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# $L^1$ CONTRACTION FOR BOUNDED (NONINTEGRABLE) SOLUTIONS OF DEGENERATE PARABOLIC EQUATIONS*

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**Abstract.** We obtain new $L^1$ contraction results for bounded entropy solutions of Cauchy problems for degenerate parabolic equations. The equations we consider have possibly strongly degenerate local or nonlocal diffusion terms. As opposed to previous results, our results apply without any integrability assumption on the solutions. They take the form of partial Duhamel formulas and can be seen as quantitative extensions of finite speed of propagation local $L^1$ contraction results for scalar conservation laws. A key ingredient in the proofs is a new and nontrivial construction of a subsolution of a fully nonlinear (dual) equation. Consequences of our results are maximum and comparison principles, new a priori estimates, and, in the nonlocal case, new existence and uniqueness results.

**Key words.** degenerate parabolic equations, $L^1$ contraction, entropy solutions, nonlocal/local equation, equations of mixed hyperbolic/parabolic type, a priori estimates, uniqueness, existence

**AMS subject classifications.** 35K65, 35B45, 35B50, 35B30, 35B51, 35D30, 35K59, 35L65, 35R11, 35R09

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## 1. Introduction.

In this paper, we consider the following Cauchy problem:

$$\begin{cases} \partial_t u + \operatorname{div} f(u) - \mathfrak{L}\,\varphi(u) = g(x,t) & \text{in} \quad Q_T := \mathbb{R}^d \times (0,T), \\ u(x,0) = u_0(x) & \text{on} \quad \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is the solution, $T > 0$, and div is the $x$-divergence. The operator $\mathfrak{L}$ will be either the $x$-Laplacian $\Delta$ or a nonlocal operator $\mathcal{L}^\mu$ defined on $C_c^\infty(\mathbb{R}^d)$ as

$$\mathcal{L}^\mu[\phi](x) := \int_{\mathbb{R}^d\setminus\{0\}} \phi(x+z) - \phi(x) - z \cdot D\phi(x)\mathbf{1}_{|z|\leq 1}\,\mathrm{d}\mu(z), \tag{1.2}$$

where $\mu$ is a nonnegative Radon measure, $D$ the $x$-gradient, and $\mathbf{1}_{|z|\leq 1}$ the characteristic function of $|z| \leq 1$. Throughout the paper we assume that

$(\mathrm{A}_f)$ $\quad f = (f_1, f_2, \ldots, f_d) \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^d)$;

$(\mathrm{A}_\varphi)$ $\quad \varphi \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ and $\varphi$ is nondecreasing ($\varphi' \geq 0$);

$(\mathrm{A}_g)$ $\quad g$ is measurable and $\displaystyle\int_0^T \|g(\cdot,t)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}t < \infty$;

$(\mathrm{A}_{u_0})$ $\quad u_0 \in L^\infty(\mathbb{R}^d)$;

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($A_\mu$) $\mu \geq 0$ is a Radon measure on $\mathbb{R}^d \setminus \{0\}$, and there is $M \geq 0$ such that

$$\int_{|z|\leq 1} |z|^2 \,\mathrm{d}\mu(z) + \int_{|z|>1} \mathrm{e}^{M|z|}\,\mathrm{d}\mu(z) < \infty.$$

($A_\mu^+$) Assumption ($A_\mu$) holds with $M > 0$.

*Remark* 1.1. Without loss of generality, we can assume $f(0) = 0$ and $\varphi(0) = 0$ (by adding constants to $f$ and $\varphi$) and $f$ and $\varphi$ are globally Lipschitz (since solutions are bounded). ($A_\mu$) implies that $\int_{|z|>0} |z|^2 \wedge 1 \,\mathrm{d}\mu(z) < \infty$ and $\mu$ is a Lévy measure.

Equation (1.1) is a degenerate parabolic equation. It can be strongly degenerate; i.e., $\varphi'$ may vanish/degenerate on sets of positive measure. Equation (1.1) can therefore be of mixed hyperbolic parabolic type. The equation is local when $\mathfrak{L} = \Delta$ and nonlocal when $\mathfrak{L} = \mathcal{L}^\mu$. In the latter case, it is an anomalous diffusion equation: When ($A_\mu$) holds, $\mathcal{L}^\mu$ is the generator of a pure jump Lévy process, and conversely, any pure jump Lévy process has a generator like $\mathcal{L}^\mu$. An example is the isotropic $\alpha$-stable process for $\alpha \in (0, 2)$. Here the generator is the fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$, which can be defined as a Fourier multiplier or, equivalently, via (1.2) with $\mathrm{d}\mu(z) = c_\alpha \frac{\mathrm{d}z}{|z|^{d+\alpha}}$ for some $c_\alpha > 0$ [6, 23]. If also ($A_\mu^+$) holds, then $\mathcal{L}^\mu$ is the generator of a tempered $\alpha$-stable process [17]. Almost all Lévy processes in finance are of this type. In this paper, this assumption is needed to ensure that the solution of a dual problem belongs to $L^1$; see the discussion on page 3959. For more details and examples of nonlocal operators, we refer the reader to [6, 17].

A large number of physical and financial problems are modeled by convection-diffusion equations like (1.1). Being very selective, we mention reservoir simulation [24], sedimentation processes [11], and traffic flow [36] in the local case; detonation in gases [16], radiation hydrodynamics [33, 34], and semiconductor growth [37] in the nonlocal case; and porous media flow [35, 20] and mathematical finance [17] in both cases.

Let us give the main references for the well-posedness of the Cauchy problem for (1.1), starting with the most classical case $\mathfrak{L} = \Delta$. For a more complete bibliography, see the books [21, 19, 35] and the references in [28]. In the hyperbolic case where $\varphi' \equiv 0$, we get the scalar conservation law $\partial_t u + \mathrm{div} f(u) = 0$. The solutions of this equation can develop discontinuities in finite time, and the weak solutions of the Cauchy problem are generally not unique. The most famous uniqueness result relies on the notion of entropy solutions introduced in [31]. In the pure diffusive case where $f' \equiv 0$, there is no more creation of shocks and the initial-value problem for $\partial_t u - \triangle\varphi(u) = 0$ admits a unique weak solution; cf. [10]. Much later, the adequate notion of entropy solutions for mixed hyperbolic parabolic equations was introduced in [12]. This paper focuses on an initial-boundary value problem. For a general well-posedness result applying to the Cauchy problem (1.1) with $\mathfrak{L} = \Delta$, we refer the reader to, e.g., [28] and [5, 32].

At the same time, there has been a large interest in nonlocal versions of these equations (where $\mathfrak{L} = \mathcal{L}^\mu$). The study of nonlocal diffusion terms was probably initiated by [8]. Now, the well-posedness is quite well understood in the nondegenerate linear case where $\varphi(u) = u$. Smooth solutions exist and are unique for subcritical equations [8, 22]; shocks can occur [4, 30] and weak solutions can be nonunique [2] for supercritical equations; and entropy solutions exist and are always unique [1, 29]; cf. also, e.g., [13] for original regularizing effects. Very recently, the well-posedness theory of entropy solutions was extended in [14] to cover the full problem (1.1), even for strongly degenerate $\varphi$. See also [20, 9] on fractional porous medium type equations.

In all the papers on entropy solutions, the authors use doubling of variables arguments inspired by Kružkov to prove $L^1$ contraction estimates. For entropy solutions $u$ and $v$, the typical estimate when $g = 0$ is

$$\int_{\mathbb{R}^d} (u(x,t) - v(x,t))^+ \,\mathrm{d}x \leq \int_{\mathbb{R}^d} (u(x,0) - v(x,0))^+ \,\mathrm{d}x. \tag{1.3}$$

From such an estimate the maximum or comparison principle follows: If $u(x,0) \leq v(x,0)$ a.e., then $u(x,t) \leq v(x,t)$ for all $t > 0$ and a.e. $x$. A priori estimates for the $L^1$, $L^\infty$, and $BV$ norms of the solutions also follow—estimates which are important, e.g., to show existence, stability, and convergence of approximations. However, due to the global nature of this contraction estimate, it applies only to entropy solutions which satisfy $(u(\cdot,0) - v(\cdot,0))^+ \in L^1(\mathbb{R}^d)$. In particular, this estimate cannot be used to obtain $L^1$ or $BV$ type estimates when $u(\cdot,0)$ and $v(\cdot,0)$ merely belong to $L^\infty$, as in this paper. Some of the previous results also need the further restriction that solutions belong to $L^1 \cap L^\infty$; see [28, 14]. In particular, prior to this paper, there were no well-posedness results for merely bounded solutions of the nonlocal variant of (1.1) when $\varphi$ is nonlinear.

In this paper, we obtain new $L^1$ contraction results for (1.1). The estimates are more local than (1.3) and take the form of a "partial Duhamel formula" (see (2.4)),

$$\int_{B(x_0,\,M)} (u(x,t) - v(x,t))^+ \,\mathrm{d}x \leq \int_{B(x_0, M+1+Lt)} \left[\tilde{\Phi}(\cdot,t) * \left(u(\cdot,0) - v(\cdot,0)\right)^+\right](x) \,\mathrm{d}x \tag{1.4}$$

for all $x_0 \in \mathbb{R}^d$ and $M > 0$, some $L$, and some integrable function $\tilde{\Phi}$. See section 2 for the precise statements. In (1.4), there is no need to take $(u(\cdot,0) - v(\cdot,0))^+ \in L^1(\mathbb{R}^d)$, and we will prove that the result applies to arbitrary bounded entropy solutions $u, v$. In addition to this new and more quantitative form of the $L^1$ contraction, we obtain as consequences maximum/comparison principles and new BV estimates for both local and nonlocal versions of (1.1), and in the nonlocal case, we obtain the first well-posedness result to hold for merely bounded entropy solution of (1.1).

Estimate (1.4) can be seen as a quantitative extension of the finite speed of propagation type of estimate that holds for scalar conservation laws [31, 19]. A similar (Duhamel type) result has already been obtained for fractional conservation laws in [1]. See also [22, 23] for more Duhamel formulas for fractional conservation laws. The proof in [1] consists in establishing a so-called Kato inequality for the equation, making a clever choice of the test function to have cancellations, and then concluding in a fairly standard way. Even if it is not written like that, the test function is chosen to be a subsolution of a sort of dual equation that appears from the Kato inequality. In [1], the principal part of the "dual equation" is the (linear) fractional heat equation which can be solved exactly using the fundamental solution. The test function is therefore defined via a Duhamel-like formula involving the fractional heat kernel (the function $\tilde{\Phi}$ in this case).

In this paper, we formalize this procedure and apply it to the more difficult problems with nonlinear degenerate diffusions. To do that, we derive Kato inequalities for bounded entropy solutions and identify the useful "dual equations" from them. In the general case, we find that the "dual equations" are fully nonlinear degenerate parabolic equations. These equations do not have smooth solutions in general, but we then prove that there exist bounded continuous generalized solutions (viscosity solutions) that belong to $L^1$. In this step, assumption $(\mathrm{A}_\mu^+)$ is needed in the nonlocal case. After several regularization procedures and Duhamel type formulas, we produce a test function that gives the necessary cancellations. Since this test function is not based on a fundamental solution, or any $\tilde{\Phi}$ which is mass preserving, we can only conclude after additional approximation steps.

In effect, we have introduced a new way of obtaining $L^1$ contraction estimates for degenerate parabolic equations. The new proof exploits a "dual equation," which in this case is pretty bad, too—a degenerate fully nonlinear equation that can be best analyzed through the theory of viscosity solutions [18]. The proof can therefore be seen as a sort of duality argument, and it is, as far we know, the first proof in which viscosity solution methods were used as a key ingredient in a contraction proof for entropy solutions.

The rest of this paper is organized as follows: In section 2, we give the definitions of entropy solutions and present and discuss our main results. Their main consequences are discussed in section 3. In section 4, we derive Kato type and other auxiliary inequalities. And finally, in section 5, we give the proofs of our main results.

**Notation.** For $x \in \mathbb{R}$, we let $x^+ = \max\{x, 0\}$ and $x^- = (-x)^+$, and $\operatorname{sign}(x)$ is $\pm 1$ for $\pm x > 0$ and $0$ for $x = 0$. We let $B(x,r) = \{y \in \mathbb{R}^d : |x-y| < r\}$, and the indicator function $\mathbf{1}_{\mathcal{A}}$ is 1 on the set $\mathcal{A}$ and 0 on the complement $\mathcal{A}^c$. By $L_\phi$ and $\operatorname{supp}\phi$ we denote the Lipschitz constant and support of a function $\phi$, derivatives are denoted by $'$, $\frac{d}{dt}$, $\partial_{x_i}$, and $D\phi$ and $D^2\phi$ denote the $x$-gradient and Hessian matrix of $\phi$. Convolution is defined as $f * g(x) = [f * g](x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,\mathrm{d}y$ (the brackets are dropped whenever the notation is not ambiguous). If $\mu$ is a Borel measure, then $\mu^*$ is defined as $\mu^*(B) = \mu(-B)$ for all Borel sets on $\mathbb{R}^d \setminus \{0\}$. The $L^2$ adjoint of an operator $A$ is denoted by $A^*$, and the reader may check that $(\mathcal{L}^\mu)^* = \mathcal{L}^{\mu^*}$.

