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Credit Risk Modelling with Expected Shortfall

A Simulation-based Portfolio Analysis

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Preface

This thesis is written at the Department of Mathematical Sciences at the Norwegian University of Science and Technology (NTNU) in the period January to June 2016. It represents a term workload and leads to the degree Master of Science.

I would like to direct great thanks to my supervisor Jacob Laading for stimulating discussions, brilliant guidance and constructive feedback.

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Abstract

The Basel Committee's minimum capital requirement function for banks' credit risk is based on a risk measure called Value at Risk (VaR). This thesis performs a statistical and economic analysis of the consequences of replacing VaR with another risk measure called Expected Shortfall (ES), a switch that has already been set in motion for market risk. The empirical analysis is carried out by means of both theoretical simulations and real data from a Norwegian savings bank group's corporate portfolio.

ES has some well known conceptual advantages compared to VaR, primarily by having a better ability to capture tail risk. ES is also sub-additive in general, so that it always reflects the positive effect of diversification. These two aspects are examined in great detail, in addition to comparing parameter sensitivity, estimation stability and backtesting methods for the two risk measures. All comparisons are conducted within the Basel Committee's minimum capital requirement framework. The findings support a switch from VaR to ES for credit risk modelling.

Sammendrag

Baselkomiteens kapitalkravfunksjon for bankenes kredittrisiko er basert på risikomålet "Value at Risk" (VaR). Oppgaven foretar en statistisk og økonomisk analyse av konsekvensene av å erstatte VaR med det alternative risikomålet "Expected Shortfall" (ES), en endring som allerede er blitt satt i gang for markedsrisiko. Den empiriske analysen gjøres ved hjelp av både teoretiske simuleringer og reelle data fra næringslivsporteføljen til en norsk sparebankgruppe.

ES har noen kjente konseptuelle fordeler sammenlignet med VaR, først og fremst ved å ha en bedre evne til å fange opp halerisiko. ES er også generelt sub-additivt, slik at det alltid reflekterer den positive effekten av diversifisering. Disse to punktene undersøkes nøye. I tillegg sammenliknes de to risikomålene med hensyn til parameterfølsomhet, estimeringsstabilitet og metoder for modellvalidering. Alle sammenligninger er utført innenfor Baselkomiteens kapitalkravrammeverk. Funnene støtter en overgang fra VaR til ES for modellering av kredittrisiko.

Contents

Chapter 1

Introduction

1.1 Why Regulate Banks?

The capital levels of banks play a key part in this thesis. Banks must meet specific minimum requirements for their capital (equity), enforced by national supervisory authorities. The regulations are in most cases binding, meaning that the required capital is higher than what the banks regard as an optimal capital structure. Unlike most other sectors, the failure of one bank tends to weaken its competitors in the short run, because banks are exposed to each other both directly and indirectly [1]. A failure of a single bank may therefore have a negative impact on a whole country's banking sector. As problems in the banking sector may cause major consequences for the overall economy, the national governments find it necessary to regulate the banks to reduce the possibility for such problems to occur. The economic challenges ensuing from the financial crisis in 2008 increased the attention paid to the issue of bank solvency.

The Modigliani–Miller theorem states that the value of a firm is independent of how the firm is financed, including dividend policy and debt ratio [2], [3]. The reasoning is that the firm cannot create value through financial decisions because the shareholders can duplicate or reverse these decisions themselves through their own transactions in the financial markets [4]. For example, a shareholder can acquire a replacement for missing dividends by selling shares, and a desire for a higher debt ratio can be obtained by taking out a private loan. The theorem is based on idealistic assumptions of a liquid and complete financial market, so that one can use arbitrage arguments for valuation. Some of these assumptions will in practice not be met, most importantly the assumption of no bankruptcy costs. Also less extreme economic strains, such as financial stress, suggest a sensible debt ratio. Firms that end up in a stressed financial position often feel forced to make short-term decisions that harm long-term value. Thus, firms will normally choose financial solutions and financial counterparties that provide low probability of financial distress and bankruptcy. Special regulations in the banking industry do however counteract some of these disciplinary factors. For example, in many countries the government insures the bank customers' deposits, thus reducing the incentives for bank customers to adequately assess the banks' financial condition. Hence, market discipline in terms of high debt ratio is partly put out of action. The fact that systemically important banks often can expect to be rescued by the government when experiencing financial problems ("too big to fail") has the same effect.

The banking industry is operating in a mixed economic context in which traditional theory of corporate finance does not have full validity. Instead, one can get some insight by studying regulation theory [5]. Using this theory one can derive optimal bank regulation from a socioeconomic perspective, taking into account that the firms know their portfolios better than the regulator (asymmetric information). Acknowledging that deposit guarantees and potential rescue of systemically important banks partly puts disciplining market forces out of play, governments have implemented direct regulation of banks' risk-weighted capital ratio.

If the regulatory minimum capital requirements exceeds the level that an individual bank deems adequate, the regulation is likely to result in increased costs for that bank. If these costs are passed on to the bank's customers, economic costs will arise. The added capital does however increase the bank's resilience to losses. More resilient banks reduce the likelihood of banking crises [6].

1.2 Thesis Content

The Basel Committee on Banking Supervision has its origins in the financial market turmoil that followed the breakdown of the Bretton Woods system of managed exchange rates in 1973. After the collapse of Bretton Woods, many banks incurred large foreign currency losses. In response to these and other disruptions in the international financial markets, the Committee was established in 1974 by the central bank governors of the G10 countries [1]. The

[1] A group consisting of the countries that agreed to participate in an agreement to provide the International Monetary Fund with additional funds to increase its lending ability, established in October 1962.

Committee was designed as a forum for regular cooperation between its member countries on banking supervisory matters. Its aim was and is to enhance financial stability by improving supervisory knowhow and the quality of banking supervision worldwide. After starting life as a G10 body, the Committee has expanded to 28 member countries. The Committee's standards are not legally enforceable, but it is expected that the individual national authorities implement them [7].

This thesis focuses on the Basel Committee's minimum capital requirement for banks' credit risk. Since 2004 banks have been allowed to calculate their minimum capital requirements using risk parameters estimated by internal models [8]. The amount of capital the banks are required to hold for each of their loans are calculated by a mathematical function using the estimated risk parameters *probability of default* (PD) and *loss given default* (LGD) as inputs. The thesis will describe in detail the mathematical models that were used to derive this function, and examine its implications by applying empirical analysis.

The minimum capital requirement function for *credit risk* is based on a risk measure called *value at risk* (VaR). In January 2016, the Basel Committee published revised standards for minimum capital requirements for *market risk* [9], which include a shift from value at risk to *expected shortfall* (ES) as the preferred risk measure.

Banks have been allowed to use internal VaR models as a basis for measuring their market risk capital requirements since 1997 [7], i.e. seven years before the same applied to capital requirements for credit risk. Internal credit risk models were not allowed at an earlier stage due to the fact that they are not a simple extension of their market risk counterparts. Data limitations is a key impediment to the design and implementation of credit risk models [10]. Most credit instruments are not listed with a market value, meaning that there are no historical prices to base future projections on. As there is no market values to compare with the book values, there is no impairment loss. Loss occurs only at default events, and the infrequent nature of these events makes it difficult to collect enough relevant data. The long time horizons also make the validation of credit risk models fundamentally more difficult than the backtesting of market risk models.

The Basel Committee has currently not considered a transition from value at risk to expected shortfall for measuring credit risk. However, as the development of credit risk models lies a few years behind the market risk models, there is reason to believe that this might be considered in a not so distant future.

The main objective of this thesis is to explore what the consequences would be if the Basel Committee were to also shift from VaR to ES for the computation of credit risk. It will include an introduction to the two different risk measures and a thorough comparison, with respect to both theoretical properties and practical use. The effects of a possible shift from VaR to ES will be measured using both simulated data and real data from a Norwegian savings bank group's corporate portfolio.

The input parameters for the capital requirement function, PD and LGD, are estimated using the banks' internal risk models. This thesis examines how sensitive the capital requirement function is to uncertainty in these estimates. This is tested by simulating values for PD and LGD, and letting the relative standard deviations represent the uncertainty in the banks' estimates. The capital requirement is calculated using the simulated values, which enables us to study the relationship between the parameters' and capital requirement's relative standard deviations.

The thesis also examines how VaR and ES values are affected by the tail properties of the loss distributions, by simulating losses from distribution functions with different tail weights. Lastly, model validation through backtesting is compared for VaR and ES.

Chapter 2

Bank Regulation

In this chapter we present both former and present versions of the Basel Committee's capital adequacy framework. We also introduce the Committee's proposals for future regulations.

The Basel Committee aims to enhance financial stability by improving the quality of banking supervision worldwide. It seeks to achieve its aims by setting minimum standards for the regulation and supervision of banks. In addition to promoting common understanding and improving cross-border cooperation, the Committee is also exchanging information on developments in the banking sector and financial markets to help identify current or emerging risks for the global financial system [7].

2.1 Basel I

There was strong recognition within the Committee of the overriding need for a multinational accord to strengthen the stability of the international banking system and to remove a source of competitive inequality arising from differences in national capital requirements [7]. The Basel Committee published its first capital measurement system for the banking sector in 1988, commonly referred to as the *Basel Capital Accord* (Basel I). The Accord introduced a minimum capital (equity) requirement given by 8 % of the *risk-weighted assets* (RWA):

$$K \geq 0.08 \cdot \sum_{i=1}^{n} RW_i A_i,$$

where K is the minimum capital requirement, RW_i is the risk weight assigned to asset i and A_i is the credit risk exposure of asset i.

The Committee introduced a standard of five different risk weight levels: 0, 10, 20, 50 and 100 % [11]. For example, cash and government bonds were assigned a risk weight of 0 %, claims on most banks were weighted with 20 %, mortgages were given a risk weight of 50 % and claims on the private sector were weighted with 100 %. Half of the required capital had to be core capital (4 % of the RWA). Common equity (share capital and retained earnings) should amount to half of the core capital (2 % of the RWA).

2.2 Basel II

In 2004 the Basel Committee published a new capital adequacy framework to replace the 1988 Accord: the *Revised Capital Framework* (Basel II). The new framework was designed to improve the way regulatory capital requirements reflect underlying risks and to better address the financial innovation that had occurred in the recent years. The new framework comprised three *pillars*. Pillar 1 represents the minimum requirements for bank capital, building on the standardised rules from Basel I. Pillar 2 is a collection of rules for supervisory review of a bank's capital adequacy and internal assessment process. Pillar 3 contains disclosure requirements of banks' activities, to strengthen market discipline and encourage sound banking practices [7].

Pillar 2 is intended to capture risk elements that are not covered or only partially covered by the capital requirements of Pillar 1. Such risks may include concentration risk, interest rate risk, currency risk and model risk. Banks are required to assess their overall capital adequacy in relation to their risk profile, and at all times keep a strategy for maintaining their capital levels [8]. Additional capital requirements can be set by the supervisory authorities if they consider that the level of the bank's capital is not sufficiently adapted to their risk profile.

2.2.1 Internal Ratings Based Approach

The introduction of Basel II also opened the possibility for banks to calculate the assets' risk weights by internal models, instead of using the given standard rates (the standardised approach). To be able to use this *internal ratings based* (IRB) approach, the bank's risk models have to be approved by the national supervisory authorities. A bank that uses this approach is called an IRB bank.

If the IRB bank's internal models implies lower risk weights than the standard rates, the minimum capital requirement will decrease. Since equity is the most expensive form of financing, the IRB banks have incentives to calculate artificially low risk weights. To remedy this issue the Basel II framework included a temporary lower limit for the risk weights. The lower limit is defined as a percentage of the original risk weights from Basel I and is therefore called the Basel I transitional floor. Since 2009 the floor level has been 80 % [8].

2.3 Basel III

Even before Lehman Brothers collapsed in September 2008, the need for a fundamental strengthening of the Basel II framework had become apparent. The banking sector had entered the financial crisis with too much leverage and inadequate liquidity buffers [7]. The third regulatory framework from the Basel Committee, published in 2010, keeps the concept of the three pillars of Basel II. The emphasis is on Pillar 1, for which the Committee has focused on qualitative requirements. The total minimum capital requirement is still 8 % of the RWA, but now 75 % of the capital (6 % of the RWA) must be core capital, and 56.25 % of the capital (4.5% of the RWA) must be common equity [12]. A comparison of the different Basel versions are shown in Table 2.1.

Basel III also requires that banks have capital buffers beyond the minimum capital requirement, so they will be prepared for periods of financial instability. A *conservation buffer* consisting of common equity should constitute 2.5 % of the RWA. The banks must also hold a *countercyclical buffer* consisting of common equity, that varies between zero and 2.5 % of the RWA. The level of this buffer should be cyclical, and is determined on an ongoing basis by national authorities. The idea behind this is that the banks should build capital in good times, so that they can better cope with recessions [12].

There is also an additional buffer requirement for banks that are identified as systemically important, both globally and nationally. Global systemically important banks must hold additional core capital corresponding to 1-3.5 % of the RWA, depending on the bank's systemic importance. This requirement will be introduced gradually from 2016, and will be in full effect from 2019 [13].

The Basel Committee has also introduced a *leverage ratio*, which is an absolute and unweighted minimum capital requirement. This minimum requirement is currently set to 3 %

of total exposure, and is scheduled to be phased in from 2018 [12].

	Basel I	**Basel II**	**Basel III**
Minimum ratio of total capital to RWAs	8 %	8 %	8 %
Minimum ratio of core capital to RWAs	4 %	4 %	6 %
Minimum ratio of common equity to RWAs	2 %	2 %	4.5 %
Supervisory review	No	Yes	Yes
Market dicipline and disclosure	No	Yes	Yes
IRB approach	No	Yes	Yes
Conservation buffer	-	-	2.5 %
Countercyclical buffer	-	-	0 - 2.5 %
Systematically importance buffer	-	-	1 - 3.5 %
Leverage ratio	-	-	3 %

Table 2.1: Comparison of the Basel Committee's three different capital adequacy frameworks.

2.4 Upcoming Regulations

The standardised approach for determining risk weights is up for revision, and the Basel Committee's second consultation on this subject was published in December 2015 [14]. The new standardised approach is intended to be more risk sensitive, while still keeping its simplicity compared to the IRB approach. The Basel Committee is also consulting on the design of a new capital floor framework based on the Basel II/III standardised approaches [15], which is supposed to complement the leverage ratio introduced as part of Basel III. This framework will replace the current Basel I transitional floor. The capital floor is part of a range of policy measures that aim to enhance reliability and comparability of risk-weighted capital ratios.

A hypothetical portfolio benchmark exercise was conducted by the Basel Committee in 2013 [16]. The purpose of this exercise was to identify the degree of "practise-based" variation in risk weights for credit risk across major international banks using the IRB approach. A total of 32 participating banks calculated a total capital requirement for the same hypothetical portfolio, using their internal models to determine the risk weights. The study found a high degree of consistency in banks' assessment of the relative riskiness of the different obligors in the portfolio. However, considerable variation in the *levels* of the estimated risk was found, as expressed by the banks' estimates for probability of default (PD) and loss given default (LGD). A separate survey of bank practices for estimating exposure at default (EAD) also found significant differences.

The Basel Committee is currently consulting on a package of proposed policy measures that will improve the comparability of risk weights that are calculated using the IRB approach for credit risk [17]. These proposals include removing the option to use the IRB approach for certain exposures, and added constraints on model parameters for exposures still eligible for the IRB approach. Loans to banks and other financial institutions, equities and large corporates (belonging to consolidated groups with total assets exceeding 50 billion euros) is proposed to be subject to the standardised approach. Banks, other financial institutions and large corporates are usually considered to be low-default exposures, which makes it difficult to obtain enough observations to reliable estimate the model parameters. For equities, the Committee argues that it is unlikely that banks will have specific knowledge concerning the issuer over and above public data.

