Gamma(t + T)$ be the loss over the time period $T > t$ and $\alpha \in (0, 1)$, then the definition of VaR(α,T) is given by

$$\text{VaR}(\alpha, T) = \inf\{\ell \in \mathbb{R} | \text{P}\left(\mathcal{L} > \ell\right) \leq 1 - \alpha\}.$$

The confidence levels are typically given as 95%, 97.5% or 99%. There are several reasons why VaR is so popular. Firstly, VaR is an easily understandable concept. Also, it is consistent across different positions and risk factors. Therefore it is easy to compare several projects/portfolios in terms of VaR. In addition, it also takes correlations between risky assets into account when calculating the loss bound. This is helpful when two or more assets/portfolios are varying together.

However, there also exist disadvantages with VaR. One of them is that it gives no information of the rest of the tail-losses. It gives what loss is expected with 95% confidence, but not how large the expected loss above VaR can be. Another disadvantage is the general violation of the subadditive property of a coherent risk measure. This means it discourages diversification. However, the risk measure introduced in the next chapter is in fact coherent, namely expected shortfall.

6.3 Expected Shortfall

The expected shortfall (ES) is the expected loss given a tail event. Again letting $\mathcal{L}$ be the loss over the time period $T > t$ with $\alpha \in (0, 1)$, then ES is given by

$$\mathrm{ES} = \mathrm{E}\{\mathcal{L} | \mathcal{L} > \mathrm{VaR}(\alpha)\} = \frac{\int_0^\alpha \mathrm{VaR}(u) du}{\alpha}.$$

From the definition we see that ES will always be greater than or equal to VaR. It also gives more information about the potential loss compared to the VaR. ES is also a consistent measure of risk across positions/portfolios and takes into account the correlations between them. Lastly, expected shortfall is a coherent risk measure and therefore satisfies all the properties in Chapter 6.1.

6.4 Calculation of VaR and ES

The value at risk and expected shortfall can be estimated both parametric and non-parametric. The parametric estimates rely on the fact that the loss distribution is assumed to be in a parametric family such as the normal distribution or the student t-distribution. For instance, assuming the losses follows a normal distribution, we obtain an easy formula for estimating both the VaR and ES

$$\hat{\mathrm{VaR}}(\alpha) = -S\left(\hat{\mu} + \Phi^{-1}(\alpha)\hat{\sigma}\right)$$

and

$$\hat{\mathrm{ES}}(\alpha) = S\left(-\hat{\mu} + \hat{\sigma}\left(\frac{\phi\{\Phi^{-1}(\alpha)\}}{\alpha}\right)\right).$$

However, when they are estimated non-parametrically we need to either look at historical data or simulated data. The simplest way is to look at what losses the portfolio has suffered over the time-horizon T in the historical data. Another way is to use Monte Carlo simulation, which is introduced in the next chapter, to acquire the distribution of the losses in the future. In this thesis we use MC-simulation to find the non-parametric estimates of VaR and ES. This is done by simulating the progression of the portfolio value and then storing each endpoint of the realizations. Furthermore, we pick the $\alpha\%$ quantile directly from the simulated loss distribution. This is the VaR(α) value. Lastly, we calculate the arithmetic mean of the $\alpha\%$ worst losses, which gives the ES.

Chapter 7

Simulation

7.1 Monte Carlo Simulation

Consider evaluating the integral of a function $f(x)$ on the unit interval, this is given by

$$\Gamma = \int_0^1 f(x)dx.$$

However, it can also be represented as the expectation of f, $E[f(U)]$, where U is uniformly distributed between 0 and 1. This means we simulate the points $U_1, U_2, ..., U_n$ independently and uniform on $[0,1]$ and then evaluate f at these points. By averaging the evaluated $f's$ we obtain the Monte Carlo estimate

$$\hat{\Gamma}_n = \frac{1}{n}\sum_{i=1}^{n} f(U_i).$$

Further, assuming f is integrable on $[0,1]$

$$P\left(\Gamma - \hat{\Gamma} > \epsilon\right) \to 0 \quad as \quad n \to \infty,$$

by the strong law of large numbers [17]. Also if f is square integrable we have the Monte Carlo error $\varepsilon_n = \hat{\Gamma}_n - \Gamma$ is normally distributed with mean zero and standard deviation $\frac{\sigma_f}{\sqrt{n}}$, where σ_f^2 is given by

$$\sigma_f^2 = \int_0^1 (f(x) - \Gamma)^2 \, dx.$$

The σ_f^2 is usually unknown, but can be estimated with

$$s_f^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(f(U_i) - \hat{\Gamma}_n \right)^2.$$

This means we have an error of the estimate, as well as the possibility to create a confidence interval for the estimate. Further, the convergence rate is $\mathcal{O}(n^{-\frac{1}{2}})$.

7.2 Interpolation

Usually we have yields for a set of maturities $T_1, T_2, ..., T_n$, but sometimes we need to find the yield in between one of these nodes. This is solved by interpolating between the two closest points. In this thesis we use linear interpolation which creates a straight line between the two closest points

$$r = r_0 + (r_1 - r_0) \frac{T - T_0}{T_1 - T_0}.$$

Here r is only valid in between it's neighborhood points $T \in [T_0, T_1]$, and (T_0, r_0) and (T_1, r_1) are always known.

7.3 Log-Normal Maximum Likelihood Estimation

To check the log-normal assumption used in simulation of LMM-rates regarding the relative change in LIBOR forward rates, we compare the historical relative change in forward rates with the log-normal distribution. This can be done by maximum likelihood estimation. First, the log-normal distribution has the following probability density

$$f(x, \mu, \sigma^2) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp \left(\frac{\log(x) - \mu}{2\sigma^2} \right).$$

The likelihood function L then takes the form

$$L = \prod_{i=1}^{n} \frac{1}{x_i\sqrt{2\pi\sigma^2}} \exp\left(\frac{\log(x_i) - \mu}{2\sigma^2}\right),$$

which means the log-likelihood function $l = \log L$ is given by

$$l = -\sum_{i=1}^{n} \left(\log(x_i) + \frac{\log(x_i) - \mu}{2\sigma^2}\right) - n\log(\sigma) - \frac{n}{2}\log(2\pi).$$

Taking partial derivatives with respect to μ and σ^2 gives the following maximum likelihood estimators

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} \log(x_i), \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} \left(\log(x_i) - \hat{\mu}\right)^2.$$

7.4 Portfolio Simulation

In order to find the mean gain, VaR and ES of the portfolios including the paid-up policies managed by the life insurance companies, we need to simulate the LIBOR rates together with the stock indexes. This is done by using a correlation matrix between the LIBOR rates and the stock index. However, the simulation forward in time needs to be done in the real statistical probability measure, i.e. we apply the real drift of the stock indexes and forward LIBOR rates. Compared to the risk-neutral world where we only use the risk-free rate as the drift of the equity, we now need to estimate the drift observed in the market. Obviously it is not possible to estimate the drift only by using a short historical time period. This would contradict the weak form of the efficient market hypothesis, and also it gives the possibility of negative drift. This is the reason why we look at a large time horizon to find the general trend of the stock index.

We also need to estimate the real drift of the forward LIBOR rates. However, interest rates have significantly different characteristics than a stock index. For instance, the stock index modeling includes an expected exponential growth. This does not fit very well with the mean-reverting features of an interest rate. The potential real drift of the interest rates will only be "short"-term events, which means they are hard to model. This is the reason why this thesis assumes zero real drift of the forward LIBOR rates.

Further, to model the portfolio we first apply an Euler discretization on log S, from Equation (2.2.1) we obtain the modelling scheme

$$S(t_{i+1}) = S(t_i)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)(t_{i+1} - t_i) + \sqrt{t_{i+1} - t_i}\sigma^2\mathbb{Z}_{i+1}^6\right). \tag{7.4.1}$$

Here μ is the historic mean of the stock index while σ is the volatility of the stock index. Further, $\mathbb{Z}_{i+1}^6$ is the sixth element of the random vector

$$\mathbb{Z}_{i+1} = \left(Z_{i+1}^1, ..., Z_{i+1}^6\right) \sim N(\mathbf{0}, \rho).$$

Here ρ is the correlation matrix of $L_1, L_2, L_3, L_4, L_5, S$, namely the correlation between the LIBOR rates and the respective stock index. Further, using the modeling scheme for the LIBOR rates defined in [12], we obtain

$$\hat{L}_n(t_{i+1}) = \hat{L}_n(t_i)\exp\left(\left(\mu_n\left(\hat{L}_n, t_i\right)t_i - \frac{1}{2}\|\sigma_n(t_i)\|^2\right)(t_{i+1} - t_i) + \sqrt{t_{i+1} - t_i}\sigma_n(t_i)\mathbb{Z}_{i+1}^n\right). \tag{7.4.2}$$

Here μ_n is given in Equation (5.2.3) and $\sigma_n(t_i)$ is defined in Chapter 5.3.3. Further, $\mathbb{Z}_{i+1}^n$ is the n'th element in the random vector

$$\mathbb{Z}_{i+1} = \left(Z_{i+1}^1, ..., Z_{i+1}^6\right) \sim N(\mathbf{0}, \rho),$$

where n = 1, 2, ..., 5.

Moreover, the paid-up policies the insurance companies gets from a customer can either be placed in equity or in fixed-income securities. In this thesis we assume the life insurance company receives 100 in the respective currency today and then places x of the 100 in the equity market, and $1-x$ of the 100 in zero-coupon bonds. This means in six years, the value of the portfolio then will be the money received from the zero-coupon bond, plus the money received/lost from the position in the equity market, minus the floor paid out by the life insurance company each year. In other words, the gain the life insurance company receives in six year is calculated by the following formula,

$$Pf = 100xE_r + 100(1-x)\text{ZCP} - \text{Floor}. \tag{7.4.3}$$

Here Pf is the portfolio value, E_r is the return of the equity in 6 years, ZCP is the money received from the zero-coupon bond in six years, while the Floor is the sum of the forward value of the money paid out to the customer annually through six years. It is evident that the value received from the ZCP is known today, since we use the six year forward LIBOR rate as the yield. This means we look at each stock and interest rate trajectory and calculates the gain for each realization. This is done by varying the floor guarantee from 0 to 0.1 and for each floor guarantee we vary x, the position in the equity market, from 0 to 1. The VaR and ES are then found by the technique discussed in Chapter 6.4.

Chapter 8

Data

8.1 Stock Index

In this thesis we have considered OSEBX (Oslo Børs Hovedindeks), SX5P (STOXX Europe 50 Index) and S&P 500 (Standard & Poor 500 Index) as a proxy for the markets in Norway, Europe and US respectively. The reason for the use of these indexes is that they include stocks of a variety of business sectors, which means they are a good indicator of how these sectors perform together in the financial market. Also they are well diversified, which is a requirement for being a proxy to the stock market.

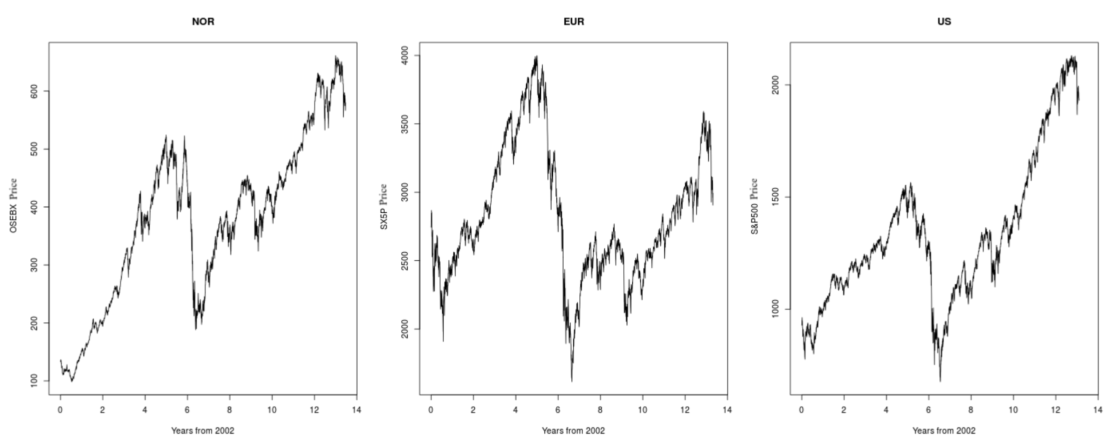

Figure 8.1: Stock Indexes in Norway, Europe and US respectively.