We use standard notation for $L^p$, $BV$, and $H^1$ spaces, and $C_b$ and $C_c^\infty$ are the spaces of bounded continuous functions and smooth functions with compact support. We use the following norm and seminorm:

$$\|\phi\|_{C([0,T];L^1(\mathbb{R}^d))} := \operatorname*{ess\,sup}_{t\in[0,T]} \int_{\mathbb{R}^d} |\phi(x,t)|\,\mathrm{d}x,$$
$$|\psi|_{BV(\mathbb{R}^d)} := \sup_{h\neq 0} \int_{\mathbb{R}^d} \frac{|\psi(x+h)-\psi(x)|}{|h|}\,\mathrm{d}x.$$

The $|\cdot|_{BV}$ seminorm is equivalent to the standard definition of total variation; see [25, Lemma A.1] or [3, Lemma A.2]. We define the spaces $C([0,T];L^1(\mathbb{R}^d))$ and $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^d))$ in the usual way; e.g., the space $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}^d))$ is the space of measurable functions $u : \mathbb{R}^d \times [0,T] \to \mathbb{R}$ satisfying $u(\cdot,t) \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for every $t \in [0,T]$, $\max_{t\in[0,T]} \int_K |u(x,t)|\,\mathrm{d}x < \infty$, and $\int_K |u(x,t)-u(x,s)|\,\mathrm{d}x \to 0$ when $t \to s$ for all compact $K \subset \mathbb{R}^d$ and $s \in [0,T]$.

For the rest of the paper, we fix three families of mollifiers $\omega_\varepsilon$, $\hat{\omega}_\varepsilon$, $\rho_\varepsilon$ defined by

$$\omega_\varepsilon(\sigma) := \frac{1}{\varepsilon}\omega\left(\frac{\sigma}{\varepsilon}\right) \tag{1.5}$$

for fixed $0 \leq \omega \in C_c^\infty(\mathbb{R})$ satisfying $\operatorname{supp}\omega \subseteq [-1,1]$, $\omega(\sigma) = \omega(-\sigma)$, $\int \omega = 1$;

$$\hat{\omega}(x) = \omega(x_1), \dots, \omega(x_d) \qquad \text{and} \qquad \hat{\omega}_\varepsilon(x) = \frac{1}{\varepsilon^d}\hat{\omega}\left(\frac{x}{\varepsilon}\right) \tag{1.6}$$

for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$; and

$$\rho_\delta(\sigma,\tau) := \frac{1}{\delta^{d+2}}\rho\left(\frac{\sigma}{\delta}, \frac{\tau}{\delta^2}\right) \tag{1.7}$$

for fixed $0 \leq \rho \in C_c^\infty(Q_T)$, $\operatorname{supp} \rho \subseteq B(0,1) \times (0,1)$, $\rho(\sigma,\tau) = \rho(-\sigma,-\tau)$, $\int \rho = 1$.

## 2. Entropy formulation and main results.

In this section, we give the definitions of entropy solutions of (1.1) and then present our main results. We will use the splitting

$$\mathcal{L}^\mu[\phi](x) = \mathcal{L}_r^\mu[\phi](x) + \mathcal{L}^{\mu,r}[\phi](x) + b^{\mu,r} \cdot D\phi(x)$$

for $\phi \in C_c^\infty(Q_T)$, $r > 0$, and $x \in \mathbb{R}^d$, where

$$\begin{aligned}
\mathcal{L}_r^\mu[\phi](x) &:= \int_{0<|z|\leq r} \phi(x+z) - \phi(x) - z \cdot D\phi \mathbf{1}_{|z|\leq 1} \, \mathrm{d}\mu(z), \\
\mathcal{L}^{\mu,r}[\phi](x) &:= \int_{|z|>r} \phi(x+z) - \phi(x) \, \mathrm{d}\mu(z), \\
b^{\mu,r} &:= -\int_{|z|>r} z \mathbf{1}_{|z|\leq 1} \, \mathrm{d}\mu(z).
\end{aligned}$$

Below we will use the Kružkov entropy-entropy flux pairs, $|u-k|$ and $\operatorname{sign}(u-k)(f(u)-f(k))$, and the corresponding semi entropy-entropy flux pairs,

$$(u-k)^\pm \qquad \text{and} \qquad \pm \operatorname{sign}(u-k)^\pm (f(u) - f(k)) \qquad \text{for all } k \in \mathbb{R}.$$

Definition 2.1. *Let $\mathfrak{L} = \Delta$. A function $u \in L^\infty(Q_T) \cap C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^d))$ is*

(a) *an* entropy subsolution *of* (1.1) *if*

(i) *for all nonnegative $\phi \in C_c^\infty(Q_T)$ and all $k \in \mathbb{R}$*

$$\begin{aligned}
&\iint_{Q_T} (u-k)^+ \phi_t + \operatorname{sign}(u-k)^+ [f(u) - f(k)] \cdot D\phi \, \mathrm{d}x \, \mathrm{d}t \\
&+ \iint_{Q_T} (\varphi(u) - \varphi(k))^+ \Delta\phi \, \mathrm{d}x \, \mathrm{d}t \\
&+ \iint_{Q_T} \operatorname{sign}(u-k)^+ g \, \phi \, \mathrm{d}x \, \mathrm{d}t \geq 0;
\end{aligned}$$

(ii) *$\varphi(u) \in L^2((0,T); H^1_{\mathrm{loc}}(\mathbb{R}^d))$;*

(iii) *$u(\cdot,0) \leq u_0$ for a.e. $x \in \mathbb{R}^d$;*

(b) *an* entropy supersolution *of* (1.1) *if*

(i) *for all nonnegative $\phi \in C_c^\infty(Q_T)$ and all $k \in \mathbb{R}$*

$$\begin{aligned}
&\iint_{Q_T} (u-k)^- \phi_t - \operatorname{sign}(u-k)^- [f(u) - f(k)] \cdot D\phi \, \mathrm{d}x \, \mathrm{d}t \\
&+ \iint_{Q_T} (\varphi(u) - \varphi(k))^- \Delta\phi \, \mathrm{d}x \, \mathrm{d}t \\
&+ \iint_{Q_T} -\operatorname{sign}(u-k)^- g \, \phi \, \mathrm{d}x \, \mathrm{d}t \geq 0;
\end{aligned}$$

(ii) *$\varphi(u) \in L^2((0,T); H^1_{\mathrm{loc}}(\mathbb{R}^d))$;*

(iii) *$u(\cdot,0) \geq u_0$ for a.e. $x \in \mathbb{R}^d$;*

(c) *an* entropy solution *of* (1.1) *if it is both an entropy subsolution and an entropy supersolution.*

Definition 2.2. *Let $\mathfrak{L} = \mathcal{L}^\mu$. A function $u \in L^\infty(Q_T) \cap C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^d))$ is*

(a) *an* entropy subsolution *of* (1.1) *if*

(i) *for all nonnegative* $\phi \in C_c^\infty(Q_T)$ *and all* $k \in \mathbb{R}$

$$\iint_{Q_T} (u-k)^+ \partial_t \phi + \operatorname{sign}(u-k)^+ [f(u)-f(k)] \cdot D\phi \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \iint_{Q_T} (\varphi(u)-\varphi(k))^+ \big(\mathcal{L}_r^{\mu^*}[\phi] + b^{\mu^*,r} \cdot D\phi\big) + \operatorname{sign}(u-k)^+ \mathcal{L}^{\mu,r}[\varphi(u)]\phi \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \iint_{Q_T} \operatorname{sign}(u-k)^+ g\,\phi \,\mathrm{d}x\,\mathrm{d}t \geq 0;$$

(ii) $u(\cdot,0) \leq u_0(\cdot)$ *for a.e.* $x \in \mathbb{R}^d$*;*

(b) *an* entropy supersolution *of* (1.1) *if*

(i) *for all nonnegative* $\phi \in C_c^\infty(Q_T)$ *and all* $k \in \mathbb{R}$

$$\iint_{Q_T} (u-k)^- \partial_t \phi - \operatorname{sign}(u-k)^- [f(u)-f(k)] \cdot D\phi \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \iint_{Q_T} (\varphi(u)-\varphi(k))^- \big(\mathcal{L}_r^{\mu^*}[\phi] + b^{\mu^*,r} \cdot D\phi\big) - \operatorname{sign}(u-k)^- \mathcal{L}^{\mu,r}[\varphi(u)]\phi \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \iint_{Q_T} -\operatorname{sign}(u-k)^- g\,\phi \,\mathrm{d}x\,\mathrm{d}t \geq 0;$$

(ii) $u(\cdot,0) \geq u_0(\cdot)$ *for a.e.* $x \in \mathbb{R}^d$*;*

(c) *an* entropy solution *of* (1.1) *if it is both an entropy subsolution and an entropy supersolution.*

*Remark* 2.3.

(a) Similar definitions are given, e.g., in [32, Definition 3.4] and [14, Definition 5.1].
(b) Since an entropy solution $u \in C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^d))$ and $u(\cdot,0) = u_0(\cdot)$ a.e., the initial condition is imposed in a strong sense: $u(\cdot,t) \to u_0(\cdot)$ in $L^1_{\mathrm{loc}}$ as $t \to 0^+$.
(c) By $(\mathrm{A}_f)$, $(\mathrm{A}_\varphi)$, and $u \in L^\infty(Q_T)$, $f(u)$ and $\varphi(u)$ are in $L^\infty(Q_T)$.
(d) By (c) and $(\mathrm{A}_g)$, all integrals in Definition 2.1 (a) and (b) are well defined.
(e) By (c) and $(\mathrm{A}_g)$, the first and third integrals in Definition 2.2 (a) and (b) are well defined. Since $\mathcal{L}_r^{\mu^*}[\phi] \in C_c^\infty(Q_T)$ for $\phi \in C_c^\infty(Q_T)$ and $\mathcal{L}^{\mu,r}[\varphi(u)] \in L^\infty(Q_T)$ for $\varphi(u) \in L^\infty(Q_T)$, by (c) the second integral is also well defined. Since $u$ is a Lebesgue measurable function, it is not immediately clear that $\varphi(u)$ is $\mu$-measurable and $\mathcal{L}^{\mu,r}[\varphi(u)]$ is pointwisely well defined. We refer the reader to Remark 2.1 and Lemma 4.2 in [3] for a discussion and proof that this is actually the case.

LEMMA 2.4. *$u(x,t)$ is an entropy solution of* (1.1) *in the sense of Definition* 2.1 *or* 2.2 *if and only if $u(x,t)$ is an entropy solution in the usual sense.*

*Proof.* Since $|u-k| = (u-k)^+ + (u-k)^-$ and $\operatorname{sign}(u-k) = \operatorname{sign}(u-k)^+ - \operatorname{sign}(u-k)^-$,

$$\text{Definition 2.1 (a) and (b)} \quad \text{or} \quad \text{Definition 2.2 (a) and (b)}$$
$$\Downarrow$$
$$|u-k|_t + \operatorname{div}\Big(\operatorname{sign}(u-k)[f(u)-f(k)]\Big) - \mathfrak{L}\,\big|\varphi(u)-\varphi(k)\big| - \operatorname{sign}(u-k)g \leq 0$$

in $\mathcal{D}'(Q_T)$, which is the usual definition in terms of Kružkov entropy-entropy fluxes.

Part (a) of Definitions 2.2 and 2.1 can be obtained from the usual definition in a similar way. First, we check that $u-k$ satisfy

$$(u-k)_t + \operatorname{div}\big(f(u)-f(k)\big) - \mathfrak{L}\big(\varphi(u)-\varphi(k)\big) - g = 0 \quad \text{in} \quad \mathcal{D}'(Q_T).$$

Then we add this equation to the entropy inequality for $u$. Since this inequality involves the Kružkov flux $|u-k|$, the result follows by the identities

$$\begin{aligned}
&|u-k| + (u-k) = 2(u-k)^+,\\
&\operatorname{sign}(u-k)(f(u)-f(k)) + \big(f(u)-f(k)\big) = 2\operatorname{sign}(u-k)^+(f(u)-f(k)),
\end{aligned}$$

and a similar one for the $\varphi(u)$-terms. The proof of part (b) is similar. ◻

**Main results.** To give the main results, we introduce the functions $\tilde{K}$ and $\Phi$. We define

$$\tilde{K}(x,t) = \mathcal{F}^{-1}(\mathrm{e}^{-t|2\pi\xi|^\alpha})(x) \quad \text{for } \alpha \in (0,2], \tag{2.1}$$

where $\mathcal{F}(\phi)(\xi) = \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i}\xi\cdot x}\phi(x)\,\mathrm{d}x$. Then $\tilde{K}$ is a fundamental solution satisfying

$$\begin{cases} \partial_t \tilde{K} - \mathfrak{L}^* \tilde{K} = 0, & t>0,\\ \tilde{K}(x,0) = \delta_0, \end{cases}$$

for $\mathfrak{L}^* = \mathfrak{L} = -(-\Delta)^{\frac{\alpha}{2}}$, where $\delta_0$ is the Dirac measure centered at the origin. Furthermore, $\Phi$ is the (nonsmooth viscosity) solution of

$$\begin{cases} \partial_t \Phi - (\mathfrak{L}^*\Phi)^+ = 0 & \text{in} \quad \mathbb{R}^d \times (0,\tilde{T}),\\ \Phi(x,0) = \Phi_0(x) & \text{on} \quad \mathbb{R}^d \end{cases} \tag{2.2}$$

for some $\Phi_0 \in C_c^\infty(\mathbb{R}^d)$.