The Committee proposes applying floors to PD, LGD and credit conversion factors used to determine EAD for off-balance sheet items. The proposed floors for corporate exposures are a 0.05 % PD floor, a 25 % LGD floor for unsecured exposures and a 0-20 % LGD floor for secured exposures, depending on collateral type. When choosing the proposed levels for the parameter floors, the Committee has taken into account that banks could get incentives to shift their exposures to higher risks to avoid the effect of the parameter floors. Consistency with the standardised approach has also been a priority. In addition to the parameter floors, the Committee has also proposed greater specification of parameter estimation practices [17].

Chapter 3

Credit Risk Modelling

In this chapter we start with a general introduction to credit risk modelling, then shifting focus to the specific model choices made by the Basel Committee when deriving a mathematical function for calculating regulatory capital under the IRB approach.

Credit risk models encompass all of the policies, procedures and practices used by a bank in estimating a credit portfolio's probability density function (PDF) of future credit losses. Such models contribute to an improvement in a bank's overall ability to identify, measure and manage risk. The models also allow banks to analyse marginal and absolute contributions to risk, and are used for determining concentration and exposure limits within a portfolio. The initial motivation for developing credit risk models was the desire to produce quantitative estimates of the amount of economic capital needed for banks to absorb substantial losses. In addition to economic capital allocation, the applications of credit risk models include risk-based pricing and performance evaluation using risk-adjusted return on capital. As credit risk models have gained a large role in banks' risk management processes, they are now also utilized for supervisory and regulatory purposes [10].

Correlations are vital in assessing risk at the portfolio level since they capture the interaction of losses on individual credits. Nearly all credit risk models assume that these correlations are driven by one or more *risk factors* that represent various influences on the credit quality of the borrower. Such risk factors can include industry, geographic region and the general state of the economy. The assumptions about the statistical processes driving these risk factors determine the overall mathematical structure of the model and the shape of the PDF. In some cases, a specific functional form for the PDF is assumed and the empirical results are calculated analytically. In other cases, simulation of the underlying risk factors is used to numerically provide a PDF [18].

Stress tests aim to overcome some of the uncertainties in credit risk models by specifying particular economic scenarios and judging the adequacy of bank capital against those scenarios, regardless of the probability that such events may occur. Scenarios covered include deterioration in credit ratings or market spreads, changes in LGDs, shifts in default probabilities and changes in correlation structures [10].

Models adopting a *bottom-up approach* measure credit risk for each loan based on an explicit evaluation of the creditworthiness of each borrower. A *top-down approach* measures credit risk for buckets of loans with similar risk profiles, where loans within each bucket are treated as statistically identical [10].

3.1 Risk Parameters

This section defines the main parameters used to model credit risk.

Probability of default (PD) is the probability that a borrower will be unable to meet the debt obligations. This probability is defined for a particular time horizon, typically one year.

Exposure at default (EAD) is the lender's outstanding exposure to the borrower in case of default.

Loss given default (LGD) is the lender's likely loss in case of default. Usually stated as a percentage of the EAD.

The *expected loss* (EL) is the average credit loss a bank can expect on its credit portfolio over the chosen time horizon. The expected loss is calculated as the mean of the loss distribution, and is typically covered by provisioning and pricing policies [19]. The expected loss of a single loan can be calculated as follows:

$$EL = PD \cdot EAD \cdot LGD. \tag{3.1}$$

Banks typically express the risk of a portfolio with the *unexpected loss* (UL), which is the amount by which the actual credit loss exceeds the expected loss. The economic capital held to support a bank's credit risk exposure is usually determined so that the estimated probability of unexpected loss exceeding economic capital is less than a target insolvency rate. The potential unexpected loss which is judged too expensive to hold capital against, is called *stress loss*, and leads to insolvency. This is illustrated in Figure 3.1. The PDF of future credit losses is the basis for calculating the unexpected loss, and the target insolvency rate is chosen so that the resulting economic capital will cover all but the most extreme events.

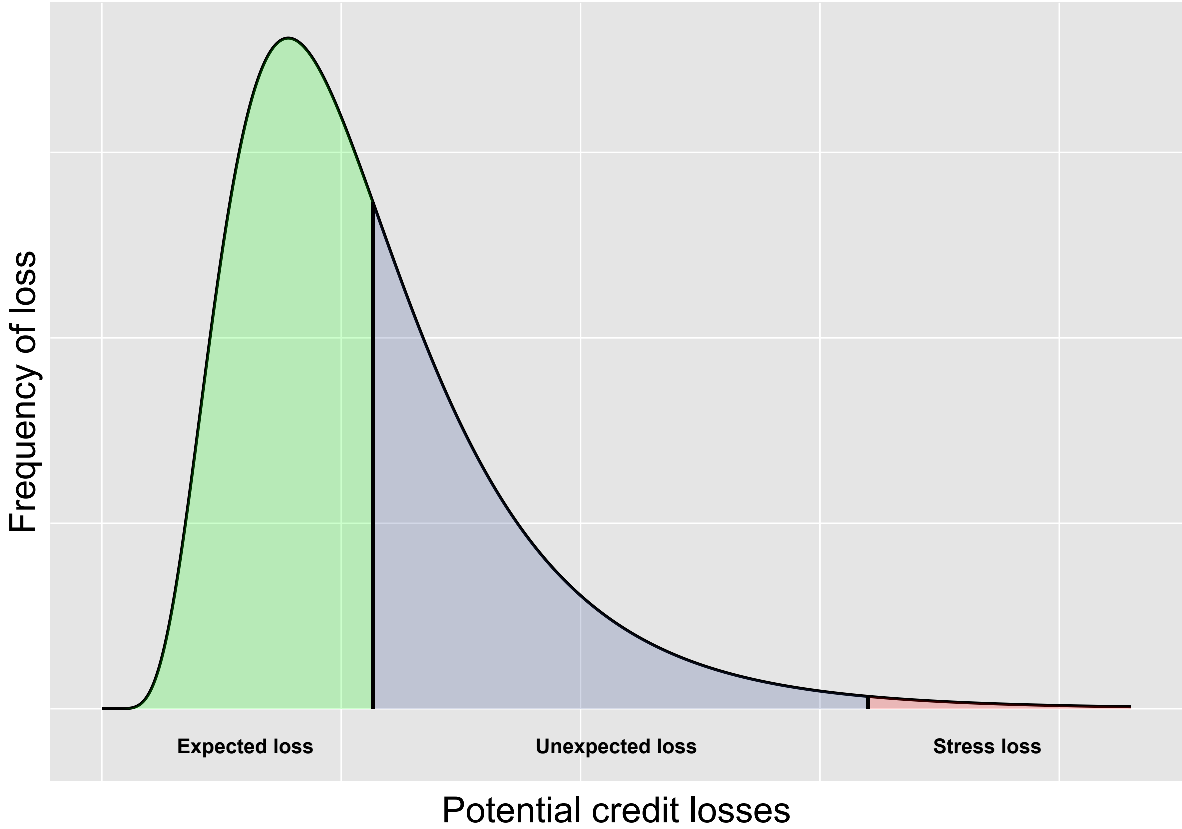


Figure 3.1: The three different types of loss in credit risk modelling.

3.2 Models Adopted by the Basel Committee

This section presents the models that have formed the basis for the derivation of the Basel Committee's mathematical function for calculating regulatory capital under the IRB approach. The Committee's preferred risk measure is also defined.

3.2.1 The Merton Model

Merton [20] models the default probability of a firm based on its assets and liabilities at the end of a given time period. The model assumes that the firm's debt is given by a zero-coupon bond with face value B and maturity T. The value of the firm's equity and debt at a given time t is denoted by S_t and B_t. Merton assumes that the firm's equity does not receive dividend, and that no new debt is issued. Also omitting transaction costs and taxes, the value of the firm's assets is given by

$$V_t = S_t + B_t, \qquad 0 \le t \le T.$$

The payout for the shareholders and debtholders at time T is given by

$$S_T = \max(V_T - B, 0) = (V_T - B)^+, \qquad B_T = \min(V_T, B) = B - (B - V_T)^+.$$

The value of the firm's equity may thus be considered as the payout of a European call option on its assets, with a strike price equalling its debt. The value of the firm's debt may in the same way be considered as the debt amount plus a short European put option on its assets, with a strike price equalling its debt [21].

Merton further assumes that the value of the firm's assets follows a diffusion model of the form

$$dV_t = \mu_V V_t dt + \sigma_V V_t dW_t, \qquad 0 \le t \le T, \tag{3.2}$$

where $\mu_V \in \mathbb{R}$ and $\sigma_V > 0$ are constants, and W_t is a standard Brownian motion. The solution of the stochastic differential equation (3.2) for time T with initial value V_0 can be found analytically: $V_T = V_0 \exp\left(\left(\mu_V - \frac{1}{2}\sigma_V^2\right) T + \sigma_V W_T\right)$. This implies that

$$\ln V_T \sim N\left(\ln V_0 + \left(\mu_V - \frac{1}{2}\sigma_V^2\right) T, \sigma_V^2 T\right).$$

The probability of default is thus given by

$$P(V_T \le B) = P(\ln V_T \le \ln B) = \Phi\left(\frac{\ln(B/V_0) - (\mu_V - \frac{1}{2}\sigma_V^2)T}{\sigma_V \sqrt{T}}\right), \tag{3.3}$$

where Φ is the cumulative distribution function of the standard normal distribution.

The Merton model provides a useful context for modelling credit risk, and is used in practical implementations by many financial institutions. However, it has admittedly also some drawbacks. For most firms the assumption that the financing consists of a one-year zero coupon bond is an oversimplification. Also, the assumption of normally distributed losses can lead to an underestimation of the potential risk in a loan portfolio. The most important shortcoming of the Merton model might be that the firm's value is not observable, which makes assigning values to it and its volatility problematic [22].

3.2.2 Vasicek's Loan Portfolio Model

Vasicek [23] derived a loan portfolio model in 1991, based on the Merton model. It models the probability of default *conditional* on a common risk factor. The model assumes a portfolio of n equally large loans, but he also shows that this assumption can be relaxed if the portfolio consists of a large number of loans where no single loan is too dominant. The probability of default on a single loan is denoted p, and is given by (3.3). The value of the borrower's assets is assumed to have a pairwise correlation ρ. All loans have the same maturity T. L_i denotes the gross loss (before recoveries) on the i-th loan, so that $L_i = 1$ if the i-th borrower defaults and $L_i = 0$ otherwise. The gross loss ratio for the portfolio is given by $L_r(n) = \frac{1}{n}\sum_{i=1}^{n} L_i$.

Vasicek uses the diffusion model (3.2), which means that the value of the assets of firm i at time T can be expressed as $\ln V_i(T) = \ln V_i + \mu_i T - \frac{1}{2}\sigma_i^2 T + \sigma_i \sqrt{T} X_i$. The variables X_i are jointly standard normal distributed with equal pairwise correlations ρ, and can therefore be represented as [1]

$$X_i = Y\sqrt{\rho} + Z_i\sqrt{1-\rho}, \tag{3.4}$$

where $Y, Z_1, Z_2, \ldots, Z_n$ are mutually independent standard normal distributed variables. The variable Y can be interpreted as a portfolio common factor, such as an economic index, over the interval $(0, T)$. The first term in (3.4) is the firm's exposure to the common factor and the second term represents the idiosyncratic risk [23].

On this basis Vasicek proves that when Y is constant, the probability for loss on a single loan conditional on Y is given by

$$p(Y) = P[L_i = 1|Y] = \Phi\left(\frac{\Phi^{-1}(p) - Y\sqrt{\rho}}{\sqrt{1-\rho}}\right). \tag{3.5}$$

A drawback with the Vasicek model is that it is a purely static, one-period model. In practice, portfolio default rates move in a predictable way from period to period, and exhibit quite well-defined time series properties. Ignoring these properties in risk and capital calculations may lead to an erroneous perception of true risk [24].

[1] See [21] for details, example 3.34, page 104.

3.2.3 Value at Risk

Value at Risk (VaR) describes the risk of holding a portfolio over a given time period. As the term indicates, VaR is a risk measure defined as the largest possible loss over the time period, provided that the probability for an even larger loss does not exceed a certain level. A formal definition [21] is the following:

Definition 1 (Value at Risk). *Given a confidence level $q \in (0,1)$, the VaR of a portfolio for the confidence level q is given by the smallest number l such that the probability for the loss L exceeding l is not larger than $(1-q)$:*

$$VaR_q(L) = \inf\left\{l \in \mathbb{R} : P(L > l) \le 1-q\right\} = \inf\left\{l \in \mathbb{R} : F_L(l) \ge q\right\},$$

where $F_L(l) = P(L \le l)$ is the cumulative distribution function of the loss variable.

This means that VaR is simply a quantile of the loss distribution. Note that by definition, VaR does not give any information about the size of the losses that occurs with a probability less than $1-q$. This can be problematic if the loss distribution is heavy-tailed.

3.2.4 The ASRF Model

The *Asymptotic Single Risk Factor* (ASRF) model was developed by Michael B. Gordy in 2003. As the name suggests it models risk using only one risk factor, which may be interpreted as reflecting the state of the global economy. The model is constructed to be *portfolio-invariant*, so that the marginal capital requirement for a loan does not depend on the properties of the portfolio in which it is held [25]. A capital charge to a loan may therefore be based only on its own characteristics. This makes the model applicable for a wide range of countries and institutions, and thus very suitable for regulatory purposes.

The ASRF model is based on two fundamental assumptions which ensure the desired portfolio-invariance:

Assumption 1. *There is only a single systematic risk factor driving correlations across borrowers.*

Assumption 2. *No exposure in a portfolio accounts for more than an arbitrarily small share of the total exposure.*

By using the strong law of large numbers, Gordy proves the following:

Proposition 1. *If Assumption 2 holds, the portfolio loss ratio conditional on $X = x$ will almost surely converge to its conditional expectation as $n \to \infty$: $L_r(n) - E[L_r(n)|x] \to 0$.*

Proposition 1 implies that as the exposure share of each asset in the portfolio goes to zero, the idiosyncratic risk in portfolio loss is diversified away. In the limit, the loss ratio converges to a fixed function of the systematic risk factor X [25].

Let $\alpha_q(X)$ denote the q^{th} quantile of the systematic risk factor X. In other words, $\alpha_q(X)$ denotes the value at risk at confidence level q. The quantiles of $E[L_n|x]$ take on a particularly simple and desirable asymptotic form when Gordy imposes an additional restriction:

Assumption 3. *The systematic risk factor X is one-dimentional.*

Gordy proves that Assumption 3 yields the following result:

Proposition 2. *If Assumption 3 is satisfied, the following applies for $n > n_0$:*
$\alpha_q(E[L_n|X]) = E[L_n|\alpha_q(X)]$.

Building on Proposition 2, Gordy further proves the following:

Proposition 3. *If Assumptions 1-3 hold, then:*
$P\left(L_n \leq E\left[L_n|\alpha_q(X)\right]\right) \to q$ *and* $|\alpha_q(L_n) - E\left[L_n|\alpha_q(X)\right]| \to 0$,
where the last part may also be written as $|VaR_q(L_n) - VaR_q(E[L_n|X])| \to 0$.

Prepositions 2 and 3 are the core of the Basel Committee's capital requirement function. It presents a portfolio invariant rule to determine capital requirements by taking the exposure-weighted average of the individual assets' conditional expected losses. However, the portfolio invariance comes along with some drawbacks as it makes recognition of diversification effects very difficult. Judging whether a loan fits well into an existing portfolio requires the knowledge of the portfolio decomposition and therefore contradicts portfolio invariance. Thus the ASRF model is based on the assumption of a well diversified portfolio [22].

3.3 The Basel Committee's Capital Requirement Function

Basel II made it possible for banks to use internal risk models to estimate PD, EAD and LGD [8]. These estimates are used as input parameters for a mathematical function that returns the capital requirement for the specific loan. This capital requirement function is based on Gordy's ASRF model, which allows the use of a bottom-up approach, as it does not depend on portfolio composition. This makes the resulting capital requirement function applicable for a wide range of countries and institutions, which of course is an important prerequisite for a global regulatory practice. As well-diversified banks is a main assumption of the ASRF model, banks are expected to address their deviations from this ideal under the Pillar 2 framework [19].

