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A Nonlinear Partial Differential Equation and Its Viscosity Solutions

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Preface

With this thesis, I complete my master's degree in industrial mathematics at NTNU.

I would like to thank my supervisor Peter Lindqvist for many enlightening discussions. His support in my work this semester has been invaluable.

Abstract

We study a nonlinear partial differential equation with Lipschitz continuous coefficient functions. Existence and uniqueness of viscosity solutions is proved by approximating with minimizers of variational integrals. The solutions are shown to satisfy a corresponding minimization property. Stability of solutions with respect to small perturbations of the coefficient functions is discussed, and proved for C^2-solutions.

Sammendrag

Vi studerer en ikke-lineær partiell differensialligning med Lipschitzkontinuerlige koeffisientfunksjoner. Eksistens og entydighet av viskositetsløsninger bevises ved å approksimere med minimerere av variasjonsintegraler. Det vises at løsningene har en lignende minimeringsegenskap. Stabilitet av løsninger med hensyn på små perturbasjoner av koeffisientfunksjonene diskuteres, og bevises for C^2-løsninger.

Contents

1 Introduction

We study the nonlinear partial differential equation

$$\Delta_{\infty,A}\, u = \sum_{i,j,k,l=1}^{n} \Big(\frac{\partial a_{ij}}{\partial x_k} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} a_{kl} \frac{\partial u}{\partial x_l} + 2 a_{ik} \frac{\partial u}{\partial x_k} \frac{\partial^2 u}{\partial x_i \partial x_j} a_{jl} \frac{\partial u}{\partial x_l} \Big) = 0, \tag{1.1}$$

which comes from the minimax problem

$$\min_{u} \max_{x\in\Omega} \Big(\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial u}{\partial x_j}(x) \Big)^{1/2},$$

among all all admissible functions u defined in a bounded domain $\Omega \subset \mathbb{R}^n$, having the same boundary values. Here a_{ij} denotes given coefficient functions. The equation is related to the infinity-Laplace equation

$$\Delta_{\infty}\, u = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \tag{1.2}$$

which is seen by setting $a_{ij} = \delta_{ij}/2$ in (1.1), where δ_{ij} is the Kronecker delta. Thus, the equation represents a generalized infinity-Laplace equation, and our strategy for showing existence and uniqueness of viscosity solutions is based on Jensen's work in [12] on the infinity-Laplace equation, and Juutinen's work in [13] on more general problems. To motivate this strategy, we give a short introduction to the infinity-Laplace equation. The equation was derived by Aronsson in 1967, and provides the best Lipschitz extension of given boundary values, see [1]. The concept of solutions of (1.2) is difficult. Aronsson demonstrated that the equation does not necessarily have classical solutions. For instance, he constructed the function

$$u(x_1, x_2) = x_1^{4/3} - x_2^{4/3},$$

which satisfies the equation in the whole plane, but does not have second derivatives on the axes. Furthermore, the equation does not have a weak formulation involving only the first derivatives. The way out of these difficulties turned out to be the concept of viscosity solutions, which was developed by Crandall, Evans, Ishii, Lions and others in the 1980's. The breakthrough came in 1993, when Jensen established uniqueness of viscosity solutions by proving the comparison principle, where the main idea is to introduce two auxiliary equations, see [12]. In the proof, viscosity solutions are constructed as limits of weak solutions of the p-Laplace equation as $p \to \infty$, via some subsequence.

The layout of this thesis is as follows: In Section 2 we present some results which will be used frequently throughout the text. Lebesgue spaces, Sobolev spaces, and various compactness results are discussed, followed by fundamental properties of quadratic forms.

In Section 3 we introduce a variational integral and state the assumptions on the coefficient functions. We derive the Euler-Lagrange equation and establish some fundamental properties of the involved operators.

Section 4 is devoted to the Euler-Lagrange equation. We prove existence and uniqueness of weak solutions, and show that these are minimizers of the variational integral.

In Section 5 we construct the limit of weak solutions of the Euler-Lagrange equation, and show that it satisfies a minimization property.

In Section 6 we introduce viscosity solutions and show some fundamental properties of these. We show that the definition of viscosity solutions can be rephrased in terms of so-called jets. Then we state Ishii's lemma - a deep result of the theory which will play a central part in proving uniqueness of viscosity solutions.

Modified versions of Jensen's two auxiliary equations are presented in Section 7. We show that viscosity solutions of these can be constructed as limits of weak solutions of two Euler-Lagrange equations, and that the difference between these limits can be made arbitrarily small. Furthermore, we conclude that the limit constructed in Section 5 is a viscosity solution of (1.1).

We prove the Comparison principle in Section 8, which implies that an arbitrary viscosity solution of (1.1) lies between the two auxiliary solutions, which in turn implies uniqueness of viscosity solutions.

In Section 9 we perturb the coefficient functions by constants, and state the assumptions these has to satisfy such that there is a unique viscosity solution of the perturbed equation. Then we show stability with respect to small perturbations for solutions in one variable, and for C^2-solutions.

2 Preliminaries

In the following we assume that the domain Ω is an open and connected subset of $\mathbb{R}^n$ with boundary $\partial\Omega$.

2.1 Ascoli's theorem

We start out with a version of Ascoli's theorem.

Definition 2.1. We say that a sequence of functions $u_k : \Omega \to \mathbb{R}$ is *equibounded*, if

$$\sup_{x\in\Omega} |u_k(x)| \le M < \infty \quad \text{for all } k \in \mathbb{N},$$

and *equicontinuous*, if for $x, y \in \Omega$,

$$|u_k(x) - u_k(y)| \le C|x-y|^\alpha \quad \text{for all } k \in \mathbb{N},$$

where $0 < \alpha \le 1$ and C is a constant.

Theorem 2.2 (Ascoli's theorem)**.** *Let (u_k) be an equibounded and equicontinuous sequence of functions. Then there exist a subsequence (u_{k_j}) and a continuous function $u : \Omega \to \mathbb{R}$ such that $u_{k_j} \to u$ locally uniformly in Ω. If Ω is bounded, the functions can be extended to be continuous in the closure $\overline{\Omega}$, where the convergence is uniform.*

Proof. Let $(q_k)_{k\in\mathbb{N}}$ be an enumeration of the rational points in Ω. By assumption, $(u_k(q_1))_k$ is bounded, and by the Bolzano–Weierstrass theorem[1] has a subsequence, denoted by $(u_{1_j}(q_1))_j$ converging at q_1. Similary, the sequence $(u_{1_j}(q_2))_j$ is bounded, so it has a subsequence $(u_{2_j}(q_2))_j$ which converges at q_1 and q_2. Continuing in this fashion, extracting subsequences of subsequences, we obtain sequences $(u_{k_j}(q_k))_j$ for all $k \in \mathbb{N}$, converging at q_1, q_2, $\dots$, q_k. Then the diagonal sequence, $(u_{j_j}(q_k))_{j_j}$, which we simply denote by $(u_j(q_k))_j$, converges at every rational point in Ω. Thus, for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$|u_j(q_k) - u_i(q_k)| < \varepsilon/2 \quad \text{for all } i, j > N.$$

Now we show that the constructed diagonal sequence converges at each point in Ω, not just the rational ones. Consider an arbitrary $x \in \Omega$. By the density of the rational points in Ω, given $\varepsilon > 0$ there is a rational point $q \in \Omega$ such that

$$2C|x-q|^\alpha < \varepsilon/2.$$

Then by the equicontinuity,

$$\begin{aligned}
|u_j(x) - u_i(x)| &\le |u_j(x) - u_j(q)| + |u_j(q) - u_i(q)| + |u_i(q) - u_i(x)| \\
&\le 2C|x-q|^\alpha + |u_j(q) - u_i(q)| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon,
\end{aligned}$$

[1] The proof can be found in most analysis books, see e.g. [6].

whenever $i,j > N$. Thus, (u_j) is a Cauchy sequence, so the sequence converges pointwise to a function denoted by u:

$$u(x) = \lim_{j\to\infty} u_j(x).$$

It remains to show that the convergence is uniform. First suppose that Ω is bounded. Then the closure $\overline{\Omega}$ is compact, and we can cover it by a finite number of open balls $B(x_m, r)$, centered at x_m with diameter $2r = \varepsilon^{1/\alpha}$, that is

$$\overline{\Omega} \subset \bigcup_{m=1}^{n} B(x_m, r).$$

Choose a rational point r_m from each ball. Since there is only a finite number of these points, given $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that

$$\max_m |u_j(r_m) - u_i(r_m)| < \varepsilon, \quad \text{for all } i,j > N_\varepsilon.$$

Consider an arbitrary $x \in \overline{\Omega}$, which must belong to some ball, say $B(x_m, r)$. Then

$$\begin{aligned} |u_j(x) - u_i(x)| &\le 2C|x - r_m|^\alpha + |u_j(r_m) - u_i(r_m)| \\ &\le 2C(2r)^\alpha + \max_m |u_j(r_m) - u_i(r_m)| \\ &\le 2C\varepsilon + \varepsilon, \quad \text{for all } i,j > N_\varepsilon. \end{aligned}$$

Notice that N_ε is independent of how we chose the point x, so the convergence is uniform in $\overline{\Omega}$. Thus, the limit function u is continuous.

If Ω is unbounded, the proof above holds for any fixed, bounded subdomain of Ω, so the convergence is locally uniform. □

The proof is based on Theorem 1 in [18].

2.2 Lebesgue spaces

Now we derive some properties of the Lebsegue spaces.

Definition 2.3. For any Lebesgue measurable function $u : \Omega \to \mathbb{R}$ we define

$$||u||_{p,\Omega} = \begin{cases} \left(\int_\Omega |u(x)|^p dx\right)^{1/p} & \text{if } p \in [1,\infty) \\ \text{ess sup}_{x\in\Omega} |u(x)| & \text{if } p = \infty, \end{cases} \tag{2.1}$$

where the essential supremum is

$$\text{ess sup}_{x\in\Omega} |u(x)| = \inf\{M : u(x) \le M \text{ for a.e. } x \in \Omega\}.$$

We say that $u \in L^p(\Omega)$ if $||u||_{p,\Omega} < \infty$. If $u \in L^p(D)$ for each open set $D \subset\subset \Omega$, we say that $u \in L^p_{\text{loc}}(\Omega)$.[2]

[2] The notation "$\subset\subset$" is explained in Definition 2.27.

When the domain Ω is evident from the context we simply write $||\cdot||_{p,\Omega} = ||\cdot||_p$. Observe that by the Lebesgue integral we have $||u||_{p,\Omega} = 0$ if and only if $u = 0$ almost everywhere in Ω, so the axiom of normed spaces is not satisfied on sets of measure zero. To get a proper normed space we have to consider equivalence classes of functions, that is, each equivalence class consisits of functions which coincide a.e. We say that a function u has a *version* $\tilde{u}$, if u and $\tilde{u}$ belong to the same equivalence class.

A fundamental property of $L^p(\Omega)$ for $1 \le p \le \infty$ is that it is a Banach space with respect to the norm defined in (2.1).

Our next result shows that the Lebesgue spaces are nested when Ω is of finite measure:

$$L^1(\Omega) \supseteq L^2(\Omega) \supseteq \cdots \supseteq L^p(\Omega) \supseteq L^q(\Omega) \supseteq \cdots \supseteq L^\infty(\Omega), \quad p \le q.$$

First we introduce the notation

$$\fint_\Omega u\,dx = \frac{1}{\mu(\Omega)} \int_\Omega u\,dx$$

for the average of a function u over a bounded domain Ω, where μ denotes n-dimensional Lebesgue measure.

Proposition 2.4. *If $\mu(\Omega) < \infty$ and $u \in L^q(\Omega)$, then*

$$\left(\fint_\Omega |u|^p dx\right)^{1/p} \le \left(\fint_\Omega |u|^q dx\right)^{1/q} \quad \text{when } 1 \le p \le q. \tag{2.2}$$

Proof. By Hölder's inequality we have

$$||u||_p^p = ||1 \cdot |u|^p||_1 \le ||1||_{q/(q-p)} |||u|^p||_{q/p} = \mu(\Omega)^{(q-p)/q} ||u||_q^p,$$

where $p \ge 1$. This yields inequality (2.2) if $p \le q$. □

In many limit procedures we rely on the fact that the norm is continuous as $p \to \infty$:

Proposition 2.5. *If $\mu(\Omega) < \infty$ and $u \in L^\infty(\Omega)$, then*

$$\lim_{p\to\infty} ||u||_p = ||u||_\infty.$$

Proof. Let $\varepsilon > 0$ and define the set

$$A = \{x \in \Omega : |u(x)| > ||u||_\infty - \varepsilon\}.$$

Then

$$\int_\Omega |u|^p dx \ge \int_A |u|^p dx \ge (||u||_\infty - \varepsilon)^p \mu(A),$$

which implies that

$$\liminf_{p\to\infty} ||u||_p \ge ||u||_\infty - \varepsilon.$$

On the other hand, by Proposition 2.4,

$$||u||_p \le \mu(\Omega)^{1/p} ||u||_\infty,$$

thus

$$\limsup_{p\to\infty} ||u||_p \leq ||u||_\infty,$$

and the conclusion

$$\lim_{p\to\infty} ||u||_p = ||u||_\infty$$

follows since $\varepsilon > 0$ was arbitrarily small. □

We mention a fundamental result in the calculus of variations[3].

Lemma 2.6 (Variational lemma)**.** *Suppose that* $u \in L^1_{\text{loc}}(\Omega)$*. If*

$$\int_\Omega u\phi\, dx = 0 \quad \textit{for all } \phi \in C_0^\infty(\Omega),$$

then $u = 0$ *a.e. in* Ω*.*

We now turn to the dual space of $L^p(\Omega)$. An explicit characterization of bounded linear functionals on $L^p(\Omega)$ is provided by Riesz' representation theorem[4].

Theorem 2.7 (Riesz' representation theorem)**.** *Let* $1 \leq p < \infty$ *and suppose that* $\Lambda : L^p(\Omega) \to \mathbb{R}$ *is a bounded linear functional. Then there exists a unique function* $v \in L^q(\Omega)$*, where* $1/p + 1/q = 1$*, such that*

$$\Lambda(u) = \int_\Omega uv\, dx,$$

for all functions $u \in L^p(\Omega)$*. Moreover,* $||\Lambda|| = ||v||_{q,\Omega}$*.*

A consequence of this result is that we identify the dual space of $L^p(\Omega)$ as $L^q(\Omega)$ for $1/p + 1/q = 1$ when $p \in [1,\infty)$, and write $L^p(\Omega)' = L^q(\Omega)$.

Working in Banach spaces requires various concepts of convergence, and one of the most frequently used in this text is the notion of weak convergence.

Definition 2.8. Let X be a Banach space. We say that a sequence $(x_n) \subset X$ converges *weakly* to $x \in X$, if for all x' in the dual space X' we have

$$\lim_{n\to\infty} x'(x_n) = x'(x),$$

and we write $x_n \rightharpoonup x$.

By Riesz' representation theorem there is an explicit characterization of weak convergence in Lebesgue spaces. Indeed, let $1 \leq p < \infty$ and suppose that $(u_n) \subset L^p(\Omega)$ converges weakly to $u \in L^p(\Omega)$, that is

$$\lim_{n\to\infty} \Lambda(u_n) = \Lambda(u)$$

for all bounded linear functionals Λ on $L^p(\Omega)$. This is equivalent to

$$\lim_{n\to\infty} \int_\Omega u_n v\, dx = \int_\Omega uv\, dx, \tag{2.3}$$

for all $v \in L^q(\Omega)$ such that $1/p + 1/q = 1$.

Now we present som key properties of weak convergence.

[3] We refer to Theorem 3.40 in [5] for a proof.

[4] The proof can be found in [6], Theorem 13.1.

Proposition 2.9. *Let X be a Banach space and suppose that the sequence $(x_n) \subset X$ converges weakly to $x \in X$. Then the sequence is uniformly bounded:*

$$\sup_n ||x_n||_X \leq M < \infty,$$

and the norm is lower semicontinuous:

$$||x||_X \leq \liminf_{n\to\infty} ||x_n||_X.$$

Proposition 2.10. *Let $1 < p < \infty$ and assume that the sequence $(u_n) \subset L^p(\Omega)$ is uniformly bounded:*

$$\sup_n ||u_n||_p \leq M < \infty.$$

Then there is a subsequence (u_{n_k}) and a function $u \in L^p(\Omega)$ such that

$$u_{n_k} \rightharpoonup u \quad \text{weakly in } L^p(\Omega).$$

Proof. Let $1/p + 1/q = 1$. Since $q \in (1,\infty)$, $L^q(\Omega)$ is separable[5]. Let (v_n) be a countable collection of simple functions, which is dense in $L^q(\Omega)$. Set

$$\Lambda_n(v_j) = \int_\Omega v_j u_n \, dx \quad \text{for each } j \in \mathbb{N}.$$

By Hölder's inequality and the uniform boundedness we have

$$|\Lambda_n(v_j)| \leq M||v_j||_q,$$

so the sequence $(\Lambda_n(v_1))_n$ is bounded, and by the Bolzano–Weierstrass theorem, we can extract a subsequence, denoted by $(\Lambda_{1_j}(v_1))_j$ converging at v_1. Similary, $(\Lambda_{1_j}(v_2))_j$ is bounded, so we can extract a subsequence, denoted by $(\Lambda_{2_j}(v_2))_j$ converging at v_1 and v_2. Continuing this procedure, we see that the diagonal sequence $(\Lambda_{j_j}(v_n))_{j_j}$ converges at every v_n. To ease the notation we denote the constructed diagonal sequence by $(\Lambda_j(v_n))_j$. Thus, for every $\varepsilon > 0$ there is an N such that

$$|\Lambda_j(v_n) - \Lambda_i(v_n)| < \varepsilon, \qquad \text{for all } i,j > N.$$

Fix $v \in L^q(\Omega)$. By density there is a simple function v_n such that

$$||v - v_n||_q < \varepsilon.$$

It follows that

$$\begin{aligned} |\Lambda_j(v) - \Lambda_i(v)| &\leq |\Lambda_j(v - v_n)| + |\Lambda_i(v_n - v)| + |\Lambda_j(v_n) - \Lambda_i(v_n)| \\ &\leq 2M||v - v_n||_q + \varepsilon \\ &\leq 2M\varepsilon + \varepsilon \quad \text{for all } i,j > N, \end{aligned}$$

which shows that $(\Lambda_j(v))_j$ is a Cauchy sequence for all $v \in L^q(\Omega)$. We denote the limit by

$$\Lambda(v) = \lim_{n\to\infty} \Lambda_n(v) \quad \text{for all } v \in L^q(\Omega),$$

[5] Consult for instance Theorem 18.1 in [6] for a proof.

which defines a bounded linear functional on $L^q(\Omega)$. Then by Riesz' representation theorem there exists a unique $u \in L^p(\Omega)$ such that

$$\Lambda(v) = \int_\Omega vu\,dx \quad \text{for all } v \in L^q(\Omega),$$

thus

$$\lim_{n\to\infty} \int_\Omega vu_n\,dx = \int_\Omega vu\,dx \quad \text{for all } v \in L^q(\Omega),$$

and we conclude that $u_n \rightharpoonup u$ weakly in $L^p(\Omega)$. □

Another concept of convergence in Banach spaces is weak-star convergence.

Definition 2.11. Let X and Y be Banach spaces such that $X = Y'$. We say that a sequence $(x_n) \subset X$ converges *weak-star* to $x \in X$, if for all $y \in Y$ we have

$$\lim_{n\to\infty} x_n(y) = x(y),$$

and we write $x_n \overset{*}{\rightharpoonup} x$.

By Riesz' representation theorem we find that the notions of weak convergence and weak-star convergence coincide in $L^p(\Omega)$ when $p \in (1,\infty)$. Furthermore, $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$ if and only if

$$\lim_{n\to\infty} \int_\Omega u_n v\,dx = \int_\Omega uv\,dx \quad \text{for all } v \in L^1(\Omega). \tag{2.4}$$

We have the following analogous results of Proposition 2.9 and Proposition 2.10.

Proposition 2.12. *Let X and Y be a Banach spaces such that $X = Y'$. Suppose that the sequence $(x_n) \subset X$ converges weak-star to $x \in X$. Then the sequence is uniformly bounded:*

$$\sup_n ||x_n||_X \le M < \infty,$$

and the norm is lower semicontinuous:

$$||x||_X \le \liminf_{n\to\infty} ||x_n||_X.$$

Theorem 2.13 (Helly's theorem)**.** *Let X and Y be Banach spaces. Suppose that $X = Y'$ and that Y is separable. Assume that the sequence $(x_n) \subset X$ is uniformly bounded:*

$$\sup_n ||x_n||_X \le M < \infty.$$

Then (x_n) has a weak-star convergent subsequence.

We refer to Theorem 2.13 in [11] for a proof of Helly's theorem. The proofs of Proposition 2.9 and Proposition 2.12 can be found in most functional analysis books, see e.g. [15].

2.3 Sobolev spaces

Now we introduce Sobolev spaces and derive some important properties of these. We begin with some definitions. Recall that Ω is a domain in $\mathbb{R}^n$.

Definition 2.14. Consider a function $\phi : \Omega \to \mathbb{R}$ which belongs to $C^\infty(\Omega)$. We define the *support* of ϕ as

$$\text{supp}(\phi) = \overline{\{x \in \Omega : \phi(x) \neq 0\}}.$$

If supp(ϕ) is bounded we define

$$C_0^\infty(\Omega) = \{\phi \in C^\infty(\Omega) : \text{supp}(\phi) \subset \Omega\}.$$

For $\phi \in C_0^\infty(\Omega)$ we define $\phi(x) = 0$ when $x \in \mathbb{R}^n \setminus \Omega$.

We make the following definition motivated by the integration by parts formula for continuously differentiable functions.

Definition 2.15. Let $u \in L^1_{\text{loc}}(\Omega)$. If there is a function $w_j \in L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega u \frac{\partial \phi}{\partial x_j}\, dx = -\int_\Omega w_j \phi\, dx \quad \text{for all } \phi \in C_0^\infty(\Omega),$$

then we say that w_j is the *weak partial derivative* of u with respect to x_j in Ω. We write $w_j = \frac{\partial u}{\partial x_j}$ and $\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$, provided that the weak derivatives exist.

Definition 2.16. Let $1 \leq p \leq \infty$. We say that $u \in W^{1,p}(\Omega)$ if u and all its weak derivatives $\frac{\partial u}{\partial x_j}$, $j = 1, \ldots, n$, belong to $L^p(\Omega)$.

Then

$$||u||_{1,p,\Omega} = \begin{cases} \left(||u||^p_{p,\Omega} + ||\nabla u||^p_{p,\Omega}\right)^{1/p} & \text{if } p \in [1, \infty) \\ ||u||_{\infty,\Omega} + ||\nabla u||_{\infty,\Omega} & \text{if } p = \infty \end{cases} \tag{2.5}$$

defines a norm on $W^{1,p}(\Omega)$, where

$$||\nabla u||_{p,\Omega} = \left(\int_\Omega |\nabla u(x)|^p dx\right)^{1/p}, \quad ||\nabla u||_{\infty,\Omega} = \text{ess sup}_{x \in \Omega} |\nabla u(x)|.$$

The space $W^{1,p}(\Omega)$ possesses many properties similar to the space $L^p(\Omega)$, the most fundamental being that it is a Banach space with respect to the norm defined in (2.5).

Definition 2.17. We define the following spaces for $1 \leq p \leq \infty$:

i) Let $W_0^{1,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the space $W^{1,p}(\Omega)$, i.e. the closure of $C_0^\infty(\Omega)$ with respect to the norm $||\cdot||_{1,p,\Omega}$.

ii) We say that $u \in W^{1,p}_{\text{loc}}(\Omega)$ if $u \in W^{1,p}(D)$ for each open set $D \subset\subset \Omega$.

We mention that if $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ and $u|_{\partial\Omega} = 0$, then $u \in W_0^{1,p}(\Omega)$. In addition, if $u, v \in W_0^{1,p}(\Omega)$, then $\max\{u, v\}, \min\{u, v\} \in W_0^{1,p}(\Omega)$.[6]

[6] Consult for instance [9].

Remark 2.18. The notation

$$\nabla u_j \rightharpoonup \nabla u \quad \text{weakly in } L^p(\Omega)$$

means that

$$\frac{\partial u_j}{\partial x_k} \rightharpoonup \frac{\partial u}{\partial x_k} \quad \text{weakly in } L^p(\Omega)$$

for each $k = 1, 2, \ldots, n$. If

$$u_j \rightharpoonup u, \quad \nabla u_j \rightharpoonup w \quad \text{weakly in } L^p(\Omega)$$

for some $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$, then $w = \nabla u$. Indeed, by (2.3) the weak convergence means

$$\lim_{j\to\infty} \int_\Omega \eta u_j \, dx = \int_\Omega \eta u \, dx,$$

$$\lim_{j\to\infty} \int_\Omega \psi \frac{\partial u_j}{\partial x_k} \, dx = \int_\Omega \psi w_k \, dx$$

for all $\eta, \psi \in L^q(\Omega)$ such that $1/p + 1/q = 1$. Furthermore, since ∇u_j is the weak gradient of u_j we have

$$\int_\Omega u_j \frac{\partial \phi}{\partial x_k} \, dx = -\int_\Omega \phi \frac{\partial u_j}{\partial x_k} \, dx \quad \text{for all } \phi \in C_0^\infty(\Omega).$$

Let $\phi \in C_0^\infty(\Omega)$. Then by the above and since

$$\phi, \frac{\partial \phi}{\partial x_k} \in L^q(\Omega)$$

we obtain

$$\int_\Omega u \frac{\partial \phi}{\partial x_k} \, dx = \lim_{j\to\infty} \int_\Omega u_j \frac{\partial \phi}{\partial x_k} \, dx = -\lim_{j\to\infty} \int_\Omega \phi \frac{\partial u_j}{\partial x_k} \, dx = -\int_\Omega \phi w_k \, dx.$$

Thus $w = \nabla u$.

The following variant of Morrey's inequality is useful.