LEMMA 2.5. *Let $\tilde{K}$ be defined by* (2.1)*; then it has the following properties:*

(a) *$\tilde{K}$ is nonnegative, smooth, and bounded for $t>\delta$ for all $\delta>0$;*

(b) $\int_{\mathbb{R}^d} \tilde{K}(x,t)\,\mathrm{d}x = 1$*;*

(c) *$\{\tilde{K}(\cdot,t)\}_{t>0}$ is an approximate unit as $t\to 0$;*

(d) *$\tilde{K}(x,t) = \tilde{K}(-x,t)$ for all $t>0$ and $x\in\mathbb{R}^d$.*

This result is classical and can be found in, e.g., [1].

LEMMA 2.6. *Assume that* ($\mathrm{A}_f$), ($\mathrm{A}_\varphi$), *and* ($\mathrm{A}_g$) *hold, that $\mathfrak{L} = \Delta$ or $\mathfrak{L} = \mathcal{L}^\mu$ and* ($\mathrm{A}_\mu^+$) *holds, and that $0 \le \Phi_0 \in C_c^\infty(Q_T)$. Let $\tilde{T} := \max\{T, L_\varphi T\}$, where $L_\varphi$ is the Lipschitz constant of $\varphi$. Then there exists a unique viscosity solution $\Phi(x,t)$ of* (2.2) *such that*

$$0 \le \Phi \in C_b(Q_{\tilde{T}}) \cap C([0,\tilde{T}]; L^1(\mathbb{R}^d)).$$

We prove this lemma in section 5. Note that viscosity solutions are the right type of weak solutions for fully nonlinear and degenerate equations like (2.2); see, e.g., [18, 26].

*Remark* 2.7.

(a) To handle bounded, nonintegrable solutions of (1.1), it is important that $\Phi$ belong to $L^1$—a nonstandard result for (2.2).

(b) As for $\tilde{K}$, we would have liked to take $\Phi_0 = \delta_0$ (Dirac measure), since this would give us better constants in the results that follow. We have not been able to do this for two reasons: (i) There is no well-posedness theory for equations like (2.2) with measure initial data, and (ii) the $L^1$ bound for $\Phi$ is obtained by comparison with a particular $L^1$ supersolution. Hence, if we let $\Phi_0$ be an approximate delta function and then take the limit, these estimates would blow up and the crucial $L^1$ property would be lost.

(c) When $\mathfrak{L}$ is self-adjoint (that is, when $\mathfrak{L} = \Delta$ or $\mathfrak{L} = \mathcal{L}^\mu$ with $\mu$ symmetric), we may assume that $\Phi(-x,t) = \Phi(x,t)$. Simply take a symmetric $\Phi_0$, and the solution of (2.2) has this property.

Before the main theorems are given, we revisit some of the known results in special cases.

THEOREM 2.8. *Assume* $(\mathrm{A}_f)$ *holds, and* $\varphi = 0$. *Let* $u$ *and* $v$ *be entropy sub- and supersolutions of* (1.1) *with initial data* $u_0, v_0 \in L^\infty(\mathbb{R}^d)$ *and measurable source terms* $g, h$ *satisfying* $\int_0^T \|g(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} + \|h(\cdot,t)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}t < \infty$. *Then for all* $t \in (0,T)$, $M > 0$, *and* $x_0 \in \mathbb{R}^d$

$$\int_{B(x_0,\,M)} (u(x,t) - v(x,t))^+\,\mathrm{d}x \le \int_{B(x_0,\,M+L_f t)} (u_0(x) - v_0(x))^+\,\mathrm{d}x + \int_0^t \int_{B(x_0,\,M+L_f(t-s))} (g(x,s) - h(x,s))^+\,\mathrm{d}x\,\mathrm{d}s,$$

*where* $L_f$ *is the Lipschitz constant of* $f$.

This is the classical local $L^1$ contraction result for scalar conservation laws; see, e.g., Dafermos [19, p. 149] for a proof. The hyperbolic finite speed of propagation property is encoded in the result.

In the linear nonlocal diffusion case, Alibaud [1] obtained the inequality

$$\int_{B(x_0,\,M)} (u(x,t) - v(x,t))^+\,\mathrm{d}x \le \int_{B(x_0,\,M+L_f t)} \big[\tilde{K}(\cdot,t) * (u_0 - v_0)^+\big](x)\,\mathrm{d}x + \int_0^t \int_{B(x_0,\,M+L_f(t-s))} \big[\tilde{K}(\cdot,t-s) * (g(\cdot,s) - h(\cdot,s))^+\big](x)\,\mathrm{d}x\,\mathrm{d}s, \tag{2.3}$$

where $L_f$ is the Lipschitz constant of $f$. We state the result along with a new result for the local case.

THEOREM 2.9. *Assume* $(\mathrm{A}_f)$, $\varphi(u) = u$, *and* $\tilde{K}$ *is defined by* (2.1). *Let* $t \in (0,T)$, $M > 0$, $x_0 \in \mathbb{R}^d$, *and* $u$ *and* $v$ *be entropy sub- and supersolutions of* (1.1) *with initial data* $u_0, v_0 \in L^\infty(\mathbb{R}^d)$ *and measurable source terms* $g, h$ *satisfying* $\int_0^T \|g(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} + \|h(\cdot,t)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}t < \infty$.

(a) *If* $\mathfrak{L} = -(-\Delta)^{\frac{\alpha}{2}}$ *for* $\alpha \in (0,2)$, *then the* $L^1$ *contraction estimate* (2.3) *holds.*

(b) *If* $\mathfrak{L} = \Delta$ $(\alpha = 2)$, *then the* $L^1$ *contraction estimate* (2.3) *holds.*

The result has the form of a partial Duhamel formula involving the fundamental solution of the parabolic part of the equation (which is linear here). The proof of (a) can be found in [1] when $g = 0$, and the extension to general $g$ is easy. Part (b) seems to be new, but essentially it follows from the argument of [1] and Proposition 4.2. The proof is given in section 5.

Now, we give our main result, which is an $L^1$ contraction estimate of the form

(2.4)
$$\int_{B(x_0,\,M)} (u(x,t)-v(x,t))^+ \,\mathrm{d}x \le \int_{B(x_0,M+1+L_f t)} \left[\Phi(-\cdot, L_\varphi t) * (u_0-v_0)^+\right](x)\,\mathrm{d}x$$
$$+\int_0^t \int_{B(x_0,M+1+L_f(t-s))} \left[\Phi(-\cdot, L_\varphi(t-s)) * (g(\cdot,s)-h(\cdot,s))^+\right](x)\,\mathrm{d}x\,\mathrm{d}s,$$

where $L_f$ and $L_\varphi$ are the Lipschitz constants of $f$ and $\varphi$, respectively.

THEOREM 2.10. *Assume* $(\mathrm{A}_f)$, $(\mathrm{A}_\varphi)$ *hold, and* $\Phi$ *is given by Lemma* 2.6. *Let* $t \in (0,T)$, $M > 0$, $x_0 \in \mathbb{R}^d$, *and* $u$ *and* $v$ *be entropy sub- and supersolutions of* (1.1) *with initial data* $u_0, v_0 \in L^\infty(\mathbb{R}^d)$ *and measurable source terms* $g, h$ *satisfying* $\int_0^T \|g(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} + \|h(\cdot,t)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}t < \infty$.

(a) *If* $\mathfrak{L} = \mathcal{L}^\mu$ *and* $(\mathrm{A}_\mu^+)$ *holds, then the* $L^1$ *contraction estimate* (2.4) *holds.*

(b) *If* $\mathfrak{L} = \Delta$, *then the* $L^1$ *contraction estimate* (2.4) *holds.*

The proof is given in section 5. These results, the $L^1$ contractions (2.3) and (2.4), encode both the finite speed of propagation of the hyperbolic term and the infinite speed of propagation of the parabolic term. As far as we know, this is the first time such a partial Duhamel type $L^1$ contraction result has been given for nonlinear diffusions.

*Remark* 2.11.

(a) By Fubini and a change of variables,[1] the $L^1$ contraction (2.4) is equivalent to an inequality involving convolutions of local $L^1$ norms and $\Phi$:

$$\|(u(\cdot,t)-v(\cdot,t))^+\|_{L^1(B(x_0,M))}$$
$$\le \int_{\mathbb{R}^d} \Phi(-y, L_\varphi t)\|(u_0-v_0)^+\|_{L^1(B(x_0-y,M+1+L_f t))}\,\mathrm{d}y$$
$$+\int_0^t \int_{\mathbb{R}^d} \Phi(-y, L_\varphi(t-s))\|(g(\cdot,s)-h(\cdot,s))^+\|_{L^1(B(x_0-y,M+1+L_f(t-s)))}\,\mathrm{d}y\,\mathrm{d}s.$$

(b) Theorem 2.10 gives a stronger $L^1$ contraction estimate than previous results [32, 5, 14]; see the discussion in the introduction and the next section.

(c) Theorem 2.10 (a) is the first $L^1$ contraction result for bounded solutions of (1.1) with nonlocal $\mathfrak{L}$.

(d) Theorem 2.10 (a) holds under assumption $(\mathrm{A}_\mu^+)$ which is discussed in the introduction. We do not know if this assumption can be relaxed. We use it to prove that $\Phi(\cdot,t)$ belongs to $L^1$, a result which is needed for (2.4) to be well defined for merely bounded initial data and source term.

(e) The $+1$-factor in $B(x_0, M+1+L_f t)$ in Theorem 2.10 depends on the choice of $\Phi$ and comes from the fact that $\Phi(x,t)$ is not an approximate unit as $t \to 0^+$. In fact, it will have increasing mass (or $L^1$ norm) in time.

---

[1] E.g.,

$$\int_{B(x_0,M+1+L_f t)} \int_{\mathbb{R}^d} \Phi(-y, L_\varphi t)(u_0-v_0)^+(x-y)\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_{\mathbb{R}^d} \Phi(-y, L_\varphi t) \int_{B(x_0,M+1+L_f t)} (u_0-v_0)^+(x-y)\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{\mathbb{R}^d} \Phi(-y, L_\varphi t) \int_{B(x_0-y,M+1+L_f t)} (u_0-v_0)^+(z)\,\mathrm{d}z\,\mathrm{d}y.$$

**3. Consequences.** Using Theorem 2.10, we now derive maximum and comparison principles, new a priori estimates, and new existence and uniqueness results for (1.1). The latter results are new only in the nonlocal case.

COROLLARY 3.1. *Assume* $(\mathrm{A}_f)$ *and* $(\mathrm{A}_\varphi)$ *hold,* $(\mathrm{A}_\mu^+)$ *holds when* $\mathfrak{L} = \mathcal{L}^\mu$, $u_0, v_0 \in L^\infty(\mathbb{R}^d)$, *and measurable* $g, h$ *satisfying*

$$\int_0^T \|g(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} + \|h(\cdot,t)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}t < \infty.$$

*Let* $M > 0$, $x_0 \in \mathbb{R}^d$, *and* $L_f$ *and* $L_\varphi$ *be the Lipschitz constants of* $f$ *and* $\varphi$, *respectively.*

(a) ($L^1$ contraction) *Let* $u$ *and* $v$ *be entropy solutions of* (1.1) *with initial data* $u_0, v_0$ *and source terms* $g, h$, *respectively. Then for all* $t \in (0,T)$,

$$\begin{aligned}\|u(\cdot,t) - v(\cdot,t)\|_{L^1(B(x_0,M))} &\leq \|\Phi(-\cdot, L_\varphi t) * |u_0 - v_0|\|_{L^1(B(x_0,M+1+L_f t))}\\ &\quad + \int_0^t \|\Phi(-\cdot, L_\varphi(t-s)) * |g(\cdot,s) - h(\cdot,s)|\|_{L^1(B(x_0,M+1+L_f(t-s)))}\,\mathrm{d}s.\end{aligned}$$

(b) ($L^1$ bound) *Let* $u$ *be an entropy solution of* (1.1). *Then for all* $t \in (0,T)$,

$$\begin{aligned}\|u(\cdot,t)\|_{L^1(B(x_0,M))} &\leq \|\Phi(-\cdot, L_\varphi t) * |u_0|\|_{L^1(B(x_0,M+1+L_f t))}\\ &\quad + \int_0^t \|\Phi(-\cdot, L_\varphi(t-s)) * |g(\cdot,s)|\|_{L^1(B(x_0,M+1+L_f(t-s)))}\,\mathrm{d}s.\end{aligned}$$

(c) (Comparison principle) *Let* $u$ *and* $v$ *be entropy sub- and supersolutions of* (1.1) *with initial data* $u_0, v_0$ *and source terms* $g, h$, *respectively. If* $u_0 \leq v_0$ *a.e. on* $\mathbb{R}^d$ *and* $g \leq h$ *a.e. in* $Q_T$, *then*

$$u(x,t) \leq v(x,t) \qquad \text{a.e. in } Q_T.$$

(d) (Maximum principle) *Let* $u$ *be an entropy solution of* (1.1). *Then*

$$\inf_{x\in\mathbb{R}^d} u_0(x) + \int_0^t \inf_{x\in\mathbb{R}^d} g(x,s)\,\mathrm{d}s \leq u(x,t) \leq \sup_{x\in\mathbb{R}^d} u_0(x) + \int_0^t \sup_{x\in\mathbb{R}^d} g(x,s)\,\mathrm{d}s$$

*a.e. in* $Q_T$.

