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The Dynamical Behaviour of some Automorphisms of C^2 that fixes the axes

Egil Aleksander Bugge

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Supervisor: Berit Stensønes, MATH

Norwegian University of Science and Technology
Department of Mathematical Sciences

Preface

This Master's thesis in Complex Analysis in Several Variables is written for the Institute of Mathematics at NTNU. It was written during the spring semester of 2016, with preliminary work done in the autumn semester of 2015. The original idea for the project is due to my supervisor Berit Stensønes.

It is assumed that the reader has at least an undergraduate level in mathematics and preferably a solid understanding of complex analysis in both one and several variables. Some familiarity with the theory on dynamical systems will also be helpful.

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Egil Aleksander Bugge

Acknowledgment

I have to extend my warmest gratitude to my supervisor professor Berit Stensønes, whose support and enthusiasm have made the last two years possible. Her dedication to her students is exemplary. I would also like to thank professor John Erik Fornæss for his helpful contributions.

Finally I would like to thank my fiancé, whose unending patience the last few months should go down in the history books.

E.A.B.

Summary and Conclusions

In this thesis we study iterations of a map in two complex variables. More precisely we study the set of points which are such that iterates of a map approaches a fixed point. It has been shown that for some such sets there exists a bijective and analytic map, a biholomorphic map, from this set to the whole two-dimensional complex space $\mathbb{C}^2$. The following question is still open: does there exist a subset of $\mathbb{C}^2$ that does not intersect the complex axes, where the points iterate to a fixed point and which has an interior biholomorphic to the whole of $\mathbb{C}^2$ itself?

In an attempt to explore this question further we construct a map from $\mathbb{C}^2$ to $\mathbb{C}^2$ and analyze if the set of points that iterate to the origin has a interior. The set of fixed points for this map is the union of the complex axes. Since all points on the complex axes are fixed for this map, none of them approach the origin as we iterate. This means that the set of points that approaches the origin under iteration does not intersect with the complex axes. If this set has an interior then we have a positive answer to the question above. The current research on the subject, however, points to that any such set will not have an open interior and is therefor not biholomorphic to $\mathbb{C}^2$.

We study the set of iterates towards the origin and show that a part of this set will be located close to a complex line. For an approximation of our maps this line will be the only part of the set where iterates approaches the origin of $\mathbb{C}^2$. Outside of this line we show that points that are sufficiently close to either of the complex axes will converge to the axis. We also show that the region of points close to the line which approach the origin will also eventually hit one of the complex axes.

Sammendrag og Konklusjon

I denne masteroppgaven vil vi studere iterasjoner av funksjoner i to komplekse variable. Vi vil studere mengden av punkter som under iterasjon konvergerer mot et fikspunkt. Det har blitt vist at for visse slikre mengder så eksisterer det en bijektiv og analytisk, en biholomorf, funksjon fra mengden til hele det to-dimensjonale komplekse rommet $\mathbb{C}^2$. Følgende spørsmål om slike mengder er fortsatt åpent: eksisterer det en undermengde av $\mathbb{C}^2$ som ikke deler punkter med de komplekse aksene, hvor punktene konvergerer under iterasjon mot et fikspunkt, har et indre som er biholomorft med hele $\mathbb{C}^2$?

I et forsøk på å studere dette spørsmålet konstruerer vi en funksjon fra $\mathbb{C}^2$ til $\mathbb{C}^2$ og forsøker å analysere om mengden med punkter som under iterasjon konvergerer til origo har et indre. For denne funksjonen er mengden av fikspunkter unionen av de komplekse aksene. Siden de komplekse aksene er fikspunkter vil ingen av de konvergere mot origo under iterasjon. Dette betyr at mengden med punkter som konvergerer under iterasjon mot origo ikke vil inneholde punkter som tilhører aksene. Hvis denne mengden viser seg å ha et indre så har vi et positivt svar på spørsmålet over. Forskningen så langt indikerer derimot at en slik mengde ikke vil ha et åpent indre og derfor ikke vil kunne være biholomorft med $\mathbb{C}^2$.

Vi vil studere mengden av punkter som under iterasjon konvergerer mot origo og vi vil vise at en del av denne mengden vil være lokalisert nærme en kompleks linje. Vi studerer deretter en tilnærming av vår funksjon nær origo hvor de eneste punktene som konvergerer mot origo ligger på akkurat denne linjen. Utenfor linjen viser vi at for den tilnærmede funksjonen vil punkter som er nærme både origo og en av aksene vil konvergere mot den respektive aksen. Vi benytter oss av dette til å vise at punkter som er veldig nærme den komplekse linjen, men ikke på den, og som beveger seg mot origo vil til slutt havne på en de komplekse aksene.

Contents

Chapter 1

Introduction

In this thesis we explore a possible candidate for a Fatou-Bieberbach domain in $\mathbb{C}^2$ which avoids the complex axes. We will do as in the classic examples of Fatou and Bieberbach and look for a basin of attraction to a fixed point of an automorphism in $\mathbb{C}^2$. We will identify a region which is a part of the stable set of this automorphism. This will also hold for a class of automorphisms which has a similar power series close to the origin. Then we will study an approximation of these automorphisms close to the origin in an attempt to identify the points which approaches the origin. In the following section we will justify the motivation behind studying our map and give a short overview of the research that has been done on the topic so far.

Chapter 2 will introduce definitions and prelimenary results. Chapter 3 will contain the main analysis of this class of automorphisms. In the last chapter, chapter 4, we will give a summary of our work and give suggestions for further work on the subject.

1.1 Motivation

In order to find the root of a complex polynomial $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ where $a_i \in \mathbb{C}$ for $i \in [0, n]$ and $a_j \neq 0$ for at least one $j \geq 1$, the traditional approach is to use a numerical method like Newton's method. Newton's method involves making an initial guess at the root. Let us denote this guess as z_0. Then we inductively create a sequence of complex numbers $\{z_n\}_{n\in\mathbb{N}}$ where $z_{n+1} = z_n - f(z_n)/f'(z_n)$. If the initial guess z_0 was close enough to a root of $f(z)$ the sequence will converge to that root. Now if we let $H(z) := z - f(z)/f'(z)$ we see that $z_k = H^k(z)$ for $z_k \in \{z_n\}_{n\in\mathbb{N}}$, where $H^k(z)$ is the function $H(z)$ iterated k-times. One can easily see that finding a limit point for the sequence $\{z_n\}_{n\in\mathbb{N}}$ is the same as finding a fixed point $z \in \mathbb{C}$ such that $H(z) = z$. This is one of the problems that motivated the study of complex dynamical systems.

Complex dynamical systems in one variable have been extensively studied and is in large parts complete. See John Milnor's "Dynamics in One Complex Variable" [6] for an extensive survey on the subject. This thesis will be concerned with dynamical systems in several complex variables. In particular we will be exploring an interesting property held by some dynamical systems, first described by Fatou and Bieberbach in the 1920's.

Fatou and Bieberbach [3] proved that there exists proper subdomains of $\mathbb{C}^2$ which are biholomorphically equivalent to $\mathbb{C}^2$ itself. A subdomain like this is called a Fatou-Bieberbach domain, or F-B domain for short. The examples they gave of such domains were basins of attraction of fixed points of some automorphisms in $\mathbb{C}^2$. Basins of attraction became the classic way of constructing F-B domains. Even though it has now been proved that not all F-B domains are basins of attraction of automorphisms [9], these still make up the majority of examples.

