0.1 GNSS

As discussed in Section ?? $\boldsymbol{P}_0$ is normally a pre-defined point on the vessel. From this point the lever arms to the different GNSS antennas are measured by surveying with laser equipment. Then the measurements from the GNSS antennas are used to find the NED position of $\boldsymbol{P}_0$ as the vessel is in operation.

When the GNSS measurements are used in a estimation algorithm to find the lever arms, the $\boldsymbol{P}_0$ position that is found is the NED-position of the rotation point of the vessel. Depending on the operation, or load condition this rotation point would move.

0.1.1 GNSS Luenberger observer design

The GNSS positions in the North-East-Down (NED) frame is given as

$$\boldsymbol{P}_{GNSSi} = \begin{bmatrix} P_{GNSSi_N} & P_{GNSSi_E} & P_{GNSSi_D} \end{bmatrix}^\top \in \mathbb{R}^3, \tag{1}$$

where i denotes a GNSS sensor. For m GNSS sensors this can be written

$$\begin{aligned} \boldsymbol{P}_{GNSS1}(t) &= \boldsymbol{P}_0(t) + \boldsymbol{R}(\boldsymbol{\Theta})\boldsymbol{l}_1 \\ \boldsymbol{P}_{GNSS2}(t) &= \boldsymbol{P}_0(t) + \boldsymbol{R}(\boldsymbol{\Theta})\boldsymbol{l}_2 \\ &\vdots \\ \boldsymbol{P}_{GNSSm}(t) &= \boldsymbol{P}_0(t) + \boldsymbol{R}(\boldsymbol{\Theta})\boldsymbol{l}_m \end{aligned} \tag{2}$$

where

$$\boldsymbol{l}_i = \begin{bmatrix} l_{xi} \\ l_{yi} \\ l_{zi} \end{bmatrix}, \quad i = 1 \dots m, \tag{3}$$

is a vector that contains the body fixed coordinates of a lever arm, and $\boldsymbol{P}_0(t)$ is the NED-position of the vessel rotation point as discussed in Section **??**. The rotation matrix $\boldsymbol{R}(\boldsymbol{\Theta})$ between the BODY and the NED frame given by Eq. (**??**) of Section **??**.

Eq. (2) can be written as

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{P}_0 \\ \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \\ \vdots \\ \boldsymbol{l}_m \end{bmatrix} \in \mathbb{R}^{3n}, \quad n = m + 1, \tag{4}$$

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{P}_{GNSS1} \\ \boldsymbol{P}_{GNSS2} \\ \vdots \\ \boldsymbol{P}_{GNSSm} \end{bmatrix} \in \mathbb{R}^{3m}, \tag{5}$$

where $\boldsymbol{x}$ consists of the states to be estimated, and $\boldsymbol{y}$ is the output.

The Euler angles (ϕ, θ, and ψ) are viewed as an input signal to the system, so it is a linear time varying (LTV) system, instead of a nonlinear system, setting $\boldsymbol{R}(\boldsymbol{\Theta}) := \boldsymbol{R}(t)$. Also, p, q, r, and $\dot{p}$, $\dot{q}$, and $\dot{r}$ are considered input signals.

The dynamics of $\boldsymbol{P}_0$ can be represented as (?)

$$\dot{\boldsymbol{P}}_0 = \boldsymbol{R}(t)\boldsymbol{\nu}(t), \tag{6}$$

where $\boldsymbol{\nu} = \begin{bmatrix} u & v & w \end{bmatrix}^\top$ is the linear velocity of the vessel in BODY-coordinates, assumed measured. The lever arms are constants, and hence

$$\dot{\boldsymbol{l}}_i = \boldsymbol{0}, \quad i = 1 \dots m. \tag{7}$$

The dynamic lever arm system can be written as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{\nu}(t), \tag{8}$$

$$\boldsymbol{y} = \boldsymbol{C}(t)\boldsymbol{x}. \tag{9}$$

The matrices of (8) and (9) are given by

$$\boldsymbol{A} = \boldsymbol{0}_{3n\times 3n}, \tag{10}$$

$$\boldsymbol{B}(t) = \begin{bmatrix} \boldsymbol{R}(t)^\top & \boldsymbol{0}_{3\times 3} & \boldsymbol{0}_{3\times 3} & \cdots & \boldsymbol{0}_{3\times 3} \end{bmatrix}^\top \in \mathbb{R}^{3n\times 3}, \tag{11}$$

$$\boldsymbol{C}(t) = \begin{bmatrix} \boldsymbol{I}_{3\times 3} & \boldsymbol{R}(t) & \boldsymbol{0}_{3\times 3} & \cdots & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{I}_{3\times 3} & \boldsymbol{0}_{3\times 3} & \boldsymbol{R}(t) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{I}_{3\times 3} & \boldsymbol{0}_{3\times 3} & \cdots & \boldsymbol{0}_{3\times 3} & \boldsymbol{R}(t) \end{bmatrix} \in \mathbb{R}^{3m\times 3n}. \tag{12}$$