Lemma 2.19 (Morrey's inequality)**.** *Let Ω be a bounded domain and suppose that $p > n$. If $u \in W_0^{1,p}(\Omega)$, then*

$$|u(x) - u(y)| \le C_p |x-y|^{1-n/p} ||\nabla u||_{p,\Omega} \quad \textit{for a.e. } x, y \in \Omega,$$

where C_p depends on p and n, and is such that $C_p \to 2^{n+1}$ as $p \to \infty$. One can redefine u in a set of measure zero and extend it to the boundary such that $u \in C(\overline{\Omega})$ and $u|_{\partial\Omega} = 0$.

Proof. We first show the inequality for functions in $C_0^\infty(\Omega)$. Let $u \in C_0^\infty(\Omega)$ and set $r > 0$. Fix $q, z \in \Omega$ such that

$$|q - z| = r.$$

Let $\xi \in B(z, r)$, where $B = B(z, r)$ is the open ball centered at z with radius r. We have

$$u(\xi) - u(q) = \int_0^1 \frac{d}{dt} u(q + t(\xi - q)) \, dt = \int_0^1 \langle \nabla u(q + t(\xi - q)), \xi - q \rangle \, dt,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^n$. By writing

$$u_B = \fint_B u(\xi)\, d\xi,$$

and integrating over B with respect to ξ we find

$$\omega_n r^n (u_B - u(q)) = \int_{B(z,r)} \int_0^1 \langle \nabla u(q + t(\xi - q)), \xi - q \rangle \, dt \, d\xi,$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. By the Cauchy–Schwarz inequality we obtain

$$\begin{aligned} \omega_n r^n |u_B - u(q)| &\le \int_{B(z,r)} \int_0^1 |\nabla u(q + t(\xi - q))||\xi - q| \, dt \, d\xi \\ &= \int_0^1 \int_{B(z,r)} |\nabla u(q + t(\xi - q))||\xi - q| \, d\xi \, dt, \end{aligned}$$

where we used Tonelli's theorem to change the order of integration. By changing the variables to

$$\eta = q + t(\xi - q), \quad d\eta = t^n d\xi,$$

we find that the new domain of integration is contained in the ball $B(q, 2rt)$. Then by Hölder's inequality,

$$\begin{aligned} &\omega_n r^n |u_B - u(q)| \\ &\le \int_0^1 \int_{B(z,r)} |\nabla u(q + t(\xi - q))||\xi - q| \, d\xi \, dt \\ &\le \int_0^1 t^{-1-n} \int_{B(q,2rt)\cap\Omega} |\nabla u(\eta)||\eta - q| \, d\eta \, dt \\ &\le \int_0^1 t^{-1-n} \Big(\int_{B(q,2rt)\cap\Omega} |\nabla u(\eta)|^p \, d\eta \Big)^{1/p} \Big(\int_{B(q,2rt)\cap\Omega} |\eta - q|^{p/(p-1)} \, d\eta \Big)^{(p-1)/p} dt \\ &\le ||\nabla u||_{p,\Omega} \int_0^1 t^{-1-n} \Big(\int_{B(q,2rt)} (2rt)^{p/(p-1)} \, d\eta \Big)^{(p-1)/p} dt \\ &= ||\nabla u||_{p,\Omega} \int_0^1 t^{-1-n} \, 2rt(\omega_n (2rt)^n)^{(p-1)/p} dt \\ &= \omega_n^{(p-1)/p} (2r)^{1+n(p-1)/p} ||\nabla u||_{p,\Omega} \int_0^1 t^{-n/p} \, dt \\ &= \omega_n^{(p-1)/p} (2r)^{1+n(p-1)/p} ||\nabla u||_{p,\Omega} \frac{p}{p-n}. \end{aligned}$$

We evaluated the last integral by using the Monotone convergence theorem, where it was needed that $p > n$. Now we have

$$|u_{B(z,r)} - u(q)| \le 2^{1+n(p-1)/p} \omega_n^{-1/p} \frac{p}{p-n} r^{1-n/p} ||\nabla u||_{p,\Omega},$$

for $z, q \in \Omega$ such that $|q - z| = r$. Fix $x, y \in \Omega$ and let

$$z = \frac{1}{2}(x + y), \quad r = |x - z| = |y - z| = \frac{1}{2}|x - y|.$$

Then

$$
\begin{aligned}
|u(x)-u(y)| &\leq |u(x)-u_{B(z,r)}|+|u_{B(z,r)}-u(y)| \\
&\leq 2\cdot 2^{1+n(p-1)/p}\omega_n^{-1/p}\frac{p}{p-n}\left(\frac{1}{2}|x-y|\right)^{1-n/p}||\nabla u||_{p,\Omega} \qquad (2.6)\\
&= 2^{n+1}\omega_n^{-1/p}\frac{p}{p-n}|x-y|^{1-n/p}||\nabla u||_{p,\Omega},
\end{aligned}
$$

which concludes the proof when $u \in C_0^\infty(\Omega)$.

Now let $u \in W_0^{1,p}(\Omega)$. Then there is a sequence $u_j \in C_0^\infty(\Omega)$ such that $u_j \to u$ in $W^{1,p}(\Omega)$. We claim that there is a subsequence such that $u_j \to u$ a.e. in Ω. Indeed, for $\varepsilon > 0$ we have

$$
\begin{aligned}
||u_j-u||_{p,\Omega}^p &= \varepsilon^p \int_\Omega \left|\frac{u_j-u}{\varepsilon}\right|^p dx \\
&\geq \varepsilon^p \int_{\{|u_j-u|\geq\varepsilon\}} \left|\frac{u_j-u}{\varepsilon}\right|^p dx \\
&\geq \varepsilon^p \mu(\{x\in\Omega : |u_j(x)-u(x)|\geq\varepsilon\}),
\end{aligned}
$$

which shows that the sequence converges in measure to the function u. Then it is known that there is a subsequence such that $u_j \to u$ a.e. in Ω. The strong convergence assures that

$$
||u_j||_{p,\Omega} \leq ||u||_{p,\Omega}+1, \quad ||\nabla u_j||_{p,\Omega} \leq ||\nabla u||_{p,\Omega}+1
$$

for sufficiently large j, and by (2.6) we find

$$
|u_j(x)-u_j(y)| \leq 2^{n+1}\omega_n^{-1/p}\frac{p}{p-n}|x-y|^{1-n/p}(||\nabla u||_{p,\Omega}+1).
$$

Hence (u_j) is equibounded and equicontinuous for large j. Then by Ascoli's theorem 2.2 there is a further subsequence and a continuous function $v \in C(\Omega)$ such that $u_j \to v$ uniformly in Ω. Thus, v is a continuous version of u, and we redefine u to be v in Ω. Then

$$
\begin{aligned}
|u(x)-u(y)| &\leq |u(x)-u_j(x)|+|u_j(x)-u_j(y)|+|u_j(y)-u(y)| \\
&\leq |u(x)-u_j(x)|+|u_j(y)-u(y)| \\
&\quad + 2^{n+1}\omega_n^{-1/p}\frac{p}{p-n}|x-y|^{1-n/p}(||\nabla u_j-\nabla u||_{p,\Omega}+||\nabla u||_{p,\Omega}).
\end{aligned}
$$

Letting $j \to \infty$ we obtain

$$
\begin{aligned}
|u(x)-u(y)| &\leq 2^{n+1}\omega_n^{-1/p}\frac{p}{p-n}|x-y|^{1-n/p}||\nabla u||_{p,\Omega} \\
&= C_p|x-y|^{1-n/p}||\nabla u||_{p,\Omega}
\end{aligned}
$$

where C_p is such that

$$
C_p = 2^{n+1}\omega_n^{-1/p}\frac{p}{p-n} \to 2^{n+1} \quad \text{as } p\to\infty.
$$

By redefining u in a set of measure zero we can extend it to the boundary such that $u \in C(\overline{\Omega})$ and $u|_{\partial\Omega} = 0$. □

Remark 2.20. We shall mostly encounter the situation when the domain Ω is bounded and $p > n$. Then since every function in $W_0^{1,p}(\Omega)$ has a continuous version, we always assume when given such a function that it is its continuous version.

Morrey's inequality suggests a connection between functions in $W^{1,p}(\Omega)$ and Hölder continuous functions. For more details see the Rellich–Kondrachov compactness theorem 2.28 later in this section.

We note a convenient inequality.

Lemma 2.21 (Friedrichs' inequality)**.** *Let Ω be a bounded domain. Suppose that $u \in W_0^{1,p}(\Omega)$, where $1 \le p \le \infty$. Then*

$$||u||_p \le \operatorname{diam}(\Omega)||\nabla u||_p.$$

Proof. It suffices to show the inequality for $u \in C_0^\infty(\Omega)$. We begin with the case when $1 \le p < \infty$. Let $u \in C_0^\infty(\Omega)$. Since Ω is bounded there are numbers $\eta_i < \xi_i$, $i = 1, 2, \ldots, n$, such that $\Omega \subset\subset Q$, where

$$Q = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \eta_i < x_i < \xi_i \text{ for each } 1 \le i \le n\}.$$

Then $u \in C_0^\infty(Q)$. We have

$$\begin{aligned} u(x_1, x_2, \ldots, x_n) &= u(\eta_1, x_2, \ldots, x_n) + \int_{\eta_1}^{x_1} u_t(t, x_2, \ldots, x_n)\, dt \\ &= \int_{\eta_1}^{x_1} u_t(t, x_2, \ldots, x_n)\, dt, \end{aligned}$$

thus by Proposition 2.4,

$$\begin{aligned} |u(x)| &\le \int_{\eta_1}^{x_1} |u_t(t, x_2, \ldots, x_n)|\, dt \\ &\le \int_{\eta_1}^{\xi_1} |u_{x_1}(x_1, x_2, \ldots, x_n)|\, dx_1 \\ &\le \fint_{\eta_1}^{\xi_1} |\nabla u(x)||\xi_1 - \eta_1|\, dx_1 \\ &\le \left(\fint_{\eta_1}^{\xi_1} |\nabla u(x)|^p |\xi_1 - \eta_1|^p dx_1 \right)^{1/p}. \end{aligned}$$

This implies that

$$|u(x)|^p \le |\xi_1 - \eta_1|^{p-1} \int_{\eta_1}^{\xi_1} |\nabla u(x)|^p dx_1.$$

Observe that the right-hand side only depends on $(x_2, x_3, \ldots, x_n)$, while the left-hand side depends on $(x_1, x_2, \ldots, x_n)$. Integrating with respect to x_1 we find

$$\int_{\eta_1}^{\xi_1} |u(x)|^p dx_1 \le |\xi_1 - \eta_1|^p \int_{\eta_1}^{\xi_1} |\nabla u(x)|^p dx_1.$$

Now integrate over the other variables to obtain

$$\int_Q |u(x)|^p dx \le |\xi_1 - \eta_1|^p \int_Q |\nabla u(x)|^p dx,$$

hence

$$\int_\Omega |u(x)|^p dx \le \operatorname{diam}(\Omega)^p \int_\Omega |\nabla u(x)|^p dx,$$

where $\operatorname{diam}(\Omega) := \sup_{x,y\in\Omega} |x-y|$.

By the continuity of the norm in Proposition 2.5, we find that the inequality is also valid for $p = \infty$. □

Now we clarify the relationship between the weak derivatives and the derivatives from calculus.

Definition 2.22. We say that $u : \Omega \to \mathbb{R}$ is *differentiable* at $x \in \Omega$ if there exists $\eta \in \mathbb{R}^n$ such that

$$u(y) = u(x) + \langle \eta, y-x\rangle + o(|y-x|) \quad \text{as } y \to x.$$

If η exists it is unique, and we denote it by $\nabla u(x)$.

So far we have denoted the weak derivatives with the same notation as the usual derivatives from calculus. Let us verify that these actually coincide when $n < p \le \infty$, so that the notation is consistent.

Theorem 2.23. *Let $n < p \le \infty$ and suppose that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$. Then u is differentiable a.e. in Ω and its weak gradient equals its gradient a.e.*

Proof. First we consider the case when $n < p < \infty$. Let ∇ denote the weak gradient. We need the following version of Lebegue's differentiation theorem, see [7] for more details. For a.e. $x \in \Omega$ we have

$$\fint_{B(x,r)} |\nabla u(z) - \nabla u(x)|^p dz \to 0 \quad \text{as } r \to 0.$$

Fix any such x and define

$$v(y) = u(y) - u(x) - \langle \nabla u(x), y-x\rangle, \quad y \in \Omega.$$

By consulting the proof of Morrey's inequality 2.19 we find that the inequality is applicable to the function v in the ball $B(x,r) \subset\subset \Omega$. With $r = |x-y|$ we find

$$\begin{aligned}
&|u(y) - u(x) - \langle \nabla u(x), y-x\rangle| \\
&= |v(y) - v(x)| \\
&\le C_p r^{1-n/p} \Big(\int_{B(x,r)} |\nabla v(z)|^p dz \Big)^{1/p} \\
&= C_p r^{1-n/p} \Big(\omega_n r^n \fint_{B(x,r)} |\nabla u(z) - \nabla u(x)|^p dz \Big)^{1/p} \\
&= C_p \omega_n^{1/p} r \Big(\fint_{B(x,r)} |\nabla u(z) - \nabla u(x)|^p dz \Big)^{1/p} \\
&= o(r) = o(|x-y|),
\end{aligned}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. If $p = \infty$, we have that $W^{1,\infty}_{\mathrm{loc}}(\Omega) \subset W^{1,p}_{\mathrm{loc}}(\Omega)$ for any $n < p < \infty$, so we can apply the argument above. □

We now turn our attention to Lipschitz continuity.

Definition 2.24. A function $u:\Omega\to\mathbb{R}$ is said to be *Lipschitz continuous* if

$$|u(x)-u(y)|\le L|x-y| \quad \text{when } x,y\in\Omega,$$

for some constant L.

The following result provides an interesting characterization of the space $W^{1,\infty}_{\text{loc}}(\Omega)$.

Theorem 2.25. *A function $u:\Omega\to\mathbb{R}$ is locally Lipschitz continuous if and only if $u\in W^{1,\infty}_{\text{loc}}(\Omega)$.*

Proof. Assume that $u\in W^{1,\infty}_{\text{loc}}(\Omega)$. Then u also belongs to $W^{1,p}_{\text{loc}}(\Omega)$ for some finite $p>n$. Let $\varepsilon>0$ and define the function

$$u_\varepsilon=\rho_\varepsilon * u,$$

where ρ_ε is Friedrichs' mollifier. Then $u_\varepsilon\in C^\infty(\Omega)$ and $u_\varepsilon\to u$ in $W^{1,p}_{\text{loc}}(\Omega)$ as $\varepsilon\to 0$. Actually, the convergence is locally uniform[7]. Let B be a subdomain such that $B\subset\subset\Omega$. Then we have

$$||\nabla u_\varepsilon||_{p,B}\le ||\nabla u||_{p,B},$$

which implies that

$$\sup_{0<\varepsilon<\delta}||\nabla u_\varepsilon||_{\infty,B}\le C<\infty,$$

for sufficiently small $\delta>0$, where C is a constant that is independent of ε. For $x,y\in B$ we have

$$u_\varepsilon(x)-u_\varepsilon(y)=\int_0^1\frac{d}{dt}u_\varepsilon(y+t(x-y))\,dt=\int_0^1\langle\nabla u_\varepsilon(y+t(x-y)),x-y\rangle\,dt,$$

which leads to

$$|u_\varepsilon(x)-u_\varepsilon(y)|\le C|x-y|,$$

by the Cauchy–Schwarz inequality and the above. Observe that

$$|u(x)-u(y)|\le|u(x)-u_\varepsilon(x)|+C|x-y|+|u_\varepsilon(y)-u(y)|.$$

Letting $\varepsilon\to 0$ we obtain

$$|u(x)-u(y)|\le C|x-y| \quad x,y\in B,$$

by the uniform convergence. Thus, u is locally Lipschitz continuous.

Now suppose that u is locally Lipschitz continuous. Once again we let $B\subset\subset\Omega$. We have that $u\in L^\infty(B)$, so we only have to show that the weak first partial derivatives are bounded. For $i=1,2,\dots,n$ write

$$D_i^h u(x)=\frac{u(x+he_i)-u(x)}{h},$$
$$D_i^{-h}u(x)=\frac{u(x)-u(x-he_i)}{h},$$

[7] See Theorem 4.40 and Theorem 5.3 in [11].

where $h > 0$ and e_i is the ith unit vector. Since u is locally Lipschitz continuous,

$$\sup_{0<h<\varepsilon} ||D_i^{-h} u||_{\infty,B} \leq M < \infty,$$

for sufficiently small $\varepsilon > 0$. Since $L^1(B)$ is separable, Helly's theorem 2.13 implies that there is a subsequence $h_k \to 0$ and a function $w_i \in L^\infty(B)$ such that

$$D_i^{-h_k} u \overset{*}{\rightharpoonup} w_i \quad \text{weak-star in } L^\infty(B)$$

for every $i = 1, 2, \ldots, n$. By (2.4) this is equivalent to

$$\lim_{k\to\infty} \int_B \psi D_i^{-h_k} u \, dx = \int_B \psi w_i \, dx \quad \text{for all } \psi \in L^1(B).$$

We can approximate functions in $L^1(B)$ by functions in $C_0^\infty(B)$ since $C_0(B)$ is dense in $L^1(B)$.[8] Let $\phi \in C_0^\infty(B)$. By the Dominated convergence theorem and the above we find

$$\int_B u\phi_{x_i} \, dx = \lim_{k\to\infty} \int_B u D_i^{h_k} \phi \, dx = -\lim_{k\to\infty} \int_B \phi D_i^{-h_k} u \, dx = -\int_B \phi w_i \, dx,$$

which holds for any $\phi \in C_0^\infty(B)$. This shows that $u_{x_i} = w_i$ in the sense of weak derivatives, for $i = 1, 2, \ldots, n$. Furthermore, by the weak-star lower semicontinuity in Proposition 2.12,

$$||u_{x_i}||_{\infty,B} = ||w_i||_{\infty,B} \leq \liminf_{k\to\infty} ||D_i^{-h_k} u||_{\infty,B} < \infty,$$

which implies that $||\nabla u||_{\infty,B} < \infty$ for any $B \subset\subset \Omega$, and we conclude that $u \in W^{1,\infty}_{\text{loc}}(\Omega)$. □

We immediately obtain the following important result.

Theorem 2.26 (Rademacher's theorem)**.** *If $u : \Omega \to \mathbb{R}$ is locally Lipschitz continuous, then u is differentiable a.e. in Ω.*

Proof. Since u is locally Lipschitz continuous, $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ by Theorem 2.25. Then it follows from Theorem 2.23 that u is differentiable a.e. in Ω. □

We introduce some notation, and seize the opportunity to mention a Sobolev embedding result. We refer to [9] for more details.

Definition 2.27. Let X and Y be Banach spaces. We say that X is *compactly embedded* in Y, and write

$$X \subset\subset Y$$

if

i) there exists a linear, continuous, and injective map $\Psi : X \to Y$;

ii) the map $\Psi(B)$ is precompact in Y for any bounded set $B \subset X$, that is $\overline{\Psi(B)}$ is compact in Y.

[8]See for instance Lemma 4.38 in [11].

Theorem 2.28 (Rellich–Kondrachov compactness theorem)**.** *Suppose that $p > n$ and let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega \in C^1$. Then*

$$W^{1,p}(\Omega) \subset\subset C^\gamma(\overline{\Omega}),$$

where $\gamma = 1 - \frac{n}{p}$.

Here $C^\gamma(\overline{\Omega})$ refers to the *Hölder space*:

$$C^\gamma(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : ||u||_{C^\gamma(\overline{\Omega})} < \infty\},$$

consisting of *Hölder continuous* functions $u : \overline{\Omega} \to \mathbb{R}$:

$$|u(x) - u(y)| \leq C|x - y|^\gamma,$$

where C is a constant and $0 < \gamma \leq 1$. The Hölder space is in fact a Banach space with respect to the norm

$$||u||_{C^\gamma(\overline{\Omega})} = ||u||_{\infty,\Omega} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^\gamma}. \tag{2.7}$$

Notice that in the above theorem, $\gamma = 1$ when $p = \infty$, so in that case the Hölder space consists of Lipschitz continuous functions, which agrees with what we found in Theorem 2.25.

The proof of Morrey's inequality 2.19 is based on Theorem 8.1 in [6], and the proof of Theorem 2.23 follows Theorem 5 in section 5.8.3 of [7]. The proof of Theorem 2.25 is based on Theorem 5 in section 4.2.3 of [8].

2.4 Quadratic forms and convex functions

We denote the space of all $n \times n$ real-valued symmetric matrices by S^n. For $B, C \in S^n$ the notation $B \geq C$ means

$$\langle B\xi, \xi\rangle \geq \langle C\xi, \xi\rangle \quad \text{for all } \xi \in \mathbb{R}^n.$$

In the following section we assume that $A \in S^n$ is such that for constants $0 < \alpha \leq \beta < \infty$,

$$\alpha|\eta|^2 \leq \langle A\eta, \eta\rangle \leq \beta|\eta|^2,$$

for all $\eta \in \mathbb{R}^n$. Thus, the matrix A is positive definite:

$$\langle A\eta, \eta\rangle > 0 \quad \text{for all nonzero } \eta \in \mathbb{R}^n.$$

We begin with an inequality.

Proposition 2.29. *If $2 \leq p < \infty$, then*

$$\Big\langle \langle A\xi, \xi\rangle^{\frac{p-2}{2}} A\xi - \langle A\psi, \psi\rangle^{\frac{p-2}{2}} A\psi, \xi - \psi \Big\rangle \geq 4\Big(\frac{\sqrt{\alpha}}{2}\Big)^p |\xi - \psi|^p \tag{2.8}$$

for all $\xi, \psi \in \mathbb{R}^n$.

Proof. Since A is symmetric it has a decomposition $A = Q^T DQ$, where D is a diagonal matrix with the eigenvalues of A on the main diagonal, and Q is an orthogonal matrix, that is $Q^T = Q^{-1}$. Moreover, since A is positive definite it has a square root $B = A^{1/2}$ defined as

$$B = A^{1/2} = Q^T D^{1/2} Q,$$

where the diagonal matrix $D^{1/2}$ consists of the square roots of the eigenvalues of A on the main diagonal. The matrix B is symmetric and $B^2 = A$. Furthermore, the square root B is unique, see [3] for more details. We have

$$\begin{aligned}
&\Big\langle \langle A\xi, \xi\rangle^{\frac{p-2}{2}} A\xi - \langle A\psi, \psi\rangle^{\frac{p-2}{2}} A\psi, \xi - \psi \Big\rangle \\
&= \Big\langle \langle B^2\xi, \xi\rangle^{\frac{p-2}{2}} B^2\xi - \langle B^2\psi, \psi\rangle^{\frac{p-2}{2}} B^2\psi, \xi - \psi \Big\rangle \\
&= \Big\langle \langle B^T B\xi, \xi\rangle^{\frac{p-2}{2}} B^T B\xi - \langle B^T B\psi, \psi\rangle^{\frac{p-2}{2}} B^T B\psi, \xi - \psi \Big\rangle \\
&= \Big\langle \langle B\xi, B\xi\rangle^{\frac{p-2}{2}} B\xi - \langle B\psi, B\psi\rangle^{\frac{p-2}{2}} B\psi, B\xi - B\psi \Big\rangle \\
&= \langle |B\xi|^{p-2} B\xi - |B\psi|^{p-2} B\psi, B\xi - B\psi\rangle \\
&= \langle |b|^{p-2} b - |a|^{p-2} a, b - a\rangle
\end{aligned} \tag{2.9}$$

where $b = B\xi$ and $a = B\psi$. We claim that

$$\langle |b|^{p-2} b - |a|^{p-2} a, b - a\rangle \geq 2^{2-p} |b - a|^p.$$

To see this first observe that

$$\begin{aligned}
\langle b, b - a\rangle &= \frac{1}{2}(|b - a|^2 + |b|^2 - |a|^2), \\
\langle a, b - a\rangle &= -\frac{1}{2}(|b - a|^2 + |a|^2 - |b|^2),
\end{aligned}$$

which leads to the identity

$$\langle |b|^{p-2} b - |a|^{p-2} a, b - a\rangle = \frac{|b|^{p-2} + |a|^{p-2}}{2} |b - a|^2 + \frac{(|b|^{p-2} - |a|^{p-2})(|b|^2 - |a|^2)}{2}.$$

Notice that the second term on the right-hand side is nonnegative for $p \geq 2$, which is easily seen by first letting $|b| \geq |a|$ and then letting $|a| \geq |b|$. Thus

$$\langle |b|^{p-2} b - |a|^{p-2} a, b - a\rangle \geq \frac{|b|^{p-2} + |a|^{p-2}}{2} |b - a|^2. \tag{2.10}$$

Furthermore, by some well known inequalities we see that

$$\frac{|b|^{p-2} + |a|^{p-2}}{2} \geq 2^{2-p}(|b| + |a|)^{p-2} \geq 2^{2-p} |b - a|^{p-2},$$

where we used that $p \geq 2$. The claim follows by inserting this into (2.10). Continuing to estimate (2.9) we find

$$\begin{aligned}
&\Big\langle \langle A\xi,\xi\rangle^{\frac{p-2}{2}} A\xi - \langle A\psi,\psi\rangle^{\frac{p-2}{2}} A\psi, \xi-\psi\Big\rangle \\
&= \langle |B\xi|^{p-2}B\xi - |B\psi|^{p-2}B\psi, B\xi - B\psi\rangle \\
&\geq 2^{2-p}|B\xi - B\psi|^p \\
&= 2^{2-p}\langle B(\xi-\psi), B(\xi-\psi)\rangle^{p/2} \\
&= 2^{2-p}\langle A(\xi-\psi), \xi-\psi\rangle^{p/2} \\
&\geq 2^{2-p}(\alpha|\xi-\psi|^2)^{p/2} \\
&= 4\Big(\frac{\sqrt{\alpha}}{2}\Big)^p |\xi-\psi|^p.
\end{aligned}$$

□

Now we introduce convex functions.

