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HELSON'S PROBLEM FOR SUMS OF A RANDOM MULTIPLICATIVE FUNCTION

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ABSTRACT. We consider the random functions $S_N(z) := \sum_{n=1}^{N} z(n)$, where $z(n)$ is the completely multiplicative random function generated by independent Steinhaus variables $z(p)$. It is shown that $\mathbb{E}|S_N| \gg \sqrt{N}(\log N)^{-0.05616}$ and that $(\mathbb{E}|S_N|^q)^{1/q} \gg_q \sqrt{N}(\log N)^{-0.07672}$ for all $q > 0$.

1. INTRODUCTION

This paper deals with the following

Question. *Do there exist absolute constants $c > 0$, $0 < \lambda < 1$ such that for every positive integer N and every interval I whose length exceeds some number depending on N, we have*

$$\left| \sum_{n=1}^{N} n^{-it} \right| \geq c\sqrt{N}$$

on a subset of I of measure larger than $\lambda|I|$?

We do not know the answer and can only conclude from our main result that we have, for every $\varepsilon > 0$ and suitable $c = c(\varepsilon)$,

$$\left| \sum_{n=1}^{N} n^{-it} \right| \geq c\sqrt{N}(\log N)^{-0.07672} \tag{1}$$

on a subset of measure $(\log N)^{-\varepsilon}|I|$ of every sufficiently large interval I.

Our question fits into the following general framework. We begin by associating with every prime p a random variable $X(p)$ with mean 0 and variance 1, and we assume that these variables are independent and identically distributed. We then define $X(n)$ by requiring it to be a completely multiplicative function for every point in our probability space. Now suppose that $a(n)$ is an arithmetic function which is either 0 or 1 for every n. We refer to the sequence

$$C_N(X) := \sum_{n=1}^{N} a(n)X(n)$$

as the arithmetic chaos associated with X and $a(n)$.

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Our question concerns the case when $X(p)$ are independent Steinhaus variables $z(p)$, i.e. the random variable $z(p)$ is equidistributed on the unit circle. When $a(n) \equiv 1$, we refer to the resulting sequence

$$S_N(z) := \sum_{n=1}^{N} z(n)$$

as arithmetic Steinhaus chaos. The relation between our question and arithmetic Steinhaus chaos is given by the well-known norm identity

$$\mathbb{E}(|S_N|^q) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \left|\sum_{n=1}^{N} n^{-it}\right|^q dt, \tag{2}$$

valid for all $q > 0$ (see [12, Section 3]).

The point of departure for our research is Helson's last paper [10] in which he conjectured that $\mathbb{E}(|S_N|) = o(\sqrt{N})$ when $N \to \infty$. This means that Helson anticipated that our question has a negative answer. Using an inequality from another paper of Helson [9], we get immediately that

$$\mathbb{E}(|S_N|) \gg \sqrt{N}(\log N)^{-1/4}. \tag{3}$$

Our attempt to settle Helson's conjecture has resulted in a reduction from $1/4$ to 0.05616 in the exponent of the logarithmic factor in (3). We note in passing that the problem leading Helson to his conjecture was solved in [11] avoiding the use of the random functions S_N.

To get a picture of what our work is about, it is instructive to return for a moment to a general arithmetic chaos $C := (C_N(X))$. To this end, let us assume that $X(p)$ is such that the moments

$$\|C_N\|_q^q := \mathbb{E}(|C_N|^q)$$

are well defined for all $q > 0$. We declare the number

$$q(C) := \inf\left\{q > 0 : \limsup_{N\to\infty} \|C_N\|_{q+\varepsilon}/\|C_N\|_q = \infty \quad \text{for every } \varepsilon > 0\right\}$$

to be the critical exponent of C, setting $q(C) = \infty$ should the set on the right-hand side be empty. A problem closely related to Helson's conjecture is that of computing the critical exponent of a given arithmetic chaos. We observe that $q(C) \geq 2$ is equivalent to the statement that there exist absolute constants $c > 0$, $0 < \lambda < 1$ such that

$$\mathbb{P}\left(|C_N| \geq c\sqrt{N}\right) \geq \lambda$$

holds for all N, cf. our question. In our case, the critical exponent is strictly smaller than 4, and then a serious obstacle for saying much more is that only even moments are accessible by direct methods.