(e) ($BV$ bound) *Let* $u$ *be an entropy solution of* (1.1), *and assume* $u_0 \in BV(\mathbb{R}^d)$, $g$ *is measurable, and* $\int_0^T |g(\cdot,t)|_{BV(\mathbb{R}^d)}\,\mathrm{d}t < \infty$. *Then for all* $t \in (0,T)$, $x_0 \in \mathbb{R}^d$, *and* $M > 0$,

$$\begin{aligned}&|u(\cdot,t)|_{BV(B(x_0,M))}\\ &\leq \sup_{h\neq 0} \frac{\|\Phi(-\cdot, L_\varphi t) * |u_0(\cdot+h) - u_0|\|_{L^1(B(x_0,M+1+L_f t))}}{|h|}\\ &\quad + \sup_{h\neq 0} \frac{\int_0^t \|\Phi(-\cdot, L_\varphi(t-s)) * |g(\cdot+h,s) - g(\cdot,s)|\|_{L^1(B(x_0,M+1+L_f(t-s)))}\,\mathrm{d}s}{|h|}.\end{aligned}$$

*Remark* 3.2.

(a) The $L^1$ and $BV$ bounds are new even in the local case.

(b) In a similar way as in Remark 2.11 (a), the bounds in (a), (b), and (e) can be expressed as convolutions of local norms; e.g., when $g = h = 0$,

$$\|u(\cdot,t)\|_{L^1(B(x_0,M))} \leq \int_{\mathbb{R}^d} \Phi(-y, L_\varphi t)\|u_0\|_{L^1(B(x_0-y,M+1+L_f t))}\, \mathrm{d}y,$$
$$|u(\cdot,t)|_{BV(B(x_0,M))} \leq \int_{\mathbb{R}^d} \Phi(-y, L_\varphi t)|u_0|_{BV(B(x_0-y,M+1+L_f t))}\, \mathrm{d}y.$$

If $|u_0|_{BV(\mathbb{R}^d)} < \infty$, then $|u(\cdot,t)|_{BV(B(x_0,M))} \leq \|\Phi(\cdot, L_\varphi t)\|_{L^1(\mathbb{R}^d)}|u_0|_{BV(\mathbb{R}^d)}$.

*Proof.* (a) By Theorem 2.10, estimate (2.4) holds. Interchanging the roles of $u, g$ and $v, h$, and using $(v-u)^+ = (u-v)^-$, etc., we see that (2.4) holds for $(u-v)^-$ as well as for $(u-v)^+$. Hence (a) follows.

(b) This part of the proof follows from (a) with $v = v_0 = h = 0$.

(c) By the contraction estimate (2.4) and the assumptions on the initial data and source terms, for all $t > 0$, $x_0 \in \mathbb{R}^d$, and $M > 0$,

$$\int_{B(x_0,\,M)} (u(x,t) - v(x,t))^+\, \mathrm{d}x \leq 0.$$

Hence $(u-v)^+ = 0$ and $u \leq v$ a.e. in $Q_T$.

(d) Note that $w(t) = \sup_{x\in\mathbb{R}^d} u_0(x) + \int_0^t \sup_{x\in\mathbb{R}^d} g(x,s)\, \mathrm{d}s$ is an entropy supersolution of (1.1), and then $u \leq w$ a.e. by part (c). In a similar way, the lower bound follows.

(e) Since (1.1) is translation invariant, both $u(x,t)$ and $u(x+h,t)$ are entropy solutions of (1.1) with initial data $u_0(x)$ and $u_0(x+h)$, and sources $g(x,t)$ and $g(x+h,t)$, respectively. By the definition of $|\cdot|_{BV}$ and part (a),

$$\begin{aligned}
&|u(\cdot,t)|_{BV(B(x_0,M))} \\
&= \sup_{h\neq 0} \frac{\|u(\cdot+h,t) - u(\cdot,t)\|_{L^1(B(x_0,M))}}{|h|} \\
&\leq \sup_{h\neq 0} \int_{B(x_0,M+1+L_f t)} \int_{\mathbb{R}^d} \Phi(-(x-y), L_\varphi t) \frac{|u_0(y+h) - u_0(y)|}{|h|}\, \mathrm{d}y\, \mathrm{d}x \\
&\quad + \sup_{h\neq 0} \int_0^t \int_{B(x_0,M+1+L_f(t-s))} \int_{\mathbb{R}^d} \Phi(-(x-y), L_\varphi(t-s)) \\
&\qquad\qquad \cdot \frac{|g(y+h,s) - g(y,s)|}{|h|}\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}s. \qquad \square
\end{aligned}$$

THEOREM 3.3 (existence and uniqueness). *Assume that* $(\mathrm{A}_f)$, $(\mathrm{A}_g)$, $(\mathrm{A}_\varphi)$, *and* $(\mathrm{A}_{u_0})$ *hold, and*

$$\mathfrak{L} = \Delta \quad \text{or} \quad \mathfrak{L} = \mathcal{L}^\mu \text{ and } (\mathrm{A}_\mu^+) \text{ holds.}$$

*Then there exists a unique entropy solution of the initial value problem* (1.1).

*Proof.* In the local case, this result was proved in [32, Theorem 3.7]. In the nonlocal case, uniqueness is an immediate consequence of Theorem 2.10 with $u_0 = v_0$ and $g = h$, and the existence result follows from existence results for $L^1\cap L^\infty$ solutions [14, 15] and the $L^1$ contraction of Corollary 3.1 (a). We perform the proof under the simplifying assumption that $g = 0$. It is not hard to extend the proof to the general case.

Take functions $u_{0,n} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ such that

$$\|u_{0,n}\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)} \text{ and } u_{0,n} \to u_0 \text{ in } L^1_{\text{loc}}(\mathbb{R}^d) \text{ and pointwise a.e.} \tag{3.1}$$

By [14, 15], there exist entropy solutions $u_m, u_n$ of (1.1) with initial data $u_{0,m}, u_{0,n}$, respectively. By Corollary 3.1 (a) and the triangle inequality,

$$\begin{aligned}
&\|u_m - u_n\|_{C([0,T];L^1(B(x_0,M)))} \\
&\leq \max_{t\in[0,T]} \|\Phi(-\cdot, L_\varphi t) * |u_{0,m} - u_0|\|_{L^1(B(x_0,M+1+L_f t))} \\
&\quad + \max_{t\in[0,T]} \|\Phi(-\cdot, L_\varphi t) * |u_{0,n} - u_0|\|_{L^1(B(x_0,M+1+L_f t))}.
\end{aligned}$$

The right-hand side of the inequality goes to zero by Lebesgue's dominated convergence theorem and (3.1) when $n, m \to \infty$ (the integrand is dominated by $2\Phi(-y, L_\varphi t)\|u_0\|_{L^\infty}$). Therefore, the sequence of entropy solutions $\{u_n\}$ is Cauchy in $C([0,T]; L^1(B(x_0,M)))$.

Since $\mathbb{R}^d$ can be covered by a countable number of such balls, a diagonal argument produces a function $u$ such that $u_\varepsilon \to u$ in $C([0,T]; L^1_{\text{loc}}(\mathbb{R}^d))$. Taking, if necessary, a further subsequence we may assume $u_n \to u$ a.e., and hence $\|u\|_{L^\infty} \leq \|u_0\|_{L^\infty}$ since $\|u_n\|_{L^\infty} \leq \|u_0\|_{L^\infty}$ by Corollary 3.1 (d). We conclude that $u$ is an entropy solution of (1.1) by passing to the limit in the entropy inequality for $u_n$; cf. Definition 2.2 (c). ∎

**4. Auxiliary results.** To establish the $L^1$ contraction estimates, we will need some auxiliary results that we derive here.

LEMMA 4.1. *Assume $r > 0$ and that $(\mathrm{A}_\mu)$ holds. Let $\phi \in W^{2,1}(\mathbb{R}^d)$; then*

$$\|\mathcal{L}^\mu_r[\phi]\|_{L^1(\mathbb{R}^d)} \leq \frac{1}{2}\|D^2\phi\|_{L^1(\mathbb{R}^d,\,\mathbb{R}^{d\times d})} \int_{0<|z|\leq r} |z|^2 \,\mathrm{d}\mu(z) \qquad \text{for } r < 1,$$

$$\|\mathcal{L}^{\mu,r}[\phi]\|_{L^1(\mathbb{R}^d)} \leq 2\|\phi\|_{L^1(\mathbb{R}^d)} \int_{|z|>r} \mathrm{d}\mu(z) \qquad \text{for } r > 1,$$

*and*

$$\|\mathcal{L}^\mu[\phi]\|_{L^1(\mathbb{R}^d)} \leq 2\|\phi\|_{W^{2,1}(\mathbb{R}^d)} \int_{\mathbb{R}^d\setminus\{0\}} \min\{|z|^2, 1\} \,\mathrm{d}\mu(z).$$

See, e.g., Lemmas 4.1 and 4.2 in [3] for proofs of the above lemmas. The main result of this section is a "Kato inequality" or a "dual equation" for (1.1).

PROPOSITION 4.2. *Assume $(\mathrm{A}_f)$ and $(\mathrm{A}_\varphi)$ hold. Let $u$ and $v$ be entropy sub- and supersolutions of (1.1) with initial data $u_0, v_0 \in L^\infty(\mathbb{R}^d)$ and measurable source terms $g, h$ satisfying $\int_0^T \|g(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} + \|h(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \,\mathrm{d}t < \infty$. If either $\mathfrak{L} = \Delta$ or $\mathfrak{L} = \mathcal{L}^\mu$ and $(\mathrm{A}_\mu)$ holds, then for all nonnegative $\psi \in C_c^\infty(Q_T)$*

$$\begin{aligned}
&\iint_{Q_T} \eta(u(x,t), v(x,t))\partial_t\psi(x,t) + q(u(x,t), v(x,t)) \cdot D\psi(x,t) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \iint_{Q_T} \eta(\varphi(u(x,t)), \varphi(v(x,t)))\mathfrak{L}^*\psi(x,t) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \iint_{Q_T} \eta(g(x,t), h(x,t))\psi(x,t) \,\mathrm{d}x \,\mathrm{d}t \geq 0,
\end{aligned} \tag{4.1}$$

*where* $\eta(u,v) = (u-v)^+$ *and* $q(u,v) = \mathrm{sign}(u-v)^+[f(u)-f(v)]$.

The proof relies on the Kružkov doubling of variables technique, and the result is new in the nonlocal case.

*Proof.* If $\mathfrak{L} = \Delta$, this is a known result; see, e.g., [32, Theorem 3.9]. The result can also be obtained by following the calculations of Karlsen and Risebro; see the proofs of Lemmas 2.3 and 2.4 and Theorem 1.1 in [28]. Our assumptions and Definition 2.1 ensure that equation (3.48) in [28] holds (with Const $= 0$ and $F(x,t,u,v) = F(u,v) = \mathrm{sign}(u-v)[f(u)-f(v)]$) when the solutions $u, v$ are in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^d)) \cap L^\infty(Q_T)$ instead of $C([0,T]; L^1(\mathbb{R}^d)) \cap L^\infty(Q_T)$.