Definition 2.30. We say that a function $F:\mathbb{R}^n \to \mathbb{R}$ is *convex*, if

$$F(t\xi + (1-t)\eta) \leq tF(\xi) + (1-t)F(\eta)$$

for all $\xi,\eta \in \mathbb{R}^n$ and $0 \leq t \leq 1$. We say that F is *strictly convex*, if

$$F(t\xi + (1-t)\eta) < tF(\xi) + (1-t)F(\eta)$$

whenever $\xi \neq \eta$ and $0 < t < 1$.

A convex function $F:\Omega \to \mathbb{R}$ is locally Lipschitz continuous, and thus differentiable almost everywhere according to Rademacher's theorem 2.26. Furthermore, the inequality

$$F(\xi) \geq F(\eta) + \langle \nabla F(\eta), \xi - \eta\rangle \tag{2.11}$$

holds for all $\xi,\eta \in \Omega$.[9]

Lemma 2.31. *The map*

$$\xi \mapsto \langle A\xi,\xi\rangle^{p/2}$$

is strictly convex for $2 \leq p < \infty$.

Proof. We first show that $\xi \mapsto \langle A\xi,\xi\rangle$ is stricly convex. Let $0<t<1$ and $\xi \neq \eta$. We calculate

$$\begin{aligned}
&\langle A(t\xi+(1-t)\eta), t\xi+(1-t)\eta\rangle \\
&= \langle A(\eta + t(\xi-\eta)), \eta + t(\xi-\eta)\rangle \\
&= \langle A\eta,\eta\rangle + 2t\langle A\eta, \xi-\eta\rangle + t^2\langle A(\xi-\eta),\xi-\eta\rangle \\
&< \langle A\eta,\eta\rangle + 2t\langle A\eta, \xi-\eta\rangle + t\langle A(\xi-\eta),\xi-\eta\rangle \\
&= \langle A\eta,\eta\rangle + t\langle A\eta, \xi-\eta\rangle + t\langle A\xi,\xi-\eta\rangle \\
&= t\langle A\xi,\xi\rangle + (1-t)\langle A\eta,\eta\rangle,
\end{aligned}$$

[9]Consult for instance [6] for proofs of these statements.

where the strict inequality follows since $t^2 < t$ and A is positive definite. Now we show that $\xi \mapsto \langle A\xi,\xi\rangle^{p/2}$ is strictly convex. Since the function

$$H(z) = z^{p/2}, \quad z \geq 0,\ p \geq 2$$

is convex, we find by writing

$$z_1 = \langle A\xi,\xi\rangle, \quad z_2 = \langle A\eta,\eta\rangle$$

that

$$\begin{aligned}
&\langle A(t\xi+(1-t)\eta), t\xi+(1-t)\eta\rangle^{p/2} \\
&< (t\langle A\xi,\xi\rangle + (1-t)\langle A\eta,\eta\rangle)^{p/2} \\
&= H(tz_1+(1-t)z_2) \\
&\leq tH(z_1)+(1-t)H(z_2) \\
&= t\langle A\xi,\xi\rangle^{p/2} + (1-t)\langle A\eta,\eta\rangle^{p/2},
\end{aligned}$$

where we used that $\xi \mapsto \langle A\xi,\xi\rangle$ is strictly convex in the first inequality. □

For the next result we assume that the symmetric $n\times n$ matrix $A = A(x) = (a_{ij}(x))$ consists of real-valued functions $a_{ij} \in L^\infty(\Omega)$, $i,j = 1,2,\ldots,n$, such that for constants $0 < \alpha \leq \beta < \infty$ we have

$$\alpha|\xi|^2 \leq \langle A(x)\xi,\xi\rangle \leq \beta|\xi|^2, \tag{2.12}$$

for all $\xi \in \mathbb{R}^n$ and all $x \in \Omega$. Then $\xi \mapsto \langle A(x)\xi,\xi\rangle^{p/2}$ is convex by the above, and we now show an important consequence of this, namely the weak lower semicontinuity of the integral

$$I(u) = \int_\Omega \langle A(x)\nabla u,\nabla u\rangle^{p/2} dx.$$

Proposition 2.32. *Let Ω be a bounded domain and suppose that $2 \leq p < \infty$. If*

$$\nabla u_j \rightharpoonup \nabla u \quad \textit{weakly in } L^p(\Omega),$$

then

$$I(u) \leq \liminf_{j\to\infty} I(u_j).$$

Proof. Since $\xi \mapsto \langle A(x)\xi,\xi\rangle^{p/2}$ is convex it follows from (2.11) that

$$\langle A(x)\nabla u_j,\nabla u_j\rangle^{p/2} \geq \langle A(x)\nabla u,\nabla u\rangle^{p/2} + p\langle A(x)\nabla u,\nabla u\rangle^{\frac{p-2}{2}}\langle A(x)\nabla u,\nabla u_j - \nabla u\rangle.$$

Integration leads to

$$\begin{aligned}
I(u_j) &= \int_\Omega \langle A(x)\nabla u_j,\nabla u_j\rangle^{p/2} dx \\
&\geq \int_\Omega \langle A(x)\nabla u,\nabla u\rangle^{p/2} dx \\
&\quad + p\int_\Omega \langle A(x)\nabla u,\nabla u\rangle^{\frac{p-2}{2}}\langle A(x)\nabla u,\nabla u_j-\nabla u\rangle\, dx \\
&= I(u) + p\int_\Omega \langle A(x)\nabla u,\nabla u\rangle^{\frac{p-2}{2}}\langle A(x)\nabla u,\nabla u_j-\nabla u\rangle\, dx \\
&= I(u) + p\sum_{k,l=1}^n \int_\Omega \langle A(x)\nabla u,\nabla u\rangle^{\frac{p-2}{2}} a_{kl}(x)\frac{\partial u}{\partial x_l}\Big(\frac{\partial u_j}{\partial x_k} - \frac{\partial u}{\partial x_k}\Big) dx.
\end{aligned} \tag{2.13}$$

Recall from (2.3) and Remark 2.18 that the weak convergence

$$\nabla u_j \rightharpoonup \nabla u \quad \text{in } L^p(\Omega)$$

means that for each $k = 1, 2, \ldots, n$,

$$\lim_{j\to\infty} \int_\Omega v\Big(\frac{\partial u_j}{\partial x_k} - \frac{\partial u}{\partial x_k}\Big) dx = 0$$

for all $v \in L^q(\Omega)$ such that $1/p + 1/q = 1$. Thus, if the function

$$\langle A(x)\nabla u, \nabla u\rangle^{\frac{p-2}{2}} a_{kl}(x) \frac{\partial u}{\partial x_l}$$

belongs to $L^q(\Omega)$ for all $k, l = 1, 2, \ldots, n$, then

$$\sum_{k,l=1}^{n} \int_\Omega \langle A(x)\nabla u, \nabla u\rangle^{\frac{p-2}{2}} a_{kl}(x) \frac{\partial u}{\partial x_l}\Big(\frac{\partial u_j}{\partial x_k} - \frac{\partial u}{\partial x_k}\Big) dx = 0,$$

and (2.13) reduces to

$$I(u_j) \geq I(u).$$

This implies that

$$\liminf_{j\to\infty} I(u_j) \geq I(u).$$

Hence, we only have to show that

$$\langle A(x)\nabla u, \nabla u\rangle^{\frac{p-2}{2}} a_{kl}(x) \frac{\partial u}{\partial x_l} \in L^q(\Omega).$$

By (2.12) we have

$$\begin{aligned}\Big|\langle A(x)\nabla u, \nabla u\rangle^{\frac{p-2}{2}} a_{kl}(x) \frac{\partial u}{\partial x_l}\Big| &\leq \max_{q,r} ||a_{qr}||_\infty (\beta|\nabla u|^2)^{\frac{p-2}{2}} \Big|\frac{\partial u}{\partial x_l}\Big| \\ &\leq \beta^{\frac{p-2}{2}} \max_{q,r} ||a_{qr}||_\infty |\nabla u|^{p-2}|\nabla u|.\end{aligned}$$

Raising to the q power we find

$$\begin{aligned}\Big|\langle A(x)\nabla u, \nabla u\rangle^{\frac{p-2}{2}} a_{kl}(x) \frac{\partial u}{\partial x_l}\Big|^q &\leq (\beta^{\frac{p-2}{2}} \max_{q,r} ||a_{qr}||_\infty)^q |\nabla u|^{(p-2)q}|\nabla u|^q \\ &= (\beta^{\frac{p-2}{2}} \max_{q,r} ||a_{qr}||_\infty)^q |\nabla u|^{p-q}|\nabla u|^q \\ &= (\beta^{\frac{p-2}{2}} \max_{q,r} ||a_{qr}||_\infty)^q |\nabla u|^{p},\end{aligned}$$

thus

$$\int_\Omega \Big|\langle A(x)\nabla u, \nabla u\rangle^{\frac{p-2}{2}} a_{kl}(x) \frac{\partial u}{\partial x_l}\Big|^q dx = (\beta^{\frac{p-2}{2}} \max_{q,r} ||a_{qr}||_\infty)^q \int_\Omega |\nabla u|^p dx < \infty,$$

and we conclude that

$$\langle A(x)\nabla u, \nabla u\rangle^{\frac{p-2}{2}} a_{kl}(x) \frac{\partial u}{\partial x_l} \in L^q(\Omega).$$

□

3 The equation

Now we introduce the variational integral and state the assumptions on the coefficient functions. Moreover, we derive the Euler-Lagrange equation

$$\Delta_{p,A}\, u = 0$$

for finite values of p, and show that it can be expressed with $\Delta_{\infty,A}\, u$. Then we show some properties of $\Delta_{p,A}\, u$ for $4 \le p \le \infty$.

3.1 Variational problem

Our starting point is the problem of minimizing the variational integral

$$I(v) = \int_\Omega \Big(\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial v}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x) \Big)^{p/2} dx$$

among all admissible functions $v : \Omega \to \mathbb{R}$ with the same boundary values g, where Ω is a bounded domain in $\mathbb{R}^n$. By introducing the $n \times n$ matrix $A(x) = (a_{ij}(x))$, we can write

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} = \langle A(x) \nabla v, \nabla v \rangle.$$

Although standard convention is to simply denote the matrix by A, we shall write $A(x)$ throughout the text to remind the reader that the elements of the matrix depend on x. We assume the following:

1. For each fixed $x \in \Omega$ we can assume without loss of generality that the matrix $A(x)$ is symmetric. Indeed, if $a_{ij} \neq a_{ji}$ we can define $\hat{a}_{ij} = (a_{ij} + a_{ji})/2$, and we find

$$\sum_{i,j=1}^{n} \hat{a}_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j}.$$

2. For constants $0 < \alpha \le \beta < \infty$ we assume that

$$\alpha |\xi|^2 \le \langle A(x) \xi, \xi \rangle \le \beta |\xi|^2 \tag{3.1}$$

for all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$. This means that for each $x \in \Omega$, the matrix $A(x)$ is positive definite:

$$\langle A(x) \xi, \xi \rangle > 0 \quad \text{for all nonzero vectors } \xi \in \mathbb{R}^n.$$

3. For $i, j, k = 1, 2, \ldots, n$ we assume that

$$\frac{\partial a_{ij}}{\partial x_k} : \overline{\Omega} \to \mathbb{R}$$

is Lipschitz continuous with constant K:

$$\Big| \frac{\partial a_{ij}}{\partial x_k}(x) - \frac{\partial a_{ij}}{\partial x_k}(y) \Big| \le K |x - y|, \quad x, y \in \overline{\Omega}.$$

4. From **3.** it follows that $a_{ij}:\overline{\Omega}\to\mathbb{R}$ is Lipschitz continuous:

$$|a_{ij}(x)-a_{ij}(y)|\le H|x-y|,\quad x,y\in\overline{\Omega}, \tag{3.2}$$

for $i,j=1,2,\dots,n$. To see this, notice that

$$\begin{aligned}|a_{ij}(x)-a_{ij}(y)| &= \left|\int_0^1 \frac{d}{dt}a_{ij}(y+t(x-y))\,dt\right|\\ &= \left|\int_0^1 \langle\nabla a_{ij}(y+t(x-y)),x-y\rangle\,dt\right|\\ &\le \int_0^1 |\langle\nabla a_{ij}(y+t(x-y)),x-y\rangle|\,dt\\ &\le |x-y|\int_0^1 |\nabla a_{ij}(y+t(x-y))|\,dt\\ &\le ||\nabla a_{ij}||_\infty|x-y|,\end{aligned}$$

by the Cauchy–Schwarz inequality. Since the domain Ω is bounded and the derivatives are Lipschitz continuous, $||\nabla a_{ij}||_\infty$ is bounded. Thus, we can define the Lipschitz constant H as

$$H=\max_{i,j}||\nabla a_{ij}||_\infty,$$

and (3.2) follows.

We now derive the Euler-Lagrange equation for the variational integral

$$I(v)=\int_\Omega\langle A(x)\nabla v,\nabla v\rangle^{p/2}dx, \tag{3.3}$$

for finite values of p. We seek to minimize the integral among all functions $v\in W^{1,p}(\Omega)$ such that $v-g\in W_0^{1,p}(\Omega)$. Suppose that u is a minimizer. Then the function

$$v=u+t\eta,\quad t\in\mathbb{R},\ \eta\in C_0^\infty(\Omega)$$

is admissible. Since u is a minimizer and $v=u+t\eta=u=g$ on $\partial\Omega$, the function $\mathfrak{I}(t)=I(u+t\eta)$ has a minimum at $t=0$, hence

$$\mathfrak{I}'(0)=0. \tag{3.4}$$

We evaluate the derivative

$$\begin{aligned}\mathfrak{I}'(t) &= \frac{d}{dt}\int_\Omega\langle A(x)\nabla(u+t\eta),\nabla(u+t\eta)\rangle^{p/2}dx\\ &= \int_\Omega p\langle A(x)\nabla(u+t\eta),\nabla(u+t\eta)\rangle^{\frac{p-2}{2}}\langle A(x)\nabla(u+t\eta),\nabla\eta\rangle\,dx,\end{aligned}$$

where the differentiation under the integral sign is justified by the Dominated convergence theorem. Using (3.4) we find that the first variation vanishes:

$$\int_\Omega\langle A(x)\nabla u,\nabla u\rangle^{\frac{p-2}{2}}\langle A(x)\nabla u,\nabla\eta\rangle\,dx=0\quad\text{for all }\eta\in C_0^\infty(\Omega). \tag{3.5}$$

Integration by parts leads to

$$\int_\Omega \operatorname{div}(\langle A(x)\nabla u, \nabla u\rangle^{\frac{p-2}{2}} A(x)\nabla u)\eta \; dx = 0.$$

Since this holds for all test functions η, we have

$$\Delta_{p,A}\, u := \operatorname{div}(\langle A(x)\nabla u(x), \nabla u(x)\rangle^{\frac{p-2}{2}} A(x)\nabla u(x)) = 0$$

by the Variational lemma 2.6. This is the Euler-Lagrange equation for the variational integral (3.3) for finite values of p, and we shall restrict our attention to $4 \le p < \infty$. If we write this out we find

$$\Delta_{p,A}\, u = \langle A(x)\nabla u, \nabla u\rangle^{\frac{p-4}{2}} \Big\{ \langle A(x)\nabla u, \nabla u\rangle \operatorname{div}(A(x)\nabla u(x)) + \Big(\frac{p-2}{2}\Big)\Delta_{\infty,A}\, u \Big\} = 0.$$

The notation means

$$\operatorname{div}(A(x)\nabla u(x)) = \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_i}\frac{\partial u}{\partial x_j} + \operatorname{trace}(A(D^2 u)),$$

where $D^2 u$ denotes the Hessian matrix of u, and the trace of an $n \times n$ matrix $B = (b_{ij})$ is defined as

$$\operatorname{trace}(B) = \sum_{i=1}^{n} b_{ii}.$$

Furthermore

$$\begin{aligned} \Delta_{\infty,A}\, u &:= \Big\langle \nabla_x \langle A(x)\nabla u, \nabla u\rangle + 2(D^2 u)A(x)\nabla u, A(x)\nabla u \Big\rangle \\ &= \Big\langle \nabla_x \langle A(x)\nabla u, \nabla u\rangle, A(x)\nabla u \Big\rangle + 2\langle (D^2 u)A(x)\nabla u, A(x)\nabla u\rangle \\ &= \sum_{i,j,k,l=1}^{n} \Big(\frac{\partial a_{ij}}{\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} a_{kl}\frac{\partial u}{\partial x_l} + 2a_{ik}\frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_i \partial x_j} a_{jl}\frac{\partial u}{\partial x_l} \Big), \end{aligned}$$

where

$$\begin{aligned} \nabla_x \langle A(x)\xi, \xi\rangle &= \Big(\frac{\partial}{\partial x_1}\langle A(x)\xi, \xi\rangle, \frac{\partial}{\partial x_2}\langle A(x)\xi, \xi\rangle, \dots, \frac{\partial}{\partial x_n}\langle A(x)\xi, \xi\rangle \Big) \\ &= \Big(\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_1}(x)\xi_i\xi_j, \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_2}(x)\xi_i\xi_j, \dots, \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_n}(x)\xi_i\xi_j \Big), \quad \xi \in \mathbb{R}^n. \end{aligned}$$

As we shall see, the equation

$$\Delta_{\infty,A}\, u = 0$$

is in some sense the limit equation of the Euler-Lagrange equations

$$\Delta_{p,A}\, u = 0 \quad \text{as } p \to \infty.$$

Furthermore, we interpret the limit equation as the Euler-Lagrange equation for the "variational problem"

$$\inf_v ||\langle A\nabla v, \nabla v\rangle^{1/2}||_\infty.$$

These statements will be justified later in the text.

Observation 3.1. The notation $\Delta_{p,A}$ is suggestive. If a_{ij} equals the Kronecker delta:

$$a_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

then A reduces to the identity matrix I. In that case the Euler-Lagrange equation for finite p becomes the *p-Laplace equation*:

$$\Delta_{p,I}\, u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \Delta_p\, u = 0.$$

In particular, if $p = 2$ we find the *Laplace equation*:

$$\Delta_{2,I}\, u = \Delta u = 0.$$

If $a_{ij} = \delta_{ij}/2$ and $p = \infty$ we obtain the *infinity-Laplace equation*:

$$\Delta_{\infty,\frac{1}{2}I}\, u = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = \Delta_\infty\, u = 0.$$

Example 3.2 (Solution in one variable)**.** In one variable the equation $\Delta_{\infty,A}\, u = 0$ can be integrated. Consider the Dirichlet problem

$$\begin{cases} \Delta_{\infty,A}\, u = a(x)a'(x)u'(x)^3 + 2a(x)^2u'(x)^2u''(x) = 0 & \text{in } (l,r) \\ u(l) = L, \quad u(r) = R, \end{cases}$$

where we assume that a and a' is Lipschitz continuous in $[l,r]$, and that

$$0 < \alpha \leq a(x) \leq \beta < \infty \quad \text{for all } x \in [l,r].$$

Observe that

$$\begin{aligned} 0 &= a(x)a'(x)u'(x)^3 + 2a(x)^2u'(x)^2u''(x) \\ &= a(x)a'(x)v(x)^3 + 2a(x)^2v(x)^2v'(x), & v &= u' \\ &= a(x)a'(x)v(x)^3 + \frac{2}{3}a(x)^2\frac{d}{dx}v(x)^3, & \frac{d}{dx}v(x)^3 &= 3v(x)^2v'(x) \\ &= a(x)a'(x)w(x) + \frac{2}{3}a(x)^2w'(x), & w &= v^3. \end{aligned}$$

By integrating the equation

$$a(x)a'(x)w(x) + \frac{2}{3}a(x)^2w'(x) = 0$$

we find

$$\begin{aligned} w(x) &= \tilde{C}a(x)^{-3/2} \\ v(x) &= Ca(x)^{-1/2} \\ u'(x) &= Ca(x)^{-1/2}. \end{aligned}$$

Integrating the last equation from l to x and using $u(r) = R$ we obtain the solution

$$u(x) = L + (R-L)\frac{\int_l^x a(t)^{-1/2}dt}{\int_l^r a(t)^{-1/2}dt}.$$

It is considerably harder to find an explicit solution in several variables.

3.2 Some properties

It is often convenient to write

$$\Delta_{p,A}\, u(x) = \mathcal{L}_{p,A}(x, \nabla u(x), D^2u(x)), \quad 4 \le p \le \infty.$$

This defines a function

$$\mathcal{L}_{p,A} : \Omega \times \mathbb{R}^n \times S^n \to \mathbb{R}.$$

For $x \in \Omega$, $\xi \in \mathbb{R}^n$ and $\mathbb{X} \in S^n$ we can now write

$$\mathcal{L}_{\infty,A}(x,\xi,\mathbb{X}) = \Big\langle \nabla_x \langle A(x)\xi,\xi\rangle + 2\mathbb{X}A(x)\xi, A(x)\xi \Big\rangle$$

and

$$\mathcal{L}_{p,A}(x,\xi,\mathbb{X}) = \langle A(x)\xi,\xi\rangle^{\frac{p-4}{2}} \Bigg\{ \langle A(x)\xi,\xi\rangle \Bigg(\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_i}(x)\xi_j + \operatorname{trace}(A(x)\mathbb{X}) \Bigg) \\ + \Big(\frac{p-2}{2}\Big) \mathcal{L}_{\infty,A}(x,\xi,\mathbb{X}) \Bigg\}, \quad 4 \le p < \infty.$$

A key property of the function $\mathcal{L}_{p,A}$ is the following.

Proposition 3.3. *Let $\mathbb{X}, \mathbb{Y} \in S^n$ be symmetric $n \times n$ matrices, $\xi \in \mathbb{R}^n$, and $x \in \Omega$. If $\mathbb{X} \ge \mathbb{Y}$, then*

$$\mathcal{L}_{p,A}(x,\xi,\mathbb{X}) \ge \mathcal{L}_{p,A}(x,\xi,\mathbb{Y}) \quad \textit{for all } 4 \le p \le \infty.$$

Proof. We remind that the notation $\mathbb{X} \ge \mathbb{Y}$ means

$$\langle \mathbb{X}\eta,\eta\rangle \ge \langle \mathbb{Y}\eta,\eta\rangle \quad \text{for all } \eta \in \mathbb{R}^n.$$

If $p = \infty$ we immediately find

$$\begin{aligned} \mathcal{L}_{\infty,A}(x,\xi,\mathbb{X}) &= \Big\langle \nabla_x \langle A(x)\xi,\xi\rangle, A(x)\xi \Big\rangle + 2\langle \mathbb{X}A(x)\xi, A(x)\xi\rangle \\ &\ge \Big\langle \nabla_x \langle A(x)\xi,\xi\rangle, A(x)\xi \Big\rangle + 2\langle \mathbb{Y}A(x)\xi, A(x)\xi\rangle \\ &= \mathcal{L}_{\infty,A}(x,\xi,\mathbb{Y}). \end{aligned}$$

If $4 \le p < \infty$ we obtain by the above,

$$\begin{aligned} &\mathcal{L}_{p,A}(x,\xi,\mathbb{X}) - \mathcal{L}_{p,A}(x,\xi,\mathbb{Y}) \\ &= \langle A(x)\xi,\xi\rangle^{\frac{p-4}{2}} \Big\{ \langle A(x)\xi,\xi\rangle \Big(\operatorname{trace}(A(x)\mathbb{X}) - \operatorname{trace}(A(x)\mathbb{Y}) \Big) \\ &\quad + \Big(\frac{p-2}{2}\Big) \Big(\mathcal{L}_{\infty,A}(x,\xi,\mathbb{X}) - \mathcal{L}_{\infty,A}(x,\xi,\mathbb{Y}) \Big) \Big\} \\ &\ge \langle A(x)\xi,\xi\rangle^{\frac{p-2}{2}} \Big(\operatorname{trace}(A(x)\mathbb{X}) - \operatorname{trace}(A(x)\mathbb{Y}) \Big). \end{aligned} \tag{3.6}$$

Since $\langle A(x)\xi,\xi\rangle \ge 0$ for all $\xi \in \mathbb{R}^n$, we only have to show that

$$\operatorname{trace}(A(x)\mathbb{X}) - \operatorname{trace}(A(x)\mathbb{Y}) \ge 0.$$

Let $A = Q^T DQ$, where $Q^{-1} = Q^T$, and $D = (\lambda_j \delta_{ji})$. Here λ_j denotes the eigenvalues of A and δ_{ji} the Kronecker delta. We have

$$\begin{aligned}
&\operatorname{trace}(A(x)\mathbb{X}) - \operatorname{trace}(A(x)\mathbb{Y}) \\
&= \operatorname{trace}(A(\mathbb{X}-\mathbb{Y})) \\
&= \operatorname{trace}\Big(Q^T DQ(\mathbb{X}-\mathbb{Y})\Big) \\
&= \operatorname{trace}\Big((Q^T D^{1/2})(D^{1/2}Q(\mathbb{X}-\mathbb{Y}))\Big), && D^{1/2} = (\lambda_j^{1/2}\delta_{ji}) \\
&= \operatorname{trace}\Big((D^{1/2}Q(\mathbb{X}-\mathbb{Y}))(Q^T D^{1/2})\Big), && \operatorname{trace}(EF) = \operatorname{trace}(FE) \\
&= \operatorname{trace}\Big((D^{1/2}Q)(\mathbb{X}-\mathbb{Y})(D^{1/2}Q)^T\Big), && D^{1/2} = (D^{1/2})^T \\
&= \operatorname{trace}(B(\mathbb{X}-\mathbb{Y})B^T), && D^{1/2}Q = B = (b_{ij}) \\
&= \sum_{j,k,l=1}^{n} b_{jl}(\mathbb{X}-\mathbb{Y})_{lk} b_{jk} \\
&= \sum_{j=1}^{n} \langle (\mathbb{X}-\mathbb{Y})b^j, b^j\rangle \geq 0, && b^j = (b_{j1}, b_{j2}, \ldots, b_{jn}),
\end{aligned}$$

where we used that $\mathbb{X} - \mathbb{Y} \geq 0$ in the last inequality. Inserting this into (3.6) we finally conclude that

$$\mathcal{L}_{p,A}(x,\xi,\mathbb{X}) - \mathcal{L}_{p,A}(x,\xi,\mathbb{Y}) \geq \langle A(x)\xi,\xi\rangle^{\frac{p-2}{2}}\Big(\operatorname{trace}(A(x)\mathbb{X}) - \operatorname{trace}(A(x)\mathbb{Y})\Big) \geq 0.$$

□

Remark 3.4. We mention that the above result is the same as saying that $-\mathcal{L}_{p,A}$ is *degenerate elliptic*. This is a common classification of second-order partial differential equations.