It is interesting to see that all of the indexes in Figure (8.1) are influenced by the dot-com bubble [19] which appeared around 2002. Further, we also see that the financial crisis of 2007-08 had a large negative impact on the indexes.

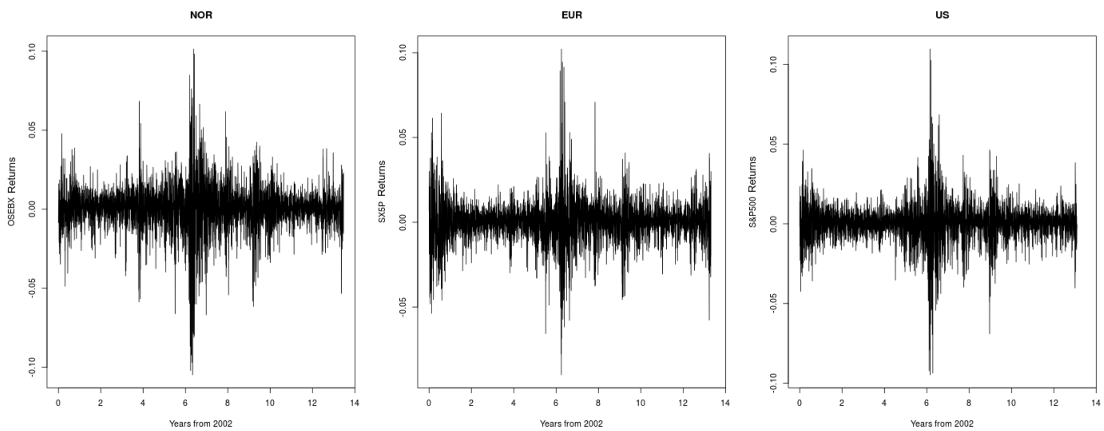

Figure 8.2: Index returns in Norway, Europe and US respectively.

The returns of the stock indexes in Figure (8.2) gives the same information as in the previous Figure. Namely that the stock indexes are unstable in distressed times. we also see that the variance is not constant, but increases significantly during the dot-com bubble and during the financial crisis. Another observation is that the stock markets in Norway, Europe and USA reacts very similarly, which means they are highly correlated.

Table 8.1: Descriptive Statistics of the Returns

	OSEBX	SX5P	S&P 500
Mean	0.00043	0.00003	0.00022
Var	0.00023	0.00016	0.00015
Skew	-0.582	0.0806	-0.2809
Kurt	6.9747	7.6803	10.1954

Further, it is also interesting to look at the distribution of the returns. In Table 8.1 we see that all three indexes have skewness close to zero, while the kurtosis values are significantly non-zero. Normal distributed returns should have skewness and kurtosis equal to zero, which means the indexes are not quite following a normal distribution. The skewness coefficients for student-t distributed returns should also be equal to zero, however the kurtosis values are allowed to be slightly positive. From the skewness coefficients we see that all three indexes could be approximated both by a normal and a student-t distribution, but the kurtosis values strongly indicates that the returns follow a more peaked distribution than the normal one.

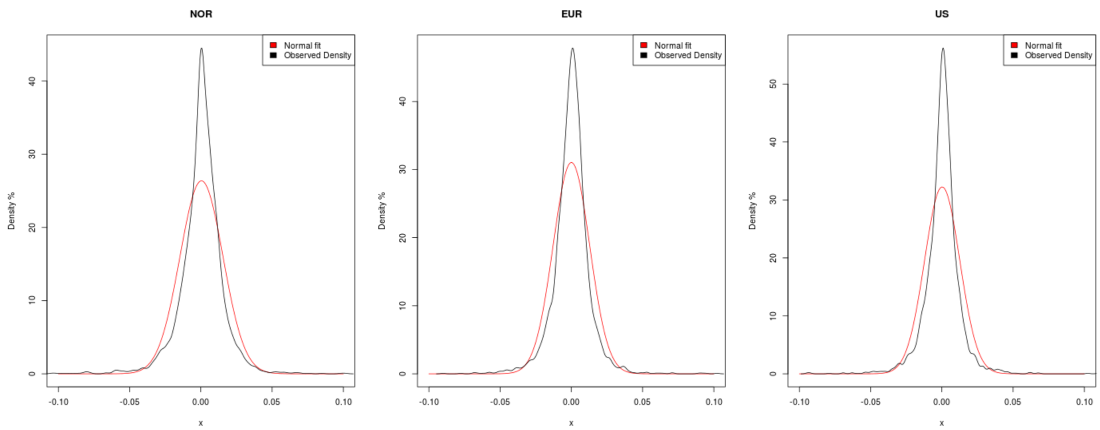

Figure 8.3: Normal density fit to index returns.

From Figure 8.3 we see that a normal fit to the index returns is performing bad. The plots also coincide with the high kurtosis coefficients listed earlier, namely that the observed densities have much higher peaks compared to the fitted normal densities. We also see that the observed density sometimes goes above the normal tails, but this is expected since the normal distribution assign very small probabilities to extreme events. Furthermore, in Figure 8.4 we see the student-t fit of the index returns. In this case the fitted densities are much closer to the observed densities, compared to the normal fits. This is expected since the student-t distribution has one extra parameter to fit compared to the normal distribution. However, we still see the observed densities have fatter tails than the student-t fit.

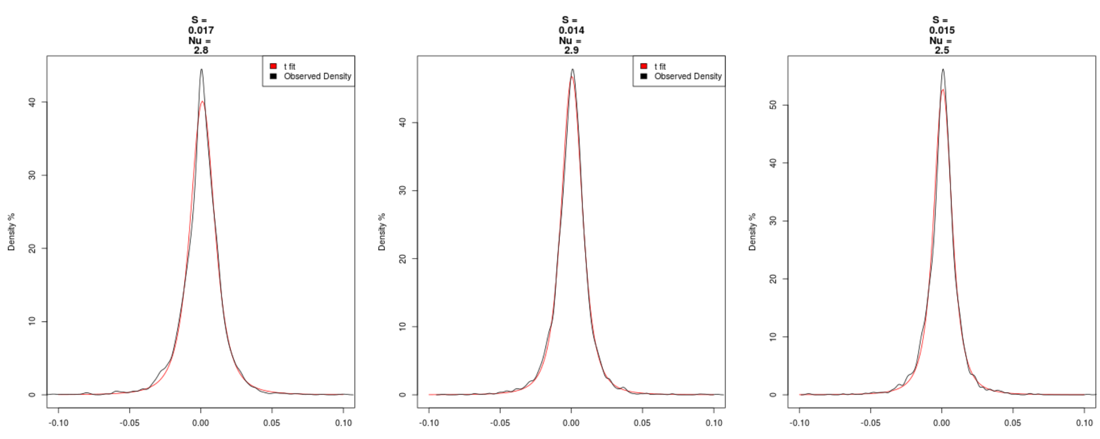

Figure 8.4: Student t density fit to index returns. The scale and degrees of freedom is listed above each plot.

Further, in Table 8.2 we see the covariance matrix of the index returns. It looks like the Norwegian and European market have about the same covariance as the European and American market, while they have both larger covariance than the covariance between the Norwegian and American market.

Table 8.2: Covariance Matrix of Returns

	OSEBX	SX5P	S&P 500
OSEBX	0.00023	0.000016	0.000006
SX5P	0.000016	0.00016	0.000012
S&P 500	0.000006	0.000012	0.00015

In Figure refstockprice2 we see the rolling volatility defined in Chapter 5, namely by using exponentially weighted moving average and simple moving average. We notice that EMWA adjusts faster than the SMA and that EMWA has more extreme peaks. SMA depends highly on the length of the period it uses to calculate the volatility and this in turn makes it react slower than the EMWA.

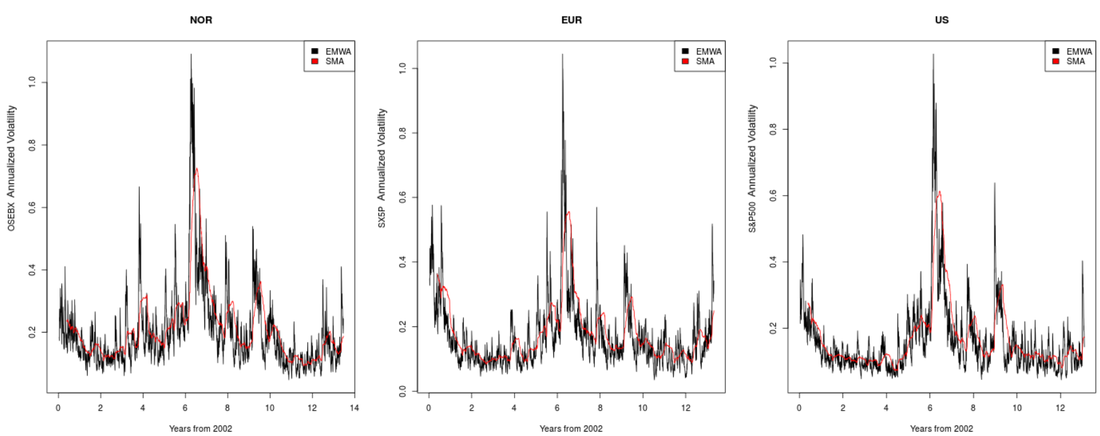

Figure 8.5: Rolling volatility of index returns.

8.2 Historical Yields Used for the Forward Rate Curve

The data is provided by DNB and it includes daily quoted yields for the Norwegian, European and American market. The data used for HJM and LMM starts at 08.01.2006 and ends at 10.14.2015 for both the European market and American market, while the Norwegian data starts at 08.15.2002 and ends at 10.14.2015.

The maturities analyzed are 1 day, 1, 2, 3, 4, 5, 7, 10, 15 and 20 years. However, we only simulate up until 6 years for the floor prices and portfolio values, which means we only apply the 1 day, 1, 2, 3, 4 and 5 years maturities in the simulation. Furthermore, if we use volatility factors in the HJM-framework and LMM based on historical data, we need to first convert historical yields to historical instantaneous forward rates and historical LIBOR forward rates, and then find the volatility of these.

8.3 Inversion from Yield Curve to Forward Rate Curve

Even though we do not have quoted yields for all of the maturities wanted, for instance year two, we obtain an estimate for the instantaneous forward rate curve by doing the following. First set the instantaneous forward rate today equal to the 1 day rate. Then interpolate the yield curve to get the yield for the following maturities 1, 2, 3, ..., 20 years. When this is done we use equation (5.1.5) to find the instantaneous forward rate for the following maturities 1, 2, 3, ..., 20 years.

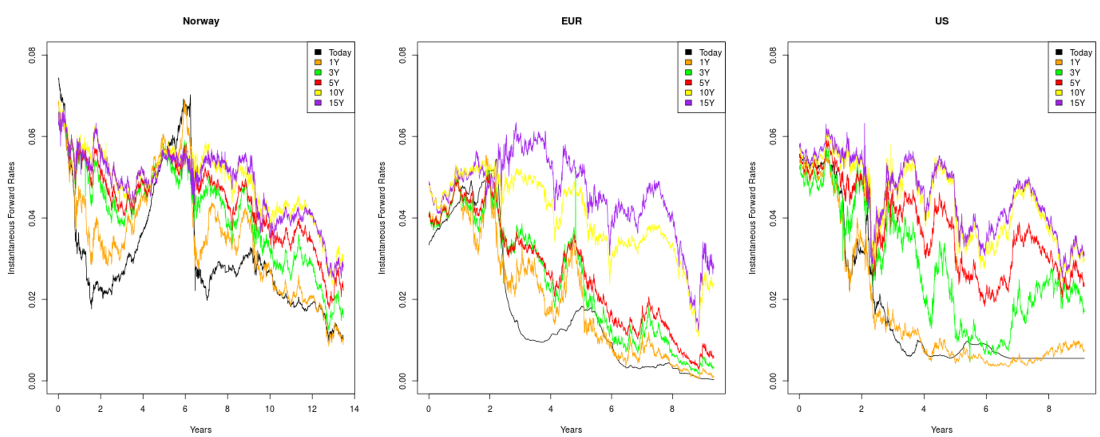

Figure 8.6: Historical instantaneous forward rates in the Norwegian, European and American market respectively. The Norwegian rates start in 2002 and last till 2015, while the European and American rates start in 2006 and last to 2015.

