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The Vacuum Polarisation Contribution to the Lamb Shift Using Non-Relativistic Quantum Electrodynamics

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Preface

This master thesis has been written as an integral part of a Master's degree in Applied Physics at the Norwegian University of Science and Technology. The objective has been to derive the Lamb shift in the hydrogen atom. During the course of this thesis, the problem was restricted to finding the vacuum polarisation contribution to the Lamb shift.

I would like to thank my supervisor Prof. Jens O. Andersen for giving me the opportunity to write a master thesis within theoretical physics, which was both challenging and educational. I am grateful for the continuous guidance throughout this thesis, both on physics and language.

I thank my friends and fellow students for many fruitful discussions during the past five years. Lastly I would like to thank my better half, Nina, for brightening my days and keeping me motivated.

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Abstract

In this thesis we calculate the vacuum polarisation contribution to the well-known Lamb shift. The Lamb shift is a correction of order α^5 to the non-relativistic energy spectrum of the hydrogen atom. We discuss the gauge symmetry of quantum electrodynamics (QED), and derive the photon propagator in both the Lorenz and Coulomb gauge. We then calculate the vacuum polarisation tensor of the photon in QED. It is calculated using dimensional regularisation and the modified minimal subtraction scheme. We show that the vacuum polarisation results in a finite correction to the photon propagator, and thus changes the two-point function of the photon field. We introduce non-relativistic QED (NRQED), an effective field theory that is popular for QED bound state calculations. Radiative and relativistic corrections are incorporated into NRQED by a matching procedure. The matching procedure enforces that the scattering amplitudes of QED coincide with the scattering amplitudes of NRQED. The vacuum polarisation of the photon is incorporated into NRQED by adding correction terms to the photon Lagrangian. The two-point function of the photon field in NRQED is matched to the two-point function of the photon field in QED at one loop. We use the corrected photon Lagrangian to calculate the vacuum polarisation contribution to the Lamb shift. This contribution is a relatively small fraction of the total Lamb shift, it shifts the level of the $2S_{\frac{1}{2}}$ -state downward by 27.1 MHz compared to the $2P_{\frac{1}{2}}$ -state. Lamb shift is an upward shift of around 1000 MHz. We would not have agreement between the theoretical result and experimental observations without the vacuum polarisation contribution. The Lamb shift lifts the degeneracy between the $2S_{\frac{1}{2}}$ -state and the $2P_{\frac{1}{2}}$ -state.

Sammendrag

I denne avhandlingen beregner vi vakuumpolarisasjonens bidrag til den velkjente Lamb-forskyvningen. Lamb-forskyvningen er en korreksjon til hydrogenatomets ikke-relativistiske energispektrum. Forskyvningen er av størrelsesorden α^5 . Vi diskuterer gaugesymmetri i kvanteelektrodynamikk (QED), og utleder fotonpropagatoren i Lorenz gauge og i Coulomb gauge. Deretter beregnes vakuumpolarisasjonstensoren for et foton i QED. Den blir beregnet ved bruk av dimensjonal regularisering. Det vises at vakuumpolarisasjonen korrigerer fotonpropagatoren, og dermed også to-punkt-funksjonen til fotonfeltet. Videre introduseres ikke-relativistisk QED (NRQED), en effektiv feltteori som er populær ved beregninger av bundne tilstander i QED. Relativistiske- og strålingskorreksjoner innarbeides i NRQED ved hjelp av matchingmetoden. Når vi bruker matchingmetoden, krever vi at spredningsamplitudene i QED og NRQED samsvarer. Vakuumpolarisasjonen til fotonet blir inkludert i NRQED gjennom korreksjonsledd i fotonets Lagrangetetthet. To-punkt-funksjonen til fotonfeltet i NRQED blir matchet mot to-punkt-funksjonen til fotonfeltet i QED, slik at vi finner størrelsen til det korrigerende fotonleddet. Det korrigerende fotonleddet i Lagrangetettheten brukes til å beregne vakuumpolarisasjonens bidrag til Lamb-forskyvningen. Dette bidraget er en relativt liten faktor av den totale Lamb-forskyvningen, det forskyver energinivået til $2S_{\frac{1}{2}}$ tilstanden negativt med 27,1 MHz sammenliknet med $2P_{\frac{1}{2}}$ tilstanden. Hele Lamb-forskyvningen er en positiv forskyvning av energinivået på omtrent 1000 MHz. Det er foretatt svært nøyaktige målinger av hydrogenatomets energispektrum i QED. Uten fotonets bidrag, ville ikke teorien vært i samsvar med eksperimentelle observasjoner. Fotonets bidrag er derfor av signifikant betydning. Lamb-forskyvningen splitter degenerasjonen mellom $2S_{\frac{1}{2}}$ tilstanden og $2P_{\frac{1}{2}}$ tilstanden.

Notation and conventions

The notation and conventions used in this thesis are presented in the following list;

- Natural units, $c = \hbar = 1$ are used, specifically the Lorentz-Heaviside convention, i.e. $\epsilon_0 = 1$, $\alpha = \frac{e^2}{4\pi}$. Notice that speed, $v \leq 1$, is dimensionless.
- Greek indices denote space-time coordinates, $x^\mu = (x^0, \mathbf{x})$, ($\mu = 0, 1, 2, 3$), while Latin indices are used for space coordinates, x^i ($i = 1, 2, 3$).
- Boldface font represents three-dimensional vectors, i.e. $\vec{r} = \mathbf{r}$.
- We write four-dimensional vectors as either k_μ or k .
- The dot product of two four-vector are written as $kx = k_\mu x^\mu = g_{\mu\nu} k^\mu x^\nu$.
- The Minkowski metric sign convention is chosen to be $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$.
- Differential operators are written in short notation, $\partial_\mu = \frac{\partial}{\partial x^\mu}$, where $\partial_\mu = (\partial_0, \nabla)$.
- The covariant derivative is defined by $D_\mu \equiv \partial_\mu + iqA_\mu$.
- The D'Alembertian operator is denoted $\square \equiv \partial_\mu \partial^\mu = \partial_t^2 - \nabla^2$.
- Repeated indices are summed over, as to follow the Einstein summation convention.
- Feynman slash notation reads $\not{A} = \gamma^\mu A_\mu$.

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Quantum mechanics will be assumed familiar throughout this thesis, as well as the basics of quantum field theory (QFT). We will remind the reader about some of these subjects in the preliminaries.

1.1 Introduction

Quantum mechanics started developing early in the twentieth century. One of the starting points was the atomic model of Bohr [1]. He postulated that the electron in the hydrogen atom was bound in stationary states.

The atomic energy spectrum was theoretically derived after Schrödinger formulated the theory of quantum mechanics based on Hamiltonian mechanics [2]. The energy spectrum derived from the Schrödinger equation resulted in a higher number of degenerate states than experiments showed. Schrödinger theory does not take into account relativistic or radiative effects, which are necessary to theoretically reproduce experimental observations.

The development of quantum mechanics took a leap in the right direction when Dirac presented a relativistic wave equation for spin- $\frac{1}{2}$ particles [3]. Contrary to the Schrödinger equation, the Dirac equation incorporates relativistic corrections. Thus, the energy spectrum is more in agreement with experimental observations than the energy spectrum derived from Schrödinger theory.

Although the Dirac equation is more in agreement with the experimentally observed energy spectrum for the hydrogen atom, it is still not exact. In 1947, Lamb and Retherford observed a splitting of the two degenerate states, $2S_{\frac{1}{2}}$ -state and $2P_{\frac{1}{2}}$ -state [4]. The energy level of the two states did not coincide. This was theoretically explained in the same year by Bethe [5]. He derived the Lamb shift by including radiative corrections, and using non-relativistic approximations.

The traditional way to solve bound state problems in quantum field theory was to use the Bethe-Salpeter equation [6]. It describes an electron-nucleous scattering using Green's functions. The equation mixes contributions from different energy scales, which make it difficult to work with. It proved to be very hard to reach beyond the leading order of the non-relativistic limit [7]. However difficult, it was the only method used in practice until effective field theory was developed.

In 1979, Caswell and Lepage developed an effective field theory called non-relativistic quantum electrodynamics (NRQED) [8]. This theory was developed as a model to systematically formulate a non-relativistic Lagrangian, taking both relativity and radiation into consideration. The NRQED Lagrangian can be constructed to find both the Lamb shift, and a smaller corrections, where the latter constitute the hyperfine structure [8–10]. The Lagrangian is systematically constructed in orders of both $\frac{p}{m} \sim v$ and α , where m is the fermion mass, v is the fermion speed and α is the fine-structure constant.

Some fundamental introduction to regularisation and renormalization in quantum field theory (QFT) will first be discussed. Gauge theory is analysed, and both the Lorenz and Coulomb gauge

of the photon propagator are derived. NRQED is used to find the $\mathcal{O}\left(\frac{\alpha^5}{m^2}\right)$ correction to the non-relativistic energy spectrum. We will derive the vacuum polarisation contribution of the Lamb shift in Chapter 4.

1.2 Quantum field theory

Quantum field theory combines two of the fundamental theories of physics; quantum mechanics and special relativity [11]. Quantum mechanics alone is not capable of describing creation and annihilation of particles, or fluctuation between energy and mass. The Schrödinger equation describes a fixed number of particles.

Quantum field theory describes particles as modes of a single particle field. Both the particle and the anti-particle can be seen through the same field, i.e. both the electron and positron is embedded in the Dirac field. Quantum field theory can be formulated, either by using a Hamiltonian formalism, or by using a Lagrangian formalism. The Lagrangian formalism will be used in this thesis. The basis of this formalism will be assumed familiar, e.g. the path integral and generating functional. In Chapter 2, we will derive the propagator for the photon, and discuss the general procedure of obtain the propagator from the two-point function of the field.

During the early development of quantum field theory, two problems arose which nearly disrupted QFT as a theory. In the 1930s, the appearance of infinities in loop integrals was discovered [11]. This was a huge problem at the time, and were not solve until the 1950s. The other problem was its unsuccessful attempts to treat the strong interaction. Perturbation theory is not capable of treating the strong coupling of strong interaction. This was eventually addressed by using effective field theory.

In the next sections, we will discuss how to treat the infinities of quantum field theory, by regularisation and renormalisation.

1.3 Regularisation

Regularisation is the process of isolating the divergent terms in a divergent Feynman integral [12]. It is thus possible to analyse the divergent term separately and find counterterms in a process called renormalisation. There are many methods of regularisation; cut-off regularisation, Pauli-Villars regularisation [13], lattice regularisation, Schwinger's proper time regularisation [14] and dimensional regularisation [15]. We will discuss dimensional regularisation in this thesis. A more detailed description of the regularisation method can be found in introductory books of quantum field theory, e.g. [16, 17]. The dimensional regularisation method is beneficial when working in QED, since conservation of current is independent on the spacetime dimensions [17]. This independence will guarantee that the vacuum polarisation tensor is transverse, and that we keep the photon massless, see Section 3.1.1.

The integral we are evaluating is of the form

$$I_n = \int \frac{d^4 p_E}{(2\pi)^4} \frac{1}{(p_E^2 + \Delta)^n}, \quad (1.1)$$

where n is an integer and the subscript E denotes Euclidean space. The variable Δ is a well behaved constant of mass dimension two. This integral is ultraviolet divergent, for $n < 3$. From

dimensional analysis we find that it should be quadratically divergent for $n = 1$, and logarithmically divergent for $n = 2$. The divergence is isolated by dimensional regularisation. We make a small deviation from the four dimensions of spacetime, and introduce the d -dimensional integral

$$I_n = \int \frac{d^d p_E}{(2\pi)^d} \frac{1}{(p_E^2 + \Delta)^n}, \quad (1.2)$$

where d is the spacetime dimension. The dimension of I_n should be kept fixed. This is achieved by introducing an auxiliary mass scale μ , which is of mass dimension one. The integral is multiplied by μ^{4-d} , which allows the dimension of our physical quantities to be fixed. Notice that the mass scale vanishes in spacetime dimension four. The dimensional integral is now written as

$$I_n = \mu^{4-d} \int \frac{d^d p_E}{(2\pi)^d} \frac{1}{(p_E^2 + \Delta)^n}. \quad (1.3)$$

This dimensional integral is calculated in Appendix A, and the result is shown in equation (A.19). We obtain

$$I_n = \mu^{4-d} \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \Delta^{d/2-n}. \quad (1.4)$$

Let us choose $n = 2$ and $d = 4 - 2\epsilon$, where ϵ is a real scalar. We will assume that ϵ is infinitesimal, and expand in powers of ϵ . Thus, equation (1.4) can be written as

$$I_2 = \frac{1}{(4\pi)^2} \Gamma(\epsilon) \left(\frac{4\pi\mu^2}{\Delta} \right)^\epsilon, \quad (1.5)$$

where we have used $\Gamma(2) = 1$, see Appendix A. We expand the last two factors in powers of ϵ , and omit terms of higher order than $\mathcal{O}(\epsilon^0)$. This yields

$$I_2 = \frac{1}{(4\pi)^2} \left[\frac{1}{\epsilon} - \gamma_E + \ln \left(\frac{4\pi\mu^2}{\Delta} \right) + \mathcal{O}(\epsilon) \right]. \quad (1.6)$$

To make this expansion, we have used equation (A.6) and $a^\epsilon = 1 + \epsilon \ln(a) + \mathcal{O}(\epsilon^2)$. γ_E denotes the Euler-Mascheroni constant. We have managed to separate the divergent term by introducing an infinitesimal deviation in the spacetime dimension. Divergent terms are often accompanied by γ_E and 4π . The integral can therefore be simplified by letting the mass scale be modified to $\mu^2 \rightarrow \frac{\mu^2 e^{\gamma_E}}{4\pi}$. This yields

$$I_2 = \frac{1}{(4\pi)^2} \left(\frac{1}{\epsilon} + \ln \left(\frac{\mu^2}{\Delta} \right) + \mathcal{O}(\epsilon) \right). \quad (1.7)$$

These divergent integrals occur when evaluating loop integrals. In the following section we will discuss how to treat divergent terms.

1.4 Renormalisation

Loop integrals are important since they make contribution to the exact propagators and tree level diagrams. They are considered higher order diagrams, since they increase the number of vertices.

In QED, the coupling constant is the electric charge. One additional loop in a QED diagram increases the power of $\alpha = \frac{e^2}{4\pi}$ by one, where α is the fine structure constant.

We want to calculate the exact propagator in $\lambda\Phi^4$ -theory as an example of regularisation and renormalisation. To calculate the exact propagator, we must take into account all propagator-like diagrams. Propagator-like diagrams are diagrams with one incoming and one outgoing line. Examples of the different propagator-like diagrams can be seen in Figure 1.1. In Figure 1.1 we distinguish between one-particle irreducible diagrams (1PI) and two-particle irreducible diagrams (2PI). We define 1PI as diagrams which cannot be divided into subdiagrams by cutting a single internal line. The 2PI can be divided into two 1PI, and so on. Let $i\Delta$ denote all 1PI diagrams. The exact propagator then reads [18]

$$\begin{aligned}
 \text{---} \circ \text{---} &= \text{---} + \text{---} \textcircled{\text{1PI}} \text{---} + \text{---} \textcircled{\text{1PI}} \textcircled{\text{1PI}} \text{---} + \dots \\
 &= \frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2} i\Delta \frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2} i\Delta \frac{i}{p^2 - m^2} i\Delta \frac{i}{p^2 - m^2} + \dots \\
 &= \frac{i}{p^2 - m^2} \sum_{n=0}^{\infty} \left(i\Delta \frac{i}{p^2 - m^2} \right)^n = \frac{i}{p^2 - m^2} \frac{1}{1 - i\Delta \frac{i}{p^2 - m^2}} = \frac{i}{p^2 - m^2 + \Delta}, \quad (1.8)
 \end{aligned}$$

where $\frac{i}{p^2 - m^2}$ is the propagator of the scalar field. We see that the propagator correction is equivalent with a shift in the mass by Δ . Thus, Δ is known as the mass counter term and is often denoted by δm . The exact propagator corresponds to the proper propagator of the theory, and the corrected mass must correspond to the physical mass m_0 . The physical mass is written as $m_0^2 = m_B^2 - \delta m$, where we have denoted the ‘‘bare’’ mass as $m \rightarrow m_B$. A bare quantity is defined to be a quantity which has not been renormalised. It is the quantity which we find in the non-renormalised Lagrangian. We obtain the dressed quantities, e.g. m_0 , by renormalising the bare quantities. Dressed quantities should be used when describing a physically correct system. The bare value absorb any infinities and yields a finite dressed value.