For $\mathfrak{L} = \mathcal{L}^\mu$ we follow the proof of Theorem 3.1 in [14] closely, sketching known estimates and focusing on new ones (which are needed since $u, v \notin L^1$). We start with the Kružkov doubling of variables technique [31, 1, 14]. Since $u$ and $v$ are sub- and supersolutions, we can take Definition 2.2 (a) with $u = u(x,t)$ and $k = v(y,s)$ and Definition 2.2 (b) with $u = v(x,t)$ and $k = u(y,s)$. Integrate the two inequalities over $(y,s) \in Q_T$, rename $(x,t,y,s)$ as $(y,s,x,t)$ in the second one, and add the two inequalities. Then note that $(v-u)^- = (u-v)^+$, $(\varphi(v)-\varphi(u))^- = (\varphi(u)-\varphi(v))^+$, and that we can manipulate (cf. [14, Proof of Theorem 3.1]) the integral with integrand $\mathrm{sign}(u-v)^+(\mathcal{L}^{\mu,r}[\varphi(u)] - \mathcal{L}^{\mu,r}[\varphi(v)])\phi$ to get the integrand of the form $(\varphi(u)-\varphi(v))^+ \tilde{\mathcal{L}}^{\mu^*,r}[\phi]$, where

$$\tilde{\mathcal{L}}^{\mu^*,r}[\phi](x,y) := \int_{|z|>r} \phi(x+z, y+z) - \phi(x,y)\, \mathrm{d}\mu^*(z).$$

Now, we let $\mathrm{d}w := \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$ and send $r \to 0$ to find that

$$\begin{aligned} &\iiiint_{Q_T\times Q_T} (u-v)^+(\partial_t + \partial_s)\phi \\ &\qquad\qquad + \mathrm{sign}(u-v)^+[f(u)-f(v)]\cdot(D_x + D_y)\phi \ \mathrm{d}w \\ &+ \iiiint_{Q_T\times Q_T} (\varphi(u)-\varphi(v))^+ \tilde{\mathcal{L}}^{\mu^*}[\phi(\cdot,t,\cdot,s)](x,y)\,\mathrm{d}w \\ &+ \iiiint_{Q_T\times Q_T} (g-h)^+\phi \ \mathrm{d}w \geq 0, \end{aligned} \tag{4.2}$$

where we have used that $\mathrm{sign}(u-v)^+(g-h) \leq (g-h)^+$. Take

$$\phi(x,t,y,s) = \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right)\omega_{\varepsilon_2}\left(\frac{t-s}{2}\right)\psi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)$$

for $\varepsilon_1, \varepsilon_2 > 0$, $\psi \in C_c^\infty(Q_T)$, where $\omega_\varepsilon$ is a mollifier (see (1.5)) and $\hat{\omega}_{\varepsilon_1}(x)$ is defined by (1.6). We insert this test function into (4.2), noting that

$$\tilde{\mathcal{L}}^{\mu^*}[\phi(\cdot,t,\cdot,s)](x,y) = \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right)\omega_{\varepsilon_2}\left(\frac{t-s}{2}\right)\mathcal{L}^{\mu^*}\left[\psi\left(\cdot, \frac{t+s}{2}\right)\right]\left(\frac{x+y}{2}\right),$$

and then we want to take the limit as $(\varepsilon_1, \varepsilon_2) \to (0,0)$.

So far the proof is quite similar to the proof of Theorem 3.1 in [14]. Taking the last limit, however, requires some attention. Some of the arguments of [14] will not hold here since the solutions are no longer in $L^1$.

The convergence as $(\varepsilon_1, \varepsilon_2) \to (0,0)$ of the local terms is well known (cf. [19, Proof of Theorem 6.2.3]), and the convergence of the source term follows from a simple computation. So here we give details only for the nonlocal term. We need to show that $M \to 0$ for

$$\begin{aligned} M := \Bigg| \iiiint_{Q_T \times Q_T} & \eta(\varphi(u(x,t)), \varphi(v(y,s))) \\ & \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right) \omega_{\varepsilon_2}\left(\frac{t-s}{2}\right) \mathcal{L}^{\mu^*}\left[\psi\left(\cdot, \frac{t+s}{2}\right)\right]\left(\frac{x+y}{2}\right) \mathrm{d}w \\ & - \iint_{Q_T} \eta(\varphi(u(x,t)), \varphi(v(x,t))) \mathcal{L}^{\mu^*}[\psi(\cdot,t)](x) \,\mathrm{d}x \,\mathrm{d}t \Bigg| \end{aligned}$$

and $\eta(a,b) = (a-b)^+$. To do that, we add and subtract

$$\begin{aligned} \iiiint_{Q_T \times Q_T} & \eta(\varphi(u(x,t)), \varphi(v(x,t))) \\ & \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right) \omega_{\varepsilon_2}\left(\frac{t-s}{2}\right) \mathcal{L}^{\mu^*}\left[\psi\left(\cdot, \frac{t+s}{2}\right)\right]\left(\frac{x+y}{2}\right) \mathrm{d}w \end{aligned}$$

and use that

$$\iint_{Q_T} \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right) \omega_{\varepsilon_2}\left(\frac{t-s}{2}\right) \mathrm{d}y \,\mathrm{d}s = 1 \tag{4.3}$$

to get that

$$\begin{aligned} M \le & \iiiint_{Q_T \times Q_T} |\eta(\varphi(u(x,t)), \varphi(v(y,s))) - \eta(\varphi(u(x,t)), \varphi(v(x,t)))| \\ & \qquad \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right) \omega_{\varepsilon_2}\left(\frac{t-s}{2}\right) \mathcal{L}^{\mu^*}\left[\psi\left(\cdot, \frac{t+s}{2}\right)\right]\left(\frac{x+y}{2}\right) \mathrm{d}w \\ & + \iiiint_{Q_T \times Q_T} \eta(\varphi(u(x,t)), \varphi(v(x,t))) \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right) \omega_{\varepsilon_2}\left(\frac{t-s}{2}\right) \\ & \qquad \left| \mathcal{L}^{\mu^*}\left[\psi\left(\cdot, \frac{t+s}{2}\right)\right]\left(\frac{x+y}{2}\right) - \mathcal{L}^{\mu^*}[\psi(\cdot,t)](x) \right| \mathrm{d}w \\ & =: M_1 + M_2. \end{aligned}$$

Since $|\eta(\varphi(u(x,t)), \varphi(v(y,s))) - \eta(\varphi(u(x,t)), \varphi(v(x,t)))| \le |\varphi(v(x,t)) - \varphi(v(y,s))|$, extensive use of adding and subtracting terms and the triangle inequality will give

$$\begin{aligned} M_1 \le \iiiint_{Q_T \times Q_T} & \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right) \omega_{\varepsilon_2}\left(\frac{t-s}{2}\right) \\ & \Bigg\{ |\varphi(v(x,t))| \left| \mathcal{L}^{\mu^*}\left[\psi\left(\cdot, \frac{t+s}{2}\right)\right]\left(\frac{x+y}{2}\right) - \mathcal{L}^{\mu^*}[\psi(\cdot,t)](x) \right| \\ & + \Big| \varphi(v(x,t)) \big| \mathcal{L}^{\mu^*}[\psi(\cdot,t)](x) \big| - \varphi(v(y,s)) \big| \mathcal{L}^{\mu^*}[\psi(\cdot,s)](y) \big| \Big| \\ & + |\varphi(v(y,s))| \left| \mathcal{L}^{\mu^*}\left[\psi\left(\cdot, \frac{t+s}{2}\right)\right]\left(\frac{x+y}{2}\right) - \mathcal{L}^{\mu^*}[\psi(\cdot,s)](y) \right| \Bigg\} \mathrm{d}w. \end{aligned}$$

Let us now show the convergence to zero of the term

$$\begin{aligned} M_2 = \iiiint_{Q_T \times Q_T} & \hat{\omega}_{\varepsilon_1}\left(\frac{x-y}{2}\right) \omega_{\varepsilon_2}\left(\frac{t-s}{2}\right) \eta(\varphi(u(x,t)), \varphi(v(x,t))) \\ & \left| \mathcal{L}^{\mu^*}\left[\psi\left(\cdot, \frac{t+s}{2}\right)\right]\left(\frac{x+y}{2}\right) - \mathcal{L}^{\mu^*}[\psi(\cdot,t)](x) \right| \mathrm{d}w. \end{aligned}$$

Note that $\mathcal{L}^\mu[\psi] \in L^1(Q_T)$ by Lemma 4.1 and that $u, v \in L^\infty(Q_T)$, and hence $\varphi(u), \varphi(v) \in L^\infty(Q_T)$ by ($\mathrm{A}_\varphi$). By a change of variables $y - x = y'$ and $s - t = s'$, changing the order of integration, Hölder's inequality, and (4.3) we get

$$M_2 \leq \|\eta(\varphi(u), \varphi(v))\|_{L^\infty(Q_T)} \sup_{|y'|\leq\varepsilon_1,\, |s'|\leq\varepsilon_2} \left\| \mathcal{L}^{\mu^*}\left[\psi\left(\cdot, t + \frac{s'}{2}\right)\right]\left(x + \frac{y'}{2}\right) - \mathcal{L}^{\mu^*}[\psi(\cdot, t)](x) \right\|_{L^1(Q_T)},$$

which goes to zero as $(\varepsilon_1, \varepsilon_2) \to (0,0)$ by the continuity of the $L^1$ translation. In a similar way, we can also show that $M_1 \to 0$, and the proof is complete. ☐

In the next section we need the following corollary of Proposition 4.2.

Corollary 4.3. *Assume* ($\mathrm{A}_f$), ($\mathrm{A}_\varphi$) *hold, and either* $\mathfrak{L} = \Delta$ *or* $\mathfrak{L} = \mathcal{L}^\mu$ *and* ($\mathrm{A}_\mu$) *holds. Let* $u$ *and* $v$ *be entropy sub- and supersolutions of* (1.1) *with initial data* $u_0, v_0 \in L^\infty(\mathbb{R}^d)$ *and measurable source terms* $g, h$ *satisfying* $\int_0^T \|g(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} + \|h(\cdot,t)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}t < \infty$. *Let* $\psi(x,t) = \Gamma(x,t)\Theta(t)$.

(a) *If* $0 < t < T$, $0 \leq \Gamma \in C_c^\infty(Q_T)$, *and* $0 \leq \Theta \in C_c^\infty((0,T))$, *then*

$$\begin{aligned}0 \leq &\iint_{Q_T} (u-v)^+(x,t)\Gamma(x,t)\Theta'(t)\,\mathrm{d}x\,\mathrm{d}t\\ &+ \iint_{Q_T} \Theta(t)(u-v)^+(x,t)\left[\partial_t\Gamma + L_f|D\Gamma| + L_\varphi\big(\mathfrak{L}^*\Gamma(x,t)\big)^+\right]\mathrm{d}x\,\mathrm{d}t\\ &+ \int_0^T \Theta(t)\int_{\mathbb{R}^d}(g-h)^+(x,t)\Gamma(x,t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

(b) *If* $\varphi(u) = u$ *and* $0 \leq \Gamma \in C([0,T]; L^1(\mathbb{R}^d)) \cap L^1((0,T); W^{2,1}(\mathbb{R}^d)) \cap C^\infty(Q_T) \cap L^\infty(Q_T)$ *satisfies*

$$\partial_t\Gamma + L_f|D\Gamma| + \mathfrak{L}^*\Gamma(x,t) \leq 0 \quad in \quad Q_T,$$

*then*

$$\begin{aligned}&\int_{\mathbb{R}^d}(u-v)^+(x,T)\,\Gamma(x,T)\,\mathrm{d}x\\ &\leq \int_{\mathbb{R}^d}(u_0-v_0)^+(x)\,\Gamma(x,0)\,\mathrm{d}x + \int_0^T\int_{\mathbb{R}^d}(g-h)^+(x,t)\Gamma(x,t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

(c) *If* $0 \leq \Gamma \in C([0,T]; L^1(\mathbb{R}^d)) \cap L^1((0,T); W^{2,1}(\mathbb{R}^d)) \cap C^\infty(Q_T) \cap L^\infty(Q_T)$ *satisfies*

$$\partial_t\Gamma + L_f|D\Gamma| + L_\varphi\big(\mathfrak{L}^*\Gamma(x,t)\big)^+ \leq 0 \quad in \quad Q_T,$$

*then*

$$\begin{aligned}&\int_{\mathbb{R}^d}(u-v)^+(x,T)\Gamma(x,T)\,\mathrm{d}x\\ &\leq \int_{\mathbb{R}^d}(u_0-v_0)^+(x)\Gamma(x,0)\,\mathrm{d}x + \int_0^T\int_{\mathbb{R}^d}(g-h)^+(x,t)\Gamma(x,t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

*Proof.* (a) Remember that $(u-v)^+ = \eta(u,v)$. The proof is a simple consequence of (4.1) and the easy estimates $|q(u,v)\cdot D\Gamma| \leq |q(u,v)||D\Gamma|$, $|q(u,v)| \leq L_f\eta(u,v)$ (see [19, p. 151]), and $\eta(\varphi(u),\varphi(v)) \leq L_\varphi\eta(u,v)$ (by ($\mathrm{A}_\varphi$)), which implies that

$$\eta(\varphi(u),\varphi(v))\mathfrak{L}^*[\Gamma] \leq L_\varphi\eta(u,v)\big(\mathfrak{L}^*[\Gamma]\big)^+.$$

(b) Since this part of the proof is similar to but easier than (c), we omit it. See also [1] for a proof when $\mathfrak{L}^* = -(-\Delta)^{\frac{\alpha}{2}}$.

