

# Fault-tolerant control allocation with actuator dynamics: finite-time control reconfiguration

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**Abstract**—This paper focuses on fault tolerant control allocation for overactuated systems with actuator dynamics. The proposed scheme for fault detection and isolation is based on unknown input observers and the main contribution of the paper consists in the presentation of a finite-time control reconfiguration technique which provides a successful recovering of system performances in spite of actuator faults. Simulation results support theoretical developments.

## I. INTRODUCTION

The main objective of control allocation is to determine how to generate a specified control effect from a redundant set of actuators and effectors. Due to input redundancy, several configurations leading to the same generalized force are admissible and for this reason the control allocation scheme commonly incorporates additional secondary objectives [2] [12] [16], such as power or fuel consumption minimization. On the other hand, usually there are also some limitation factors to be accounted for [3] [10] [15] [21]: actuators/effectors dynamics, input saturation and other physical or operational constraints. One further advantage of actuator and effector redundancy is the possibility to reconfigure the control in order to cope with unexpected changes on the system dynamics, such as failures or malfunctions: in particular if the set of actuators and effectors is partially affected by faults, one can modify the control allocation scheme by preventing the use of inefficient/ineffective devices in the generation of control effect or compensating for the loss of efficiency. However, one key point for successfully re-allocating the control is the availability of adequate information about the faults that have occurred; indeed, some accurate fault estimation and/or a correct identification of the faulty actuators or effectors are necessary to address the reconfiguration. Recent results toward fault tolerant control allocation are based on sliding-mode techniques [7] [14], adaptive control strategies [5] [18] [19] and unknown input observers [8] [9]. Further investigations on this topic, with a more application-oriented character, are proposed for reconfiguration in flight control [4], [22], [20] and fault accommodation in automated underwater vehicles [17]. An interesting survey on the general problem of fault-tolerant control reconfiguration is provided in [23].

The aim of this paper is to present the extension of the results

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on fault detection/isolation/accommodation proposed in [8] [9] to systems with a redundant set of inputs in the presence of actuator dynamics. Faults are modeled as multiplicative terms which lead to slow response of actuators, i.e. rate limitation. The paper is structured as follows. In Section II the basic setup of control allocation with actuator dynamics is introduced, while Section III is devoted to the presentation of the proposed method for fault detection/isolation based on unknown input observers, assuming that the actuator state is not directly measured; for sake of simplicity, the output of the plant is assumed to be the whole state. However, such assumption can be relaxed and methods proposed in [9] can be used to tackle the case of systems with unmeasured states and inputs. Section IV reports the main results of the paper: first the steady-state effects of the faults are estimated, then a finite-time actuator reconfiguration strategy is defined and finally the fault accommodation of the overall system is proved. In Section V the validation of theoretical results is provided by numerical tests.

## II. CONTROL ALLOCATION SETUP

Let us consider the linear system  $\Sigma = (A, B, G)$  :

$$\begin{cases} \dot{x}(t) = Ax(t) + B\tau(t) \\ x(0) = x_0 \end{cases} \quad (1)$$

with

$$\tau(t) = Gu(t), \quad (2)$$

where  $x \in \mathbb{R}^n$ ,  $\tau \in \mathbb{R}^k$ ,  $u \in \mathbb{R}^m$ ,  $k < m$  and  $B, G$  are assumed to be full-rank. The vector  $u(t)$  represents the redundant control input and  $\tau(t)$  is the generalized control effect or virtual input; the two variables are related through the matrix  $G \in \mathbb{R}^{k \times n}$ . The input  $u(t)$  (actuator state) is assigned by the following actuator dynamics:

$$\dot{u}(t) = Z(t)u(t) + v(t), \quad (3)$$

where  $Z = \text{diag}(z_1, \dots, z_m)$  and  $v(\cdot)$  is a free parameter to be tuned (actuator input).

**Assumption II.1** *The actuator dynamics is asymptotically stable, i.e.  $Z$  is Hurwitz.*

**Assumption II.2** *The state  $x(t)$  of the system is supposed to be measured. The actuator state  $u(t)$  is instead assumed to be not known and only the value of the initial configuration  $u(0)$  is available together with the actuator input  $v(t)$ .*

**Remark II.1** *Since the actuator dynamics is asymptotically stable, possible errors on the initial configuration  $u(0)$  can be easily handled in the proposed approach as they only provide transient terms and do not affect the asymptotic response of the actuators.*