We can write the Lagrangian in terms of the physical mass, it reads

$$\mathcal{L}_{\lambda\Phi^4} = \frac{1}{2} (\partial_\mu \Phi)^2 - \frac{1}{2} m_0^2 \Phi^2 - \frac{1}{2} \Delta \Phi^2 - \frac{1}{4} \lambda \Phi^4. \quad (1.9)$$

The Lagrangian has been renormalised with respect to the mass. To fully renormalise the Lagrangian, both the coupling constant, λ , and the field, Φ , must be renormalised. The reminding part is to identify Δ .

If we assume that the coupling constant, λ , is small, we can use the self-energy diagram to approximate the 1PI diagrams. With this approximation, the correction to the coupling constant is neglected, and we only find correction to the mass. This is the diagram which contributes the most to the propagator, as each vertex add an additional power to the coupling constant. We use the Feynman rules to express the self-energy diagram mathematically. It reads [18]

$$\text{---} \textcircled{\text{self-energy}} \text{---} \propto i\Delta^{(1)} = i \frac{\lambda}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - m^2} = i \frac{\lambda}{2} \int \frac{d^4 p_E}{(2\pi)^4} \frac{1}{p_E^2 + m^2}, \quad (1.10)$$

where the factor $\frac{1}{2}$ is due to symmetry, and the superscript (1) denotes that we only keep terms of $\mathcal{O}(\lambda)$. In the last step we performed a Wick rotation, see Appendix A. The integral has a superficial degree of divergence of two, meaning that it is at worst quadratically divergent.

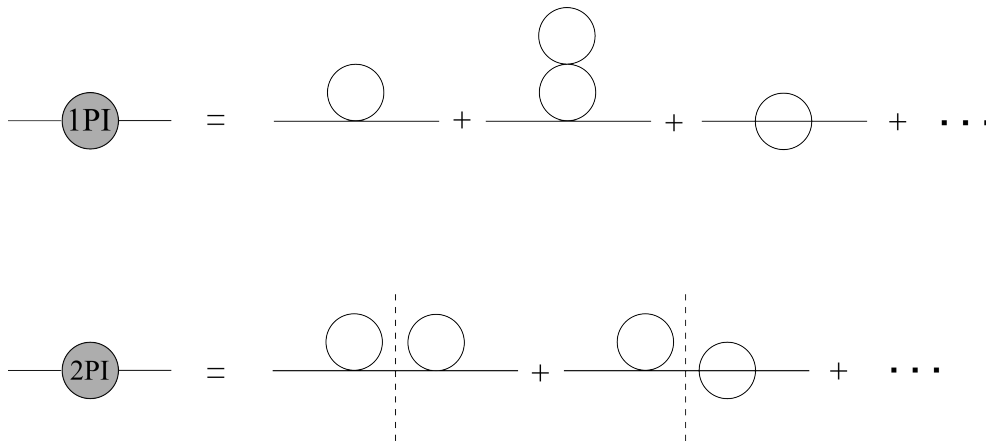


Figure 1.1: The figure represents some basic one-particle irreducible (1PI) diagrams in scalar field theory. The dotted line in the two-particle irreducible diagrams denotes where the diagram can be split into two 1PI.

In order to isolate the divergence, we apply dimensional regularisation as before, and find

$$i\Delta^{(1)} = -i\frac{\lambda}{2} \frac{m^2}{(4\pi)^2} \left(1 + \ln\left(\frac{\mu^2}{m^2}\right) + \frac{1}{\epsilon} + \mathcal{O}(\epsilon) \right). \quad (1.11)$$

1.5 Ward's identity

Ward's identity,

$$k^\mu M_\mu = 0, \quad (1.12)$$

is a special case of a more general identity known as the Ward-Takahashi identity [19]. Here k^μ is the momentum of an external photon, and M_μ is a Feynman amplitude for a QED process. We will not derive the Ward-Takahashi identity here, as this is done in great detail in several text books, e.g. [12, 16, 20].

Ward's identity can be used to show that the probability for a longitudinal and a time-like photon cancel, and thus, the photon polarisation is transverse [16]. Another consequence of the identity is that the renormalisation of charge is universal, i.e. the renormalised charge of a muon particle has the same magnitude as the renormalised charge of the electron [16, 18]. We apply the Ward's identity later, to argue that the photon polarisation must be transverse.

Gauge theory in quantum electrodynamics

In this chapter we will discuss the gauge invariance of QED, and derive the photon propagator in different gauges. The choice of our gauge must not affect the physical values in the theory. We are going to focus mainly on the gauge invariance of the photon propagator. We first remind the reader of the QED Lagrangian.

2.1 QED Lagrangian

The QED Lagrangian should be familiar. It reads

$$\begin{aligned}\mathcal{L}_{\text{QED}} &= \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Maxwell}} + \mathcal{L}_{\text{Interactions}} \\ &= \bar{\Psi} (i\not{\partial} - m) \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e \bar{\Psi} \gamma^\mu \Psi A_\mu,\end{aligned}\quad (2.1)$$

where $\bar{\Psi} = \Psi^\dagger \gamma^0$, i.e. the Dirac adjoint spinor. It constitutes three different contributions, where the Dirac Lagrangian describes the behaviour of the Dirac field, $\Psi(x)$, the Maxwell Lagrangian describes the behavior of the Maxwell field, A_μ , and the interaction Lagrangian describes the interaction between the fields. The Maxwell field is often called the gauge field, or the gauge vector field. It is not uniquely defined, and is thus a physically unobservable quantity. The electrodynamic field strength, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, is the physically observable quantity.

We will demonstrate a simple way of constructing the QED Lagrangian. This is done by demanding that the free Dirac field is invariant under a local $U(1)$ symmetry, and not only a global $U(1)$ symmetry. The difference between the two is that a local symmetry contains transformation operators which are dependent on the space-time point x_μ [21].

2.2 Gauge invariance in QED

The symmetry of QED is fundamentally necessary in understanding the different phenomena in the theory, such as the universality of charge and the massless property of the photon. QED respects the local $U(1)$ symmetry group, and is an abelian gauge theory.¹ The symmetries of a physical system often plays an essential role when constructing the Lagrangian density. For instance, if we want to have a relativistic model of the photon propagator, we must enforce Lorentz invariance.

Let us stipulate that, in QED, the Dirac field and the gauge vector field transform as

$$\Psi(x) \rightarrow e^{i\alpha(x)} \Psi(x), \quad A_\mu \rightarrow A_\mu - \frac{1}{e} \partial_\mu \alpha(x), \quad (2.2)$$

without changing the appearance of the QED Lagrangian. We will begin with the free Dirac Lagrangian, which is only invariant under a global $U(1)$ symmetry. By applying a local transformation to the Dirac field, we get an additional term in the Lagrangian due to the dynamical

¹In an abelian symmetry, the transformation operators are commutative.

properties of the Dirac field,

$$\mathcal{L}_{\text{Dirac}} \rightarrow \mathcal{L}_{\text{Dirac}} - (\partial_\mu \alpha) \bar{\Psi} \gamma^\mu \Psi. \quad (2.3)$$

An additional term must be added to the free Dirac Lagrangian, in order to get rid of the additional term in the transformed Lagrangian. The additional term should couple a new field to the Dirac field. We see that the interaction Lagrangian in equation (2.1) will result in a cancellation, if the new field transforms like A_μ . Thus, we add $\mathcal{L}_{\text{Interactions}}$ to the free Dirac Lagrangian. The "new" field, i.e. A_μ , must also have a free Lagrangian, such that the effective Lagrangian is complete. We therefore add the Maxwell Lagrangian to the complete Lagrangian, since it respects the condition of the gauge invariance in equation (2.2). The conclusion is that, when demanding that a global symmetry should be a local symmetry, we are forced to introduce a massless vector field. It is essential that the field is massless. If it was not, the gauge invariance would have been broken.

The photon field is a four vector, which implies that the photon has four degrees of freedom. We know that the photon only has two polarised states, the transverse states. We will see how we can fix this redundancy in the next section.

2.3 Photon propagator

Looking at the path integral formalism of QFT, the propagator of a field is defined as the inverse of the operator associated with the quadratic term of said field in the Lagrangian. This can be shown by looking at the two-point Greens function of the field. The calculation regarding the two-point Greens function will not be derived, as this can be found in introductory books to QFT, e.g. [16, 18]. We will adopt the method of obtaining the propagator to the field using the Lagrangian. In the case of the photon propagator, the quadratic term of the Maxwell field is found in the free Maxwell Lagrangian. The free Maxwell Lagrangian, or photon Lagrangian, is

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (2.4)$$

where $F_{\mu\nu}$ is the electromagnetic field strength and reads

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2.5)$$

The Lagrangian can be expressed by the photon field, A_μ , by multiplying the two electromagnetic field strengths. We get

$$\begin{aligned} \frac{1}{4} F_{\mu\nu} F^{\mu\nu} &= \frac{1}{4} (\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu) \\ &= \frac{1}{2} \partial_\mu A_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu). \end{aligned} \quad (2.6)$$

In the last step, the two last terms have had their indices changed, i.e. $\mu \rightarrow \nu$ and $\nu \rightarrow \mu$. We can perform a partial integration to the Lagrangian. This yields

$$\mathcal{L}_{\text{Maxwell}} = \frac{1}{2} A_\nu (\square g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu = \frac{1}{2} A_\mu (D^{-1})^{\mu\nu} A_\nu, \quad (2.7)$$

where we have omitted the surface term. It is admissible to omit the surface term as long as we assume that the field is zero at the boundary [12]. The Lagrangian can be Fourier transformed to momentum space. We obtain

$$\mathcal{L}_{\text{Maxwell}} = \frac{1}{2} A_\nu (k^2 g^{\mu\nu} - k^\mu k^\nu) A_\nu = \frac{1}{2} A_\mu (D^{-1})^{\mu\nu} A_\nu. \quad (2.8)$$

The propagator is the inverse of the operator $D_{\mu\nu}^{-1}$ by definition. We denote the photon propagator as $D^{\mu\sigma}$, and thus have the relation

$$D_{\mu\nu}^{-1} D^{\mu\sigma}(x-y) = \delta_\nu^\sigma \delta^4(x-y), \quad (2.9)$$

where x and y are two space-time points. The Dirac delta function, $\delta^4(x-y)$, is in four dimensions, and δ_ν^σ is the Kronecker delta. We must find an inverse of the 4×4 matrix, $D_{\mu\nu}^{-1} = (\square g_{\mu\nu} - \partial_\mu \partial_\nu)$. Unfortunately, the 4×4 matrix is singular, meaning that the determinant is zero and not invertible. This is shown by letting the operator act on an eigenvector, k^ν . In momentum space, the operator can be written as

$$D_{\mu\nu}^{-1} = (k^2 g_{\mu\nu} - k_\mu k_\nu). \quad (2.10)$$

By acting the operator on the eigenvector k^ν , we find

$$(k^2 g_{\mu\nu} - k_\mu k_\nu) k^\nu = 0, \quad (2.11)$$

which is equivalent with stating that the determinant of the operator is zero. The singular property of the operator is a consequence of gauge invariance of the Maxwell field. When looking at the generating functional, we see that it contains an integration over all possible A_μ configurations, many of which correspond to the same physical system. The integration is done over a redundant number of configurations. We have to choose a gauge, i.e. fix the gauge field in one configuration. The gauge field is fixed in one configuration by identifying a proper constraint of the field, and adding a gauge-fixing term to the Lagrangian. A constraint will be equivalent to a line in configuration space, which must satisfy the constraint equation. This can be done more rigorously by the Faddeev-Popov procedure [22], where the gauge dependency of A_μ is factored out [16]. We will not use the Faddeev-Popov procedure, but will add a gauge-fixing term directly to the Lagrangian. First we will mention two gauge configurations which are commonly used in QED, the Lorenz gauge and the Coulomb gauge.² The constraint associated with the Lorenz gauge is

$$\partial_\mu A^\mu = 0. \quad (2.12)$$

One of the components of the photon field is fixed by the constraint equation. There are still three degrees of freedom of A^μ . The advantage of the Lorenz gauge is that it is manifest Lorentz invariant. When doing calculations over physical quantities, e.g. on-shell elements of the scattering matrix, the gauge choice can not affect the calculations. The physical quantities do not depend on the choice of gauge. Thus, when performing on-shell scattering calculations, the gauge dependent terms must cancel and the photon must have two degrees of freedom.

²The Lorenz gauge is named in honour of L.V. Lorenz, and not H.A. Lorentz [23] It is often misspelled as Lorentz gauge.

Although the Coulomb gauge is not manifest Lorentz invariant, the advantage comes from the fact that it separates the zeroth component of the photon field, A_0 , from the spatial part of the photon field, A_i . The constraint introduced in the Coulomb gauge is

$$\nabla \cdot \mathbf{A} = 0. \quad (2.13)$$

We see that the constraint fixes one of the spatial components of field. The zeroth component of the photon field is also fixed by using the Gauss's law [20].

Choosing a gauge is arbitrary when doing physical calculations, it can be chosen by preference and convenience. When doing QFT calculations, the Feynman gauge might be simpler, since the propagator can be expressed in a very shorthand notation. In NRQED, the Coulomb gauge is often used, since it separates the zeroth component of the photon propagator from the spatial components of the photon propagator. The zeroth component of the photon propagator is associated with the Coulomb interaction. In the scope of this thesis, the Coulomb interaction is an instantaneous interaction.

2.3.1 Lorenz gauge

As we mentioned above, the Lorenz gauge is manifest Lorentz invariant. The free Maxwell Lagrangian is manifest Lorentz invariant by construction, and the gauge-fixing term in the Lorenz gauge is also manifest Lorentz invariant. We construct the gauge-fixing term by using the constraint in equation (2.12). We see that the constraint sums over the four indices, and leaves us with a Lorentz invariant scalar constraint. We can multiply the scalar constraint by an arbitrary constant, $\frac{1}{2\xi}$. The gauge-fixing term added to the Lagrangian is therefore [12]

$$\mathcal{L}_{\text{GF}} = \frac{1}{2\xi} (\partial_\mu A^\mu)^2. \quad (2.14)$$

By performing a partial integration on the Lagrangian, including the gauge-fixing term, we can find a new operator $(D^{-1})_{\mu\nu}$, equivalent to equation (2.7). The operator reads

$$D_{\mu\nu}^{-1} = \left[\square g_{\mu\nu} + \left(\frac{1}{\xi} - 1 \right) \partial_\mu \partial_\nu \right]. \quad (2.15)$$

Plugging this operator into equation (2.9) and Fourier transforming it to momentum space, we find

$$\int \frac{d^4 k}{(2\pi)^4} \left(k^2 g^{\mu\nu} + \left(\frac{1}{\xi} - 1 \right) k^\mu k^\nu \right) D_{\mu\sigma} e^{ik(x_1-x_2)} = \int \frac{d^4 k}{(2\pi)^4} \delta_\nu^\sigma e^{ik(x_1-x_2)}. \quad (2.16)$$

By equating the integrand in the expression above, we find an equation for the photon propagator,

$$\left(k^2 g^{\mu\nu} + \left(\frac{1}{\xi} - 1 \right) k^\mu k^\nu \right) D_{\mu\sigma} = \delta_\nu^\sigma. \quad (2.17)$$

The tensor structure of the propagator must be a linear combination of $k_\mu k_\sigma$ and $g_{\mu\sigma}$. These two second rank tensors constitutes the general Lorentz invariant rank two tensors which are

available. By inverting this matrix we find³

$$D^{\mu\nu} = \frac{1}{k^2} [g^{\mu\nu} - (1 - \xi)k^\mu k^\nu]. \quad (2.18)$$

The most commonly used values for ξ are,

$$\xi = 0, \quad \text{Feynman gauge} \quad (2.19)$$

$$\xi = 1, \quad \text{Landau gauge.} \quad (2.20)$$

We can choose ξ arbitrarily, the gauge choice does not affect physical quantities such as on-shell S-matrix elements.⁴ The ξ is chosen in such a way that the calculations are as easy as possible. We are free to change the gauge for isolated calculations, i.e. different scattering calculations.