(c) Since $C_c^\infty(Q_T)$ is dense in

$$E = \{w : w \in C([0,T]; L^1(\mathbb{R}^d)) \cap L^1((0,T); W^{2,1}(\mathbb{R}^d)) \text{ and } \partial_t w \in L^1(Q_T)\}$$

(cf. [1, p. 159]), there is a sequence of functions $\Gamma_\varepsilon \in C_c^\infty(Q_T)$ such that

$$\Gamma_\varepsilon, \partial_t\Gamma_\varepsilon, |D\Gamma_\varepsilon|, \mathfrak{L}^*\Gamma_\varepsilon \to \Gamma, \partial_t\Gamma, |D\Gamma|, \mathfrak{L}^*\Gamma \quad \text{in } L^1(Q_T),$$

when $\varepsilon \to 0^+$. Here we used that $\|\mathfrak{L}^*\Gamma_\varepsilon\|_{L^1(Q_T)} \leq c\|\Gamma_\varepsilon\|_{L^1((0,T);W^{2,1}(\mathbb{R}^d))}$ by the definition of $\Delta$ and by Lemma 4.1. Part (a) holds with $\Gamma_\varepsilon$ replacing $\Gamma$, and then also for $\Gamma$ by sending $\varepsilon \to 0^+$.

By (a) and the extra assumption on $\Gamma$ we see that

$$\begin{aligned} &\iint_{Q_T} (u-v)^+(x,t)\Gamma(x,t)\Theta'(t)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \Theta(t)\int_{\mathbb{R}^d}(g-h)^+(x,t)\Gamma(x,t)\,\mathrm{d}x\,\mathrm{d}t \geq 0. \end{aligned} \tag{4.4}$$

Let $0 \leq \Theta \in C_c^\infty((0,T))$ be defined by

$$\Theta(t) = \Theta_\varepsilon(t) = \int_{-\infty}^t \omega_\varepsilon(s-t_1) - \omega_\varepsilon(s-t_2)\,\mathrm{d}s,$$

where $0 < t_1 < t_2 < T$. For $\varepsilon > 0$ small enough, $\Theta_\varepsilon(t)$ is supported in $[0,T]$ and is a smooth approximation to a square pulse which is one between $t = t_1$ and $t = t_2$ and zero otherwise. By (4.4), we get

$$\begin{aligned} &\iint_{Q_T} (u-v)^+(x,t)\Gamma(x,t)\omega_\varepsilon(t-t_2)\,\mathrm{d}x\,\mathrm{d}t \\ &\leq \iint_{Q_T} (u-v)^+(x,t)\Gamma(x,t)\omega_\varepsilon(t-t_1)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T \Theta_\varepsilon(t)\int_{\mathbb{R}^d}(g-h)^+(x,t)\Gamma(x,t)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Since $\eta(u,v) \in L^\infty(Q_T)$ and $\Gamma \in C([0,T]; L^1(\mathbb{R}^d))$, a direct argument and using the continuity of the $L^1$ translation show the convergence of the integrals involving $(u-v)^+\Gamma\omega_\varepsilon$ as $\varepsilon \to 0^+$. Moreover, since $\int_{\mathbb{R}^d}(g-h)^+(x,t)\Gamma(x,t)\,\mathrm{d}x$ is finite, Lebesgue's dominated convergence theorem will give convergence of the integral involving $\Theta_\varepsilon(g-h)^+\Gamma$ as $\varepsilon \to 0^+$. Thus, we end up with

$$\begin{aligned} &\int_{\mathbb{R}^d} (u-v)^+(x,t_2)\Gamma(x,t_2)\,\mathrm{d}x \\ &\leq \int_{\mathbb{R}^d} (u-v)^+(x,t_1)\Gamma(x,t_1)\,\mathrm{d}x \\ &\quad + \int_{t_1}^{t_2}\int_{\mathbb{R}^d}(g-h)^+(x,t)\Gamma(x,t)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Finally, the conclusion can be obtained by letting $t_2 \to T^-$ and $t_1 \to 0^+$. Since $u, v \in C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^d))$ and $\Gamma \in C([0,T]; L^1(\mathbb{R}^d))$, we can use Fatou's lemma on the left-hand side (the integrand is nonnegative) as $t_2 \to T^-$. The computations as $t_1 \to 0^+$ of the first integral on the right-hand side are shown in the following:

$$\begin{aligned}&\|(u-v)^+(\cdot,t_1)\Gamma(\cdot,t_1) - (u-v)^+(\cdot,0)\Gamma(\cdot,0)\|_{L^1(\mathbb{R}^d)}\\ &\leq \|(u-v)^+\|_{L^\infty(Q_T)}\|\Gamma(\cdot,t_1)-\Gamma(\cdot,0)\|_{L^1(\mathbb{R}^d)}\\ &\quad + \|((u-v)^+(\cdot,t_1)-(u-v)^+(\cdot,0))\Gamma(\cdot,0)\|_{L^1(\mathbb{R}^d)},\end{aligned}$$

where the first term goes to zero as $t_1 \to 0^+$ since $\Gamma \in C([0,T];L^1(\mathbb{R}^d))$. The second term, however, needs a more refined argument. By Definition 2.1 or 2.2 (a) it follows that as $t \to 0^+$, $u(\cdot,t) \to u(\cdot,0)$ in $L^1_{\text{loc}}(\mathbb{R}^d)$ and hence also pointwise a.e. (along a subsequence). Moreover, $|(u-v)^+(x,t_1)-(u-v)^+(x,0)|\Gamma(x,0)$ is dominated by $2\|(u-v)^+\|_{L^\infty(Q_T)}\Gamma(x,0) \in L^1(\mathbb{R}^d)$. Hence, Lebesgue's dominated convergence theorem ensures that the second term also goes to zero when $t_1 \to 0^+$.

We conclude by using Lebesgue's dominated convergence theorem on the integral involving $(g-h)^+\Gamma$ as $t_2 \to T^-$ and $t_1 \to 0^+$, and by noting that $(u-v)^+(x,0) \leq (u_0-v_0)^+(x)$ by Definition 2.1 or 2.2 (a) and (b). ☐

**5. Proof of Theorems 2.9 and 2.10.** In previous proofs of $L^1$ contractions (see, e.g., [19, 1]), even if it was not written in that way, the idea was essentially to prove a result like Corollary 4.3 (b) and then construct a suitable $\Gamma$ to conclude. In a similar way, we will construct $\Gamma$'s for Corollary 4.3 (b) and (c) and then conclude. Note that since (2.2) is fully nonlinear and degenerate, this task will be much more difficult than in [1], where $\mathfrak{L} = -(-\Delta)^{\frac{\alpha}{2}}$ and $\varphi(u) = u$.

As in [1], we will build $\Gamma$ by the convolution of subsolutions of simpler problems, but first we give an auxiliary result.

LEMMA 5.1. *If $\phi \in L^1(\mathbb{R}^d)$ is nonnegative and $f \in C_b(\mathbb{R}^d)$, then*

$$(\phi * f)^+ \leq \phi * f^+ \quad \text{and} \quad |\phi * f| \leq \phi * |f|.$$

*Proof.* The proofs are easy and similar, so we do only one case. Since

$$0 \leq \int_{\mathbb{R}^d} \phi(x-y)\max\{f(y),0\}\,\mathrm{d}y$$

and

$$\int_{\mathbb{R}^d} \phi(x-y)f(y)\,\mathrm{d}y \leq \int_{\mathbb{R}^d} \phi(x-y)\max\{f(y),0\}\,\mathrm{d}y,$$

the proof is immediate. ☐

LEMMA 5.2. *Assume that $\mathfrak{L} = \Delta$ or $\mathfrak{L} = \mathcal{L}^\mu$ and $(\mathrm{A}_\mu)$ holds, assume that $0 \leq \phi(x,t) \in C^\infty(Q_T) \cap C([0,T];L^1(\mathbb{R}^d)) \cap L^\infty(Q_T)$ solves*

$$\partial_t\phi(x,t) + L_f|D\phi(x,t)| \leq 0 \quad \text{in} \quad Q_T,$$

*and define $\Gamma(x,t) = [\psi(\cdot,t) * \phi(\cdot,t)](x)$.*

(a) *If $0 \leq \psi(x,t) \in C^\infty(Q_T) \cap C([0,T];L^1(\mathbb{R}^d)) \cap L^\infty(Q_T)$ solves*

$$\partial_t\psi(x,t) + \mathfrak{L}^*\psi(x,t) \leq 0 \quad \text{in} \quad Q_T,$$

*then $0 \leq \Gamma \in C([0,T];L^1(\mathbb{R}^d)) \cap C^\infty(Q_T)$ and solves*

$$\partial_t\Gamma(x,t) + L_f|D\Gamma(x,t)| + \mathfrak{L}^*\Gamma(x,t) \leq 0 \quad \text{in} \quad Q_T.$$

(b) *If* $0 \le \psi(x,t) \in C^\infty(Q_T) \cap C([0,T];L^1(\mathbb{R}^d)) \cap L^\infty(Q_T)$ *solves*

$$\partial_t \psi(x,t) + L_\varphi(\mathfrak{L}^* \psi(x,t))^+ \le 0 \quad \text{in} \quad Q_T,$$

*then* $0 \le \Gamma \in C([0,T];L^1(\mathbb{R}^d)) \cap C^\infty(Q_T)$ *and solves*

$$\partial_t \Gamma(x,t) + L_f|D\Gamma(x,t)| + L_\varphi\big(\mathfrak{L}^*\Gamma(x,t)\big)^+ \le 0 \quad \text{in} \quad Q_T.$$

*Remark* 5.3. If $\mathfrak{L}^* = \mathfrak{L} = -(-\Delta)^{\frac{\alpha}{2}}$, $\alpha \in (0,2]$, then Lemma 5.2 (a) is satisfied with $\psi(x,t) = \tilde{K}(x,\tau - t)$ for $0 \le t \le \tau$, where $\tilde{K}$ is defined by (2.1).

*Proof.* We prove only (b) since the proof of (a) is similar but easier. By Lemma 5.1 and properties of convolutions

$$\partial_t \Gamma(x,t) = \big[\partial_t \psi(\cdot,t) * \phi(\cdot,t)\big](x) + \big[\psi(\cdot,t) * \partial_t \phi(\cdot,t)\big](x),$$

$$|D\Gamma(x,t)| \le \big[\psi(\cdot,t) * |D\phi(\cdot,t)|\big](x),$$

and

$$(\mathfrak{L}^*\Gamma(x,t))^+ = \big[\phi(\cdot,t) * \mathfrak{L}^*\psi(\cdot,t)\big]^+(x) \le \big[\phi(\cdot,t) * (\mathfrak{L}^*\psi(\cdot,t))^+\big](x).$$

An easy computation using the assumptions on $\phi$ and $\psi$ then gives the result. ◻

To find a $\psi$ for Lemma 5.2, we take the (viscosity) solution of (2.2) and mollify it. We start by several auxiliary results and the proof of Lemma 2.6.

LEMMA 5.4. *Assume that* $\mathfrak{L} = \Delta$ *or* $\mathfrak{L} = \mathcal{L}^\mu$ *and* ($\mathrm{A}_\mu$) *holds. If* $\Phi \in C_b(Q_T)$ *is a viscosity solution of* (2.2) *and* $\rho_\delta$ *is a mollifier satisfying* (1.7)*, then*

$$\Phi_\delta(x,t) := [\Phi * \rho_\delta](x,t) = \iint_{\mathbb{R}^d \times \mathbb{R}} \Phi(x-y,t-s)\rho_\delta(y,s)\,\mathrm{d}y\,\mathrm{d}s \tag{5.1}$$

*is a classical supersolution of* (2.2):

$$\partial_t \Phi_\delta(x,t) \ge (\mathfrak{L}^*\Phi_\delta(x,t))^+. \tag{5.2}$$

*Remark* 5.5. As usual, $\lim_{\delta \to 0^+} \Phi_\delta = \Phi$ pointwise.

*Outline of proof.* To understand the idea behind the proof, let $\Phi(y,s)$ be a classical solution of (2.2). Multiply the equation by $\rho_\delta(x-y,t-s)$, integrate over $\mathbb{R}^d \times \mathbb{R}$ w.r.t. $(y,s)$, and use Lemma 5.1 to conclude

$$\begin{aligned}
0 &= \int_{\mathbb{R}}\int_{\mathbb{R}^d} \partial_t \Phi(y,s)\rho_\delta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad - \int_{\mathbb{R}}\int_{\mathbb{R}^d} (\mathfrak{L}^*\Phi(y,s))^+ \rho_\delta(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s \\
&\le \partial_t[\Phi * \rho_\delta](x,t) - (\mathfrak{L}^*[\Phi * \rho_\delta](x,t))^+ \\
&= \partial_t \Phi_\delta - (\mathfrak{L}^*\Phi_\delta)^+.
\end{aligned}$$

We refer the reader to [7, Theorem 3.1 (a)] for a proof in the case $\mathfrak{L} = \Delta$ and to [27, Theorem 6.4] for how to adapt this proof when $\mathfrak{L} = \mathcal{L}^\mu$. ◻

We state some well-known results for (2.2); see, e.g., [18, 26] for proofs.