In Figure 8.6 we see the instantaneous forward rates for the following maturities, 1 day, 1, 3, 5, 10 and 15 years. We clearly see that a constant rate assumption in the Black-Scholes framework does not hold very well for longer periods. Also we see that in normal times the rate is increasing with increasing maturity, while in distressed times it tends to be the inverse relation. Further, we see that the time series are not stationary, i.e. the variance tends to vary across time periods. This is why the PCA is done on daily changes in rates rather on the rates themselves. Lastly, it looks like the rates vary less with increasing maturity and that the American and Norwegian markets are slightly more volatile than the European one. We also notice that the European rates go to a lower level in the end, compared to the Norwegian and American rates.

Table 8.3: Descriptive statistics of the forward rates in Norway

Maturity	Mean	Std	Mean daily change	Std daily change
1 day	0.03164	0.01517	-1.888627e-05	0.00045
1Y	0.03583	0.01397	-1.747201e-05	0.00063
3Y	0.04131	0.01159	-1.465822e-05	0.00067
5Y	0.04449	0.00992	-1.240424e-05	0.00052
10Y	0.04858	0.00847	-1.007661e-05	0.00063
15Y	0.04877	0.00841	-1.029817e-05	0.00085

In Tables 8.3, 8.4 and 8.5 we see that the mean of the rates in general increases with increasing maturity. We also see that the standard deviation is decreasing with increasing maturity. The mean and the standard deviation of the daily changes in rates also look more similar across maturities, than the mean and standard deviation of the rates themselves.

Table 8.4: Descriptive statistics of the forward rates in Europe

Maturity	Mean	Std	Mean daily change	Std daily change
1 day	0.01765	0.01701	-1.403119e-05	0.00012
1Y	0.01836	0.01666	-1.609415e-05	0.00047
3Y	0.02416	0.01464	-1.582485e-05	0.00084
5Y	0.02605	0.01342	-1.470279e-05	0.00045
10Y	0.04563	0.00990	-9.077714e-06	0.00076
15Y	0.04019	0.00988	-1.027208e-05	0.00054

Table 8.5: Descriptive statistics of the forward rates in America

Maturity	Mean	Std	Mean daily change	Std daily change
1 day	0.01698	0.01701	-2.160551e-05	0.00032
1Y	0.01666	0.016124	-2.030988e-05	0.00051
3Y	0.02786	0.01398	-1.553168e-05	0.00105
5Y	0.03688	0.01056	-1.410588e-05	0.00080
10Y	0.0440	0.00914	-1.236303e-05	0.0010
15Y	0.04553	0.00884	-1.192208e-05	0.00085

To confirm the non-stationary assumption, Table 8.6 contains the p-values for the Phillops-Perron unit root test [20]. All of the p-values are big which means it can not reject the null hypothesis, which is that the data are non-stationary. We also see that the p-values are less for bigger maturity. This makes sense since larger maturity means less variance in the data.

Table 8.6: P-values for the stationary test

Maturity	P-value (NOR)	P-value (EUR)	P-value (US)
1 day	0.8323	0.9395	0.9311
10Y	0.223	0.3289	0.1263

8.4 Inversion From Yield Curve to Forward LIBOR Rate Curve

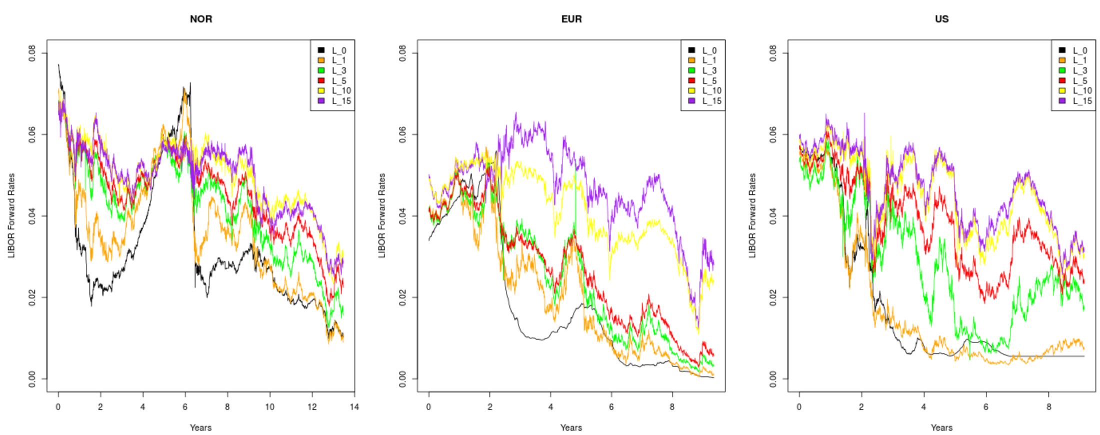

Figure 8.7: Historical LIBOR forward rates in the Norwegian, European and American market respectively. The Norwegian rates start in 2002 and last till 2015, while the European and American rates start in 2006 and last to 2015.

The rates that are modeled in the LMM-framework are directly observable in the market and it's volatility factors have a one-to-one relation with it's traded contracts. On the contrary, the instantaneous forward rates modeled in the HJM-framework are completely abstract. For the LIBOR forward rate we use the maturities $\{T_0, T_1, T_2, ..., T_{20}\} = \{1 \text{ day}, 1, 2, ..., 20 \text{ years}\}$, and to find the historical $L_n(0)$ we use equation (5.2.6) with the listed set of maturities. In Figure 8.7 we see the historical LIBOR forward rates with the following maturities

$\{T_0, T_1, T_3, T_5, T_{10}, T_{15}\}$. Here we also notice the same characteristics as with the instantaneous forward rates. Namely that the mean of the rates tend to increase with increasing maturities, while the standard deviation has the opposite relation to increasing maturities. Also, we see that the LIBOR forward rates are not stationary, which means we need to find the daily changes in order to use PCA for the volatility factors. Lastly, we notice that also the European LIBOR forward rates goes below both the Norwegian and American rates in the end.

Table 8.7: Descriptive statistics of the forward rates in Norway

Maturity	Mean	Std	Mean daily change	Std daily change
1 day	0.03227	0.01579	-1.970665e-05	0.00047
1Y	0.03658	0.01451	-1.8165e-05	0.00066
3Y	0.04225	0.01205	-1.526984e-05	0.00070
5Y	0.04555	0.01034	-1.295993e-05	0.00054
10Y	0.04981	0.00890	-1.056203e-05	0.00067
15Y	0.04939	0.00890	-1.078462e-05	0.00089

In Tables 8.7, 8.8 and 8.9 we see the same tendency as with the instantaneous forward rates, namely that the mean of the rates increases with increasing maturity, while the same inverse relation between standard deviation and maturity. Also we see that PCA should be done on daily changes if the volatility factors are found by diagonalizing.

Table 8.8: Descriptive statistics of the forward rates in Europe

Maturity	Mean	Std	Mean daily change	Std daily change
1 day	0.017953	0.01708	-1.426883e-05	0.00013
1Y	0.018679	0.01704	-1.640589e-05	0.00048
3Y	0.02457	0.01501	-1.617403e-05	0.00087
5Y	0.02649	0.01377	-1.504542e-05	0.00046
10Y	0.04674	0.01032	-9.43205e-06	0.00079
15Y	0.04106	0.01026	-1.064587e-05	0.00056

Table 8.9: Descriptive statistics of the forward rates in America

Maturity	Mean	Std	Mean daily change	Std daily change
1 day	0.01727	0.01750	-2.227215e-05	0.00033
1Y	0.01694	0.01657	-2.094337e-05	0.00052
3Y	0.02835	0.01441	-1.608244e-05	0.00109
5Y	0.03763	0.01097	-1.467383e-05	0.00084
10Y	0.04502	0.00954	-1.291269e-05	0.00104
15Y	0.04662	0.00925	-1.246485e-05	0.00089

Further, to also confirm the non-stationary assumption in the LIBOR forward rate case, Table 8.10 contains the p-values for the non-stationary test. All of the p-values are also big in the LIBOR case, which means it can not reject the null hypothesis, which is that the data are non-stationary.

Table 8.10: P-values for the stationary test

Maturity	P-value (NOR)	P-value (EUR)	P-value (US)
1 day	0.8968	0.9793	0.9593
10Y	0.2092	0.4252	0.12751

8.5 Distribution of Historical LIBOR Forward Rates

From Chapter 5.2.1 we have that the LIBOR Market Model is modeling the relative change in the forward rate as log-normal. This assumption can be investigated by plotting the density of the historical relative change in the forward rate $\frac{L_{n+\delta}}{L_n}$, with the fitted log-normal distribution using the maximum likelihood parameters given in Chapter 7.5. Here δ is the the period of business days between each rate observation. For the following plots δ is fixed to 1 day.

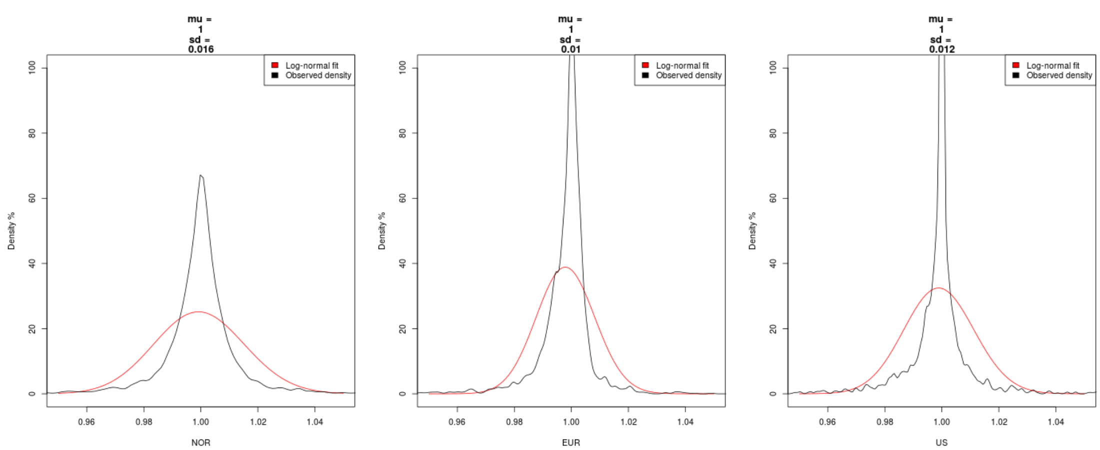

Figure 8.8: Log-normal fit to historical relative changes in the 1 day forward rate in Norway, Europe and America respectively. The fitted mean and standard deviation are listed above each plot.

From Figure 8.8 we see that modeling the relative change in the 1 day forward rate as log-normal, is not a very good idea. This seems to be the general idea in all of the markets. However, the log-normal fit for the relative change in the 1 day forward rate is slightly better in the Norwegian market compared to the other markets. Further, in Figure 8.9 we see the log-normal fit of the relative change in the 5 year forward rate. In this case a log-normal fit seems more reasonable for all of the markets, but the observed density still has higher peaks than the fitted density, aswell as fatter tails. However, the log-normal assumption of the relative change in the forward rate seems to improve with longer maturities.

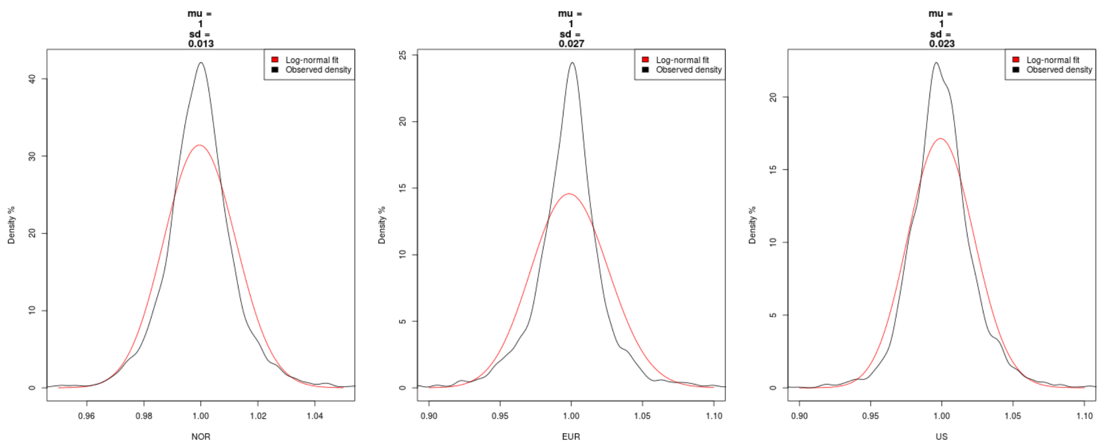

Figure 8.9: Log-normal fit to historical relative changes in the 5 years forward rate in Norway, Europe and America respectively. The fitted mean and standard deviation are listed above each plot.