2.3.2 Coulomb gauge

In the coulomb gauge, the photon propagator is separated in two terms. One term is associated with the coulomb photon field, and one term is associated with the spatial photon field. This makes it easier to evaluate corrections to the non-relativistic Hamiltonian, which constitutes the different shifts in energy. We find the gauge-fixing term in a similar manner as for the Lorenz gauge. In the Coulomb gauge, the constraint is shown in equation (2.13). The gauge-fixing term added to the Lagrangian is [18]

$$\mathcal{L}_{\text{GF}} = \frac{1}{2\xi} (\partial_i A^i)^2 = \frac{1}{2\xi} (\partial_\mu A^\mu - n^\nu \partial_\nu n_\mu A^\mu)^2, \quad (2.21)$$

where $n_\mu = (1, 0, 0, 0)$. We add the gauge-fixing term to the free photon Lagrangian, and partial integrating the Lagrangian as before. This yields an operator, which reads

$$D_{\mu\nu}^{-1} = \left[\square g_{\mu\nu} + \left(\frac{1}{\xi} - 1 \right) \partial_\mu \partial_\nu - \frac{1}{\xi} \partial_0 \partial_\mu n_\nu - \frac{1}{\xi} \partial_0 n_\mu \partial_\nu + \frac{1}{\xi} n_\mu n_\nu \partial_0^2 \right]. \quad (2.22)$$

By using the same procedure as for the Lorenz gauge, i.e. Fourier transforming into momentum space and inverting the operator, we get a propagator of the form

$$D^{\mu\nu} = \begin{bmatrix} \frac{1}{\mathbf{k}^2} & \mathbf{0} \\ \mathbf{0} & \frac{1}{k^2} \left(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2} \right) \end{bmatrix}. \quad (2.23)$$

We had to choose $\xi = 0$ to get this simple expression for the photon propagator in the Coulomb gauge. The bold faced zeroes indicate null-vectors, and $\frac{1}{k^2} \left(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2} \right)$ is a 3×3 -matrix. The propagator can also be written in four-component notation as

$$D^{\mu\nu} = \frac{1}{k^2} \left(\Theta^{\mu\nu} + \frac{k^2}{\mathbf{k}^2} n^\mu n^\nu \right), \quad (2.24)$$

where $\Theta^{0\nu} = \Theta^{\mu 0} = 0$ and $\Theta^{ij} = \left(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2} \right)$.

³We have used computational tools to invert the operator.

⁴S-matrix elements (scattering-matrix elements) are associated with a scattering process. They relate the initial and final states.

Vacuum polarisation of the photon

In this chapter we are going to derive a radiative correction to the photon propagator. The exact photon propagator must take into account all possible propagator-like diagrams. There are infinitely many diagrams contributing to the photon propagator. Let $i\Pi_{\mu\nu}$ denote the sum of all one-particle-irreducible (1PI) diagrams contributing to the photon propagator. $i\Pi_{\mu\nu}$ is known as the vacuum polarisation tensor. We will try to predict how this tensor should look like, before we calculate it. Throughout this derivation, we follow the approach of Peskin and Schroeder [16].

3.1 Vacuum polarisation tensor

We can write the exact photon propagator as a sum of all contributing diagrams, analogous to $\lambda\Phi^4$ -theory in the preliminaries. The correction to the photon propagator is not as simple as the correction to the propagator in the $\lambda\Phi^4$ -theory. One of the reasons is the vector-like properties of the photon field reflected in the indices of the 1PI, $i\Pi_{\mu\nu}$. The most general form for the vacuum polarisation tensor must be in agreement with the symmetries otherwise found in QED. We have that the photon propagator must be Lorentz invariant, which tells us that the tensor must be a linear combination of $g_{\mu\nu}$ and $k_\mu k_\nu$. Here $g_{\mu\nu}$ denotes the metric tensor, and k_μ is the four momentum of the photon. The Ward's identity, Section 1.5, suggests that the vacuum polarisation tensor must be transverse, i.e. $k^\mu \Pi_{\mu\nu} = 0$. We use the projection matrix, $P_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$, to write the polarisation tensor [24]. The projection matrix has some simple properties, $P_{\mu\nu} P_{\sigma\nu} = P_{\mu\sigma}$ and $k^\mu P_{\mu\nu} = 0$, guaranteeing a transverse polarisation tensor. Thus, we write the vacuum polarisation tensor as

$$i\Pi_{\mu\nu} = iP_{\mu\nu}\Pi(k^2). \quad (3.1)$$

The scalar $\Pi(k^2)$ is of mass dimension two. It will be shown that this scalar can be made dimensionless by factorising out k^2 . This simplifies the equation for the exact photon propagator,

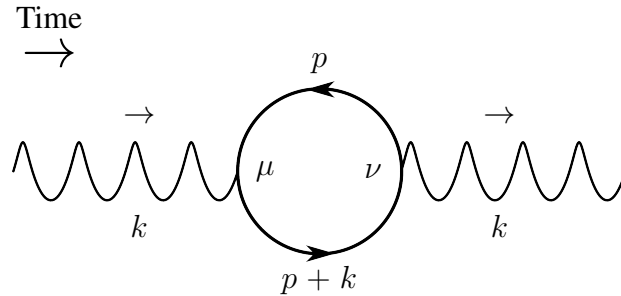


Figure 3.1: The Feynman diagram represents the vacuum polarisation. It is a one-loop correction to the photon propagator. The wavy lines are photon propagators. We use solid lines for the fermion loop. The k is the photon four-momentum, and p and $p+k$ is the fermion four-momentum. The μ and ν indicates the vertices. The arrow indicates the direction of propagation.

as we will show below. We factor out k^2 and write the vacuum polarisation tensor as

$$i\Pi_{\mu\nu} = ik^2 P_{\mu\nu} \Pi(k^2). \quad (3.2)$$

The photon propagator will be written in the Coulomb gauge, see equation (2.24). By using the Coulomb gauge, the exact photon propagator is written as

$$\begin{aligned} \text{---}\bigcirc\text{---} &= \text{---}\text{---} + \text{---}\bigcirc\text{---} + \text{---}\bigcirc\bigcirc\text{---} + \dots \\ &= iD^{\mu\nu} + iD^{\mu\rho}i\Pi_{\rho\sigma}iD^{\sigma\nu} + iD^{\mu\rho}i\Pi_{\rho\sigma}iD^{\sigma\lambda}i\Pi_{\lambda\eta}iD^{\eta\nu} + \dots \\ &= iD^{\mu\nu} + iD^{\mu\rho}ik^2P_{\rho\sigma}iD^{\sigma\nu}\Pi + k^4iD^{\mu\rho}iP_{\rho\sigma}iD^{\sigma\lambda}iP_{\lambda\eta}iD^{\eta\nu}\Pi^2 + \dots \\ &= iD^{\mu\nu} (1 + \Pi + \Pi^2 + \dots) \\ &= iD^{\mu\nu} \left(\frac{1}{1 - \Pi} \right), \end{aligned} \quad (3.3)$$

where we in the last step summed the geometric series. The contraction between the indices of the propagator and polarisation tensor reads

$$\begin{aligned} \frac{1}{\Pi(k^2)} D^{\mu\rho} \Pi_{\rho\sigma} D^{\sigma\nu} &= \frac{1}{k^2} \left(\Theta^{\mu\rho} + \frac{k^2}{\mathbf{k}^2} n^\mu n^\rho \right) (k^2 g_{\rho\sigma} - k_\rho k_\sigma) D^{\sigma\nu} \\ &= \left(\delta_j^i - \frac{k^i k_j}{\mathbf{k}^2} - \frac{k^2}{\mathbf{k}^2} n^\mu n_\sigma + \frac{n^\mu k_\sigma k^0}{\mathbf{k}^2} \right) D^{\sigma\nu} \\ &= \frac{1}{k^2} \left(\Theta_\sigma^\mu - \frac{k^2}{\mathbf{k}^2} n^\mu n_\sigma + \frac{n^\mu k_\sigma k^0}{\mathbf{k}^2} \right) \left(\Theta^{\sigma\nu} + \frac{k^2}{\mathbf{k}^2} n^\sigma n^\nu \right) \\ &= D^{\mu\nu}. \end{aligned} \quad (3.4)$$

For a more rigorous derivation, see Appendix A. The contraction between the indices have not created any mass term to the photon. It would have been a dramatic failure if our prediction resulted in a massive photon. We see that the photon propagator is simply scaled by the factor $\frac{1}{1-\Pi}$. The photon propagator in the Feynman gauge is calculated by Peskin & Schroeder [16]. In the Feynman gauge, the exact propagator receives a term proportional to $k^\mu k^\nu$. This term is not present in the original photon propagator in the Feynman gauge, see equation (2.18). The $k^\mu k^\nu$ -term is called a gauge term, since the scaling of this term depends on the choice of gauge. Since it is gauge dependent, it does not affect physical quantities, such as on-shell S-matrix elements. In any real calculation, the photon is virtual, and connects to at least one conserved current. The conserved current in QED is a fermion line, which respects the equation $\partial_\mu j^\mu = 0$ [19]. In momentum space it reads $k_\mu j^\mu = 0$, which means that the gauge term have not contribution to on-shell scattering calculations. The vacuum polarisation tensor will be calculated in the next section, and the scalar $\Pi(k^2)$ will be identified.

3.1.1 Calculation of $\Pi_{\mu\nu}$

There are an infinite number of 1PI diagrams contributing to the exact photon propagator. We must thus try to approximate the correction with the diagram which contributes the most. Since

each vertex contributes with an additional power to the coupling constant, we only include the diagram with the lowest number of vertices possible. We will only consider the one-loop contribution to the 1PI diagrams when we calculate the polarisation tensor. This is the lowest order contribution to the photon propagator, and the diagram is shown in Figure 3.1. The diagram illustrates fermion pair production and annihilation in vacuum. This effect was first observed experimentally in 1947 [4], before a theoretical explanation was developed. The calculation of the vacuum polarisation is useful in itself, as it is a common correction in many tree diagrams. By calculating it once, the result can be used in many contexts where this diagram must be accounted for, such as scattering diagrams [21].

We obtain the following equation by applying the Feynman rules of QED to the vacuum polarisation tensor [12, 16],

$$i\Pi_{\mu\nu}(k^2) = (-1) \int \frac{d^4p}{(2\pi)^4} \text{Tr} \left[(-ie\gamma_\mu) \frac{1}{\not{p} - m} (-ie\gamma_\nu) \frac{1}{\not{p} + \not{k} - m} \right]. \quad (3.5)$$

Three-momentum has been conserved in the vertices. The trace is taken over the Dirac space, i.e. the trace affects the gamma matrices. We get the factor (-1) due to the fermion loop. The trace can be rewritten by factorising a common denominator. We obtain

$$\text{Tr}(\dots) = \frac{\text{Tr} [\gamma_\mu(\not{p} + m)\gamma_\nu(\not{p} + \not{k} + m)]}{(p^2 - m^2)[(p+k)^2 - m^2]}, \quad (3.6)$$

where we have used

$$\not{p}^2 = \gamma_\mu\gamma_\nu p^\mu p^\nu = \frac{1}{2} \{\gamma_\mu, \gamma_\nu\} p^\mu p^\nu = g_{\mu\nu} p^\mu p^\nu = p^2. \quad (3.7)$$

The trace can be written out by using trace identities, which can be found in many text books, e.g. [16, 21]. We only get contribution from even numbers of gamma matrices, and thus have

$$\begin{aligned} \text{Tr}[\dots] &= \text{Tr} [\gamma_\mu \not{p} \gamma_\nu (\not{p} + \not{k}) + m^2 \gamma_\mu \gamma_\nu] \\ &= 4 [2p_\mu p_\nu + (m^2 - p \cdot k - p^2)g_{\mu\nu} + p_\mu k_\nu + p_\nu k_\mu]. \end{aligned} \quad (3.8)$$

The two factors in the denominator in equation (3.6) can be collected in a single factor by using Feynman parametrisation, see equation (A.8). This yields

$$\frac{1}{(p^2 - m^2)[(p+k)^2 - m^2]} = \int_0^1 dx \frac{1}{[(p+kx)^2 + k^2x(1-x) - m^2]^2}. \quad (3.9)$$

We make the substitution $p' \rightarrow p + kx$, such that the denominator becomes even in p' . By using equation (3.8) and (3.9), the polarisation tensor, equation (3.5), now reads

$$i\Pi_{\mu\nu}(k^2) = -e^2 \int_0^1 dx \int \frac{d^4p'}{(2\pi)^4} \frac{N}{[p'^2 + k^2x(1-x) - m^2]^2}, \quad (3.10)$$

where the numerator N is

$$N = 4 \left[(m^2 + k^2x(1-x))g_{\mu\nu} + 2p'_\mu p'_\nu - p'^2 g_{\mu\nu} - 2k_\mu k_\nu x(1-x) \right] + \text{terms linear in } (p').$$

Since the denominator is even in p' , we only get contribution from even terms in p' in the numerator. This is because we integrate over space-time.

We must take some care before evaluating the integral. The denominator contains a pole for $m^2 > k^2/4$, which might be integrated over.¹ Notice that $k^2 = (2m)^2$ is the threshold energy for creating real electron-positron pairs. We can avoid the pole by evaluating our system for small values of k^2 .

By performing a Wick rotation, see Appendix A, we make the change from Minkowski space to Euclidean space. This results in the following transformation of the integrating variable, $p^0 \rightarrow ip_E^0$ and $\mathbf{p}' \rightarrow \mathbf{p}_E$. We will now drop the subscript E in p_E in further calculations. Equation (3.10) can be written in Euclidean space, and reads

$$i\Pi_{\mu\nu}(k^2) = -4ie^2 \int_0^1 dx \int \frac{d^4p}{(2\pi)^4} \times \frac{[(m^2 + k^2x(1-x))g_{\mu\nu} - 2p_\mu p_\nu + p^2 g_{\mu\nu} - 2k_\mu k_\nu x(1-x)]}{[p^2 - k^2x(1-x) + m^2]^2}. \quad (3.11)$$

To solve this expression we use dimensional regularisation, see Chapter 1. This method allows us to isolate the divergent terms. Our denominator is $\mathcal{O}(p^4)$ and numerator is $\mathcal{O}(p^4)$ and $\mathcal{O}(p^6)$, meaning that we should expect a quadratic ultraviolet divergent term and a logarithmic ultraviolet divergent term.

In order to apply dimensional regularisation, we must introduce a dimensionful scale, analogous to our example in Chapter 1. We integrate in d dimensions and multiply the integral with μ^{4-d} . Equation (3.11) can now be separated into two parts, one containing the quadratically divergent terms, and one containing the logarithmic divergent terms. We write the integral as a d -dimensional integral,

$$i\Pi_{\mu\nu}(k^2) = -4i\mu^{4-d}e^2 \int_0^1 dx \int \frac{d^d p}{(2\pi)^d} \frac{N_2 + N_{\ln}}{[p^2 - k^2x(1-x) + m^2]^2}, \quad (3.12)$$

where we have denoted

$$N_2 = [p^2 + m^2 - k^2x(1-x)] g_{\mu\nu} - \frac{2}{d} g_{\mu\nu} p^2 \quad (3.13)$$

$$N_{\ln} = 2x(1-x) (k^2 g_{\mu\nu} - k_\mu k_\nu). \quad (3.14)$$

In the calculations from equation (3.11) to equation (3.12), we have used that $p_\mu p_\nu = \frac{g_{\mu\nu}}{d} p^2$. By using equation (A.19) and (A.27), we will show that that the two terms in equation (3.13) cancel. We have

$$i\Pi_{\mu\nu}^{N_2} = \frac{[m^2 - k^2x(1-x)]^{\frac{d}{2}-1}}{(4\pi)^{\frac{d}{2}}} \Gamma\left(1 - \frac{d}{2}\right) - \frac{[m^2 - k^2x(1-x)]^{\frac{d}{2}-(2-1)}}{(4\pi)^{\frac{d}{2}}} \Gamma\left(2 - 1 - \frac{d}{2}\right), \quad (3.15)$$

¹The largest value of $x(1-x)$, for $x \in [0, 1]$ is $\frac{1}{4}$.

where we can see that the two terms cancel. The vacuum polarisation tensor can now be written as a single integral of the logarithmically divergent term,

$$i\Pi_{\mu\nu}(k^2) = -4i\mu^{4-d}e^2 \int_0^1 dx \int \frac{d^d p}{(2\pi)^d} \frac{2x(1-x)(k^2 g_{\mu\nu} - k_\mu k_\nu)}{[p^2 - k^2 x(1-x) + m^2]^2}. \quad (3.16)$$

Notice that the numerator is independent of p . We can therefore use equation (A.19) to write the polarisation tensor as a function of a Gamma function. The dimension is set to $d = 4 - 2\epsilon$ as in the preliminaries, and the polarisation tensor now reads

$$i\Pi_{\mu\nu}(k^2) = -\frac{8ie^2}{(4\pi)^2} \int_0^1 dx \left[x(1-x)(k^2 g_{\mu\nu} - k_\mu k_\nu) \left(\frac{4\pi\mu^2}{m^2 - k^2 x(1-x)} \right)^\epsilon \Gamma(\epsilon) \right], \quad (3.17)$$

We now series expand the Gamma function and the factor $\left(\frac{4\pi\mu^2}{m^2 - k^2 x(1-x)} \right)^\epsilon$ in powers of ϵ , see Appendix A. The series expansion yields

$$\begin{aligned} \left(\frac{4\pi\mu^2}{m^2 - k^2 x(1-x)} \right)^\epsilon &\approx 1 + \epsilon \ln \left(\frac{4\pi\mu^2}{m^2 - k^2 x(1-x)} \right) + \mathcal{O}(\epsilon^2) \\ \Gamma(\epsilon) &\approx \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \\ \left(\frac{4\pi\mu^2}{m^2 - k^2 x(1-x)} \right)^\epsilon \Gamma(\epsilon) &\approx \frac{1}{\epsilon} - \gamma_E + \ln \left(\frac{4\pi\mu^2}{m^2 - k^2 x(1-x)} \right) + \mathcal{O}(\epsilon), \end{aligned} \quad (3.18)$$

where γ_E is the Euler-Mascheroni constant.