LEMMA 5.6. *Assume that* $\mathfrak{L} = \Delta$ *or* $\mathfrak{L} = \mathcal{L}^\mu$ *and* ($\mathrm{A}_\mu$) *holds.*

(a) *If* $u_0 \in C_b(\mathbb{R}^d)$*, then there exists a unique viscosity solution* $u \in C_b(Q_T)$ *of* (2.2)*.*

(b) *If* $u$ *and* $v$ *are viscosity sub- and supersolutions of* (2.2) *and* $u_0 \le v_0$ *on* $\mathbb{R}^d$*, then* $u \le v$ *in* $Q_T$*.*

(c) *If* $u$ *is a solution of* (2.2) *with initial data* $u_0 \in W^{1,\infty}(\mathbb{R}^d)$*, then*

$$|u(x,t) - u(y,s)| \le C(|x-y| + |t-s|^{\frac{1}{2}}) \qquad \textit{for} \qquad (x,t),(y,s) \in Q_T.$$

(d) *If* $u$ *is a classical subsolution (supersolution) of* (2.2)*, then* $u$ *is a viscosity subsolution (supersolution) of* (2.2)*.*

*Proof of Lemma* 2.6. Since $\Phi_0(x)$ belongs to $C_c^\infty(\mathbb{R}^d)$ (and hence$W^{1,\infty}(\mathbb{R}^d)$) by assumption, there exists a unique viscosity solution $\Phi \in C_b(Q_{\tilde{T}})$ of (2.2) by Lemma 5.6 (a). Furthermore, since $0 \le \Phi_0(x)$, $0 \le \Phi(x,t)$ by Lemma 5.6 (b).

We claim that there are $C > 0$, $k > 0$, $K > 0$ such that for all $|\xi| = 1$,

$$\Phi(x,t) \le w(x,t) := C\mathrm{e}^{Kt}\mathrm{e}^{k\xi\cdot x} \quad \text{in } Q_{\tilde{T}}.$$

If this is the case, then $\Phi(x,t) \le C\mathrm{e}^{Kt}\mathrm{e}^{-k|x|}$ (take $\xi = -\frac{x}{|x|}$ for $x \neq 0$) and $\Phi \in L^\infty(0,\tilde{T};L^1(\mathbb{R}^d))$. Moreover, $\Phi \in C([0,\tilde{T}];L^1(\mathbb{R}^d))$ since by Lebesgue's dominated convergence theorem (the integrand is dominated by $2C\mathrm{e}^{K\tilde{T}}\mathrm{e}^{-k|x|}$),

$$\lim_{h\to 0}\int_{\mathbb{R}^d} |\Phi(x,t+h) - \Phi(x,t)|\,\mathrm{d}x = 0 \qquad \text{for all} \qquad t \in [0,\tilde{T}].$$

To complete the proof, it remains only to prove the claim.

Let $\mathfrak{L}^* = \mathcal{L}^{\mu^*}$, and assume that $(\mathrm{A}_\mu^+)$ holds. Note that $\partial_t w = Kw$ and

$$\begin{aligned}
&\mathcal{L}^{\mu^*}[w(\cdot,t)](x)\\
&= \int_{|z|>0} w(x+z,t) - w(x,t) - z\cdot Dw(x,t)\mathbf{1}_{|z|\le 1}\,\mathrm{d}\mu^*(z)\\
&= w(x,t)\left[\int_{0<|z|\le 1} \mathrm{e}^{k\xi\cdot z} - 1 - k\xi\cdot z \;\mathrm{d}\mu^*(z) + \int_{|z|>1} \mathrm{e}^{k\xi\cdot z} - 1 \;\mathrm{d}\mu^*(z)\right].
\end{aligned}$$

Take $k \le M$, where $M$ is defined in $(\mathrm{A}_\mu^+)$. Then by Taylor's theorem and $(\mathrm{A}_\mu^+)$,

$$\mathcal{L}^{\mu^*}[w(\cdot,t)](x) \le C_k w(x,t),$$

where

$$C_k := \frac{\mathrm{e}^k}{2}k^2 \int_{0<|z|\le 1} |z|^2\,\mathrm{d}\mu^*(z) + \int_{|z|>1} \mathrm{e}^{M|z|}\,\mathrm{d}\mu^*(z) \in (0,\infty).$$

It then follows that

$$\partial_t w - (\mathcal{L}^{\mu^*}[w])^+ = \partial_t w + \min\{-\mathcal{L}^{\mu^*}[w],0\} \ge w(K - C_k).$$

We take $K$ such that $K - C_k \ge 0$ in order to make $w$ a supersolution. Now, choose $C$ such that $\Phi_0 \le w(\cdot,0)$. Then Lemma 5.6 (d) shows that $w$ is a viscosity supersolution, and Lemma 5.6 (b) ensures that $\Phi(x,t) \le w(x,t)$.

When $\mathfrak{L}^* = \Delta$, the argument is similar. We take any $k > 0$ and a $C$ such that $\Phi_0 \le w(\cdot,0)$, and then we observe that

$$\partial_t w - (\Delta w)^+ = w(K - k^2).$$

If $K-k^2 \geq 0$, then Lemma 5.6 (d) and (b) ensure that $\Phi(x,t) \leq w(x,t)$ as before. ∎

PROPOSITION 5.7. *Let $\Phi$ be the function given by Lemma* 2.6, $\tilde{T} = \max\{T, L_\varphi T\}$, *and let $L_\varphi$ be the Lipschitz constant of $\varphi$. Then $\Phi_\delta(x,t)$ defined by* (5.1) *solves* (5.2) *and satisfies*

$$0 \leq \Phi_\delta \in C([0,\tilde{T}];L^1(\mathbb{R}^d)) \cap C^\infty(Q_{\tilde{T}}) \cap L^\infty(Q_{\tilde{T}}),$$

*and*

$$\|\Phi_\delta(\cdot,0) - \Phi_0\|_{L^\infty(\mathbb{R}^d)} \leq C\delta, \tag{5.3}$$

*where $C$ is some constant independent of $\delta > 0$.*

*Proof.* First note that $\Phi$, $\rho_\delta$, and, hence, $\Phi_\delta$ are nonnegative and bounded and $\rho_\delta$ and $\Phi_\delta$ are smooth. Moreover, by Tonelli's theorem $\Phi_\delta \in C([0,\tilde{T}];L^1(\mathbb{R}^d))$ since

$$\int_{\mathbb{R}^d} \Phi_\delta(x,t)\,\mathrm{d}x = \iint_{\mathbb{R}^d\times\mathbb{R}} \rho_\delta(y,s) \int_{\mathbb{R}^d} \Phi(x-y,t-s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \leq \max_{t\in[0,\tilde{T}]} \|\Phi(\cdot,t)\|_{L^1(\mathbb{R}^d)}.$$

By Lemma 5.4, $\Phi_\delta$ is a classical supersolution of (2.2) and hence solves (5.2).

We use simple computations, the compact support of $\rho_\delta$, and Lemma 5.6 (c) to obtain

$$\begin{aligned}
&|\Phi_\delta(x,0) - \Phi_0(x)| \\
&\leq \iint_{\mathbb{R}^d\times\mathbb{R}} \left(|\Phi(x-y,0-s) - \Phi_0(x-y)| + |\Phi_0(x-y) - \Phi_0(x)|\right) \rho_\delta(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&\leq \iint_{\mathbb{R}^d\times\mathbb{R}} C(|s|^{\frac{1}{2}} + |y|)\rho_\delta(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&\leq C\left(\sup_{s\in(0,\delta^2)} |s|^{\frac{1}{2}} + \sup_{y\in(-\delta,\delta)^d} |y|\right) \iint_{\mathbb{R}^d\times\mathbb{R}} \rho_\delta(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&= C\delta,
\end{aligned}$$

and hence (5.3) holds. ∎

COROLLARY 5.8. *Let $\Phi_\delta$ be the function given by Proposition* 5.7, $\tilde{T} = \max\{T, L_\varphi T\}$, $0 < \tau < \tilde{T}$ *and* $0 \leq t \leq \tau$, *and let*

$$K_\delta(x,t) := \Phi_\delta(x, L_\varphi(\tau - t)),$$

*where $L_\varphi$ is the Lipschitz constant of $\varphi$. Then*

$$0 \leq K_\delta \in C([0,\tilde{T}];L^1(\mathbb{R}^d)) \cap C^\infty(Q_{\tilde{T}}) \cap L^\infty(Q_{\tilde{T}})$$

*solves*

$$\partial_t K_\delta + L_\varphi(\mathfrak{L}^* K_\delta)^+ \leq 0 \quad \textit{in } Q_{\tilde{T}}$$

*and satisfies*

$$\|K_\delta(\cdot,\tau) - \Phi_0\|_{L^\infty(\mathbb{R}^d)} \leq C\delta,$$

*where $C$ is a constant independent of $\delta > 0$.*

To complete the collection of lemmas needed to prove Theorems 2.9 and 2.10, we now show how to choose $\phi$ in Lemma 5.2.

LEMMA 5.9. *Let $L_f$ be the Lipschitz constant of $f$, $0 < \tau < T$, $0 \le t \le \tau$, $R > L_f T + 1$, $\tilde{\delta} > 0$, $x_0 \in \mathbb{R}^d$, and*

$$\gamma_{\tilde{\delta}}(x,t) := \left[\mathbf{1}_{(-\infty,R]} * \omega_\varepsilon\right]\left(\sqrt{\tilde{\delta}^2 + |x - x_0|^2} + L_f t\right), \tag{5.4}$$

*where $\omega_\varepsilon$ is a mollifier (defined by* (1.5)*). Then $\gamma_{\tilde{\delta}} \in C_c^\infty(Q_T)$ and*

$$\partial_t \gamma_{\tilde{\delta}}(x,t) + L_f |D\gamma_{\tilde{\delta}}(x,t)| \le 0.$$

Since $\left[\mathbf{1}_{(-\infty,R]} * \omega_\varepsilon\right]' \le 0$ in $\mathbb{R}_+$, the proof is a straightforward computation.

*Proof of Theorem* 2.10. Let $0 < \tau < T$, $R > L_f T + 1$, $x_0 \in \mathbb{R}^d$, and $\varepsilon, \delta, \tilde{\delta} > 0$, and $\gamma_{\tilde{\delta}}$ be defined by (5.4). Define

$$\gamma(x,t) := \lim_{\tilde{\delta} \to 0^+} \gamma_{\tilde{\delta}}(x,t) = \left[\mathbf{1}_{(-\infty,R]} * \omega_\varepsilon\right](|x - x_0| + L_f t)$$

and

$$\Gamma(x,t) = \left[K_\delta(\cdot,t) * \gamma_{\tilde{\delta}}(\cdot,t)\right](x) \qquad \text{for} \qquad 0 \le t \le \tau,$$

where $K_\delta$ is given by Corollary 5.8. By the properties of $K_\delta$, and since $0 \le \gamma_{\tilde{\delta}} \in C_c^\infty(Q_T)$,

$$0 \le \Gamma \in C([0,\tau]; L^1(\mathbb{R}^d)) \cap L^1(0,\tau; W^{2,1}(\mathbb{R}^d)) \cap C^\infty(Q_\tau) \cap L^\infty(Q_\tau).$$

By Lemma 5.2 (with $\phi = \gamma_{\tilde{\delta}}$ and $\psi = K_\delta$) and Corollary 4.3 (c), it then follows that

$$\int_{\mathbb{R}^d} (u-v)^+(x,\tau)\,\Gamma(x,\tau)\,\mathrm{d}x \le \int_{\mathbb{R}^d} (u_0 - v_0)^+(x)\,\Gamma(x,0)\,\mathrm{d}x + \int_0^\tau \int_{\mathbb{R}^d} (g-h)^+(x,t)\,\Gamma(x,t)\,\mathrm{d}x\,\mathrm{d}t,$$

or

$$\begin{aligned} &\int_{\mathbb{R}^d} (u-v)^+(x,\tau)\left[K_\delta(\cdot,\tau) * \gamma_{\tilde{\delta}}(\cdot,\tau)\right](x)\,\mathrm{d}x \\ &\le \int_{\mathbb{R}^d} (u_0 - v_0)^+(x)\left[K_\delta(\cdot,0) * \gamma_{\tilde{\delta}}(\cdot,0)\right](x)\,\mathrm{d}x \\ &\quad + \int_0^\tau \int_{\mathbb{R}^d} (g-h)^+(x,t)\left[K_\delta(\cdot,t) * \gamma_{\tilde{\delta}}(\cdot,t)\right](x)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{5.5}$$