8.6 PCA on Instantaneous Forward Rate Curve

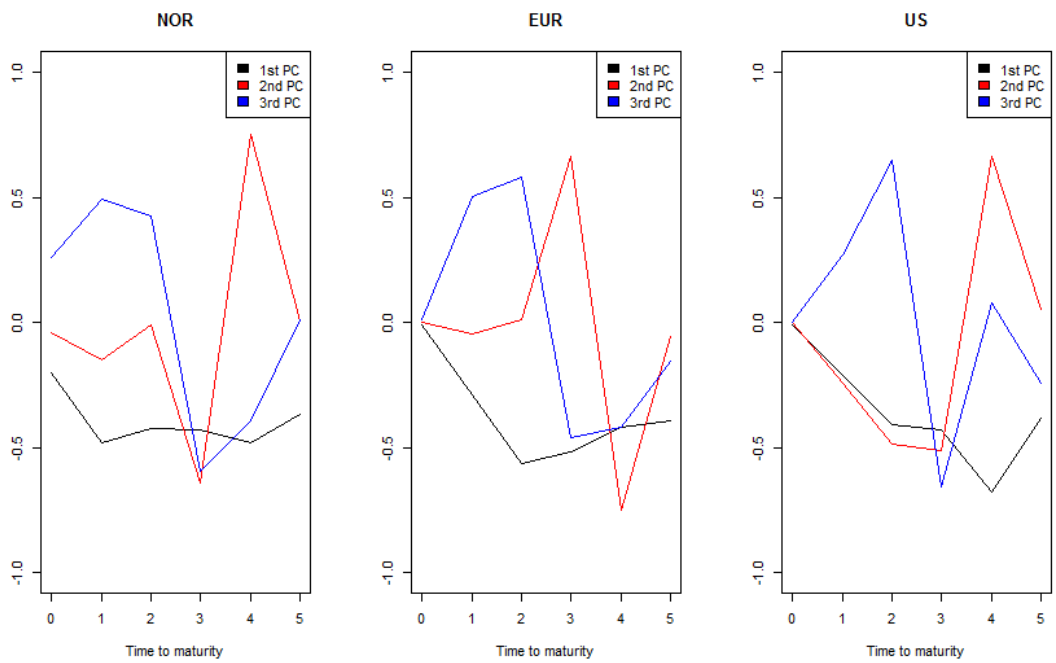

Figure 8.10: The three first principal components for Norway, Europe and the US.

As mentioned earlier, the volatility factors in the HJM-framework is based on principal component analysis on historical instantaneous forward rates. We simulate six years into the future, which means we have six different maturities. This in turn makes the covariance matrix a 6 x 6 matrix. After the PCA is done, we have six different principal components where the first component explains most of the variance. In Figure 8.10 we see the three first principal components plotted for Norway, Europe and US. We see that the first PC for all of the markets is flat compared to the other components, and it's elements are all of the same sign. This tells us that a parallel shift in the forward rate curve is the dominant movement. Further, the second PC changes sign only once for both the Norwegian and American market, which means it describes a twisting of the forward rate curve. However, the second PC for the European market changes sign twice, which means in this case the second most dominant move is caused by a bending. Lastly, the third PC is different in all of the three markets. The third PC in Norway changes sign twice, which tells us that presumably the third largest movement in the forward rate curve is caused by a bending. However, for the European market the third PC changes sign only once, which means the movement in the forward rate curve is caused by a twisting. Furthermore, the third PC in US changes sign three times, which means it's economical meaning is not straight forward. These principal component results from the forward rate curve give less intuitive economical meanings than the results obtained by applying the same procedure on the yield curve.

Table 8.11: PCA for the Norwegian market

Eigenvalues	Value	Cum variation explained
λ_1	2.715829e-04	0.4956323
λ_2	1.034839e-04	0.6844879
λ_3	5.656838e-05	0.7877238
λ_4	4.972145e-05	0.8784643
λ_5	3.862965e-05	0.9489625
λ_6	2.796612e-05	1

In Tables 8.11, 8.12 and 8.13 we see the respective eigenvalues for the PCA of the Norwegian, European and American market. We clearly see that decay of eigenvalues are slower in the Norwegian market, compared to the European and American market. We also see that the US market has the fastest decay in eigenvalues. This suggests that the Norwegian interest market is more illiquid. Also, we see that λ_6 in the US market gives approximately no contribution to the analysis, which means it can be secluded.

Table 8.12: PCA for the European market

Eigenvalues	Value	Cum variation explained
λ_1	2.972935e-04	0.5063628
λ_2	1.899770e-04	0.8299396
λ_3	7.133703e-05	0.9514438
λ_4	2.010044e-05	0.9856797
λ_5	4.515412e-06	0.9933706
λ_6	3.892250e-06	1

Table 8.13: PCA for the US market

Eigenvalues	Value	Cum variation explained
λ_1	8.394414e-04	0.6407358
λ_2	2.942566e-04	0.8653384
λ_3	9.893878e-05	0.9408572
λ_4	4.709726e-05	0.9768060
λ_5	3.038694e-05	1
λ_6	2.077477e-11	1

Further, to find the volatility factors used in HJM we use equation (5.1.8). The factors are illustrated in Figure 8.11. We clearly see that the first factors are contributing the most. Furthermore, we see that the US has the biggest volatility factors.

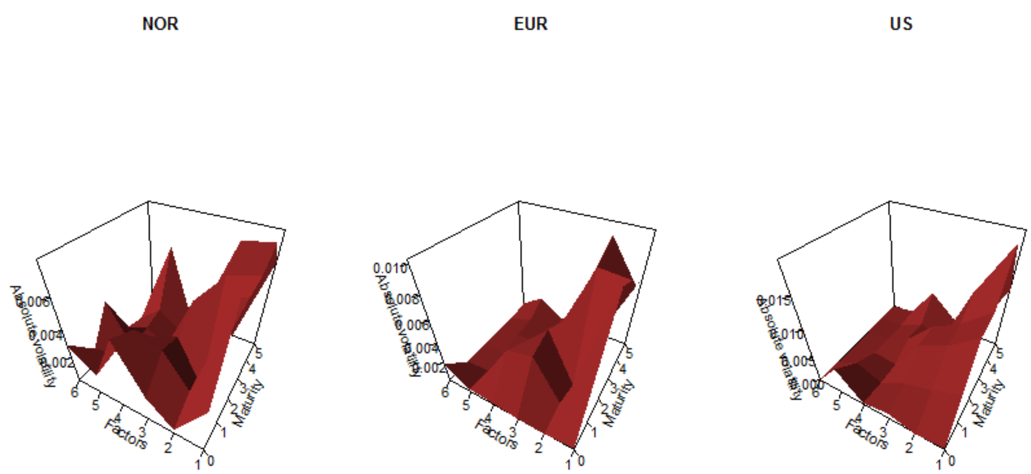

Figure 8.11: The absolute value of the volatility structure for the HJM-framework.

8.7 Volatility of Forward LIBOR Rate Curve

In Figure 8.12 we see the rolling volatility of the L_5 returns in the Norwegian, European and US market. It gives the same results as seen in Chapter 8.1, namely that the simple moving average reacts slower and less aggressive to the market volatility compared to the exponentially weighted moving average. It also shows that the financial crisis creates large peaks in the volatility. However, from the rolling volatility we see that the European and American L_5 reacts more to the financial crisis, compared to the Norwegian L_5.

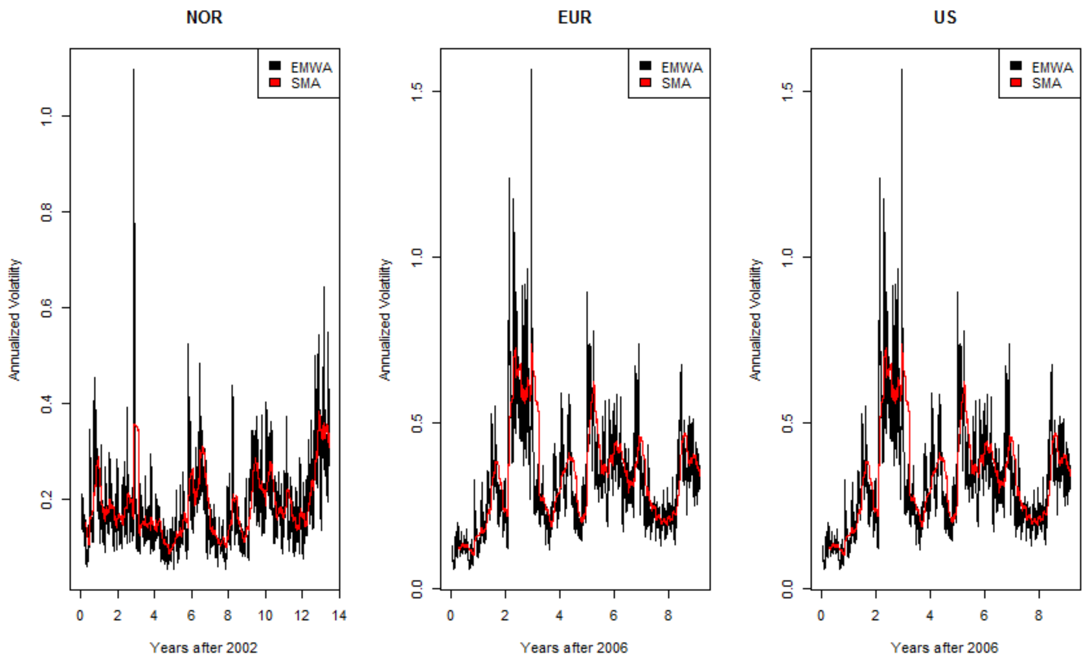

Figure 8.12: Rolling volatility of the log returns of the L_5-rate.

8.8 Correlation in Portfolio Simulation

In order to simulate the life insurance portfolios, we need to know whether the respective stock indexes and the Libor rates are independent or not. This is the reason why the Kendall's tau test is used to check for statistical dependence [18]. In Table 8.14 we see the P-values of Kendall's tau coefficient test, tested on the respective stock indexes versus the LIBOR rates. The null hypothesis is that the data are independent, which means we reject the null hypothesis for all of the LIBOR rates in Norway with regards to a 10% significance level. However, in Europe we fail to reject the independence between the stock index and the 1 day, 1 year and 2 years LIBOR rates. In US, we fail to reject the independence between the stock index and the 1 day, 4 years and 5 years LIBOR rates with a 10% significance level.

Table 8.14: P-values of Kendall's tau coefficient test.

	1d Libor	1y Libor	2y Libor	3y Libor	4y Libor	5y Libor
Norway	$< 2.22e-16$	$< 2.22e-16$	$< 2.22e-16$	$< 2.22e-16$	$< 2.22e-16$	$< 2.22e-16$
Europe	0.2744	0.2265	0.9570	0.0581	0.03943	0.0434
US	0.2973	0.0006	0.0003	0.0041	0.1663	0.1597

Since the Kendall's tau coefficient test failed to reject independence between several of the stock indexes and the LIBOR rates, we simulate the stock index and the rates through a correlation matrix. This is estimated by using the EMWA technique explained in Chapter 5.3.3. In Table 8.15 we see the estimated correlation matrix of the Norwegian market. We observe that the correlations between OSEBX and the LIBOR rates are significant, and the correlation coefficients are all positive. This also coincide with the results from the Kendall's tau test, namely that the dependency is significant.

Table 8.15: Correlation matrix of the Norwegian market

	1d Libor	1y Libor	2y Libor	3y Libor	4y Libor	5y Libor	OSEBX
1d Libor	1	0.2066	0.4594	0.2973	0.2455	0.2562	0.1174
1y Libor	0.2066	1	0.5860	0.4842	0.4070	0.4910	0.1737
2y Libor	0.4594	0.5860	1	0.4772	0.5069	0.5721	0.2258
3y Libor	0.2973	0.4842	0.4772	1	0.3059	0.5310	0.1641
4y Libor	0.2455	0.4070	0.5069	0.3059	1	0.4898	0.1671
5y Libor	0.2562	0.4910	0.5721	0.5310	0.4898	1	0.1867
OSEBX	0.1174	0.1737	0.2258	0.1641	0.1671	0.1867	1

In Table 8.16 we see the estimated correlation matrix of the European market. Again we observe the correlation matrix coincides with the Kendall's tau test results, namely that 3y, 4y and 5 years LIBOR rate have a significantly correlation with the stock index. However, the Kendall's tau test fails to reject the independence between the 1 year LIBOR rate and the stock index on a 10% significance level, even though the absolute value of its correlation coefficient is of equal size as the coefficient between the 3 years LIBOR rate and the stock index. We also notice that the 1d, 1y and 2 years LIBOR rate have positive correlation with the stock index, while the 3y, 4y and 5 years LIBOR rates have negative correlation with the stock index.