We will prove the following result about arithmetic Steinhaus chaos.

Theorem 1. *We have*

$$\|S_N\|_q \gg_q \sqrt{N}(\log N)^{-0.07672} \tag{4}$$

for all $q > 0$.

This estimate is of course of interest only for small q; our method allows us to improve (4) for each individual $0 < q < 2$ as will be demonstrated in the last section of the paper. In the range $q > 2$, we note that the L^4 norm has an interesting number theoretic interpretation and has been estimated with high precision [1]:

$$\|S_N\|_4^4 = \frac{12}{\pi^2} N^2 \log N + cN^2 + O\big(N^{19/13}(\log N)^{7/13}\big)$$

with c a certain number theoretic constant. This means in particular that the critical exponent of arithmetic Steinhaus chaos $S := (S_N)$ satisfies $q(S) < 4$. We mention without proof that, applying the Hardy–Littlewood inequality from [2] to S_N^2, we have been able to verify that in fact $q(S) \le 8/3$. A further elaboration of our methods could probably lower this estimate slightly, but this would not alter the main conclusion that it remains unknown whether $q(S)$ is positive.

Before turning to the proof of Theorem 1, we mention the following simple fact: There exists a constant $c < 1$ such that $\|S_N\|_1 \le c\|S_N\|_2$ when $N \ge 2$. To see this, we apply the Cauchy–Schwarz inequality to the product of $(1-\varepsilon z(2))S_N$ and $(1-\varepsilon z(2))^{-1}$ to obtain

$$\|S_N\|_1^2 \le \frac{1}{(1-\varepsilon^2)} \cdot \big((1-\varepsilon)^2[N/2] + (1+\varepsilon^2)[(N+1)/2]\big) \le \frac{N-(\varepsilon-\varepsilon^2)(N-1)}{1-\varepsilon^2}$$

for every $0 < \varepsilon < 1$. Choosing a suitable small ε, we obtain the desired constant $c < 1$.

2. Proof of Theorem 1

Our proof starts from a decomposition of S_N into a sum of homogeneous polynomials. To this end, we set

$$E_{N,m} := \{n \le N : \ \Omega(n) = m\},$$

where $\Omega(n)$ is the number of prime factors of n, counting multiplicities. Correspondingly, we introduce the homogeneous polynomials

$$S_{N,m}(z) := \sum_{n \in E_{N,m}} z(n)$$

so that we may write

$$S_N(z) = \sum_{m \le (\log N)/\log 2} S_{N,m}(z). \tag{5}$$

We need two lemmas. The first is a well-known estimate of Sathe; the standard reference for this result is Selberg's paper [13]. To formulate this lemma, we introduce the function

$$\Phi(z) := \frac{1}{\Gamma(z+1)} \prod_p \big(1-1/p)\big)^z \big(1-z/p\big)^{-1},$$

where the product runs over all prime numbers p. This function is meromorphic in $\mathbb{C}$ with simple poles at the primes and zeros at the negative integers.

Lemma 2. *When $N \ge 3$ and $1 \le m \le (2-\varepsilon)\log\log N$ for $0 < \varepsilon < 1$, we have*

$$|E_{N,m}| = \frac{N}{\log N} \Phi\left(\frac{m}{\log\log N}\right) \frac{(\log\log N)^{m-1}}{(m-1)!}\left(1 + O\left(\frac{1}{\log\log N}\right)\right),$$

where the implied constant in the error term only depends on ε.