A major result on this topic was proved by Rosay and Rudin in 1988 [7]. In this paper they prove the following theorem.

Theorem 1.1. *Suppose that* $F \in \mathrm{Aut}(\mathbb{C}^n)$ *fixes a point* $p \in \mathbb{C}^n$ *and that all the eigenvalues* $\lambda_1, \dots, \lambda_n$ *of* $F'(p)$ *satisfy* $|\lambda_i| < 1$ *for* $\forall i \in \{1, 2, \dots, n\}$. *Let* $\Omega(p) = \left\{ z \in \mathbb{C}^n \mid \lim_{k\to\infty} F^k(z) = p \right\}$. *Then there exists a biholomorphic map* Ψ *from* $\Omega(p)$ *onto* $\mathbb{C}^n$.

In the language we introduce in the next chapter, Theorem 1.1 says that $\Omega_F(p)$ is a F-B domain for all attracting automorphisms $F \in \mathrm{Aut}(\mathbb{C}^n)$. Rosay and Rudin gives several examples of such F-B domains. For example in 9.7 of [7] they construct the following automorphism of $\mathbb{C}^2$. Choose some $\alpha \in \mathbb{C}$, $0 < |\alpha| < 1$, and find an entire function $f : \mathbb{C} \to \mathbb{C}$ such that

$$e^{f(0)} = \frac{1}{\alpha}, \quad f'(0) = 0, \quad f(1) = 0, \quad f'(1) = (1+\alpha^2)/(1-\alpha^2).$$

Then define the automorphism $F(z,w)$ to be

$$F(z,w) = \left(1-\alpha^2+\alpha z e^{f(zw)},\ w e^{-f(zw)}\right).$$

The point (1,1) is a fixed point of F and the eigenvalues of $F'((1,1))$ are $\lambda = \pm\alpha i$. It follows from Theorem 1.1 that $\Omega((1,1))$ is a F-B domain. It can be easily shown that $\overline{\Omega}((1,1))$ does not intersect the line $\left\{(z,w)\in\mathbb{C}^2 \mid w=0\right\}$. What is constructed here is a F-B domain whose closure misses a complex line.

It was proved in 1972 by Green [4] that a holomorphic map $F:\mathbb{C}^n\to\mathbb{CP}^1$ that omits any 3 hyperplanes has an image lying in a proper projective linear subspace. Here $\mathbb{CP}^1$ denotes the complex projective space of $\mathbb{C}^2$, which is the Riemann Sphere. This result means that we have cannot have non-degenerate automorphism of $\mathbb{C}^2$ which avoids three complex lines. The work of Green, together with the example described above, raises the natural question which Rosay and Rudin posit at the end of their paper [7]:

Is there a biholomorphic map from $\mathbb{C}^2$ into the set $\{zw\neq 0\}$, i.e., into the complement of the union of two intersecting complex lines?

Seeing as basins of attraction of automorphisms are the classical way of constructing F-B domains, it is only natural to check those for a possible positive answer to this question. In [8] Vivas studies a particular set of automorphisms of $\mathbb{C}^2$ which has the following properties

(i) $F(0)=0$ and $DF(0)=\mathrm{Id}$;

(ii) $F(z,0)=(z',0)$ and $F(0,w)=(0,w')$ for $\forall z,z',w,w'\in\mathbb{C}$.

We say that F is *tangent to the identity* at the origin and that F fixes the coordinate axes. The definitions will become clear in the next chapter. For these automorphisms Vivas states that a positive answer to either of the following gives a positive answer to the question posed by Rosay and Rudin:

(a) There exists an attracting fixed point for F.

(b) There exists a non-degenerate characteristic direction v of F at the origin such that $\text{Re}A(v) > 0$, where $A(v)$ is a number associated to the direction v.

If we have (a) then Theorem 1.1 ensures that the basin of attraction to that point will be a F-B domain. That (b) gives a positive answer is a bit more technical for us at this point, but it follows from Theorem 5.1 of [5] that there exists some basin of attraction associated to the direction v. For both cases there exists some basin of attraction $\Omega(p) \subset \left\{(z,w) \in \mathbb{C}^2 \mid zw \neq 0\right\}$.

The main result of [8] is the following proposition, which states that neither (a) or (b) is possible for such an automorphism. It is to be noted that the proof of the proposition relies on the following conjecture which, by the time of this thesis, still has not been proved.

Conjecture 1.1. *If F is an automorphism of $\mathbb{C}^* \times \mathbb{C}^*$, where $\mathbb{C}^*$ denotes the punctured plane $\mathbb{C} \setminus \{0\}$, then F preserves the form:*

$$\frac{dz \wedge dw}{zw}$$

Proposition 1.1. *If Conjecture 1.1 is valid, and F is an automorphism of $\mathbb{C}^2$ tangent to the identity that fixes the coordinate axes, then (a) and (b) are both false.*

An interesting consequence of Conjecture 1.1 that Vivas show in the same paper is that any automorphism of $\mathbb{C}^2$ that has property (i) and (ii) above, will be of the form

$$F(z,w) = (ze^{wg(z,w)}, we^{zh(z,w)}).$$

where

$$g(z,w) = \sum_{\alpha+\beta\geq k} c_{(\alpha,\beta)} z^\alpha w^\beta \quad \text{and} \quad h(z,w) = \sum_{\alpha+\beta\geq k} d_{(\alpha,\beta)} z^\alpha w^\beta$$

with

$$c_{\alpha-1,\beta} = -\frac{\alpha}{\beta} d_{\alpha,\beta-1}$$

for $k \leq \alpha+\beta \leq 2k$.

We choose $\beta \equiv 0$ and for $\alpha \geq 0$ we let $c_{(\alpha,0)} = \frac{1}{\alpha!}$. Now let $d_{(0,0)} = 1$ and $d_{(\alpha,0)} = 0$ for $\forall \alpha \geq 1$.

Then $h(z,w) = 1$ and

$$g(z,w) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k = e^z.$$

We now have the following automorphism of $\mathbb{C}^2$,

$$f(z,w) = \left(ze^{we^z}, we^z\right). \tag{1.1}$$

This automorphism serves as the inspiration for this thesis. It was constructed to fix the origin, and to have a region attracted by the origin that does not touch the complex axes. A most interesting outcome would be if this region gave a basin of attraction. This would provide a positive answer to Rosay and Rudins question. Our analysis in chapter 3 does not give a global description about the stable set of this automorphism, but at least close to the origin the stable set approaches a complex line.

1.2 Overview

In the next chapter we will describe the basic definitions used for dynamical systems. We will also present results for complex dynamical systems in several variables which are relevant for the analysis that follows in chapter 3.

Our analysis of the automorphisms (1.1) is done in chapter 3. After identifying the fixed points and expanding the power series (1.1) in the first section, we use methods based on [5] in order to find a region of $\mathbb{C}^2$ where iterations of (1.1) will approach the origin. This is done by identifying a particular direction in the projective space of $\mathbb{C}^2$ and using a blow-up of the origin to show that there exists a connected set where iterations of the map approaches the origin along this direction.