Table 8.16: Correlation matrix of the European market

	1d Libor	1y Libor	2y Libor	3y Libor	4y Libor	5y Libor	SXP5
1d Libor	1	0.2145	0.1004	0.1062	0.0836	0.1154	0.0054
1y Libor	0.2145	1	0.7305	0.4589	0.3115	0.4464	0.0361
2y Libor	0.1004	0.7305	1	0.7288	0.5068	0.7319	0.0061
3y Libor	0.1062	0.4589	0.7288	1	0.4336	0.8051	-0.0354
4y Libor	0.0836	0.3115	0.5068	0.4336	1	0.7849	-0.0299
5y Libor	0.1154	0.4464	0.7319	0.8051	0.7849	1	-0.0365
SXP5	0.0054	0.0361	0.0061	-0.0354	-0.0299	-0.0365	1

In Table 8.17 we see the estimated correlation matrix of the US market. Here we notice that all of the correlations coefficients between the LIBOR rates and the stock index are negative, opposite of what is the case in the Norwegian and the US market. We also see that the 1y, 2y and 3 years LIBOR rates have the largest correlation with the stock index, and this also coincides with the Kendall's tau test results. Further, we notice that in all of the markets the correlation between the LIBOR rates are in general larger than the correlation between them and the respective stock index. If we ignore the diagonal elements, the largest correlation coefficient in the Norwegian market is between the 1y and 2 years LIBOR rates. The largest correlation in the European market is between the 4y and 5 years LIBOR rates. This is also the case in the US market.

Table 8.17: Correlation matrix of the US market

	1d Libor	1y Libor	2y Libor	3y Libor	4y Libor	5y Libor	S&P 500
1d Libor	1	0.1573	-0.0237	0.0645	0.0275	0.0793	-0.0026
1y Libor	0.1573	1	0.6656	0.6593	0.4717	0.5520	-0.0637
2y Libor	-0.0237	0.6656	1	0.5601	0.3270	0.3568	-0.0417
3y Libor	0.0645	0.6593	0.5601	1	0.5081	0.6189	-0.0391
4y Libor	0.02759	0.4717	0.3270	0.5081	1	0.7806	-0.0293
5y Libor	0.0793	0.5520	0.3568	0.6189	0.7806	1	-0.0318
S&P 500	-0.0026	-0.0637	-0.0417	-0.0391	-0.0293	-0.0318	1

Chapter 9

Results

9.1 HJM-Framework

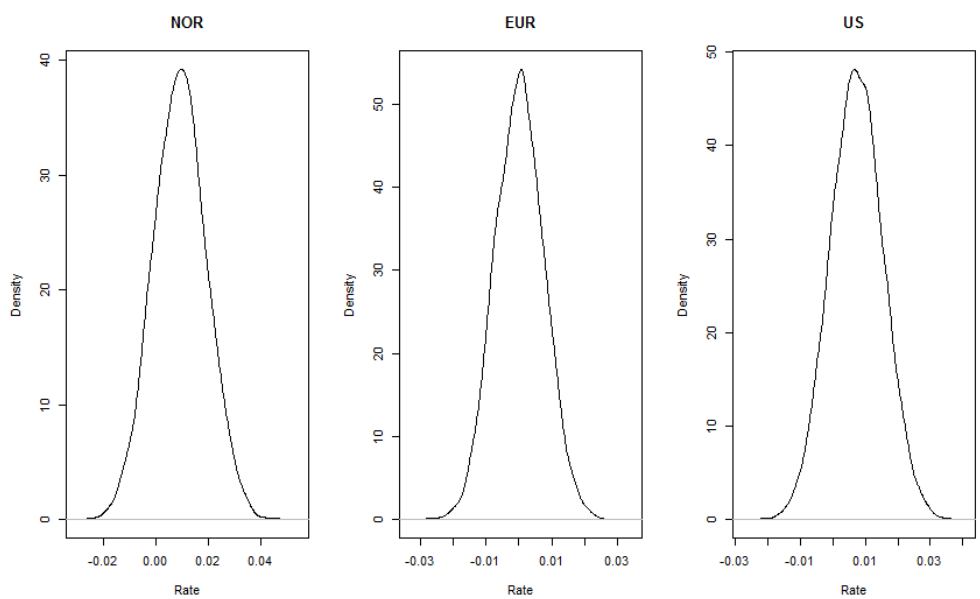

Figure 9.1: Spot rate simulated in 1 year from 10.14.2015 in Norway, Europe and US.

From equation (5.1.6) we have simulated the forward rates in Norway, Europe and US up until six years into the future, with 30000 Monte Carlo simulations. In Figure 9.1 we see the result of the spot rate simulated in 1 year. We clearly see that the simulated rate is dependent upon its initial value. Also we notice that the European market has more exposure to negative rates, while the Norwegian market predicts the largest rates. We further see that all of the distributions look normal, as they should be.

Further, in Figure 9.2 we see the result of the spot rate simulated in five years. We still see that the European spot rate is symmetric about its initial value, but with larger tails. However, both the Norwegian and American rate has more exposure to positive rates than negative rates. Further, we notice the spot rate in 5 years covers a wider rate interval compared to the 1 year case. This is logical because further into the future means more uncertainty.

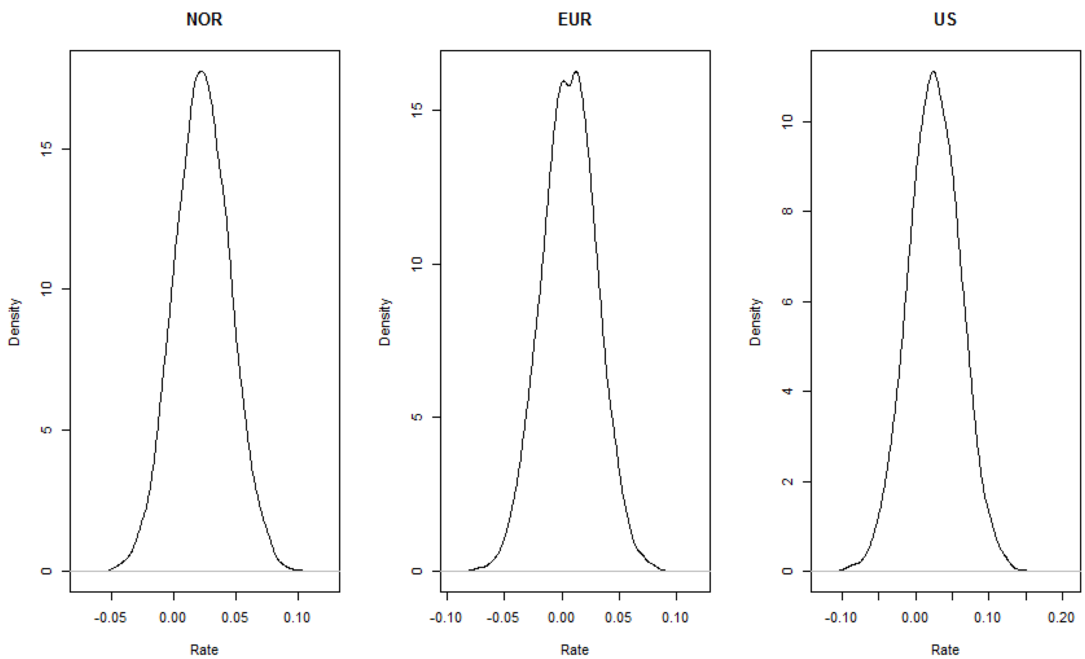

Figure 9.2: Spot rate simulated in 5 year from 10.14.2015 in Norway, Europe and US.

The means and standard deviations of the simulated spot rate for Norway, Europe and US is given in Table 9.1. From the table we see that Europe has a much lower spot rate simulated both in 1 year and 5 years copmared to Norway and US. This makes sense since Europe at today's date has a significantly lower rate level than in the Norwegian and American market. We further see that the simulated spot rate in the US market includes more variance compared to the rate in the Norwegian market, which means the US spot rate has fatter tails as can be seen in Figures 9.1 and 9.2.

Table 9.1: Means and standard deviation of the simulated spot rate in Norway, Europe and US

	1 year mean	5 years mean	1 year std	5 years std
Norway	0.00921	0.02307	0.01013	0.02242
Europe	0.00021	0.00646	0.00752	0.02402
US	0.00732	0.02517	0.02402	0.03591

The HJM-framework uses Gaussian variables which means it is a significant probability that the rates can turn negative. From Figures 9.1 and 9.2 we notice that there is a chance the rates go negative both in 1 year and 5 years forward in time. This probability is significantly bigger in the European market because of a lower rate level compared to the Norwegian and American market. The Norwegian market is further less likely to develop negative rates compared to the European and American one. This is confirmed in Figure 9.3 where the rates in Europe after the financial crisis can very well turn negative. We also notice that the probability of negative rates in all of the markets is in general decreasing with longer maturities, which means the model predicts larger rates in the future.

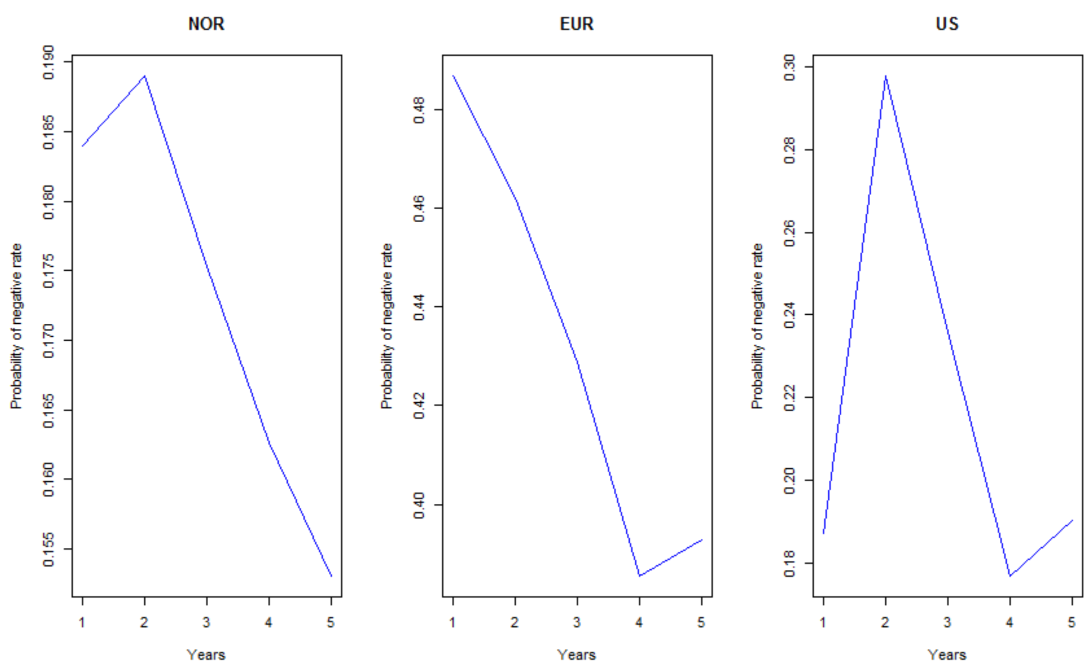

Figure 9.3: Probability of negative spot rate simulated from 10.14.2015 in Norway, Europe and US.

9.2 LIBOR Rate

For the LIBOR market model we have simulated LIBOR forward rates in Norway, Europe and US using equation (5.2.5) up until six years into the future with 30000 Monte Carlo simulations. As in the HJM case, we clearly see in 9.4 that $L_1(1)$, namely the LIBOR forward rate in one year between year 1 and 2 is concentrated around its initial values, but with much less variance compared to the HJM case. Also we notice that the densities look lognormal with small variance. The lognormal distribution implies that the rates cannot go negative, which again means that the probability of LIBOR forward rates below zero is zero. We also see that the LIBOR rate in Europe is significantly smaller than in the Norwegian and American market.

