0.1 MRU

By differentiating the Equation **??** for the GNSS equation twice,

$$GNSS : \boldsymbol{P}_1 = \boldsymbol{P}_0 + \boldsymbol{R}(\boldsymbol{\Theta})\boldsymbol{l}_1, \tag{1}$$

$$Velocity : \boldsymbol{V}_2 = \boldsymbol{V}_0 + \boldsymbol{R}(\boldsymbol{\Theta})\boldsymbol{S}(\boldsymbol{\omega})\boldsymbol{l}_2, \tag{2}$$

$$MRU : \boldsymbol{A}_3 = \boldsymbol{A}_0 + \boldsymbol{R}(\boldsymbol{\Theta})\boldsymbol{S}(\boldsymbol{\omega})^2\boldsymbol{l}_3 + \boldsymbol{R}(\boldsymbol{\Theta})\boldsymbol{S}(\dot{\boldsymbol{\omega}})\boldsymbol{l}_3. \tag{3}$$

Where $\boldsymbol{A}_3$ is the linear NED-acceleration, and $\boldsymbol{A}_0$ is the linear NED-acceleration at the reference point. The MRU measure a body fixed acceleration, so this acceleration will have to transformed into NED-coordinates in order to be used in the above equation. In the same way as for the GNSS equations of Eq. (**??**), the equations for several MRUs can be written as

$$\begin{aligned}
\boldsymbol{A}_1(t) &= \boldsymbol{A}_0(t) + \boldsymbol{R}(\boldsymbol{\Theta})\boldsymbol{S}(\boldsymbol{\omega})^2\boldsymbol{l}_1 + \boldsymbol{R}(\Theta)\boldsymbol{S}(\dot{\boldsymbol{\omega}})\boldsymbol{l}_1 \\
\boldsymbol{A}_2(t) &= \boldsymbol{A}_0(t) + \boldsymbol{R}(\boldsymbol{\Theta})\boldsymbol{S}(\boldsymbol{\omega})^2\boldsymbol{l}_2 + \boldsymbol{R}(\boldsymbol{\Theta})\boldsymbol{S}(\dot{\boldsymbol{\omega}})\boldsymbol{l}_2 \\
&\vdots \\
\boldsymbol{A}_m(t) &= \boldsymbol{A}_0(t) + \boldsymbol{R}(\boldsymbol{\Theta})\boldsymbol{S}(\boldsymbol{\omega})^2\boldsymbol{l}_3 + \boldsymbol{R}(\boldsymbol{\Theta})\boldsymbol{S}(\dot{\boldsymbol{\omega}})\boldsymbol{l}_m
\end{aligned} \tag{4}$$

In the same manner as with the GNSS problem from Section **??** the rotation matrix $\boldsymbol{R}$, and the other signals, such as the angular rates, and the linear velocities are treated as input signals to the system, such that $\boldsymbol{R}(\boldsymbol{\Theta}) = \boldsymbol{R}(t)$, $\boldsymbol{S}(\boldsymbol{\omega} = \boldsymbol{S}(t)$, and $\boldsymbol{S}(\dot{\boldsymbol{\omega}}) = \dot{\boldsymbol{S}}(t)$. In this way the system is treated as a time varying linear system, instead of a time invariant nonlinear system.

0.1.1 MRU Luenberger observer design

Eq. (4) can be written on a state space form as

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{A}_0 \\ \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \\ \vdots \\ \boldsymbol{l}_m \end{bmatrix} \in \mathbb{R}^{3n}, \quad n = m + 1, \tag{5}$$

$$y = \begin{bmatrix} \boldsymbol{A}_{MRU1} \\ \boldsymbol{A}_{MRU2} \\ \vdots \\ \boldsymbol{A}_{MRUm} \end{bmatrix} \in \mathbb{R}^{3m}. \tag{6}$$

From Eq. (**??**)

$$\dot{\boldsymbol{P}}_{\boldsymbol{0}} = \boldsymbol{R}(t)\boldsymbol{\nu} = \boldsymbol{V}_0(t),$$

and by (time) differentiating $\boldsymbol{P}_0$ twice $\boldsymbol{A}_0$ can be found as

$$\dot{\boldsymbol{V}}_{\mathbf{0}} = \boldsymbol{R}(t)\boldsymbol{S}(t)\boldsymbol{\nu}(t) + \boldsymbol{R}(t)\dot{\boldsymbol{\nu}}(t) = \boldsymbol{A}_0(t) \tag{7}$$

$$\dot{\boldsymbol{A}}_{\mathbf{0}} = (\boldsymbol{R}(t)\boldsymbol{S}^2(t) + \boldsymbol{R}(t)\dot{\boldsymbol{S}}(t))\boldsymbol{\nu}(t) + \boldsymbol{R}(t)\boldsymbol{S}(t)\dot{\boldsymbol{\nu}}(t) + \boldsymbol{R}(t)\ddot{\boldsymbol{\nu}}(t) \tag{8}$$