After we have found such a region we show via an approximation of (1.1) close to the origin that points not on a particular complex line which approaches the origin will eventually hit one of the complex axes. As mentioned, this result will hold for the class of automorphisms of $\mathbb{C}^2$ which have a power series expansion similiar to (3.3) close to the origin.

Chapter 2

Definitions and Preliminaries

This chapter will give a brief introduction to the notions and techniques used in this thesis. There will mostly be definitions and notation from [5], [2] and [9]. We will also be discussing some results that are relevant for this thesis, mainly from Hakim's paper.

2.1 Basic Definitions

In this section we will cover the majority of definitions for complex dynamical systems relevant for this thesis. Even though a lot of the definitions and techniques that follow are defined for more general maps of complex manifolds, we will concern ourselves mostly with automorphisms of $\mathbb{C}^n$. For the more general definitions we direct the reader to [2].

Definition 2.1. The set $\mathrm{Aut}(\mathbb{C}^n)$ consists of all holomorphic mappings $F : \mathbb{C}^n \to \mathbb{C}^n$ which are such that

(i) F is one-to-one

(ii) F is onto.

The elements of $\mathrm{Aut}(\mathbb{C}^n)$ are called the *automorphisms* of $\mathbb{C}^n$.

The theory of dynamical systems is mainly concerned with studying the behaviour of a map when it is iterated. If $F \in \mathrm{Aut}(\mathbb{C}^n)$ we define the k-th iterate of F to be $F^k = F \circ F^{k-1}$. It is natural to define F^0 to be the identity map. Given some $z_0 \in \mathbb{C}^n$ we will sometimes write $z_1 = F(z_0)$,

$z_2 = F^2(z_0), \dots, z_n = F^n(z_0)$. The sequence $\{z_k\}_{k\in\mathbb{N}}$ given by these iterations is called the *forward orbit* of z_0. The *backward orbit* of z_0 is simply the forward orbit of z_0 with respect to the inverse map F^{-1}.

If the forward orbit of z_0 converges to a limit point p, then we call p a *fixed point* for the map F. The point p is fixed by F since if $\lim_{n\to\infty} F^n(z_0) = p$,then $\lim_{n\to\infty} F^{n+1}(z_0) = F(p)$ so $F(p) = p$.

Definition 2.2. Let $F \in \mathrm{Aut}(\mathbb{C}^n)$. Let $p \in \mathbb{C}^n$. The set $\left\{z \in \mathbb{C}^n \mid F^n(z) \to p \text{ as } n \to \infty\right\}$ is called the *stable set* of p. The interior of the stable set of p is denoted by $\Omega(p)$ and is called the *basin of attraction* of p. Sometimes it is necessary to specify for which map the basin of attraction of a point p is defined for. For a map F we will then write $\Omega_F(p)$.

For dynamical systems in one dimension it is common to classify the behaviour of an automorphism F near a fixed point z_0 by the value $\lambda = F'(z_0)$. We call λ the *multiplier* of z_0.

(i) If $|\lambda| \neq 1$ then we say that z_0 is a *hyperbolic* fixed point. A hyperbolic fixed point is *attracting* if $0 \leq |\lambda| < 1$ and *repelling* if $|\lambda| > 1$.

(ii) If $|\lambda| = 1$ and $\lambda^n = 1$ for some $n \in \mathbb{N}$, then we say that z_0 is a *parabolic* fixed point.

(ii) If λ is not a root of unity, we say that the fixed point point z_0 is *elliptic*.

When $n > 1$ we classify the automorphisms by the eigenvalues of the differential at a fixed point $z_0 \in \mathbb{C}^n$ instead. This classification will mirror that of the one dimensional case.

Definition 2.3. Let $F \in \mathrm{Aut}(\mathbb{C}^n)$ with $z_0 \in \mathbb{C}^n$ a fixed point of F. Then

(i) if all eigenvalues of $DF(z_0)$ have modulus different from 1, we call the fixed point *hyperbolic*. If all eigenvalues have modulus less than 1, we say that the fixed point is *attracting*. If all eigenvalues have modulus greater than 1, we say that the fixed point is *repelling*.

(ii) if all eigenvalues of $DF(z_0)$ are roots of unity, we call the fixed point *parabolic*. In particular, if $DF(z_0) = \mathrm{Id}$ we say the F is *tangent to the identity*.

(iii) if all eigenvalues of $DF(z_0)$ have modulus 1, but none are roots of unity we call the fixed point *elliptic*.

A very useful technique in the theory of dynamical systems is the idea of changing coordinates by way of conjugation.

Definition 2.4. Let $F, G \in \mathrm{Aut}(\mathbb{C}^n)$. We say that F is *conjugated* to G if there exists a biholomorphic map $\varphi \in \mathrm{Aut}(\mathbb{C}^n)$ such that $G \circ \varphi = \varphi \circ F$.

The usefulness of conjugation is the fact that it respects iteration and fixed points. Both of these facts are easy to see. Rewrite $G = \varphi \circ F \circ \varphi^{-1}$. Now, $G^2 = G \circ G = \varphi \circ F \circ \varphi^{-1} \circ \varphi \circ f \circ \varphi^{-1} = \varphi \circ F^2 \circ \varphi^{-1}$. It follows by induction that $G^n = \varphi \circ F^n \circ \varphi^{-1}$. If p is fixed point for F then $G \circ \varphi(p) = \varphi \circ F(p) = \varphi(p)$ so $\varphi(p)$ is a fixed point for G. It follows from these two facts that conjugation will map $\Omega(p)$ into $\Omega(\varphi(p))$.

2.2 The Cauliflower Set

Here we describe a classic dynamical system in one variable known as the cauliflower set. See [6] and [1] for a more detailed description. The reason we bring up this set will become apparent in the next chapter. The cauliflower set is the basin of attraction of the origin, $\Omega(0)$, of the map $g(z) = z + z^2$. The set $\Omega(0)$ is shaped much like a cauliflower, see figure 2.1, hence the name. It is easy to see that the only fixed point of $g(z)$ is the origin.

Outside of $\overline{\Omega}(0)$ all orbits tend to infinity at an exponential rate. The boundary $\partial\Omega(0)$ is the so-called *Julia set* of g, for more detail see [6]. The Julia set is the set of points which have chaotic behaviour, or *sensitive dependence on initial conditions.* From the theory on Julia sets for quadratic polynomials we know that $\partial\Omega(0)$ is closed, g-invariant and that it contains the origin. The following two lemmata describes the behaviour of orbits inside $\Omega(0)$ and are from [1].

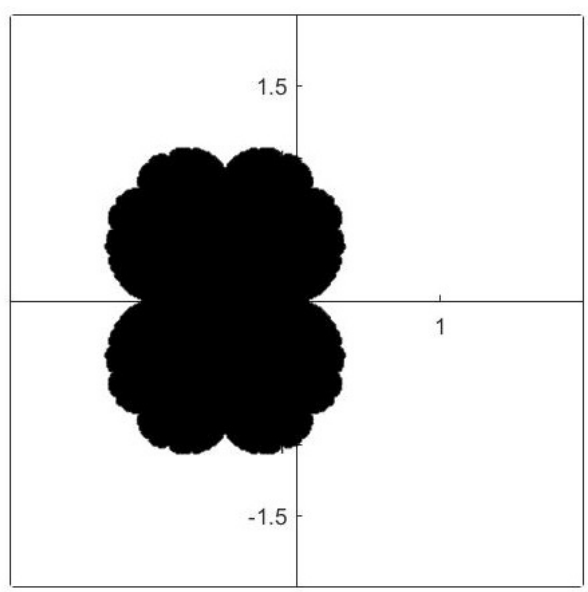


Figure 2.1: The filled Julia set of $g(z)$

Lemma 2.1. *Let $u_0 \in \mathbb{R}$ and $u_n = g^n(u_0)$.*

(i) For all $u_0 \in \mathbb{R}\backslash\{0,-1\}$ the sequence $\{u_n\}$ is strictly increasing.