We now restrict our attention to the function $\mathcal{L}_{\infty,A}$. First we note an auxiliary estimate.

Lemma 3.5. *Let $\mathbb{X} \in S^n$ be a symmetric $n \times n$ matrix, $\xi \in \mathbb{R}^n$, and $x, y \in \Omega$. Then*

$$\Big|\Big\langle \nabla_x \langle A(x)\xi,\xi\rangle, A(x)\xi\Big\rangle - \Big\langle \nabla_x \langle A(y)\xi,\xi\rangle, A(y)\xi\Big\rangle\Big| \leq \kappa|x-y||\xi|^3,$$

where

$$\kappa = n^3(K \max_{q,r} ||a_{qr}||_\infty + H^2).$$

Proof. Write $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$. We have

$$\begin{aligned}
&\Big|\Big\langle \nabla_x \langle A(x)\xi,\xi\rangle, A(x)\xi\Big\rangle - \Big\langle \nabla_x \langle A(y)\xi,\xi\rangle, A(y)\xi\Big\rangle\Big| \\
&= \Bigg|\sum_{i,j,k,l=1}^{n}\Big(\frac{\partial a_{ij}}{\partial x_k}(x)\xi_i\xi_j a_{kl}(x)\xi_l - \frac{\partial a_{ij}}{\partial x_k}(y)\xi_i\xi_j a_{kl}(y)\xi_l\Big)\Bigg| \\
&\leq \sum_{i,j,k,l=1}^{n}\Big|\frac{\partial a_{ij}}{\partial x_k}(x)a_{kl}(x) - \frac{\partial a_{ij}}{\partial x_k}(y)a_{kl}(y)\Big||\xi_i\xi_j\xi_l|.
\end{aligned} \tag{3.7}$$

By the Lipschitz continuity we find

$$\begin{aligned}
&\Big|\frac{\partial a_{ij}}{\partial x_k}(x)a_{kl}(x)-\frac{\partial a_{ij}}{\partial x_k}(y)a_{kl}(y)\Big|\\
&=\Big|\Big(\frac{\partial a_{ij}}{\partial x_k}(x)-\frac{\partial a_{ij}}{\partial x_k}(y)\Big)a_{kl}(x)+\Big(a_{kl}(x)-a_{kl}(y)\Big)\frac{\partial a_{ij}}{\partial x_k}(y)\Big|\\
&\le\Big|\frac{\partial a_{ij}}{\partial x_k}(x)-\frac{\partial a_{ij}}{\partial x_k}(y)\Big|\Big|a_{kl}(x)\Big|+\Big|a_{kl}(x)-a_{kl}(y)\Big|\Big|\frac{\partial a_{ij}}{\partial x_k}(y)\Big|\\
&\le K|x-y|\max_{q,r}||a_{qr}||_\infty+H|x-y|\max_{q,r}||\nabla a_{qr}||_\infty\\
&=(K\max_{q,r}||a_{qr}||_\infty+H^2)|x-y|.
\end{aligned}$$

Furthermore, by applying Young's inequality twice on $|\xi_i\xi_j\xi_l|$:

$$|\xi_i\xi_j\xi_l|\le\frac{|\xi_i|^3}{3}+\frac{|\xi_j\xi_l|^{3/2}}{3/2}\le\frac{|\xi_i|^3}{3}+\frac{2}{3}\Big(\frac{|\xi_j|^3}{2}+\frac{|\xi_l|^3}{2}\Big)=\frac{|\xi_i|^3}{3}+\frac{|\xi_j|^3}{3}+\frac{|\xi_l|^3}{3}$$

we obtain

$$\sum_{i,j,k,l=1}^n|\xi_i\xi_j\xi_l|\le\sum_{i,j,k,l=1}^n\Big(\frac{|\xi_i|^3}{3}+\frac{|\xi_j|^3}{3}+\frac{|\xi_l|^3}{3}\Big)=n^3\sum_{i=1}^n|\xi_i|^3. \tag{3.8}$$

We claim that

$$\sum_{i=1}^n|\xi_i|^3\le\Big(\sum_{i=1}^n\xi_i^2\Big)^{3/2}.$$

To see this we first calculate

$$\Big(\sum_{i=1}^n\xi_i^2\Big)^3=\sum_{i=1}^n\xi_i^6+\sum_{j=1}^n\sum_{\substack{i=1\\i\neq j}}^n\xi_j^4\xi_i^2+P,$$

where P is a sum of nonnegative numbers, then we apply Young's inequality such that

$$\begin{aligned}
\Big(\sum_{i=1}^n|\xi_i|^3\Big)^2&=\sum_{i=1}^n\xi_i^6+\sum_{j=1}^n\sum_{\substack{i=1\\i\neq j}}^n|\xi_j|^3|\xi_i|^3\\
&=\sum_{i=1}^n\xi_i^6+\sum_{j=1}^n\sum_{\substack{i=1\\i\neq j}}^n(\xi_j^2|\xi_i|)(|\xi_j|\xi_i^2)\\
&\le\sum_{i=1}^n\xi_i^6+\frac{1}{2}\sum_{j=1}^n\sum_{\substack{i=1\\i\neq j}}^n\xi_j^4\xi_i^2+\frac{1}{2}\sum_{j=1}^n\sum_{\substack{i=1\\i\neq j}}^n\xi_j^2\xi_i^4\\
&=\sum_{i=1}^n\xi_i^6+\sum_{j=1}^n\sum_{\substack{i=1\\i\neq j}}^n\xi_j^4\xi_i^2\\
&\le\sum_{i=1}^n\xi_i^6+\sum_{j=1}^n\sum_{\substack{i=1\\i\neq j}}^n\xi_j^4\xi_i^2+P\\
&=\Big(\sum_{i=1}^n\xi_i^2\Big)^3.
\end{aligned}$$

Inserting this into (3.8) we now have

$$\sum_{i,j,k,l=1}^{n} |\xi_i \xi_j \xi_l| \leq n^3 \Big(\sum_{i=1}^{n} \xi_i^2 \Big)^{3/2} = n^3 |\xi|^3. \tag{3.9}$$

Continuing to estimate (3.7) we obtain

$$\begin{aligned}
&\Big| \Big\langle \nabla_x \langle A(x)\xi,\xi\rangle, A(x)\xi \Big\rangle - \Big\langle \nabla_x \langle A(y)\xi,\xi\rangle, A(y)\xi \Big\rangle \Big| \\
&\leq \sum_{i,j,k,l=1}^{n} \Big| \frac{\partial a_{ij}}{\partial x_k}(x) a_{kl}(x) - \frac{\partial a_{ij}}{\partial x_k}(y) a_{kl}(y) \Big| |\xi_i \xi_j \xi_l| \\
&\leq (K \max_{q,r} ||a_{qr}||_\infty + H^2)|x-y| \sum_{i,j,k,l=1}^{n} |\xi_i \xi_j \xi_l| \\
&\leq n^3 (K \max_{q,r} ||a_{qr}||_\infty + H^2)|x-y||\xi|^3.
\end{aligned} \tag{3.10}$$

□

From this Lemma we immediately obtain the following continuity property of $\mathcal{L}_{\infty,A}$.

Corollary 3.6. *Let $\mathbb{X}, \mathbb{Y} \in S^n$ be symmetric $n \times n$ matrices. Then the estimate*

$$\begin{aligned}
&\mathcal{L}_{\infty,A}(x,\xi,\mathbb{X}) - \mathcal{L}_{\infty,A}(y,\xi,\mathbb{Y}) \\
&\leq \kappa |x-y||\xi|^3 + 2\left\{ \langle \mathbb{X}A(x)\xi, A(x)\xi\rangle - \langle \mathbb{Y}A(y)\xi, A(y)\xi\rangle \right\}
\end{aligned}$$

is valid for $\xi \in \mathbb{R}^n$ and $x, y \in \Omega$.

Proof. By Lemma 3.5 we find

$$\begin{aligned}
&\mathcal{L}_{\infty,A}(x,\xi,\mathbb{X}) - \mathcal{L}_{\infty,A}(y,\xi,\mathbb{Y}) \\
&= \Big\langle \nabla_x \langle A(x)\xi,\xi\rangle + 2\mathbb{X}A(x)\xi, A(x)\xi \Big\rangle - \Big\langle \nabla_x \langle A(y)\xi,\xi\rangle + 2\mathbb{Y}A(y)\xi, A(y)\xi \Big\rangle \\
&= \Big\langle \nabla_x \langle A(x)\xi,\xi\rangle, A(x)\xi \Big\rangle - \Big\langle \nabla_x \langle A(y)\xi,\xi\rangle, A(y)\xi \Big\rangle \\
&\quad + 2\left\{ \langle \mathbb{X}A(x)\xi, A(x)\xi\rangle - \langle \mathbb{Y}A(y)\xi, A(y)\xi\rangle \right\} \\
&\leq \Big| \Big\langle \nabla_x \langle A(x)\xi,\xi\rangle, A(x)\xi \Big\rangle - \Big\langle \nabla_x \langle A(y)\xi,\xi\rangle, A(y)\xi \Big\rangle \Big| \\
&\quad + 2\left\{ \langle \mathbb{X}A(x)\xi, A(x)\xi\rangle - \langle \mathbb{Y}A(y)\xi, A(y)\xi\rangle \right\} \\
&\leq \kappa |x-y||\xi|^3 + 2\left\{ \langle \mathbb{X}A(x)\xi, A(x)\xi\rangle - \langle \mathbb{Y}A(y)\xi, A(y)\xi\rangle \right\}.
\end{aligned} \tag{3.11}$$

□

We mention that one can obtain various estimates of

$$\langle \mathbb{X}A(x)\xi, A(x)\xi\rangle - \langle \mathbb{Y}A(y)\xi, A(y)\xi\rangle,$$

but we omit these since they will not be needed later in the text.

4 The equation for finite p

We now study existence, uniqueness and other fundamental properties of solutions of the Dirichlet problem

$$\begin{cases} \Delta_{p,A}\, u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases}$$

for finite values of p. For the remaining part of the text we assume that

$$\Omega \text{ is a bounded domain in } \mathbb{R}^n, \quad p > n, \quad p \geq 4.$$

In this section we also assume that

$$p < \infty.$$

We remind that the equation reads

$$\Delta_{p,A}\, u = \operatorname{div}(\langle A(x)\nabla u(x), \nabla u(x)\rangle^{\frac{p-2}{2}} A(x)\nabla u(x)) = 0.$$

Motivated by (3.5) we make the following definition.

Definition 4.1. We say that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a *weak solution* of the equation $\Delta_{p,A}\, u = 0$ in Ω, if

$$\int_\Omega \langle A(x)\nabla u, \nabla u\rangle^{\frac{p-2}{2}} \langle A(x)\nabla u, \nabla \eta\rangle\, dx = 0 \quad \text{for all } \eta \in C_0^\infty(\Omega).$$

We have the following fundamental result.

Proposition 4.2. *Suppose that $u \in W^{1,p}(\Omega)$. Then u is a minimizer of the variational integral*

$$I(u) = \int_\Omega \langle A(x)\nabla u, \nabla u\rangle^{p/2} dx$$

if and only if u is a weak solution of the equation $\Delta_{p,A}\, u = 0$.

Proof. In Section 3.1 we showed in the calculations from (3.3) to (3.5) that if u is a minimizer, then u is also a weak solution.

Now suppose that u is a weak solution, and that v is admissible, both having the same boundary values. By Lemma 2.31, the function $\xi \mapsto \langle A(x)\xi, \xi\rangle^{p/2}$ is convex, thus by (2.11),

$$\langle A(x)\nabla v, \nabla v\rangle^{p/2} \geq \langle A(x)\nabla u, \nabla u\rangle^{p/2} + p\langle A(x)\nabla u, \nabla u\rangle^{\frac{p-2}{2}} \langle A(x)\nabla u, \nabla(v-u)\rangle.$$

Integration leads to

$$\begin{aligned} I(v) &= \int_\Omega \langle A(x)\nabla v, \nabla v\rangle^{p/2} dx \\ &\geq \int_\Omega \langle A(x)\nabla u, \nabla u\rangle^{p/2} dx + p\int_\Omega \langle A(x)\nabla u, \nabla u\rangle^{\frac{p-2}{2}} \langle A(x)\nabla u, \nabla(v-u)\rangle\, dx \\ &= I(u) + p\int_\Omega \langle A(x)\nabla u, \nabla u\rangle^{\frac{p-2}{2}} \langle A(x)\nabla u, \nabla(v-u)\rangle\, dx. \end{aligned}$$

Now let $\eta = v - u$. Since η belongs to $W_0^{1,p}(\Omega)$, it can be approximated by functions in $C_0^\infty(\Omega)$, which implies that the last integral vanishes. Hence

$$I(v) \geq I(u),$$

and we conclude that u is a minimizer. □

The term *variational solutions* is frequently used for weak solutions/minimizers.

4.1 Existence and uniqueness

The main result of this section is the following.

Theorem 4.3. *Assume that $g \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$. Then there exists a unique weak solution $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ of the equation $\Delta_{p,A}\, u = 0$ with boundary values g.*

Proof. In view of Proposition 4.2 it is enough to show that there exists a unique minimizer. We first show that the minimizer exists by using the Direct method in the calculus of variations, see [5] for more details. Let

$$I_0 = \inf_v I(v),$$

where the infimum is taken over all $v \in W^{1,p}(\Omega)$ with boundary values g. Since g is admissible we have

$$I_0 \leq I(g) = \int_\Omega \langle A(x)\nabla g, \nabla g\rangle^{p/2} dx \leq \int_\Omega \beta^{p/2} |\nabla g|^p dx = \beta^{p/2} ||\nabla g||_p^p =: M < \infty,$$

where we used (3.1). Moreover, since

$$I(v) = \int_\Omega \langle A(x)\nabla v, \nabla v\rangle^{p/2} dx \geq \int_\Omega \alpha^{p/2} |\nabla v|^p dx \geq 0$$

for all admissible v, we have

$$\inf_v I(v) = I_0 \geq 0.$$

Thus

$$0 \leq I_0 \leq M < \infty,$$

so there is a sequence (u_j) of admissible functions such that

$$\lim_{j\to\infty} I(u_j) = I_0,$$

and we may assume that $I(u_j) < I_0 + 1$ for all $j \in \mathbb{N}$. Since

$$\int_\Omega \alpha^{p/2} |\nabla u_j|^p dx \leq I(u_j) < I_0 + 1$$

we see that the sequence (∇u_j) is uniformly bounded:

$$||\nabla u_j||_p < \alpha^{-1/2} (I_0 + 1)^{1/p} =: C_1 < \infty \quad \text{for all } j \in \mathbb{N}.$$

Furthermore, since $u_j - g \in W_0^{1,p}(\Omega)$ we find

$$\begin{aligned}||u_j||_p &\leq ||u_j - g||_p + ||g||_p \\ &\leq \mathrm{diam}(\Omega)||\nabla(u_j - g)||_p + ||g||_p \\ &\leq \mathrm{diam}(\Omega)(||\nabla u_j||_p + ||\nabla g||_p) + ||g||_p \\ &\leq \mathrm{diam}(\Omega)(C_1 + ||\nabla g||_p) + ||g||_p \\ &=: C_2 < \infty \quad \text{for all } j \in \mathbb{N},\end{aligned}$$

by Friedrichs' inequality 2.21. This shows that the sequence (u_j) is uniformly bounded. Then by Proposition 2.10 there exists a subsequence (u_{j_k}) and a function $u \in W^{1,p}(\Omega)$ such that

$$u_{j_k} \rightharpoonup u \quad \text{and} \quad \nabla u_{j_k} \rightharpoonup \nabla u \quad \text{weakly in } L^p(\Omega).$$

Since $u - g \in W_0^{1,p}(\Omega)$ and $p > n$ we conclude by Morrey's inequality 2.19 that $u \in C(\overline{\Omega})$ with $u|_{\partial\Omega} = g$. By Proposition 2.32 we find

$$I(u) \leq \liminf_{k\to\infty} I(u_{j_k}) = I_0.$$

On the other hand, since u is admissible we have that $I(u) \geq I_0$. Thus $I(u) = I_0$, and we conclude that there exists a minimizer $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$.

To establish uniqueness, we assume by contradiction that there are two minimizers u_1 and u_2. Then $(u_1 + u_2)/2$ is admissible, and by the strict convexity in Lemma 2.31 we have

$$\begin{aligned}&\Big\langle A(x)\Big(\frac{\nabla u_1 + \nabla u_2}{2}\Big), \frac{\nabla u_1 + \nabla u_2}{2}\Big\rangle^{p/2} \\ &< \frac{1}{2}\langle A(x)\nabla u_1, \nabla u_1\rangle^{p/2} + \frac{1}{2}\langle A(x)\nabla u_2, \nabla u_2\rangle^{p/2} \quad \text{when } \nabla u_1 \neq \nabla u_2.\end{aligned} \tag{4.1}$$

This leads to the contradiction

$$\begin{aligned}I(u_1) &\leq I\Big(\frac{u_1 + u_2}{2}\Big) \\ &= \int_\Omega \Big\langle A(x)\Big(\frac{\nabla u_1 + \nabla u_2}{2}\Big), \frac{\nabla u_1 + \nabla u_2}{2}\Big\rangle^{p/2} dx \\ &< \frac{1}{2}\int_\Omega \langle A(x)\nabla u_1, \nabla u_1\rangle^{p/2} dx + \frac{1}{2}\int_\Omega \langle A(x)\nabla u_2, \nabla u_2\rangle^{p/2} dx \\ &= \frac{1}{2}I(u_1) + \frac{1}{2}I(u_2) \\ &= I(u_1),\end{aligned}$$

unless $\nabla u_1 = \nabla u_2$ a.e. in Ω. Thus we must have $u_1 = u_2$. □

4.2 Comparison principle

The following two results will be useful. We begin with the Maximum principle, which states that the difference of two weak solutions attains its maximum on the boundary.

Lemma 4.4 (Maximum principle)**.** *Suppose that $u, v \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ are two weak solutions of $\Delta_{p,A}\, w = 0$ in Ω. Then*

$$\max_{\overline{\Omega}}(u-v) = \max_{\partial\Omega}(u-v).$$

Proof. Assume by contradiction that

$$c = \max_{\Omega}(u-v) > \max_{\partial\Omega}(u-v) = d.$$

Define the open set

$$G = \left\{ x \in \Omega : u(x) - v(x) > \frac{c+d}{2} \right\}.$$

Then $G \subset\subset \Omega$ and $u = v + (c+d)/2$ on ∂G. Furthermore, since u and v are weak solutions we have

$$\Delta_{p,A}\, u = 0 \quad \text{in } G,$$

and

$$\begin{aligned}\Delta_{p,A}\Big(v + \frac{c+d}{2}\Big) &= \operatorname{div}\Big\{\Big\langle A(x)\nabla\Big(v+\frac{c+d}{2}\Big), \nabla\Big(v+\frac{c+d}{2}\Big)\Big\rangle^{\frac{p-2}{2}} A(x)\nabla\Big(v+\frac{c+d}{2}\Big)\Big\} \\ &= \operatorname{div}(\langle A(x)\nabla v, \nabla v\rangle^{\frac{p-2}{2}} A(x)\nabla v) = \Delta_{p,A}\, v = 0 \quad \text{in } G.\end{aligned}$$

Thus, u and $v + (c+d)/2$ are both weak solutions in G with the same boundary values. Then by Theorem 4.3 we find

$$u = v + \frac{c+d}{2} \quad \text{in } G.$$

This shows that the set G is empty, hence

$$u - v \le \frac{c+d}{2} \quad \text{in } \Omega,$$

which leads to the contradiction

$$\max_{\Omega}(u-v) \le \frac{c+d}{2} < \frac{c+c}{2} = \max_{\Omega}(u-v).$$

Thus we must have

$$\max_{\overline{\Omega}}(u-v) = \max_{\partial\Omega}(u-v).$$

□

Now we introduce weak supersolutions and weak subsolutions.

Definition 4.5. We say that $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a *weak supersolution* of the equation $\Delta_{p,A}\, u = 0$ in Ω, if

$$\int_{\Omega} \langle A(x)\nabla u, \nabla u\rangle^{\frac{p-2}{2}} \langle A(x)\nabla u, \nabla \eta\rangle\, dx \ge 0,$$

for all nonnegative $\eta \in C_0^\infty(\Omega)$. For *weak subsolutions* the inequality is reversed.

A weak solution is clearly both a weak supersolution and a weak subsolution. Our next result establishes a comparison principle when $n < p < \infty$.

Proposition 4.6 (Comparison principle)**.** *If u is a weak subsolution and v is a weak supersolution of $\Delta_{p,A}\, w = 0$, then $u \le v$ a.e. in Ω.*

Proof. By subtracting

$$\int_\Omega \langle A(x)\nabla v, \nabla v\rangle^{\frac{p-2}{2}} \langle A(x)\nabla v, \nabla \eta\rangle\, dx \ge 0$$

from

$$\int_\Omega \langle A(x)\nabla u, \nabla u\rangle^{\frac{p-2}{2}} \langle A(x)\nabla u, \nabla \eta\rangle\, dx \le 0,$$

and inserting the function

$$\eta = (u-v)^+ = \max\{u-v, 0\} \in W_0^{1,p}(\Omega)$$

we find

$$\int_\Omega \Big\langle \langle A(x)\nabla u, \nabla u\rangle^{\frac{p-2}{2}} A(x)\nabla u - \langle A(x)\nabla v, \nabla v\rangle^{\frac{p-2}{2}} A(x)\nabla v, \nabla (u-v)^+ \Big\rangle\, dx \le 0.$$

Then by Proposition 2.29,

$$4\Big(\frac{\sqrt{\alpha}}{2}\Big)^p \int_\Omega |\nabla (u-v)^+|^p dx \le 0,$$

which implies that $\nabla (u-v)^+ = 0$ a.e. in Ω, and consequently $u \le v$ a.e. in Ω. □

5 Limit of solutions as $p \to \infty$

In the previous section we established the existence and uniqueness of minimizers u_p of the variational integral

$$\int_\Omega \langle A(x)\nabla v, \nabla v\rangle^{p/2} dx.$$

These are weak solutions of the corresponding Euler-Lagrange equation

$$\Delta_{p,A}\, u_p = 0.$$

In this section we show that the uniform limit

$$\lim_{p_j \to \infty} u_{p_j} = u_\infty$$

exists for some subsequence p_j, and that u_∞ minimizes the norm

$$||\langle A\nabla v, \nabla v\rangle^{1/2}||_\infty$$

in some sense. Two important questions arise: how is u_∞ related to the equation

$$\Delta_{\infty,A}\, v = 0,$$

and is it unique, or does it depend on the particular choice of subsequence p_j? It turns out that we are not able to answer these questions using only variational techniques - we have to introduce some new tools. We postpone these issues and turn to the task at hand. Since the weak solution u_p of

$$\Delta_{p,A}\, u_p = 0$$

belongs to the space $W^{1,p}(\Omega)$, we expect that the limit

$$\lim_{p\to\infty} u_p$$

belongs to $W^{1,\infty}(\Omega)$. In Theorem 2.25 we found that locally the Sobolev space $W^{1,\infty}(\Omega)$ consists of Lipschitz continuous functions. Thus, we consider Lipschitz continuous boundary values, and we begin by extending these to the whole domain. Let $g : \partial\Omega \to \mathbb{R}$ be Lipschitz continuous:

$$|g(y_1) - g(y_2)| \leq L|y_1 - y_2|, \quad y_1, y_2 \in \partial\Omega,$$

and suppose that $w : \overline{\Omega} \to \mathbb{R}$ has boundary values g and is Lipschitz continuous with the same constant:

$$|w(x) - w(y)| \leq L|x - y|, \quad x, y \in \overline{\Omega}.$$

By some manipulations it follows that

$$\max_{y\in\partial\Omega}(g(y) - L|x-y|) \leq w(x) \leq \max_{y\in\partial\Omega}(g(y) + L|x-y|).$$

These two bounds are themselves Lipschitz continuous extensions of g as a function of x with constant L. Extending g by for instance the upper bound, we have that $g \in$

$C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$, where we use the same notation for the extension. Furthermore by Rademacher's theorem 2.26, g is differentiable a.e. in Ω, thus

$$||\nabla g||_{\infty,\Omega} \leq L.$$

We remind that Ω is a bounded domain in $\mathbb{R}^n$ and that $p > n$.

Theorem 5.1. *Let $g \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ be Lipschitz continuous. Then there exists a function $u_\infty \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ with boundary values g having the following minimizing property in each subdomain $D \subset \Omega$: if $v \in C(\overline{D}) \cap W^{1,\infty}(D)$ is such that $v = u_\infty$ on ∂D, then*

$$||\langle A\nabla u_\infty, \nabla u_\infty\rangle^{1/2}||_{\infty,D} \leq ||\langle A\nabla v, \nabla v\rangle^{1/2}||_{\infty,D}.$$

The function u_∞ can be obtained as the uniform limit

$$\lim_{p_j \to \infty} u_{p_j} = u_\infty \quad \text{in } \overline{\Omega},$$

where u_{p_j} is the weak solution of the equation

$$\Delta_{p_j,A}\, u_{p_j} = 0$$

with $u_{p_j}|_{\partial\Omega} = g$.