Without loss of generality, the desired control effect  $\tau_c(t)$  is assumed to be given as a linear feedback:

$$\tau_c(t) = Kx + Lw, \quad (4)$$

where  $w$  is the state of a suitable exogenous system to be tracked:

$$\dot{w} = Sw. \quad (5)$$

In particular  $K, L$  can be determined by

$$L = Y - K\Pi,$$

where  $(\Pi, Y)$  is the solution to the Francis equations [11] associated to the regulation of the output  $y = Cx + Dw$ :

$$\begin{aligned} \Pi S &= A\Pi + BY \\ 0 &= C\Pi + D \end{aligned}$$

A control allocation strategy is defined such that, whenever it is possible, the actuator input  $v(t)$  in (3) ensures that the actuator state  $u(t)$  satisfies

$$Gu(t) \simeq \tau_c(t). \quad (6)$$

Although the linear equation  $Gu_c = \tau_c$  always admits (uncountable) exact solutions when  $\text{rank}(G) = k$ , a simple solution can be obtained using the right pseudo-inverse matrix [13]:

$$u_c = G^{-R}\tau_c, \quad G^{-R} := G^T(GG^T)^{-1}. \quad (7)$$

In this paper we consider the class of faults acting on effectors and actuators efficiency by changing their effectiveness: these can be modeled by a multiplicative term  $\Delta(t)$  which limit the rate of the actuators, i.e.  $\Delta(t) = \text{diag}(\delta_1(t), \dots, \delta_m(t))$  for some unknown functions  $\delta_i(t) \in [0, 1]$ , which are called efficiency factors. The overall actuator dynamics to be considered is therefore

$$\dot{u}(t) = \Delta(t)Zu(t) + v \quad (8)$$

It follows that, whenever  $\delta_i(t) \equiv 1 \forall i = 1, \dots, m$ , the controller operates with nominal conditions and hence  $u(t) = u_c(t)$ , i.e.

$$\tau(t) = Gu(t) \simeq \tau_c(t)$$

On the other hand if one of the actuators is subject to a loss of effectiveness or complete failure, i.e. if  $\delta_i(t) \neq 1$  for some  $i$ , the designed control law may no longer be able to ensure the desired effect, this meaning that, in the case of fault presence, one may have  $\tau(t) \neq \hat{\tau}_c(t)$  with a consequent deterioration of system performances. Such problems can be avoided by accommodating the fault effects if a suitable control reconfiguration policy is considered. Moreover, it can be noticed that the case of faults affecting the actuator input  $v(t)$ , as well as actuator total failures, can be tackled using the framework proposed in [9].

Summarizing, throughout the paper we will deal with the system resulting from the coupling of (1), (2), (5) and (8) with desired control effect assigned by (4).

### III. FAULT DETECTION AND ISOLATION

Define the input observer

$$\begin{cases} \dot{\hat{u}}(t) = Z\hat{u}(t) + v(t) \\ \hat{u}(0) = u(0) \end{cases} \quad (9)$$

Let us suppose that the actuators  $i_1, \dots, i_q$  are faulty, that is  $\delta_j(t) \neq 1$  for  $j = i_1, \dots, i_q$ ; due to the diagonal form of the system (8), the following identity can be verified

$$u(t) - \hat{u}(t) = [0_{i_1-1} \star 0_{i_2-i_1-1} \star \dots \star 0_{i_q-i_{q-1}-1} \star 0_{m-i_q}]^T,$$

where  $\star$  denotes a non-null quantity. In particular the fault-free components of  $u(t)$  are equal to those of  $\hat{u}(t)$  and moreover the matrix  $W = BG$  satisfies

$$\begin{aligned} B(Gu(t) - G\hat{u}(t)) &= W(u(t) - \hat{u}(t)) \\ &= \sum_{\ell=1}^q W_{i_\ell}(u_{i_\ell}(t) - \hat{u}_{i_\ell}(t)), \end{aligned} \quad (10)$$

where  $W_j$  stands for the  $j^{\text{th}}$  column of the matrix  $W$ .

Consider the following Unknown Input Observer UIO [6] of the state  $x(t)$ :

$$\begin{cases} \dot{z}(t) = Fz(t) + RBG\hat{u}(t) + Nx(t) \\ \hat{x}(t) = z(t) + Hx(t) \end{cases}$$

Thanks to (10), exploiting the observer structure and setting  $N = N_1 + N_2$ , if the following conditions are satisfied

$$R = I_{n \times n} - H \quad (11)$$

$$F = RA - N_1, \quad \sigma(F) \in \mathbb{C}^- \quad (12)$$

$$N_2 = FH \quad (13)$$

then the equation of the error  $e(t) = x(t) - \hat{x}(t)$  reduces to

$$\dot{e}(t) = Fe(t) + R \sum_{\ell=1}^q W_{i_\ell}(u_{i_\ell}(t) - \hat{u}_{i_\ell}(t)), \quad (14)$$

where  $\sigma(\cdot)$  stands for the spectrum of a matrix and the set  $\mathbb{C}^-$  in the left open complex half-plane.