(ii) If $u_0 \in [-1,0]$ then $u_n \to 0$. Otherwise $u_n \to +\infty$.

(iii) If $u_0 \in (-1,0)$, then for all $n \geq 1$

$$-\frac{1}{n} \leq u_n \leq \frac{u_1}{n},$$

or in other words $|u_n| = O(1/n)$.

As we see, if $u_0 \in (-1,0)$, then u_n approaches the orgin as $1/n$. The next lemma tells us that if we begin with any point in $z_0 \in \Omega(0)$ then as $n \to \infty$ the iterates will approach the origin along the negative real axis.

Lemma 2.2. *For all $z_0 \in \Omega(0)$, let $z_n = u_n + i v_n = g^n(z_0)$. Then*

$$\lim_{n\to\infty} n z_n = \lim_{n\to\infty} n u_n = -1 \quad and \quad \lim_{n\to\infty} n v_n = 0.$$

More precisely, there exists $c_1, c_2 > 0$ dependent on z_0 such that

$$|1 + n u_n| \leq |1 + n z_n| \leq \frac{c_1}{n} \log n$$

and

$$|v_n| \leq \frac{c_2}{n^2}\left(1 + \frac{c_1}{n} \log n\right)$$

for all $n \geq 1$.

If we want to visualize the set $\Omega(0)$ it helps to conjugate $g(z)$. A very useful property of quadratic polynomials is that we can always conjugate them into the form $f(z) = z^2 + c$ for some $c \in \mathbb{C}$. Conjugating $g(z) = z + z^2$ with the function $\gamma(z) = z + 1/2$ we get

$$\widetilde{g}(z) = \gamma \circ g \circ \gamma^{-1}(z) = z^2 + \frac{1}{4}.$$

We plot the *filled* Julia set of g in figure 2.1. In order to do this we plot the filled Julia set of $\widetilde{g}$ which is easier to work with and simply move all points $1/2$ in the negative real direction.

2.3 Parabolic Fixed Points Tangent to the Identity

Let $F \in \mathrm{Aut}(\mathbb{C}^n)$ be an automorphism that has a parabolic fixed point at the origin which is tangent to the identity. This means that $DF(0) = \mathrm{Id}$. It follows then that F at $z \in \mathbb{C}^n$ is given by a convergent series of the form

$$F(z) = z + P_k(z) + P_{k+1}(z) + \cdots \tag{2.1}$$

where $k \in \mathbb{N}$ and $k \geq 2$. We have for $\forall h \in \mathbb{N}$ that P_{k+h} is a homogenous polynomial map of degree $k+h$ from $\mathbb{C}^n$ to $\mathbb{C}^n$. We call k the *order* of F.

Automorphisms that are tangent to the identity are studied by Hakim in [5]. Her paper proves a useful theorem for the class of automorphisms that has order 2, but before we write down the theorem we need some more definitions.

Definition 2.5. Let F be as in (2.1). A *characteristic direction* is a direction $v \neq 0$ in $\mathbb{C}^n$ such that $P_k(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. A *nondegenerate* characteristic direction is a characteristic direction v such that $P_k(v) \neq 0$.

Definition 2.6. A *parabolic curve* or an *invariant piece of curve* for F at the origin is an injective holomorphic map $h : \Delta \to \mathbb{C}^n$ satisfying the following properties:

(i) Δ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$;

(ii) h is continuous at the origin, and $h(0) = O$;

(iii) $h(\Delta)$ is invariant under F, and $\left(F^n|_{h(\Delta)}\right) \to O$ as $n \to \infty$

We say h is *tangent to* $[v]$ *at the origin* if $[h(\xi)] \to [v] \in \mathbb{CP}^{n-1}$ as $\xi \to 0$. Here $\mathbb{CP}^{n-1}$ is the complex projective space of $\mathbb{C}^n$ and $[\cdot]$ is the projection of $\mathbb{C}^n \setminus \{0\}$ onto $\mathbb{CP}^{n-1}$. The main theorem of [5] is the following.

Theorem 2.7. *Let F be a germ of analytic transformation from $\mathbb{C}^n$ to $\mathbb{C}^n$ which fixes the origin and is tangent to the identity. For every nondegenerate characteristic direction v of F, there exists an invariant piece of curve, tangent to v at the origin, attracted by the origin.*

Theorem 2.7 is an attempt at generalizing the one dimensional Leau-Fatou Flower theorem, see [6]. What theorem 2.7 tells us is that if F has a nondegenerate characteristic direction, then there exists some connected set with the origin in its boundary where F is both invariant and approaches the origin along the direction v. This gives us a lot of information on the local behavior of the stable set of the origin for such a map. We will use this in the next chapter.

Chapter 3

Analysing the Stable Set

As stated in the introduction, the map we will be studying is given by (1.1). For the reader's convenience we will include the map again. Let $F \in \mathrm{Aut}(\mathbb{C}^2)$ be an automorphism given by

$$F(z,w) = \left(f_1(z,w), f_2(z,w)\right) = \left(ze^{we^z}, we^z\right).$$

The main goal of this chapter is to explore if any automorphism which is similar to (1.1) has a basin of attraction or not. This is the same as saying that the interior of the stable set is open or not. We do not, however, give a definitive answer to this question.

3.1 Prelimenary Analysis

We start this section by identifying the fixed points of (1.1). If $(z,w) \in \mathbb{C}^2$ is a fixed point of (1.1), then

$$(z,w) = \left(ze^{we^z}, we^z\right)$$

which gives that

$$z = ze^{we^z} \quad \text{and} \quad w = we^z.$$

The set of $(z,w) \in \mathbb{C}^2$ which solves these equations are the punctured complex planes $X = \left\{(z,0) \in \mathbb{C}^2 \mid z \in \mathbb{C} \setminus \{0\}\right\}$ and $Y = \left\{(0,w) \in \mathbb{C}^2 \mid w \in \mathbb{C} \setminus \{0\}\right\}$ together with the origin $(0,0)$. This means that (1.1) fixes the origin and both of the complex axes. We write $\{zw \neq 0\}$ as shorthand for the set $\mathbb{C}^2 \setminus \{X \cup Y \cup (0,0)\} = \left\{(z,w) \in \mathbb{C}^2 \mid zw \neq 0\right\}$.