Observability assessment

In this section an observability criterion for the system given by Eq. (8) - (9) will be investigated. Because the $\boldsymbol{A}$ matrix of (8) is zero, the state transition matrix is identity. This gives the following observability gramian for the system as (?),

$$\boldsymbol{W}_0(t_0, t_1) = \int_{t_0}^{t_1} \boldsymbol{C}(\tau)^\top \boldsymbol{C}(\tau) \mathrm{d}\tau, \tag{13}$$

and if $\boldsymbol{W}_0(t_0, t_1)$ is nonsingular, the system is observable (?).

$\boldsymbol{C}(t)^\top$ for (8) - (9) is

$$\boldsymbol{C}(t)^\top = \begin{bmatrix} \boldsymbol{I}_{3\times 3} & \boldsymbol{I}_{3\times 3} & \cdots & \boldsymbol{I}_{3\times 3} \\ \boldsymbol{R}(t)^\top & \boldsymbol{0}_{3\times 3} & \cdots & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{0}_{3\times 3} & \boldsymbol{R}(t)^\top & \ddots & \vdots \\ \vdots & \ddots & \ddots & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{0}_{3\times 3} & \cdots & \boldsymbol{0}_{3\times 3} & \boldsymbol{R}(t)^\top \end{bmatrix} \in \mathbb{R}^{3n\times 3m}, \tag{14}$$

and $\boldsymbol{C}(t)^\top \boldsymbol{C}(t)$ is given as

$$\boldsymbol{C}(t)^\top \boldsymbol{C}(t) \quad = \begin{bmatrix} (n-1)\boldsymbol{I}_{3\times3} & \boldsymbol{R}(t) & \cdots & \boldsymbol{R}(t) & \boldsymbol{R}(t) \\ \boldsymbol{R}(t)^\top & \boldsymbol{I}_{3\times3} & \mathbf{0}_{3\times3} & \cdots & \mathbf{0}_{3\times3} \\ \vdots & \mathbf{0}_{3\times3} & \ddots & \ddots & \vdots \\ \boldsymbol{R}(t)^\top & \vdots & \ddots & I_{3\times3} & \mathbf{0}_{3\times3} \\ \boldsymbol{R}(t)^\top & \mathbf{0}_{3\times3} & \cdots & \mathbf{0}_{3\times3} & \boldsymbol{I}_{3\times3} \end{bmatrix} \in \mathbb{R}^{3n\times3n}. \tag{15}$$

From (12) it can be found that if $\boldsymbol{R}(t)$ has full rank, then $\boldsymbol{C}(t)$ and $\boldsymbol{C}(t)^\top$ also have full rank, i.e. $3m$. From **?** it is found that the rank of $\boldsymbol{C}(t)^\top \boldsymbol{C}(t) \in \mathbb{R}^{3n\times3n}$ will at most be $3m$, as

$$rank(\boldsymbol{AB}) \leq min(\text{rank}(\boldsymbol{A}), \text{rank}(\boldsymbol{B})) \tag{16}$$

where both $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices. In other words the matrix is not full rank in the first place, and need to "add rank" as time increases (**?**). To find the observability condition(s), Theorem 6.O12 from **?** is used.

Theorem 1 (Thm 6.O12 (**?**))**.** *Let $\boldsymbol{A}(t)$ and $\boldsymbol{C}(t)$ be continously differentiable, then the n-dimensional pair ($\boldsymbol{A}(t)$, $\boldsymbol{C}(t)$) is observable at t_0 if there exists a finite $t_1 > t_0$ such that*

$$rank \begin{bmatrix} \boldsymbol{N}_0 \\ \boldsymbol{N}_1 \\ \vdots \\ \boldsymbol{N}_{n-1} \end{bmatrix} = n$$

where $\boldsymbol{N}_{m+1} = \boldsymbol{N}_m(t)\boldsymbol{A}(t) + \frac{d}{dt}\boldsymbol{N}_m(t) \quad m = 0, 1 \ldots n-1$
with $\boldsymbol{N}_0 = \boldsymbol{C}(t)$.