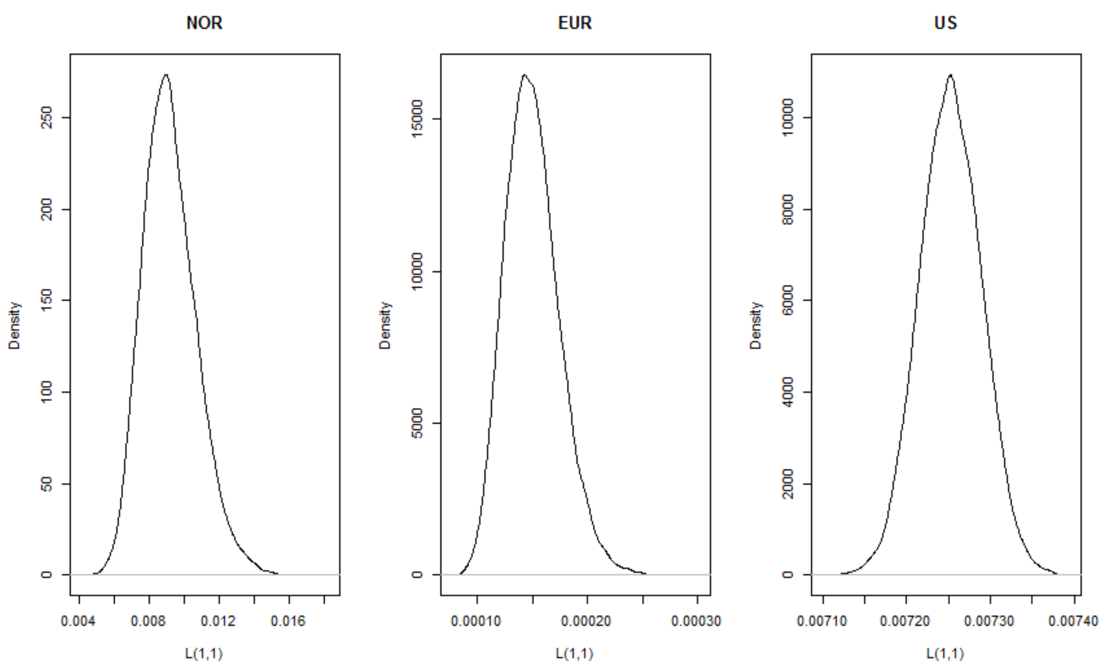

Figure 9.4: $L_1(1)$ simulated in Norway, Europe and US from 10.14.2015.

Further, in Figure 9.5 we see the result of the LIBOR forward rates simulated in 5 years, namely $L_5(5)$. We still observe that the European $L_5(5)$ is lower than in the Norwegian and US market. Also we notice that the simulated European and US $L_5(5)$ are very similar, with the US predicting a LIBOR forward rate a little larger than the Norwegian one. Further, we see the $L_5(5)$ covers a wider rate interval compared to the 1 year case. This is also logical because further into the future means more uncertainty.

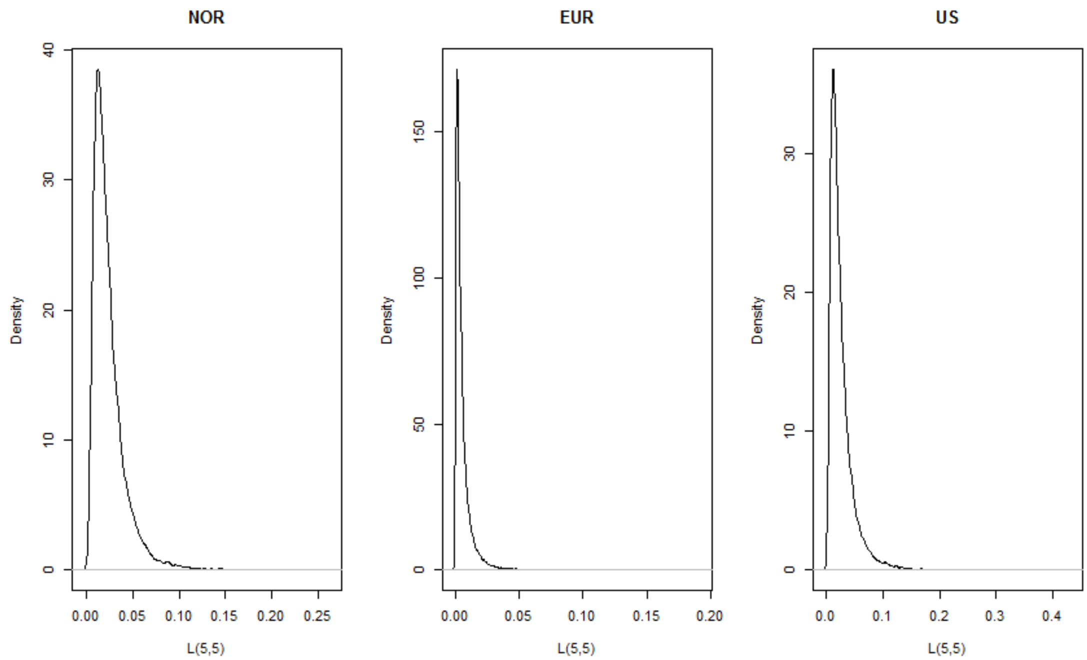

Figure 9.5: $L_5(5)$ simulated in Norway, Europe and US from 10.14.2015.

The means and standard deviations of the simulated $L_1(1)$ and $L_5(5)$ in Norway, Europe and US are given in Table 9.2. Again, we see the same pattern as in the HJM case, namely that the European rates have much lower values compared to the Norwegian and American ones. The table values also confirm what we can see from Figure 9.5, namely that the simulated $L_5(5)$ is very similar in both the Norwegian and American market.

Table 9.2: Means and standard deviation of the simulated $L_1(1)$ and $L_5(5)$ in Norway, Europe and US.

	$L_1(1)$ mean	$L_5(5)$ mean	$L_1(1)$ std	$L_5(5)$ std
Norway	0.00916	0.02405	0.00153	0.01804
Europe	0.00015	0.00590	2.4642e-05	0.00775
US	0.00725	0.02574	3.6827e-05	0.02338

9.3 Valuation of Floors

For the Heat-Jarrow-Morton model we simulate the short rate forward in the future annually up to six years. Further, as explained in Chapter 5.1.3 we assume a floorlet on every grid point, namely each year. We saw earlier that there is significant probability for a negative spot rate on these simulation points, and this will again strongly affect the price of the floor. The question whether to allow negative interest rates or not creates a big dilemma here. In other words, the HJM-framework allows negative rates, but if this is applicable to the real world it would change most of the financial models used today. Further, the LMM does not allow negative rates, which means it would dramatically under-price the floor values if negative rates are possible. Aswell as with the HJM-framework, the LMM-framework also assumes floorlets annually up till six years. These floorlets are priced accordingly to Chapter 5.2.2.

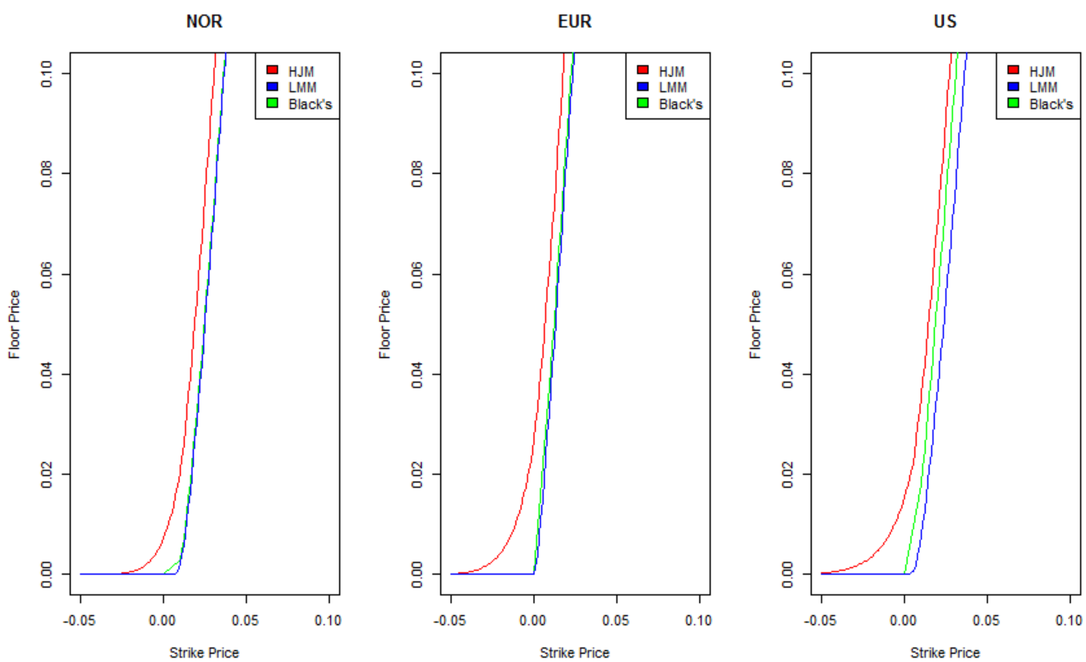

Figure 9.6: Floor prices for six annually floorlets in Norway, Europe and US calculated from 10.14.2015.

In Figure 9.6 we see the floor prices in the Norwegian, European and US market priced in the HJM-setting, LMM-setting and by Black's model. In the Black's model the price is calculated by using the same time grid as in the LMM, with the following LIBOR forward rate between the periods. Firstly, we see the price of the floor increases with increasing strike price for all the models, which it should. Further, it is evident that the greatest difference between the pricing models is that the HJM-model gives non-zero prices for floors having a strike price below zero, while this is not the case for the LMM and the Black's model. We also see that the floor prices in Europe are significantly higher than in the Norwegian and US market. This makes sense since lower rates should imply higher floor prices. We also observe that in Figure 9.6 the price calculated from the HJM-model deviates a lot from the LMM price in the interval $[-0.02, 0.02]$, especially reaching its maximum around the current spot rate, and then starts to get closer to the others again with increasing strike price. This is due to HJM having more of the rates in this interval, while modeling the rate with LMM increases the probability that the floor would end up in the money with increasing strike price, compared to HJM.

9.3.1 Floor Price Dependency on Number of Lets

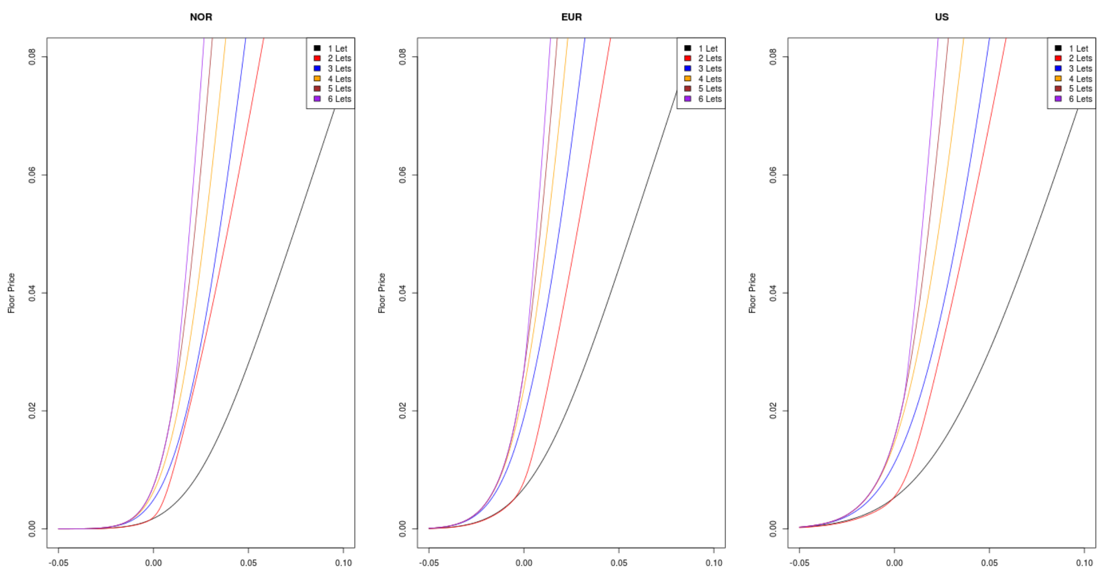

Figure 9.7: Floor prices calculated in the HJM-framework with varying number of floor lets in Norway, Europe and US simulated from 10.14.2015.