We recognise the projection tensor in equation (3.17), $(k^2 g_{\mu\nu} - k_\mu k_\nu) = k^2 P_{\mu\nu}$. This can be factored out and the result is a vacuum polarisation tensor on the form that we predicted, i.e.

$$i\Pi_{\mu\nu}(k^2) = ik^2 P_{\mu\nu} \Pi(k^2). \quad (3.19)$$

As we can see, the vacuum polarisation tensor is transverse and Lorentz invariant. The scalar quantity has been identified and reads

$$\begin{aligned} \Pi(k^2) &= -\frac{8e^2}{(4\pi)^2} \int_0^1 dx \left\{ x(1-x) \left[\frac{1}{\epsilon} - \gamma + \ln \left(\frac{4\pi\mu^2}{m^2 - k^2 x(1-x)} \right) \right] \right\} \\ &= -\frac{4e^2}{3(4\pi)^2} \left\{ \frac{1}{\epsilon} + 6 \int_0^1 dx \left[x(1-x) \ln \left(\frac{\mu^2}{m^2 - k^2 x(1-x)} \right) \right] \right\}. \end{aligned} \quad (3.20)$$

The auxiliary mass scale μ was modified in the last step, $\mu^2 \rightarrow \frac{\mu^2 e^{\gamma_E}}{4\pi}$. The modification simplifies the notation, and is commonly performed when using the modified minimal subtraction renormalisation scheme, or the $\overline{\text{MS}}$ -scheme. We use renormalisation to treat the divergent term, $\propto \frac{1}{\epsilon}$, in the polarisation tensor.

3.2 Renormalisation

Physicists have developed tools to treat divergences in quantum field theory, e.g. regularisation and renormalisation. Divergent integrals are common in quantum field calculations, and it is no surprise that the calculation of $\Pi(k^2)$ resulted in a divergent term. The divergent term was isolated by using dimensional regularisation. We still have to choose a renormalisation scheme. There are various choices of renormalisation schemes available, e.g. on-shell renormalisation, momentum subtraction, minimal subtraction [25, 26] (MS) and modified minimal subtraction ($\overline{\text{MS}}$). Each scheme has their own way of treating the divergent term, which is isolated by regularisation. In this thesis we will use the $\overline{\text{MS}}$ -scheme.

We must return to the Lagrangian in order to interpret the result obtained from calculating the polarisation tensor. Let us introduce a superscript B to original quantities in the Lagrangian, e.g. $A_\mu \rightarrow A_\mu^{\text{B}}$. This is introduced in order to separate the bare values from the dressed values, as discussed in Chapter 1. The dressed quantities will be written without any extra notation, i.e. A_μ .

We begin by looking at the exact propagator, equation (3.3). The divergent result affects the photon propagator. We absorb the divergent term into the photon field by a factor $Z_3^{\frac{1}{2}}$. The Z_3 is known as the field strength renormalisation since it is absorbed into the bare photon field. We write the renormalised photon field as

$$A_\mu = Z_3^{-\frac{1}{2}} A_\mu^{\text{B}}. \quad (3.21)$$

Recall that the propagator is found by inverting the quadratic term in the Lagrangian, we thus have to invert the scalar correction, $Z_3^{\frac{1}{2}}$, when going back to the Lagrangian. Similar field strength renormalisation is introduced to the Dirac field, i.e.

$$\Psi = Z_2^{-\frac{1}{2}} \Psi. \quad (3.22)$$

The Lagrangian now reads

$$\mathcal{L}_{\text{QED}} = Z_2 \bar{\Psi} (i\rlap{/}\partial - m^{\text{B}}) \Psi - \frac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu} - e^{\text{B}} Z_2 Z_3^{\frac{1}{2}} \bar{\Psi} \gamma^\mu \Psi A_\mu, \quad (3.23)$$

where e^{B} denotes the bare coupling constant, and m^{B} denotes the bare mass of the fermion. Although we have rescaled the fields, the Lagrangian is still dependent on bare quantities. We can write the Lagrangian purely by dressed quantities, if we introduce counterterms. The counterterms are introduced to renormalise the masses and coupling constants. The Lagrangian is written as

$$\mathcal{L}_{\text{QED}} = \bar{\Psi} (i\rlap{/}\partial - m) \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e \bar{\Psi} \gamma^\mu \Psi A_\mu + \mathcal{L}_{\text{ct}}, \quad (3.24)$$

where

$$\mathcal{L}_{\text{ct}} = \bar{\Psi} (i\delta_{Z_2} \rlap{/}\partial - \delta_m) \Psi - \frac{1}{4} \delta_{Z_3} F_{\mu\nu} F^{\mu\nu} - \delta_e \bar{\Psi} \gamma^\mu \Psi A_\mu. \quad (3.25)$$

We can easily check that the coefficients must be

$$\begin{aligned}
 \delta_{Z_2} &= Z_2 - 1 \\
 \delta_{Z_3} &= Z_3 - 1 \\
 \delta_m &= Z_2 m^B - m \\
 \delta_e &= Z_2 Z_3^{\frac{1}{2}} e^B - e.
 \end{aligned} \tag{3.26}$$

The counterterms act as corrections in a similar way as the 1PI-diagrams, e.g. the vacuum polarisation. It is common to represent the corrections graphically as in Figure 3.2. The photon propagator now contains an additional term due to the counterterm, δ_{Z_3} . We obtain

$$i\bar{\Pi}^{\mu\nu}(k^2) = ik^2 P^{\mu\nu} [\Pi(k^2) - \delta_{Z_3}] = ik^2 P^{\mu\nu} \bar{\Pi}(k^2). \tag{3.27}$$

This is an effective vacuum polarisation, and can be made finite with an adequate choice of δ_{Z_3} . The choice of δ_{Z_3} depends on the renormalisation scheme. We use the $\overline{\text{MS}}$ -scheme in this calculation. This is a particularly convenient scheme to apply, as we only subtract the divergent term.² Another advantage with this scheme is that the subtracted term is mass independent, unlike the on-shell renormalisation scheme [18]. The divergent term can be found in equation (3.20) and is canceled by the counterterm. Thus, the coefficient associated with the counterterm must be

$$\delta_{Z_3} = -\frac{4e^2}{3(4\pi)^2} \frac{1}{\epsilon}, \tag{3.28}$$

in the $\overline{\text{MS}}$. Notice that the vacuum polarisation is finite after the renormalisation. Recall that the vacuum polarisation is associated with the one particle irreducible diagrams. The vacuum polarisation is the leading order in α of the one particle irreducible diagrams. We have

$$\begin{aligned}
 \text{---} \textcircled{\text{1PI}} \text{---} &= \text{---} \textcircled{} \text{---} + \text{---} \text{X} \text{---} + \dots \\
 &\approx i\bar{\Pi}^{\mu\nu}(k^2).
 \end{aligned} \tag{3.29}$$

The only contribution included is the vacuum polarisation and its counterterm. To get the exact propagator, we must add all propagator like diagrams, as in Section 3.1. We use equation (3.3) to express the corrected photon propagator, but substitutes the divergent scalar Π with the finite scalar $\bar{\Pi}$. The photon propagator now reads

$$\tilde{D}^{\mu\nu} = D^{\mu\nu} \frac{1}{1 - \bar{\Pi}(k^2)} = D^{\mu\nu} \left\{ 1 + \frac{8e^2}{(4\pi)^2} \int_0^1 dx \left[x(1-x) \ln \left(\frac{\mu^2}{m^2 - k^2 x(1-x)} \right) \right] \right\}^{-1}. \tag{3.30}$$

In Chapter 4, we will show that this correction modifies the Hamiltonian in non-relativistic quantum electrodynamics (NRQED), and alters the energy spectrum of a the hydrogen atom.

²The modified MS-scheme also subtract constant terms when modifying the auxiliary mass scale.

$$\text{---}\times\text{---} = ik^2 P^{\mu\nu}$$

Figure 3.2: The diagram represents the counterterm in the Lagrangian for the vacuum polarisation of the photon. Figure is adapted from Peskin & Schroeder [16].

3.3 Non-relativistic limit

In bound state physics, the rest mass is much larger than the kinetic energy, i.e. $k^2 \ll m^2$. This is because the binding energy of a bound state system is very small compared to the rest energy of an electron. The binding for the ground state of the hydrogen atom is 13.6eV [21], while the rest energy of an electron is 511keV. We can thus approximate the four-potential to a fermion line to be slightly off-shell $p_\mu = (E_0, \mathbf{p})$, where $E_0 \approx m$. The zeroth component of the four-momentum associated with the photon is zero, as a result of the approximation on the four-momentum of the fermion. In QED, the photon will always connect with a fermion line. The four-momentum of the photon is determined by the momentum transfer of the fermion, i.e. $k_\mu = p_\mu - p'_\mu = (0, \mathbf{p} - \mathbf{p}')$, i.e. $k^2 = -\mathbf{k}^2$.

Lets evaluate the integral in the second term in equation (3.30) by first expanding the logarithm in powers of $\frac{\mathbf{k}^2}{m^2}$, and use $k_0 = 0$. We find

$$\begin{aligned} \ln \left(\frac{\mu^2}{m^2 \left[1 + \frac{\mathbf{k}^2}{m^2} x(1-x) \right]} \right) &= \ln \left(\frac{\mu^2}{m^2} \right) + \ln \left(\frac{1}{\left[1 + \frac{\mathbf{k}^2}{m^2} x(1-x) \right]} \right) \\ &\approx \ln \left(\frac{\mu^2}{m^2} \right) - \ln \left[1 + \frac{\mathbf{k}^2}{m^2} x(1-x) \right] \\ &\approx \ln \left(\frac{\mu^2}{m^2} \right) - \frac{\mathbf{k}^2}{m^2} x(1-x), \end{aligned} \quad (3.31)$$

where terms of higher order than $\mathcal{O}\left(\frac{\mathbf{k}^2}{m^2}\right)$, have been omitted. The integral in equation (3.30) can thus be solved analytically by using the approximation above. This yields

$$\begin{aligned} \int_0^1 dx \left[x(1-x) \ln \left(\frac{\mu^2}{m^2 + \mathbf{k}^2 x(1-x)} \right) \right] &\approx \\ \int_0^1 dx \left\{ x(1-x) \left[\ln \left(\frac{\mu^2}{m^2} \right) - \frac{\mathbf{k}^2}{m^2} x(1-x) \right] \right\} &= \frac{1}{6} \ln \left(\frac{\mu^2}{m^2} \right) - \frac{1}{30} \frac{\mathbf{k}^2}{m^2}. \end{aligned} \quad (3.32)$$

The photon propagator in the non-relativistic limit can be written as

$$\tilde{D}^{\mu\nu} = D^{\mu\nu} \left[1 + \frac{\alpha}{3\pi} \ln \left(\frac{\mu^2}{m^2} \right) - \frac{\alpha}{15\pi} \frac{\mathbf{k}^2}{m^2} \right]^{-1}, \quad (3.33)$$

where $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant.

NRQED is an effective field theory developed by Caswell and Lepage [8]. It systematically incorporates relativistic and radiative corrections into the non-relativistic Hamiltonian. The Hamiltonian is constructed in powers of α and $\frac{p}{m} \sim v$, where α is the fine-structure constant, m is the fermion mass, p is the momentum, and v the speed of the fermion in units of speed of light. Since its introduction, NRQED has been repeatedly reviewed. The paper [27], by Kinoshita and Nio, is one of the more thorough reviews, and is relevant for this thesis. It is the first paper in a series of papers that outline the theory. In the paper, Kinoshita and Nio derive the Lagrangian including corrections of $\mathcal{O}(v^4)$ and $\alpha^2(Z\alpha)$. Kinoshita and Nio derive the hyperfine structure, a correction to the non-relativistic energy spectrum which is about 1000 times smaller than the Lamb shift correction [21].

The construction of the Lagrangian in NRQED will be outlined, using the same notation as Kinoshita and Nio [27]. We will first present a transformation called the Foldy-Wouthuysen transformation [28]. This transformation decouples the upper components from the lower components of the Dirac field. The upper and lower component of the Dirac field describes the fermion and the anti-fermion sector respectively. If we successfully decouple the two sectors, we can use the two-component Pauli spinor to represent the fermion field in the Lagrangian.

4.1 Foldy-Wouthuysen transformation

If we are able to decouple the upper from the lower components of the Dirac spinor, the particle and anti-particle sectors can be analysed separately. We will show that the decoupling can be performed to a certain order in v by applying a canonical unitary transformation to the Lagrangian. The transformation is named after Foldy and Wouthuysen, who in 1950 introduced the transformation [28].

The operators in the transformed Lagrangian can not have off-diagonal block elements in the Pauli-Dirac basis.¹ This basis is the choice of convention of the gamma matrices, we use the same basis as is presented in [29]. Diagonal operators in this basis are called even operators, as opposed to odd operators which contains off-diagonal block elements. The product between two odd or even operators yields an even operator, while the product between one odd and one even operator, produces an odd operator. For instance, the generalised spin operators, $\Sigma_i = \text{diag}(\sigma_i, \sigma_i)$, where σ_i are the Pauli matrices, are even operators. The gamma matrices, γ^i , are odd operators. We apply minimal coupling to the Dirac Lagrangian. It reads

$$\mathcal{L}_{\text{DL}} = \bar{\Psi} (i\cancel{D} - m) \Psi, \quad (4.1)$$

where we have used the Feynman slash notation on the covariant derivative. The spatial part of the operator $\cancel{D} = \gamma^\mu D_\mu = \gamma^\mu (\partial_\mu - ieA_\mu)$ is odd, due to the gamma matrices. We can find the

¹By block elements, we mean the 2×2 matrices which constitute the 4×4 -matrices in QED, e.g. the Pauli matrices are block elements in the gamma matrices.

equation of motion, by varying the action, $S = \int d^4x \mathcal{L}_{\text{DL}}$, with respect to $\bar{\Psi}$. This yields

$$\frac{\partial \mathcal{L}_{\text{DL}}}{\partial \bar{\Psi}} = (i\not{D} - m) \Psi = 0. \quad (4.2)$$

By substituting the covariant derivative into the equation above, we find

$$i\partial_t \Psi = (\gamma^0 m + i\alpha^i \partial_i + e\alpha^i A_i - eA_0) \Psi = H_{\text{D}} \Psi, \quad (4.3)$$

where H_{D} is the Dirac Hamiltonian and $\alpha^i = \gamma^0 \gamma^i$.

we can find an expression for the transformed Hamiltonian, H'_{D} , by using the equation of motion, equation (4.3). Let us denote the unitary transformation by $U_{\text{FW}} = e^{iS}$, where S is a time-dependent operator. We transform the Dirac field by $\Psi \rightarrow U_{\text{FW}} \Psi$, and find

$$H_{\text{D}} \Psi = i\partial_t \Psi = i\partial_t e^{-iS} \Psi' = i(\partial_t e^{-iS}) \Psi' + i e^{-iS} \partial_t \Psi'. \quad (4.4)$$

The transformed Dirac Hamiltonian is found by

$$i\partial_t \Psi' = e^{iS} (H_{\text{D}} - i\partial_t) e^{-iS} \Psi' \equiv H'_{\text{D}} \Psi'. \quad (4.5)$$

We can now perform the Foldy-Wouthuysen transformation to the Dirac Hamiltonian. In general, the transformation is time dependent. Thus, we can not find one transformation which cancels all odd operators. The transformation will only cancel the leading order odd operator in powers of v . We will only sketch the procedure, since this is done rigorously by Bjorken & Drell [30], and in the specialisation project in advance of this thesis [31].