For the system of (8) - (9) both $\boldsymbol{C}(t)$ and $\boldsymbol{A}(t)$ are continuously differentiable, and $\boldsymbol{C}(t)$ has rank $3m$ given that $\boldsymbol{R}(t)$ has full rank. Hence, given $\boldsymbol{C}(t)$ full row rank, a rank of 3 is lacking to fulfil rank n of theorem 6.O12 ($\boldsymbol{C}(t)$ has three more columns than rows). Thus, $\boldsymbol{N}_1, ...\boldsymbol{N}_{n-1}$ must contribute with 3 more independent rows for the observability condition to be satisfied.

$\boldsymbol{N}_1$:

$$\boldsymbol{N}_1 = \boldsymbol{N}_0(t)\boldsymbol{A}(t) + \frac{d}{dt}\boldsymbol{N}_0(t) \tag{17}$$

$$= \mathbf{0}_{3\times3} + \frac{d}{dt}\boldsymbol{C}(t), \tag{18}$$

$$\dot{C} = \begin{bmatrix} \mathbf{0}_{3\times3} & \boldsymbol{R}(t)\boldsymbol{S}(t) & \mathbf{0}_{3\times3} & \cdots & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \boldsymbol{R}(t)\boldsymbol{S}(t) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \cdots & \mathbf{0}_{3\times3} & \boldsymbol{R}(t)\boldsymbol{S}(t) \end{bmatrix}, \tag{19}$$

where it has been used that $\dot{\boldsymbol{R}}(t) = \boldsymbol{R}(t)\boldsymbol{S}(t)$ from Eq. **??**, and $\boldsymbol{S}(t)$ is given by Eq. **??**.

$\boldsymbol{N}_2$**:**

$$\boldsymbol{N}_2 = \boldsymbol{N}_1(t)\boldsymbol{A}(t) + \frac{d}{dt}\boldsymbol{N}_1(t) \tag{20}$$

$$= \ddot{\boldsymbol{C}}(t), \tag{21}$$

$$\ddot{\boldsymbol{C}} = \begin{bmatrix} \mathbf{0}_{3\times3} & \boldsymbol{R}(t)[\boldsymbol{S}(t)^2 + \dot{\boldsymbol{S}}(t)] & \mathbf{0}_{3\times3} & \cdots & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \boldsymbol{R}(t)[\boldsymbol{S}(t)^2 + \dot{\boldsymbol{S}}(t)] & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \cdots & \mathbf{0}_{3\times3} & \boldsymbol{R}(t)[\boldsymbol{S}(t)^2 + \dot{\boldsymbol{S}}(t)] \end{bmatrix}. \tag{22}$$

giving

$$\begin{bmatrix} \boldsymbol{N}_0 \\ \boldsymbol{N}_1 \\ \boldsymbol{N}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_{3\times3} & \boldsymbol{R}(t) & \mathbf{0}_{3\times3} & \cdots & \mathbf{0}_{3\times3} \\ \boldsymbol{I}_{3\times3} & \mathbf{0}_{3\times3} & \boldsymbol{R}(t) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathbf{0}_{3\times3} \\ \boldsymbol{I}_{3\times3} & \mathbf{0}_{3\times3} & \cdots & \mathbf{0}_{3\times3} & \boldsymbol{R}(t) \\ \mathbf{0}_{3\times3} & \boldsymbol{R}(t)\boldsymbol{S}(t) & \mathbf{0}_{3\times3} & \cdots & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \boldsymbol{R}(t)\boldsymbol{S}(t) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \cdots & \mathbf{0}_{3\times3} & \boldsymbol{R}(t)\boldsymbol{S}(t) \\ \mathbf{0}_{3\times3} & \boldsymbol{R}(t)[\boldsymbol{S}(t)^2 + \dot{\boldsymbol{S}}(t)] & \mathbf{0}_{3\times3} & \cdots & 0_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \boldsymbol{R}(t)[\boldsymbol{S}(t)^2 + \dot{\boldsymbol{S}}(t)] & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \cdots & \mathbf{0}_{3\times3} & \boldsymbol{R}(t)[\boldsymbol{S}(t)^2 + \dot{\boldsymbol{S}}(t)] \end{bmatrix}, \tag{23}$$

where

$$\dot{\boldsymbol{S}} = \begin{bmatrix} 0 & -\dot{r} & \dot{q} \\ \dot{r} & 0 & -\dot{p} \\ -\dot{q} & \dot{p} & 0 \end{bmatrix}. \tag{24}$$