The second lemma is a general statement about the decomposition of a holomorphic function into a sum of homogeneous polynomials. For simplicity, we consider only an arbitrary holomorphic polynomial $P(z)$ in d complex variables $z = (z_1, ..., z_d)$. Such a polynomial has a unique decomposition

$$P(z) = \sum_{m=0}^{k} P_m(z),$$

where k is the degree of P and

$$P_m(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$$

is a homogeneous polynomial of degree m. Here we use standard multi-index notation, which means that $\alpha = (\alpha_1, ..., \alpha_d)$, where α_1, ..., α_d are nonnegative integers,

$$z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d},$$

and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. At this point, the reader should recognize that if we represent an arbitrary integer $n \le N$ by its prime factorization $p_1^{\alpha_1} \cdots p_d^{\alpha_d}$ (here $d = \pi(N)$) and set $\alpha(n) = (\alpha_1, ..., \alpha_d)$, then we may write

$$S_N(z) = \sum_{n=1}^{N} z^{\alpha(n)}.$$

Hence, as already pointed out, (5) is the decomposition of S_N into a sum of homogeneous polynomials, and we also see that $|\alpha(n)| = \Omega(n)$.

We let μ_d denote normalized Lebesgue measure on $\mathbb{T}^d$ and define

$$\|P\|_q^q := \int_{\mathbb{T}^d} |P(z)|^q \, d\mu_d(z)$$

for every $q > 0$. The variables z_1,..., z_d can be viewed as independent Steinhaus variables so that $\|S_N\|_q$ has the same meaning as before.

Lemma 3. *There exists an absolute constant C, independent of d, such that*

$$\|P_m\|_q \le \begin{cases} \|P\|_q, & q \ge 1 \\ Cm^{1/q-1}\|P\|_q, & 0 < q < 1 \end{cases}$$

holds for every holomorphic polynomial P of d complex variables.

Proof. We introduce the transformation $z_w = (wz_1, ..., wz_d)$, where w is a point on the unit circle $\mathbb{T}$. We may then write

$$P(z_w) = \sum_{m=0}^{k} P_m(z) w^m.$$

It follows that we may consider the polynomials $P_m(z)$ as the coefficients of a polynomial in one complex variable. Then a classical coefficient estimate (see [4, p. 98]) shows that

$$|P_m(z)|^q \le \begin{cases} \int_{\mathbb{T}} |P_m(z_w)|^q \, d\mu_1(w), & q \ge 1 \\ Cm^{1-q} \int_{\mathbb{T}} |P_m(z_w)|^q \, d\mu_1(w), & 0 < q < 1. \end{cases}$$

Integrating this inequality over $\mathbb{T}^d$ with respect to $d\mu_d(z)$ and using Fubini's theorem, we obtain the desired estimate. $\square$

We now turn to the proof of Theorem 1. The idea of the proof can be related to an interesting study of Harper [6] from which it can be deduced that, asymptotically, the square-free part of the homogeneous polynomial $S_{N,m}/\sqrt{N}$ has a Gaussian distribution when $m = o(\log\log N)$. When $m = \beta \log\log N$ for β bounded away from 0, this is no longer so, but what we will use, is a much weaker statement: When β is small enough, the L^2 and L^4 norms are comparable. The proof will consist in identifying for which β this holds.

To this end, we first observe that

$$\|S_{N,m}\|_2^2 = |E_{N,m}|. \tag{6}$$

To estimate $\|S_{N,m}\|_4^4$, we begin by noting that

$$|S_{N,m}|^2 = |E_{N,m}| + \sum_{k=0}^{m} \sum_{a,b\in E_{N,k},(a,b)=1} |E_{N/\max(a,b),m-k}| z(a)\overline{z(b)}.$$

Squaring this expression and taking expectation, we obtain

$$\begin{aligned}
\|S_{N,m}\|_4^4 &= |E_{N,m}|^2 + 2\sum_{k=0}^{m} \sum_{a,b\in E_{N,k},(a,b)=1,a<b} |E_{N/b,m-k}|^2 \\
&\le |E_{N,m}|^2 + 2\sum_{k=0}^{m} \sum_{b\in E_{N,k}} |E_{b,k}|\cdot|E_{N/b,m-k}|^2 \\
&\le 5|E_{N,m}|^2 + \sum_{k=1}^{m-1} \sum_{b\in E_{N,k}} |E_{b,k}|\cdot|E_{N/b,m-k}|^2. \qquad (7)
\end{aligned}$$