Proof. First we prove existence. We aim at using Ascoli's theorem 2.2. From Theorem 4.3 we know that there exists a unique minimizer $u_p \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ with boundary values g. Since u_p is a minimizer and g is Lipschitz continuous with constant L we have

$$\begin{aligned}
\int_\Omega \alpha^{p/2}|\nabla u_p|^p dx &\leq \int_\Omega \langle A\nabla u_p, \nabla u_p\rangle^{p/2} dx \\
&\leq \int_\Omega \langle A\nabla g, \nabla g\rangle^{p/2} dx \\
&\leq \int_\Omega \beta^{p/2}|\nabla g|^p dx \\
&\leq \beta^{p/2} L^p \mu(\Omega),
\end{aligned}$$

where μ is n-dimensional Lebesgue measure. Thus

$$||\nabla u_p||_p \leq \sqrt{\frac{\beta}{\alpha}} L\mu(\Omega)^{1/p}. \tag{5.1}$$

Furthermore, since $u_p - g \in W_0^{1,p}(\Omega)$ and $p > n$ we have by Morrey's inequality 2.19,

$$\begin{aligned}
|u_p(x) - u_p(y)| &\leq |g(x) - g(y)| + |(u_p(x) - g(x)) - (u_p(y) - g(y))| \\
&\leq L|x-y| + C_p|x-y|^{1-n/p}||\nabla(u_p - g)||_p \\
&\leq L|x-y| + C_p|x-y|^{1-n/p}(||\nabla u_p||_p + ||\nabla g||_p) \\
&\leq L|x-y| + C_p\Big(\sqrt{\frac{\beta}{\alpha}} L\mu(\Omega)^{1/p} + L\mu(\Omega)^{1/p}\Big)|x-y|^{1-n/p},
\end{aligned}$$

where C_p is such that $C_p \to 2^{n+1}$ as $p \to \infty$. By Friedrichs' inequality 2.21 and the above we obtain

$$\begin{aligned}||u_p||_p &\le ||u_p - g||_p + ||g||_p \\ &\le \operatorname{diam}(\Omega)||\nabla(u_p - g)||_p + ||g||_p \\ &\le \operatorname{diam}(\Omega)(||\nabla u_p||_p + ||\nabla g||_p) + ||g||_p \\ &\le \operatorname{diam}(\Omega)\left(\sqrt{\frac{\beta}{\alpha}}L\mu(\Omega)^{1/p} + L\mu(\Omega)^{1/p}\right) + ||g||_p.\end{aligned}$$

Thus, the sequence (u_p) is equicontinuous and equibounded for $p > n$. Then by Ascoli's theorem there is a subsequence (u_{p_j}) and a function $u_\infty \in C(\overline{\Omega})$ satisfying $u_\infty|_{\partial\Omega} = g$, such that

$$\lim_{p_j \to \infty} u_{p_j} = u_\infty \quad \text{uniformly in } \overline{\Omega}.$$

Now we show that $u_\infty \in W^{1,\infty}(\Omega)$. Let $p_j > s > n$. Proposition 2.4 and (5.1) yields

$$\left(\fint_\Omega |\nabla u_{p_j}|^s dx\right)^{1/s} \le \left(\fint_\Omega |\nabla u_{p_j}|^{p_j} dx\right)^{1/p_j} \le \sqrt{\frac{\beta}{\alpha}}L, \tag{5.2}$$

so the sequence (∇u_{p_j}) is uniformly bounded:

$$\sup_{p_j} ||\nabla u_{p_j}||_s \le \sqrt{\frac{\beta}{\alpha}}L\mu(\Omega)^{1/s}.$$

By Proposition 2.10 we conclude that

$$\nabla u_{p_j} \rightharpoonup \nabla u_\infty \quad \text{weakly in } L^s(\Omega),$$

for some subsequence, still denoted by the index p_j. By a diagonalization procedure we can extract a single subsequence such that

$$\nabla u_{p_j} \rightharpoonup \nabla u_\infty \quad \text{weakly in } L^s(\Omega) \text{ for all } s \in (n, \infty).$$

Furthermore, the weak lower semicontinuity

$$||\nabla u_\infty||_s \le \liminf_{p_j \to \infty} ||\nabla u_{p_j}||_s,$$

combined with (5.2) yields

$$\left(\fint_\Omega |\nabla u_\infty|^s dx\right)^{1/s} \le \sqrt{\frac{\beta}{\alpha}}L.$$

Since s was arbitrarily large, letting $s \to \infty$ we obtain

$$||\nabla u_\infty||_\infty \le \sqrt{\frac{\beta}{\alpha}}L, \tag{5.3}$$

thus $u_\infty \in W^{1,\infty}(\Omega)$. This concludes the existence.

Now we show the minimizing property. Let $D \subset \Omega$ and let v_{p_j} be the weak solution of

$$\Delta_{p_j,A}\, v_{p_j} = 0 \quad \text{in } D$$

with $v_{p_j} = u_\infty$ on ∂D. Then v_{p_j} is minimizing:

$$\int_D \langle A(x)\nabla v_{p_j}, \nabla v_{p_j}\rangle^{p_j/2} dx \le \int_D \langle A(x)\nabla v, \nabla v\rangle^{p_j/2} dx, \tag{5.4}$$

where $v \in C(\overline{D}) \cap W^{1,\infty}(D)$ is such that $v = u_\infty$ on ∂D. Now we show that $v_{p_j} \to u_\infty$ uniformly in $\overline{D}$. Notice that

$$||v_{p_j} - u_\infty||_{\infty,D} \le ||v_{p_j} - u_{p_j}||_{\infty,D} + ||u_{p_j} - u_\infty||_{\infty,D},$$

and recall that $u_{p_j} \to u_\infty$ uniformly in $\overline{D}$. Since v_{p_j} and u_{p_j} are weak solutions in D we find for the first term on the right-hand side that

$$\max_{\overline{D}}(v_{p_j} - u_{p_j}) = \max_{\partial D}(v_{p_j} - u_{p_j}) = \max_{\partial D}(u_\infty - u_{p_j}) \le ||u_\infty - u_{p_j}||_{\infty,\overline{D}},$$

by the Maximum principle 4.4. Doing the same for $u_{p_j} - v_{p_j}$ we obtain

$$\max_{\overline{D}} |v_{p_j} - u_{p_j}| \le ||u_\infty - u_{p_j}||_{\infty,\overline{D}},$$

and we conclude that $v_{p_j} \to u_\infty$ uniformly in $\overline{D}$. Let $p_j > s > n$. By Proposition 2.4 and (5.4) we have

$$\begin{aligned}
\left(⨍_D \alpha^{s/2}|\nabla v_{p_j}|^s dx\right)^{1/s} &\le \left(⨍_D \langle A(x)\nabla v_{p_j}, \nabla v_{p_j}\rangle^{s/2} dx\right)^{1/s}\\
&\le \left(⨍_D \langle A(x)\nabla v_{p_j}, \nabla v_{p_j}\rangle^{p_j/2} dx\right)^{1/p_j}\\
&\le \left(⨍_D \langle A(x)\nabla v, \nabla v\rangle^{p_j/2} dx\right)^{1/p_j}\\
&\le ||\langle A\nabla v, \nabla v\rangle^{1/2}||_{\infty,D},
\end{aligned}$$

hence the sequence (∇v_{p_j}) is uniformly bounded. By Proposition 2.10 there is a subsequence such that $\nabla v_{p_j} \rightharpoonup \nabla u_\infty$ weakly in $L^s(\Omega)$, and we can extract a single subsequence so that $\nabla v_{p_j} \rightharpoonup \nabla u_\infty$ weakly in $L^s(\Omega)$ for all $s \in (n, \infty)$. By Proposition 2.32 we find

$$\begin{aligned}
\left(⨍_D \langle A(x)\nabla u_\infty, \nabla u_\infty\rangle^{s/2} dx\right)^{1/s} &\le \liminf_{p_j \to \infty} \left(⨍_D \langle A(x)\nabla v_{p_j}, \nabla v_{p_j}\rangle^{s/2} dx\right)^{1/s}\\
&\le ||\langle A\nabla v, \nabla v\rangle^{1/2}||_{\infty,D},
\end{aligned}$$

and the conclusion

$$||\langle A\nabla u_\infty, \nabla u_\infty\rangle^{1/2}||_{\infty,D} \le ||\langle A\nabla v, \nabla v\rangle^{1/2}||_{\infty,D}$$

follows by letting $s \to \infty$. □

6 Viscosity solutions

To motivate the following concept we begin with a result by Aronsson on the infinity-Laplace equation

$$\Delta_\infty u = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0,$$

which in our setting corresponds to the situation

$$A = (a_{ij}) = (\delta_{ij}/2) = \frac{1}{2}I, \quad \Delta_{\infty,\frac{1}{2}I}\, u = \Delta_\infty u,$$

where δ_{ij} is the Kronecker delta and I is the identity matrix.

Second derivatives are present in the equation, thus a reasonable candidate for a class of solutions is C^2. However, Aronsson showed in [2] that for $u \in C^2(\Omega)$ satisfying $\Delta_\infty u = 0$ in Ω, either $\nabla u \neq 0$ or u is constant in Ω. Hence, one can not have smoothness at the critical points. This leads to the problem that the second derivatives do not always exist when they are needed. Furthermore, it seems to be impossible to derive a weak formulation involving only the first derivatives. To see this, multiply the equation with a test function η and integrate.

We mention that one can establish uniqueness of classical solutions of the infinity-Laplace equation without critical points. In the general case a construction by Jensen [12] is needed, where auxiliary equations are introduced to avoid the critical points. The absence of second derivatives requires a "doubling of variables" argument. In this procedure viscosity solutions is the appropriate setting. For the equation

$$\begin{aligned}\Delta_{\infty,A}\, u &= \Big\langle \nabla_x \langle A(x)\nabla u, \nabla u\rangle + 2(D^2 u)A(x)\nabla u, A(x)\nabla u \Big\rangle \\ &= \sum_{i,j,k,l=1}^{n} \Big(\frac{\partial a_{ij}}{\partial x_k}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} a_{kl}\frac{\partial u}{\partial x_l} + 2a_{ik}\frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_i \partial x_j} a_{jl}\frac{\partial u}{\partial x_l}\Big).\end{aligned}$$

we shall follow this procedure.

We refer to [18] for a thorough discussion of the above.

Definition 6.1. Let $n < p \leq \infty$. We say that $u \in C(\Omega)$ is a *viscosity supersolution* of the equation $\Delta_{p,A}\, u = 0$ in Ω, if

$$\Delta_{p,A}\, \phi(x_0) \leq 0$$

whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $\phi(x_0) = u(x_0)$ and $\phi(x) < u(x)$ for $x \in \Omega \setminus \{x_0\}$, that is, ϕ touches u from below at x_0.

We say that $v \in C(\Omega)$ is a *viscosity subsolution* of the equation $\Delta_{p,A}\, v = 0$ in Ω, if

$$\Delta_{p,A}\, \psi(x_0) \geq 0$$

whenever $x_0 \in \Omega$ and $\psi \in C^2(\Omega)$ are such that $\psi(x_0) = v(x_0)$ and $\psi(x) > v(x)$ for $x \in \Omega \setminus \{x_0\}$, that is, ψ touches v from above at x_0.

We call $w \in C(\Omega)$ a *viscosity solution* of the equation $\Delta_{p,A}\, w = 0$ in Ω, if it is both a viscosity supersolution and a viscosity subsolution.

Instead of the condition that ϕ touches u from below at x_0, we note that an equivalent definition for viscosity supersolutions is to require that $u-\phi$ has a strict minimum at x_0. Similary for viscosity subsolutions, we can replace the touching from above condition by requiring that $v-\psi$ has a strict maximum at x_0. We also note that for each point in Ω, there is a corresponding family of test functions satisfying the requirements in the definition above. This family may be empty, and in that case there is no requirement.

Let us verify that for a function in $C^2(\Omega)$, the concepts of classical solutions and viscosity solutions agree.

Proposition 6.2. *Let $n<p\leq\infty$. A function $u\in C^2(\Omega)$ is a viscosity solution of $\Delta_{p,A}\, u = 0$ if and only if $\Delta_{p,A}\, u(x)=0$ holds pointwise in Ω.*

Proof. We show this for subsolutions. The proof in the case of supersolutions is similar. We remind that for a classical subsolution $u\in C^2(\Omega)$,

$$\Delta_{p,A}\, u(x)\geq 0 \quad \text{for all } x\in\Omega.$$

First assume that $u\in C^2(\Omega)$ is a viscosity subsolution. Consider the function

$$\psi(x)=u(x)+|x-x_0|^4,$$

where $x_0\in\Omega$. Then $\psi\in C^2(\Omega)$ satisfies

$$\psi(x_0)=u(x_0), \quad \psi(x)>u(x) \quad \text{when } x\neq x_0,$$

so ψ is a test function as in the definition, thus

$$0\leq\Delta_{p,A}\,\psi(x_0)=\Delta_{p,A}\, u(x_0).$$

Now suppose that $u\in C^2(\Omega)$ is a classical subsolution. Let $\psi\in C^2(\Omega)$ be such that $u-\psi$ has a strict maximum at some point x_0. Then

$$\begin{aligned}\nabla\psi(x_0)&=\nabla u(x_0),\\ D^2\psi(x_0)&\geq D^2u(x_0)\end{aligned}$$

by the infinitesimal calculus. Using the alternative notation

$$\begin{aligned}\Delta_{p,A}\,\psi(x_0)&=\mathcal{L}_{p,A}(x_0,\nabla\psi(x_0),D^2\psi(x_0)),\\ \Delta_{p,A}\, u(x_0)&=\mathcal{L}_{p,A}(x_0,\nabla u(x_0),D^2u(x_0))\end{aligned}$$

we find by Proposition 3.3,

$$\begin{aligned}\Delta_{p,A}\,\psi(x_0)&=\mathcal{L}_{p,A}(x_0,\nabla\psi(x_0),D^2\psi(x_0))\\ &=\mathcal{L}_{p,A}(x_0,\nabla u(x_0),D^2\psi(x_0))\\ &\geq\mathcal{L}_{p,A}(x_0,\nabla u(x_0),D^2u(x_0))\\ &=\Delta_{p,A}\, u(x_0)\geq 0,\end{aligned}$$

which shows that u is a viscosity subsolution. □

Now we show that weak solutions are viscosity solutions.

Lemma 6.3. *Let $n < p < \infty$. If u is a weak solution of $\Delta_{p,A}\, u = 0$, then u is also a viscosity solution.*

Proof. We show the result for supersolutions. Let u be a weak supersolution and assume by contradiction that u is not a viscosity supersolution. Then at some point $x_0 \in \Omega$ there is a $\phi \in C^2(\Omega)$ touching u from below at x_0 such that

$$\Delta_{p,A}\, \phi(x_0) > 0.$$

By continuity, there is some $\delta > 0$ so that $\Delta_{p,A}\, \phi(x) > 0$ when $|x - x_0| < 2\delta$. Thus, ϕ is a classical subsolution in the ball $B(x_0, 2\delta)$. Now define the function

$$\psi(x) = \phi(x) + \frac{1}{2} \min_{\partial B(x_0,\delta)} (u - \phi).$$

Observe that $\psi < u$ on $\partial B(x_0,\delta)$ and $\psi(x_0) > u(x_0)$. Consider the open set

$$D_\delta = \{\psi > u\} \cap B(x_0,\delta),$$

and note that $\psi = u$ on ∂D_δ. Since u is a weak supersolution and ψ is a weak subsolution in D_δ, we have by continuity and the Comparison principle 4.6 that $\psi \leq u$ everywhere in D_δ. This leads to the contradiction $\psi(x_0) \leq u(x_0) < \psi(x_0)$, which concludes the proof. □

6.1 Equivalent definition

An equivalent definition of viscosity solutions can be formulated in terms of *jets*.

Definition 6.4. Let $u : \Omega \to \mathbb{R}$. We define the *superjet* $J^{2,+}u(x)$ at the point $x \in \Omega$ as the set of all $(\xi, \mathbb{X}) \in \mathbb{R}^n \times S^n$ satisfying

$$u(y) \leq u(x) + \langle \xi, y - x\rangle + \frac{1}{2}\langle \mathbb{X}(y-x), y-x\rangle + o(|y-x|^2) \quad \text{as } \Omega \ni y \to x.$$

We define the *subjet* $J^{2,-}u(x)$ at the point $x \in \Omega$ as the set of all $(\xi, \mathbb{X}) \in \mathbb{R}^n \times S^n$ satisfying

$$u(y) \geq u(x) + \langle \xi, y - x\rangle + \frac{1}{2}\langle \mathbb{X}(y-x), y-x\rangle + o(|y-x|^2) \quad \text{as } \Omega \ni y \to x.$$

We mention that if

$$J^{2,+}u(x) \cap J^{2,-}u(x) \neq \emptyset,$$

then $\nabla u(x)$ and $D^2u(x)$ exist and

$$J^{2,+}u(x) \cap J^{2,-}u(x) = \{(\nabla u(x), D^2u(x))\}.$$

The following shows that even if the jet is empty at a given point, there are always nearby points at which the jet is nonempty.

Proposition 6.5. *If $u \in C(\Omega)$ and $x \in \Omega$, there are points $x_k \in \Omega$ such that*

$$x_k \to x \quad \text{as } k \to \infty, \quad J^{2,+}u(x_k) \neq \emptyset \quad \text{when } k \in \mathbb{N}.$$

The same holds true for subjets.

Proof. Fix $x \in \Omega$ and set $r > 0$ such that $B(x,r) \subset\subset \Omega$, where $B(x,r)$ denotes the open ball centered at x with radius r. Let $k \in \mathbb{N}$. Since u is continuous there is a point $x_k \in \overline{B(x,r)}$ such that

$$\max_{y \in \overline{B(x,r)}} (u(y) - k|y-x|^2) = u(x_k) - k|x_k - x|^2.$$

By setting $y = x$ in the inequality

$$u(y) - k|y-x|^2 \leq u(x_k) - k|x_k - x|^2 \quad \text{for all } y \in \overline{B(x,r)},$$

we obtain

$$|x_k - x|^2 \leq \frac{1}{k}(u(x_k) - u(x)).$$

Thus

$$x_k \to x \quad \text{as } k \to \infty.$$

Furthermore

$$\begin{aligned} u(y) &\leq u(x_k) + k(|y-x|^2 - |x_k - x|^2) \\ &= u(x_k) + \langle 2k(x_k - x), y - x_k \rangle + \frac{1}{2}\langle 2kI(y - x_k), y - x_k \rangle, \end{aligned}$$

hence

$$(2k(x_k - x), 2kI) \in J^{2,+}u(x_k)$$

for all $k \in \mathbb{N}$. □

Definition 6.6. The *closure of the superjet* $\overline{J^{2,+}}u(x)$ is the set of all $(\xi, \mathbb{X}) \in \mathbb{R}^n \times S^n$ such that there exists a sequence

$$(x_k, u(x_k), \xi_k, \mathbb{X}_k) \to (x, u(x), \xi, \mathbb{X}) \quad \text{as } k \to \infty,$$

where $(\xi_k, \mathbb{X}_k) \in J^{2,+}u(x_k)$. The *closure of the subjet* $\overline{J^{2,-}}u(x)$ is defined in a similar way.

We remind that

$$\mathcal{L}_{\infty,A}(x,\xi,\mathbb{X}) = \Big\langle \nabla_x \langle A(x)\xi, \xi \rangle + 2\mathbb{X}A(x)\xi, A(x)\xi \Big\rangle$$

and

$$\begin{aligned} \mathcal{L}_{p,A}(x,\xi,\mathbb{X}) = \langle A(x)\xi,\xi\rangle^{\frac{p-4}{2}} \Big\{ \langle A(x)\xi,\xi\rangle \Big(\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_i}(x)\xi_j + \operatorname{trace}(A(x)\mathbb{X}) \Big) \\ + \Big(\frac{p-2}{2}\Big) \mathcal{L}_{\infty,A}(x,\xi,\mathbb{X}) \Big\}, \quad 4 \leq p < \infty, \end{aligned}$$

where $x \in \Omega$, $\xi \in \mathbb{R}^n$ and $\mathbb{X} \in S^n$.

Proposition 6.7. *Let $n < p \leq \infty$. For $u \in C(\Omega)$ the following are equivalent:*

i) u is a viscosity subsolution.

ii) If $x \in \Omega$ and $(\xi, \mathbb{X}) \in J^{2,+}u(x)$, then $\mathcal{L}_{p,A}(x, \xi, \mathbb{X}) \geq 0$.

iii) If $x \in \Omega$ and $(\xi, \mathbb{X}) \in \overline{J^{2,+}}u(x)$, then $\mathcal{L}_{p,A}(x, \xi, \mathbb{X}) \geq 0$.

The same holds true for viscosity supersolutions, where $J^{2,+}u(x)$ is replaced by $J^{2,-}u(x)$ and the inequalites are reversed.

We refer to Proposition 2.6 in [14] for a proof. Now we state two results which will be play a key role in "doubling of variables" arguments in Section 8 and 9.

Lemma 6.8. *Let $u, v \in C(\overline{\Omega})$ and suppose that there exists a point $(x_j, y_j) \in \overline{\Omega} \times \overline{\Omega}$ for which the supremum*

$$M_j = \sup_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} \left(u(x) - v(y) - \frac{j}{2}|x - y|^2 \right)$$

is attained. Then

i) $\lim_{j \to \infty} j|x_j - y_j|^2 = 0$.

ii) $\lim_{j \to \infty} M_j = u(\hat{x}) - v(\hat{x}) = \sup_{x \in \overline{\Omega}} (u(x) - v(x))$ *whenever $\hat{x}$ is a limit point of x_j as $j \to \infty$.*

Proof. Suppose that

$$x_j \to \hat{x}, \quad y_j \to \hat{y} \quad \text{as } j \to \infty.$$

Since

$$u(x) - v(y) - \frac{j}{2}|x - y|^2 \leq u(x_j) - v(y_j) - \frac{j}{2}|x_j - y_j|^2 \quad \text{for all } (x, y) \in \overline{\Omega} \times \overline{\Omega},$$

we find by setting $x = y = y_j$ that

$$|x_j - y_j|^2 \leq \frac{2}{j}(u(x_j) - u(y_j)).$$

The right-hand side has limit equal to zero as $j \to \infty$, thus $\hat{x} = \hat{y}$. Now we find

$$\lim_{j \to \infty} (u(x_j) - u(y_j)) = u(\hat{x}) - u(\hat{x}) = 0,$$

hence

$$\lim_{j \to \infty} j|x_j - y_j|^2 = 0.$$

This shows *i)*, and *ii)* follows directly. □

The main result is a special case of what is often refered to as Ishii's lemma or the Theorem of sums.

Theorem 6.9 (Ishii's lemma)**.** *Let $u, v \in C(\Omega)$ and suppose that there exists an interior point $(x_j, y_j) \in \Omega \times \Omega$ for which the maximum*

$$\max_{(x,y)\in\Omega\times\Omega} \left(u(x) - v(y) - \frac{j}{2}|x-y|^2 \right)$$

is attained. Then there exist symmetric matrices $\mathbb{X}_j, \mathbb{Y}_j \in S^n$ such that

$$(j(x_j - y_j), \mathbb{X}_j) \in \overline{J^{2,+}}u(x_j),$$

$$(j(x_j - y_j), \mathbb{Y}_j) \in \overline{J^{2,-}}v(y_j),$$

and

$$-3j \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} \mathbb{X}_j & 0 \\ 0 & -\mathbb{Y}_j \end{pmatrix} \leq 3j \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{6.1}$$

We refer to the standard text on viscosity solutions [4] for a proof of Ishii's lemma. In particular we have from (6.1) that

$$\begin{aligned} \langle \mathbb{X}_j \xi, \xi \rangle - \langle \mathbb{Y}_j \eta, \eta \rangle &= \left\langle \begin{pmatrix} \mathbb{X}_j & 0 \\ 0 & -\mathbb{Y}_j \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle \\ &\leq 3j \left\langle \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle \\ &= 3j|\xi - \eta|^2, \end{aligned} \tag{6.2}$$

for all $\xi, \eta \in \mathbb{R}^n$, so by choosing $\xi = \eta$ we find that $\mathbb{X}_j \leq \mathbb{Y}_j$.

The proofs of Proposition 6.2 and Lemma 6.3 are based on Section 4 in [18], and the proof of Proposition 6.5 is based on Proposition 2.5 in [14].

7 Auxiliary equations

The uniqueness of viscosity solutions of the equation $\Delta_{\infty,A}\, u = 0$ follows from a comparison principle originally due to Jensen [12], who proved it for the infinity-Laplace equation. Juutinen [13] later established the result for more general problems. The idea is to introduce two auxiliary equations with parameter $\varepsilon > 0$:

$$\begin{aligned} \max\Big\{\varepsilon - \langle A(x)\nabla u^+, \nabla u^+\rangle^{1/2},\ \Delta_{\infty,A}\, u^+\Big\} &= 0 \qquad \text{Upper equation}\\ \Delta_{\infty,A}\, u &= 0 \qquad \text{Equation}\\ \min\Big\{\langle A(x)\nabla u^-, \nabla u^-\rangle^{1/2} - \varepsilon,\ \Delta_{\infty,A}\, u^-\Big\} &= 0 \qquad \text{Lower equation.} \end{aligned}$$

The comparison principle states that the functions are ordered: $u^- \le u \le u^+$ when they have the same boundary values, where u^+ is a viscosity supersolution of the Upper equation and u^- is a viscosity subsolution of the Lower equation. This is the content of Section 8. The virtue of the result comes from the fact that the difference $u^+ - u^-$ can be made arbitrarily small, which we show in this section. Furthermore, we show that the constructed limit u_∞ of weak solutions u_{p_j} of $\Delta_{p_j,A}\, u_{p_j} = 0$ is a viscosity solution of $\Delta_{\infty,A}\, u_\infty = 0$.