For the system to be observable, the requirement is that there exists a time t_1 such that

$$\text{rank}(\begin{bmatrix} \boldsymbol{N}_0 \\ \boldsymbol{N}_1 \\ \boldsymbol{N}_2 \end{bmatrix}) = 3n.$$

Since $\boldsymbol{S}(t)$ is a skew-symmetric matrix, it is always singular. The observability requirement is therefore that there has to exists a time t_1 where rank $(\boldsymbol{R}(t)[\boldsymbol{S}(t)^2 + \dot{\boldsymbol{S}}(t)]) = 3$. Since $\det \boldsymbol{R}(t) = 1$, $\boldsymbol{R}(t)$ is always nonsingular (**?**), and the observability requirement reduce to that $[\boldsymbol{S}(t)^2 + \dot{\boldsymbol{S}}(t)]$ need to have full rank. This is summarized in the proposition below.

Proposition 1. *For the system of (8) and (9) to be observable at time t_0, there have to exist a time $t_1 > t_0$ where*

$$rank[\boldsymbol{S}(t)^2 + \dot{\boldsymbol{S}}(t)] = 3. \tag{25}$$

Example 1. *For a constant yaw rate $r = 1$, $p, q, \dot{r}, \dot{q} = 0$, and $\dot{p} = 0.1$ at t_1, $\boldsymbol{S}(t)^2 + \dot{\boldsymbol{S}}(t)$ becomes*

$$\boldsymbol{S}(t)^2 + \dot{\boldsymbol{S}}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -0.1 \\ 0 & 0.1 & 0 \end{bmatrix}$$

which has full rank.

By trying to maneuver the vessel with a constant yaw rate r, observability will be assured if there also exists some acceleration in roll or pitch. This will in practice always be the case.

Luenberger observer design and stability analysis

For the system dynamics of equations (8), and (9) the following observer is proposed

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}} &= \boldsymbol{B}(t)\boldsymbol{\nu}(t) + \boldsymbol{W}\boldsymbol{C}(t)^\top \tilde{\boldsymbol{y}} \\ &= \boldsymbol{B}(t)\boldsymbol{v}(t) + \boldsymbol{W}\boldsymbol{C}(t)^\top \boldsymbol{C}(t)\tilde{\boldsymbol{x}},\end{aligned} \tag{26}$$

$$\tilde{\boldsymbol{y}} = \boldsymbol{C}(t)\tilde{\boldsymbol{x}}, \tag{27}$$

where $\boldsymbol{W} = \boldsymbol{W}^\top > 0 \in \mathbb{R}^{3n\times 3n}$, $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \hat{\boldsymbol{x}}, \tilde{\boldsymbol{y}} = \boldsymbol{y} - \hat{\boldsymbol{y}}$, and with closed loop error dynamics as

$$\dot{\tilde{\boldsymbol{x}}} = -\boldsymbol{W}\boldsymbol{C}(t)^\top \boldsymbol{C}(t)\tilde{\boldsymbol{x}}. \tag{28}$$

In order to evaluate the stability properties of the observer design, the following lemma from **?** is applied.
Lemma 1. *Exponential stability of LTV system (**?**)*
For a system given by $\dot{\boldsymbol{x}} = \boldsymbol{F}(t)\boldsymbol{x}$, and the function $\boldsymbol{F}(\cdot)$ is locally integrable. Suppose there exits a positive definite matrix $\boldsymbol{P} = \boldsymbol{P}^\top > 0$ such that

$$\boldsymbol{P}\boldsymbol{F}(t) + \boldsymbol{F}(t)^\top \boldsymbol{P} \leq -\boldsymbol{N}(t)^\top \boldsymbol{N}(t) \tag{29}$$

for some matrix function $\boldsymbol{N}(\cdot)$ and all t. Then $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{t})\boldsymbol{x}$ is uniformly stable in the sense of Lyapunov.

If, further the pair $[\boldsymbol{F}(t), \boldsymbol{N}(t)]$ is uniformly completely observable, that is, writing $\boldsymbol{\phi}(t,\tau)$ as the transition function of $\dot{\boldsymbol{x}} = \boldsymbol{F}(t)\boldsymbol{x}$, there exits $T > 0,\ \beta > 0,\ \alpha > 0$ such that

$$\beta \boldsymbol{I} \geq \int_t^{t+T} \boldsymbol{\phi}(t,\tau)^\top \boldsymbol{C}(\tau)^\top \boldsymbol{C}(\tau)\boldsymbol{\phi}(t,\tau)\mathrm{d}\tau \geq \alpha \boldsymbol{I}$$

then $\dot{\boldsymbol{x}} = \boldsymbol{F}(t)\boldsymbol{x}$ is exponentially stable.