The probability of default conditional on the systematic risk factor X is calculated by Vasicek's adaptation of the Merton model (3.5):

$$PD(X) = \Phi\left(\frac{\Phi^{-1}(PD) - X\sqrt{R}}{\sqrt{1-R}}\right),$$

where R is the Basel Committee's notation for the correlation constant.

By choosing a realization of the systematic risk factor equal to the q^{th} quantile $\alpha_q(X)$, we obtain the following expression as X is assumed to be normally distributed:

$$PD(\alpha_q(X)) = PD(\Phi^{-1}(1-q)) = PD(-\Phi^{-1}(q)) = \Phi\left(\frac{\Phi^{-1}(PD) + \Phi^{-1}(q)\sqrt{R}}{\sqrt{1-R}}\right). \tag{3.6}$$

The capital requirement is expressed as a percentage of the exposure at default. The expected loss for each loan is thus calculated with (3.1) without the EAD-factor. Based on Proposition 2, the q^{th} quantile of the expected loss conditional on the systematic risk factor X is calculated as follows:

$$\alpha_q(E[L|X]) = E[L|\alpha_q(X)] = PD(\alpha_q(X)) \cdot LGD. \tag{3.7}$$

The LGD value used in (3.7) must reflect economic downturn conditions in circumstances where loss severities are expected to be higher during cyclical downturns than during typical business conditions [19]. This so-called *"downturn"* LGD value is not computed with a mapping function similar to that used for the PD value. Instead, the Basel Committee has decided to let the banks provide downturn LGD values based on their internal assessments. The reason for this is the evolving nature of bank practices in the area of LGD quantification.

The Basel Committee's capital requirement only considers the unexpected loss. As the ASRF model delivers the entire capital amount, the expected loss $PD \cdot LGD$ has to be subtracted from (3.7). When finally inserting (3.6) for $PD(\alpha_q(X))$, we arrive at the Basel Committee's capital requirement function:

$$K = LGD \cdot \Phi\left(\frac{\Phi^{-1}(PD) + \Phi^{-1}(0,999) \cdot \sqrt{R}}{\sqrt{1-R}}\right) - PD \cdot LGD, \tag{3.8}$$

where the Committee has chosen the confidence level $q = 0.999$. This means that losses on a loan should exceed the capital requirement only once in a thousand years. The reason why the confidence level is set so high is partly to protect against inevitable estimation error in the banks' internal models [19].

Under the Basel III regulation, banks must multiply (3.8) by a factor of 1.06, based on an impact study of Basel II conducted by the Basel Committee [26]. The capital requirement function is also multiplied by an adjustment factor for the maturity of the loan, as long-term credits have higher risk than short-term credits. The maturity adjustment MA is given by

$$MA = \frac{1 + (M - 2.5) \cdot b(PD)}{1 - 1.5 \cdot b(PD)},$$

where M is years to maturity and $b(PD) = (0.11852 - 0.05478 \cdot \ln(PD))^2$.

As mentioned above, R is the loan's correlation with the systematic risk factor, and it is determined from information about the borrower. For loans to states, institutions and large enterprises (annual revenues above 50 million euros) [19] the following formula applies:

$$R = 0.24 - 0.12\left(\frac{1 - e^{-50 \cdot PD}}{1 - e^{-50}}\right). \tag{3.9}$$

We see that the R value will lie in the interval $[0.12 - 0.24]$. If the enterprise's annual revenue is less than 5 million euros, the R value is decreased by 0.04. If the annual revenue is between 5 and 50 million euros, the R value is decreased by $0.04 \cdot (1 - (S-5)/45)$, where S is the annual revenue.

For retail exposures the maturity adjustment MA is not included, and the correlation R is calculated as follows:

$$R = 0.16 - 0.13\left(\frac{1 - e^{-35 \cdot PD}}{1 - e^{-35}}\right). \tag{3.10}$$

The R value for residential mortgages and qualifying revolving retail exposures is not calculated by (3.10), but are defined as constant values of 0.15 and 0.04, respectively [19].

For financial institutions whose total assets are greater than 70 billion euros, or is lacking supervision, the R value is multiplied by 1.25 [12].

As the capital requirement (3.8) is expressed as a percentage of total exposure, one must multiply by EAD to get the capital requirement stated as a money amount. As this amount shall constitute 8 % of the risk-weighted assets, the risk-weighted assets are calculated by multiplying the capital requirement with 12.5 $(1/0.08)$.

Thus, to conclude, the risk-weighted assets are calculated as:

$$RWA = 12.5 \cdot K \cdot EAD.$$

Chapter 4

Coherent Risk Measure

In this chapter we introduce the concept of *coherent* risk measures. We explain why value at risk is not a coherent risk measure, and further elaborate on its undesirable properties as a consequence of this fact. Thereafter, the coherent risk measure *expected shortfall* is proposed as an alternative to value at risk. Finally, we derive a version of the Basel Committee's capital requirement function that is based on expected shortfall.

In the paper *Thinking Coherently*[27] from 1997, Artzner et al. defined what properties a statistic of a portfolio should have in order to be considered a sensible risk measure. The following four properties should be fulfilled by such a coherent risk measure:

Definition 2 (Coherent risk measure)**.** *A risk measure ρ assigns a number $\rho(X)$ to a random variable X representing asset returns (positive numbers for losses). This number indicates the riskiness of the position. For each pair of random variables X and Y (dependent or not) as well as for each number n and for each positive number t, a coherent risk measure must satisfy all the following properties:*

(i) $\rho(X+Y) \leq \rho(X)+\rho(Y)$ *[sub-additivity]*

(ii) $\rho(t \cdot X) = t \cdot \rho(X)$ *[homogeneity]*

(iii) $\rho(X) \leq \rho(Y)$, *if* $X \leq Y$ *[monotonicity]*

(iv) $\rho(X-n) = \rho(X)-n$ *[risk-free condition]*

Property (i) ensures that the risk measure behaves reasonably when adding two positions. It reflects the positive effect of diversification. Property (ii) states that multiplied loss means multiplied risk. Properties (i) and (ii) together imply the convexity of the function ρ [27]. Property (iii) simply states that portfolios with smaller losses are assigned smaller risk. Property (iv) states logically that any amount of certain earnings/losses results in the risk decreasing/increasing by the same amount.

These four properties of coherence are all logical, and corresponds to most people's concept of risk. Acerbi and Tasche [28] have even stated that speaking of non-coherent risk measures is useless and dangerous. If a risk measure is not coherent, they choose to not call it a risk measure at all. For them, the coherence properties define the concept of risk itself via the characterization of the possible operative ways to measure it.

4.1 Value at Risk

Value at risk is not a coherent risk measure, as it has been shown [27] that it is not sub-additive in general. Thus, a merger of two portfolios may have a greater VaR than the sum of the VaR of the individual portfolios. This contradicts basic diversification theory, and is considered as one of the biggest flaws of VaR. Another property of VaR that is often pointed out as a weakness is that it does not give any information about the size of the losses that occurs with a probability less than $1-q$. This can be particularly problematic if the loss distribution is heavy-tailed, and is commonly referred to as tail risk. Assets with higher potential for large losses may appear less risky than assets with lower potential for large losses.

However, VaR is sub-additive if the loss distribution belongs to the elliptical distribution family and has finite variance, making it a coherent risk measure in these cases [29]. This includes the normal distribution, Student's t distribution (for $v > 2$) and Pareto distribution (for $\alpha > 2$). For these distributions, VaR becomes a scalar multiple of the distribution's standard deviation, which satisfies sub-additivity.

Even though value at risk is not sub-additive in general, it still remains the most widely used risk measure. The reason seems to be that its practical advantages are perceived to outweigh its theoretical shortcomings. Value at risk is considered to have smaller data requirements, easier backtesting and in some cases easier calculation than alternative risk measures [30]. Value at risk is also popular because of its conceptual simplicity. The economic capital calculated by VaR at a confidence level q corresponds to the capital needed to keep the firm's default probability below $100 \cdot (1-q)$ %.

4.2 Expected Shortfall

Artzner et al. [27] proposed an alternative risk measure for value at risk, which satisfied all four coherence axioms. This risk measure is called *tail conditional expectation* (TCE), and is closely related to value at risk:

Definition 3 (Tail Conditional Expectation)**.** *Given a confidence level* $q \in (0,1)$*, tail conditional expectation is defined as*

$$TCE_q(L) = E\left[L | L \geq VaR_q(L)\right],$$

where $VaR_q(L)$ *is the value at risk at the same confidence level.*

The tail conditional expectation is thus the expectation of loss, given that the loss is beyond the VaR level. However, the TCE is only a coherent risk measure when restricted to continuous distribution functions. For general distributions, TCE may violate sub-additivity. Acerbi and Tasche [28] later proposed a more advanced version of TCE that is coherent also for general distributions:

Definition 4 (Expected Shortfall)**.** *Given a confidence level* $q \in (0,1)$*, expected shortfall is defined as*

$$ES_q(L) = E\left[L | L \geq VaR_q(L)\right] + \left(E\left[L | L \geq VaR_q(L)\right] - VaR_q(L)\right)\left(\frac{P\left[L \geq VaR_q(L)\right]}{1-q} - 1\right),$$

where $VaR_q(L)$ *is the value at risk at the same confidence level.*

When $P\left[L \geq \text{VaR}_q(L)\right] = 1 - q$, as is always the case if the probability distribution is continuous, the last term from Definition 4 vanishes and it is easy to see that the ES equals the TCE in this case.

By using the definition of conditional probability and a change of variables, the expected shortfall can also be written as an integral over the VaR values for all confidence levels $u \geq q$:

$$\text{ES}_q(L) = \frac{1}{1-q} \int_{u=q}^{1} \text{VaR}_u(L)\, du. \tag{4.1}$$

From Definition 4 and (4.1) it is clear that expected shortfall does not have the same degree of tail risk as value at risk. Unlike VaR, ES can distinguish between two distributions of future net worth that have the same quantile but differ otherwise.

A critique of ES is the fact that tail behaviour is taken into account through an averaging procedure. Medina and Munari [31] claim that averages are poor indicators of risk, thus making ES a potentially deceiving measure of risk.

4.2.1 Capital Requirement Function Using Expected Shortfall

The ASRF model is also applicable for expected shortfall. Gordy has proved an equivalent of Proposition 3 regarding expected shortfall [25]:

Proposition 4. *If Assumptions 1-3 hold, then* $|ES_q(L_n) - ES_q(E[L_n|X])| \to 0$.

Proposition 4 implies that ES-based capital charges are portfolio invariant under the same assumptions as VaR-based capital charges. It is thus possible to derive a version of the Basel Committee's capital requirement function (3.8) that is based on expected shortfall [32].

Recall from Chapter 3 that the expected loss conditional on the q^{th} quantile of the systematic risk factor X is given by:

$$VaR_q(L) = LGD \cdot \Phi\left(\frac{\Phi^{-1}(PD) + \Phi^{-1}(q)\sqrt{R}}{\sqrt{1-R}}\right). \tag{4.2}$$

To derive a corresponding equation for expected shortfall, we apply (4.1) to (4.2):

$$ES_q(L) = \frac{1}{1-q}\int_{u=q}^{1} LGD \cdot \Phi\left(\frac{\Phi^{-1}(PD) + \Phi^{-1}(u)\sqrt{R}}{\sqrt{1-R}}\right) du. \tag{4.3}$$

Using the substitution $x := -\Phi^{-1}(u)$ so that $du/dx = -\phi(x)$,$x(u = q) = -\Phi^{-1}(q)$ and $x(u = 1) = -\Phi^{-1}(1) = -\infty$ [32], (4.3) leads to:

$$\begin{aligned} ES_q(L) &= \frac{LGD}{1-q}\int_{x=-\Phi^{-1}(q)}^{-\infty} \Phi\left(\frac{\Phi^{-1}(PD) - x\sqrt{R}}{\sqrt{1-R}}\right) \cdot (-1) \cdot \phi(x)\,dx \\ &= \frac{LGD}{1-q}\int_{x=-\infty}^{-\Phi^{-1}(q)} \Phi\left(\frac{\Phi^{-1}(PD) - x\sqrt{R}}{\sqrt{1-R}}\right) \cdot \phi(x)\,dx. \end{aligned} \tag{4.4}$$

By applying the identity [1]

$$\int_{-\infty}^{c} \Phi(ax+b)\phi(x)dx = \Phi_2\left(\frac{b}{\sqrt{1+a^2}}, c; \frac{-a}{\sqrt{1+a^2}}\right),$$

where $\Phi_2(\cdot)$ stands for the bivariate cumulative normal distribution function, (4.4) can be expressed as

$$ES_q(L) = \frac{LGD}{1-q}\Phi_2\left(\Phi^{-1}(PD), -\Phi^{-1}(q); \sqrt{R}\right). \tag{4.5}$$

The bivariate cumulative normal distribution function is defined as

$$\Phi_2(x, y, \rho^2) = P(X \le x, Y \le y) = \int_{u=-\infty}^{x} \int_{v=-\infty}^{y} \frac{1}{2\pi\sqrt{1-\rho^2}} \cdot \exp\left(-\frac{1}{2}\frac{u^2 - 2\rho uv + v^2}{1-\rho^2}\right) dv\, du,$$

where X and Y are standard normal distributed random variables, with a correlation of ρ.

By replacing (4.2) with (4.5), the expected shortfall version of the Basel Committee's capital requirement function (3.8) becomes

$$K = \frac{LGD}{1-q}\Phi_2\left(\Phi^{-1}(PD), -\Phi^{-1}(q); \sqrt{R}\right) - PD \cdot LGD, \tag{4.6}$$

where q is the confidence level and R is the correlation factor for the systematic risk factor.

[1] See (30.c) in [33]

Chapter 5

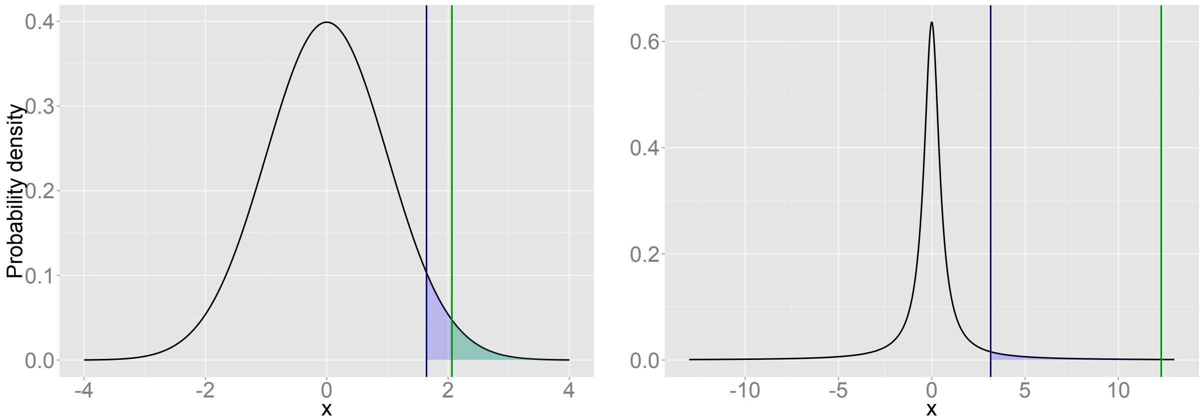
Value at Risk Versus Expected Shortfall

In January 2016, the Basel Committee published revised standards for minimum capital requirements for *market risk* [9], which include a shift from value at risk (VaR) to expected shortfall (ES) as the preferred risk measure. The Committee stated that the former market risk framework's reliance on VaR as a quantitative risk metric stems largely from historical precedent and common industry practice. This has been reinforced over time by the requirement to use VaR for regulatory capital purposes. However, the Committee recognized that a number of weaknesses have been identified with VaR, including its inability to capture tail risk [34].