Here we used that, plainly,

$$\sum_{b\in E_{N,0}} |E_{N/b,m}|^2 = |E_{N,m}|^2 \quad \text{and} \quad \sum_{b\in E_{N,m}} |E_{b,m}| \le |E_{N,m}|^2.$$

To estimate the sum over b in (7), we begin by observing that Lemma 2 implies that

$$\begin{aligned}
|E_{N/b,m-k}| &\ll b^{-1}|E_{N,m-k}|, \quad b\le\sqrt{N}, \\
|E_{b,k}| &\ll bN^{-1}|E_{N,k}|, \quad \sqrt{N}<b\le N. \qquad (8)
\end{aligned}$$

We split correspondingly the sum into two parts:

$$\sum_{b\in E_{N,k}} |E_{b,k}|\cdot|E_{N/b,m-k}|^2 \ll |E_{N,m-k}|^2 \sum_{b\in E_{\sqrt{N},k}} b^{-2}|E_{b,k}| + |E_{N,k}| \sum_{b\in E_{N,k}\setminus E_{\sqrt{N},k}} bN^{-1}|E_{N/b,m-k}|^2. \tag{9}$$

To deal with the first of the two sums in (9), we begin by using Lemma 2 so that we get smooth terms in the sum:

$$\sum_{b\in E_{\sqrt{N},k}} b^{-2}|E_{b,k}| \ll \sum_{b\in E_{\sqrt{N},k}\setminus\{1,2\}} \frac{(\log\log b)^{k-1}}{b(\log b)(k-1)!} = \sum_{2<b\le\sqrt{N}} g(b)\frac{(\log\log b)^{k-1}}{b(\log b)(k-1)!},$$

where $g(n)$ is the characteristic function of the set $E_{\sqrt{N},k}$. We apply Abel's summation formula to the latter sum and obtain, using also Lemma 2,

$$\begin{aligned}\sum_{b\in E_{\sqrt{N},k}} b^{-2}|E_{b,k}| &\ll \left|E_{\sqrt{N},k}\right| N^{-1/2}\frac{(\log\log N)^{k-1}}{(\log N)(k-1)!}+\int_3^N \left|E_{x,k}\right|\frac{(\log\log x)^{k-1}}{x^2(\log x)(k-1)!}dx\\ &\ll \frac{(\log\log N)^{2(k-1)}}{(\log N)^2((k-1)!)^2}+\int_3^N\frac{(\log\log x)^{2(k-1)}}{x(\log x)^2((k-1)!)^2}dx\\ &\ll \frac{1}{((k-1)!)^2}\int_0^\infty y^{2(k-1)}e^{-y}dy=\frac{(2k-2)!}{((k-1)!)^2}\ll\frac{2^{2k}}{\sqrt{k}}.\end{aligned}\tag{10}$$

Arguing in a similar fashion, using Abel's summation formula and again (8), we get

$$\begin{aligned}\sum_{b\in E_{N,k}\setminus E_{\sqrt{N},k}} bN^{-1}|E_{N/b,m-k}|^2 &\ll N\int_{\sqrt{N}}^{N/3}\left|E_{x,k}\right|\frac{\left(\log\log\frac{N}{x}\right)^{2(m-k-1)}}{x^2\left(\log\frac{N}{x}\right)^2((m-k-1)!)^2}dx\\ &\ll \frac{|E_{N,k}|}{((m-k-1)!)^2}\int_0^\infty y^{2(m-k-1)}e^{-y}dy\\ &\ll |E_{N,k}|\cdot\frac{2^{2(m-k)}}{\sqrt{m-k}}.\end{aligned}\tag{11}$$

Inserting (10) and (11) into (9), we obtain

$$\sum_{b\in E_{N,k}}|E_{b,k}|\cdot|E_{N/b,m-k}|^2\ll|E_{N,m-k}|^2\cdot\frac{2^{2k}}{\sqrt{k}}+|E_{N,k}|^2\cdot\frac{2^{2(m-k)}}{\sqrt{m-k}}.$$