Mimicking the approaches presented in [8] and [9], a family of unknown input observers for system (1) can be designed in order to detect and isolate the faults affecting the actuator dynamics (8). To this purpose, with respect to the matrix  $W = BG \in \mathbb{R}^{n \times m}$ , we call *uniform sub-rank* the integer  $k_0 \leq k$  computed as follows:

$$k_0 := \max \left\{ p \leq k : \text{rank}[W_{j_1} \dots W_{j_p}] = p, \right. \\ \left. \forall J = (j_1, \dots, j_p) : j_\ell \in \{1, \dots, m\} \right\}. \quad (15)$$

**Theorem III.1** [9] *Let  $k_0 \leq k$  be the uniform sub-rank of the matrix  $W = BG$  and set  $s_0 = \binom{m}{k_0}$ .*

i) *For any multi-index  $J_h$  of length  $k_0$ ,  $h = 1, \dots, s_0$ , an unknown input observer  $\{\mathcal{O}_h\} = \{F^{(h)}, R^{(h)}, N^{(h)}, H^{(h)}\}$  can be designed such that conditions (11)-(13) are satisfied and moreover*

$$R^{(h)}W_{J_h} = 0, \quad R^{(h)}W \neq 0. \quad (16)$$

ii) *Combining together the information provided by all residuals  $e^{(h)}(t)$ ,  $h = 1, \dots, s_0$ , it is possible to detect and isolate up to  $k_0 - 1$  faults simultaneously affecting the system.*

The residual  $e^{(h)}(t)$  is said to be active, this meaning that some faults have occurred, if it overpass a known threshold  $\epsilon^{(h)}$ , i.e.

$$\begin{cases} \|e^{(h)}(t)\| \leq \epsilon^{(h)} \Leftarrow \text{no faults} \\ \|e^{(h)}(t)\| > \epsilon^{(h)} \Rightarrow \text{presence of faults.} \end{cases}$$

The threshold  $\epsilon^{(h)}$  can be computed or tuned based on known bounds for observer initialization error as well as for possible disturbance or noise terms affecting the plant [9].

#### IV. CONTROL RECONFIGURATION

Suppose that, through the FDI module described in Theorem III.1, faults in the actuators  $i_1, \dots, i_q$  have been identified. Let us group into the reduced-order vector  $\tilde{u}(t) \in \mathbb{R}^{m-q}$  the fault-free entries of  $u(t)$ , corresponding to any  $u_i(t)$  with  $i \neq i_\ell$ ,  $\ell = 1, \dots, q$ . Assume that  $k \leq m - q = \tilde{m}$  and define the reduced-order matrices  $\tilde{Z} \in \mathbb{R}^{\tilde{m} \times \tilde{m}} = \text{diag}(\tilde{z}_1, \dots, \tilde{z}_{\tilde{m}})$ ,  $\tilde{G} \in \mathbb{R}^{k \times \tilde{m}}$  obtained from  $Z$  and  $G$  by neglecting the columns corresponding to the positions  $i_1, \dots, i_q$ . Assuming that  $\text{rank}(\tilde{G}) = \text{rank}(G) = k$ , the desired actuator state can be updated as follows:

$$\tilde{u}_c(t) = \tilde{G}^{-R} \tau_c(t). \quad (17)$$

We have to deal with the following decoupled dynamics

$$\dot{u}_{i_\ell}(t) = \delta_{i_\ell}(t) z_{i_\ell} u_{i_\ell}(t) + v_{i_\ell}(t), \quad \ell = 1, \dots, q$$

and

$$\dot{\tilde{u}}(t) = \tilde{Z}\tilde{u}(t) + \tilde{v}(t).$$

The basic idea is to compensate the effect of the inputs  $u_{i_\ell}(t)$  while simultaneously the input  $\tilde{u}$  tracks the desired actuator state  $\tilde{u}_c$  in finite-time [1]. We set

$$v_{i_\ell}(t) = 0 \quad (18)$$

and, for  $r \in (0, 1)$ ,

$$\tilde{v}(t) = -\tilde{Z}\tilde{u}_c(t) + \dot{\tilde{u}}_c(t) - \xi \frac{\tilde{u}(t) - \tilde{u}_c(t)}{\|\tilde{u}(t) - \tilde{u}_c(t)\|^{1-r}}, \quad (19)$$

with  $\xi > 0$  and where, by construction, the exact knowledge of  $\tilde{u}(t)$  is provided by the observer (9). It is worth to note that, due to (4)-(5) and (17), the derivative  $\dot{\tilde{u}}_c(t)$  can be computed without differentiating signals.