The Basel Committee has currently not considered a transition from VaR to ES for measuring *credit risk*. However, as the development of credit risk models lies a few years behind the market risk models, there is reason to believe that this might be considered in a not so distant future. In this chapter we will compare VaR and ES as credit risk measures. In addition to elaborate on the implications of the fact that only one of them is a coherent risk measure, we will examine how the Basel Committee's capital requirement function is affected by the choice of its underlying risk measure. There will be a particular focus on the confidence level calibration of the ES version.

5.1 Sub-additivity and Tail Risk

As pointed out in Chapter 4, value at risk satisfies sub-additivity when the loss distribution belongs to the elliptical distribution family and has finite variance. In these cases, value at risk actually provides the same information about the tail loss as expected shortfall. The reason being that both risk measures becomes a scalar multiple of the loss distribution's standard deviation [29].



(a) Normal(0,1) (b) Cauchy(0,0.5)

Figure 5.1: Comparison of the 95 % VaR (blue) and the 95 % ES (green) for a standard normal distribution and a Cauchy(0,0.5) distribution.

Figure 5.1 shows a comparison between the 95 % VaR and 95 % ES for a standard normal distribution and the more heavy-tailed Cauchy(0,0.5) distribution. Both distributions belongs to the elliptical distribution family, but the Cauchy distribution's variance is undefined. Thus, value at risk is not sub-additive for the Cauchy distribution. Expected shortfall is by definition exceeding value at risk for equal confidence levels. The extent of the difference between these two risk measures depends on the loss distribution. We see that the difference for the Cauchy distribution is substantial compared to the difference for the normal distribution.

Yamai and Yoshiba provide a simple example [1] of how the tail risk of VaR may result in serious practical problems in credit portfolios. A modified version of this example follows: first, suppose a bank holds a credit portfolio consisting of 100 corporate loans to different firms, each with a one year default probability of 1 percent, and a recovery rate of zero (LGD=100 %). The exposure at default is $1 million for each loan. For simplicity, it is assumed that the occurrences of defaults are mutually independent. From (3.1) we have that the expected loss for each loan is $ 10000. Assuming a 1 % net lending margin ($10000), each loan is thus priced at $20000. This means that the bank earns $20000 for each firm not defaulting, while it loses $1 million for each defaulting firm. Thus, the bank loses money if more than one firm defaults in one year, making the probability of loss approximately 26 %

[1] Example 2 in [29]

$(1 - 0.99^{100} - 100 \cdot 0.99^{99} \cdot 0.01)$. As the probability of loss exceeds 5 %, the 95 % VaR for this diversified investment will have a positive value.

Second, we consider the bank investing the same total amount of $100 million in a large loan to only one of the firms. For this concentrated investment the probability of loss is only 1 % and the 95 % VaR is thus -$2 million: the loan price. As the probability of default is below 5 %, the potential of default is disregarded at the 95 % confidence level. We also observe that value at risk is not sub-additive in this case as the VaR of the diversified portfolio is larger than the VaR for the concentrated portfolio. Table 5.1 shows the value at risk and expected shortfall for both the diversified and the concentrated investment. We see that ES is able to detect the tail risk, resulting in correctly pointing out the concentrated investment as the most risky investment.

	95 % VaR	95 % ES
100 loans	$1.06 million	$1.52 million
1 loan	-$2.0 million	$18.4 million

Table 5.1: 95 % value at risk and expected shortfall for a diversified investment and a concentrated investment. Positive numbers correspond to loss, negative numbers indicate profit.

This example shows how value at risk can disregard the increase of potential loss due to credit concentration. One should therefore always ensure that credit concentration is limited by complementary measures when using VaR for risk management. In the Basel Committee's regulatory framework, this issue is addressed in Pillar 2.

5.2 Confidence Level

Since the purpose of regulatory capital requirements is to ensure that banks hold sufficient capital to withstand significant losses, a very high confidence level seems reasonable. However, other considerations offset this to some degree. The capital guidelines are meant to be *minimum* regulatory standards, and safe and prudent banks will almost certainly be expected to hold actual capital amounts higher than these minimums. If this is the case, then it would be desirable to establish a confidence level that are lower than the ones that safe and prudent banks apply for internal purposes[18]. The confidence level for value at risk is set as high as 99.9 % for credit risk, to protect against estimation errors and other model uncertainties.

The expected shortfall version of the Basel Committee's capital requirement function (4.6) was derived using the same assumptions as for the VaR version (3.8). Namely, the assumption of a normal distribution for the systematic risk factor, which leads to the loss distribution also being normal. To really benefit from a change to the more tail risk sensitive ES, one would possibly assume a more heavy-tailed loss distribution. In that case, it could be justifiable to apply a confidence level resulting in a slightly smaller capital requirement, as one could argue the increased tail risk sensitivity reduces the model risk.

Although the derived ES version of the capital requirement function is based on the same assumptions as the VaR version, the difference between the two risk measures is significant enough that the two functions behave quite differently. We now try to determine if it is possible to choose a confidence level for the ES version that makes it behave like the 99.9 % VaR version. Given the definition of ES, this confidence level must be lower than 99.9 %. Figure 5.2 shows the ES capital charge calculated by (4.6) for confidence levels 99.5-99.9 %, compared to the 99.9 % VaR capital charge.

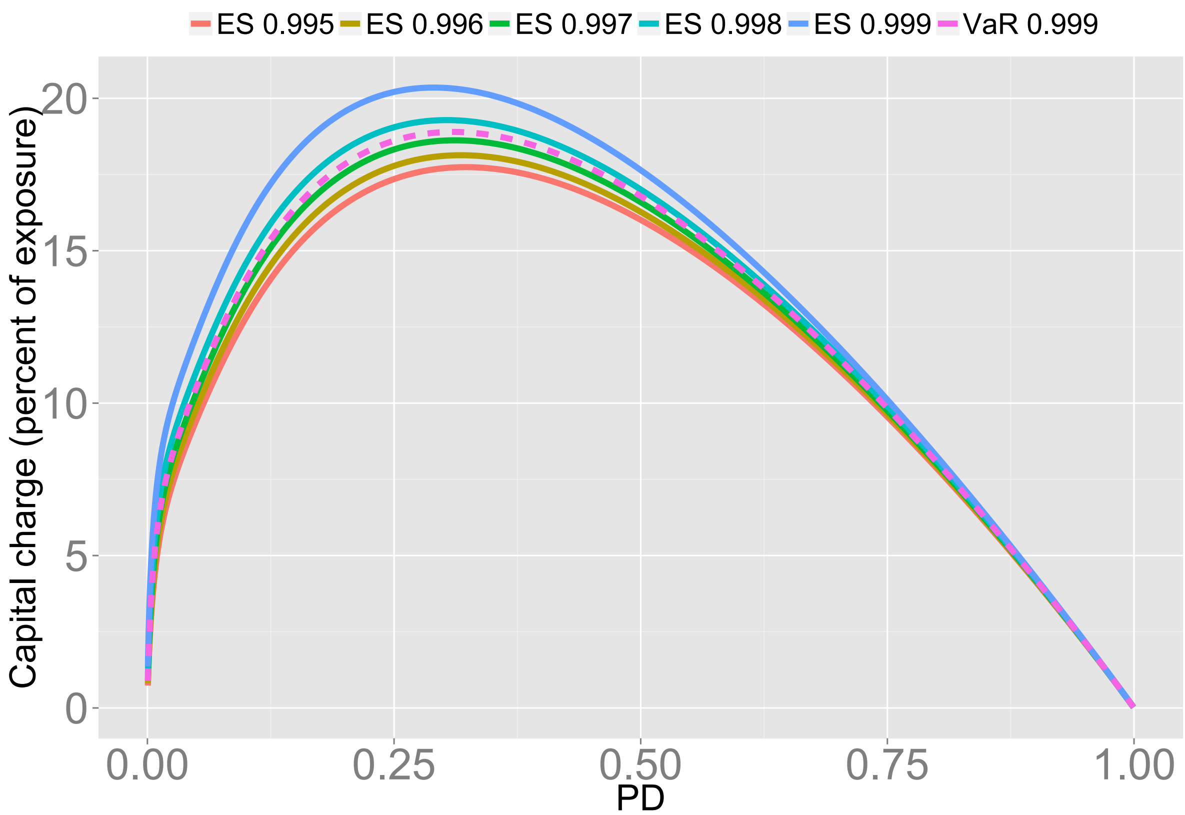


Figure 5.2: Comparison of the Basel Committee's capital requirement function (dotted line) and the expected shortfall version of this function. Five different confidence levels is used for the ES version.

Conducting a least squares fit over the interval $PD \in (0,1)$, we found that the confidence level 99.742 % made the ES version most similar to the 99.9 % VaR version. There is however considerable differences for the smallest PD values, as shown in Figure 5.3. Table 5.2 shows the resulting confidence level for least squares fits over different PD intervals. We see that the confidence level is noticeably lower for the intervals only containing small PD values. By choosing the confidence level that gives the best fit over the whole (0,1) interval, the ES version of the capital requirement function will slightly increase capital charges for loans with low probabilities of default, and slightly decrease capital charges for loans with high probabilities of default. As the Basel Committee has proposed to apply floors to the PD estimation [17], this may be considered a good thing.

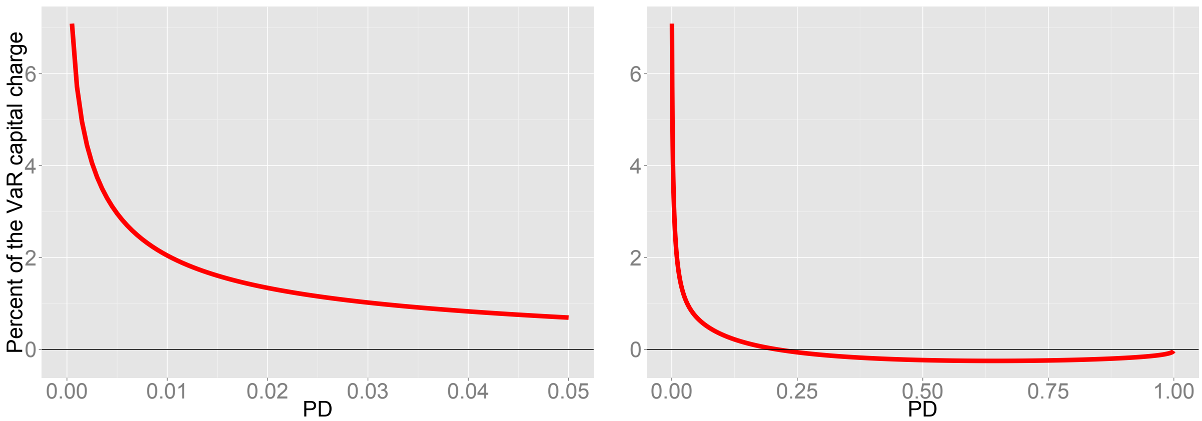


Figure 5.3: The difference between the calculated capital requirement from the ES version with confidence level 99.742 % and the standard 99.9 % VaR version. Positive y-values mean that the ES version results in a higher capital charge. The left graph gives a detailed view for small PD values, while the right graph shows the whole (0,1) interval.

PD interval	ES confidence level
(0,0.005)	99.708 %
(0,0.05)	99.726 %
(0,1)	99.742 %

Table 5.2: The confidence level that makes the ES version of the capital requirement function most similar to the 99.9 % VaR version, for PD values in the given interval. Found by least squares fit.

Figure 5.4 shows the difference from the 99.9 % VaR version when the ES confidence level is chosen to 99.708 %: the level that minimizes the difference for PD values less than 0.005. At this confidence level, the capital charge is still increasing for the smallest PD values and decreasing for large PD values. The change from increase to decrease do however occur at $PD \approx 0.003$, compared to $PD \approx 0.21$ for the confidence level fitted to the whole (0,1) interval.

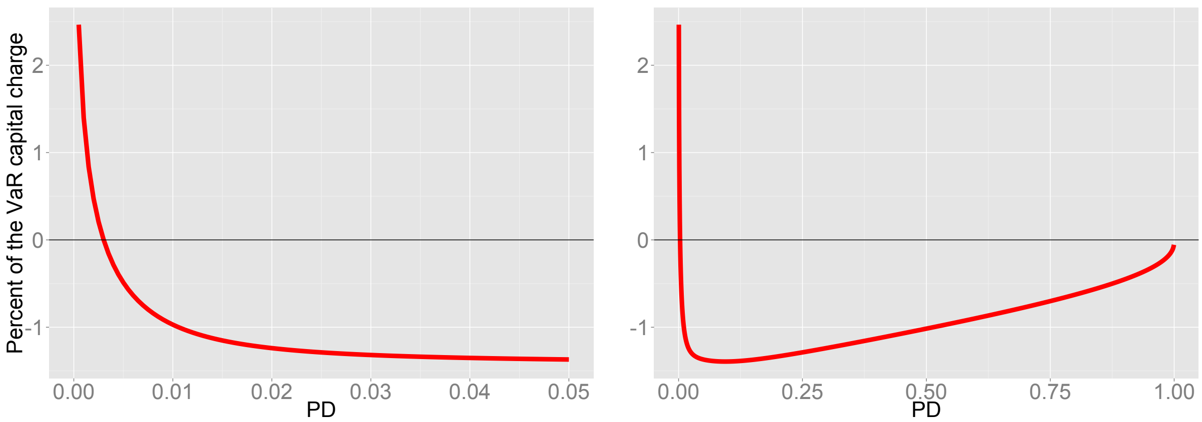


Figure 5.4: The difference between the calculated capital requirement from the ES version with confidence level 99.708 % and the standard 99.9 % VaR version. Positive y-values mean that the ES version results in a higher capital charge. The left graph gives a detailed view for small PD values, while the right graph shows the whole (0,1) interval.

5.2.1 Data Set

In addition to simulated values, this thesis will also make use of a data set. The data set contains information about corporate loans issued by a Norwegian savings bank group from March 2015 to January 2016. The data set contains about a fifth of the group's total corporate portfolio from this period, picked randomly. This amounts to a total of 109045 loans. For each loan, the data set contains numbers for EAD, LGD and PD. The loans' correlation to the systematic risk factor are also included.

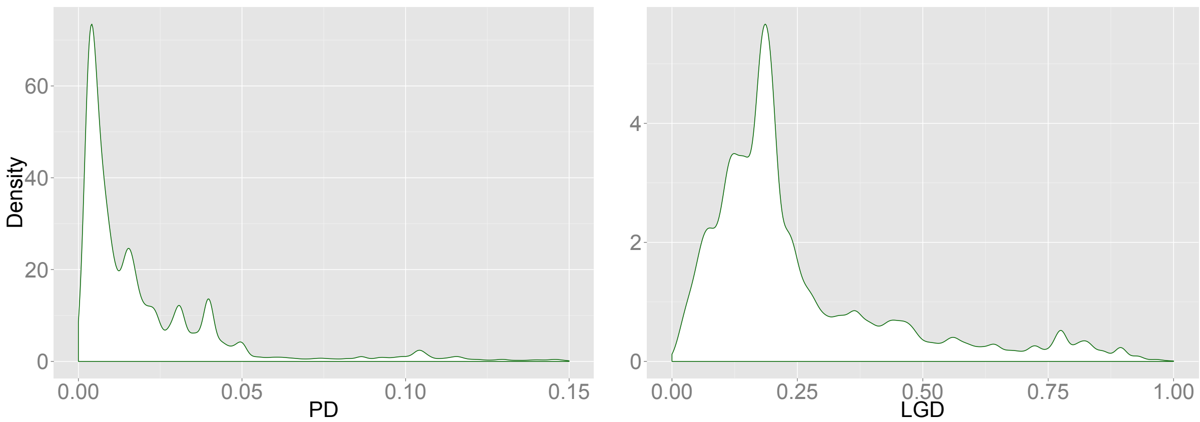


Figure 5.5: The distribution of LGD and PD values in the data set.