Returning to (7) and using (6), we therefore find that

$$\|S_{N,m}\|_4^4\ll\|S_{N,m}\|_2^4\Big(1+\sum_{k=1}^{m-1}\frac{|E_{N,m-k}|^2}{|E_{N,m}|^2}\cdot\frac{2^{2k}}{\sqrt{k}}\Big).$$

Applying again Lemma 2, we get

$$\begin{aligned}\sum_{k=1}^{m-1}\frac{|E_{N,m-k}|^2}{|E_{N,m}|^2}\cdot\frac{2^{2k}}{\sqrt{k}} &\ll\sum_{k=1}^{m-1}\frac{1}{\sqrt{k}}\cdot\left(\frac{(m-1)!}{(m-k-1)!}\right)^2\cdot\left(\frac{2}{\log\log N}\right)^{2k}\\ &\ll\sum_{k=1}^{m-1}\frac{1}{\sqrt{k}}\cdot e^{-\frac{k^2}{m}}\cdot\left(\frac{2m}{\log\log N}\right)^{2k}.\end{aligned}\tag{12}$$

It follows that the two norms are comparable whenever $m=\frac{e^{-\varepsilon}}{2}\log\log N$ for $\varepsilon>0$, in which case Hölder's inequality yields

$$\|S_{N,m}\|_2\ll_\varepsilon\|S_{N,m}\|_q\tag{13}$$

for $0<q<2$. By (6), Lemma 2, and Stirling's formula, we have

$$\|S_{N,m}\|_2=|E_{N,m}|^{1/2}\asymp\sqrt{N}(\log N)^{-\delta(\varepsilon)}m^{-1/4}\tag{14}$$

when $m \le \frac{e^{-\varepsilon}}{2}\log\log N$, where

$$\delta(\varepsilon) := (2 - e^{-\varepsilon}(1+\log 2+\varepsilon))/4 = (1-\log 2)/4 + O(\varepsilon)$$

when $\varepsilon \to 0$. Combining this with (13) and applying Lemma 3, we infer that

$$\sqrt{N}(\log N)^{-\delta(\varepsilon)}(\log\log N)^{-1/4} \ll \|S_{N,m}\|_q \ll (\log\log N)^{\max(1/q-1,0)}\|S_N\|_q$$

when $0 < q < 2$. Theorem 1 follows since $(1-\log 2)/4 < 0.07672$.

3. Concluding remarks

1. We will now deduce (1) from Theorem 1. Since $t \mapsto \sum_{n=1}^N n^{-it}$ is an almost periodic function, it suffices to consider the interval $I = [0,T]$ for some large T. Moreover, by (2), it amounts to the same to estimate the measure of the subset

$$\mathscr{E} := \left\{z :\ |S_N(z)| \ge c\sqrt{N}(\log N)^{-0.07672}\right\}$$

of $\mathbb{T}^{\pi(N)}$ for a suitable c depending on ε. We find that

$$\begin{aligned}\|S_N\|_q^q &\le c^q N^{q/2}(\log N)^{-0.07672q} + \int_{\mathscr{E}} |S_N(z)|^q\, d\mu_{\pi(N)}(z)\\ &\le c^q N^{q/2}(\log N)^{-0.07672q} + \|S_N\|_2^q |\mathscr{E}|^{1-q/2},\end{aligned}$$

where we in the last step used Hölder's inequality. Using Theorem 1 to estimate $\|S_N\|_q^q$ from below and recalling that $\|S_N\|_2 = \sqrt{N}$, we therefore get

$$\kappa_q(\log N)^{-0.07672q} \le c^q(\log N)^{-0.07672q} + |\mathscr{E}|^{1-q/2}, \tag{15}$$

where κ_q is a constant depending on q. Given $\varepsilon > 0$, we now choose q such that $\varepsilon = 0.07672q/(1-q/2)$ and $c^q = \kappa_q/2$. Then (15) yields

$$|\mathscr{E}| \ge (\kappa_q/2)^{(1-q/2)^{-1}}(\log N)^{-\varepsilon}.$$

2. We may improve (4) in the following way. If $m = \frac{e^y}{2}\log\log N$ with $y > 0$, then we see from (12) that

$$\sum_{k=1}^{m-1} \frac{|E_{N,m-k}|^2}{|E_{N,m}|^2}\cdot\frac{2^{2k}}{\sqrt{k}} \ll_y e^{my^2}$$