The Jacobian of (1.1) is

$$DF(z,w) = \begin{bmatrix} \left(e^{we^z} + zwe^{z+we^z}\right) & \left(zwe^{z+we^z}\right) \\ we^z & e^z \end{bmatrix}$$

and it follows that at the origin we have

$$DF(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The map (1.1) is therefore tangent to the identity at the origin. On the sets X and Y we get the following Jacobians

$$DF(z,0) = \begin{bmatrix} 1 & 0 \\ 0 & e^z \end{bmatrix} \quad \text{and} \quad DF(0,w) = \begin{bmatrix} e^w & 0 \\ w & 1 \end{bmatrix}.$$

In terms of proposition 1.1 we see that (a) is false for all fixed points since none of them give eigenvalues which all have modulus less than 1. In other words, (1.1) does not have an attractive fixed point. This gives that theorem 1.1 does not apply for any of the fixed points of (1.1).

It is useful to expand (1.1) into its power series. Since

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

we have

$$\begin{aligned} f_1(z,w) &= z\exp\left(w + wz + \frac{wz^2}{2!} + O\left(wz^3\right)\right) \\ &= z\left(1 + w + wz + \frac{wz^2}{2} + w^2 + 2w^2z + O\left(wz^3, w^2z^2, w^3\right)\right) \\ &= z + wz + wz^2 + w^2z + O\left(wz^3, w^2z^2, w^3z\right) \\ &= z + wz + O\left(wz^2, w^2z, w^2z^2\right) \end{aligned} \tag{3.1}$$

and for $f_2(z,w)$ we have

$$f_2(z,w) = w + wz + \frac{wz^2}{2!} + O\big(wz^3\big) = w + wz + O\big(wz^2\big). \tag{3.2}$$

Everything we prove in the following sections will hold for any automorphism of $\mathbb{C}^2$ which has the same form as

$$F(z,w) = \begin{cases} z_1 = f_1(z,w) = z + wz + O\big(wz^2, w^2z, w^2z^2\big) \\ w_1 = f_2(z,w) = w + wz + O\big(wz^2\big) \end{cases} \tag{3.3}$$

close to the origin.

3.2 The Stable Set

A lot of the existing theory on the global behavior of dynamical systems in two variables or more rely on fixed points being isolated, see [2]. This is clearly not the case here. To describe the stable set of the origin for (1.1) we will have to use a more local theory. From theorem 2.7 we have that close to the origin the stable set is situated around the complex line $\big\{(z,w) \in \mathbb{C}^2 \mid z = w\big\}$. We write $\{z = w\}$ as shorthand for this line. We dedicate this section to making the reason for this more clear.

From the power series of (1.1) we see that the map has order 2 and that $P_2(z,w) = (zw, zw)$. The nondegenerate direction is then a direction $v = (v_1, v_2)$ such that $P_2(v_1, v_2) = \lambda(v_1, v_2)$ for some $\lambda \in \mathbb{C}$. Since $P_2(v_1, v_2) = (v_1v_2, v_1v_2)$ the only nondegenerate characteristic direction of (1.1) is the complex line $\{z = w\}$. This line in the complex projective space $\mathbb{CP}^1$ is given by $[v] = (1,1)$. For this $[v]$ we have $P_2(1,1) = (1,1)$. Thus we have that $P_2(\lambda v) = \lambda v$ for all $\lambda v \in \{z = w\}$.

Hakim proves theorem 2.7 in [5] by using several linear transformations to transform any general nondegenerate characteristic direction $v = (v_1, v_2)$ into the form (1,0). This is done to get the map into a particular form which she then uses to prove the existence of an invariant curve which is tangent to (1,0). Hakim does this for $\mathbb{C}^n$ in general, but since we are working in

$n = 2$ we will present the results for that.

The process begins by choosing new coordinates (x, y) in $\mathbb{C} \times \mathbb{C}$ such that $v = (1, u)$ where $u \in \mathbb{C}$. Since for our map the characteristic direction is simply $v = (1, 1)$, we are already in this situation.

For these new coordinates, if the forward orbit $(x_n, y_n) = F^n(x_0, y_0)$ converges to the origin in such a way that $\lim_{n\to\infty}[(x_n, y_n)] = [v]$, then it can be shown that v is a characteristic direction. See proposition 2.3 in [5].

Writing $P_2(x, y) = \big(p_2(x, y), q_2(x, y)\big)$ we have that the behavior of x_n as $n \to \infty$ is

$$x_n \sim \frac{1}{n p_2(1, u)}.$$

Since we have assumed that (x_n, y_n) approaches the origin tangentially to $(1, u)$ we have that

$$\lim_{n\to\infty} \frac{y_n}{x_n} = u.$$

This fact motivates the change of variable into $u_n \in \mathbb{C}$ which is such that $y_n = u_n x_n$ for all $n \in \mathbb{N}$. This is called a blow-up of the origin. The blow-up is simply a map such as the following

$$\begin{aligned} \gamma : \mathbb{C}^2 &\to \mathbb{C}^2 \\ (x, u) &\mapsto (x, ux) := (x, y). \end{aligned}$$

In what follows we will use linear transformations to transform the characteristic direction into $(1, 0)$ and then study the new map in the (x, u) coordinates instead.

In order to transform the characteristic direction into $(1, 0)$ we use a conjugation with the linear transformation

$$\phi = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Conjugating with (1.1) we get the map

$$G(x,y) = \phi \circ F \circ \phi^{-1}$$

which looks like this

$$G(x,y) = \begin{cases} x_1 = x + (x+y)x + (x+y)^2 x + (x+y)x^2 + O((x+y)x^3, (x+y)^2 x^2, (x+y)^2 x) \\ y_1 = y - \frac{1}{2}(x+y)x^2 - (x+y)^2 x + O((x+y)x^3, (x+y)^2 x^2, (x+y)^2 x). \end{cases}$$

Applying the blow-up $y = ux$ and calculating $u_1 = \frac{y_1}{x_1}$ gives us the following map

$$\widetilde{G}(x,u) = \begin{cases} x_1 = x + (1+u)x^2 + (1+u)^2 x^3 + (1+u)x^3 + O(ux^4, u^2x^4, u^2x^2) \\ u_1 = u - ux + O(u^2x, ux^2, u^2x^2) + x^2\psi_1(x). \end{cases}$$

This map has $v = (1,0)$ as its characteristic direction. The function $\psi_1(x)$ is the polynomial given by the pure x-terms. Hakim does another conjugation in order to get $\widetilde{G}(x,u)$ on a particular form. This is done in order to prove proposition 3.1.

We conjugate with the linear transformation

$$\varphi = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

to get the map $H(x,u) = \varphi \circ \widetilde{G} \circ \varphi^{-1}$ which is as follows

$$H(x,u) = \begin{cases} x_1 = f(x,u) = x - x^2 + O(ux^2, u^2x^2, x^3) \\ u_1 = \Theta(x,u) = u + ux + O(u^2x, ux^2, u^2x^2) + x^2\psi_1(x). \end{cases} \tag{3.4}$$

Finding an analytic invariant curve which is tangent to $u = 0$ is equivalent to finding the function $g(x) = u$ which is analytic in a neighborhood of zero, $g(0) = 0$, and which is such that

$$g(f(x,g(x))) = \Theta(x,g(x)). \tag{3.5}$$

In order to find such a function we define a sequence of polynomials which converges uniformly to a polynomial with these properties. The proof of existence and uniqueness of this sequence is identical to the proof of Proposition 3.1 of [5]. It is simply a special case of that proposition with $A(v) = \alpha = -1$. Without going into detail on the function $A(v)$ we simply mention that since $A(v) = \mathrm{Re}A(v) = -1 < 0$ we have that (b) from proposition 1.1 is also false for our map.