In Figures 9.7 and 9.8 we can see the floor price dependency on number of floor lets in Norway, Europe and US, both by using the HJM-framework and the LMM-framework respectively. We see that the floor prices increase in general with increasing number of lets, both in the HJM and the LMM-setting. This makes sense since the floor price formula depends on the maximum function. Further, we observe that the largest price jump in the high strike region is caused by going from one floor let to two floor lets. However in the HJM-setting, for low strikes the largest price jump is caused by going from two lets to three lets. In the HJM-setting for high strike prices we see that in Norway and US each floor let after the second one contributes to approximately the same jump in the floor price, while in the European market the second and third floor let creates greater jumps in the floor price compared to the fourth, fifth and sixth. However, in the LMM-setting we see the same trend for all of the markets in the high strike price region, namely that the jumps in floor price decreases with increasing number of floor lets. Again, we observe that the HJM-framework gives higher floor prices than the LMM-framework, which is due to HJM's exposure to negative interest rates.

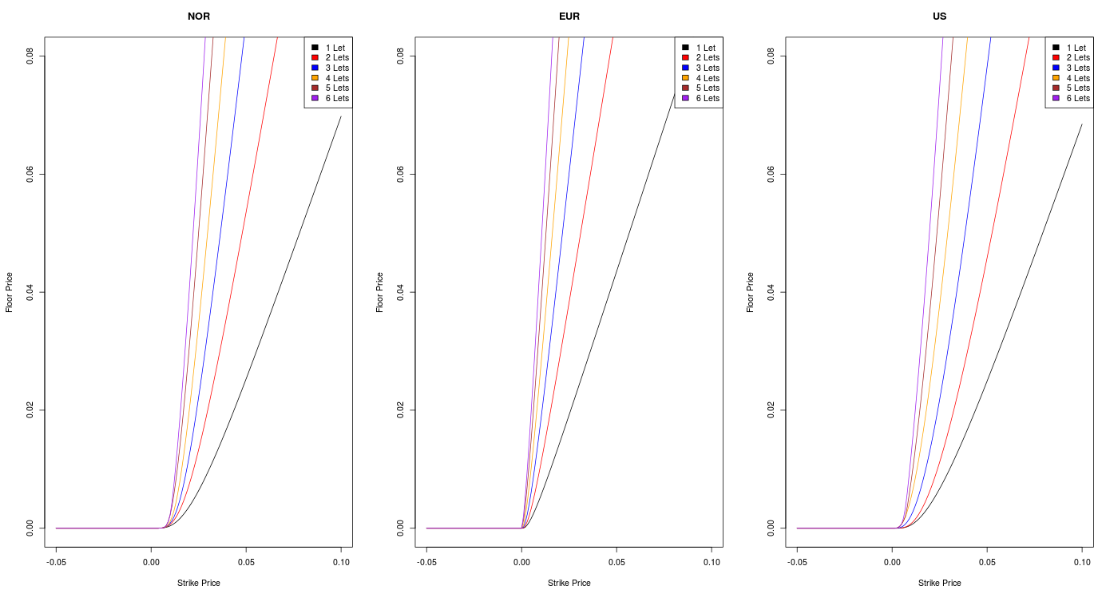

Figure 9.8: Floor prices calculated in the LMM-framework with varying number of floor lets in Norway, Europe and US simulated from 10.14.2015.

9.4 Paid-Up Policy Portfolios

In Figure 9.9 we can see the result of the paid-up policy portfolios held by the life insurance companies six years in the future. We have simulated the LIBOR forward rates together with the stock index 30000 times. In this chapter we also have multiplied the VaR and ES estimates explained in Chapter 6, with minus 1 because of easier interpretation. This means that a lower VaR estimate implies more money at risk. We see that for all of the markets, the mean gain increases with increasing share in the stock index, but it decreases with an increasing floor guarantee. This makes sense since the life insurance companies need to pay out more with higher floors, and also the positive drift of the equity increases the mean of the portfolio. Further, we see that the Norwegian market has the highest mean gain, while the European market has a much higher probability of achieving negative mean gain. This is evident because of the low-rate environment in the European market. An interesting observation is that the Norwegian market gives a significantly higher mean gain than the US market, even though the rate level in the two economies are about the same. The reason for this is because the observed drift of the Norwegian stock index is twice the size of the US stock index, as we can see in Table 8.1.

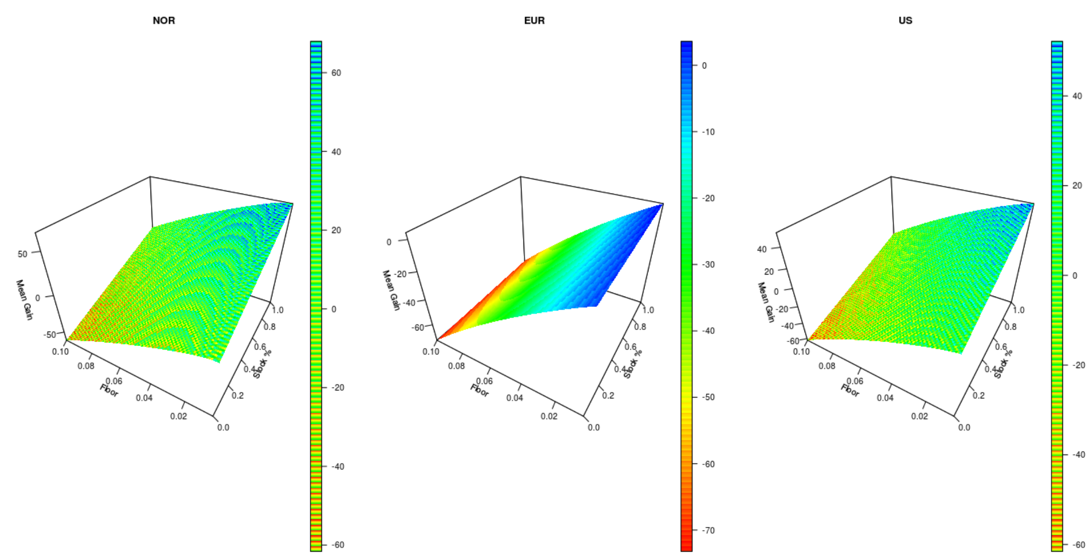

Figure 9.9: Mean gain of the paid-up policy portfolios held by the life insurance companies in Norway, Europe and US simulated 6 years in the future from 10.14.2015.

Further, in Figure 9.10 we can see the 95% VaR of paid-up policy portfolios held by the life insurance companies 6 years in the future. Again, we notice that the European market offers a much lower VaR estimate compared to the Norwegian and US market. We also see that in the Norwegian and US market we obtain a positive VaR if the life insurance offer a low enough floor with a low enough share in the stock index. This is not the case for the European market. This situation is not very applicable in the Norwegian and US market however, since customers would rather put the money in the bank than buy such a low floor.

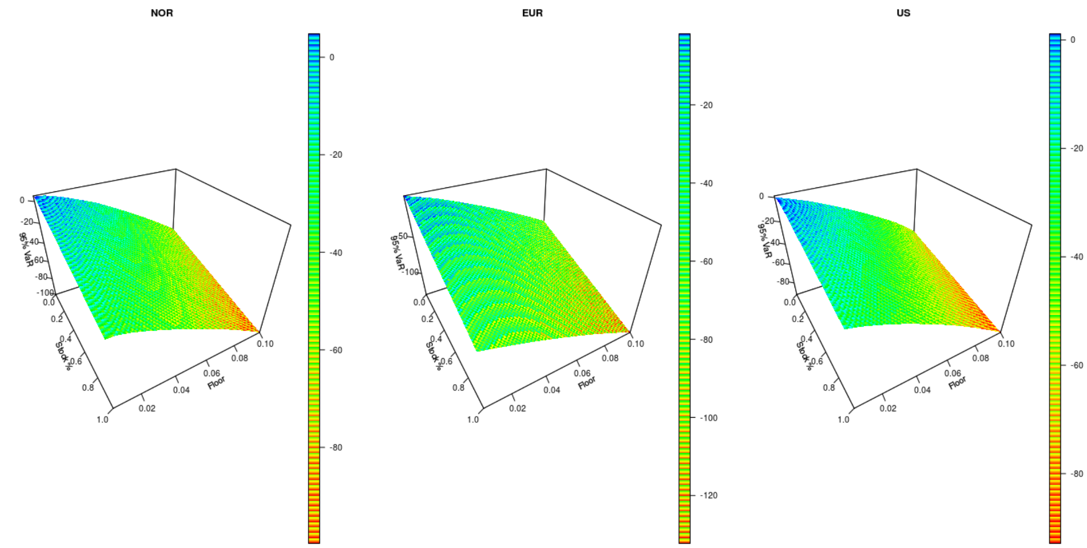

Figure 9.10: 95% VaR of the paid-up policy portfolios held by insurance companies in Norway, Europe and US simulated 6 years in the future from 10.14.2015.

Furthermore, it is also interesting to look at the difference in gain densities between offering a high floor guarantee versus a low floor guarantee, while keeping the stock share constant. In Figure 9.11 we can see the simulated density of keeping a 30% stock share with a 3% floor. We notice that the gain density in the Norwegian market has a little bit more variance compared to the European and US market, but also that the gain in the Norwegian and US market are much higher than in the European market. We also see the 95% VaR and 95% ES estimates in each of the markets. We see that it is much more likely for the insurance companies in Europe to loose money compared to the Norwegian and US market, assuming the same floor guarantee and stock share. We also notice that the ES estimates are less than the VaR estimates, which coincides with theory.

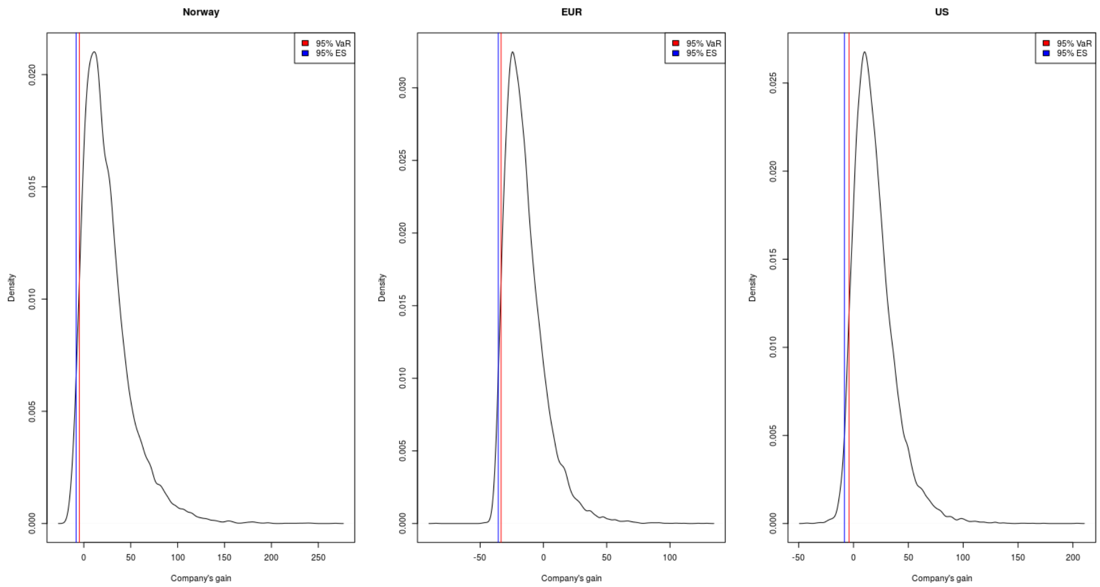

Figure 9.11: Gain density of the paid-up policy portfolios held by the life insurance companies simulated 6 years in the future from 10.14.2015. Based on 30% stock share and a 3% floor.

Evidently, when the life insurance companies offers a lower floor guarantee, the mean gain will increase. This is confirmed in Figure 9.12. Here we can see the gain densities with still a 30% stock share, but now with a 1% floor. We notice that the densities has shifted to the right compared to the 3% floor densities. Now the 95% VaR and 95% ES estimates in both the Norwegian and US markets are positive, while the European estimates are still significantly negative. Again we observe more variability in the Norwegian density compared to the other markets.