which in turn implies that $\|S_{N,m}\|_2/\|S_{N,m}\|_4 \gg_y e^{-my^2/4}$. By Hölder's inequality,

$$\|S_{N,m}\|_2 \le \|S_{N,m}\|_q^{\frac{q}{4-q}}\|S_{N,m}\|_4^{\frac{4-2q}{4-q}},$$

and we therefore get

$$\|S_{N,m}\|_q \gg_y \|S_{N,m}\|_2 e^{-my^2(2/q-1)/2} = \|S_{N,m}\|_2(\log N)^{-e^y y^2(2/q-1)/4}.$$

We also observe that (14) now takes the form

$$\|S_{N,m}\|_2 = |E_{N,m}|^{1/2} \asymp \sqrt{N}(\log N)^{(e^y(1+\log 2-y)-2)/4} m^{-1/4}.$$

We find that the exponent of $\log N$ in the lower bound for $\|S_{N,m}\|_q$ becomes optimal if we choose y as the positive solution to the quadratic equation $(2/q-1)y^2+(4/q-1)y-\log 2=0$; when $q=1$, we get for instance $y=0.21556...$ and hence, after a numerical calculation,

$$\|S_N\|_1 \gg \sqrt{N}(\log N)^{-0.05616}.$$

3. Our proof shows that we essentially need $\beta \le 1/2$ for the projection

$$P_\beta S_N := \sum_{m\le\beta\log\log N} S_{N,m}$$

to have comparable L^2 and L^4 norms. To use our method of proof to show that Helson's conjecture fails, it would suffice to know that the projection $P_1 S_N$ has comparable L^2 and L^q norms for some $q>2$, because in that case $\|P_1 S_N\|_2 \ge (1+o(1))\sqrt{N}$. However, we see no reason to expect that such a q exists.

4. A careful examination of our proof, including a detailed estimation of the last sum in (12), shows that

$$\|S_{N,m}\|_4/\|S_{N,m}\|_2 \asymp (\log\log N)^{1/16} \tag{16}$$

when $m=\frac{1}{2}\log\log N + O(\sqrt{\log\log N})$. This means that $\frac{1}{2}\log\log N$ is indeed the critical degree of homogeneity and, moreover, that the two norms fail to be comparable in the limiting case.

5. Helson's problem makes sense for other distributions; an interesting case is when $X(p)$ are independent Rademacher functions $\epsilon(p)$ taking values $+1$ and -1 each with probability $1/2$. If we set $a(n)=|\mu(n)|$ (here $\mu(n)$ is the Möbius function), then we obtain arithmetic Rademacher chaos:

$$R_N(\epsilon) := \sum_{n=1}^{N} |\mu(n)|\epsilon(n).$$

Rademacher chaos was first considered by Wintner [14] and has been studied by many authors, see e.g. [5, 6]. Here it is of interest to note that Chatterjee and Soundararajan showed that $R_{N+y}-R_N$ is approximately Gaussian when $y=o(N/\log N)$ [3], which means that the analogue of Helson's conjecture is false in short intervals $[N,N+y]$.

6. While we were preparing a revision of this paper, further progress on Helson's problem was announced by Harper, Nikeghbali, and Radziwiłł [8]. By a completely different method, relying on Harper's lower bounds for sums of random multiplicative functions [7], these authors obtained the lower bound $\sqrt{N}(\log\log N)^{-3+o(1)}$ for both $\mathbb{E}|R_N|$ and $\mathbb{E}|S_N|$. In view of this result, it seems reasonable to conjecture that $\|S_{N,m}\|_2/\|S_{N,m}\|_1$ is bounded whenever $m=e^{-\varepsilon}\log\log N$ for $\varepsilon>0$ and that $m=\log\log N$ is the limiting case for the boundedness of this ratio. Comparing with (16) and taking into account Remark 3 above, one might wonder if the ratio $\|S_N\|_2/\|S_N\|_1$ does indeed grow as a power of $\log\log N$.

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