The asymptotic recovery of the system performances will be proved in three steps:

- The steady-state of the faulty inputs is computed with the scope of designing a correction factor  $\tau_\delta(t)$ .
- The input (19) is proved to guarantee finite-time convergence of the reconfigured actuator state  $\tilde{u}(t)$  to the reconfigured desired actuator state  $\tilde{u}_c(t)$ .
- The fault accommodation of the total system is shown to be enforced: the state of the faulty plant converges to the state of the nominal (fault-free) system.

#### A. Evaluation of faulty inputs steady-state

Setting  $v_{i_\ell} = 0$  as in (18), the dynamics of faulty actuators reduces to the free evolution:

$$\dot{u}_{i_\ell}(t) = \delta_{i_\ell}(t) z_{i_\ell} u_{i_\ell}(t), \quad \ell = 1, \dots, q. \quad (20)$$

Two possible behaviors for the steady-state of  $u_{i_\ell}(t)$  exist, depending on the rate of the fault  $\delta_{i_\ell}(t)$ . To this end, we give the following definition.

**Definition IV.1** *The efficiency factor  $\delta(t) \geq 0$  is said to be a low-degrade fault if*

$$\int_0^{+\infty} \delta(\sigma) d\sigma = +\infty.$$

*Conversely, if the above integral is a finite quantity, we will refer to  $\delta(t)$  as to a high-degrade fault.*

**Proposition IV.1** *Let us consider the free-input actuator dynamics (20) with  $u(0) \neq 0$ . Two cases are admissible:*

- if the factor  $\delta_{i_\ell}(t)$  is a low-degrade fault then  $\lim_{t \rightarrow +\infty} u_{i_\ell}(t) = 0$ ;*
- if  $\delta_{i_\ell}(t)$  is a high-degrade fault then there exists  $\bar{u}_{i_\ell} \in \mathbb{R}$ ,  $\bar{u}_{i_\ell} \neq 0$ , such that  $\lim_{t \rightarrow +\infty} u_{i_\ell}(t) = \bar{u}_{i_\ell}$ .*

*Proof:* Let us prove case a) first. The solution of equation (20) can be expressed by

$$u_{i_\ell}(t) = e^{z_{i_\ell} \int_0^t \delta_{i_\ell}(\sigma) d\sigma} u_{i_\ell}(0);$$

since by definition  $z_i < 0 \forall i = 1, \dots, m$ , if  $\delta_{i_\ell}(t)$  is a low-degrade fault, the exponent  $z_{i_\ell} \int_0^t \delta_{i_\ell}(\sigma) d\sigma$  in the above formula tends to  $-\infty$  and, consequently,  $u_{i_\ell}(t)$  converges to zero. On the other hand, in the case b) of high-degrade fault one has

$$\int_0^{+\infty} \delta_{i_\ell}(\sigma) d\sigma = c_\ell > 0$$

and hence the identity  $\lim_{t \rightarrow +\infty} u_{i_\ell}(t) = \bar{u}_{i_\ell}$  holds with  $\bar{u}_{i_\ell} = e^{z_{i_\ell} c_\ell} u_{i_\ell}(0)$ . ■

**Remark IV.1** *A low-degrade fault can be interpreted as a partial loss of actuator efficiency, while actuators affected by high-degrade faults are subject to complete failure.*

**Remark IV.2** *It is worth to note that, as the actuator dynamics is ruled by the Hurwitz matrix  $Z$ , the stability of the total system may not be compromised by the transient of faulty actuators.*

The first issue of the proposed FTC scheme to be addressed is the estimation of the steady-state  $\{\bar{u}_{i_1}, \dots, \bar{u}_{i_q}\}$  of fault-inputs. This can be achieved using the information provided by the residual signals  $e^{(h)}(t)$  in (14), (16); in particular, according to Theorem III.1, let us fix  $h \in \{1, \dots, s_0\}$  such that

$$R^{(h)} W_{i_\ell} \neq 0 \quad \forall \ell = 1, \dots, q. \quad (21)$$

The input  $u_{i_\ell}(t)$  can be decomposed as  $u_{i_\ell}(t) = \bar{u}_{i_\ell} + u_{i_\ell}^\dagger(t)$ , where  $u_{i_\ell}^\dagger(t)$  is the transient. Let us set  $\bar{q}$  as

$$\bar{q} = \text{rank}[W_{i_1} \ \cdots \ W_{i_q}], \quad \bar{q} \leq q.$$

**Proposition IV.2** Suppose that condition (21) holds true and choose  $\bar{W} \in \mathbb{R}^{n \times \bar{q}}$  such that  $\text{Im}(\bar{W}) = \text{span}(W_{i_1} \cdots W_{i_q})$ . Then there exist a constant vector  $\bar{u}^* \in \mathbb{R}^{\bar{q}}$  such that

$$\sum_{\ell=1}^q G_{i_\ell} \bar{u}_{i_\ell} = B^{-L} \bar{W} \bar{u}^*,$$

where  $B^{-L}$  stands for the left pseudo-inverse of  $B$ .