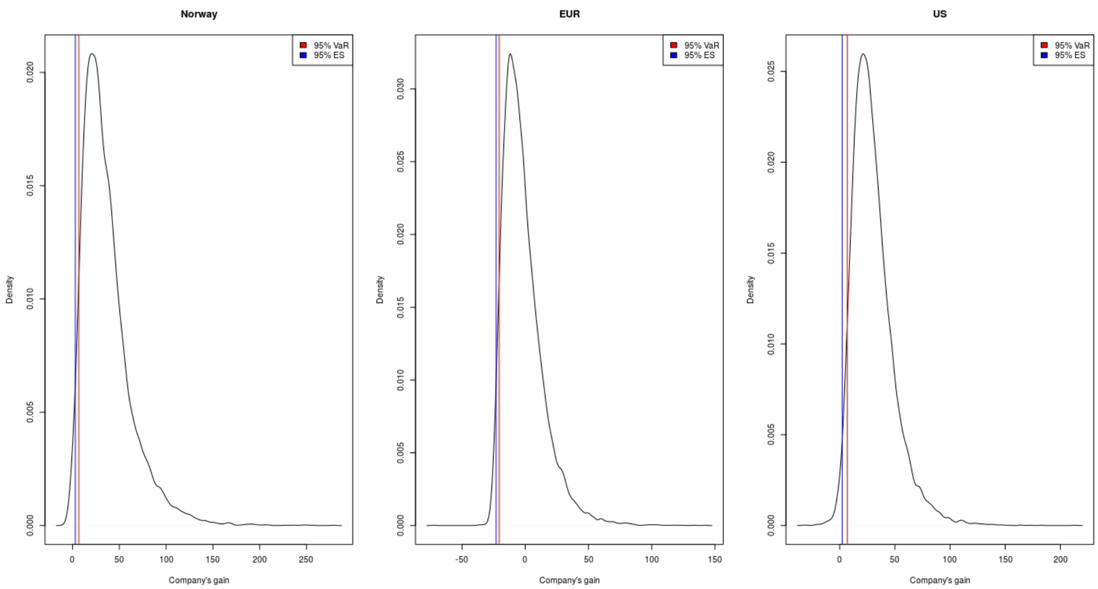

Figure 9.12: Gain density of the paid-up policy portfolios held by the life insurance companies simulated 6 years in the future from 10.14.2015. Based on 30% stock share and a 1% floor.

In the end we want to look into how big of a share we need to put into the equity market, in order to have a positive mean gain. In Table 9.3 we can see the the necessary share in all of the markets, given a floor guarantee. We see that the Norwegian market requires less equity share in order to have positive mean gain, compared to the US and European market. Also we observe that with a 2% or higher floor we need more than 100% of the money in the equity market to achieve a positive mean gain in Europe. The reason why we need a less equity share in the Norwegian market compared to in the US market, is again because of the higher observed drift in the Norwegian equity market.

Table 9.3: Stock share needed for positive mean gains

Floor	Stock Share (NOR)	Stock Share (EUR)	Stock Share (US)
1%	0 %	76 %	0 %
2%	0 %	> 100 %	0 %
3%	7 %	> 100 %	13 %
4%	19 %	> 100 %	28 %
5%	31 %	> 100 %	44 %

Further, in Table 9.4 we can see the VaR estimates given a % share in the equity market and a floor guarantee in the Norwegian market. The numbers marked in red corresponds to the respective stock shares listed in Table 9.3. Since the European market only had one equity share in Table 9.4 less than 100%, we only list the VaR estimate of this position, which is -48.0119. This is still a very low VaR estimate, which stems from the fact that the rates are much lower in the European market compared to the Norwegian and US markets.

Table 9.4: 95% VaR estimates for Norwegian life insurance companies

Floor	0 % Stock	7 % Stock	19 % Stock	31 % Stock
1%	3.6852	3.3012	-0.8216 %	-5.2836
2%	-0.4196	-1.3073	-5.6622 %	-10.2160
3%	-5.9074	-7.5581	-12.0682	-16.6047
4%	-11.8665	-14.5223	-19.0710	-23.6392
5%	-19.2398	-21.9409	-26.5158	-31.0444

In Table 9.5 we see the VaR estimates given a % share in the equity market and a floor guarantee in the European market. The numbers marked in red again corresponds to the respective US stock shares listed in Table 9.3. We see that the VaR estimates in the US table is slightly less than in the Norwegian table. This is partly due to higher stock share, and also because of less observed drift in the US equity market compared to the Norwegian one. Again, we also notice that the VaR estimates in all of the markets decreases with increasing stock share, and increasing floor guarantee.

Table 9.5: 95% VaR estimates for US life insurance companies

Floor	0 % Stock	13 % Stock	28 % Stock	44 % Stock
1%	-1.8346	-0.6708	-4.0164 %	-8.7400
2%	-6.3721	-5.23995	-8.9848 %	-13.6955
3%	-11.4840	-10.76521	-14.8840	-19.6773
4%	-17.3030	17.0515	-21.3892	-26.2438
5%	-23.4894	-23.9849	-28.5019	-33.3635

Chapter 10

Discussion and Conclusion

From the stock index analysis chapter it is evident that all of the equity markets fell drastically in value during the dot-com bubble and the financial crisis. This shows some of the disadvantages related to creating a portfolio with stocks. In other words, the uncertainty in the stock price development can cause critical adjustments into the financial policy within a firm. We could also see from the fitted normal and student-t densities that they were not perfect, but with the student-t fit performing better than the normal fit. This makes sense since the student-t distribution has one more parameter to be fitted compared to the normal distribution, as well as the student-t distribution assigns higher probability to extreme events compared to the normal one. However, the observed density had fatter tails than both of the fitted densities. Another interesting observation is that the observed drift of the Norwegian equity market is twice the size of the US equity market, as well as the European market had a significantly lower observed drift than both of the other markets.

In most of the introductory financial courses we learn that receiving money now is preferred over receiving money in the future, i.e. there exists a risk free interest rate larger than zero. However, the low-rate environment particularly in Europe today is challenging the classical concept of interest rates. The densities of the simulated spot rate in Figures 9.1 and 9.2 confirms this. These interest rates are modeled through the HJM-framework, while we can see the simulated LMM rates in Figures 9.4 and in 9.5. From these figures we also notice the biggest difference between the two frameworks, namely that HJM produces both positive and negative rates, while LMM only allows positive rates. This is because LMM simulates interest rates under a lognormal assumption. Further, through the HJM-framework we see that the probability of negative rates are significant. However, the probability of negative rates are greater in the European market than in the

Norwegian or US market. It is evident that the closer the today's rate is to zero, the higher the probability is of a negative rate in the future.

The HJM and LMM frameworks have several differences, but they also share one important detail. Namely they both have to be calibrated to the market in order to give correct interest rate option prices. It is evident that the volatility factors determines the models in both cases, and this means that market calibration should be done on the volatility factors. In this thesis the HJM-model has been calibrated by using historical data with PCA, while the LMM-model has been calibrated by historical data using EMWA. The reason for using historical volatility is that it gives a good insight into the risks we face when trading in the market, based on previous events.

From the Monte Carlo simulation we notice a large price gap around the current spot rate for the HJM, LMM and Black's pricing model. Because of negative rates in the HJM-framework, it prices floors larger for negative and zero rates compared to the LMM-framework. However, the LMM-framework produces more in-the-money floors for higher strike prices, which is the reason the LMM floor price gets closer to the HJM floor prices for increasing strike prices. Furthermore, it would be a poor decision to price low-strike floors with the lognormal framework LMM. This is because zero-strike floors indeed has a value in some markets. In other words, if an investor only expects positive rates, he potentially can loose a lot of money. On the other side, for an old-style market with large positive rates we ought to be careful with pricing the floors through the HJM-framework. This is because zero-strike floors might very well be worthless in this case. Further, from the floor price dependency chapter we see that the floor price increases in all of the markets with increasing number of lets, as well as the largest price jump is caused by going from one floorlet to two floorlets.

The European market has significantly lower rates compared to the other markets, and combined with a lower observed drift of the equity market, this in turn makes it much harder to offer the same floor as in the Norwegian or US market. For instance, in order to keep the mean gain of the paid-up policy portfolios positive when offering a floor guarantee of 1%, the European life insurance need to place 76% of their money in equity, while the Norwegian and US life insurance do not need any shares in the equity market. Another interesting result is that even though the Norwegian and US market have about the same rate level, the Norwegian life insurance needs to place less of their money in equity compared to the US life insurance, in order to achieve the same mean gain.

To sum up, it is evident that the low-rate environment we now experience are pushing the life insurance companies into new territories. The floors offered by the life insurance companies are now priced higher than before, and because of the low rate-level the life insurance companies receives less money from their bank deposits than earlier. This means the life insurance companies need to increase their stock share, in order to prevent a negative mean gain from their portfolios. However, the problem with this form of yield hunting is that the risk of loosing money greatly increases with increasing share in the equity market. This in turn increases the importance of having well-diversified stock portfolios. We also notice that the portfolio's VaR increases with increasing floor guarantee, as expected.

For further work it would be interesting to perform a sensitivity analysis of the value of λ, used in the EWMA framework. Another possibility would be to capture the historic volatility using a GARCH model. However, since these models are based on historic data, it will always react to the market with a delay as we could see in Chapter 8.1. This could be improved by estimating the real implied volatility observed in the markets, calculated from observed floor prices today. Furthermore, there are several regulations regarding how much of the money the life insurance companies may place in risky assets. This means it would also be interesting to optimize the portfolio gain/VaR with regards to restrictions of the VaR or the equity share of the company. Lastly, we could also simulate the portfolios through another framework than the LMM-framework. The reason for this is that we saw the lognormal assumption of the historical relative changes in the forward rate did not hold very well with short maturities in Chapter 8.5.

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Appendix A

Probability Theory

A.1 Probability Space

Definition 1: Measure

A measure μ assigns positive numbers to sets $A : \mu(A) \in \Re$

Definition 2: Algebra

A collection $\mathbf{A}$ of subsets of the space $\mathbf{X}$ is an algebra if

$$\mathbf{X} \in \mathbf{A},$$

$$A \in \mathbf{A} \Rightarrow A^c \in \mathbf{A},$$

$$A, B \in \mathbf{A} \Rightarrow A \cup B \in \mathbf{A}.$$

Further, $\mathbf{A}$ is closed under finitely many set operations.

Definition 3: σ -algebra

$\mathbf{A}$ is a σ-algebra if it is an algebra and for $A_n \in \mathbf{A}, n \in \mathbb{N}$, we have $\cup A_n \in \mathbf{A}$

Further, **A** is closed under countably many set operations.

Definition 4: Probability Space

The triple $(\Omega, \mathbf{A}, P)$ is a probability space, where Ω is a set, **A** is a σ-algebra of subsets of Ω, and P is a probability measure, $P(\Omega) = 1$, on **A**. See [21] for a more thorough introduction.

A.2 Ito's Lemma

If we have a stochastic differential equation of the form

$$dS = u(t, S)dt + w(t, S)dX(t), \tag{A.2.1}$$

then given $f(S)$, with $dX^2 \to dt$ as $dt \to 0$ with probability 1, then Ito's lemma says that

$$df = w\frac{df}{dS}dX + (u\frac{df}{dS} + \frac{1}{2}w^2\frac{d^2f}{dS^2})dt. \tag{A.2.2}$$

See [5] for proof.

A.3 Brownian Motion

A stochastic process X_t is Brownian motion if the following conditions hold

- $X_0 = 0$.
- X_t has independent increments, which means if $d < t' \leq t < u$, then $X_u - X_t$ and $X_{t'} - X_d$ are independent stochastic variables.
- For $d < t$, $X_t - X_d$ is normally distributed with $\mathrm{E}[X_t - X_d] = 0$ and $\mathrm{VAR}[X_t - X_d] = t - d$.
- X_t is almost surely continuous.

A.4 Lognormal Walk

Assuming the geometric Brownian motion

$$dS(t) = \mu S(t)dt + \sigma S(t)dX(t),$$

by investigating the process $Z_t = f(t, S_t) = \ln S_t$ with Ito's lemma we obtain

$$dZ(t) = \frac{1}{S(t)}dS(t) - \frac{1}{2}\frac{1}{S(t)^2}(dS(t))^2,$$

$$dZ(t) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dX(t),$$

$$\int_0^T dZ(t) = \int_0^T (\mu - \frac{1}{2}\sigma^2)dt + \int_0^T \sigma dX(t),$$

$$Z(T) - Z(0) = \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\left(X(T) - X(0)\right),$$

$$\ln S(t) = \ln S(0) + \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma X(t),$$

which leads to the lognormal walk

$$S(t) = S(0)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma X(t)\right).$$

That is

$$\ln\frac{S(T)}{S(0)} \backsim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)T, \sqrt{T}\right).$$

A.5 Central Limit Theorem

Let $(X_1, ..., X_n)$ be a random, independent and identically distributed sample from a specific distribution with mean μ and finite, nonzero variance σ^2. Then the limit

$$\lim_{n \to \infty} \frac{X_1 + ... + X_n}{n} \tag{A.5.1}$$

approaches the normal distribution with mean μ and variance $\frac{\sigma^2}{n}$.

See [17] for proof.