A Taylor expansion is applied to the unitary transformation, U_{FW} . This results in commutators of the operator S and the operators in the Hamiltonian. The operator S can be chosen to cancel the leading order odd operator. Each transformation removes odd operators at next-to-leading order in $\frac{v}{m} = v$. Notice that v is small in the non-relativistic limit, it is a fraction of the speed of light when using natural units.

The Foldy-Wouthuysen transformation yields the following transformed Dirac Hamiltonian [30–33]

$$\begin{aligned} H'''_{\text{D}} = & \left(m - \frac{\mathbf{D}^2}{2m} - \frac{\mathbf{D}^4}{8m^3} - eA_0 + \frac{e}{2m} \boldsymbol{\sigma} \cdot \mathbf{B} \right) \psi \\ & + \left[\frac{e}{4m^2} \boldsymbol{\sigma} \cdot (\mathbf{E} \times \mathbf{p}) + \frac{e}{8m^2} \nabla \cdot \mathbf{E} + \frac{ie}{8m^2} \boldsymbol{\sigma} \cdot (\nabla \times \mathbf{E}) \right] \psi + \dots, \end{aligned} \quad (4.6)$$

where ψ is the two-component Pauli spinor, $\boldsymbol{\sigma}$ is the Pauli matrices, \mathbf{E} and \mathbf{B} are the electric and magnetic field respectively and $\mathbf{D} = i(\mathbf{p} - e\mathbf{A})$. We had to transform The Hamiltonian three times to get the final result, therefore it is denoted by three apostrophes.

4.2 NRQED Lagrangian

One of the many advantages of NRQED is that it is manifest gauge invariant. By construction, we can enforce that the NRQED Hamiltonian respects gauge invariance. NRQED, as any effective field theory, must respect the same symmetries as the underlying theory. Another advantage

of NRQED is that the Lagrangian is written in terms of the two-component Pauli spinor, as opposed to the four-component Dirac spinor. We can freely focus on the particle sector alone, since the particle and anti-particle sectors are decoupled. The decoupling of the particle sector from the anti-particle sector is limited to a certain power of $\frac{p}{m}$, which is evident from the Foldy-Wouthuysen transformation. We will later see that the NRQED Lagrangian contains operators which are not found by the Foldy-Wouthuysen transformation, e.g. the vacuum polarisation of the photon. This is a radiative correction, which is not incorporated in the Foldy-Wouthuysen transformation. Radiative corrections are not found in classical physics. The photon is treated the same way as in QED, because it is relativistic of nature [27]. NRQED does not have vacuum polarisation diagrams in the same way as QED. The vacuum polarisation of the photon is incorporated through additional terms in the photon Lagrangian [32].

We will follow the paper by Paz [34] when constructing the NRQED Lagrangian. The symmetries of NRQED are gauge and parity invariance, time reversal, hermiticity, and invariant under Galilean transformations. NRQED is a non-relativistic theory, and is thus invariant under Galilean transformations, while QED is relativistic and invariant under Lorentz transformations. Thus, the Lagrangian can not contain any vector-like terms. The NRQED Lagrangian will be written in terms containing the Pauli spinor, ψ , the electric field, \mathbf{E} , the magnetic field, \mathbf{B} , and the Pauli matrices, σ_i . We can represent the electric and magnetic field by using the components of A_μ . To remind the reader, the electric and magnetic field can be written in terms of the four-potential, i.e.

$$\begin{aligned}\mathbf{E} &= -\nabla A_0 - \partial_t \mathbf{A} \\ \mathbf{B} &= \nabla \times \mathbf{A}.\end{aligned}\tag{4.7}$$

Hermiticity is respected by using Hermitian operators. The eigenvalues of any Hermitian operator are real. This guarantees real energy eigenvalues. We will use the components of the covariant derivative, $D_\mu = \partial_\mu - ieA_\mu$, to construct the NRQED Lagrangian.

Parity invariance and time reversal symmetry can be analysed by looking at the different terms we want to use when constructing the Lagrangian. The electromagnetic vector potential and the position vector are classical vectors which are odd under parity transformation. The scalar potential and time variable are scalar quantities which transform even under parity. With this, we have that the spatial part of the covariant derivative, $\mathbf{D} = \nabla - ie\mathbf{A}$, and the electric field must be parity odd. The magnetic field and the zeroth component of the covariant derivative, $D_t = \partial_t - ieA_0$, must be parity even. Spin is an axial vector, thus the Pauli matrices are even under parity.

As for the time reversal symmetry, we know that time is odd under time reversal, and position is even. The time reversal also leads to complex conjugation [30], because the transformation is anti-unitary. Thus we have that $iD_t = i\partial_t + eA_0$ is even under time reversal, and $i\mathbf{D} = i\nabla + e\mathbf{A}$ is odd under time reversal. This implies that the vector potential, \mathbf{A} , is odd, whilst the scalar potential, A_0 , is even under time reversal. The Pauli matrices are odd under time reversal. Both time reversal and parity invariance are extensively discussed by Bjorken and Drell [20].

The Lagrangian must be constructed with terms respecting even parity and time reversal symmetry, in agreement with QED. When constructing the NRQED Lagrangian, there are two corrections we must take into account, namely the relativistic and radiative corrections. The relativistic corrections are incorporated in a series expansion in powers of $\frac{p}{m}$ or equivalently, in

powers of v . The radiative corrections are incorporated through a power series in α , which are found by taking loop diagrams into consideration. Loop diagrams add two vertices with coupling constant e , thus each additional loop diagram contributes with an additional power in α .

We use the energy-momentum relation to indicate to which order in v the operators are. The energy-momentum relation reads

$$\begin{aligned}
 E &= \sqrt{\mathbf{p}^2 + m^2} = m \left(1 + \frac{\mathbf{p}^2}{2m^2} - \frac{\mathbf{p}^4}{8m^4} + \dots \right) \\
 &= m + \left(\frac{\mathbf{p}^2}{2m} - \frac{\mathbf{p}^4}{8m^3} + \dots \right) \\
 &= m + \left[\frac{\mathbf{p}^2}{2m} \left(1 - \frac{\mathbf{p}^2}{4m^2} + \dots \right) \right]. \tag{4.8}
 \end{aligned}$$

It should be familiar that the energy for a free non-relativistic particle is $m + \frac{\mathbf{p}^2}{2m}$. Therefore the relativistic correction, $\frac{\mathbf{p}^4}{8m^3}$, is a correction of $\mathcal{O}(v^2)$, relative to the free particle.

Recall that minimal coupling is $\mathbf{p} \rightarrow \mathbf{p}' = \mathbf{p} + e\mathbf{A} = i\mathbf{D}$. We know that the momentum is associated with the derivative. Thus, the \mathbf{D}^4 is accompanied by a factor $\frac{1}{m^3}$. This is shown in the Foldy-Wouthuysen transformation and in the energy-momentum relation. The term \mathbf{D}^4 is of $\mathcal{O}(v^4)$, i.e. a $\mathcal{O}(v^2)$ correction. Terms of higher order than \mathbf{D}^4 will be omitted. Notice that $\frac{\mathbf{D}^2}{2m}$ and $\frac{\mathbf{D}^4}{8m^3}$ were derived from the expansion of the energy-momentum relation. These terms have coefficients which are unaffected by radiative corrections [27]. The results from the Foldy-Wouthuysen transformation indicates other types of operators we can expect to find. The Foldy-Wouthuysen transformation contains all the operators of the fermion NRQED Lagrangian at $\mathcal{O}(v^2)$. These operators can be found by using combinations of the covariant derivative, and the symmetries of NRQED. This is done rigorously in [34], and will not be shown here.

Operators which are not found from the series expansion of the relativistic energy-momentum relation will be accompanied by a coefficient written as a power series in α . The power series incorporates radiative corrections. We will normalise the coefficients in such a way that every power series start at unity. The leading order of the coefficients, the zeroth order in α , is equivalent to the coefficients found from the Foldy-Wouthuysen transformation.

We should point out that the Lagrangian is not uniquely defined, but defined up to a surface term. One can always perform a partial integration to one of the terms in the Lagrangian. If the partial integration of one operator result in a new operator, the two operators are considered equivalent. The partial integration will also result in a surface term, which can in most cases be set to zero. We can omit the surface terms if we assume that the fields are zero at the boundary [12]. Two equivalent operators are not independent of one another, as one can be derived from the other. For instance, the term containing the operator $\nabla \cdot \mathbf{E}$, can be partially integrated to obtain the operator $-\mathbf{E} \cdot \nabla$, plus a surface term. Thus, the two operators are equivalent up to a surface term, and connected through a partial integration. The two operators have the same coefficient. In addition to the fermion NRQED Lagrangian, we will later see that the photon NRQED Lagrangian has additional terms in the NRQED Lagrangian. The photon NRQED Lagrangian contains corrections, which are not obtained using the Foldy-Wouthuysen transformation.

We now understand that the relativistic aspect of bound state physics is manifested in a series expansion in v . Relativistic corrections are taken into account during the construction of the

Lagrangian. The Lamb shift is an $\mathcal{O}(v^2\alpha^5)$ correction [9]. We identify the power series in α by using power counting rule. We will elaborate on the power counting rules later.

Recall that the QED Lagrangian can be written as

$$\mathcal{L}_{\text{QED}} = \bar{\Psi} (iD_\mu \gamma^\mu - m) \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (4.9)$$

where $\bar{\Psi} = \Psi^\dagger \gamma^0$. This is expressed using a four-component Dirac field, containing both particle and anti-particle components. The Foldy-Wouthuysen transformation showed that we can express the Dirac Lagrangian by a non-relativistic expansion in powers of $\frac{p}{m} \sim v$, where the upper and lower components decouple. Thus, we can express the NRQED Lagrangian in terms dependent on the two-component Pauli spinor. We can replace the Dirac Lagrangian by the NRQED Lagrangian, which reads [9, 34]

$$\mathcal{L}_{\text{NRQED}} = \mathcal{L}_0 + \mathcal{L}_{\text{rel}} + \mathcal{L}_{\text{F}} + \mathcal{L}_{\text{D}} + \mathcal{L}_{\text{SO}} + \mathcal{L}_\gamma, \quad (4.10)$$

where

$$\begin{aligned} \mathcal{L}_0 &= \psi^\dagger \left(iD_t + \frac{\mathbf{D}^2}{2m} \right) \psi \\ \mathcal{L}_{\text{rel}} &= \psi^\dagger \delta_{\text{rel}} \frac{\mathbf{D}^4}{8m^3} \psi \\ \mathcal{L}_{\text{F}} &= -\psi^\dagger \delta_{\text{F}} \frac{e\boldsymbol{\sigma} \cdot \mathbf{B}}{2m} \psi \\ \mathcal{L}_{\text{D}} &= -\psi^\dagger \delta_{\text{D}} \frac{e(\mathbf{D} \cdot \mathbf{E} - \mathbf{E} \cdot \mathbf{D})}{8m^2} \psi \\ \mathcal{L}_{\text{SO}} &= -\psi^\dagger \delta_{\text{SO}} \frac{ie\boldsymbol{\sigma} \cdot (\mathbf{D} \times \mathbf{E} - \mathbf{E} \times \mathbf{D})}{8m^2} \psi, \end{aligned} \quad (4.11)$$

and ψ is the two-component Pauli spinor. The last term, \mathcal{L}_γ , is the term associated with the photon. We will discuss the photon Lagrangian later in this section. Notice that every operator in the fermion Lagrangian is also present in the Foldy-Wouthuysen transformation. This follows naturally, given that the Foldy-Wouthuysen transformation incorporates relativistic corrections. The terms in the fermion Lagrangian can all be treated as perturbations, and we can use Rayleigh-Schrödinger perturbation theory to find the energy correction [10]. This is not the case for the photon. The photon terms can not be treated as perturbations.

The coefficients, δ_i , where $i = [\text{rel}, \text{D}, \text{F}, \text{SO}]$, are found by using a matching procedure. This procedure matches the scattering diagrams in NRQED to the scattering diagrams in QED, or the two-point function in NRQED to the two-point function in QED. The matching procedure enforces that the scattering amplitudes in the two theories must agree. Matching the NRQED Lagrangian at tree level, results in coefficients equal to the coefficients from the Foldy-Wouthuysen transformation. Going beyond tree level, we must add radiative corrections. The addition of a loop correction gives a contribution which has an additional power in α , due to the addition of two coupling constants e . This gives a power series in α .

We will perform NRQED calculations in the Coulomb gauge. This is advantageous since it isolates the Coulomb photon. The Coulomb gauge separates the Coulomb photon from the

spatial photon field [9]. In the Coulomb gauge, two new photon interactions are added to the photon Lagrangian [27].

We can derive the two new photon interactions by adding a correction to the electromagnetic field tensor. The correction must be Lorentz invariant, Hermitian and respect the same parity and time symmetry as the electromagnetic field tensor, $F_{\mu\nu}$. We can add the D'Alembertian operator between the electromagnetic field tensors in the Maxwell Lagrangian. In QED, the photon field has mass dimension one, which implies that the operator must be of mass dimension two. The action in the path integral formalism must be dimensionless. Thus, in four dimensions the Lagrangian must be of mass dimension four. This is also mentioned in earlier discussion, when we added the auxiliary mass μ in Chapter 3. The D'Alembertian operator must be multiplied by a factor $\frac{1}{m^2}$. By using the approximation $k_0 = 0$,² and Fourier transforming the Lagrangian to momentum space, we find

$$\mathcal{L}_\gamma = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\bar{\delta}_{\text{VP}}}{m^2}F_{\mu\nu}\mathbf{k}^2F^{\mu\nu} \quad (4.12a)$$

$$= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \mathcal{L}_{\text{VPt}} + \mathcal{L}_{\text{VPs}}, \quad (4.12b)$$

where

$$\mathcal{L}_{\text{VPt}} = \delta_{\text{VP}} \frac{\alpha}{15\pi} A^0(\mathbf{k}) \frac{\mathbf{k}^4}{m^2} A^0(\mathbf{k}) \quad (4.13)$$

$$\mathcal{L}_{\text{VPs}} = -\delta_{\text{VP}} \frac{\alpha}{15\pi} A^i(k) \frac{\mathbf{k}^4}{m^2} A^j(k) \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} \right). \quad (4.14)$$

The t and s in the subscript stands for time and spatial part respectively. The steps between equation (4.12a) and (4.12b) are similar to the mathematical steps shown in Section 2.3. We have changed the notation from $\bar{\delta}_{\text{VP}} \rightarrow \delta_{\text{VP}}$. We factored out the leading order in α , such that δ_{VP} is normalised to be one for the leading order. The leading order contribution is the vacuum polarisation of the photon.

4.2.1 NRQED Hamiltonian

To calculate the energy shift, we must determine the Hamiltonian density. The NRQED Hamiltonian can be found by performing a Legendre transformation to the NRQED Lagrangian. We first identify the conjugated momentum for the photon field, $\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu}$, and for the fermion field $\pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}}$. The photon field yields three conjugated momentum, since the Lagrangian is independent of \dot{A}^0 . Thus, the zeroth component of the photon field is not a dynamical variable. We find that the conjugated momentum for the photon and the fermion field are

$$\begin{aligned} \pi_{A_i} &= \frac{\partial \mathcal{L}_{\text{NRQED}}}{\partial \dot{A}_i} = F^{0i} = E^i \\ \pi_\psi &= \frac{\partial \mathcal{L}_{\text{NRQED}}}{\partial \dot{\psi}} = i\psi^\dagger. \end{aligned} \quad (4.15)$$

²Justified by the non-relativistic limit, see Section 3.3.