*Proof:* The proof is straightforward observing that, by construction,  $\text{span}(W_{i_1} \cdots W_{i_q}) = \text{span}(BG_{i_1} \cdots BG_{i_q})$  and hence  $\bar{G} \in \mathbb{R}^{k \times \bar{q}}$  can be found such that  $\bar{W} = B\bar{G}$ . ■

**Theorem IV.1** Let  $\bar{u}^* \in \mathbb{R}^{\bar{q}}$  be the constant vector defined in Proposition IV.2 and denote by  $t_{\#} > 0$  the time instant of control reconfiguration initialization. Then the following identity holds

$$\lim_{t \rightarrow +\infty} Q(t, t_{\#})^{-L} \left( \epsilon^{(h)}(t) - e^{F^{(h)}t} \epsilon^{(h)}(t_{\#}) \right) = \bar{u}^*, \quad (22)$$

where

$$Q(t, t_{\#}) = \left( \int_{t_{\#}}^t e^{F^{(h)}(t-\sigma)} d\sigma \right) R^{(h)} \bar{W}.$$

*Proof:* Let us omit the superscript  $(h)$  in order to improve the readability. We notice that, by construction, the overall input matrix  $\bar{W} \in \mathbb{R}^{n \times \bar{q}}$  associated to  $\bar{u}^*$  is full-rank, and therefore the left pseudo-inverse  $Q(t, t_{\#})^{-L}$  is well-defined. Since  $u_{c,i_\ell}(t) = 0$  for  $t \geq t_{\#}$ , integrating equation (14), the error  $\epsilon^{(h)}(t) = \epsilon(t)$  can be expressed as

$$\begin{aligned} \epsilon(t) &= e^{Ft} \epsilon(t_{\#}) + \int_{t_{\#}}^t e^{F(t-\sigma)} R \sum_{\ell=1}^q W_{i_\ell} u_{i_\ell}(\sigma) d\sigma \\ &= e^{Ft} \epsilon(t_{\#}) + \int_{t_{\#}}^t e^{F(t-\sigma)} R^{(h)} \bar{W} \bar{u}^* d\sigma \\ &\quad + \int_{t_{\#}}^t e^{F(t-\sigma)} R \sum_{\ell=1}^q W_{i_\ell} u_{i_\ell}^\dagger(\sigma) d\sigma \\ &= e^{Ft} \epsilon(t_{\#}) + Q(t, t_{\#}) \bar{u}^* \\ &\quad + \int_{t_{\#}}^t e^{F(t-\sigma)} R \sum_{\ell=1}^q W_{i_\ell} u_{i_\ell}^\dagger(\sigma) d\sigma. \end{aligned}$$

By construction  $F$  is a Hurwitz matrix and  $u_{i_\ell}^\dagger(t)$  converges to zero, and hence the last term in the right-hand side is asymptotically null. Observing that  $Q(t, t_{\#})^{-L}$  is bounded, the identity (22) is proved. ■

### B. Finite-time convergence

Consider the control law (19). By construction, using (4), (5) and (1) with  $\tau(t) = \tau_c(t)$ , the derivative of  $\tilde{u}_c(t)$  can be expressed as

$$\dot{\tilde{u}}_c(t) = K_1 x(t) + L_1 w(t) + B_1 \tilde{u}(t), \quad (23)$$

where, for sake of clarity, we have set

$$K_1 := G^{-R} K A, \quad L_1 := G^{-R} L S, \quad B_1 = G^{-R} K B G. \quad (24)$$

**Theorem IV.2** Assume  $u_{i_\ell}(t) \equiv 0$  for  $t \geq t_{\#}$ ,  $\ell = 1, \dots, q$ . Given the exogenous signal  $w(t)$ , the control law (19), (23)-(24) guarantees that  $\tilde{u}(t)$  converges to  $\tilde{u}_c(t)$  in finite time. In particular, for any initial condition  $\tilde{u}(t_{\#})$ , there exists  $T > t_{\#}$  such that the input  $\tilde{u}(t)$  driven by the control

$$\begin{aligned} \tilde{v}(t) &= -\tilde{Z} \tilde{u}_c(t) + K_1 x(t) + L_1 w(t) \\ &\quad + B_1 \tilde{u}(t) - \xi \frac{\tilde{u}(t) - \tilde{u}_c(t)}{\|\tilde{u}(t) - \tilde{u}_c(t)\|^{1-r}} \end{aligned} \quad (25)$$