Proposition 3.1. *Let (f, Θ) be the analytic transformation (3.4). Then there exists a unique sequence $\{P_k\}_{k\in\mathbb{N}}$ of polynomials P_k of degree k such that $P_k(0) = 0$, and*

$$\Theta(x, P_k(x)) = P_k(f(x, P_k(x))) + x^{k+2}\psi_{k+1}(x). \tag{3.6}$$

Also

$$P_{k+1}(x) = P_k(x) + c_{k+1}x^{k+1}$$

where c_{k+1} is given by

$$c_{k+1} = \frac{\psi_{k+1}(0)}{-(k+2)}.$$

Proof. This proposition is proved by induction. For $k = 1$ we have $P_1 = c_1 x$. We simply need to find a c_1 such that it solves the equation

$$c_1 f(x, c_1 x) = \Theta(x, c_1 x) + O(x^3). \tag{3.7}$$

Inserting $u = c_1 x$ into (3.4) we get (3.7) the following equation

$$x - x^2 + O(x^3) = c_1 x(1+x) + O(x^3) + x^2\psi_1(x).$$

Expanding $\psi_1(x)$ around zero and rearranging terms we have

$$x - x^2 - c_1 x(1+x) - x^2\psi_1(0) = O(x^3).$$

From this we see that c_1 solves (3.7) if and only if

$$c_1 = \frac{\psi_1(0)}{-2}$$

Assume we have found a unique polynomial of degree k that satisfies (3.6). We want to find a polynomial P_{k+1} of degree $k+1$ such that

$$P_{k+1}(f(x,P_{k+1}(x))) - \Theta(x,P_{k+1}(x)) = O(x^{k+3}).$$

Write

$$P_{k+1}(x) = p_k + c_{k+1}x^{k+1}$$

for some polynomial p_k of degree k. We want to show that $p_k = P_k$. From (3.4) we see that

$$P_{k+1}(f(x,P_{k+1}(x))) = p_k(f(x,p_k(x))) + c_{k+1}x^{k+1}(1-(k+1)x) + O(x^{k+3})$$

and

$$\Theta(x,p_k(x) + c_{k+1}x^{k+1}) = \Psi(x,p_k(x)) + c_{k+1}x^{k+1}(1+x) + O(x^{k+3}).$$

Putting these results together, we get that (3.6) with P_{k+1} becomes

$$\Theta(x,P_{k+1}(x)) - P_{k+1}(f(x,P_{k+1}(x))) = \Theta(x,p_k(x)) + c_{k+1}x^{k+2}(k+2) - p_k(f(x,p_k(x))) + O(x^{k+3}).$$

We see that p_k is necessarily a solution to (3.6) and by the induction hypothesis p_k is equal to P_k. This gives that

$$\Theta(x,P_{k+1}(x)) - P_{k+1}(f(x,P_{k+1}(x))) = c_{k+1}x^{k+2}(k+2) + x^{k+2}\psi_{k+1}(x) + O(x^{k+3}).$$

Expanding $\psi_{k+1}(x)$ around zero we get that P_{k+1} solves (3.6) if and only if

$$c_{k+1} = \frac{\psi_{k+1}(0)}{-(k+2)}.$$

□

Proposition 3.1 shows the existence of a sequence of polynomials $\{P_k\}_{k\in\mathbb{N}}$ and the convergence can be read as formal at this point. Hakim does not directly prove that P_k converges uniformly to a P, but instead proves the existence of a map with the property given by (3.5) in general. Hakim does this by defining a operator T which is such that a $g(x)$ is a fixed point for T

if and only if it satisfies (3.5). She then proceeds to show that on a closed convex set of a suitable Banach space, the operator T is a contraction and thus has a fixed point.

3.3 The Dynamical Behaviour of our Map

Trying to describe the behaviour of (1.1) under iteration will quickly become too complex. We need a more well behaved map. If $(z, w) \in \mathbb{C}^2$ are close enough to the origin we can safely ignore the higher order terms of the power series of (1.1). Given this, it follows from (3.3) that (1.1) is approximately equal to the map

$$\widetilde{F}(z, w) = \left(\widetilde{f}_1(z, w), \widetilde{f}_2(z, w)\right) = (z(1+w), w(1+z)). \tag{3.8}$$

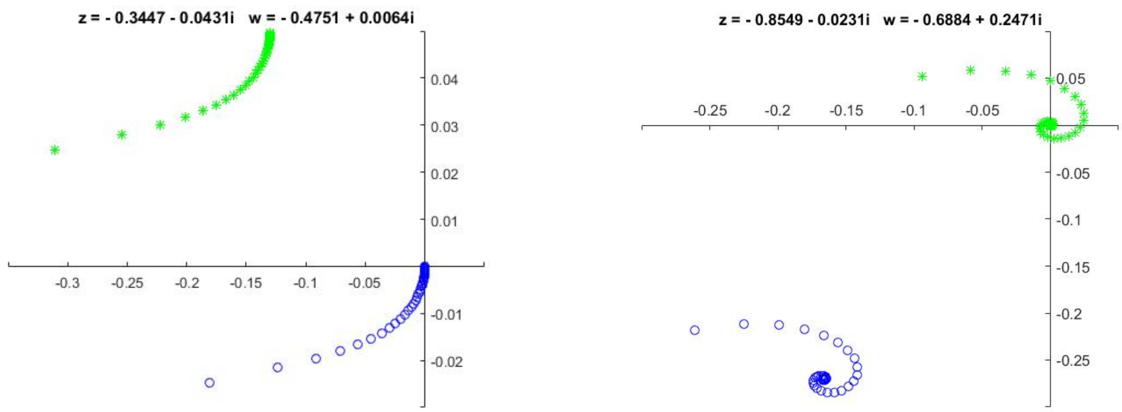
The map (3.8) has the benefit of being symmetrical and shares much of the same behaviour as the main map (1.1). The set of fixed points for (3.8) is $X \cup Y \cup (0,0)$. This is easy to see. On the line $\{z = w\}$ we have that (3.8) takes on the values

$$\widetilde{F}(z, z) = (z + z^2, z + z^2).$$

This means that on the line $\{z = w\}$ the map (3.8) admits a basin of attraction of zero in each complex direction. From section 2.2 we know that both of these basins of attraction are cauliflower sets. For all z outside of the closure of this set, the iterates of $\widetilde{f}_i^n(z)$, $i \in \{1, 2\}$, will diverge to infinity. In the rest of this section we will be mainly concerned with points in the following region of $\mathbb{C}^2$

$$U = \left\{ z \in \mathbb{C} \,|\, |z| < 1, \pi - a < \operatorname{Arg} z < \pi + a \right\} \times \left\{ w \in \mathbb{C} \,|\, |w| < 1, \pi - a < \operatorname{Arg} w < \pi + a \right\}$$

for some $a \in \mathbb{R}$ with $0 \le a \le \pi/4$. If $|z| < 1/R$ for some large $R \in \mathbb{R}$, the behaviour of $z + z^2$ inside the region can be described by changing coordinates $1/s = z$ and looking at $s_n = 1/\widetilde{f}_1^n(1/s, 1/s)$.