Figure 5.5 provides some insight about the data set, by displaying density plots of the LGD and PD values. We see that most of the loans are deemed to have low risk. In fact, 90.8 % of the loans have been assigned a probability of default of 0.05 or less. Only 2.1 % of the loans have a PD value greater than 0.15. The majority of the values for loss given default is also in the low end of the scale, with 68 % of the loans having a LGD value of 0.25 or less. There are however also a substantial number of loans that have high LGD values, unlike what is the case for the PD values.

Figure 5.6 shows a histogram of the capital charges calculated using the loan parameters from the data set. We see that the 99.742 % expected shortfall version results in a smaller proportion of loans in the left part of the histogram than for the 99.9 % VaR version. This corresponds well with Figure 5.3. In total, the 99.742 % ES results in a 1.4 % greater capital charge for the loans in the data set. However, the PD values from the data set are not uniformly distributed on the interval (0,1). Since most of the PD values are 0.05 or less, we see from Table 5.2 that a lower confidence level will probably result in a capital charge more similar to the 99.9 % VaR version. A least squares fit reveals that a 99.726 % ES version would be the closest fit to the 99.9 % VaR version for this data set, which results in a capital charge increase of 0.2 %.

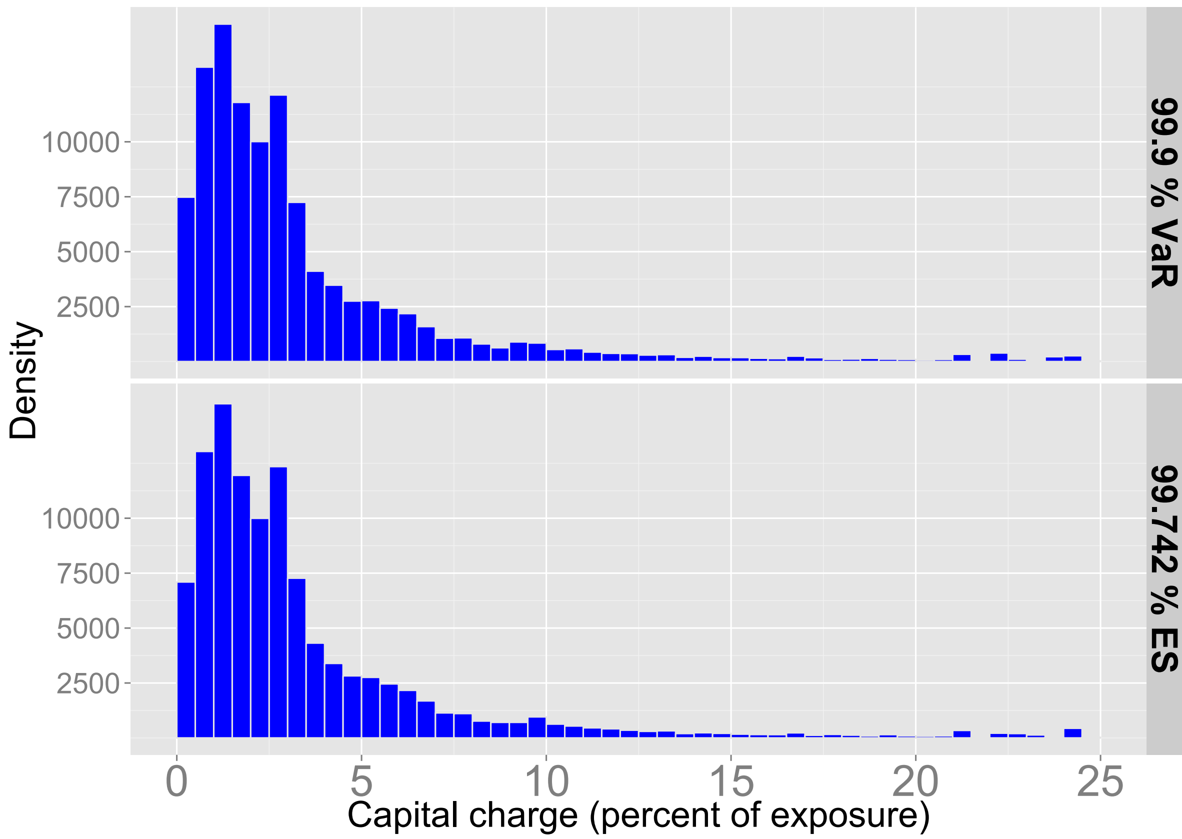


Figure 5.6: Histogram showing the distribution of the capital charges calculated using parameter values from the data set, using both 99.9 % VaR (top) and 99.742 % ES (bottom).

For market risk, the Basel Committee has replaced a 99 % VaR with a 97.5 % ES [9]. This is consistent with a least squares fit over the whole interval $PD \in (0,1)$, as seen in Table 5.3.

This table is the market risk equivalent to Table 5.2, with a 99 % confidence level used for VaR. If the Committee introduces a shift to ES for credit risk models, it is thus most likely that the chosen confidence level will be close to 99.742. As mentioned, this would result in a 1.4 % greater capital charge for the data set portfolio. It would imply a capital charge constituting 8.12 % of the risk-weighted assets, compared to the current 8 %. The increase will be smaller for a portfolio with a larger proportion of loans with high default probability. Moreover, the 0.12 % increase of the capital ratio is small compared to the various capital buffers introduced by Basel III.

PD interval	ES confidence level
(0,0.005)	96.939 %
(0,0.05)	97.225 %
(0,1)	97.463 %

Table 5.3: The confidence level that makes the ES version of the capital requirement function most similar to a 99 % VaR version, for PD values in the given interval. Found by least squares fit.

Chapter 6

Parameter Sensitivity

In this chapter we will examine how the uncertainty of the banks' parameter estimates affects the output from the Basel Committee's capital requirement function, using both value at risk and expected shortfall. This is carried out by simulating *LGD* and *PD* values. The estimation uncertainty of these two parameters is represented by the relative standard deviation of the probability distributions they are sampled from.

6.1 Simulating LGD Values

The simulation method used for the *LGD* values is based on a model for recovery rates developed by Jon Frye [35]. Frye's model adapts some of Michael Gordy's work, and lets the recovery rate depend on the systematic risk factor X. *LGD* values are related to the recovery rates R in the following simple way:

$$LGD = 1 - R.$$

Frye's model is similar to Vasicek's asset value model (3.4). The recovery rate for firm j is expressed as

$$R_j = \mu_j + \sigma q X + \sigma\sqrt{1-q^2} Z_j, \tag{6.1}$$

where Z_j is a standard normal variable independent of X. This implies that R_j is normally distributed with expected value $\mu_j + \sigma q X$ and variance $\sigma^2(1-q^2)$. The parameter q is the correlation between the recovery rate R_j and the systematic risk factor X.

Following a proposal from Schönbucher [1], we apply a logistic transformation $F(Y) = \frac{\exp(Y)}{1+\exp(Y)}$ on (6.1), to limit the R values (and thus also the LGD values) to the interval $[0,1]$.

The simulations use the same confidence level for X as the capital requirement function (3.8), but with opposite sign as the recovery rate decreases in financial recession:

$x = -\Phi^{-1}(0,999) = \Phi^{-1}(0,001)$. This gives the following distribution for the LGD values:

$$\begin{aligned} Y &\sim N\left(\mu + \sigma q \Phi^{-1}(0,001), \sigma^2\left(1-q^2\right)\right), \\ \widehat{LGD} &= 1 - R = 1 - F(Y) = 1 - \frac{\exp(Y)}{1+\exp(Y)}. \end{aligned} \tag{6.2}$$

6.2 Simulating PD Values

Over a time period, a firm either meets the loan terms or defaults. This makes it natural to model the number of default occurrences m by applying a binomial distribution:

$$m \sim Bin(n, PD),$$

where PD is the bank's estimated value for the probability of default, and n is the number of simulations.

After completing the n simulations, the probability of default is estimated as

$$\widehat{PD} = \frac{m}{n}.$$

The values for $\widehat{PD}$ will thus be centered around the expected value PD, and these simulated values are thus a representation of the uncertainty in the bank's PD estimate. The variance of $\widehat{PD}$ is inversely proportional to the number of simulations:

$$\mathrm{Var}\left(\widehat{PD}\right) = \mathrm{Var}\left(\frac{m}{n}\right) = \frac{1}{n^2}\mathrm{Var}(m) = \frac{PD(1-PD)}{n}.$$

[1] See [36], pages 147-150.

6.3 Calculation of Parameter Sensitivity

To simulate $\widehat{LGD}$ values representing different degrees of estimation uncertainty, five different values are used for σ in (6.2). The chosen values range from 0.05 to 0.45, with increments of 0.1. q is kept constant at 0.2. The $\widehat{PD}$ values are simulated in a similar way, where five different values of n represent varying degrees of estimation uncertainty. For each of the five σ values there are simulated N different $\widehat{LGD}$ values, and N different $\widehat{PD}$ values are simulated for each of the five n values. These five values for n are chosen so that the five different series of $\widehat{PD}$ values have the same relative standard deviations as the corresponding five different series of $\widehat{LGD}$ values . This is achieved by selecting n values that satisfy the following equation:

$$\frac{\sqrt{\text{Var}\left(\widehat{PD}\right)}}{PD} = \sqrt{\frac{1-PD}{nPD}} = \sigma_{\widehat{LGD}\text{rel}} \Longrightarrow n = \frac{1-PD}{\left(\sigma_{\widehat{LGD}\text{rel}}\right)^2 PD},$$

where $\sigma_{\widehat{LGD}\text{rel}}$ is the relative standard deviation of a $\widehat{LGD}$ value series, and PD is the expected value of $\widehat{PD}$. Figure 6.1 shows how the distributions of the simulated values depends on σ and n.

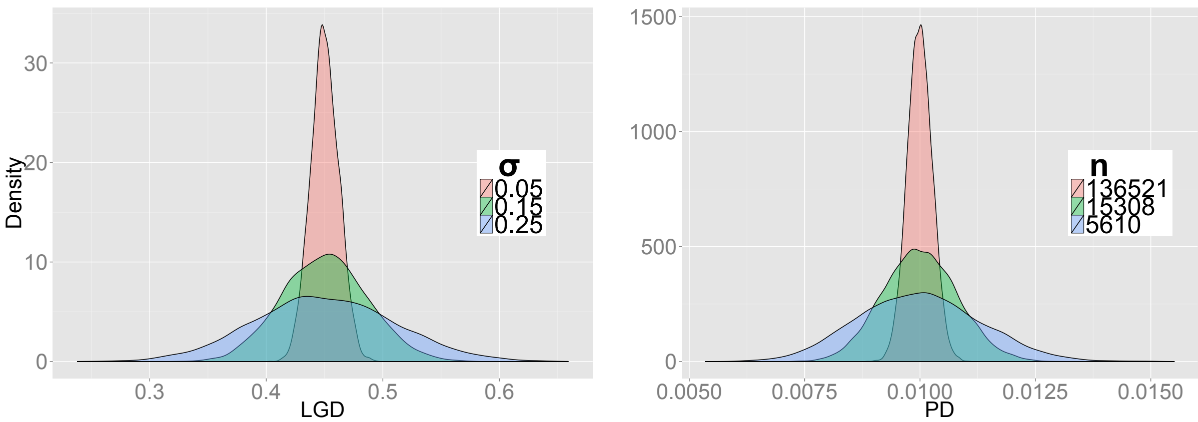


Figure 6.1: The distributions of 10000 simulated LGD and PD values, with expected values of respectively 0.45 and 0.01. The LGD values shown are simulated with three lowest σ values, and the PD values are simulated with the three n values that result in the same relative standard deviations.

The simulated $\widehat{LGD}$ and $\widehat{PD}$ values are used pairwise to calculate the corresponding capital requirement values. The loan maturity is chosen to one year, so that the adjustment factor MA equals one. The correlation factor is chosen to be calculated as for loans to firms with annual revenue above 50 million euros, given by (3.9). These choices result in the capital requirement function (3.8) taking the following form

$$\widehat{K} = \widehat{LGD} \cdot \Phi \left(\frac{\Phi^{-1}(\widehat{PD}) + \Phi^{-1}(0,999) \cdot \sqrt{0.24 - 0.12\left(\frac{1-e^{-50\cdot\widehat{PD}}}{1-e^{-50}}\right)}}{\sqrt{0.76 - 0.12\left(\frac{1-e^{-50\cdot\widehat{PD}}}{1-e^{-50}}\right)}} \right) - \widehat{PD} \cdot \widehat{LGD}. \tag{6.3}$$

The corresponding version of the expected shortfall capital requirement function (4.6) takes the form

$$\widehat{K} = \frac{\widehat{LGD}}{1-q} \cdot \Phi_2 \left(\Phi^{-1}(\widehat{PD}), -\Phi^{-1}(q); \sqrt{0.24 - 0.12\left(\frac{1-e^{-50\cdot\widehat{PD}}}{1-e^{-50}}\right)} \right) - \widehat{PD} \cdot \widehat{LGD}. \tag{6.4}$$

The capital requirements (6.3) and (6.4) are calculated for all 25 possible combinations of σ and n. At last, the relative standard deviation of the calculated capital requirements for each of these combinations are computed:

$$\sigma_{K\text{rel}} = \sqrt{\frac{\sum_{i=1}^{N}\left(K_i - \overline{K}\right)^2}{N-1}} \Big/ \overline{K}.$$

Since (6.3) and (6.4) are proportional to $\widehat{LGD}$, it is not interesting to vary the expected value of $\widehat{LGD}$, as it will only result in a linear scaling of the capital requirement's variation. The expected value of the simulated $\widehat{LGD}$ values is set to 0.45 for all the different σ values. To achieve this, for each different σ value in (6.2), we must calculate a μ which gives this desired expectation value. However, several different expectation values will be used for the $\widehat{PD}$ simulations, to see how this impacts the resulting standard deviations of the calculated capital requirements.

6.4 Results

Figure 6.2 shows the relative standard deviation of the simulated capital requirement for 99.9 % value at risk, for different expectation values for PD. For each of the different expectation values, there are 25 different values for the standard deviation of the capital requirement, which corresponds to different combinations of the five levels of uncertainty for both the PD and LGD simulations. As one would expect, the capital requirement's uncertainty increases with increasing parameter uncertainty. What we are interested in, is which of the two parameters' uncertainty that affects the capital requirement the most.