7.1 Variational problem

We shall see that the functions u^+, u^- can be constructed as uniform limits

$$u_p^+ \to u^+, \quad u_p^- \to u^- \quad \text{in } \overline{\Omega},$$

where u_p^+, u_p^- are minimizers of the variational integrals

$$\begin{aligned} J^+(u_p^+) &= \int_\Omega \Big(\frac{2}{p}\langle A(x)\nabla u_p^+, \nabla u_p^+\rangle^{p/2} - \varepsilon^{p-1} u_p^+\Big)dx,\\ J^-(u_p^-) &= \int_\Omega \Big(\frac{2}{p}\langle A(x)\nabla u_p^-, \nabla u_p^-\rangle^{p/2} + \varepsilon^{p-1} u_p^-\Big)dx. \end{aligned}$$

Now the Euler-Lagrange equations are

$$\Delta_{p,A}\, u_p^+ = -\varepsilon^{p-1}, \quad \Delta_{p,A}\, u_p^- = +\varepsilon^{p-1},$$

which in weak form reads

$$\begin{aligned} \int_\Omega \langle A(x)\nabla u_p^+, \nabla u_p^+\rangle^{\frac{p-2}{2}} \langle A(x)\nabla u_p^+, \nabla\eta\rangle\, dx &= +\varepsilon^{p-1}\int_\Omega \eta\, dx,\\ \int_\Omega \langle A(x)\nabla u_p^-, \nabla u_p^-\rangle^{\frac{p-2}{2}} \langle A(x)\nabla u_p^-, \nabla\eta\rangle\, dx &= -\varepsilon^{p-1}\int_\Omega \eta\, dx, \end{aligned}$$

for all test functions $\eta \in C_0^\infty(\Omega)$. As before we define functions that belong to $W^{1,p}_{\mathrm{loc}}(\Omega)$ and satisfy this for all test functions to be weak solutions of the Euler-Lagrange equations. By an argument similar to the one in the proof of Proposition 4.2, we once more find that weak solutions and minimizers are equivalent. We begin with a familiar procedure.

Lemma 7.1. *Let $n < p < \infty$ and assume that $g \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$. Then there exist unique weak solutions $u, w \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ of the equations*

$$\Delta_{p,A}\, u = -\varepsilon^{p-1}, \quad \Delta_{p,A}\, w = +\varepsilon^{p-1}$$

with $u|_{\partial\Omega} = w|_{\partial\Omega} = g$.

Proof. The proofs are similar for both equations, and we only show the existence of a unique minimizer of the variational integral

$$J^+(v) = \int_\Omega \Big(\frac{2}{p}\langle A(x)\nabla v, \nabla v\rangle^{p/2} - \varepsilon^{p-1} v\Big)\, dx.$$

As before we use the Direct method in the calculus of variations. Let

$$J_0^+ = \inf_v J^+(v),$$

where the infimum is taken over all $v \in W^{1,p}(\Omega)$ with boundary values $g \in C(\overline{\Omega})$. Since g is admissible we have

$$\begin{aligned} J_0^+ &\le J^+(g) \\ &= \int_\Omega \Big(\frac{2}{p}\langle A(x)\nabla g, \nabla g\rangle^{p/2} - \varepsilon^{p-1} g\Big)\, dx \\ &\le \int_\Omega \Big(\frac{2}{p}\beta^{p/2}|\nabla g|^p - \varepsilon^{p-1} g\Big)\, dx, \end{aligned}$$

thus

$$|J_0^+| \le \frac{2}{p}\beta^{p/2}\int_\Omega |\nabla g|^p dx + \varepsilon^{p-1}\int_\Omega |g| dx =: M < \infty,$$

so there is a sequence (u_j) of admissible functions such that

$$\lim_{j\to\infty} J^+(u_j) = J_0^+,$$

and we can assume that $J^+(u_j) < J_0^+ + 1$ for all $j \in \mathbb{N}$. Now we show that the sequences (u_j) and (∇u_j) are uniformly bounded. Since

$$\begin{aligned} \int_\Omega \Big(\frac{2}{p}\alpha^{p/2}|\nabla u_j|^p - \varepsilon^{p-1} u_j\Big) dx &\le \int_\Omega \Big(\frac{2}{p}\langle A(x)\nabla u_j, \nabla u_j\rangle^{p/2} - \varepsilon^{p-1} u_j\Big) dx \\ &= J^+(u_j) < J_0^+ + 1, \end{aligned}$$

it follows that

$$\int_\Omega |\nabla u_j|^p dx \le \frac{p}{2}\alpha^{-p/2}\Big\{(J_0^+ + 1) + \varepsilon^{p-1}\int_\Omega |u_j|\, dx\Big\}. \tag{7.1}$$

Let $\lambda > 0$. For the last term on the right-hand side we have the estimate

$$\begin{aligned}\int_\Omega |u_j|\,dx &\le \int_\Omega \lambda^{-1}|u_j-g|\lambda\,dx + \int_\Omega |g|\,dx\\ &\le \frac{\lambda^{-p}}{p}\int_\Omega |u_j-g|^p dx + \frac{\lambda^q}{q}\mu(\Omega) + \int_\Omega |g|\,dx\\ &\le \frac{\lambda^{-p}}{p}\mathrm{diam}(\Omega)^p\int_\Omega |\nabla u_j - \nabla g|^p dx + \frac{\lambda^q}{q}\mu(\Omega) + \int_\Omega |g|\,dx\\ &\le 2^{p-1}\frac{\lambda^{-p}}{p}\mathrm{diam}(\Omega)^p\Big\{\int_\Omega |\nabla u_j|^p dx + \int_\Omega |\nabla g|^p dx\Big\}\\ &\quad + \frac{\lambda^q}{q}\mu(\Omega) + \int_\Omega |g|\,dx,\end{aligned}$$

where $1/p+1/q=1$. We used Young's inequality and Friedrichs' inequality 2.21. If we insert this into (7.1) we find

$$\begin{aligned}\int_\Omega |\nabla u_j|^p dx \le{}& \alpha^{-p/2}\varepsilon^{p-1}2^{p-2}\lambda^{-p}\mathrm{diam}(\Omega)^p\Big\{\int_\Omega |\nabla u_j|^p dx + \int_\Omega |\nabla g|^p dx\Big\}\\ &+ \frac{p}{2}\alpha^{-p/2}\Big\{(J_0^+ + 1) + \varepsilon^{p-1}\frac{\lambda^q}{q}\mu(\Omega) + \varepsilon^{p-1}\int_\Omega |g|\,dx\Big\}.\end{aligned} \tag{7.2}$$

By setting λ such that

$$\alpha^{-p/2}\varepsilon^{p-1}2^{p-2}\lambda^{-p}\mathrm{diam}(\Omega)^p = \frac{1}{2}$$

we obtain

$$\frac{1}{2}\int_\Omega |\nabla u_j|^p dx \le \frac{1}{2}\int_\Omega |\nabla g|^p dx + \frac{p}{2}\alpha^{-p/2}\Big\{(J_0^+ + 1) + \varepsilon^{p-1}\frac{\lambda^q}{q}\mu(\Omega) + \varepsilon^{p-1}\int_\Omega |g|\,dx\Big\},$$

hence

$$\int_\Omega |\nabla u_j|^p dx \le \int_\Omega |\nabla g|^p dx + p\alpha^{-p/2}\Big\{(J_0^+ + 1) + \varepsilon^{p-1}\frac{\lambda^q}{q}\mu(\Omega) + \varepsilon^{p-1}\int_\Omega |g|\,dx\Big\} =: N_1 < \infty.$$

Since this holds for all j we conclude that

$$\sup_j ||\nabla u_j||_p \le N_1^{1/p} < \infty,$$

which shows that the sequence (∇u_j) is uniformly bounded. It remains to show that (u_j) is uniformly bounded. By the above we find

$$\begin{aligned}||u_j||_p &\le ||u_j - g||_p + ||g||_p\\ &\le \mathrm{diam}(\Omega)||\nabla(u_j - g)||_p + ||g||_p\\ &\le \mathrm{diam}(\Omega)(||\nabla u_j||_p + ||\nabla g||_p) + ||g||_p\\ &\le \mathrm{diam}(\Omega)(N_1^{1/p} + ||\nabla g||_p) + ||g||_p\\ &=: N_2 < \infty \quad \text{for all } j,\end{aligned}$$

where we once more used Friedrichs' inequality 2.21. It follows that the sequence (u_j) is uniformly bounded:

$$\sup_j ||u_j||_p \le N_2 < \infty.$$

Then by Proposition 2.10 there is a subsequence (u_{j_k}) and a function $u \in W^{1,p}(\Omega)$ such that

$$u_{j_k} \rightharpoonup u, \quad \nabla u_{j_k} \rightharpoonup \nabla u \quad \text{weakly in } L^p(\Omega).$$

Since $u - g \in W_0^{1,p}(\Omega)$ and $p > n$ we have by Morrey's inequality 2.19 that $u \in C(\overline{\Omega})$. By the weak lower semicontinuity of the integral in Proposition 2.32,

$$\int_\Omega \langle A(x)\nabla u, \nabla u\rangle^{p/2} dx \le \liminf_{k\to\infty} \int_\Omega \langle A(x)\nabla u_{j_k}, \nabla u_{j_k}\rangle^{p/2} dx.$$

Furthermore, since the domain Ω is bounded, the constant function 1 belongs to $L^q(\Omega)$. Thus by (2.3) we find

$$\int_\Omega u \cdot 1 \; dx = \lim_{k\to\infty} \int_\Omega u_{j_k} \cdot 1 \; dx.$$

From this we obtain

$$\begin{aligned}
J^+(u) &= \int_\Omega \Big(\frac{2}{p}\langle A(x)\nabla u, \nabla u\rangle^{p/2} - \varepsilon^{p-1} u\Big) dx \\
&= \int_\Omega \frac{2}{p}\langle A(x)\nabla u, \nabla u\rangle^{p/2} dx - \varepsilon^{p-1} \lim_{k\to\infty} \int_\Omega u_{j_k} dx \\
&\le \liminf_{k\to\infty} \int_\Omega \frac{2}{p}\langle A(x)\nabla u_{j_k}, \nabla u_{j_k}\rangle^{p/2} dx - \varepsilon^{p-1} \lim_{k\to\infty} \int_\Omega u_{j_k} dx \\
&= \liminf_{k\to\infty} \int_\Omega \Big(\frac{2}{p}\langle A(x)\nabla u_{j_k}, \nabla u_{j_k}\rangle^{p/2} - \varepsilon^{p-1} u_{j_k}\Big) dx \\
&= \liminf_{k\to\infty} J^+(u_{j_k}) = J_0^+.
\end{aligned}$$

On the other hand we also have $J^+(u) \ge J_0^+$ since u is admissible. Thus $J^+(u) = J_0^+$, and we conclude that there exists a minimizer $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$.

To establish uniqueness we proceed as in the proof of Theorem 4.3. Assume by contradiction that there are two minimizers u_1 and u_2. Then $(u_1 + u_2)/2$ is admissible. By the strict convexity in Lemma 2.31 we have

$$\begin{aligned}
&\Big\langle A(x)\Big(\frac{\nabla u_1 + \nabla u_2}{2}\Big), \frac{\nabla u_1 + \nabla u_2}{2}\Big\rangle^{p/2} \\
&< \frac{1}{2}\langle A(x)\nabla u_1, \nabla u_1\rangle^{p/2} + \frac{1}{2}\langle A(x)\nabla u_2, \nabla u_2\rangle^{p/2} \quad \text{when } \nabla u_1 \ne \nabla u_2.
\end{aligned}$$

Hence

$$\begin{aligned}
J^+(u_1) &\le J^+\Big(\frac{u_1 + u_2}{2}\Big) \\
&= \int_\Omega \Big\{\frac{2}{p}\Big\langle A(x)\Big(\frac{\nabla u_1 + \nabla u_2}{2}\Big), \frac{\nabla u_1 + \nabla u_2}{2}\Big\rangle^{p/2} - \varepsilon^{p-1}\Big(\frac{u_1 + u_2}{2}\Big)\Big\} dx \\
&< \int_\Omega \Big\{\frac{2}{p}\Big(\frac{1}{2}\langle A(x)\nabla u_1, \nabla u_1\rangle^{p/2} + \frac{1}{2}\langle A(x)\nabla u_2, \nabla u_2\rangle^{p/2}\Big) - \frac{\varepsilon^{p-1}}{2} u_1 - \frac{\varepsilon^{p-1}}{2} u_2\Big\} dx \\
&= \frac{1}{2} J^+(u_1) + \frac{1}{2} J^+(u_2) = J^+(u_1).
\end{aligned}$$

To avoid the contradiction we must have $\nabla u_1 = \nabla u_2$ a.e. in Ω. Thus $u_1 = u_2$, and we conclude that the minimizer is unique. □

By an argument as in Lemma 6.3 we find that weak solutions of the Euler-Lagrange equations

$$\Delta_{p,A}\, u = -\varepsilon^{p-1}, \quad \Delta_{p,A}\, w = +\varepsilon^{p-1}$$

also are viscosity solutions.

Lemma 7.2. *Let $n < p < \infty$. If u is a weak solution of $\Delta_{p,A}\, u = -\varepsilon^{p-1}$, then u is also a viscosity solution, meaning that*

$$\Delta_{p,A}\, \phi(x_0) \leq -\varepsilon^{p-1}$$

holds for a function $\phi \in C^2(\Omega)$ touching u from below at x_0, and that

$$\Delta_{p,A}\, \psi(x_0) \geq -\varepsilon^{p-1}$$

holds for a function $\psi \in C^2(\Omega)$ touching u from above at x_0. The same holds true for $\Delta_{p,A}\, w = +\varepsilon^{p-1}$ with $-\varepsilon^{p-1}$ replaced by $+\varepsilon^{p-1}$ in the above.

7.2 Limit procedure

Now we show that the limit of the weak solutions constructed in Lemma 7.1 exists.

Lemma 7.3. *Assume that $g \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ is Lipschitz continuous. Then there exist functions $u^+, u^- \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ with $u^+|_{\partial\Omega} = u^-|_{\partial\Omega} = g$, which are the uniform limits*

$$\lim_{p_j\to\infty} u_{p_j} = u^+, \quad \lim_{p_j\to\infty} w_{p_j} = u^- \quad \text{in } \overline{\Omega},$$

where u_{p_j} and w_{p_j} are weak solutions of the equations

$$\Delta_{p_j,A}\, u_{p_j} = -\varepsilon^{p_j-1}, \qquad \Delta_{p_j,A}\, w_{p_j} = +\varepsilon^{p_j-1}$$

with $u_{p_j}|_{\partial\Omega} = w_{p_j}|_{\partial\Omega} = g$.

Proof. We only show the existence of the uniform limit

$$\lim_{p_j\to\infty} u_{p_j} = u^+.$$

Let $u_p \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ be the unique minimizer of the variational integral

$$J^+(u_p) = \int_\Omega \Big(\frac{2}{p}\langle A(x)\nabla u_p, \nabla u_p\rangle^{p/2} - \varepsilon^{p-1} u_p\Big)\,dx.$$

As before we aim at using Ascoli's theorem 2.2. First we show that the sequence (u_p) is equibounded. Since u_p is minimizing and g is admissible we have

$$\begin{aligned}
\int_\Omega \Big(\frac{2}{p}\alpha^{p/2}|\nabla u_p|^p - \varepsilon^{p-1}u_p\Big)dx &\leq \int_\Omega \Big(\frac{2}{p}\langle A(x)\nabla u_p, \nabla u_p\rangle^{p/2} - \varepsilon^{p-1}u_p\Big)dx \\
&\leq \int_\Omega \Big(\frac{2}{p}\langle A(x)\nabla g, \nabla g\rangle^{p/2} - \varepsilon^{p-1}g\Big)dx \\
&\leq \int_\Omega \Big(\frac{2}{p}\beta^{p/2}|\nabla g|^p - \varepsilon^{p-1}g\Big)dx.
\end{aligned}$$

By rearranging we find

$$\int_\Omega |\nabla u_p|^p dx \le \alpha^{-p/2}\beta^{p/2}\int_\Omega |\nabla g|^p dx + \frac{p}{2}\alpha^{-p/2}\varepsilon^{p-1}\int_\Omega (u_p - g)\,dx. \tag{7.3}$$

Let $\lambda > 0$. For the last term on the right-hand side the estimate

$$\begin{aligned}\left|\int_\Omega (u_p-g)\,dx\right| &\le \int_\Omega \lambda|u_p-g|\lambda^{-1}dx\\ &= \frac{\lambda^p}{p}\int_\Omega |u_p-g|^p dx + \frac{\lambda^{-q}}{q}\mu(\Omega)\\ &\le \frac{\lambda^p}{p}\mathrm{diam}(\Omega)^p\int_\Omega |\nabla u_p - \nabla g|^p dx + \frac{\lambda^{-q}}{q}\mu(\Omega)\\ &\le 2^{p-1}\frac{\lambda^p}{p}\mathrm{diam}(\Omega)^p\Big\{\int_\Omega |\nabla u_p|^p dx + \int_\Omega |\nabla g|^p dx\Big\} + \frac{\lambda^{-q}}{q}\mu(\Omega)\end{aligned}$$

follows. We used Young's inequality and Friedrichs' inequality 2.21. By inserting this into (7.3) we obtain

$$\begin{aligned}\int_\Omega |\nabla u_p|^p dx \le{}& \alpha^{-p/2}\beta^{p/2}\int_\Omega |\nabla g|^p dx + \frac{p}{2}\alpha^{-p/2}\varepsilon^{p-1}\frac{\lambda^{-q}}{q}\mu(\Omega)\\ &+ \alpha^{-p/2}\varepsilon^{p-1}2^{p-2}\lambda^p\mathrm{diam}(\Omega)^p\Big\{\int_\Omega |\nabla u_p|^p dx + \int_\Omega |\nabla g|^p dx\Big\}.\end{aligned}$$

If we set λ such that

$$\alpha^{-p/2}\varepsilon^{p-1}2^{p-2}\lambda^p\mathrm{diam}(\Omega)^p = \frac{1}{2},$$

and divide by the measure $\mu(\Omega)$ we find

$$\begin{aligned}\frac{1}{2}⨍_\Omega |\nabla u_p|^p dx &\le \Big(\Big(\frac{\beta}{\alpha}\Big)^{p/2} + \frac{1}{2}\Big)⨍_\Omega |\nabla g|^p dx + \frac{1}{q}\Big(\frac{\mathrm{diam}(\Omega)}{\sqrt{\alpha}}\Big)^q p\Big(\frac{\varepsilon}{\sqrt{\alpha}}\Big)^p\\ &\le 2\max\Big\{\Big(\Big(\frac{\beta}{\alpha}\Big)^{p/2} + \frac{1}{2}\Big)⨍_\Omega |\nabla g|^p dx,\ \frac{1}{q}\Big(\frac{\mathrm{diam}(\Omega)}{\sqrt{\alpha}}\Big)^q p\Big(\frac{\varepsilon}{\sqrt{\alpha}}\Big)^p\Big\}\\ &=: 2m.\end{aligned}$$

If

$$m = \Big(\Big(\frac{\beta}{\alpha}\Big)^{p/2} + \frac{1}{2}\Big)⨍_\Omega |\nabla g|^p dx$$

then

$$\Big(\frac{1}{2}⨍_\Omega |\nabla u_p|^p dx\Big)^{1/p} \le 2^{1/p}\Big(\Big(\frac{\beta}{\alpha}\Big)^{p/2} + \frac{1}{2}\Big)^{1/p}\Big(⨍_\Omega |\nabla g|^p dx\Big)^{1/p},$$

and we find

$$\limsup_{p\to\infty} ||\nabla u_p||_p \le \sqrt{\frac{\beta}{\alpha}}||\nabla g||_\infty \le \sqrt{\frac{\beta}{\alpha}}L.$$

On the other hand, if

$$m = \frac{1}{q}\Big(\frac{\mathrm{diam}(\Omega)}{\sqrt{\alpha}}\Big)^q p\Big(\frac{\varepsilon}{\sqrt{\alpha}}\Big)^p = \Big(\frac{\mathrm{diam}(\Omega)}{\sqrt{\alpha}}\Big)^q (p-1)\Big(\frac{\varepsilon}{\sqrt{\alpha}}\Big)^p$$

then

$$\left(\frac{1}{2}\fint_\Omega |\nabla u_p|^p dx\right)^{1/p} \le 2^{1/p}\left(\frac{\operatorname{diam}(\Omega)}{\sqrt{\alpha}}\right)^{1/(p-1)}(p-1)^{1/p}\left(\frac{\varepsilon}{\sqrt{\alpha}}\right),$$

thus

$$\limsup_{p\to\infty} ||\nabla u_p||_p \le \frac{\varepsilon}{\sqrt{\alpha}}.$$

We see that the norm is bounded in both cases, and we denote the bound by M:

$$\limsup_{p\to\infty} ||\nabla u_p||_p \le M < \infty. \tag{7.4}$$

By Friedrichs' inequality we find

$$\begin{aligned}||u_p||_p &\le ||u_p - g||_p + ||g||_p\\ &\le \operatorname{diam}(\Omega)||\nabla(u_p - g)||_p + ||g||_p\\ &\le \operatorname{diam}(\Omega)(||\nabla u_p||_p + ||\nabla g||_p) + ||g||_p,\end{aligned}$$

hence

$$\limsup_{p\to\infty} ||u_p||_p \le \operatorname{diam}(\Omega)(M+L) + ||g||_\infty,$$

which shows that the sequence (u_p) is equibounded for $p > n$. By Morrey's inequality 2.19 and the Lipschitz continuity of g we have

$$\begin{aligned}|u_p(x) - u_p(y)| &\le |(u_p(x) - g(x)) - (u_p(y) - g(y))| + |g(x) - g(y)|\\ &\le C_p|x-y|^{1-n/p}||\nabla(u_p - g)||_p + L|x-y|\\ &\le C_p|x-y|^{1-n/p}\operatorname{diam}(\Omega)(||\nabla u_p||_p + ||\nabla g||_p) + L|x-y|.\end{aligned}$$

Since $C_p \to 2^{n+1}$ as $p \to \infty$ we obtain

$$\limsup_{p\to\infty} |u_p(x) - u_p(y)| \le 2^{n+1}\operatorname{diam}(\Omega)(M+L)|x-y| + L|x-y|.$$

Thus, the sequence is equicontinuous for $p > n$. Then by Ascoli's theorem 2.2 there is a subsequence and a function $u^+ \in C(\overline{\Omega})$ with $u^+|_{\partial\Omega} = g$, such that

$$\lim_{p_j\to\infty} u_{p_j} = u^+ \quad \text{uniformly in } \overline{\Omega}.$$

Now we show that $u^+ \in W^{1,\infty}(\Omega)$. Let $p_j > s > n$. By Proposition 2.4 and (7.4) we have

$$\left(\fint_\Omega |\nabla u_{p_j}|^s dx\right)^{1/s} \le \left(\fint_\Omega |\nabla u_{p_j}|^{p_j} dx\right)^{1/p_j} \le M, \tag{7.5}$$

so the sequence (∇u_{p_j}) is uniformly bounded, and by Proposition 2.10 we conclude that

$$\nabla u_{p_j} \rightharpoonup \nabla u^+ \quad \text{weakly in } L^s(\Omega),$$

for some subsequence. By a diagonalization procedure, we can extract a single subsequence such that

$$\nabla u_{p_j} \rightharpoonup \nabla u^+ \quad \text{weakly in } L^s(\Omega) \text{ for all } s \in (n,\infty).$$

By weak lower semicontinuity and (7.5) we find

$$\left(\fint_\Omega |\nabla u^+|^s dx\right)^{1/s} \le \liminf_{p_j\to\infty}\left(\fint_\Omega |\nabla u_{p_j}|^s dx\right)^{1/s} \le M.$$

Since s was arbitrary large, letting $s\to\infty$ we obtain

$$||\nabla u^+||_\infty \le M,$$

and we conclude that $u^+ \in W^{1,\infty}(\Omega)$. □

Now we show that the difference between the functions constructed in the previous lemma can be made arbitrarily small.

Lemma 7.4. *Let u^+ and u^- be the constructed functions in Lemma 7.3. Then*

$$||u^+ - u^-||_{\infty,\Omega} \le C\varepsilon,$$

where the constant C depends on α and Ω.

Proof. Let $g \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ be Lipschitz continuous. Suppose that u_p^+, u_p^- are the unique weak solutions of

$$\Delta_{p,A}\, u_p^+ = -\varepsilon^{p-1}, \quad \Delta_{p,A}\, u_p^- = +\varepsilon^{p-1},$$

both with boundary values g. By the previous Lemma we have

$$\lim_{p\to\infty} u_p^+ = u^+, \quad \lim_{p\to\infty} u_p^- = u^- \quad \text{uniformly in } \overline{\Omega}$$

for some subsequence, still denoted by the index p, where $u^+|_{\partial\Omega} = u^-|_{\partial\Omega} = g$. By subtracting the weak formulation

$$\int_\Omega \Big\langle \langle A(x)\nabla u_p^-, \nabla u_p^-\rangle^{\frac{p-2}{2}} A(x)\nabla u_p^-, \nabla\eta\Big\rangle\, dx = -\varepsilon^{p-1}\int_\Omega \eta\, dx$$

from

$$\int_\Omega \Big\langle \langle A(x)\nabla u_p^+, \nabla u_p^+\rangle^{\frac{p-2}{2}} A(x)\nabla u_p^+, \nabla\eta\Big\rangle\, dx = +\varepsilon^{p-1}\int_\Omega \eta\, dx$$

and by setting $\eta = u_p^+ - u_p^- \in W_0^{1,p}(\Omega)$, we obtain

$$\begin{aligned}
&2\varepsilon^{p-1}\int_\Omega (u_p^+ - u_p^-)\, dx \\
&= \int_\Omega \Big\langle \langle A(x)\nabla u_p^+, \nabla u_p^+\rangle^{\frac{p-2}{2}} A(x)\nabla u_p^+ - \langle A(x)\nabla u_p^-, \nabla u_p^-\rangle^{\frac{p-2}{2}} A(x)\nabla u_p^-, \nabla u_p^+ - \nabla u_p^-\Big\rangle\, dx \\
&\ge 4\Big(\frac{\sqrt{\alpha}}{2}\Big)^p \int_\Omega |\nabla u_p^+ - \nabla u_p^-|^p dx,
\end{aligned}$$

where we used the inequality in Proposition 2.29. Rearranging and raising to the $1/p$ power yields

$$||\nabla u_p^+ - \nabla u_p^-||_p \le \frac{2\varepsilon}{\sqrt{\alpha}}\Big(\frac{1}{2\varepsilon}\int_\Omega (u_p^+ - u_p^-)\, dx\Big)^{1/p},$$

thus

$$\lim_{p\to\infty} ||\nabla u_p^+ - \nabla u_p^-||_p \le \frac{2\varepsilon}{\sqrt{\alpha}}.$$

By Friedrichs' inequality 2.21 we find

$$\begin{aligned} ||u^+ - u^-||_\infty &\le ||u^+ - u_p^+||_\infty + ||u_p^+ - u_p^-||_\infty + ||u_p^- - u^-||_\infty \\ &\le ||u^+ - u_p^+||_\infty + \operatorname{diam}(\Omega) \lim_{p\to\infty} ||\nabla u_p^+ - \nabla u_p^-||_p \\ &\quad + ||u_p^- - u^-||_\infty. \end{aligned}$$

The bound above and the uniform convergence implies that

$$||u^+ - u^-||_\infty \le \frac{2\varepsilon}{\sqrt{\alpha}} \operatorname{diam}(\Omega).$$

□

7.3 Viscosity solutions

It remains to show that the constructed functions u^+, u_∞ and u^- satisfies the Upper equation, the Equation, and the Lower equation, respectively in the viscosity sense. The proof relies on the following result[10].