We use Tonelli's theorem to rewrite the right-hand side,

$$\begin{aligned} &\int_{\mathbb{R}^d} (u_0 - v_0)^+(x) \int_{\mathbb{R}^d} K_\delta(x-y,0)\gamma_{\tilde{\delta}}(y,0)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \gamma_{\tilde{\delta}}(y,0) \int_{\mathbb{R}^d} (u_0 - v_0)^+(x) K_\delta(x-y,0)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \gamma_{\tilde{\delta}}(x,0)\left[K_\delta(-\cdot,0) * (u_0 - v_0)^+\right](x)\,\mathrm{d}x, \end{aligned} \tag{5.6}$$

and similarly,

$$\begin{aligned}
&\int_0^\tau \int_{\mathbb{R}^d} (g-h)^+(x,t) \left[K_\delta(\cdot,t) * \gamma_{\tilde{\delta}}(\cdot,t)\right](x) \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_0^\tau \int_{\mathbb{R}^d} \gamma_{\tilde{\delta}}(x,t) \left[K_\delta(-\cdot,t) * (g(\cdot,t)-h(\cdot,t))^+\right](x) \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}$$

With the above manipulation in mind, we take the limit inferior of (5.5) as $\tilde{\delta} \to 0^+$ using Fatou's lemma on the left-hand side (the integrand is nonnegative) and Lebesgue's dominated convergence theorem on the right-hand side since the integrands are dominated by $2\left[\mathbf{1}_{(-\infty,2R]} * \omega_\varepsilon\right](|x-x_0| + L_f t) K_\delta(-y,t) M(t)$ for $M(t) = \|u_0\|_{L^\infty(\mathbb{R}^d)} + \|v_0\|_{L^\infty(\mathbb{R}^d)} + \|g(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} + \|h(\cdot,t)\|_{L^\infty(\mathbb{R}^d)}$. Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^d} (u-v)^+(x,\tau) \left[K_\delta(\cdot,\tau) * \gamma(\cdot,\tau)\right](x) \,\mathrm{d}x \\
&\le \int_{\mathbb{R}^d} \gamma(x,0) \left[K_\delta(-\cdot,0) * (u_0-v_0)^+\right](x) \,\mathrm{d}x \\
&\quad + \int_0^\tau \int_{\mathbb{R}^d} \gamma(x,t) \left[K_\delta(-\cdot,t) * (g(\cdot,t)-h(\cdot,t))^+\right](x) \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned} \tag{5.7}$$

By Hölder's inequality and Corollary 5.8,

$$\begin{aligned}
&\left|\left[K_\delta(\cdot,\tau) * \gamma(\cdot,\tau)\right](x) - \left[\Phi_0 * \gamma(\cdot,\tau)\right](x)\right| \\
&\le \|K_\delta(\cdot,\tau) - \Phi_0\|_{L^\infty(\mathbb{R}^d)} \|\gamma(\cdot,\tau)\|_{L^1(\mathbb{R}^d)} \\
&= C\delta.
\end{aligned}$$

Hence, taking the limit inferior as $\delta \to 0^+$ in (5.7) using Fatou's lemma gives

$$\begin{aligned}
&\int_{\mathbb{R}^d} (u-v)^+(x,\tau) \left[\Phi_0 * \gamma(\cdot,\tau)\right](x) \,\mathrm{d}x \\
&\le \liminf_{\delta\to 0^+} \int_{\mathbb{R}^d} \gamma(x,0) \left[K_\delta(-\cdot,0) * (u_0-v_0)^+\right](x) \,\mathrm{d}x \\
&\quad + \liminf_{\delta\to 0^+} \int_0^\tau \int_{\mathbb{R}^d} \gamma(x,t) \left[K_\delta(-\cdot,t) * (g(\cdot,t)-h(\cdot,t))^+\right](x) \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned} \tag{5.8}$$

Now, let $C_c^\infty(\mathbb{R}^d) \ni \Phi_0(x) := \hat{\omega}_{\tilde{\varepsilon}}(x-x_0)$ (see (1.6)). Note that $[\Phi_0 * \gamma(\cdot,\tau)] \ge 0$ and that $[\Phi_0 * \gamma(\cdot,\tau)](x) = 1$ when $|x-x_0| < R - L_f\tau - \varepsilon - \tilde{\varepsilon}$. Hence, if $\varepsilon + \tilde{\varepsilon} < 1$, then

$$[\Phi_0 * \gamma(\cdot,\tau)](x) \ge \mathbf{1}_{|x-x_0| \le R - L_f\tau - 1},$$

and hence we have the following lower bound for the left-hand side of (5.8):

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbf{1}_{|x-x_0| \le R - L_f\tau - 1} (u-v)^+(x,\tau) \,\mathrm{d}x \\
&\le \int_{\mathbb{R}^d} (u-v)^+(x,\tau) \left[\Phi_0 * \gamma(\cdot,\tau)\right](x) \,\mathrm{d}x.
\end{aligned}$$

Observe that we cannot send $\tilde{\varepsilon} \to 0^+$ here because this will violate the inequality $w(x,0) \ge \Phi_0$ in the proof of Proposition 5.7, and we would lose the $L^1$ bound on $K_\delta$.

Consider the first term on the right-hand side of (5.8). Note that $\gamma(x,0) = \left[\mathbf{1}_{(-\infty,R]} * \omega_\varepsilon\right](|x-x_0|)$ and $K_\delta(-\cdot,0) = \Phi_\delta(-\cdot, L_\varphi\tau)$, and define

$$\begin{aligned} M := \Bigg| &\int_{\mathbb{R}^d} \left[\mathbf{1}_{(-\infty,R]} * \omega_\varepsilon\right](|x-x_0|)\left[\Phi_\delta(-\cdot, L_\varphi\tau) * (u_0-v_0)^+\right](x)\,\mathrm{d}x \\ &- \int_{\mathbb{R}^d} \left[\mathbf{1}_{(-\infty,R]} * \omega_\varepsilon\right](|x-x_0|)\left[\Phi(-\cdot, L_\varphi\tau) * (u_0-v_0)^+\right](x)\,\mathrm{d}x\Bigg| \\ \le &\int_{\mathbb{R}^d} \left[\mathbf{1}_{(-\infty,R]} * \omega_\varepsilon\right](|x-x_0|) \\ &\qquad \left|\left[\Phi_\delta(-\cdot, L_\varphi\tau) * (u_0-v_0)^+\right](x) - \left[\Phi(-\cdot, L_\varphi\tau) * (u_0-v_0)^+\right](x)\right|\,\mathrm{d}x. \end{aligned}$$

We will show that $M \to 0$ as $\delta \to 0^+$, a result which follows from Lebesgue's dominated convergence theorem if

$$\tilde{M} := \left|\left[\Phi_\delta(-\cdot, L_\varphi\tau) * (u_0-v_0)^+\right](x) - \left[\Phi(-\cdot, L_\varphi\tau) * (u_0-v_0)^+\right](x)\right| \to 0$$

a.e. as $\delta \to 0^+$. By the definitions of $\Phi_\delta$ and $\rho_\delta$ ((5.1) and (1.7)), interchanging the order of integration, and Hölder's inequality, we find that

$$\begin{aligned} \tilde{M} \le &\left(\|u_0\|_{L^\infty(\mathbb{R}^d)} + \|v_0\|_{L^\infty(\mathbb{R}^d)}\right) \\ &\iint_{\mathbb{R}^d\times\mathbb{R}} \rho_\delta(\xi,s)\,\|\Phi(-\xi-\cdot, L_\varphi\tau - s) - \Phi(-\cdot, L_\varphi\tau)\|_{L^1(\mathbb{R}^d)}\,\mathrm{d}\xi\,\mathrm{d}s. \end{aligned}$$

The triangle and Hölder inequalities and the compact support of $\rho_\delta$ then give

$$\begin{aligned} \tilde{M} \le\ &\left(\|u_0\|_{L^\infty(\mathbb{R}^d)} + \|v_0\|_{L^\infty(\mathbb{R}^d)}\right) \\ &\cdot\Bigg\{\sup_{|s|<\delta^2} \|\Phi(-\cdot, L_\varphi\tau - s) - \Phi(-\cdot, L_\varphi\tau)\|_{L^1(\mathbb{R}^d)} \\ &+ \sup_{|\xi|<\delta} \|\Phi(-\xi-\cdot, L_\varphi\tau) - \Phi(-\cdot, L_\varphi\tau)\|_{L^1(\mathbb{R}^d)}\Bigg\}. \end{aligned}$$

The two suprema (and hence also $\tilde{M}$ and $M$) converge to zero since $\Phi \in C([0,T]; L^1(\mathbb{R}^d))$ and by the continuity of the $L^1$ translation, respectively.

The second term on the right-hand side of (5.8) can be estimated by similar arguments (note that $K_\delta(x,t) = \Phi_\delta(x, L_\varphi(\tau - t))$), and when we combine all the estimates we find the following inequality:

$$\begin{aligned} &\int_{\mathbb{R}^d} \mathbf{1}_{|x-x_0|\le R-L_f\tau-1}(u-v)^+(x,\tau)\,\mathrm{d}x \\ &\le \int_{\mathbb{R}^d} \left[\mathbf{1}_{(-\infty,R]} * \omega_\varepsilon\right](|x-x_0|)\left[\Phi(-\cdot, L_\varphi\tau) * (u_0-v_0)^+\right](x)\,\mathrm{d}x \\ &\quad + \int_0^\tau \int_{\mathbb{R}^d} \left[\mathbf{1}_{(-\infty,R]} * \omega_\varepsilon\right](|x-x_0| + L_f t) \\ &\qquad\qquad \left[\Phi(-\cdot, L_\varphi(\tau-t)) * (g(\cdot,t) - h(\cdot,t))^+\right](x)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

The integrands on the right-hand side are dominated by $2\mathbf{1}_{(-\infty,2R]}(|x-x_0| + L_f t)$ $\Phi(-y, L_\varphi(\tau-t))M(t)$, where $M(t) = \|u_0\|_{L^\infty(\mathbb{R}^d)} + \|v_0\|_{L^\infty(\mathbb{R}^d)} + \|g(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} +$

$\|h(\cdot,t)\|_{L^\infty(\mathbb{R}^d)}$, so we can use Lebesgue's dominated convergence theorem to send $\varepsilon \to 0^+$ and obtain

$$\begin{aligned}&\int_{B(x_0,R-L_f\tau-1)}(u(x,\tau)-v(x,\tau))^+\,\mathrm{d}x\\&\le\int_{B(x_0,R)}\left[\Phi(-\cdot,L_\varphi\tau)*(u_0-v_0)^+\right](x)\,\mathrm{d}y\,\mathrm{d}x\\&\quad+\int_0^\tau\int_{B(x_0,R-L_ft)}\left[\Phi(-\cdot,L_\varphi(\tau-t))*(g(\cdot,t)-h(\cdot,t))^+\right](x)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

For any $M>0$, we set $R=M+1+L_f\tau$. Since $\tau\in(0,T)$ is arbitrary, the proof of Theorem 2.10 is complete. ◻

*Proof of Theorem* 2.9. We sketch the proof in the case when $g=0$. We proceed as in the proof of Theorem 2.10, this time with the choice $\psi(x,t)=\tilde{K}(x,\tau-t)$ for $0\le t\le\tau$ (see Remark 5.3). We obtain an inequality like (5.5), take the limit as $t\to\tau^-$ in (5.5), and find that

$$\begin{aligned}&\lim_{t\to\tau^-}\int_{\mathbb{R}^d}(u-v)^+(x,\tau)\left[\tilde{K}(\cdot,\tau-t)*\gamma_{\tilde{\delta}}(\cdot,\tau)\right](x)\,\mathrm{d}x\\&\le\int_{\mathbb{R}^d}(u_0-v_0)^+(x)\left[\tilde{K}(\cdot,\tau)*\gamma_{\tilde{\delta}}(\cdot,0)\right](x)\,\mathrm{d}x.\end{aligned}$$

Following (5.6) (using Lemma 2.5 (d)), using that $\tilde{K}$ is an approximate delta function in time, and taking the limit as $\tilde{\delta}\to 0^+$, we get

$$\begin{aligned}&\int_{\mathbb{R}^d}\left[\mathbf{1}_{(-\infty,R]}*\omega_\varepsilon\right](|x-x_0|+L_f\tau)(u(x,\tau)-v(x,\tau))^+\,\mathrm{d}x\\&\le\int_{\mathbb{R}^d}\left[\mathbf{1}_{(-\infty,R]}*\omega_\varepsilon\right](|x-x_0|)\,\left[\tilde{K}(\cdot,\tau)*(u_0-v_0)^+\right](x)\,\mathrm{d}x\end{aligned}$$

by Fatou's lemma, Lebesgue's dominated convergence theorem, and Lemma 2.5 (c). Taking the limit as $\varepsilon\to 0^+$ (using Lemma 2.5 (b), Fatou's lemma, and Lebesgue's dominated convergence theorem) yields for any $M>0$ with $R=M+L_f\tau$

$$\int_{B(x_0,M)}(u(x,\tau)-v(x,\tau))^+\,\mathrm{d}x\le\int_{B(x_0,M+L_f\tau)}\left[\tilde{K}(\cdot,\tau)*(u_0-v_0)^+\right](x)\,\mathrm{d}x.\qquad ◻$$

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