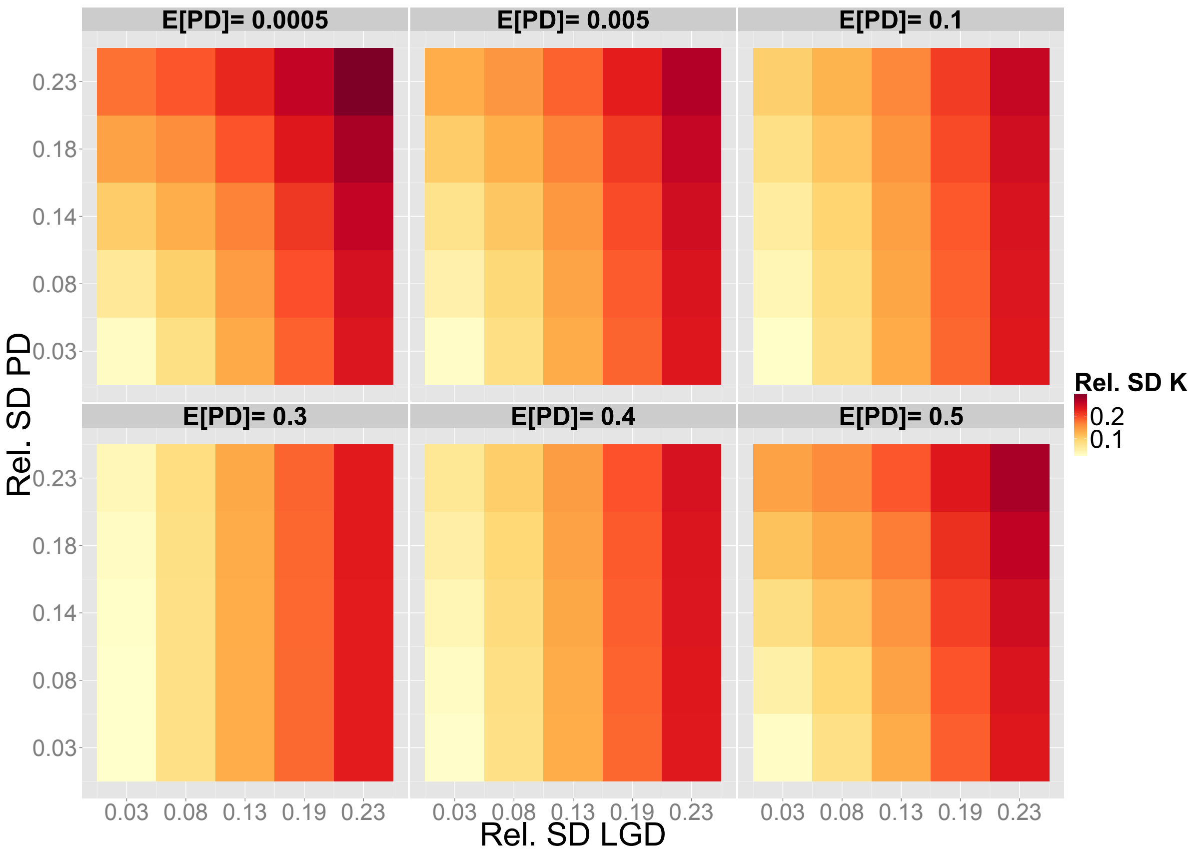


Figure 6.2: Relative standard deviation of the 99.9 % VaR capital requirement (6.3), given the relative standard deviation for PD and LGD. Calculated for six different expectation values of PD, with the expected value of LGD equal to 0.45. Using $N = 10000$ simulations for each calculation.

We see from Figure 6.2 that the uncertainty of the capital requirement is heavily influenced by the expected value of the simulated PD values, especially when the standard deviation of the simulated PD values are at a high level. When the expected probability of default is close to 0.3, the capital requirement uncertainty is almost only influenced by the uncertainty of the simulated LGD values. The uncertainty of the PD values plays a greater role when the expectation of PD is either small or above 0.4. But even for $E[PD] = 0.0005$ the LGD uncertainty is most influential, as we see that the rightmost column is a slightly darker red than the upper row. However, when the expected value of the probability of default exceeds 0.7, the PD uncertainty is extremely influential, and the standard deviation of the capital requirement increases significantly. This can be seen in Figure 6.3, where the expected value of the simulated PD values are 0.8. Note that the colors in this figure correspond to larger relative standard deviations than in Figure 6.2.

Looking at (6.3) it is clear that $e^{-50\cdot\widehat{PD}}$ is the part of the capital requirement function that

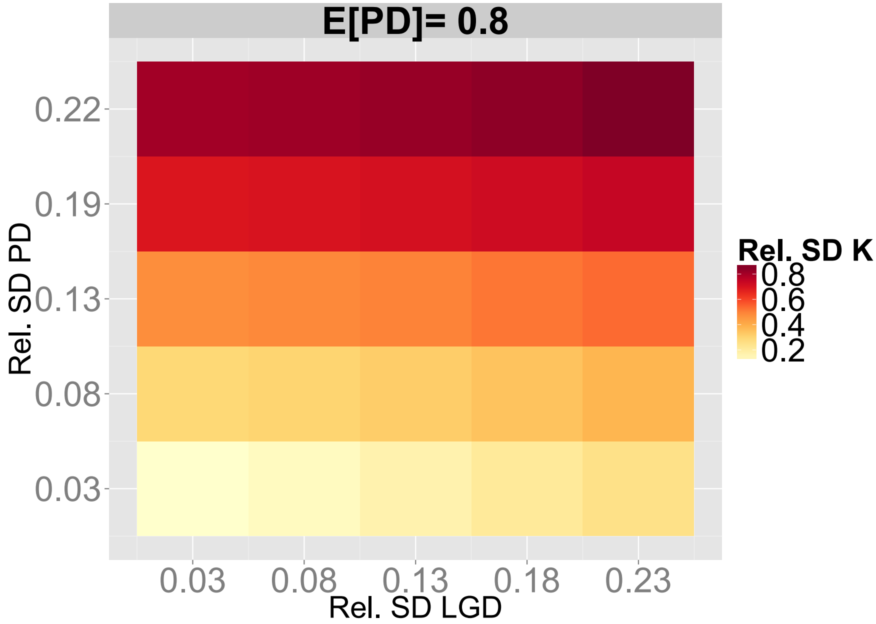


Figure 6.3: Relative standard deviation of 99.9 % VaR capital requirement (6.3), given the relative standard deviation for PD and LGD. Calculated for $E[PD] = 0.8$ and $E[LGD] = 0.45$, using $N = 10000$ simulations.

explains the influence of the PD values' uncertainty when the expectation of PD is small. For expectation values of PD close to zero, this part of the function is sensitive to very small changes in the PD value. This sensitivity gradually becomes smaller for larger PD values, and for PD values larger than 0.1 this part of the function will remain approximately constant. The function part $\Phi^{-1}(\widehat{PD})$ is particularly sensitive when the expected value of PD is close to zero or one.

The major impact on the capital requirement's uncertainty for large expectation values of PD is due to the last term in (6.3). For large PD values, the value of the last term becomes large enough so that its variation affects the variation of the whole function.

When the same simulation method was carried out using expected shortfall, the confidence level was chosen to 99.742 %, as it was shown in Chapter 5 that this confidence level results in the capital requirement closest to the 99.9 % VaR. For large expected values of PD, there were virtually no difference in the capital requirement's uncertainty between the ES and VaR approach. For smaller expected values of PD the ES approach resulted in reduced uncertainty, as shown in Figure 6.4. We see that the relative reduction is largest when the LGD uncertainty is low and the PD uncertainty is high. This reduction is however only a few percent, so there is not that much of a difference between the two approaches regarding the parameter sensitivity.

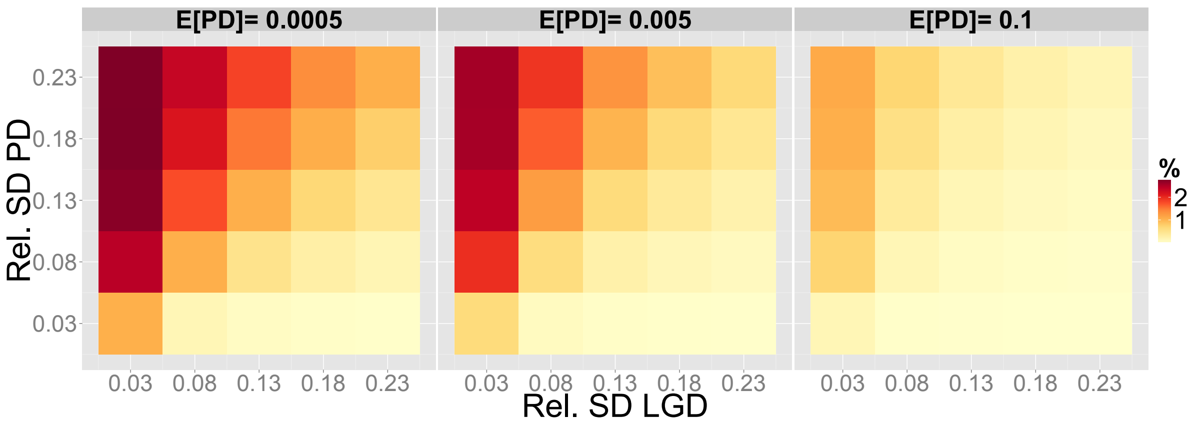


Figure 6.4: Percentage reduction in the relative standard deviation of the capital requirement by switching from 99.9 % VaR (6.3) to 99.742 % ES (6.4), given the relative standard deviation for PD and LGD. Calculated for three different expectation values of PD, with the expected value of LGD equal to 0.45. Using $N = 10000$ simulations for each calculation.

Because the relative reduction of the capital requirement's relative standard deviation is largest for the combination of the lowest LGD uncertainty and the highest PD uncertainty, we decide to calculate the reduction for this case using different confidence levels for the expected shortfall. Figure 6.5 shows the results for confidence levels ranging from 99.4 % to 99.9 %. We see that the relative uncertainty reduction is largest for the combination of high confidence levels and low PDs. The combination of large PDs and smaller confidence levels also stands out. We see that switching to ES also increases the uncertainty in a few cases, especially for large PD values at the 99.9 % confidence level.

As both the VaR version (6.3) and the ES version (6.4) of the capital requirement function are based on the same assumptions and models, the only distinction between the two versions is the risk measure. The results in this section thus show how a credit model's parameter sensitivity can depend on the chosen risk measure. In Chapter 5 we saw from Figure 5.3 that the 99.742 % ES version resulted in a higher capital charge than the 99.9 % VaR version for PD values below 0.21, with considerable increasing relative difference for PD values below 0.01. Figure 6.4 shows that the lowest PD values also cause the most notable difference between the VaR and ES version when it comes to the relative standard deviation of the capital requirement. This is of course no coincidence. As the difference between the ES and VaR version increases for the small PD values, the ES version will result in a smaller change in capital charge than the VaR version for the same change in these PD values. The ES version thus makes the capital charge depend on these values in a more stable manner, thereby reducing the relative standard deviation.

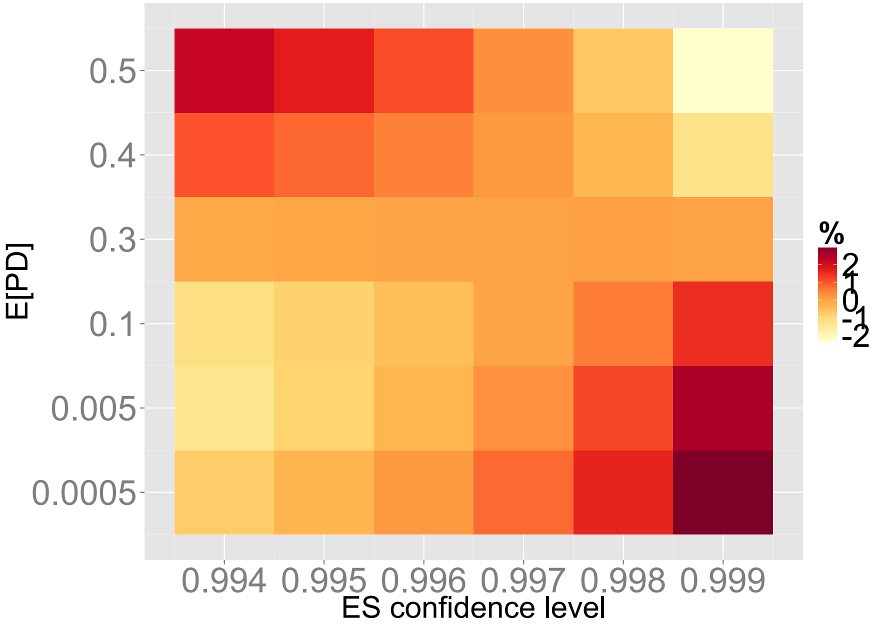


Figure 6.5: Percentage reduction in the capital requirement's relative standard deviation by switching from 99.9 % VaR to ES with different confidence levels. The results are shown for different expectation values for the probability of default. The relative standard deviation of the simulated PD and LGD values is set constant to 23 % and 3 % respectively. The expected value of LGD is equal to 0.45. Using $N = 10000$ simulations for each calculation.

There could be both advantages and disadvantages with a capital requirement function that is less sensitive to the PD parameter in the lowest end of the scale. On the plus side, one could argue that this to some degree reduces the banks' incentive to estimate artificially low PD values. At the same time this might be viewed as counterproductive, since the fundamental idea behind the IRB approach is a more risk sensitive capital charge. In case of a change of risk measure from VaR to ES, one would probably also make changes to the model itself. To really benefit from a change to the more tail risk sensitive ES, one could assume a more heavy-tailed loss distribution.

Chapter 7

Loss Distributions

In this chapter we will examine how value at risk and expected shortfall depend on the tail of the loss distribution. Using parameter values from the data set mentioned in Chapter 5, we will simulate loss realizations by assuming different loss distributions. The simulated loss values are used to create VaR and ES estimates for different confidence levels. Both the level and the uncertainty of these estimates are compared. It is also tested how the results depend on the number of simulations.

7.1 Simulation

As mentioned in Chapter 3, the Basel Committee's capital requirement function calculates the probability of default conditional on the systematic risk factor X using Vasicek's adaptation of the Merton model (3.5):

$$PD(X) = \Phi\left(\frac{\Phi^{-1}(PD) - X\sqrt{R}}{\sqrt{1-R}}\right), \tag{7.1}$$

where Φ is the cumulative distribution function of the standard normal distribution and R is the asset correlation expressing the borrowers exposure to the systematic risk factor X.

We simulate conditional PD values by using simulated X values in (7.1). The PD and R values used in this calculation are obtained from the data set. For each simulated X value, the conditional PD is calculated for the data set's 109045 loans. We want to simulate loss distributions with different tail weights. This is achieved by simulating the X values by drawing from different probability distributions. Figure 7.1 shows the probability density function of the distributions that will be used to simulate the X values. The standard normal distribution is used as a baseline. The Cauchy distribution is chosen as it provides different tails weights by changing the scale parameter. The scale parameters 0.5, 1, 1.5, 2 and 2.5 are used.

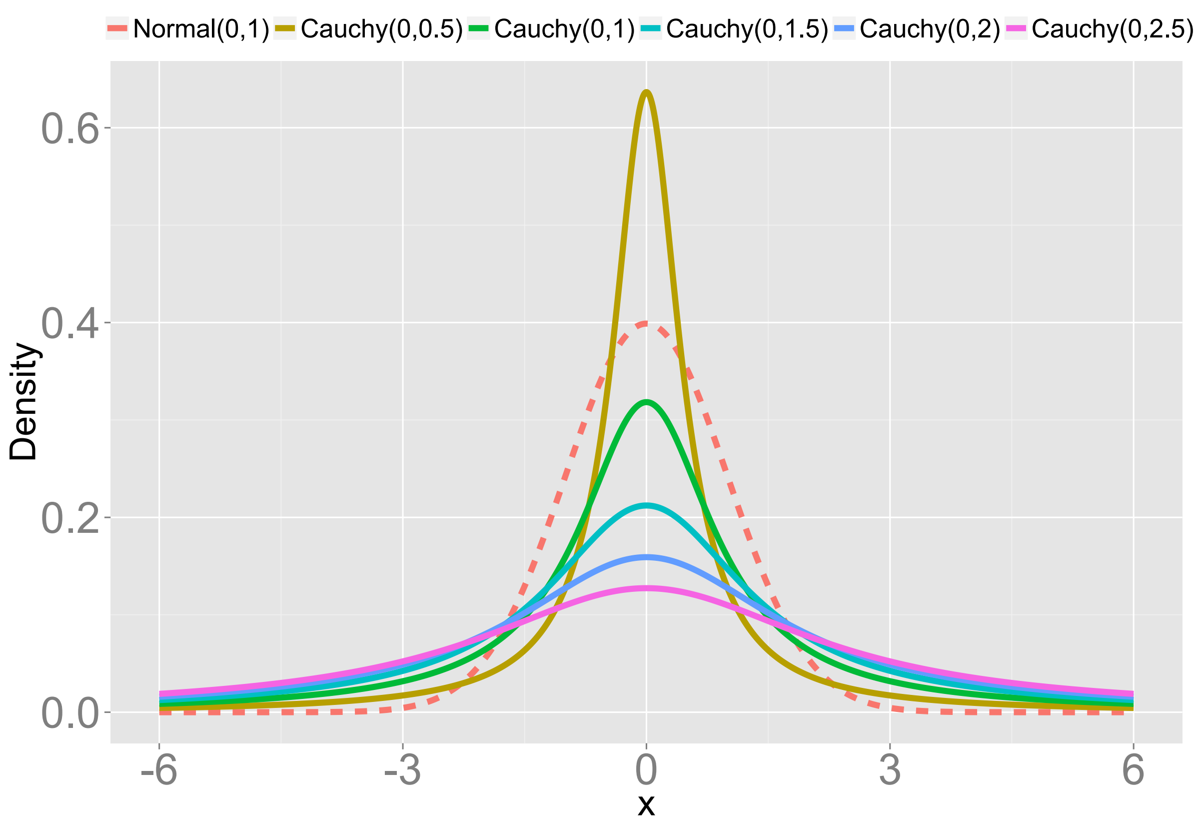


Figure 7.1: Probability density function for the Cauchy distribution with five different scale parameters ranging from 0.5 to 2.5. The probability density function of the standard normal distribution is also included, with dotted lines.