The Hamiltonian is found by the Legendre transformation [9, 20],

$$\mathcal{H}_{\text{NRQED}} = \psi^\dagger \left(\frac{\mathbf{p}^2}{2m} + qA_0 \right) \psi + \mathcal{H}_{\text{rel}} + \mathcal{H}_{\text{F}} + \mathcal{H}_{\text{D}} + \mathcal{H}_{\text{SO}} + \mathcal{H}_{\gamma}, \quad (4.16)$$

where each of the \mathcal{H}_i is its negative equivalence to \mathcal{L}_i . There is one exception to the latter statement, i.e.

$$\mathcal{H}_{\gamma} = \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) + \mathcal{H}_{\text{VPt}} + \mathcal{H}_{\text{VPs}}, \quad (4.17)$$

where we have used that $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)$ [20]. The last two terms in the equation above are $\mathcal{H}_{\text{VPt}} = -\mathcal{L}_{\text{VPt}}$ and $\mathcal{H}_{\text{VPs}} = -\mathcal{L}_{\text{VPs}}$. Notice that when performing the Legendre transformation, the interaction terms are only affected by a change in sign. This is also true in QED.

The Feynman rules are presented in Figure B.1. We have not derived the Feynman rules in this thesis, we adapt the Feynman rules from the paper by Kinoshita and Nio [27]. It is the vacuum polarisation contribution of the Lamb shift which is of interest, and only the Feynman rules for the Coulomb photon is presented in the figure. The Feynman rules for the fermion and spatial part of the photon can be found graphically in the paper by Kinoshita and Nio [27]. For the interaction terms, the Feynman rules could have been directly read from the Hamiltonian, see equation (4.16). We will next introduce the matching method used for identifying the coefficients.

4.3 Matching NRQED with QED

The matching procedure between the NRQED and QED involves scattering diagrams and the two-point functions of the fields. This procedure enforces that the scattering amplitudes of QED coincide with the scattering amplitudes of NRQED. To incorporate higher order corrections in powers of α , we must calculate higher order loop diagrams in QED and include higher order diagrams in NRQED.

In this thesis, we calculate corrections to order v^2 in the Lagrangian. For the fermion field, we are content with matching at tree level. At tree level, the coefficients in NRQED is equivalent to the coefficients of the Foldy-Wouthuysen transformed Dirac Hamiltonian equation (4.6). The tree-level matching does not reproduce the Lamb shift, it only reproduces the fine structure [31]. As mentioned, to find the correction with higher powers of α , we must include radiative corrections with loop diagrams in QED.

The Lamb shift is an α^5 -correction [9]. We will not derive the power counting rules in this thesis. A derivation has been worked out by Labelle in [10]. The result of the counting rules is that the leading order radiative corrections are of $\mathcal{O}(\alpha^5)$. Diagrams which contribute to leading order are discussed in [9, 27]. For the photon field, we only get contribution from the instantaneous Coulomb interaction describes by the zeroth component of the photon propagator [35]. If we were to include the next-to-leading order in α , we get retardation effects [10]. This effect account for the finite speed of a photon propagator. At the energy level of the Lamb shift, the photon interaction can be said to be instantaneous.

The power counting rules also show that the photon term can not be treated as perturbation. We can add any number of Coulomb interactions without changing the net power of α , because the Coulomb interaction is independent of the mass [36]. Each addition of a Coulomb interaction

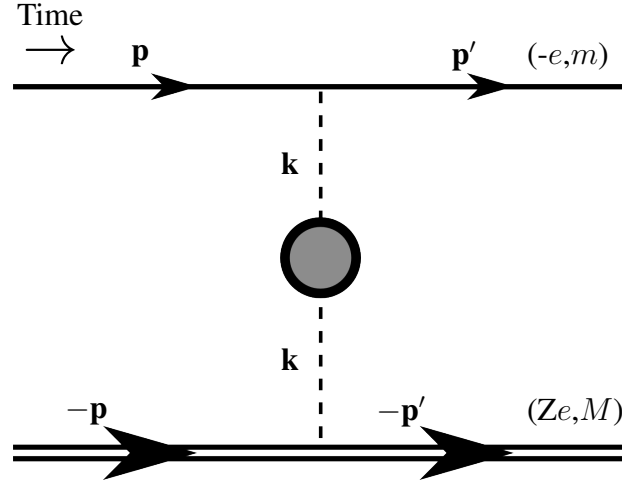


Figure 4.1: The Feynman diagram represents the vacuum polarisation contribution to the coulomb interaction in NRQED. We use solid and double solid lines to represent the fermion and the nucleus line respectively. The blob represents the correction due to vacuum polarisation, and the dotted lines are the Coulomb-photon propagators. The nucleus has mass M and charge Ze , while the fermion has mass m and charge $-e$. We adapted the figure from Labelle and Zebarjad [9].

will also decrease the power of v . Since we can add any number of Coulomb interaction, we must sum over all. We will associate the external lines with the Schrödinger wave function in the Coulomb potential. This implies that we sum over the Coulomb interactions.

The loop correction to the photon propagator will be graphically represented as a blob, see Figure 4.1. Recall that NRQED does not have loop corrections, this is simply an additional correction to the photon Hamiltonian in NRQED. The diagram in Figure 4.1 is the diagram which must be calculated to find the vacuum polarisation contribution to the Lamb shift. We use the NRQED Feynman rules, shown in Figure B.1, to calculate the diagram in the next section.

Before we perform the calculations, the coefficient, δ_{VP} , is shown to be unity to leading order correction in α . We enforce the two-point function of NRQED to match the two-point function of QED. Recall that the photon Lagrangian in NRQED is treated analogous to the photon Lagrangian in QED [27]. We find

$$\mathcal{L}_\gamma = \mathcal{L}_{\gamma\text{QED}}, \quad (4.18)$$

where we disregard the common term $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. The two Lagrangian reads

$$\mathcal{L}_\gamma = \delta_{VP} \frac{\alpha}{15\pi} \left[A^0(\mathbf{k}) \frac{\mathbf{k}^4}{m^2} A^0(\mathbf{k}) - A^i(\mathbf{k}) \frac{\mathbf{k}^4}{m^2} A^j(\mathbf{k}) \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} \right) \right], \quad (4.19)$$

and

$$\mathcal{L}_{\gamma\text{QED}} = A^\mu \left(-\mathbf{k}^2 g_{\mu\nu} - k_\mu k_\nu \right) \left[-\frac{\alpha}{15\pi} \frac{\mathbf{k}^2}{m^2} + \frac{\alpha}{3\pi} \ln \left(\frac{\mu^2}{m^2} \right) \right] A^\nu. \quad (4.20)$$

We can separate the zeroth component from the three spatial components in equation (4.20), this yields

$$\mathcal{L}_{\gamma\text{QED}} = A^0 \mathbf{k}^2 \left(\frac{\alpha}{15\pi} \frac{\mathbf{k}^2}{m^2} \right) A^0 - A^i \frac{\alpha}{15\pi} \frac{\mathbf{k}^2}{m^2} \left(\mathbf{k}^2 \delta_{ij} - k_i k_j \right) A^j, \quad (4.21)$$

where we have omitted the dependence on the auxiliary mass scale, μ . This will be elaborated below. By equating the two equations (4.19) and (4.21) we find that $\delta_{\text{VP}} = 1$.

The term which is dependent on the auxiliary mass scale, μ , were omitted from the two-point function. This term should not affect the physical energy levels since it is not a physical quantity. The auxiliary mass scale is canceled by a field redefinition. We can factor out $-\frac{\alpha}{15\pi} \frac{\mathbf{k}^2}{m^2}$ from the square bracket in equation (4.20). The square bracket now reads $\left[1 - \frac{5m^2}{\mathbf{k}^2} \ln\left(\frac{\mu^2}{m^2}\right)\right]$. We redefine the photon field by $A_\mu \rightarrow A'_\mu = \left[1 - \frac{5m^2}{\mathbf{k}^2} \ln\left(\frac{\mu^2}{m^2}\right)\right] A_\mu$.

The other coefficients introduced in the NRQED Lagrangian (4.10) can be identified by adding loop corrections to the tree level diagrams for the vertex factor in QED. By matching the coefficient at higher loop corrections of the vertex factor, we could get contributions of higher order than the leading order in α .

4.4 Lamb shift

The calculation of the energy corrections is the final part of the calculations. We will evaluate the diagram for the leading order correction to the Coulomb interaction, see in Figure 4.1. The Feynman rules in Figure B.1 are applied to the diagram [9]. The diagram is known as the Uehling potential [37], and is a correction to the Coulomb potential. We choose $Z = 1$, thus the diagram reads

$$\begin{aligned}
\Delta E_{\text{VP}} &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} \psi^*(\mathbf{p}') \left[(-e) \frac{1}{\mathbf{k}^2} \left(-\delta_{\text{VP}} \frac{\alpha}{15\pi} \frac{\mathbf{k}^4}{m^2} \right) \frac{1}{\mathbf{k}^2} (+e) \right] \psi(\mathbf{p}) \\
&= \delta_{\text{VP}} \frac{4\alpha^2}{15m^2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} \psi^*(\mathbf{p}') \psi(\mathbf{p}) \\
&= \delta_{\text{VP}} \frac{4\alpha^2}{15m^2} |\psi(0)|^2 \\
&= \delta_{\text{VP}} \frac{4\alpha^2}{15m^2} \frac{1}{\pi} \left(\frac{m\alpha}{n} \right)^3 \delta_{l0} \\
&= \delta_{\text{VP}} \frac{4m\alpha^5}{15\pi n^3} \delta_{l0},
\end{aligned} \tag{4.22}$$

where we have evaluated the wave function at the origin [29]. The integration is recognised as a Fourier transformation of the wave function from momentum to position space, for the wave function evaluated at origin. We get a delta function due to the fact that the radial function is only non-zero at the origin for the S-states. The quantum number n and l is known as the principal quantum number and the azimuthal quantum number respectively. Notice that the result shown in equation (4.22) is of order $\mathcal{O}(\alpha^5)$ as it should be. The coefficient δ_{VP} is determined by the matching procedure, and would be different from unity if we included higher order loop corrections.

Notice that the energy correction in equation (4.22) is positive. This is not in agreement with literature or experimental observations [12, 16, 33, 38–40]. The disagreement suggests a sign error in the calculation above, which we were unable to correct. As far as we can tell, the sign error is also present in the paper by Labelle and Zebarjad [9], but missed in the end result. We

make reservations that we miss crucial information in the thesis, and that there is an explanation to why Labelle and Zebarjad [9] got the correct sign in the final result. When we calculated the Lamb shift above, we used Feynman rules that coincide with the paper by Kinoshita and Nio [27].

We will demonstrate the traditional way of finding the energy shift caused by the vacuum polarisation of the photon. The vacuum polarisation corrects the Coulomb potential, and the correction is known as the Uehling potential [37]. We will not derive the Coulomb potential in a rigorous manner, as this is done by Peskin & Schroeder [16].

The Coulomb potential can be obtained by evaluating the electron-positron scattering, see Figure 4.2. As mentioned earlier, it is more convenient to do QED calculations in Feynman gauge, see equation (2.18). The scattering diagram yields the following scattering amplitude,

$$i\mathcal{M} = \bar{u}(p')(-ie)\gamma^\mu u(p) \frac{-ig_{\mu\nu}}{(p' - p)^2} \bar{v}(q)(ie)\gamma^\nu v(q'), \quad (4.23)$$

where we have used a plane wave solution for the Dirac field, i.e. $\Psi = u(p)e^{-ipx}$ for the electron, and $\Psi = v(q)e^{iqx}$ for the positron. Momentum-conservation have been enforced in the vertices, i.e. $k = p' - p$. We want to do the calculation in the non-relativistic limit, which yields $(p' - p)^2 \approx -(\mathbf{p}' - \mathbf{p})^2 = -\mathbf{k}^2$. We can write the spinors in the non-relativistic limit as [16]

$$u(p) \approx \sqrt{m} \begin{bmatrix} \xi \\ \xi \end{bmatrix} \quad (4.24)$$

$$v(q) \approx \sqrt{m} \begin{bmatrix} \xi \\ -\xi \end{bmatrix}, \quad (4.25)$$

where ξ denotes a two-component spinor. The spinors have been approximated to zeroth order in $\frac{p}{m}$. The spinor ξ depends on the spin configuration of the particle. We can now evaluate the spinors, u and v , in the approximation above. This yields

$$\bar{u}(p')\gamma^i u(p) = 0 \quad (4.26)$$

$$\bar{v}(q)\gamma^i v(q') = 0 \quad (4.27)$$

$$\bar{u}(p')\gamma^0 u(p) = 2m\xi^{\dagger}\xi \quad (4.28)$$

$$\bar{v}(q)\gamma^0 v(q') = 2m\xi^{\dagger}\xi', \quad (4.29)$$

where we have used $\bar{u}(p') = u^{\dagger}\gamma^0$. Thus, the scattering amplitude in the non-relativistic limit reads

$$i\mathcal{M} = \frac{ie^2}{\mathbf{k}^2} (2m\xi^{\dagger}\xi)_u (2m\xi^{\dagger}\xi')_v \quad (4.30)$$

The scattering amplitude can now be compared with the scattering amplitude from the Born approximation [11, 29]. We use the Born approximation for a scattering potential. The potential reads

$$V(\mathbf{x}) = - \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{e^2}{\mathbf{k}^2} = -\frac{\alpha}{\mathbf{x}}, \quad (4.31)$$

which we recognise as the Coulomb potential. We refer to Adkins [41] for a detailed calculation of the integral. By including the vacuum polarisation diagram, we can find a correction to the

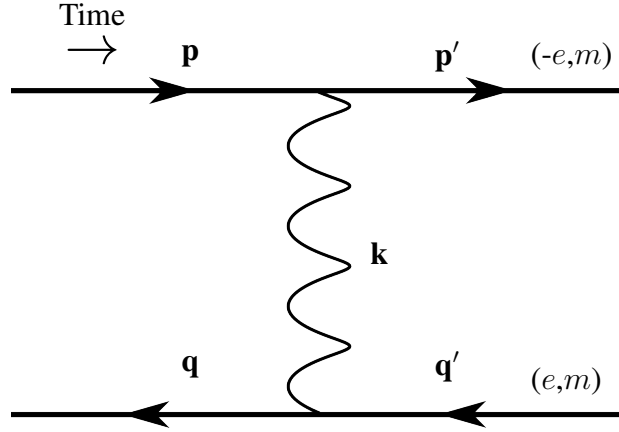


Figure 4.2: This figure illustrates the electron-positron scattering. We denote the electron, positron and photon momentum by \mathbf{p} , \mathbf{q} and \mathbf{k} respectively. The positron propagates in negative time direction. We have adapted the figure from Peskin & Schroeder [16].

Coulomb potential. We include the vacuum polarisation diagram, Figure 3.1, by inserting it into the photon propagator. This gives the next-to-leading order correction in α . The vacuum polarisation was calculated in Chapter 3, and resulted in a corrected photon propagator, see (3.33). We expand the correction in powers of $\frac{\mathbf{k}^2}{m^2}$. The zeroth component reads

$$D_{00} = \frac{1}{\mathbf{k}^2} \left[1 + \frac{\alpha}{15\pi} \frac{\mathbf{k}^2}{m^2} + \dots \right], \quad (4.32)$$

where the auxiliary mass scale is absorbed in a field redefinition, analogous to Section 4.3. We have omitted terms of higher order than $\frac{\mathbf{k}^2}{m^2}$. The scattering amplitude in equation (4.30) is now scaled by the vacuum polarisation contribution. By comparing it to the scattering amplitude of the Born approximation, the potential reads

$$V(\mathbf{x}) = (-e^2) \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left(\frac{1}{\mathbf{k}^2} + \frac{\alpha}{15\pi} \frac{1}{m^2} \right). \quad (4.33)$$

We are interested in the correction term, and not the Coulomb potential. The correction term is a scalar in momentum space. Thus, the Fourier transform result in a three-dimensional delta function. We obtain

$$\mathcal{U}(\mathbf{x}) = -\frac{4\alpha^2}{15m^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} = -\frac{4\alpha^2}{15m^2} \delta^{(3)}(\mathbf{x}). \quad (4.34)$$

This is the famous Uehling potential [37], which we denote by $\mathcal{U}(\mathbf{x})$. We can now apply Rayleigh-Schrödinger perturbation theory on the Schrödinger wave function, using the Uehling potential as the perturbation [29]. This gives

$$\Delta E_U = \left\langle \psi \left| \left[-\frac{4\alpha^2}{15m^2} \delta^{(3)}(\mathbf{x}) \right] \right| \psi \right\rangle = \left[-\frac{4\alpha^2}{15m^2} \right] |\psi(0)|^2 = -\frac{4m\alpha^5}{15\pi n^3} \delta_{l0}, \quad (4.35)$$

where $|\psi(0)|^2 = \frac{1}{\pi} \left(\frac{m\alpha}{n} \right)^2 \delta_{l0}$, and $\psi(0)$ is the Schrödinger wave function in the Coulomb potential evaluated at the origin [29]. We see that the energy correction is of the right magnitude and

negative. By using the traditional method, we have calculated the vacuum polarisation contribution to the Lamb shift, which is in agreement with literature and experimental observations.