(a) (b)

Figure 3.1: Plots of iterates of (3.8) for two points outside of $\{z = w\}$. Both z_n (blue) and the w_n (green) is plotted on the same complex plane. The initial point is given above the plot.

We have that s_1 is of the following form

$$\begin{aligned}\frac{1}{\widetilde{f}_1(1/s, 1/s)} &= 1/\left(1/s + 1/s^2\right)\\ &= s\left(1 - \frac{1}{s} + \mathrm{O}\left(\frac{1}{s^2}\right)\right)\\ &= s - 1 + \mathrm{O}\left(\frac{1}{s}\right).\end{aligned}$$

We have that $\mathrm{Re}(s) < -|R'|$ for some large $|R'|$ and from above we see that s_n is of the form

$$\frac{1}{\widetilde{f}_1^n(1/s, 1/s)} = s - n + \mathrm{O}\left(\frac{1}{s}\right).$$

This gives us that s_n approaches $-\infty$ almost as $-n$ and from this we see that $|z_n|$ approaches the origin almost as $1/n$.

In figure 3.1 we have plotted some iterations of (3.8) with initial points outside of the line $\{z = w\}$. We see that the iterates seem to move with a constant distance apart. We will show that this is always the case for the approximated map.

This section is dedicated to exploring the dynamical behaviour of (3.8) when we are close to the origin. We begin by looking at a region close to the origin where we are also close to either X or Y. We define the two regions R_z and R_w in the following way,

$$R_z = \left\{ (z_0, w_0) \in \mathbb{C}^2 \mid \exists\, 0 < c < 1/2,\ 0 < b_0 < c/4 \text{ where } |z_0 + c| < b_0 \text{ and } |w_0| < a_0 < c/16 \right\}.$$

We define R_w in the same way, just z_0 interchanged with w_0.

We will show that if we have initial points inside either R_z or R_w, then the iterates will always approach the complex axes. We begin by the proving the following preliminary lemma.

Lemma 3.1. *Let* $0 < a, b, c < 1$. *Assume* $|z_0 + c| < b$ *and* $|w_0| < a$. *Then we have that*

$$|z_1 + c| < b + (b + c)a \quad \text{and} \quad |w_1| < a(1 - c + b). \tag{3.9}$$

Proof.

$$\begin{aligned} |z_1 + c| &= |z_0 + z_0 w_0 + c| \\ &\le b + |z_0 w_0| < b + (b + c)a \\ |w_1| &= |w_0||1 + z_0| \\ &< a|1 - c + (z_0 + c)| \\ &< a(1 - c + b) \end{aligned}$$

□

We now use lemma 3.1 to get estimates for the orbit $(z_n, w_n) = \widetilde{F}^n(z_0, w_0)$ when $(z_0, w_0) \in R_z$.

Lemma 3.2. *Suppose that* $(z_0, w_0) \in R_z$. *Then we have the following inductive estimate on the orbit* $(z_n, w_n) = \widetilde{F}^n(z_0, w_0)$*:*

$$|z_n + c| < b_0 + 2ca_0 \sum_{j=0}^{n} \left(1 - \frac{c}{2}\right)^j \quad \text{and} \quad |w_n| < a_0 \left(1 - \frac{c}{2}\right)^n. \tag{3.10}$$

Proof. By assumption the estimate holds for $n = 0$. Suppose that it valid for (z_n, w_n). Let

$$b = b_0 + 2ca_0 \sum_{j=0}^{n} \left(1 - \frac{c}{2}\right)^j \quad and \quad a = a_0 \left(1 - \frac{c}{2}\right)^n$$

and apply (3.9). Observe that

$$b \leq b_0 + 2ca_0 \frac{1}{1 - \left(1 - \frac{c}{2}\right)} = b_0 + 4a_0 < c/2.$$

Hence we get that

$$\begin{aligned} |z_{n+1} + c| &< b_0 + 2ca_0 \sum_{j=0}^{n} \left(1 - \frac{c}{2}\right)^j + \frac{3c}{2} a_0 \left(1 - \frac{c}{2}\right)^n \\ &< b_0 + 2ca_0 \sum_{j=0}^{n+1} \left(1 - \frac{c}{2}\right)^j \end{aligned}$$

and

$$|w_{n+1}| < a_0 \left(1 - \frac{c}{2}\right)^n \left(1 - c + \frac{c}{2}\right) = a_0 \left(1 - \frac{c}{2}\right)^{n+1}$$

which completes the proof. □

Lemma 3.2 gives us the following corollary. What it tells us is that given $(z_0, w_0) \in R_z$ we have $|z_n| > 0$ for $\forall n \in \mathbb{N}$, while $|w_n| \to 0$ as $n \to \infty$.

Corollary 3.1. *If $(z_0, w_0) \in R_z$, then $|z_n + c| < c/2$ and $|w_n| \to 0$ as $n \to \infty$*

Proof. This follows almost directly from the lemma above. That $|w_n| \to 0$ is trivial. As $n \to \infty$ we see that

$$|z_n + c| < b_0 + 2ca_0 \sum_{j=0}^{n} \left(1 - \frac{c}{2}\right)^j \longrightarrow b_0 + 2ca_0 \frac{2}{c} < c/4 + c/4 = c/2$$

□

Now if $(z_0, w_0) \in R_w$, then $(z_n, w_n) \to (0, w')$ for some $w' \in \mathbb{C} \setminus \{0\}$ when $n \to \infty$. The argument is identical as the one for R_z, just change z_0 and w_0.

Our hypothesis is that if we are close to the line $\{z = w\}$ (but not on it) and sufficiently close to the origin, then iterates of (3.8) will always eventually reach a point inside either R_z or R_w.

We mentioned above that the iterates of the (3.8) seemed to move in a constant distance apart when we did simulations. We will now show that if $\lambda = w_0 - z_0$, then $\lambda = w_n - z_n$ for $\forall n \in \mathbb{N}$. If $\lambda = w_0 - z_0$, then

$$w_1 - z_1 = w_0 + z_0 w_0 - z_0 - z_0 w_0 = w_0 - z_0 = \lambda.$$

Now it follows by induction that this holds for $\forall n > 0$. If it holds for $n-1$, then we have for n that

$$w_n - z_n = w_{n-1} + z_{n-1} w_{n-1} - z_{n-1} - z_{n-1} w_{n-1} = w_{n-1} - z_{n-1} = \lambda.$$

The value of λ is a way of measuring how far we are from the line $\{z = w\}$. If $|\lambda|$ is very small, but non-zero, when does z_n or w_n approach zero? In order to simplify the study of when $z_n \to 0$ we conjugate (3.8) with the linear transformation

$$\phi = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

We see that $\phi(z_0, w_0) = (z_0, \lambda)$. Conjugating we get the following map

$$\begin{aligned} \widetilde{G}(z_0, \lambda) &= \phi \circ \widetilde{F} \circ \phi^{-1}(z_0,\ \lambda\ ) \\ &= \phi \circ (z_0 + z_0(\lambda + z_0),\ \lambda + z_0 + z_0(\lambda + z_0)\ ) \\ &= \left(z_0(1+\lambda) + z_0^2,\ \lambda\ \right). \end{aligned} \tag{3.11}$$

The conjugation done in (3.11) is a way of rotating the line $\{z = w\}$ so that it coincides with the complex plane $X \cup (0,0)$. We see that iterations of $\widetilde{G}$ will only change the z-coordinate so we let $z_n = \widetilde{g}^n(z_0) = z_{n-1}(1+\lambda) + z_{n-1}^2$.