From equations (27) and (28), $F(t) = -\boldsymbol{W}\boldsymbol{C}(t)^\top \boldsymbol{C}(t)$ and $\boldsymbol{P} = \boldsymbol{W}^{-1}$ gives

$$\begin{aligned}\boldsymbol{P}\boldsymbol{F}(t) + \boldsymbol{F}(t)^\top \boldsymbol{P} &= -2\boldsymbol{C}(t)^\top \boldsymbol{C}(t) \\ &= -\boldsymbol{N}(t)^\top \boldsymbol{N}(t),\end{aligned} \tag{30}$$

where

$$\boldsymbol{N}(t) = \sqrt{2}\boldsymbol{C}(t), \tag{31}$$

The discussion above is summarized in Theorem 2 below.
Theorem 2. *The observer design of (26) is exponentially stable given that the observability condition of Proposition 1 holds, according to Lemma 1.*

0.1.2 GNSS adaptive observer design

Another way to solve the lever arm estimation problem is an adaptive solution. This set up will remove $\boldsymbol{P}_0$ as a variable to be estimated, and thus have full state measurements available. The value of $\boldsymbol{P}_0$ can then be found once the lever arms have converged, from Eq. (2) as

$$\boldsymbol{P}_0 = \boldsymbol{P}_{GNSSi}(t) - \boldsymbol{R}(\boldsymbol{\Theta})\boldsymbol{l}_i. \tag{32}$$

By taking the time derivative of (2), using (6) and $\boldsymbol{R}(\Theta) = \boldsymbol{R}(t),\ \boldsymbol{S}(\omega) = \boldsymbol{S}(t)$, the lever arm problem can be formulated as

$$\begin{aligned}\dot{\boldsymbol{P}}_{GNSS1}(t) &= \boldsymbol{R}(t)\boldsymbol{\nu}(t) + \boldsymbol{R}(t)\boldsymbol{S}(t)\boldsymbol{l}_1 \\ \dot{\boldsymbol{P}}_{GNSS2}(t) &= \boldsymbol{R}(t)\boldsymbol{\nu}(t) + \boldsymbol{R}(t)\boldsymbol{S}(t)\boldsymbol{l}_2 \\ &\vdots \\ \dot{\boldsymbol{P}}_{GNSSn}(t) &= \boldsymbol{R}(t)\boldsymbol{\nu}(t) + \boldsymbol{R}(t)\boldsymbol{S}(t)\boldsymbol{l}1_n,\end{aligned} \tag{33}$$

$$\boldsymbol{x}(t) = \begin{bmatrix}\boldsymbol{P}_{GNSS1} & \boldsymbol{P}_{GNSS2} & \cdots & \boldsymbol{P}_{GNSSn}\end{bmatrix}^\top, \tag{34}$$

$$\boldsymbol{y}(t) = \boldsymbol{I}_{3n\times 3n}\boldsymbol{x}. \tag{35}$$

Written in state space form

$$\dot{\boldsymbol{x}} = \boldsymbol{B}(t)\boldsymbol{\nu}(t) + \boldsymbol{\Omega}(t)\boldsymbol{\varphi}, \tag{36}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}, \tag{37}$$

where

$$\boldsymbol{B}(t) = \begin{bmatrix} \boldsymbol{R}(t) \\ \boldsymbol{R}(t) \\ \vdots \\ \boldsymbol{R}(t) \end{bmatrix} \in \mathbb{R}^{3n\times 3}, \tag{38}$$

$$\boldsymbol{C} = \boldsymbol{I}_{3n\times 3n}, \tag{39}$$

$$\boldsymbol{\varphi} = \begin{bmatrix} \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \\ \cdots \\ \boldsymbol{l}_n \end{bmatrix} \in \mathbb{R}^{3n}, \tag{40}$$

$$\boldsymbol{\Omega}(t) = \begin{bmatrix} \boldsymbol{R}(t)\boldsymbol{S}(t) & \boldsymbol{0}_{3\times 3} & \cdots & \boldsymbol{0}_{3\times 3} \\ 0_{3\times 3} & \boldsymbol{R}(t)\boldsymbol{S}(t) & \ddots & \vdots \\ \vdots & & \ddots & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{0}_{3\times 3} & \cdots & \boldsymbol{0}_{3\times 3} & \boldsymbol{R}(t)\boldsymbol{S}(t) \end{bmatrix} \in \mathbb{R}^{3n\times 3n}. \tag{41}$$