satisfies

$$\|\tilde{u}(t) - \tilde{u}_c(t)\| \equiv 0 \quad \forall t \geq T.$$

*Proof:* Set  $h(t) := \tilde{u}(t) - \tilde{u}_c(t)$  and consider the Lyapunov function  $V(h) = \frac{1}{2} h^T h$ . Computing the derivative and using (25) one gets

$$\dot{V}(h(t)) = \dot{h}(t)^T h(t) \leq -\xi \|h(t)\|^{1+r} = -2^{\frac{1+r}{2}} \xi (V(t))^{\frac{1+r}{2}}.$$

By a standard argument from the theory of finite-time stabilization, the time horizon  $T$  can be evaluated as follows:

$$T = t_{\#} + \frac{2^{\frac{1-r}{2}}}{(1-r)\xi} (V(t_{\#}))^{\frac{1-r}{2}}. \quad \blacksquare$$

### C. Fault accommodation

In order to compensate for the overall input supply of faulty actuators, the control effect is updated as follows:

$$\tilde{\tau}_c(t) = \tau_c(t) + \tau_\delta(t),$$

with  $\tau_\delta(t)$  to be tuned. Referring to (22), let us set

$$\eta(t) := Q(t, t_{\#})^{-L} \left( \epsilon^{(h)}(t) - e^{F^{(h)}t} \epsilon^{(h)}(t_{\#}) \right)$$

and define

$$\tau_\delta(t) := B^{-L} \bar{W} \eta(t), \quad u_\delta(t) = \tilde{G}^{-R} \tau_\delta(t).$$

Let us consider a copy of the original system as a reference for the fault-free nominal dynamics:

$$\begin{cases} \dot{x}^*(t) = A x^*(t) + B \tau_c^*(t) \\ x^*(t_{\#}) = x(t_{\#}) \end{cases}$$

with  $\tau_c^*(t) := K x^*(t) + L w(t)$ .

**Theorem IV.3** Set the actuator inputs commanded to the faulty actuators equal to zero, i.e.  $v_{i_\ell} \equiv 0$  for  $t \geq t_{\#}$ ,  $\ell = 1, \dots, q$ . Define

$$\tilde{v}_b(t) = (\gamma - 1) \tilde{Z} (\tilde{u}(t) - \tilde{u}_c(t)) + (\gamma \tilde{Z} + B_1) u_\delta(t) + \tilde{v}(t),$$

where  $\gamma > 0$ ,  $B_1$  is the matrix defined in (24) and  $\tilde{v}(t)$  is given by (25). Then, referring to the system (1),(2),(4),(8) where the actuators are driven by  $\tilde{v}_b(t)$  and choosing  $\gamma$  such that  $\det(\gamma \tilde{Z} + B_1) \neq 0$ , the following condition is satisfied:

$$\lim_{t \rightarrow +\infty} \|x(t) - x^*(t)\| = 0.$$

*Proof:* For sake of simplicity and without loss of generality it can be assumed  $\gamma = 1$ : see Remark IV.3 for further details. Set

$$s(t) = x(t) - x^*(t), \quad r(t) = \tilde{u}(t) - \tilde{G}^{-R} \tau_c^*(t).$$

Now by construction one has

$$\begin{aligned} \dot{r}(t) &= \tilde{Z}\tilde{u}(t) + \tilde{v}_b(t) - \tilde{G}^{-R}\dot{\tau}_c^*(t) \\ \tilde{G}^{-R}\dot{\tau}_c^*(t) &= K_1x^*(t) + L_1w(t) + B_1\tilde{G}^{-R}\tau_c^*(t). \end{aligned}$$

Let us denote by  $\tilde{G}_{ext} \in \mathbb{R}^{k \times m}$  the matrix obtained from  $\tilde{G}$  by adding null-columns in the positions corresponding to the faults, i.e.  $i_1, \dots, i_q$ ; accordingly, we extend the vector  $\tilde{u}_{ext}$  by adding null entries. We define the vector of faulty inputs

$$p(t) = u(t) - \tilde{u}_{ext}(t),$$

with  $(G - \tilde{G}_{ext})p(t) = Gp(t)$ . One has

$$\dot{s}(t) = As(t) + B(Gp(t) + \tilde{G}_{ext}\tilde{u}_{ext}(t) - \tilde{G}\tilde{G}^{-R}\tau_c^*(t)),$$

where  $\tilde{G}_{ext}\tilde{u}_{ext}(t) = \tilde{G}\tilde{u}(t)$ . Setting  $X(t) = [s(t) \ r(t) \ p(t)]$  we are led to consider the system

$$\begin{aligned} \dot{X}(t) &= \begin{bmatrix} A & B\tilde{G} & BG \\ K_1 - \tilde{Z}\tilde{G}^{-R}K & \tilde{Z} + B_1 & 0 \\ 0 & 0 & \Delta(t)Z \end{bmatrix} X(t) \\ &+ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \left( (\tilde{Z} + B_1)u_\delta(t) - \xi \frac{\tilde{u}(t) - \tilde{u}_c(t)}{\|\tilde{u}(t) - \tilde{u}_c(t)\|^{1-r}} \right). \end{aligned} \quad (26)$$