The behaviour of $\widetilde{g}^n(z_0)$ depends on the value of λ. This means that the behaviour of z_n is very dependent on the initial orientation between z_0 and w_0. We describe different situations for different values of λ.

(i) If $|1+\lambda| = 1$ then $\lambda = 0$ and we are back in the situation $\tilde{g}(z_0) = (z_0 + z_0^2)$. Here $\tilde{g}^n(z_0) \to 0$ given that z_0 is inside the cauliflower set.

(ii) For $|1+\lambda| \notin \{0,1\}$ we have from Koenigs Linerization Theorem, see [6], that there exists a linear change of coordinate $x = \mu(z)$ with $\mu(0) = 0$, such that $\mu \circ \tilde{g} \circ \mu^{-1}$ is the linear map $x \mapsto (1+\lambda)x$ for all x in some neighborhood of the origin. We see that inside this neighborhood we have $\mu \circ g^n \circ \mu^{-1}(x) = (1+\lambda)^n x$. Depending on if $|1+\lambda|$ is less than 1 or greater than 1 the iterates inside this neighborhood will be attracted or repelled by the origin respectively.

(iii) In the special case where $\lambda = -1$ we have that $\tilde{g}(z_0) = z_0^2$. This is a well understood map, as the Julia set is simply the unit circle. For $\forall z_0 \in \mathbb{C} \setminus \overline{\Delta}(0,1)$, where $\Delta(0,1)$ is the unit disk, the iterates diverge to infinity. For $\forall z_0 \in \Delta(0,1)$, however, the iterates converge to zero.

Analysing the dynamical behaviour of w_n takes a very similar approach, it is a simple matter of transposing the linear transformation ϕ and conjugating (3.8) with this new linear transformation.

As mentioned, a major question for us has been to identify the set close to $\{z = w\}$ where the iterates approach either of the complex axes.

We have identified some points which will converge to either X or Y though. Assume $|z_0| > |w_0|$ with $\operatorname{Re}\lambda < 0$. We then have that $|1+\lambda| < 1$ if

$$\begin{aligned}(1 - |\operatorname{Re}\lambda|)^2 + \operatorname{Im}\lambda^2 &< 1\\ 1 - 2|\operatorname{Re}\lambda| + |\lambda|^2 &< 1\\ |\operatorname{Re}\lambda| &> \frac{1}{2}|\lambda|^2.\end{aligned}$$

When $|1+\lambda|<1$ we have that $|z_{n+1}|<|z_n|$ when

$$|z(1+\lambda)+z^2|<|z|$$
$$|z(1+\lambda)+z^2|\le|z||1+\lambda|+|z|^2<|z|$$
$$|z|<1-|1+\lambda|.$$

This motivates the definition of the following set in $\mathbb{C}^2$,

$$L=\left\{(z,w)\in\mathbb{C}^2\,|\,\lambda=w-z\,,\,0<|\lambda|<\frac{1}{2}\,,\mathrm{Re}\,\lambda<0\,,\,|\mathrm{Re}\,\lambda|>\frac{1}{2}|\lambda|^2\,,\,|z|<(1-|1+\lambda|)\right\}.$$

It is easy to see that inside this set $z_n\to 0$. Furthermore, we have that there exists some $n\in\mathbb{N}$ such that $(z_n,w_n)\to(0,w')$ for some $w'\in\mathbb{C}\setminus\{0\}$. This is what we show in the following lemma.

Lemma 3.3. *Let* $(z_0,w_0)\in L$. *Then* $z_n\to 0$ *and* $w_n\to w'\neq 0$.

Proof. The proof follows almost directly from $\lambda=w_n-z_n$. Since $z_n\to 0$ there exists some $N\in\mathbb{N}$ such that for $\forall n\ge N$ we have that $|z_n|<\frac{|\lambda|}{16}$. Then we have that $|w_n-\lambda|<\frac{|\lambda|}{16}$ and by simply rotating with some $e^{i\theta}$ such that $e^{i\theta}\lambda=|\lambda|e^{i\pi}$ we are inside R_w. Then we know from lemma 3.1 that $z_n\to 0$ and $w_n\to e^{i\theta}w'\neq 0$. □

By the exact same procedure we can prove for the set

$$K=\left\{(z,w)\in\mathbb{C}^2\,|\,\lambda'=z-w\,,\,0<|\lambda'|<\frac{1}{2}\,,\mathrm{Re}\,\lambda'<0\,,\,|\mathrm{Re}\,\lambda'|>\frac{1}{2}|\lambda'|^2\,,\,|w|<\left(1-|1+\lambda'|\right)\right\}.$$

that as $w_n\to 0$ we have that $z_n\to z'\in\mathbb{C}\setminus\{0\}$ when $n\to\infty$.

The proof of lemma 3.3 is valid if either z_n or w_n approaches zero. Since λ is constant we cannot have them both reach zero if $z\neq w$. We have, however, situations for points close to the line $\{z=w\}$ where the techniques above does not immediately show that we approach either X or Y. If, for example, $\lambda=iy$ we have even for some very small $y\in\mathbb{R}$ that $|1+iy|>1$ and from (ii) above we have that in some neighborhood of zero the iterates of $\tilde{g}(z_0)$ will be repelled by zero. Since $\lambda'=-\lambda$, we have $|1-iy|>1$ and the same applies for w_n.

Chapter 4

Summary and Recommendations for Further Work

In this final chapter we give a brief overview of our results and discuss suggestions for future work.

4.1 Summary and Conclusions

We have shown that for any class of automorphisms of the form (3.3) the stable set will be concentrated around the complex line $\{z = w\}$ when we are close enough to the origin. To show this we closely followed the approach of Hakim. This involved conjugating our map by several linear transformations which gave us a new map in a very particular form. This new map was constructed to be able to show the existence of an analytic invariant curve which was tangent to the characteristic direction.

In the next section we studied an approximation of our original map. This approximation will be valid when we are sufficiently close to the origin. On the line $\{z = w\}$ this approximation gives a cauliflower set in each complex direction where points will approach the origin. Just outside of this line the behaviour is different. An interesting feature of this map is that if we are close to one of the complex axes we have shown that iterates will always approach this axis. We have also shown that for points close to, but not on the line $\{z = w\}$, that if either of the coordinates approach zero we will eventually be inside the region where iterates will approach one of the complex axes.

4.2 Recommendations for Further Work

Here we list some suggestions for further work

- The natural next step for the approximated map is exploring if all points (z, w) where $|\lambda|$ is small will always approach one of the complex axes when we are sufficiently close to the origin. We have only shown it when we know that either $z_n \to 0$ or $w_n \to 0$. Our techniques does not show if either of these iterates always will approach zero when we are close to the origin.
- Analysing the stable set of the origin of any germ of the form (3.3) globally. This would decide if the automorphism (1.1) admits a basin of attraction or not at the origin. If it has a basin of attraction this would give a positive answer to the question of Rosay and Rudin described in chapter 1.
- Showing explicitly that the sequence of polynomials $\{P_k\}$ from proposition 3.1 converges uniformly to a polynomial P in a neighborhood of the origin.

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