Adaptive observer design and stability analysis

Let a state observer be given as

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}} &= \boldsymbol{B}(t)\boldsymbol{\nu}(t) + \boldsymbol{\Omega}(t)\hat{\boldsymbol{\varphi}} + \boldsymbol{L}\boldsymbol{y} - \boldsymbol{L}\boldsymbol{C}\hat{\boldsymbol{x}} \\ &= \boldsymbol{B}(t)\boldsymbol{\nu}(\boldsymbol{t}) + \boldsymbol{\Omega}(t)\hat{\boldsymbol{\varphi}} + \boldsymbol{L}\boldsymbol{C}\tilde{\boldsymbol{x}}, \end{aligned} \tag{42}$$

where $\boldsymbol{L} \in \mathbb{R}^{3n\times 3n}$ and $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \hat{\boldsymbol{x}}$. Let $\tilde{\boldsymbol{\varphi}} = \boldsymbol{\varphi} - \hat{\boldsymbol{\varphi}}$, and

$$\dot{\tilde{\boldsymbol{x}}} = \dot{\boldsymbol{x}} - \dot{\hat{\boldsymbol{x}}} = -\boldsymbol{L}\boldsymbol{C}\tilde{\boldsymbol{x}} + \boldsymbol{\Omega}\tilde{\boldsymbol{\varphi}}, \tag{43}$$

$$\dot{\tilde{\boldsymbol{\varphi}}} = \dot{\boldsymbol{\varphi}} - \dot{\hat{\boldsymbol{\varphi}}} = -\dot{\hat{\boldsymbol{\varphi}}}. \tag{44}$$

Define the following Control Lyapunov Function (CLF) (**?**)

$$V = \frac{1}{2}\tilde{\boldsymbol{x}}^\top\tilde{\boldsymbol{x}} + \frac{1}{2}\tilde{\boldsymbol{\varphi}}^\top\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\varphi}}, \tag{45}$$

where the constant matrix $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^\top > 0$. This gives

$$\dot{V} = \tilde{\boldsymbol{x}}^\top[-\boldsymbol{L}\boldsymbol{C}\tilde{\boldsymbol{x}} + \boldsymbol{\Omega}\tilde{\boldsymbol{\varphi}}] + \tilde{\boldsymbol{\varphi}}^\top\boldsymbol{\Gamma}^{-1}(-\dot{\hat{\boldsymbol{\varphi}}}) \tag{46}$$

$$= -\tilde{\boldsymbol{x}}^\top\boldsymbol{L}\boldsymbol{C}\tilde{\boldsymbol{x}} + \tilde{\boldsymbol{x}}^\top\boldsymbol{\Omega}\tilde{\boldsymbol{\varphi}} - \tilde{\boldsymbol{\varphi}}^\top\boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\varphi}}}. \tag{47}$$

For the following update law for $\hat{\boldsymbol{\varphi}}$

$$\dot{\hat{\boldsymbol{\varphi}}} := \boldsymbol{\Gamma}\boldsymbol{\Omega}^\top\tilde{\boldsymbol{x}}, \tag{48}$$

$\dot{V}$ becomes

$$\dot{V} = -\tilde{\boldsymbol{x}}^\top \boldsymbol{L}\boldsymbol{C}\tilde{\boldsymbol{x}} \leq 0. \tag{49}$$

Let $\boldsymbol{LC} > 0$, then $\dot{V}$ is negative semidefinite. Barbalat's lemma (**?**) is applied, and restated here for convenience.

Lemma 2. *(Barbalat's lemma) If the differentiable function f(t) has a finite limit as $t \to \infty$, and if $\dot{f}$ is uniformly continuous, then $f(t) \to 0$ as $t \to \infty$.*

To assess uniform continuity, **?** states that a sufficient condition for a differentiable function is that its derivative is bounded. For the above CLF, this implies that $\ddot{V}$ should be bounded, and $\ddot{V}$ is given as

$$\ddot{V} = -2\tilde{\boldsymbol{x}}^\top \boldsymbol{L}\boldsymbol{C}\dot{\tilde{\boldsymbol{x}}} \tag{50}$$