The conditional probabilities of default for the Cauchy distribution is calculated using a modified version of (7.1):

$$PD(X) = F_C\left(\frac{F_C^{-1}(PD) - X\sqrt{R}}{\sqrt{1-R}}\right),$$

where F_C is the cumulative distribution function of the Cauchy(0,1) distribution.

To simulate losses, loan j is considered defaulted if a uniformly distributed random number $U_j \in [0,1]$ is smaller than or equal to the simulated conditional PD value. For the defaulted loans, the conditional PDs are multiplied with the associated LGD and EAD values from the data set to obtain the money amount lost:

$$L(X) = \sum_{j=1}^{J} \mathbf{1}_{\{PD_j(X) \geq U_j\}} \cdot LGD_j \cdot EAD_j, \tag{7.2}$$

where $\mathbf{1}_{\{\}}$ is the indicator function and $J = 109045$.

The simulation of loss as shown in (7.2) is then repeated N times, so that the N different loss values constitute a representation of the assumed loss distribution. Following Yamai and Yoshiba [37] we take the VaR estimator at confidence level α as the $(N \cdot (\alpha - 1) + 1)$th largest loss value, and the ES estimator as the mean of the $(N \cdot (\alpha - 1) + 1)$ largest loss values.

We simulate M sets of N different loss values, to obtain better estimates for VaR and ES, namely the means of the two sets of M different estimators. This way we can also study the standard deviations of the two final estimates. The whole procedure is carried out for five different confidence levels.

The size of the data set makes this process quite time consuming for big values of N and M. The simulation code is written in R [38]. Multicore computer processing was enabled to speed up the process, using the packages `foreach`, `parallel` and `doParallel`.

7.2 Results

Figure 7.2 shows three different simulated loss distributions, each consisting of N loss values simulated using (7.2). The systematic risk factor X has been drawn from, respectively, the Normal(0,1), the Cauchy(0,1) and the Cauchy(0,2.5) probability distributions. We see that the Cauchy distributed risk factors result in distributions with much heavier tails than for the normal distributed risk factor. For the Cauchy distributed risk factors, the scale parameter does not seem to have a big impact on the tail length. The bigger scale parameter does however result in a noticeably heavier tail. Figure 7.3 shows the means and relative standard deviations of $M = 100$ VaR and ES estimates produced using the simulation method described above, in addition to the percentage difference between the corresponding results for ES and VaR. Each estimate is calculated using $N = 5000$ simulated loss values, and is calculated at five different confidence levels for each of the six loss distributions.

As one would expect, we see from Figure 7.3 that the VaR and ES estimates are most dependent on the confidence level for the most heavy-tailed loss distributions. This applies especially to the ES estimates, as they are affected by the whole tail regardless of confidence level. The gap between the VaR and ES values is decreasing for higher confidence levels, as this causes VaR to take into account a greater part of the distribution function. Since an increase in scale parameter for the Cauchy distribution results in a noticeably heavier tail,

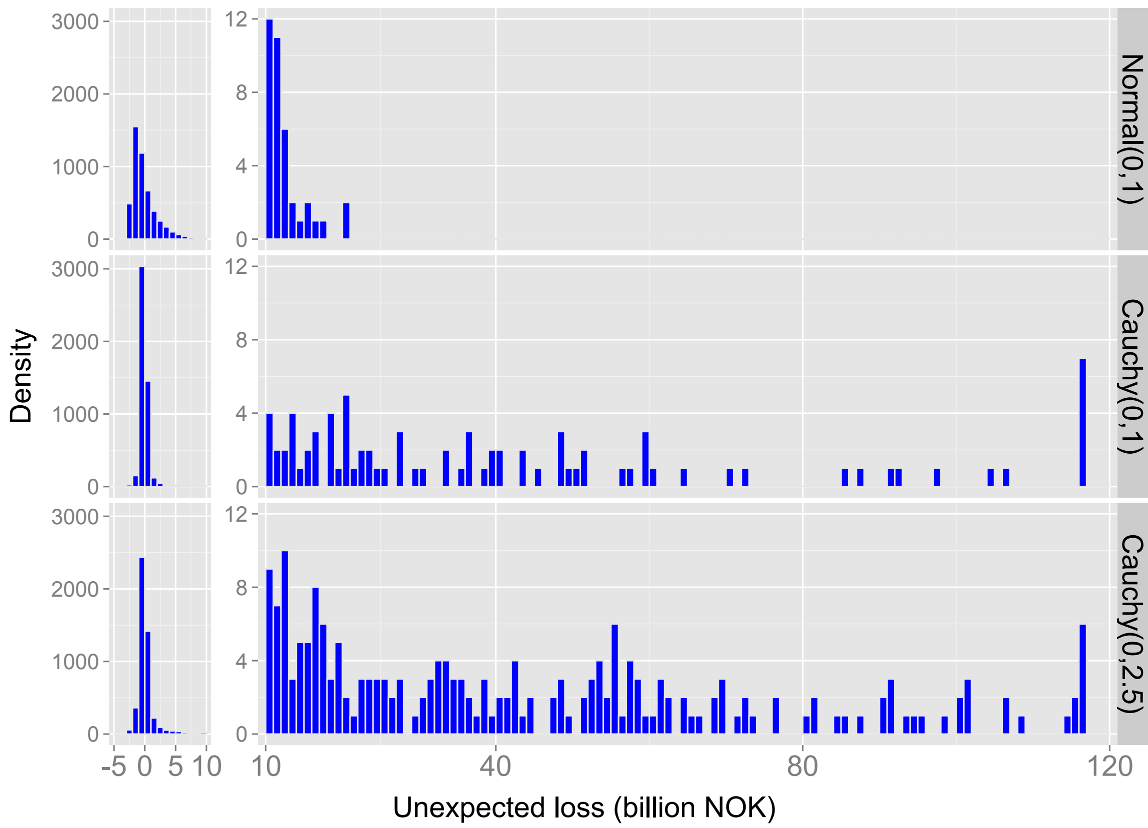


Figure 7.2: Simulated loss distributions given different probability distributions for the systematic risk factor. From the top: Standard normal distribution, Cauchy distribution with scale parameter 1 and scale parameter 2.5. All three distributions consist of $N = 5000$ simulated loss values. Two different y-axis are used to make the unlikely tail events more visible.

but not a longer tail, the difference between the two risk measures is actually decreasing when increasing the scale parameter. Note that the largest scale parameters used do result in loss distributions that are probably more heavy-tailed than what one would realistically expect.

When it comes to the estimates' relative standard deviation, the results are more varying. Only the standard normal loss distribution leads to increasing relative SD for higher confidence levels. This is also the only loss distribution that results in the ES estimates having the highest relative SD for all five confidence levels. This implies that the losses beyond the VaR quantile level varies more than the quantile level itself, meaning that the ES estimates require a larger sample size to ensure the same level of accuracy as the VaR estimates. This is not the case where we have negative values in Figure 7.3c, as this means the VaR estimate has the highest relative SD. This is the case for all the Cauchy loss distributions when the confidence level is 99 % or higher. The reason being that these loss distributions are so long-tailed that the largest simulated losses do not vary much between each simulation set.

The most notable result from Figure 7.3 is that the difference between value at risk and expected shortfall is highly dependent on the assumed loss distribution. As mentioned in Chapter 5, the closest equivalent to the 99.9 % VaR is a 99.742 % ES. For the 99 % VaR, the closest equivalent is 97.465 %. Thus, it does not make sense to compare the mean and relative standard deviation of the 99.9 % VaR with the corresponding numbers for the 99.9 % ES, and so on. Considering this, ES has the lowest relative SD also for the standard normal loss distribution.

VaR	Mean (billion NOK)					Relative SD				
Dist\CL	0.95	0.975	0.99	0.997	0.999	0.95	0.975	0.99	0.997	0.999
Normal(0,1)	7.150	8.844	11.266	14.528	17.367	0.020	0.025	0.038	0.046	0.083
Cauchy(0,0.5)	3.407	4.232	9.969	49.370	87.391	0.010	0.047	0.161	0.203	0.168
Cauchy(0,1)	4.216	7.916	28.724	78.021	111.764	0.033	0.094	0.192	0.138	0.071
Cauchy(0,1.5)	5.703	12.699	47.093	95.243	116.444	0.061	0.123	0.150	0.100	0.029
Cauchy(0,2)	7.627	20.165	59.256	104.599	118.500	0.069	0.162	0.108	0.079	0.010
Cauchy(0,2.5)	9.804	28.666	68.016	111.612	119.038	0.067	0.132	0.077	0.051	0.004

(a) Value at Risk

ES	Mean (billion NOK)					Relative SD				
Dist\CL	0.95	0.975	0.99	0.997	0.999	0.95	0.975	0.99	0.997	0.999
Normal(0,1)	9.764	11.619	14.163	17.504	20.256	0.024	0.030	0.041	0.066	0.092
Cauchy(0,0.5)	12.017	20.228	40.833	80.496	105.006	0.103	0.120	0.136	0.115	0.093
Cauchy(0,1)	20.506	35.335	66.161	102.409	116.822	0.092	0.100	0.096	0.070	0.026
Cauchy(0,1.5)	28.172	47.888	80.862	110.684	118.442	0.081	0.085	0.070	0.040	0.009
Cauchy(0,2)	34.977	57.992	89.703	115.064	119.077	0.084	0.083	0.061	0.025	0.003
Cauchy(0,2.5)	41.637	66.734	96.889	117.241	119.248	0.068	0.065	0.050	0.017	0.001

(b) Expected Shortfall

100 · (ES-VaR)/VaR	Mean					Relative SD				
Dist\CL	0.95	0.975	0.99	0.997	0.999	0.95	0.975	0.99	0.997	0.999
Normal(0,1)	36.6	31.4	25.7	20.5	16.6	19.5	21.7	9.8	43.4	11.4
Cauchy(0,0.5)	252.7	378.0	309.6	63.0	20.2	987.2	154.9	-15.9	-43.2	-44.8
Cauchy(0,1)	386.3	346.4	130.3	31.3	4.5	176.8	6.9	-50.1	-49.7	-63.7
Cauchy(0,1.5)	394.0	277.1	71.7	16.2	1.7	34.1	-30.8	-53.7	-60.3	-68.7
Cauchy(0,2)	358.6	187.6	51.4	10.0	0.5	21.9	-48.5	-43.2	-68.7	-67.1
Cauchy(0,2.5)	324.7	132.8	42.4	5.0	0.2	1.1	-51.1	-34.5	-67.9	-69.5

(c) Percentage difference

Figure 7.3: The mean and relative standard deviation of $M = 100$ VaR and ES estimates, for different confidence levels (CL). Each estimate is calculated using $N = 5000$ simulated loss values, simulated using the probability distribution indicated in the leftmost column for the systematic risk factor. The percentage difference between the corresponding ES and VaR results is also shown.

Figure 7.4 shows how the number of simulations, N, affects the relative standard deviations of the VaR and ES estimates. As one would expect, the relative SD decreases when you increase the number of simulations. The size of this reduction appears to be about the same for both the VaR and ES estimates, even if the levels differ. The relative reduction thus being largest for the estimates with the smallest relative SD. However, do still keep in mind that ES must have a smaller confidence level than VaR if the two risk measures are to result in the same capital charge.

Figure 7.5 is a graphic representation of a selection of the results in Figure 7.4, for the standard normal distribution and the Cauchy(0,2.5) distribution. We compare the VaR and ES estimates for confidence levels that result in approximately the same capital charge. We see that the 99.7 % and 97.5 % ES estimates for the normal distribution have slightly smaller relative SD for all four N values, compared respectively to the 99.9 % and 99 % VaR. The decrease in relative SD as a function of N appears to be linear for both VaR and ES, and the corresponding VaR and ES lines have about the same slope. However, the 99.9 % VaR estimate for $N = 5000$ deviates from this. For the more heavy-tailed Cauchy distribution, the decrease rate in relative SD is slowing down for large N, especially for the ES estimates. The 99.9 % VaR estimates are approaching zero for large N, thereby reaching lower relative SD than the 99.7 % ES estimates.

To summarize, this chapter shows that the difference between value at risk and expected shortfall is highly dependent on the assumed loss distribution. It also confirms that the relative standard deviations of both VaR and ES estimates decrease when the number of simulations increase, and that the rate of this decrease is roughly the same for both risk measures.

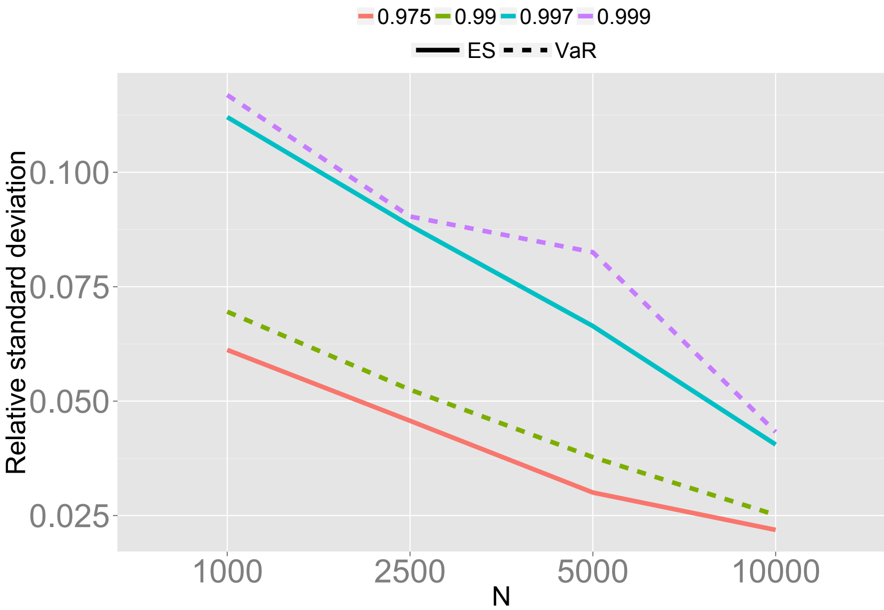
VaR	Normal(0,1)					Cauchy(0,0.5)				
N\CL	0.95	0.975	0.99	0.997	0.999	0.95	0.975	0.99	0.997	0.999
1000	0.041	0.056	0.070	0.088	0.117	0.020	0.105	0.397	0.532	0.430
2500	0.030	0.038	0.053	0.067	0.090	0.014	0.060	0.320	0.321	0.255
5000	0.020	0.025	0.038	0.046	0.083	0.010	0.047	0.161	0.203	0.168
10000	0.013	0.019	0.025	0.037	0.043	0.007	0.038	0.120	0.168	0.125
	Cauchy(0,1)					Cauchy(0,1.5)				
N\CL	0.95	0.975	0.99	0.997	0.999	0.95	0.975	0.99	0.997	0.999
1000	0.078	0.259	0.396	0.292	0.228	0.133	0.376	0.373	0.243	0.170
2500	0.051	0.164	0.327	0.173	0.149	0.083	0.170	0.192	0.139	0.084
5000	0.033	0.094	0.192	0.138	0.071	0.061	0.123	0.150	0.100	0.029
10000	0.025	0.062	0.139	0.104	0.036	0.039	0.088	0.109	0.077	0.019
	Cauchy(0,2)					Cauchy(0,2.5)				
N\CL	0.95	0.975	0.99	0.997	0.999	0.95	0.975	0.99	0.997	0.999
1000	0.147	0.274	0.238	0.193	0.117	0.176	0.288	0.192	0.130	0.089
2500	0.100	0.201	0.144	0.113	0.058	0.098	0.172	0.120	0.077	0.029
5000	0.069	0.162	0.108	0.079	0.010	0.067	0.132	0.077	0.051	0.004
10000	0.046	0.104	0.055	0.057	0.005	0.059	0.100	0.055	0.035	0.001