We can add this correction to the fine structure of the hydrogen atom. The fine structure was calculated in the specialisation project in advance of the thesis [31], and the energy level now reads

$$E = m \left[1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4} \left(\frac{n}{j + \frac{1}{2}} - \frac{3}{4} \right) - \frac{4\alpha^5}{15n^3} \delta_{l0} \right] = E_{\text{NR}} + \Delta E_{\text{FS}} + \Delta E_{\text{U}}, \quad (4.36)$$

where j is the quantum number for the total angular momentum. E_{NR} is the non-relativistic approximation to the binding energy, ΔE_{FS} is the fine structure correction, and ΔE_{U} is the vacuum polarisation contribution of the Lamb shift. We notice that the Lamb shift lifts the degeneracy between the $2S_{\frac{1}{2}}$ -state and the $2P_{\frac{1}{2}}$ -state. The energy levels are quantitatively presented in Figure 4.3.

We have calculated the vacuum polarisation contribution to the Lamb shift of the $2S_{\frac{1}{2}}$ -state, which numerically is of the order 10^{-7} eV. This is only a small fraction of the total Lamb shift [16]. By using the Planck-Einstein relation, $E = h\nu$, where h is the Planck constant and ν is frequency, we can express the energy shift in hertz. The photon contribution results in a downward shift of the energy level for the $2S_{\frac{1}{2}}$ state by 27.1 MHz [12], while the observation by Lamb and Retherford in 1947 was an upward shift of about 1000 MHz [4]. This would suggest that the larger part of the contribution to the Lamb shift is due to the corrections in the fermion NRQED Lagrangian. The next-to-leading order in α for the fermion field could have been calculated by finding higher order corrections to the coefficients in equation (4.10).

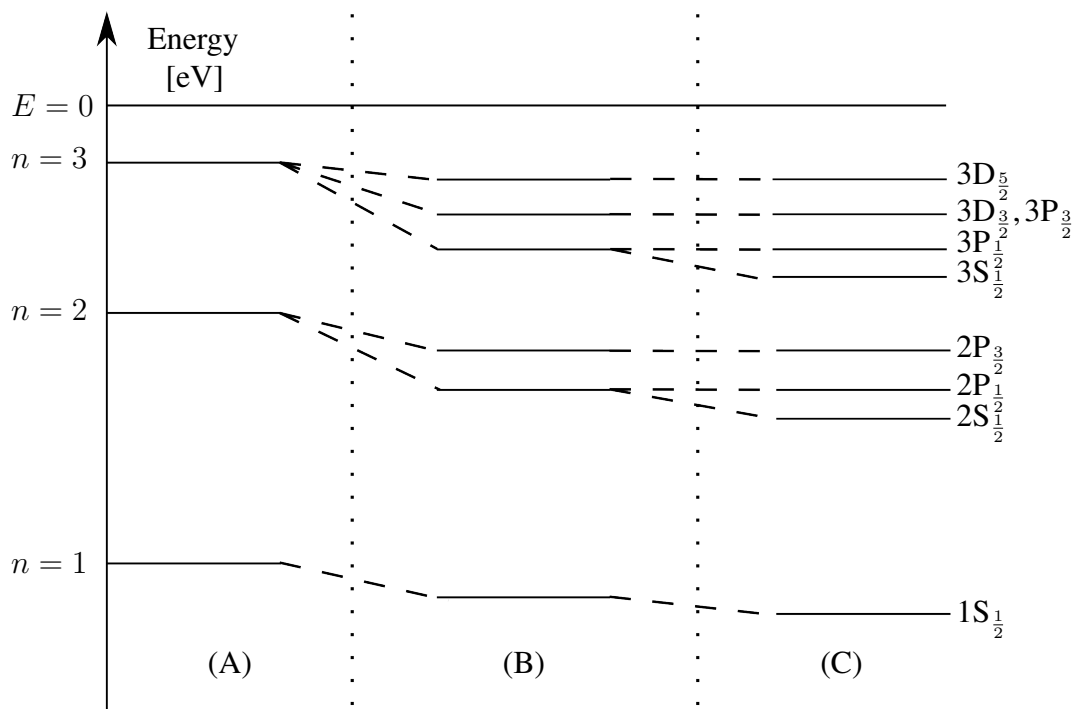


Figure 4.3: This figure shows, qualitatively, how the energy spectrum of bound states changes with the different corrections. The non-relativistic energy levels (A) is shifted by the fine structure (B). Some of the degenerate states in (A) is lifted by the effects in (B). The vacuum polarisation contribution of the Lamb shift (C), only affects the S-states, and lifts the degeneracy of some of the states from (B). S,P,D is associated with $l =$ one, two and three respectively. The fraction denotes the value of j , i.e. the total angular momentum. This figure is a reworked figure from the specialisation project in advance of this thesis [31].

Conclusion and outlook

In this thesis, we have reviewed some of the basic concepts of quantum field theory. After a brief review of regularisation and renormalisation, we discussed gauge symmetry in QED. We derived the photon propagator in both Lorenz and Coulomb gauge. The vacuum polarisation diagram for the photon was derived in QED. It resulted in a finite correction to the photon propagator. We calculated the diagram by using dimensional regularisation and the modified minimal subtraction scheme. The correction to the photon propagator did not result in a massive photon, i.e. the pole of the photon propagator remained located at $k^2 = m_\gamma^2 = 0$.

We introduced an effective field theory known as non-relativistic QED. We discussed how one systematically constructs the NRQED Lagrangian by identifying the symmetries of the theory. The Lagrangian is constructed with operators in an expansion in powers of v . In this thesis, we omitted operators which are of higher order than v^4 . The terms in the NRQED Lagrangian were multiplied by coefficients. These coefficients are power series in α . The power series incorporates radiative corrections into the theory. They are determined by a matching procedure. The matching procedure enforces that the scattering amplitudes in QED coincide with the scattering amplitudes in NRQED. We match at tree-level for the terms in the fermion Lagrangian. Tree-level matching is not sufficient for calculating the Lamb shift, but reproduces the fine structure of the hydrogen energy spectrum. The coefficients at tree level, had the same value as the coefficients of the Foldy-Wouthuysen transformed Dirac Hamiltonian. This is not surprising, given that the Foldy-Wouthuysen transformation only incorporates relativistic corrections, and not radiative corrections.

The terms in the photon Lagrangian were matched at one loop. We matched the two-point function of the photon field in NRQED with the two-point function of the photon field in QED at one loop. The two-point function of QED was corrected by the vacuum polarisation of the photon field. By using the one-loop correction, we calculated a correction to the energy spectrum of the hydrogen atom. We found that the calculation resulted in an energy shift written as $\frac{4m\alpha^5}{15\pi n^3} \delta_{l0}$. The fact that the result is a positive energy shift is a clear indication that the result is wrong. The sign error was attempted solved, but without success. We think that the sign error is caused by using wrong Feynman rules when finding the energy correction, and suggest that it might have been solved by using different Feynman rules. In QED, the Feynman rules are multiplied by i . The Feynman rules were not derived in this thesis, due to lack of time, and we can not state that the Feynman rules by Kinoshita and Nio [27] are incorrect. We make reservation that the sign error may be caused by lack of physical insight with respect to the Feynman diagram drawn, but suggest that it may be caused by incorrect Feynman rules.

We provided the reader with an alternative derivation of the energy shift caused by the vacuum polarisation of the photon. This is done by the traditional method of finding the energy shift. It was derived by identifying a correction to the Coulomb potential known as the Uehling potential. By applying perturbation theory to the Schrödinger wave function, using the Uehling potential as perturbation, we derived the correct energy shift of the hydrogen atom. The energy shift is part of the well known Lamb shift, and was found to be $-\frac{4m\alpha^5}{15\pi n^3} \delta_{l0}$. We could have done the calculation of the energy correction using an arbitrary nucleus with general mass and charge, M and Ze

respectively. In our calculations we chose $Z = 1$, which corresponds to the hydrogen atom.

The calculated contribution only constitute a small fraction of the Lamb shift. However, the vacuum polarisation resulted in an important feature, as it lifts the degeneracy between the $2S_{\frac{1}{2}}$ -state and the $2P_{\frac{1}{2}}$ -state. The $2S_{\frac{1}{2}}$ -state was shifted by 27.1 MHz downward, compared to the $2P_{\frac{1}{2}}$ -state. The total Lamb shift is an upward shift of 1057.9 MHz [12]. Even though the energy shift, which comes from the vacuum polarisation of the photon, is small, it is essential to have agreement between the experimentally measured and the theoretically calculated energy spectrum of the hydrogen atom. The energy spectrum of the hydrogen atom has been measured to very high precision [38–40], and the correction from the vacuum polarisation is significant.

5.1 Outlook

Let us first address the sign error. We make reservations that we lack the insight to how the sign error can be avoided. This must be further studied, and it is essential that the sign error is accounted for. Since we suggested that the sign error is caused by incorrect Feynman rule, we would have wanted to derive the Feynman rules in greater detail. We could not find any articles where the Feynman rules have been rigorously derived. The Feynman rules obtained in this theory should be studied and derived in a systematically and pedagogically manner.

Chapter 4 suggests that there is still much work to be done. The calculation involving the photon Lagrangian was more demanding than anticipated. Thus, we only calculated the contribution to the Lamb shift which came from vacuum polarisation of the photon. The total Lamb shift can be derived by including higher-order corrections to the power series in α in NRQED. Matching the NRQED scattering amplitudes to the scattering amplitudes of QED at one-loop, will reproduce the famous Lamb shift [9].

There are even smaller corrections, which contributes to the hyperfine structure. These corrections can be calculated by evaluating NRQED for muonium or positronium [27, 36, 42]. The calculation includes corrections of higher order than α^5 . We must include contact terms the NRQED Lagrangian at this precision, to make NRQED equivalent to QED. These contact terms incorporates interactions between the particles in the bound state, and are need to find the hyperfine structure. The theoretical work on the hyperfine structure is used as a precision test of QED. For instance, the hyperfine structure has been measure to a very high precision using muonium and measuring the muon magnetic moment [43, 44]. The theoretical work on the muonium energy spectrum is in agreement with experimental observations to a very high precision.

The photon propagates at a finite speed. Thus, the photon interaction is not instantaneous. This effect is know as retardation. We can incorporate the retardation effect through a multipole expansion [10]. The multipole expansion is performed on the vertices of NRQED. For instance, let us consider the interaction $\frac{e}{2m}\psi^\dagger(\mathbf{p} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{p})\psi$, which comes from the term proportional to \mathbf{D}^2 in the Lagrangian. We replace the fields by plane waves, i.e. $\psi(\mathbf{p}) = e^{i\mathbf{p}\cdot\mathbf{x}}$ and $\mathbf{A}(\mathbf{k}) = e^{i\mathbf{k}\cdot\mathbf{x}}$. The exponentials are associated with three-momentum conservation. In multipole expansion, we expand the photon exponential in powers of $\mathbf{k} \cdot \mathbf{r}$. We then find that $\mathbf{p}' = \mathbf{p}$, as opposed to $\mathbf{p}' = \mathbf{p} - \mathbf{k}$. This implies that by using a multipole expansion, we do not conserve three-momentum in the vertices. Thus, the fermion momenta is unchanged by absorption or emission of a photon. This changes the expression of the photon propagator [10]. One example of the physical implication of the retardation effect is the change of the $\frac{1}{r}$ dependence of the interaction potential

from the Van der Waals force [33]. The usual non-relativistic law, disregarding retardation effect, is of $\mathcal{O}\left(\frac{1}{r^6}\right)$. By incorporating retardation effects, we find that the interaction potential is of $\mathcal{O}\left(\frac{1}{r^7}\right)$. The Van der Waals force is caused by a two-photon exchange between two neutral spinless particles [45]. We could not find any references where the Van der Waal interaction is derived by applying NRQED. It would be interesting to test the scope of this effective field theory by determining the the Van der Waal interaction.

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General derivations and mathematical tools

This chapter introduces some mathematical tools used in the thesis, as well as some derivations which were found to be too extensive for the main text. Some of the mathematical formulas are collected from [46]. We mostly follow derivations from Ryder [12], Peskin & Schroeder [16], and Kachreiss [18].

A.1 Gaussian integral

Gaussian integral recur often in quantum field theory. The Gaussian integral is

$$\int_{q=0}^{\infty} dq e^{-a^2 q^2} = \frac{\sqrt{\pi}}{2a}. \quad (\text{A.1})$$

Other variations in the integrand can use this result by rewriting it into a similar form. Let a and b be real constants, and let the integrand be $\exp(-a^2 q^2 \pm bq)$. If we complete the square in the argument of the exponential, the argument becomes $\left[-\left(aq \mp \frac{b}{2a}\right)^2 + \frac{b^2}{4a^2}\right]$. The second term have no q dependence, and can be factorised out. The integral results in

$$\int_{q=0}^{\infty} dq e^{-a^2 q^2 \pm bq} = e^{\frac{b^2}{4a^2}} \frac{\sqrt{\pi}}{a} \quad (\text{A.2})$$

A.2 Gamma and beta function

The gamma function is denoted by $\Gamma(x)$, and is defined by the Euler integral which reads

$$\Gamma(x) = \int_0^{\infty} dt t^{x-1} e^{-t}. \quad (\text{A.3})$$

Below is some of the properties associated to $\Gamma(x)$

$$\Gamma(x+1) = x\Gamma(x) \quad (\text{A.4a})$$

$$\Gamma(1) = \Gamma(2) = 1 \quad (\text{A.4b})$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad (\text{A.4c})$$

$$\Gamma(n+1) = n!, \quad (\text{A.4d})$$

where $n \in \mathbb{N}$. The beta function is denoted by $B(q_1, q_2)$, and is defined by

$$B(q_1, q_2) = \int_0^1 dt t^{q_1-1} (1-t)^{q_2-1} = \int_0^{\infty} dt \frac{t^{q_1-1}}{(1+t)^{q_1+q_2}}, \quad (\text{A.5})$$

for q_1 and q_2 positive. It is convenient to look at series expansion for a gamma function for small arguments. We have

$$\lim_{x \rightarrow 0} \Gamma(x) = \frac{1}{x} - \gamma_E + \mathcal{O}(x), \quad (\text{A.6})$$

where $\gamma_E \approx 0.5772$ is the Euler-Mascheroni constant.

The beta function can also be expressed using the gamma functions. This yields

$$B(q_1, q_2) = \frac{\Gamma(q_1)\Gamma(q_2)}{\Gamma(q_1 + q_2)}. \quad (\text{A.7})$$

From this equation we see that the beta function is invariant when changing the order of q_1 and q_2 .