Lemma 7.5. *Let $u_j, u \in C(\overline{\Omega})$ and suppose that $u_j \to u$ uniformly in $\overline{\Omega}$. Assume that $\phi \in C^2(\Omega)$ is such that $\phi(x_0) = u(x_0)$ for some $x_0 \in \Omega$, and that $\phi(x) < u(x)$ otherwise in Ω. Then there are points $x_j \in \Omega$ such that*

$$x_j \to x_0, \quad u_j(x_j) - \phi(x_j) = \min_\Omega (u_j - \phi),$$

for some subsequence.

Proof. Since $u_j - \phi = (u - \phi) + (u_j - u)$ and $u_j \to u$,

$$\inf_{\Omega \setminus B(x_0,r)} (u_j - \phi) \ge \frac{1}{2} \inf_{\Omega \setminus B(x_0,r)} (u - \phi) > 0,$$

for sufficiently large j, where $B(x_0,r)$ is a ball centered at x_0 with small radius $r > 0$. Notice that $u_j(x_0) - \phi(x_0)$ approaches zero, so by the above we have

$$\inf_{\Omega \setminus B(x_0,r)} (u_j - \phi) > u_j(x_0) - \phi(x_0)$$

when $j > j_r$. This implies that there is a $x_j \in \overline{B(x_0,r)}$ such that

$$\min_\Omega (u_j - \phi) = u_j(x_j) - \phi(x_j),$$

when $j > j_r$. Finally, via some subsequence we can let $r \to 0$. □

[10] The proof is taken from Lemma 11 in [18].

Theorem 7.6. *Let u_∞ and u^+, u^- be the constructed functions in Theorem 5.1 and Lemma 7.3, respectively. Then*

i) u^+ is a viscosity supersolution of the Upper equation

$$\max\Big\{\varepsilon - \langle A(x)\nabla u^+, \nabla u^+\rangle^{1/2},\ \Delta_{\infty,A}\, u^+\Big\} = 0,$$

meaning that

$$\varepsilon - \langle A(x_0)\nabla\phi(x_0), \nabla\phi(x_0)\rangle^{1/2} \le 0, \quad \Delta_{\infty,A}\,\phi(x_0) \le 0$$

holds for a function $\phi \in C^2(\Omega)$ touching u^+ from below at x_0.

ii) u_∞ is a viscosity solution of the Equation

$$\Delta_{\infty,A}\, u_\infty = 0.$$

iii) u^- is a viscosity subsolution of the Lower equation

$$\min\Big\{\langle A(x)\nabla u^-, \nabla u^-\rangle^{1/2} - \varepsilon,\ \Delta_{\infty,A}\, u^-\Big\} = 0,$$

meaning that

$$\langle A(x_0)\nabla\psi(x_0), \nabla\psi(x_0)\rangle^{1/2} - \varepsilon \ge 0, \quad \Delta_{\infty,A}\,\psi(x_0) \ge 0$$

holds for a function $\psi \in C^2(\Omega)$ touching u^- from above at x_0.

Proof. The proofs are similar, and we only show *i)*. Let u_p^+ be the weak solution of

$$\Delta_{p,A}\, u_p^+ = -\varepsilon^{p-1}.$$

By Lemma 7.2, u_p^+ is also a viscosity solution, and by Lemma 7.3,

$$\lim_{p\to\infty} u_p^+ = u^+ \quad \text{uniformly in } \overline{\Omega}$$

for some subsequence, which we still denote by the index p. Now suppose that $\phi \in C^2(\Omega)$ touches u^+ from below at some point $x_0 \in \Omega$. Then by Lemma 7.5 there is a sequence $x_k \to x_0$ and a subsequence so that $u_{p_k}^+ - \phi$ attains its minimum at x_k. Since $u_{p_k}^+$ is a viscosity supersolution, this is equivalent to

$$\Delta_{p_k,A}\,\phi(x_k) \le -\varepsilon^{p_k-1}.$$

Written out this becomes

$$\begin{aligned}&\langle A(x_k)\nabla\phi(x_k), \nabla\phi(x_k)\rangle^{\frac{p_k-4}{2}}\Big\{\langle A(x_k)\nabla\phi(x_k), \nabla\phi(x_k)\rangle\,\mathrm{div}(A(x_k)\nabla\phi(x_k))\\ &+\Big(\frac{p_k-2}{2}\Big)\Delta_{\infty,A}\,\phi(x_k)\Big\} \le -\varepsilon^{p_k-1}.\end{aligned}$$

If $\varepsilon \ne 0$ then

$$\langle A(x_k)\nabla\phi(x_k), \nabla\phi(x_k)\rangle \ne 0,$$

so we can divide to obtain

$$\begin{aligned}&2\langle A(x_k)\nabla\phi(x_k),\nabla\phi(x_k)\rangle\frac{\operatorname{div}(A(x_k)\nabla\phi(x_k))}{p_k-2}+\Delta_{\infty,A}\,\phi(x_k)\\&\le -\frac{2\varepsilon^3}{p_k-2}\left(\frac{\varepsilon}{\langle A(x_k)\nabla\phi(x_k),\nabla\phi(x_k)\rangle^{1/2}}\right)^{p_k-4}.\end{aligned}$$

By continuity the left-hand side approaches $\Delta_{\infty,A}\,\phi(x_0)$ as $p_k\to\infty$. If

$$\langle A(x_0)\nabla\phi(x_0),\nabla\phi(x_0)\rangle^{1/2}<\varepsilon,$$

we get the contradiction

$$\Delta_{\infty,A}\,\phi(x_0)=-\infty,$$

thus

$$\langle A(x_0)\nabla\phi(x_0),\nabla\phi(x_0)\rangle^{1/2}\ge\varepsilon.$$

This implies that the right-hand side of the above expression approaches zero, so we have

$$\Delta_{\infty,A}\,\phi(x_0)\le 0.$$

If $\varepsilon=0$ and

$$\langle A(x_0)\nabla\phi(x_0),\nabla\phi(x_0)\rangle=0$$

there is nothing to prove. If $\varepsilon=0$ and

$$\langle A(x_0)\nabla\phi(x_0),\nabla\phi(x_0)\rangle\ne 0$$

we find

$$2\langle A(x_k)\nabla\phi(x_k),\nabla\phi(x_k)\rangle\frac{\operatorname{div}\Big(A(x_k)\nabla\phi(x_k)\Big)}{p_k-2}+\Delta_{\infty,A}\,\phi(x_k)\le 0$$

for large indices k, and as $p_k\to\infty$ we once more have the inequality

$$\Delta_{\infty,A}\,\phi(x_0)\le 0.$$

□

8 Comparison principle

As before we let u^+ and u^- be the constructed supersolution of the Upper equation and subsolution of the Lower equation, respectively.

We begin with the comparison principle for viscosity subsolutions[11].

Lemma 8.1. *If u is a viscosity subsolution of the equation $\Delta_{\infty,A}\, u = 0$ and if $u \le u^+ = g$ on $\partial\Omega$, then $u \le u^+$ in Ω.*

Proof. By adding the same constant to u^+ and g, we may assume that $u^+ > 0$ and $g > 0$. Assume by contradiction that

$$\max_{\Omega}(u - u^+) > \max_{\partial\Omega}(u - u^+).$$

We shall construct a strict supersolution $w = f(u^+)$ of the Upper equation such that

$$\max_{\Omega}(u - w) > \max_{\partial\Omega}(u - w),$$

and

$$\Delta_{\infty,A}\, w \le -\mu < 0$$

in the viscosity sense. This will lead to a contradiction.

To approximate the identity we use the function

$$f(t) = \ln(1 + B(e^t - 1)),$$

where $B > 1$ and $t > 0$. Notice that

$$0 < f(t) - t < B - 1$$
$$0 < f'(t) - 1 < B - 1$$
$$f''(t) = -(B-1)B^{-1}e^{-t} f'(t)^2.$$

Now set $w = f(u^+)$. A formal calculation shows that

$$\frac{\partial w}{\partial x_i} = f'(u^+)\frac{\partial u^+}{\partial x_i}$$
$$\frac{\partial^2 w}{\partial x_i \partial x_j} = f''(u^+)\frac{\partial u^+}{\partial x_i}\frac{\partial u^+}{\partial x_j} + f'(u^+)\frac{\partial^2 u^+}{\partial x_i \partial x_j},$$

thus

$$\begin{aligned}
\Delta_{\infty,A}\, w &= \Big\langle \nabla_x \langle A(x)\nabla w, \nabla w\rangle + 2(D^2 w)A(x)\nabla w, A(x)\nabla w \Big\rangle \\
&= \sum_{i,j,k,l=1}^{n} \left(\frac{\partial a_{ij}}{\partial x_k}\frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} a_{kl}\frac{\partial w}{\partial x_l} + 2a_{ik}\frac{\partial w}{\partial x_k}\frac{\partial^2 w}{\partial x_i \partial x_j} a_{jl}\frac{\partial w}{\partial x_l} \right) \\
&= f'(u^+)^3 \sum_{i,j,k,l=1}^{n} \left(\frac{\partial a_{ij}}{\partial x_k}\frac{\partial u^+}{\partial x_i}\frac{\partial u^+}{\partial x_j} a_{kl}\frac{\partial u^+}{\partial x_l} + 2a_{ik}\frac{\partial u^+}{\partial x_k}\frac{\partial^2 u^+}{\partial x_i \partial x_j} a_{jl}\frac{\partial u^+}{\partial x_l} \right) \qquad (8.1) \\
&\quad + 2f'(u^+)^2 f''(u^+) \sum_{i,j,k,l=1}^{n} a_{ik}\frac{\partial u^+}{\partial x_i}\frac{\partial u^+}{\partial x_k} a_{jl}\frac{\partial u^+}{\partial x_j}\frac{\partial u^+}{\partial x_l} \\
&= f'(u^+)^3 \Delta_{\infty,A}\, u^+ + 2f'(u^+)^2 f''(u^+)\langle A(x)\nabla u^+, \nabla u^+\rangle^2.
\end{aligned}$$

[11]The proof is based on [20].

Multiplying the Upper equation for supersolutions

$$\max\Big\{\varepsilon - \langle A(x)\nabla u^+, \nabla u^+\rangle^{1/2},\ \Delta_{\infty,A}\, u^+\Big\} \le 0$$

by $f'(u^+)^3$ we find

$$\begin{aligned}\Delta_{\infty,A}\, w &\le 2f'(u^+)^2 f''(u^+)\langle A(x)\nabla u^+, \nabla u^+\rangle^2\\ &= -2(B-1)B^{-1}e^{-u^+}f'(u^+)^4\langle A(x)\nabla u^+, \nabla u^+\rangle^2\\ &\le -2\varepsilon^4(B-1)B^{-1}e^{-\|u^+\|_\infty},\end{aligned}$$

where we used

$$f'(u^+) > 1, \quad \langle A(x)\nabla u^+, \nabla u^+\rangle^{1/2} \ge \varepsilon.$$

To obtain the desired construction, given $\varepsilon > 0$ we fix $B > 1$ so close to 1 that

$$0 < w - u^+ = f(u^+) - u^+ < B - 1 < \delta,$$

where $\delta > 0$ is so small that

$$\max_{\Omega}(u - w) > \max_{\partial\Omega}(u - w). \tag{8.2}$$

By setting

$$\mu = 2\varepsilon^4(B-1)B^{-1}e^{-\|u^+\|_\infty}$$

we now have

$$\Delta_{\infty,A}\, w \le -\mu < 0. \tag{8.3}$$

The procedure was formal. In the calculations above we replace u^+ by a test function ϕ touching u^+ from below at a point x_0, and we replace w by a test function $\psi = f(\phi)$ touching w from below at x_0. Now we have

$$\Delta_{\infty,A}\, \psi(x_0) \le -\mu, \quad \langle A(x_0)\nabla\psi(x_0), \nabla\psi(x_0)\rangle^{1/2} \ge \varepsilon,$$

whenever ψ touches w from below at x_0.

In order to use Ishii's lemma 6.9, we double the variables writing

$$M_\nu = \sup_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} \Big(u(x) - w(y) - \frac{\nu}{2}|x-y|^2\Big).$$

The supremum is attained at some point (x_ν, y_ν), and by compactness there is a subsequence such that

$$x_\nu \to \hat{x}, \quad y_\nu \to \hat{y} \quad \text{as } \nu \to \infty.$$

By Lemma 6.8, $\hat{x} = \hat{y}$, and

$$\max_{\overline{\Omega}}(u - w) = u(\hat{x}) - w(\hat{x}).$$

Thus by (8.2), $\hat{x}$ is an interior point. Furthermore, (x_ν, y_ν) belongs to the interior for sufficiently large indices ν. Then by Ishii's lemma 6.9 there exist symmetric matrices $\mathbb{X}_\nu, \mathbb{Y}_\nu \in S^n$ such that

$$\begin{aligned}&(\nu(x_\nu - y_\nu), \mathbb{X}_\nu) \in \overline{J^{2,+}}u(x_\nu),\\ &(\nu(x_\nu - y_\nu), \mathbb{Y}_\nu) \in \overline{J^{2,-}}w(y_\nu).\end{aligned} \tag{8.4}$$

We need the following bound.

Lemma 8.2. *We have*

$$\frac{\varepsilon}{\sqrt{\beta}} \le \nu|x_\nu - y_\nu| \le C_\varepsilon,$$

where $C_\varepsilon = 2B||\nabla u^+||_\infty$.

Proof. We first show the upper bound. We have

$$\begin{aligned} u(x_\nu) - w(y_\nu) - \frac{\nu}{2}|x_\nu - y_\nu|^2 &= \max_{(x,y)\in\Omega\times\Omega}\left(u(x) - w(y) - \frac{\nu}{2}|x-y|^2\right) \\ &\ge u(x) - w(y) - \frac{\nu}{2}|x-y|^2 \end{aligned}$$

for all $(x,y) \in \Omega\times\Omega$. In particular, if $x = y = x_\nu$ we find

$$u(x_\nu) - w(y_\nu) - \frac{\nu}{2}|x_\nu - y_\nu|^2 \ge u(x_\nu) - w(x_\nu),$$

thus

$$\begin{aligned} \frac{\nu}{2}|x_\nu - y_\nu|^2 &\le w(x_\nu) - w(y_\nu) \\ &\le ||\nabla w||_\infty |x_\nu - y_\nu| \\ &= ||f'(u^+)\nabla u^+||_\infty |x_\nu - y_\nu| \\ &\le B||\nabla u^+||_\infty |x_\nu - y_\nu|. \end{aligned}$$

The upper bound

$$\nu|x_\nu - y_\nu| \le C_\varepsilon$$

follows, where $C_\varepsilon = 2B||\nabla u^+||_\infty$. The lower bound follows from the Upper equation for supersolutions. By the inequality

$$u(x_\nu) - w(y_\nu) - \frac{\nu}{2}|x_\nu - y_\nu|^2 \ge u(x_\nu) - w(y) - \frac{\nu}{2}|x_\nu - y|^2$$

we see that the test function

$$\psi(y) = w(y_\nu) + \frac{\nu}{2}|x_\nu - y_\nu|^2 - \frac{\nu}{2}|x_\nu - y|^2$$

touches w from below at the point $y = y_\nu$, hence

$$\varepsilon - \langle A(y_\nu)\nabla\psi(y_\nu), \nabla\psi(y_\nu)\rangle^{1/2} \le 0.$$

Since

$$(\nu(x_\nu - y_\nu), \mathbb{Y}_\nu) \in \overline{J^{2,-}}w(y_\nu)$$

we find by Proposition 6.7,

$$\begin{aligned} \varepsilon &\le \langle A(y_\nu)\nu(x_\nu - y_\nu), \nu(x_\nu - y_\nu)\rangle^{1/2} \\ &\le (\beta|\nu(x_\nu - y_\nu)|^2)^{1/2} \\ &= \sqrt{\beta}\nu|x_\nu - y_\nu|. \end{aligned}$$

□

We return to (8.4). Since u is a viscosity subsolution we have

$$\mathcal{L}_{\infty,A}(x_\nu,\nu(x_\nu-y_\nu),\mathbb{X}_\nu)\geq 0$$

by Proposition 6.7, and we can rewrite equation (8.3) as

$$\mathcal{L}_{\infty,A}(y_\nu,\nu(x_\nu-y_\nu),\mathbb{Y}_\nu)\leq -\mu.$$

By subtracting the last equation from the first, and applying Corollary 3.6 we obtain

$$\begin{aligned}\mu &\leq \mathcal{L}_{\infty,A}(x_\nu,\nu(x_\nu-y_\nu),\mathbb{X}_\nu)-\mathcal{L}_{\infty,A}(y_\nu,\nu(x_\nu-y_\nu),\mathbb{Y}_\nu)\\ &\leq \kappa|x_\nu-y_\nu||\nu(x_\nu-y_\nu)|^3\\ &\quad +2\{\langle\mathbb{X}_\nu A(x_\nu)\nu(x_\nu-y_\nu),A(x_\nu)\nu(x_\nu-y_\nu)\rangle\\ &\quad -\langle\mathbb{Y}_\nu A(y_\nu)\nu(x_\nu-y_\nu),A(y_\nu)\nu(x_\nu-y_\nu)\rangle\}\\ &\leq \kappa|x_\nu-y_\nu|C_\varepsilon^3\\ &\quad +2\{\langle\mathbb{X}_\nu A(x_\nu)\nu(x_\nu-y_\nu),A(x_\nu)\nu(x_\nu-y_\nu)\rangle\\ &\quad -\langle\mathbb{Y}_\nu A(y_\nu)\nu(x_\nu-y_\nu),A(y_\nu)\nu(x_\nu-y_\nu)\rangle\},\end{aligned}\tag{8.5}$$

where we used Lemma 8.2 in the last inequality. We need an estimate on the difference in the brackets. We use the notation

$$\nu(x_\nu-y_\nu)=(\nu(x_\nu-y_\nu)_1,\nu(x_\nu-y_\nu)_2,\dots,\nu(x_\nu-y_\nu)_n).$$

By (6.2) and the Cauchy–Schwarz inequality we have

$$\begin{aligned}&2\{\langle\mathbb{X}_\nu A(x_\nu)\nu(x_\nu-y_\nu),A(x_\nu)\nu(x_\nu-y_\nu)\rangle\\ &\quad -\langle\mathbb{Y}_\nu A(y_\nu)\nu(x_\nu-y_\nu),A(y_\nu)\nu(x_\nu-y_\nu)\rangle\}\\ &\leq 2\cdot 3\nu|A(x_\nu)\nu(x_\nu-y_\nu)-A(y_\nu)\nu(x_\nu-y_\nu)|^2\\ &=6\nu|(A(x_\nu)-A(y_\nu))\nu(x_\nu-y_\nu)|^2\\ &=6\nu\sum_{i=1}^n\Big\{\sum_{j=1}^n(a_{ij}(x_\nu)-a_{ij}(y_\nu))\nu(x_\nu-y_\nu)_j\Big\}^2\\ &\leq 6\nu\sum_{i=1}^n\Big\{\Big(\sum_{j=1}^n(a_{ij}(x_\nu)-a_{ij}(y_\nu))^2\Big)^{1/2}\Big(\sum_{k=1}^n\nu(x_\nu-y_\nu)_k^2\Big)^{1/2}\Big\}^2\\ &=6\nu|\nu(x_\nu-y_\nu)|^2\sum_{i,j=1}^n(a_{ij}(x_\nu)-a_{ij}(y_\nu))^2\\ &\leq 6\nu C_\varepsilon^2\sum_{i,j=1}^n H^2|x_\nu-y_\nu|^2\\ &=6n^2H^2C_\varepsilon^2\nu|x_\nu-y_\nu|^2\\ &\leq 6n^2H^2C_\varepsilon^3|x_\nu-y_\nu|,\end{aligned}\tag{8.6}$$

where we also used Lemma 8.2 and the Lipschitz continuity of a_{ij}. Inserting this into (8.5) we find

$$\begin{aligned}\mu &\leq \kappa|x_\nu-y_\nu|C_\varepsilon^3\\ &\quad +2\{\langle\mathbb{X}_\nu A(x_\nu)\nu(x_\nu-y_\nu),A(x_\nu)\nu(x_\nu-y_\nu)\rangle\\ &\quad -\langle\mathbb{Y}_\nu A(y_\nu)\nu(x_\nu-y_\nu),A(y_\nu)\nu(x_\nu-y_\nu)\rangle\}\\ &\leq \kappa|x_\nu-y_\nu|C_\varepsilon^3+6n^2H^2C_\varepsilon^3|x_\nu-y_\nu|.\end{aligned}\tag{8.7}$$

Since all the constants are bounded and $|x_\nu - y_\nu| \to 0$, we conclude that the last line has limit equal to zero as $\nu \to \infty$. This leads to a contradiction because $\mu > 0$. Thus, the assumption was false, so we must have

$$u \le u^+ \quad \text{in } \Omega.$$

□

By a similar argument we find the analogous comparison for viscosity supersolutions.

Lemma 8.3. *If u is a viscosity supersolution of the equation $\Delta_{\infty,A}\, u = 0$ and if $u^- = g \le u$ on $\partial\Omega$, then $u^- \le u$ in Ω.*

In this case we use the inverse function f^{-1} of $f(t) = \ln(1 + B(e^t - 1))$ as an approximation of the identity, which is obtained by simply replacing B by B^{-1} in the formula for f. The essential properties is that f^{-1} is strictly convex and $(f^{-1})' > B^{-1}$.

From these two Lemmas we immediately have the following.

Corollary 8.4. *If $u \in C(\overline{\Omega})$ is an arbitrary viscosity solution of the equation $\Delta_{\infty,A}\, u = 0$ with $u = g$ on $\partial\Omega$, then $u^- \le u \le u^+$ in Ω.*

This result implies uniqueness, which we show below. First we state the general version of the comparison principle[12].

Theorem 8.5 (Comparison principle)**.** *Suppose that u is a viscosity subsolution and that v is a viscosity supersolution of $\Delta_{\infty,A}\, w = 0$ in Ω. If at each point $z \in \partial\Omega$,*

$$\limsup_{x \to z} u(x) \le \liminf_{x \to z} v(x),$$

and if both sides are not simultaneously ∞ or $-\infty$, then $u \le v$ in Ω.

Proof. Let $\delta > 0$ and define the set

$$\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}.$$

We claim that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$u < v + \varepsilon \quad \text{in } \Omega \setminus \Omega_\delta.$$

Indeed, if such a δ does not exist, there are points $x_j \in \Omega$ such that

$$x_j \to z \in \partial\Omega \quad \text{as } j \to \infty, \quad u(x_j) \ge v(x_j) + \varepsilon \quad \text{for all } j \in \mathbb{N}.$$

Thus

$$\limsup_{x \to z} u(x) \ge \liminf_{x \to z} v(x) + \varepsilon,$$

which contradicts the assumption since both sides are not simultaneously ∞ or $-\infty$. Now construct the auxiliary solutions u^+, u^- in the domain Ω_δ such that

$$u^- = v + \varepsilon = u = u^+ \quad \text{on } \partial\Omega_\delta.$$

[12]We follow the proof of Corollary 4.31 in [13].

Then by Lemma 8.1 and Lemma 8.3 we have

$$u^- \le v + \varepsilon, \quad u \le u^+ \quad \text{in } \Omega_\delta.$$

Moreover, by Lemma 7.4,

$$u^+ - u^- \le C\varepsilon$$

which implies that

$$v + \varepsilon \ge u^- \ge u^+ - C\varepsilon \ge u - C\varepsilon.$$

Thus $v \ge u$ in Ω_δ because $\varepsilon > 0$ was arbitrarily small, hence $v \ge u$ in Ω. □

8.1 Uniqueness

We are now ready to establish the uniqueness of viscosity solutions of the equation $\Delta_{\infty,A}\, u = 0$. We summarize what we have found in the following.

Theorem 8.6. *Let Ω be a bounded domain in $\mathbb{R}^n$. Given a Lipschitz continuous function $g \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$, there exists a unique viscosity solution $u \in C(\overline{\Omega})$ of the equation*

$$\Delta_{\infty,A}\, u = 0 \quad \text{in } \Omega$$

with boundary values g. Moreover, the function u belongs to $W^{1,\infty}(\Omega)$ and has the following minimizing property in each subdomain $D \subset \Omega$: if $v \in C(\overline{D}) \cap W^{1,\infty}(D)$ is such that $v = u$ on ∂D, then

$$||\langle A\nabla u, \nabla u\rangle^{1/2}||_{\infty,D} \le ||\langle A\nabla v, \nabla v\rangle^{1/2}||_{\infty,D}.$$

Proof. To show uniqueness, suppose that there are two viscosity solutions u_1 and u_2, both with boundary values g. Then by Corollary 8.4 and Lemma 7.4 we have

$$u^- \le u_1 \le u^+, \quad u^- \le u_2 \le u^+, \quad ||u^+ - u^-||_{\infty,\Omega} \le C\varepsilon,$$

thus

$$-C\varepsilon \le u^- - u^+ \le u_1 - u_2 \le u^+ - u^- \le C\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary small, it follows that $u_1 = u_2$.