(a) Value at Risk

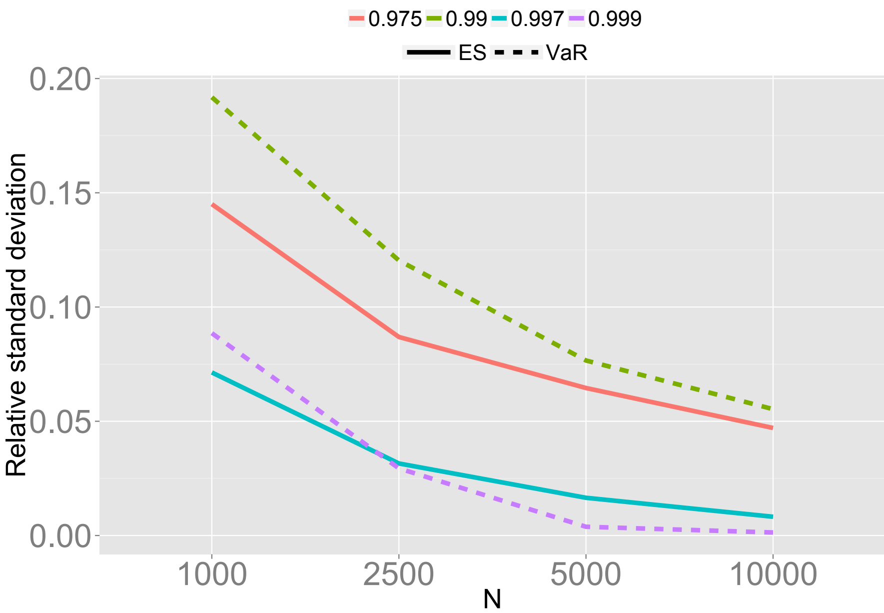
ES	Normal(0,1)					Cauchy(0,0.5)				
N\CL	0.95	0.975	0.99	0.997	0.999	0.95	0.975	0.99	0.997	0.999
1000	0.049	0.061	0.081	0.112	0.140	0.266	0.310	0.348	0.325	0.293
2500	0.037	0.046	0.062	0.088	0.116	0.172	0.200	0.219	0.190	0.162
5000	0.024	0.030	0.041	0.066	0.092	0.103	0.120	0.136	0.115	0.093
10000	0.017	0.022	0.029	0.041	0.057	0.089	0.104	0.114	0.087	0.060
	Cauchy(0,1)					Cauchy(0,1.5)				
N\CL	0.95	0.975	0.99	0.997	0.999	0.95	0.975	0.99	0.997	0.999
1000	0.204	0.218	0.205	0.166	0.138	0.215	0.223	0.192	0.137	0.108
2500	0.152	0.164	0.147	0.109	0.075	0.104	0.110	0.089	0.069	0.037
5000	0.092	0.100	0.096	0.070	0.026	0.081	0.085	0.070	0.040	0.009
10000	0.066	0.072	0.069	0.047	0.012	0.059	0.064	0.052	0.032	0.005
	Cauchy(0,2)					Cauchy(0,2.5)				
N\CL	0.95	0.975	0.99	0.997	0.999	0.95	0.975	0.99	0.997	0.999
1000	0.157	0.161	0.133	0.093	0.061	0.156	0.145	0.110	0.071	0.049
2500	0.111	0.112	0.088	0.053	0.024	0.092	0.087	0.065	0.032	0.013
5000	0.084	0.083	0.061	0.025	0.003	0.068	0.065	0.050	0.017	0.001
10000	0.055	0.056	0.043	0.019	0.002	0.052	0.047	0.037	0.008	0.000

(b) Expected Shortfall

Figure 7.4: The relative standard deviations of $M = 100$ VaR and ES estimates, for different confidence levels (CL) and different number of simulated loss values, N, used for each estimate. The loss values are simulated using the probability distribution indicated in the table headers for the systematic risk factor.



(a) Normal(0,1)



(b) Cauchy(0,2.5)

Figure 7.5: Relative standard deviation of the VaR (dotted lines) and ES (solid lines) estimates, for N simulations. 99.9 % and 99 % confidence levels are used for VaR, while 99.7 % and 97.5 % confidence levels are used for ES. The estimates are calculated from simulated loss values, from both a standard normal distribution and a Cauchy(0,2.5) distribution.

Chapter 8

Backtesting

Backtesting is a method used for model validation, where statistical procedures are used to compare actual losses to former risk measure forecasts. In this chapter we will compare backtesting for value at risk and expected shortfall, with respect to both theoretical properties and practical implementation. Lastly, we backtest both the VaR and ES version of the Basel Committee's capital requirement function, using the simulated loss values from Chapter 7 as loss realizations.

8.1 Elicitability

Gneiting [39] proved in 2010 that expected shortfall is not elicitable, as opposed to value at risk. This discovery led many to erroneously conclude that ES would not be backtestable, see for instance [40]. Elicitability is defined as follows [41]:

Definition 5 (Elicitability)**.** *A statistic $\phi(Y)$ of a random variable Y is said to be elicitable if it minimizes the expected value of a scoring function S:*

$$\phi(Y) = \arg\min_x E[S(x, Y)].$$

If you want to compare different forecasting procedures, this is typically done by using a scoring function (error measure), such as the absolute error or the squared error, which is averaged over forecast cases. Thus, the performance criterion takes the form [39]

$$\bar{S} = \frac{1}{n} \sum_{i=1}^{n} S(x_i, y_i), \tag{8.1}$$

where x_i are point forecasts, the y_i are the corresponding realizations and S is the scoring function. Most scoring functions are negatively oriented, that is, the smaller, the better. Thus, we favour the forecasting procedure that minimizes (8.1).

In simple terms, Definition 5 says that a statistic is elicitable if there exists a scoring function S that makes this statistic the best forecasting procedure according to (8.1). The mean and the median represent popular examples, minimizing the mean square and absolute error, respectively. The q^{th} quantile, hence VaR, is elicitable with the scoring function $S(x,y) = (\mathbf{1}_{\{x \geq y\}} - q)(x-y)$, where $\mathbf{1}_{\{\}}$ is the indicator function [41].

8.2 Backtesting Value at Risk

A popular backtesting method for value at risk is based on the following *violation process*:

$$I_t(q) = \mathbf{1}_{\{L(t) > \text{VaR}_q(L(t))\}}, \tag{8.2}$$

where $\mathbf{1}_{\{\}}$ is the indicator function and t denotes the time period.

Christoffersen [42] shows that VaR forecasts are valid if and only if the violation process $I_t(q)$ satisfies the unconditional coverage hypothesis: $E[I_t(q)] = 1-q$ in addition to $I_t(q)$ and $I_s(q)$ being independent for $s \neq q$. Under these two conditions, the violations are independent and identically distributed Bernoulli random variables with success probability $1-q$. Hence, the number of violations has a Binomial distribution.

To test the unconditional coverage hypothesis, we compare the fraction of violations to the VaR confidence level. Let T_1 denote the number of violations ($I_t(q) = 1$), let T_0 denote the number of cases where the VaR is not exceeded by the loss ($I_t(q) = 0$) and let T denote the number of observations ($T_1 + T_0$). For a violation fraction T_1/T, the hypothesis $T_1/T = 1-q$ can be tested using a likelihood ratio (LR) test:

$$\begin{aligned} LR &= -2\ln[\mathscr{L}(q)/\mathscr{L}(T_1/T)] = -2\ln\left[q^{T_0}(1-q)^{T_1}/(T_0/T)^{T_0}(T_1/T)^{T_1}\right] \\ &= -2\ln\left[\left(\frac{q}{T_0/T}\right)^{T_0}\left(\frac{1-q}{T_1/T}\right)^{T_1}\right]. \end{aligned} \tag{8.3}$$

As the number of observations goes to infinity, (8.3) will be distributed as a χ^2 with one degree of freedom. The null hypothesis is less likely to be true for larger LR values [42].

8.3 Backtesting Expected Shortfall

It turns out that even if ES is not elicitable, it is still '2nd order' elicitable in the following sense [43]:

Definition 6 (Conditional Elicitability). *A statistic $\phi(Y)$ of a random variable Y is called conditionally elicitable if there exist two statistics $\tilde{\pi}(Y)$ and $\pi(Y)$ such that*

$$\phi(Y) = \pi\left(Y, \tilde{\pi}(Y)\right),$$

where $\tilde{\pi}(Y)$ is elicitable and $\pi(Y)$ is such that $\pi(Y,c)$ is elicitable for all $c \in \mathbb{R}$.

Conditional elicitability is a helpful concept for the forecasting of risk measures which are not elicitable. Due to the elicitability of $\tilde{\pi}(Y)$ we can first forecast $\tilde{\pi}(Y)$ and then, in a second step, regard this result as fixed and forecast $\pi(Y,c)$ due to the elicitability of $\pi(Y)$. With regard to backtesting and forecast comparison, conditional elicitability offers a way of splitting up a forecast method into two component methods and separately backtesting and comparing their forecast performances [43]. This applies to ES, as it is simply a mean of quantiles, and both the quantiles and the mean are elicitable.

Tasche et al. [43] proposes a backtesting method for ES that is as simple as the VaR violation method, based on the following approximation:

$$\begin{aligned} \mathrm{ES}_q(L) &= \frac{1}{1-q}\int_{u=q}^{1} \mathrm{VaR}_u(L)\,du \\ &\approx \frac{1}{4}\left[\mathrm{VaR}_q(L) + \mathrm{VaR}_{0.75q+0.25}(L) + \mathrm{VaR}_{0.5q+0.5}(L) + \mathrm{VaR}_{0.25q+0.75}(L)\right]. \end{aligned} \tag{8.4}$$

If the four different VaR values in (8.4) are successfully backtested, then also the estimate of $\mathrm{ES}_q(L)$ can be considered reliable subject to careful manual inspection of the observations exceeding $\mathrm{VaR}_{0.25q+0.75}(L)$. These tail observations must at any rate be manually inspected in order to separate data outliers from genuine fair tail observations.

Acerbi and Szekely [41] have recently argued that elicitability has to do with model *selection* and not with model *testing*, and is therefore irrelevant for the choice of a regulatory risk standard. They show that expected shortfall is directly backtestable, by introducing three model-free, nonparametric backtesting methods for ES. These tests generally require more storage of information than typical VaR tests, but introduce no conceptual limitations or computational difficulties of any sort. Compared to these test procedures, the simple backtesting method based on (8.4) has the advantage of not relying on Monte Carlo simulation for the statistical test [43].

Backtesting of VaR tests the validity of a given model by comparing the frequency of the loss beyond estimated VaR with the confidence level of VaR. On the other hand, the backtesting using expected shortfall must compare the average of realized losses beyond the VaR level with the estimated expected shortfall. This requires more data than the VaR backtesting, since the loss beyond the VaR level is infrequent, thus the average of them is hard to estimate accurately [29]. The backtesting approach based on (8.4) is attractive not only for its simplicity but also because it illustrates this fact. For market risk, the Basel Committee uses a similar backtesting approach for a 97.5 % ES, which is based on testing VaR violations for the 97.5 % and 99 % confidence levels [9].

8.4 Results

Table 8.1 shows results for the VaR violation process (8.2) for 100000 simulated normal loss values as loss realizations, using the simulation method from Chapter 7. This backtesting method is conducted for four different confidence levels, which are chosen from (8.4) as the four VaR confidence levels that approximates a 99.7 % expected shortfall.

CL	T_0/T	LR	p
0.997	0.99694	0.120	0.723
0.99775	0.99774	0.004	0.947
0.9985	0.9984	0.653	0.419
0.99925	0.99916	1.040	0.308

Table 8.1: Proportion of simulated loss values not exceeding the Basel Committee's Capital Requirement, the value of the corresponding likelihood ratio test and the test's p-value. This is calculated for the four different VaR confidence levels (CL) that approximates a 99.7 % expected shortfall. 100000 simulated loss values is used for each confidence level.

We see from Table 8.1 that the proportion T_0/T of simulated loss values not exceeding the Basel Committee's Capital Requirement (3.8) is approximately equal to the confidence levels. The values for the likelihood ratio test are all low, resulting in high p-values. The null hypothesis $E[I_t(q)] = 1 - q$ is thus accepted for all four VaR confidence levels, at all reasonable confidence levels for the LR test. These results indicate that also the estimate of $\text{ES}_{0.997}(L)$ calculated by (4.6) could be considered reliable.

Note that the results in Table 8.1 are produced by comparing simulated loss values to the very model they are simulated from. Naturally, the backtesting method thus concludes that the model we are trying to validate is reliable. Of course, backtesting with actual loss data will in most cases result in significantly smaller p-values. In practise, backtesting of credit risk models can also be quite problematic. The infrequent nature of default events makes it difficult to collect enough relevant data, especially for the tail of the loss distribution. The long time horizons further complicate the data collection. The purpose of this theoretical backtesting example is merely to illustrate how model validation works, and to show that backtesting ES does not have to be more complicated than backtesting VaR.

Chapter 9

Conclusion

The Basel Committee's minimum capital requirement function for banks' credit risk is based on a risk measure called Value at Risk (VaR). This thesis performs a statistical and economic analysis of the consequences of replacing VaR with another risk measure called Expected Shortfall (ES), a switch that has already been set in motion for market risk. This is accomplished by using both theoretical simulations and real data from a Norwegian savings bank group's corporate portfolio.

By correctly calibrating the ES confidence level, it will produce approximately the same capital requirement for credit risk as with VaR, where the largest difference occurs for loans with low default probability. A switch from VaR to ES will involve some clear conceptual improvements, primarily a better ability to accurately capture tail risk. ES is also sub-additive in general, unlike VaR, so that it always reflects the positive effect of diversification. There has been some uncertainty regarding the backtesting abilities of ES, but the thesis shows that backtesting of ES does not have to be more complicated than backtesting VaR. The parameter sensitivity and estimation stability of ES have also been examined, and appear to be similar as for VaR, if not slightly less sensitive and more stable.

This thesis shows that the difference between ES and VaR is highly dependent on the assumed loss distribution. Since ES considers the entire loss distribution, it is more suitable for credit risk models that assume more heavy-tailed loss distributions than the normal distribution. For such distributions, this thesis shows that the estimation stability of ES is clearly better than for VaR.

The advantages of switching to ES must be weighed against costs and challenges associated with a transition to this risk measure, especially concerning practical implementation.

However, as this risk measure switch has already been set in motion for market risk, banks are going to have practical experience with ES before this switch potentially also happens for credit risk. In addition, ES is after all based on VaR, so we are talking about adjusting the existing system, not creating a new system from scratch. Taking all this into consideration, the conclusion is that the findings of this thesis support a switch from VaR to ES for credit risk modelling.

Appendix A

Acronyms

ASRF Asymptotic Single Risk Factor

CL Confidence Level

EAD Exposure At Default

EBA European Banking Authority

EL Expected Loss

ES Expected Shortfall

IRB Internal Ratings-Based

LGD Loss Given Default

LR Likelihood Ratio

PD Probability of Default

PDF Probability Density Function

RWA Risk-Weighted Assets

TCE Tail Conditional Expectation

UL Unexpected Loss

VaR Value at Risk

Appendix B

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