A.3 Feynman parametrisation

The Feynman parametrisation is often used in QED calculations. It is used to get propagators under the same common denominator. It is presented in many textbooks, e.g. [16, 18], and reads

$$\frac{1}{AB} = \int_0^1 dz \frac{1}{[zA + (1-z)B]^2}. \quad (\text{A.8})$$

A.4 Wick rotation

A Wick rotation is a rotation of the integration path in the complex plane [16]. Let us start with the following integral in Minkowski space,

$$I_n = \int \frac{d^4 p}{(2\pi)^4} \frac{1}{(p^2 - \Delta)^n}, \quad (\text{A.9})$$

where Δ is a variable which behaves as a constant with respect to the integration variable, and n is an integer. The integral is taken over $p_0 \in [-\infty, \infty]$, meaning that we integrate over a pole when $p_0 = \pm\sqrt{\mathbf{p}^2 + \Delta}$. By using the Feynman prescription, adding $i\epsilon$ in the denominator, we avoid the pole by shifting it off the real axes. The pole is now located at $p^0 = \pm\sqrt{\mathbf{p}^2 + \Delta} \mp i\epsilon$, and we can rotate the integration path counter-clock wise by 90° , this is illustrated in figure A.1. The integration path is now $p_0 \in [-i\infty, i\infty]$. We have gone from an integral in Minkowski space to an integral in Euclidean space by rotating the integration path. This corresponds to a substitution; $p_0 \rightarrow ip_{0E}$. In Euclidean space we have $p^2 = p_0^2 + \mathbf{p}^2$.¹ The integral in equation (A.9) can now be written as

$$I_n = i(-1)^n \int \frac{d^4 p_E}{(2\pi)^4} \frac{1}{(p_E^2 + \Delta)^n}. \quad (\text{A.10})$$

The integral does no longer contain a pole, this is important when we derive a result for a d -dimensional integral.

¹The subscript E is usually dropped, when it is clear from context that we work in Euclidean space.

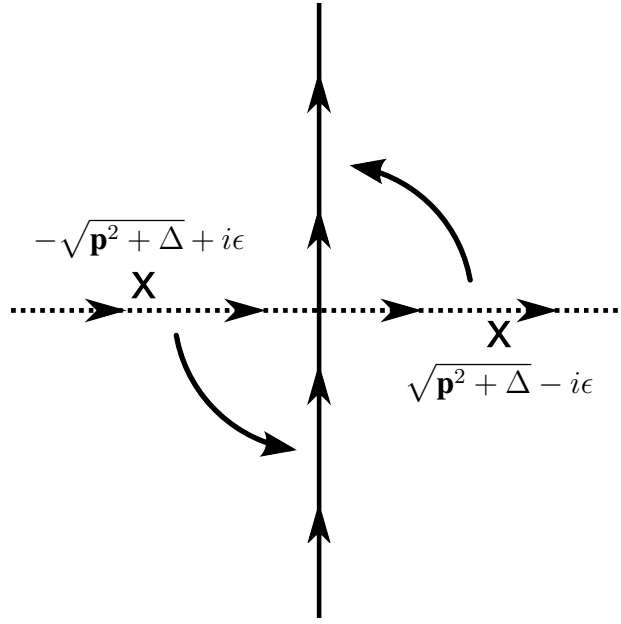


Figure A.1: This graph illustrates how the integration path is rotated. The dotted line indicate the original integration path, while the solid line indicates the rotated integration path. Each one of the crosses indicates the location of the singularities. Notice that the rotation avoids contact with the singularities. The figure is adapted from Peskin & Schröder [16].

A.5 d -dimensional integral

Equation (A.10) is ultraviolet divergent for $n \leq 2$. The integral often appears in quantum field theory and is thus useful to evaluate. It is evaluated in d space-time dimensions, where d can be any real number. The dimensional integral can be written as

$$I_n = \int \frac{d^d p_E}{(2\pi)^d} \frac{1}{(p_E^2 + \Delta)^n} \quad (\text{A.11})$$

where we have introduced an arbitrary $d = 4 - 2\epsilon$. The term 2ϵ represents a small deviation from the original four-dimensions in space-time. In d dimensions the polar coordinates reads

$$(p_0, r, \phi, \theta_1, \theta_2, \dots, \theta_{d-3}), \quad (\text{A.12})$$

where $r = |\mathbf{p}|$. The differential of p in polar coordinates reads,

$$d^d p = dp_0 r^{d-2} dr d\phi \sin(\theta_1) d\theta_1 \sin^2(\theta_2) d\theta_2 \dots \sin^{d-3}(\theta_{d-3}) d\theta_{d-3} \quad (\text{A.13})$$

Using these polar coordinates, the integral reads

$$\begin{aligned}
I_n &= \int \frac{dp_0}{(2\pi)^d} \int dr \frac{r^{d-2}}{(p^2 + \Delta)^n} \int_0^{2\pi} d\Phi \int_0^\pi \prod_{k=1}^{d-3} d\theta_k \sin^k \theta_k \\
&= \int \frac{dp_0}{(2\pi)^{d-1}} \int dr \frac{r^{d-2}}{(p^2 + \Delta)^n} \prod_{k=1}^{d-3} \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+2}{2})} \\
&= \frac{\pi^{\frac{d-3}{2}}}{\Gamma(\frac{d-1}{2})} \int \frac{dp_0}{(2\pi)^{d-1}} \int dr \frac{r^{d-2}}{(p^2 + \Delta)^n}, \tag{A.14}
\end{aligned}$$

where we have used

$$\int_0^\pi d\theta \sin^k \theta = \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+2}{2})}. \tag{A.15}$$

The integral over r in equation (A.14) can be performed by equating it to the integral in equation (A.5), but first we will massage equation (A.14) into a simpler expression. By using that $p^2 = p_0^2 + \mathbf{p}^2 = p_0^2 + r^2$, and making the substitution $x^2 = \frac{r^2}{p_0^2 + \Delta}$, we find

$$\begin{aligned}
I_n &= (p_0^2 + \Delta)^{(\frac{d-1}{2}-n)} \int dx \frac{x^{d-2}}{(1+x^2)^n} \\
&= (p_0^2 + \Delta)^{(\frac{d-1}{2}-n)} \int \frac{d(x^2)}{2x} \frac{x^{d-2}}{(1+x^2)^n} \\
&= \frac{1}{2} (p_0^2 + \Delta)^{(\frac{d-1}{2}-n)} B\left(\frac{d-1}{2}, n - \frac{d-1}{2}\right). \tag{A.16}
\end{aligned}$$

Equation (A.14) is now written as

$$I_n = \frac{\pi^{\frac{d-3}{2}} \Gamma(n - \frac{d-1}{2})}{2 \Gamma(n)} \int \frac{dp_0}{(2\pi)^{d-1}} (p_0^2 + \Delta)^{(\frac{d-1}{2}-n)}. \tag{A.17}$$

The integral over p_0 can be performed analogously to what we did for the integral over r . This yields

$$\int \frac{dp_0}{(2\pi)^{d-1}} (p_0^2 + \Delta)^{(\frac{d-1}{2}-n)} = \frac{\Delta^{\frac{d}{2}-n}}{(2\pi)^{d-1}} B\left(\frac{1}{2}, n - \frac{d}{2}\right). \tag{A.18}$$

With this last equation, the d -dimensional integral reads

$$I_n = \int \frac{d^d p_E}{(2\pi)^d} \frac{1}{(p_E^2 + \Delta)^n} = \frac{\Delta^{\frac{d}{2}-n} \Gamma(n - \frac{d}{2})}{(4\pi)^{\frac{d}{2}} \Gamma(n)}. \tag{A.19}$$

This integral often occurs in QFT calculations. There are variations of this integral which also frequently appears. These variations can be found by modifying the integral in equation (A.19). It is not as general as the derivation above, since we make the assumption that $\Delta = \Delta_0 - k^2$. The assumption to Δ introduces a variable Δ_0 which is independent on both k^2 and p^2 . In many

cases, the assumed k dependence of Δ is in agreement with the expression we want to evaluate. By making the substitution $p \rightarrow p + k$, we find

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{[(p+k)^2 + \Delta]^n} = \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 + 2p \cdot k + \Delta_0)^n}. \quad (\text{A.20})$$

We now take the derivative with respect to k_μ , this yields

$$\frac{dI_n}{dk_\mu} = \int \frac{d^d p}{(2\pi)^d} \frac{-2np^\mu}{(p^2 + 2p \cdot k + \Delta_0)^{n+1}} = \left(\frac{d}{2} - n\right) (-2k^\mu) \frac{\Delta^{\frac{d}{2}-(n+1)} \Gamma\left(n - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}} \Gamma(n)}. \quad (\text{A.21})$$

We make the substitution $n \rightarrow n - 1$ and get

$$\begin{aligned} \frac{dI_{n-1}}{dk_\mu} = I'_n &= \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu}{(p^2 + 2p \cdot k + \Delta_0)^n} = -k^\mu \frac{\Delta^{\frac{d}{2}-n} (n-1 - \frac{d}{2}) \Gamma\left(n-1 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}} (n-1)\Gamma(n-1)} \\ &= -k^\mu \frac{\Delta^{\frac{d}{2}-n} \Gamma\left(n - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}} \Gamma(n)}. \end{aligned} \quad (\text{A.22})$$

We can analogously modify this integral, and thus get

$$\begin{aligned} \frac{dI'_{n-1}}{dk_\nu} = I''_n &= \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{(p^2 + 2p \cdot k + \Delta_0)^n} \\ &= k^\mu k^\nu \frac{\Delta^{\frac{d}{2}-n} \Gamma\left(n - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}} \Gamma(n)} + \frac{g^{\mu\nu} \Delta^{\frac{d}{2}-(n-1)} \Gamma\left(n-1 - \frac{d}{2}\right)}{2 (4\pi)^{\frac{d}{2}} \Gamma(n)}. \end{aligned} \quad (\text{A.23})$$

Since we work in d space-time dimensions, we must adjust the properties of the metric tensor and gamma matrices. Some of the properties are presented in P&S [16]. The one we want to use reads

$$g_{\mu\nu} g^{\mu\nu} = d. \quad (\text{A.24})$$

Equation (A.23) can be rewritten by using the identity above. This yields

$$\begin{aligned} g_{\mu\nu} I''_n &= \int \frac{d^d p}{(2\pi)^d} \frac{p^2}{(p^2 + 2p \cdot k + \Delta_0)^n} \\ &= k^2 \frac{\Delta^{\frac{d}{2}-n} \Gamma\left(n - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}} \Gamma(n)} + \frac{d \Delta^{\frac{d}{2}-(n-1)} \Gamma\left(n-1 - \frac{d}{2}\right)}{2 (4\pi)^{\frac{d}{2}} \Gamma(n)}. \end{aligned} \quad (\text{A.25})$$

The first term is recognised as $k^2 I_n$. We make the substitution $p \rightarrow p - k$ which yields,

$$g_{\mu\nu} I''_n = \int \frac{d^d p}{(2\pi)^d} \frac{(p-k)^2}{(p^2 + \Delta)^n} = k^2 \frac{\Delta^{\frac{d}{2}-n} \Gamma\left(n - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}} \Gamma(n)} + \frac{d \Delta^{\frac{d}{2}-(n-1)} \Gamma\left(n-1 - \frac{d}{2}\right)}{2 (4\pi)^{\frac{d}{2}} \Gamma(n)}. \quad (\text{A.26})$$

Finally, the integral we want can be read off the expression above,

$$g_{\mu\nu} I''_n = \int \frac{d^d p}{(2\pi)^d} \frac{p^2}{(p^2 + \Delta)^n} = \frac{d \Delta^{\frac{d}{2}-(n-1)} \Gamma\left(n-1 - \frac{d}{2}\right)}{2 (4\pi)^{\frac{d}{2}} \Gamma(n)}, \quad (\text{A.27})$$

where we have used that terms linear in p gives zero contribution as the denominator is even in p .

A.6 Tensor contraction in the Coulomb gauge

This is ment as a supplement calculation to Section 3.1. The different tensors can be found in Section 3.1. The calculations are done below.

$$\frac{1}{\Pi(k^2)} D^{\mu\rho} \Pi_{\rho\sigma} D^{\sigma\nu} = \frac{1}{k^2} \left(\Theta^{\mu\rho} + \frac{k^2}{\mathbf{k}^2} n^\mu n^\rho \right) (k^2 g_{\rho\sigma} - k_\rho k_\sigma) D^{\sigma\nu}.$$

We use that $\Theta^{0\nu} = \Theta^{\mu 0} = 0$, $\Theta^{il} = \left(\delta^{il} - \frac{k^i k^l}{\mathbf{k}^2} \right)$, and $g_{lj} = -\delta_{lj}$. This yields,

$$\begin{aligned} \frac{1}{\Pi(k^2)} D^{\mu\rho} \Pi_{\rho\sigma} D^{\sigma\nu} &= \left(\delta_j^i - \frac{k^i k_j}{\mathbf{k}^2} - \frac{k^2}{\mathbf{k}^2} n^\mu n_\sigma + \frac{n^\mu k_\sigma k^0}{\mathbf{k}^2} \right) D^{\sigma\nu} \\ &= \frac{1}{k^2} \left(\Theta_\sigma^\mu - \frac{k^2}{\mathbf{k}^2} n^\mu n_\sigma + \frac{n^\mu k_\sigma k^0}{\mathbf{k}^2} \right) \left(\Theta^{\sigma\nu} + \frac{k^2}{\mathbf{k}^2} n^\sigma n^\nu \right). \end{aligned}$$

Notice that $n_\mu = (1, 0, 0, 0)$ contracted with $\Theta^{\mu\nu}$ is zero. We find

$$\begin{aligned} \frac{1}{\Pi(k^2)} D^{\mu\rho} \Pi_{\rho\sigma} D^{\sigma\nu} &= \frac{1}{k^2} \left[\Theta_\sigma^\mu \Theta^{\sigma\nu} - \frac{k^2}{\mathbf{k}^2} n^\mu n_\sigma + \frac{k^0 n^\mu k_\sigma \Theta^{\sigma\nu}}{\mathbf{k}^2} - \left(\frac{k^2}{\mathbf{k}^2} \right)^2 n^\mu n^\nu + \frac{k^2 (k^0)^2}{\mathbf{k}^4} n^\mu n^\nu \right] \\ &= \frac{1}{k^2} \left[\left(\delta_l^i - \frac{k^i k_l}{\mathbf{k}^2} \right) \left(\delta^{lj} - \frac{k^l k^j}{\mathbf{k}^2} \right) + \frac{k^2}{\mathbf{k}^2} n^\mu n^\nu \right] \\ &= \frac{1}{k^2} \left[\left(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2} \right) + \frac{k^2}{\mathbf{k}^2} n^\mu n^\nu \right] \\ &= \frac{1}{k^2} \left(\Theta^{\mu\nu} + \frac{k^2}{\mathbf{k}^2} n^\mu n^\nu \right) \\ &= D^{\mu\nu}. \end{aligned}$$

We thus see that the calculation results in a single tensor. This is useful for calculations in Section 3.1.

Feynman rules NRQED

In this appendix we present the Feynman rules in NRQED, for the coulomb photon. These Feynman rules are relevant when we find the energy correction which comes from vacuum polarisation of the photon in Section 4.4. The rules are derived from the NRQED Hamiltonian for the photon [27].

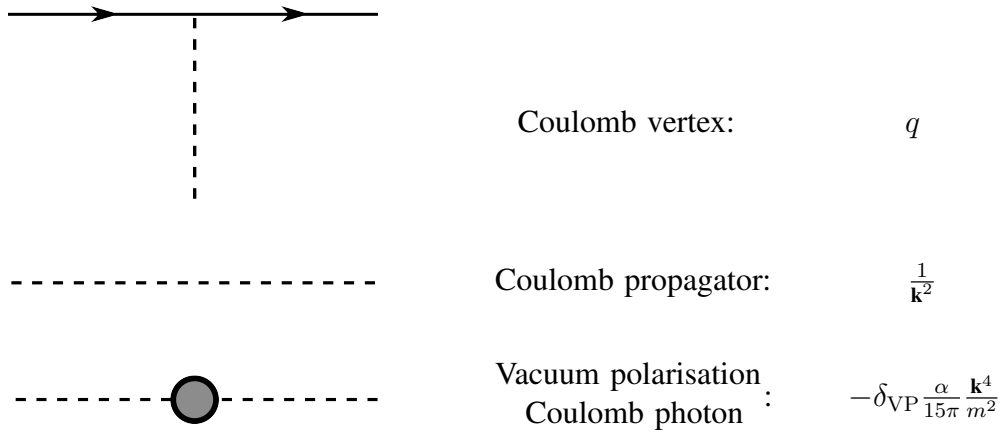


Figure B.1: The Feynman rules for the NRQED Hamiltonian. The rules are valid in the Coulomb gauge. The dotted and solid lines represent the Coulomb photon propagator and the fermion propagator respectively. q is the charge of the fermion. \mathbf{k} is the momentum of the photon. The Feynman rules are adapted from Kinoshita and Nio [27].