By Theorem 7.6, every uniform limit

$$\lim_{p_j \to \infty} u_{p_j} = u \quad \text{in } \overline{\Omega},$$

where u_{p_j} is a weak solution of $\Delta_{p_j,A}\, u_{p_j} = 0$ with $u_{p_j}|_{\partial\Omega} = g$, is also a viscosity solution of $\Delta_{\infty,A}\, u = 0$, thus they are also unique. The existence and the minimization property was settled in Theorem 5.1. □

Observation 8.7. We have found that the uniform limit u of weak solutions u_{p_j} of the Euler-Lagrange equations

$$\Delta_{p_j,A}\, u_{p_j} = 0$$

is unique and satisfies the equation

$$\Delta_{\infty,A}\, u = 0$$

in the viscosity sense. Thus, it really is the limit equation. Furthermore, since this unique viscosity solution satisfies the minimization property

$$||\langle A\nabla u, \nabla u\rangle^{1/2}||_{\infty,D} \leq ||\langle A\nabla v, \nabla v\rangle^{1/2}||_{\infty,D},$$

we interpret the limit equation as the Euler-Lagrange equation for the ”variational problem”

$$\min_u \max_x \Big(\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial u}{\partial x_j}(x) \Big)^{1/2}.$$

9 Stability

Having established the existence of a unique viscosity solution of the equation

$$\Delta_{\infty,A}\, u = \sum_{i,j,k,l=1}^{n} \left(\frac{\partial a_{ij}}{\partial x_k} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} a_{kl} \frac{\partial u}{\partial x_l} + 2a_{ik} \frac{\partial u}{\partial x_k} \frac{\partial^2 u}{\partial x_i \partial x_j} a_{jl} \frac{\partial u}{\partial x_l} \right) = 0, \tag{9.1}$$

where the matrix $A(x)$ satisfies the assumptions in Section 3, we now ask the question: if we perturbate the matrix $A(x)$ a little bit, how much will the solution differ from the solution of the original problem? More precisely stated, suppose that we are given an $n \times n$ matrix $A(x) = (a_{ij}(x))$ satisfying the conditions:

1. $A(x)$ is symmetric.

2. $\alpha|\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \beta|\xi|^2$ for $0 < \alpha \leq \beta < \infty$.

3. $\frac{\partial a_{ij}}{\partial x_k}$ is Lipschitz continuous.

4. a_{ij} is Lipschitz continuous.

We let the permutation of $A(x)$ be

$$\tilde{A}(x) = A(x) + \Lambda = (a_{ij}(x) + c_{ij}),$$

where $\Lambda = (c_{ij})$ is a symmetric $n \times n$ matrix with real-valued constant entries c_{ij}. We immediately find that **1**, **3** and **4** holds for $\tilde{A}(x)$. In order for **2** to be satisfied, we have to put some restrictions on the matrix Λ. Let μ_j, $j = 1, 2, \ldots, n$ denote the eigenvalues of Λ. We find

$$\langle \tilde{A}(x)\xi, \xi \rangle = \langle A(x)\xi, \xi \rangle + \langle \Lambda\xi, \xi \rangle \geq (\alpha + \min_j \mu_j)|\xi|^2,$$

thus we require

$$\min_j \mu_j > -\alpha.$$

Similarly we must have

$$\max_j \mu_j < \infty.$$

Then by setting

$$\tilde{\alpha} = \alpha + \min_j \mu_j, \quad \tilde{\beta} = \beta + \max_j \mu_j,$$

we obtain

$$\tilde{\alpha}|\xi|^2 \leq \langle \tilde{A}(x)\xi, \xi \rangle \leq \tilde{\beta}|\xi|^2$$

for $0 < \tilde{\alpha} \leq \tilde{\beta} < \infty$, so **2** is satisfied for $\tilde{A}(x)$. Now $A(x)$ and $\tilde{A}(x)$ both satisfies **1-4**. Then by Theorem 8.6, given a Lipschitz continuous function $g \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$, there exists a unique viscosity solution $u \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ of the equation

$$\begin{aligned} \Delta_{\infty,A}\, u &= \Big\langle \nabla_x \langle A(x)\nabla u, \nabla u \rangle + 2(D^2 u)A(x)\nabla u, A(x)\nabla u \Big\rangle \\ &= \sum_{i,j,k,l=1}^{n} \left(\frac{\partial a_{ij}}{\partial x_k} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} a_{kl} \frac{\partial u}{\partial x_l} + 2a_{ik} \frac{\partial u}{\partial x_k} \frac{\partial^2 u}{\partial x_i \partial x_j} a_{jl} \frac{\partial u}{\partial x_l} \right) = 0, \end{aligned}$$

and there exists a unique viscosity solution $\tilde{u} \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ of the equation

$$\begin{aligned}
\Delta_{\infty,\tilde{A}}\, \tilde{u} &= \Big\langle \nabla_x \langle \tilde{A}(x)\nabla\tilde{u}, \nabla\tilde{u}\rangle + 2(D^2\tilde{u})\tilde{A}(x)\nabla\tilde{u}, \tilde{A}(x)\nabla\tilde{u} \Big\rangle \\
&= \Big\langle \nabla_x \langle A(x)\nabla\tilde{u}, \nabla\tilde{u}\rangle + 2(D^2\tilde{u})(A(x)+\Lambda)\nabla\tilde{u}, (A(x)+\Lambda)\nabla\tilde{u} \Big\rangle \\
&= \sum_{i,j,k,l=1}^{n} \Big(\frac{\partial a_{ij}}{\partial x_k}\frac{\partial \tilde{u}}{\partial x_i}\frac{\partial \tilde{u}}{\partial x_j}(a_{kl}+c_{kl})\frac{\partial \tilde{u}}{\partial x_l} \\
&\quad + 2(a_{ik}+c_{ik})\frac{\partial \tilde{u}}{\partial x_k}\frac{\partial^2 \tilde{u}}{\partial x_i \partial x_j}(a_{jl}+c_{jl})\frac{\partial \tilde{u}}{\partial x_l}\Big) = 0,
\end{aligned}$$

both with boundary values $u|_{\partial\Omega} = \tilde{u}|_{\partial\Omega} = g$. By stability we mean the following: we seek a bound of the norm

$$||\tilde{u} - u||_{\infty,\Omega}$$

that depends on constants and the elements c_{ij} of the matrix Λ, which can be made arbitrarily small by setting c_{ij} small. The following motivates why we seek this bound.

9.1 Stability in one variable

In Example 3.2 we found the solution

$$u(x) = L + (R-L)\frac{\int_l^x a(t)^{-1/2}dt}{\int_l^r a(t)^{-1/2}dt}$$

of the Dirichlet problem

$$\begin{cases} \Delta_{\infty,A}\, u = a(x)a'(x)u'(x)^3 + 2a(x)^2u'(x)^2u''(x) = 0 \quad \text{in } (l,r) \\ u(l) = L, \quad u(r) = R, \end{cases}$$

where a and a' is Lipschitz continuous in $[l,r]$, and

$$0 < \alpha \le a(x) \le \beta < \infty \quad \text{for all } x \in [l,r].$$

Similarly we find the solution

$$\tilde{u}(x) = L + (R-L)\frac{\int_l^x (a(t)+c)^{-1/2}dt}{\int_l^r (a(t)+c)^{-1/2}dt}$$

of the perturbated problem

$$\begin{cases} \Delta_{\infty,\tilde{A}}\, \tilde{u} = (a(x)+c)a'(x)\tilde{u}'(x)^3 + 2(a(x)+c)^2\tilde{u}'(x)^2\tilde{u}''(x) = 0 \quad \text{in } (l,r) \\ \tilde{u}(l) = L, \quad \tilde{u}(r) = R, \end{cases}$$

where

$$0 < \tilde{\alpha} \le a(x) + c \le \tilde{\beta} < \infty \quad \text{for all } x \in [l,r].$$

It is evident that the difference

$$\tilde{u}(x)-u(x)=(R-L)\left\{\frac{\int_l^x (a(t)+c)^{-1/2}dt}{\int_l^r (a(t)+c)^{-1/2}dt}-\frac{\int_l^x a(t)^{-1/2}dt}{\int_l^r a(t)^{-1/2}dt}\right\}$$

can be made arbitrarily small by setting c small.

9.2 Stability of C^2-solutions

The problem is unresolved for viscosity solutions in several variables. To our knowledge, the only work on stability problems related to this one is by Lindgren and Lindqvist in [16]. They studied the stability of viscosity solutions of the infinity-Laplace equation with variable exponent. Applying a procedure similar to the one in [16] leads to a bound with uncontrollable terms. We suspect that this can be fixed by a clever choice of test function in the general version of Ishii's lemma, see Theorem 3.2 in [4]. This is a task which is beyond the scope of this thesis. We shall instead establish the desired bound for C^2-solutions. Although we have not shown existence of C^2-solutions of $\Delta_{\infty,A}\, u=0$, the bound at least supports the belief that also viscosity solutions are stable.

We remind that the matrices

$$A(x)=(a_{ij}(x)),\quad \tilde{A}(x)=A(x)+\Lambda=(a_{ij}(x)+c_{ij})$$

satisfy the assumptions in the beginning of this section.

Theorem 9.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose that $\tilde{u}\in C^2(\overline{\Omega})$ is a solution of*

$$\Delta_{\infty,\tilde{A}}\, \tilde{u}=0$$

and that $u\in C^2(\overline{\Omega})$ is a solution of

$$\Delta_{\infty,A}\, u=0$$

both having the same boundary values g. Then

$$||\tilde{u}-u||_{\infty,\Omega}\le \kappa\{(\max_{q,r}|c_{qr}|)^{1/5}+(\max_{q,r}|c_{qr}|)^{2/5}\},$$

for some constant κ.

Proof. First suppose that $\tilde{u}\in C^2(\overline{\Omega})$ is a classical subsolution of

$$\Delta_{\infty,\tilde{A}}\, \tilde{u}=0$$

and that $u\in C^2(\overline{\Omega})$ is a classical supersolution of

$$\Delta_{\infty,A}\, u=0,$$

both with boundary values g. In other words

$$\Delta_{\infty,\tilde{A}}\, \tilde{u}(x)\ge 0,\quad \Delta_{\infty,A}\, u(x)\le 0$$

holds at each point $x \in \Omega$. Furthermore assume that $u^+ \in C^2(\overline{\Omega})$ is a classical supersolution of the Upper equation:

$$\max\Big\{\varepsilon - \langle A(x)\nabla u^+, \nabla u^+\rangle^{1/2},\ \Delta_{\infty,A}\, u^+\Big\} \leq 0$$

and that $u^- \in C^2(\overline{\Omega})$ is classical subsolution of the Lower equation:

$$\min\Big\{\langle A(x)\nabla u^-, \nabla u^-\rangle^{1/2} - \varepsilon,\ \Delta_{\infty,A}\, u^-\Big\} \geq 0$$

both with boundary values g, such that

$$u^- \leq u \leq u^+, \quad ||u^+ - u^-||_\infty \leq C\varepsilon.$$

As in the proof of Lemma 8.1, we shall construct a strict supersolution $w = f(u^+)$ of the Upper equation:

$$\Delta_{\infty,A}\, w \leq -\mu < 0.$$

We use the function

$$f(t) = \frac{1}{\gamma}\ln(1 + B(e^{\gamma t} - 1)), \quad B > 1,\ \gamma > 0,\ t > 0$$

as an approximation of the identity. We have

$$\begin{aligned}
&0 < f(t) - t < \frac{B-1}{\gamma}\\
&0 < f'(t) - 1 < B - 1\\
&f''(t) = -\gamma(B-1)B^{-1}e^{-\gamma t}f'(t)^2.
\end{aligned}$$

By adding the same constant to $u^+, u, \tilde{u}$ and g, we may assume that $u^+ > 0$ and $g > 0$. Let $\varepsilon > 0$ and set $w = f(u^+)$. The estimate

$$\begin{aligned}
\tilde{u} - u &= (\tilde{u} - w) + (w - u^+) + (u^+ - u)\\
&< (\tilde{u} - w) + \frac{B-1}{\gamma} + C\varepsilon
\end{aligned} \tag{9.2}$$

follows. The last two terms can be made arbitrarily small by choosing B and ε, but the first term also depends on these, so this requires some work.

Lemma 9.2. *We have*

$$\max_{\Omega}(\tilde{u} - w) \leq \frac{C_1}{\varepsilon^4}\{\max_{q,r}|c_{qr}| + (\max_{q,r}|c_{qr}|)^2\},$$

for some constant C_1.

Proof. We proceed as in the proof of Lemma 8.1. In (8.1) we found

$$\Delta_{\infty,A}\, w = f'(u^+)^3\Delta_{\infty,A}\, u^+ + 2f'(u^+)^2 f''(u^+)\langle A(x)\nabla u^+, \nabla u^+\rangle^2.$$

Multiplying the Upper equation for supersolutions

$$\max\Big\{\varepsilon - \langle A(x)\nabla u^+, \nabla u^+\rangle^{1/2},\ \Delta_{\infty,A}\, u^+\Big\} \leq 0$$

by $f'(u^+)^3$ we obtain

$$\begin{aligned}\Delta_{\infty,A}\, w &\leq 2f'(u^+)^2 f''(u^+)\langle A(x)\nabla u^+, \nabla u^+\rangle^2 \\ &= -2\gamma(B-1)B^{-1}e^{-\gamma u^+} f'(u^+)^4\langle A(x)\nabla u^+, \nabla u^+\rangle^2 \\ &\leq -2\varepsilon^4\gamma(B-1)B^{-1}e^{-\gamma\|u^+\|_\infty} =: -\mu < 0,\end{aligned} \tag{9.3}$$

where we used

$$f'(u^+) > 1, \quad \langle A(x)\nabla u^+, \nabla u^+\rangle^{1/2} \geq \varepsilon.$$

Now let

$$\sigma = \max_{\overline{\Omega}}(\tilde{u} - w).$$

If the maximum is attained on the boundary, then

$$\sigma = \max_{\partial\Omega}(\tilde{u} - w) = \max_{\partial\Omega}(\tilde{u} - f(u^+)) = g - f(g) < 0,$$

and there is nothing to prove. Assume thus that the maximum is attained at some interior point $x_0 \in \Omega$:

$$\sigma = \max_{\Omega}(\tilde{u} - w) = \tilde{u}(x_0) - w(x_0).$$

Then

$$\begin{aligned}\nabla\tilde{u}(x_0) &= \nabla w(x_0) \\ D^2\tilde{u}(x_0) &\leq D^2 w(x_0)\end{aligned}$$

by the infinitesimal calculus. Recall the notation

$$\begin{aligned}\tilde{A}(x) &= A(x) + \Lambda = (a_{ij}(x) + c_{ij}) \\ \Delta_{\infty,A}\, v(x) &= \mathcal{L}_{\infty,A}(x, \nabla v(x), D^2 v(x)).\end{aligned}$$

At the maximum point x_0 we have

$$\begin{aligned}0 &\leq \Delta_{\infty,\tilde{A}}\, \tilde{u}(x_0) \\ &= \mathcal{L}_{\infty,\tilde{A}}(x_0, \nabla\tilde{u}(x_0), D^2\tilde{u}(x_0)) \\ &= \mathcal{L}_{\infty,\tilde{A}}(x_0, \nabla w(x_0), D^2\tilde{u}(x_0)) \\ &\leq \mathcal{L}_{\infty,\tilde{A}}(x_0, \nabla w(x_0), D^2 w(x_0)) \\ &= \mathcal{L}_{\infty,A}(x_0, \nabla w(x_0), D^2 w(x_0)) + \Big\langle \nabla_x\langle A(x_0)\nabla w(x_0), \nabla w(x_0)\rangle, \Lambda\nabla w(x_0)\Big\rangle \\ &\quad + 4\langle D^2 w(x_0)A(x_0)\nabla w(x_0), \Lambda\nabla w(x_0)\rangle + 2\langle D^2 w(x_0)\Lambda\nabla w(x_0), \Lambda\nabla w(x_0)\rangle,\end{aligned}$$

by Proposition 3.3. Moreover

$$-\mu \geq \Delta_{\infty,A}\, w(x_0) = \mathcal{L}_{\infty,A}(x_0, \nabla w(x_0), D^2 w(x_0)).$$

By subtracting the last equation from the first we obtain

$$\begin{aligned}\mu \le &\Big\langle \nabla_x \langle A(x_0)\nabla w(x_0), \nabla w(x_0)\rangle, \Lambda \nabla w(x_0)\Big\rangle \\ &+ 4\langle D^2 w(x_0)A(x_0)\nabla w(x_0), \Lambda\nabla w(x_0)\rangle \\ &+ 2\langle D^2 w(x_0)\Lambda\nabla w(x_0), \Lambda\nabla w(x_0)\rangle.\end{aligned} \tag{9.4}$$

We estimate each term on the right-hand side separately. For the first we find

$$\begin{aligned}&\Big|\Big\langle \nabla_x \langle A(x_0)\nabla w(x_0), \nabla w(x_0)\rangle, \Lambda \nabla w(x_0)\Big\rangle\Big| \\ &= \Big|\sum_{i,j,k,l=1}^{n} \frac{\partial a_{ij}}{\partial x_k}(x_0)\frac{\partial w}{\partial x_i}(x_0)\frac{\partial w}{\partial x_j}(x_0)c_{kl}\frac{\partial w}{\partial x_l}(x_0)\Big| \\ &\le \max_{q,r}|\nabla a_{qr}(x_0)|\max_{q,r}|c_{qr}|\sum_{i,j,k,l=1}^{n}\Big|\frac{\partial w}{\partial x_i}(x_0)\frac{\partial w}{\partial x_j}(x_0)\frac{\partial w}{\partial x_l}(x_0)\Big| \\ &\le n^3 \max_{q,r}|\nabla a_{qr}(x_0)|\max_{q,r}|c_{qr}||\nabla w(x_0)|^3,\end{aligned}$$

where we used (3.9). For the second term we have

$$\begin{aligned}&|\langle D^2 w(x_0)A(x_0)\nabla w(x_0), \Lambda\nabla w(x_0)\rangle| \\ &= \Big|\sum_{i,j,k,l=1}^{n} a_{ik}(x_0)\frac{\partial w}{\partial x_k}(x_0)\frac{\partial^2 w}{\partial x_i \partial x_j}(x_0)c_{jl}\frac{\partial w}{\partial x_l}(x_0)\Big| \\ &\le \max_{q,r}|a_{qr}(x_0)|\max_{q,r}\Big|\frac{\partial^2 w}{\partial x_q\partial x_r}(x_0)\Big|\max_{q,r}|c_{qr}|\sum_{i,j,k,l=1}^{n}\Big|\frac{\partial w}{\partial x_k}(x_0)\frac{\partial w}{\partial x_l}(x_0)\Big| \\ &\le n^3\max_{q,r}|a_{qr}(x_0)|\max_{q,r}\Big|\frac{\partial^2 w}{\partial x_q\partial x_r}(x_0)\Big|\max_{q,r}|c_{qr}||\nabla w(x_0)|^2,\end{aligned}$$

by Young's inequality. The estimate for the third term is identical with the above, except that a_{ik} is replaced by c_{ik} in the sum, hence

$$\begin{aligned}&|\langle D^2 w(x_0)\Lambda\nabla w(x_0), \Lambda\nabla w(x_0)\rangle| \\ &\le n^3(\max_{q,r}|c_{qr}|)^2\max_{q,r}\Big|\frac{\partial^2 w}{\partial x_q\partial x_r}(x_0)\Big||\nabla w(x_0)|^2.\end{aligned}$$

Inserting these three estimates into (9.4) gives

$$\begin{aligned}\mu \le n^3\Big(&\max_{q,r}|\nabla a_{qr}(x_0)||\nabla w(x_0)|^3 \\ &+ 4\max_{q,r}|a_{qr}(x_0)|\max_{q,r}\Big|\frac{\partial^2 w}{\partial x_q\partial x_r}(x_0)\Big||\nabla w(x_0)|^2\Big)\max_{q,r}|c_{qr}| \\ &+ 2n^3\max_{q,r}\Big|\frac{\partial^2 w}{\partial x_q\partial x_r}(x_0)\Big||\nabla w(x_0)|^2(\max_{q,r}|c_{qr}|)^2.\end{aligned}$$

Recall that

$$\mu = 2\varepsilon^4\gamma(B-1)B^{-1}e^{-\gamma||u^+||_\infty}.$$

Set

$$\gamma = \frac{1}{||u^+||_\infty},$$

then fix $B > 1$ such that

$$\frac{B-1}{\gamma} = \sigma,$$

where we had

$$\sigma = \max_\Omega(\tilde{u} - w).$$

Now we have

$$\mu = \frac{2\varepsilon^4\sigma}{Be||u^+||^2_\infty},$$

thus

$$\begin{aligned}\sigma \leq \frac{Be||u^+||^2_\infty n^3}{2\varepsilon^4}\Big(&\max_{q,r}|\nabla a_{qr}(x_0)||\nabla w(x_0)|^3 \\ &+ 4\max_{q,r}|a_{qr}(x_0)|\max_{q,r}\left|\frac{\partial^2 w}{\partial x_q \partial x_r}(x_0)\right||\nabla w(x_0)|^2\Big)\max_{q,r}|c_{qr}| \\ &+ \frac{Be||u^+||^2_\infty n^3}{\varepsilon^4}\max_{q,r}\left|\frac{\partial^2 w}{\partial x_q \partial x_r}(x_0)\right||\nabla w(x_0)|^2(\max_{q,r}|c_{qr}|)^2 \\ \leq \frac{C_1}{\varepsilon^4}\{&\max_{q,r}|c_{qr}| + (\max_{q,r}|c_{qr}|)^2\}.\end{aligned}$$

where the constant C_1 can be choosen to be independent of ε. □

Continuing to estimate (9.2) we obtain

$$\begin{aligned}\tilde{u} - u &\leq \sigma + \sigma + C\varepsilon \\ &\leq \frac{2C_1}{\varepsilon^4}\{\max_{q,r}|c_{qr}| + (\max_{q,r}|c_{qr}|)^2\} + C\varepsilon,\end{aligned}$$

by the previous lemma. If $\max_{q,r}|c_{qr}| \leq 1$ we can find a constant C_2 such that

$$\tilde{u} - u \leq \frac{C_2}{\varepsilon^4}\max_{q,r}|c_{qr}| + C\varepsilon.$$

Now we let $\varepsilon > 0$ be such that the right-hand side attains its minimum. Thus

$$\varepsilon = \left(\frac{4C_2}{C}\max_{q,r}|c_{qr}|\right)^{1/5}$$

by the infinitesimal calculus. Inserting this into the above yields a bound like

$$\tilde{u} - u \leq \kappa_1(\max_{q,r}|c_{qr}|)^{1/5},$$

for some constant κ_1. Similarly, if $\max_{q,r}|c_{qr}| > 1$ we find a bound of the form

$$\tilde{u} - u \leq \kappa_2(\max_{q,r}|c_{qr}|)^{2/5},$$

where κ_2 is a constant. Combining the two cases we arrive at the desired bound

$$\tilde{u} - u \leq \kappa\{(\max_{q,r} |c_{qr}|)^{1/5} + (\max_{q,r} |c_{qr}|)^{2/5}\},$$

for some new constant κ. This concludes the proof when $\tilde{u}$ is a subsolution and u is a supersolution.

Now suppose that $\tilde{u} \in C^2(\overline{\Omega})$ is a classical supersolution:

$$\Delta_{\infty,\tilde{A}}\, \tilde{u}(x) \leq 0$$

and that $u \in C^2(\overline{\Omega})$ is a classical subsolution:

$$\Delta_{\infty,A}\, u(x) \geq 0$$

both with boundary values g. We could have done the same procedure as above for $u - \tilde{u}$, by using the Lower equation. Instead, observe that

$$u - \tilde{u} = (k - \tilde{u}) - (k - u),$$

where the constant k is so large that $k - u > 0$. We find

$$\Delta_{\infty,\tilde{A}}\, (k - \tilde{u}(x)) \geq 0$$
$$\Delta_{\infty,A}\, (k - u(x)) \leq 0,$$

and we have reduced the problem to the previous case. This concludes the proof. □

Bibliography

[1] G. Aronsson. Extension of functions satisfying Lipschitz conditions. *Arkiv för Matematik*, 6(28):551–561, 1967.

[2] G. Aronsson. On the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}$. *Arkiv för Matematik*, 7(28):397–425, 1968.

[3] R. Bellman. *Introduction to matrix analysis.* McGraw-Hill, New York, 1960.

[4] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society*, 27:1–67, 1992.

[5] B. Dacorogna. *Direct methods in the calculus of variations.* Springer, Berlin, 1989.

[6] E. DiBenedetto. *Real analysis.* Birkhäuser, Boston, 2002.

[7] L. C. Evans. *Partial differential equations.* American Mathematical Society, Providence, R.I, 1998.

[8] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions.* CRC press, Boca Raton, 1992.

[9] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order, 2nd edition.* Springer, Berlin, 1983.

[10] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations.* Clarendon Press, Oxford, 1993.

[11] H. Holden. Tools from the toolbox. Functional analysis for partial differential equations. Trondheim, 2014.

[12] R. Jensen. Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient. *Archive for Rational Mechanics and Analysis*, 123(1):51–74, 1993.

[13] P. Juutinen. *Minimization problems for Lipschitz functions via viscosity solutions.* Annales Academiae Scientiarum Fennicae Mathematica. Dissertationes 115. Suomalainen tiedeakatemia, Helsinki, 1998.

[14] S. Koike. *A beginner's guide to the theory of viscosity solutions.* MSJ memoirs, 13, Mathematical Society of Japan, Tokyo, 2004.

[15] P. D. Lax. *Functional analysis.* Wiley-Interscience, New York, 2002.

[16] E. Lindgren and P. Lindqvist. Stability for the infinity-Laplace equation with variable exponent. *Differential And Integral Equations*, 25:589–600, 2012.

[17] P. Lindqvist. *Notes on the p-Laplace equation.* University of Jyväskylä, Department of Mathematics and Statistics, Jyväskylä, 2006.

[18] P. Lindqvist. Notes on the infinity-Laplace equation. Trondheim, 2014.

[19] P. Lindqvist. Regularity of supersolutions. Trondheim, 2015.

[20] P. Lindqvist and T. Lukkari. A curious equation involving the ∞-Laplacian. *Advances in Calculus of Variations*, 3:409–421, 2010.

[21] M. H. Protter and H. F. Weinberger. *Maximum principles in differential equations.* Springer, New York, 1984.