By construction, condition (4) and Theorem IV.2 ensure that the actuator input  $\tilde{v}(t)$  defined in (25) guarantees asymptotic stability of the closed-loop dynamics of  $[x(t) \ \tilde{u}(t) - \tilde{u}_c(t)]$  in the absence of exogenous signal  $w$ : this fact implies that the matrix

$$\mathcal{H}_1 = \begin{bmatrix} A & B\tilde{G} \\ K_1 - \tilde{Z}\tilde{G}^{-R}K & \tilde{Z} + B_1 \end{bmatrix}$$

is Hurwitz stable; on the other hand one has  $p(t) = \bar{p} + p^\dagger(t)$ , where by definition the transient  $p^\dagger(t)$  converges to zero and it does not influences the steady-state response of the system. Moreover, by definition, the steady-state  $\bar{p}$  verifies

$$(G - \tilde{G}_{ext})\bar{p} = G\bar{p} = B^{-L}\bar{W}\bar{u}^*,$$

and hence

$$\begin{aligned} u_\delta(t) &= \tilde{G}^{-R}B^{-L}\bar{W}(\bar{u}^* + (\eta(t) - \bar{u}^*)) \\ &= \tilde{G}^{-R}G\bar{p} + \tilde{G}^{-R}B^{-L}\bar{W}(\eta(t) - \bar{u}^*), \end{aligned}$$

where  $(\eta(t) - \bar{u}^*)$  converges to zero.

In conclusion, neglecting the null convergent terms, the steady-state of the system  $[s(t) \ r(t)]^T$  is given by

$$\begin{bmatrix} \bar{s} \\ \bar{r} \end{bmatrix} = \lim_{t \rightarrow \infty} \left( \int_{t_0}^t e^{\mathcal{H}_1(t-s)} \begin{bmatrix} BG \\ (\tilde{Z} + B_1)\tilde{G}^{-R}G \end{bmatrix} \bar{p} ds + E(t) \right),$$

where  $E(t)$  is the response of the system to the finite-time controller. In particular, since by construction the presence of  $E(t)$  helps to reduce the effect of the constant input  $\bar{p}$  and  $\lim_{t \rightarrow \infty} \|E(t)\| = 0$ , the following estimate can be deduced by integration:

$$\|\bar{s}\| \leq \left\| [I \ 0] \mathcal{H}_1^{-1} \begin{bmatrix} BG \\ (\tilde{Z} + B_1)\tilde{G}^{-R}G \end{bmatrix} \bar{p} \right\|.$$

On the other hand, recalling that by assumption  $(\tilde{Z} + B_1)$  is invertible and using the formula for the inverse of a partitioned matrix, one gets

$$[I \ 0] \mathcal{H}_1^{-1} = \hat{\mathcal{H}}^{-1} [I \ -B\tilde{G}(\tilde{Z} + B_1)^{-1}],$$

with  $\hat{\mathcal{H}} = (A - B\tilde{G}(\tilde{Z} + B_1)^{-1}(K_1 - \tilde{Z}\tilde{G}^{-R}K))$ . Finally, observing that

$$[I \ -B\tilde{G}(\tilde{Z} + B_1)^{-1}] \begin{bmatrix} BG \\ (\tilde{Z} + B_1)\tilde{G}^{-R}G \end{bmatrix} = 0,$$

the conclusion follows:

$$\bar{s} = \lim_{t \rightarrow +\infty} \|x(t) - x^*(t)\| = 0. \quad \blacksquare$$

**Remark IV.3** *If the choice  $\gamma = 1$  is not feasible, setting  $\gamma \neq 1$  with  $\det(\gamma\tilde{Z} + B_1) \neq 0$  yields an expression for  $\dot{X}(t)$  equal to (26) where  $\tilde{Z}$  is replaced by  $\gamma\tilde{Z}$ , and therefore the proof can be straightforward adapted.*