From Eq. (49) it is shown that $\dot{V} \leq 0$, and therefore V is bounded, and from (45) it can be concluded that both $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{\varphi}}$ are bounded. From (43), since both $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{\varphi}}$ are bounded, $\dot{\tilde{\boldsymbol{x}}}$ is bounded, and hence is $\ddot{V}$ bounded. So $\dot{V}$ is uniformly continuous, and by Barbalat's lemma $\dot{V} \to 0$ as $t \to \infty$, and hence $\tilde{\boldsymbol{x}} \to \boldsymbol{0}$ as $t \to \infty$. This implies that $\dot{\tilde{\boldsymbol{x}}} \to \boldsymbol{0}$ as $t \to \infty$, and from (43) it follows that $\boldsymbol{\Omega}\tilde{\boldsymbol{\varphi}} \to \boldsymbol{0}$ as $t \to \infty$.

Percistence of excitation

Consider $\boldsymbol{\Omega}\tilde{\boldsymbol{\varphi}} = \boldsymbol{0}$. In order to have $\tilde{\boldsymbol{\varphi}} = \boldsymbol{0}$ as the only solution $\boldsymbol{\Omega}$ need to be persistently excited. For a update law of the form (48), then persistence of excitation (PE) can be formulated as (**?**)

Theorem 3. *(Persistence of excitation) The matrix Ω is persistently excited if there exists $\alpha, T > 0$ such that $\forall t$*

$$\int_t^{t+T} \boldsymbol{\Omega}(\tau)^\top \boldsymbol{\Omega}(\boldsymbol{\tau})\mathrm{d}\tau > \alpha \boldsymbol{I} \tag{51}$$

Seeing that $\boldsymbol{\Omega}(t)$ is diagonal, condition (51) will only depend on $\boldsymbol{R}(t)\boldsymbol{S}(t)$, and PE criterion can be evaluated based on $\boldsymbol{R}(t)\boldsymbol{S}(t)$ as the integrand (with $\boldsymbol{I} := \boldsymbol{I}_{3\times 3}$ in (51)). Looking at the integrand

$$(\boldsymbol{R}(t)\boldsymbol{S}(t))^\top \boldsymbol{R}(t)\boldsymbol{S}(t) = \boldsymbol{S}(t)^\top \boldsymbol{R}(t)^\top \boldsymbol{R}(t)\boldsymbol{S}(t) \tag{52}$$

$$= \boldsymbol{S}(t)^\top \boldsymbol{S}(t). \tag{53}$$

This result is valid due to $\boldsymbol{R}^\top \boldsymbol{R} = \boldsymbol{I}$, which is a fundamental property of the rotation matrix (**?**). The expression for $\boldsymbol{S}(t)^\top \boldsymbol{S}(t)$ is given as

$$\boldsymbol{S}(t)^\top \boldsymbol{S}(t) = \begin{bmatrix} r(t)^2 + q(t)^2 & -p(t)q(t) & -p(t)r(t) \\ -p(t)q(t) & r(t)^2 + p(t)^2 & -q(t)r(t) \\ -p(t)r(t) & -q(t)r(t) & p(t)^2 + q(t)^2 \end{bmatrix}, \tag{54}$$

and the PE criteria can be written as

$$
\begin{aligned}
&\int_t^{t+T} \boldsymbol{S}(\tau)^\top \boldsymbol{S}(\tau)\mathrm{d}\tau = \\
&\begin{bmatrix}
\int_t^{t+T}[q(\tau)^2+r(\tau)^2]\mathrm{d}\tau & -\int_t^{t+T}p(\tau)q(\tau)\mathrm{d}\tau & -\int_t^{t+T}p(\tau)r(\tau)\mathrm{d}\tau \\
-\int_t^{t+T}p(\tau)q(\tau)\mathrm{d}\tau & \int_t^{t+T}[p(\tau)^2+r(\tau)^2]\mathrm{d}\tau & -\int_t^{t+T}q(\tau)r(\tau)\mathrm{d}\tau \\
-\int_t^{t+T}p(\tau)r(\tau)\mathrm{d}\tau & -\int_t^{t+T}q(\tau)r(\tau)\mathrm{d}\tau & \int_t^{t+T}[p(\tau)^2+q(\tau)^2]\mathrm{d}\tau
\end{bmatrix} \\
&> \alpha \boldsymbol{I}_{3\times 3},
\end{aligned}
\tag{55}
$$

A matrix is positive definite if the leading principal minors are positive (**?**). This gives the following three conditions for Eq. (55) to be satisfied