## V. SIMULATION RESULTS

Let us consider the following input-redundant linear plant

$$\begin{aligned} \dot{x} &= Ax + B\tau \\ \tau &= Gu \end{aligned}$$

with

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & -1 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ -1 & 2.5 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \\ G &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 1.5 & -1 \end{bmatrix}. \end{aligned}$$

The actuator dynamics is supposed to be governed by a drift matrix  $Z$  given by

$$Z = \text{diag}[-0.03, -0.03, -0.07, -0.04]$$

and matrices  $K, L$  are designed such that the system output  $y(t) = x_1(t)$  tracks the reference signal  $w = 10 \sin(0.7t)$ . The actuator  $u_1(t)$  is supposed to be affected by a high-degrade fault  $\delta_1(t) = e^{-0.006t}$  and the initial configuration of the actuators is assumed to be null,  $u(0) = 0$ . Figures 1-2 show the effects of the fault on the output tracking system performances and on the actuators, respectively. A family of unknown input observers  $\{F^{(h)}, R^{(h)}, N^{(h)}, H^{(h)}\}_{h=1}^4$  has been designed such that

$$R^{(h)}BG_h = 0, \quad h = 1, \dots, 4,$$

and the associated residual  $e^{(h)}(t)$  turns out to be insensitive to faults affecting the actuator  $u_h$ . Figure 3 shows that all residuals except  $e^{(1)}(t)$  exceed a suitable detection threshold and therefore the fault can be correctly isolated for  $t \geq 6 \text{sec}$ . The control reconfiguration policy is applied for  $t \geq 10 \text{sec}$  and the nominal system performances are recovered, as shown in Figure 4. The behavior of reconfigured actuators is depicted in Figure 5: the faulty actuator state  $u_1(t)$  converges to the steady-state  $\bar{u}_1 \approx 0.39$ .

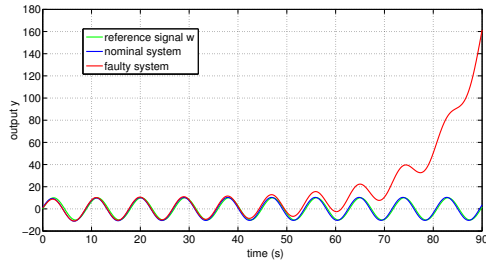


Fig. 1. System output vs reference signal: faulty case

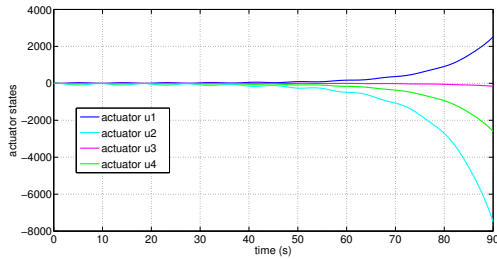


Fig. 2. Actuator states: faulty case

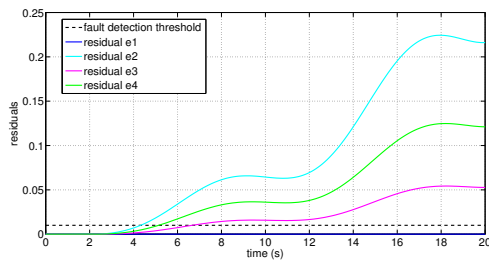


Fig. 3. Fault detection/isolation: residual signals

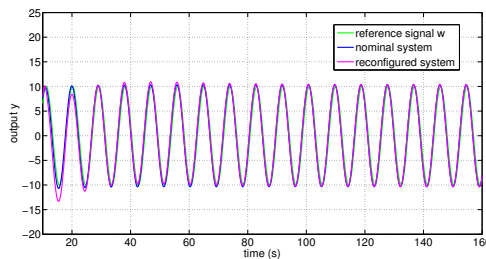


Fig. 4. System output vs reference signal: fault accommodation

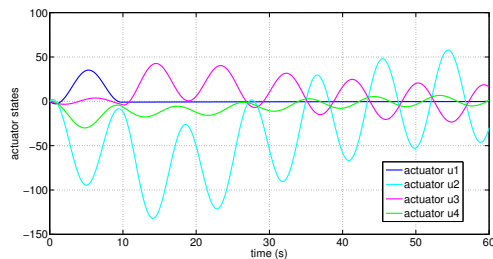


Fig. 5. Actuator states: control reconfiguration

## CONCLUSIONS

This paper, which is part of a wide research project initiated with [8], is devoted to the design of fault-tolerant control allocation schemes for overactuated linear systems with actuator dynamics. Extending some results obtained by the authors [9], faulty actuators can be identified through an UIO-based fault detection and isolation module. A finite-time control reconfiguration strategy has been proposed with the aim of compensating for faulty actuators steady-states and recovering the desired system performances.

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