1.

$$\int_t^{t+T}[q(\tau)^2+r(\tau)^2]\mathrm{d}\tau > 0 \tag{56}$$

2.

$$
\begin{aligned}
&\int_t^{t+T}[q(\tau)^2+r(\tau)^2]\mathrm{d}\tau \int_t^{t+T}[p(\tau)^2+r(\tau)^2]\mathrm{d}\tau \\
&-\left[\int_t^{t+T}p(\tau)q(\tau)\mathrm{d}\tau\right]^2 > 0
\end{aligned}
\tag{57}
$$

3.

$$
\begin{aligned}
&\int_t^{t+T}[q(\tau)^2+r(\tau)^2]\mathrm{d}\tau \left\{\int_t^{t+T}[p(\tau)^2+r(\tau)^2]\mathrm{d}\tau \int_t^{t+T}[p(\tau)^2+q(\tau)^2]\mathrm{d}\tau - \right. \\
&\left.\left[\int_t^{t+T}q(\tau)r(\tau)\mathrm{d}\tau\right]^2\right\} - \\
&\int_t^{t+T}p(\tau)q(\tau)\mathrm{d}\tau \left\{\int_t^{t+T}p(\tau)q(\tau)\mathrm{d}\tau \int_t^{t+T}[p(\tau)^2+q(\tau)^2]\mathrm{d}\tau + \right. \\
&\left.\int_t^{t+T}q(\tau)r(\tau)\mathrm{d}\tau \int_t^{t+T}p(\tau)r(\tau)\mathrm{d}\tau\right\} - \\
&\int_t^{t+T}p(\tau)r(\tau)\mathrm{d}\tau \left\{\int_t^{t+T}p(\tau)q(\tau)\mathrm{d}\tau \int_t^{t+T}q(\tau)r(\tau)\mathrm{d}\tau + \right. \\
&\left.\int_t^{t+T}p(\tau)r(\tau)\mathrm{d}\tau \int_t^{t+T}[p(\tau)^2+r(\tau)^2]\mathrm{d}\tau\right\} > 0
\end{aligned}
\tag{58}
$$

Proposition 2 (PE of adaptive observer)**.** *The adaptive observer of Eq. (43) and (48) is persistently excited if the conditions of (56) - (58) are satisfied.*

Remark 1. *If $p = 0 \; \forall \; \bar{t} \in [t, t+T]$, the minimum requirement is that $r \neq 0$ for some time $\bar{t}$ in $[t, t+T]$, and that*

$$\int_t^{t+T} r(\tau)^2 \mathrm{d}\tau \int_t^{t+T} q(\tau)^2 \mathrm{d}\tau > \left[\int_t^{t+T} q(\tau) r(\tau) \mathrm{d}\tau \right]^2, \tag{59}$$

is satisfied. If $q = 0 \; \forall \; \bar{t} \in [t, t+T]$, the minimum requirement is that $r \neq 0$ for some time $\bar{t}$ in $[t, t+T]$, and that

$$\int_t^{t+T} r(\tau)^2 \mathrm{d}\tau > \left[\int_t^{t+T} p(\tau) r(\tau) \mathrm{d}\tau \right]^2, \tag{60}$$

is satisfied.

Example 2. *For a constant yaw rate (steady turn) r, that is nonzero for a time T, assume pitch motion is zero, and roll motion oscillates with a zero mean value. Let*

$$p(t) = \sin(t), \tag{61}$$

$$q(t) = 0, \tag{62}$$

$$r(t) = c_r \neq 0 \;\; \forall t \in [t, t+T], \tag{63}$$

where c_r is a constant. The left side of Eq. (60) from Remark 1 gives

$$\int_t^{t+T} c_r^2 \mathrm{d}\tau = c_r^2 T,$$

and the right side gives

$$\begin{aligned} \left[\int_t^{t+T} c_r \sin(\tau) \mathrm{d}\tau \right]^2 &= c_r^2 (\cos(t) - cos(t+T))^2 \\ &\leq 4 c_r^2, \end{aligned}$$

such that for $T > 4$,

$$c_r^2 T > \left[\int_t^{t+T} c_r \sin(\tau) \mathrm{d}\tau \right]^2 = c_r^2 (\cos(t) - cos(t+T))^2, \tag{64}$$

and PE is satisfied.

The discussion above in concluded in Theorem 4 below.

Theorem 4. *The adaptive observer design of (42) and (48) is globally asymptotically stable if Proposition 2 is satisfied.*