As with the GNSS-setup, the system description can be written as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{u}(t), \tag{9}$$

$$\boldsymbol{y} = \boldsymbol{C}(t)\boldsymbol{x}, \tag{10}$$

where

$$\boldsymbol{A} = \mathbf{0}_{3n\times 3n}, \tag{11}$$

$$\boldsymbol{B}(t) = \begin{bmatrix} \boldsymbol{R}(t)\boldsymbol{S}(t)^2 + \boldsymbol{R}(t)\dot{\boldsymbol{S}}(t) & \boldsymbol{R}(t)\boldsymbol{S}(t) & \boldsymbol{R}(t) \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ & \vdots & \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \end{bmatrix} \in \mathbb{R}^{3n\times 3}, \tag{12}$$

$$\boldsymbol{u}(t) = \begin{bmatrix} \boldsymbol{\nu}(t) \\ \dot{\boldsymbol{\nu}}(t) \\ \ddot{\boldsymbol{\nu}}(t) \end{bmatrix}, \tag{13}$$

and

$$\boldsymbol{C}(t) = \begin{bmatrix} \boldsymbol{I}_{3\times 3} & \boldsymbol{R}(t)\boldsymbol{S}(t)^2 + \boldsymbol{R}(t)\dot{\boldsymbol{S}}(t) & \mathbf{0}_{3\times 3} & \cdots & \mathbf{0}_{3\times 3} \\ \boldsymbol{I}_{3\times 3} & \mathbf{0}_{3\times 3} & \boldsymbol{R}(t)\boldsymbol{S}(t)^2 + \boldsymbol{R}(t)\dot{\boldsymbol{S}}(t) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathbf{0}_{3\times 3} \\ \boldsymbol{I}_{3\times 3} & \mathbf{0}_{3\times 3} & \cdots & \mathbf{0}_{3\times 3} & \boldsymbol{R}(t)\boldsymbol{S}(t)^2 + \boldsymbol{R}(t)\dot{\boldsymbol{S}}(t) \end{bmatrix} \in \mathbb{R}^{3m\times 3n}. \tag{14}$$

Observability assessment

Observability will be investigated in a similar way as with the GNSS set up of Section **??**, and Theorem **??** will be used. As discussed in Section **??**, since the $\boldsymbol{A}(t)$ matrix of (**??**) is zero, only $\dot{\boldsymbol{C}}$, $\ddot{\boldsymbol{C}}$, and so on are relevant for the analysis. In fact, it is sufficient to include $\dot{\boldsymbol{C}}$, and this is shown by a simple example (in Example 1).

The time derivative of $\boldsymbol{R}(t)\boldsymbol{S}(t)^2 + \boldsymbol{R}(t)\dot{\boldsymbol{S}}(t)$ gives

$$\begin{aligned} \frac{d}{dt}[\boldsymbol{R}(t)\boldsymbol{S}(t)^2 + \boldsymbol{R}(t)\dot{\boldsymbol{S}}(t)] &= \boldsymbol{R}(t)\boldsymbol{S}(t)^3 + \boldsymbol{R}(t)\dot{\boldsymbol{S}}(t)\boldsymbol{S}(t) + 2\boldsymbol{R}(t)\boldsymbol{S}(t)\dot{\boldsymbol{S}}(t) + \boldsymbol{R}(t)\ddot{\boldsymbol{S}}(t) \\ &=: \boldsymbol{R}(t)\boldsymbol{G}(t), \end{aligned} \tag{15}$$

and $\dot{\boldsymbol{C}}$ can be written

$$\dot{C}(t) = \begin{bmatrix} \mathbf{0}_{3\times 3} & \boldsymbol{R}(t)\boldsymbol{G}(t) & \mathbf{0}_{3\times 3} & \cdots & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \boldsymbol{R}(t)\boldsymbol{G}(t) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \cdots & \mathbf{0}_{3\times 3} & \boldsymbol{R}(t)\boldsymbol{G}(t) \end{bmatrix} \in \mathbb{R}^{3m\times 3n}. \tag{16}$$

Then, by Theorem **??**

$$
\begin{bmatrix} \boldsymbol{N}_0 \\ \boldsymbol{N}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{C} \\ \dot{\boldsymbol{C}} \end{bmatrix}
$$

$$
= \begin{bmatrix}
\boldsymbol{I}_{3\times3} & \boldsymbol{R}(t)\boldsymbol{S}(t)^2 + \boldsymbol{R}(t)\dot{\boldsymbol{S}}(t) & \boldsymbol{0}_{3\times3} & \cdots & \boldsymbol{0}_{3\times3} \\
\boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{R}(t)\boldsymbol{S}(t)^2 + \boldsymbol{R}(t)\dot{\boldsymbol{S}}(t) & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \boldsymbol{0}_{3\times3} \\
\boldsymbol{I}_{3\times3} & \boldsymbol{0}_{3\times3} & \cdots & \boldsymbol{0}_{3\times3} & \boldsymbol{R}(t)\boldsymbol{S}(t)^2 + \boldsymbol{R}(t)\dot{\boldsymbol{S}}(t) \\
\boldsymbol{0}_{3\times3} & \boldsymbol{R}(t)\boldsymbol{G}(t) & \boldsymbol{0}_{3\times3} & \cdots & \boldsymbol{0}_{3\times3} \\
\boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{R}(t)\boldsymbol{G}(t) & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \boldsymbol{0}_{3\times3} \\
\boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & \cdots & \boldsymbol{0}_{3\times3} & \boldsymbol{R}(t)\boldsymbol{G}(t)
\end{bmatrix}. \tag{17}
$$

By the same argument as in Section **??** the observability requirement is that there has to exists a time t_1 where rank $[\boldsymbol{G}(t)] = 3$. This is summarized in the proposition below.

Proposition 1. *For the system of Eq. (9) and (10) to be observable at time t_0, there have to exist a time $t_1 > t_0$ where*

$$
\mathit{rank}[\boldsymbol{S}(t)^3 + \dot{\boldsymbol{S}}(t)\boldsymbol{S}(t) + 2\boldsymbol{S}(t)\dot{\boldsymbol{S}}(t) + \ddot{\boldsymbol{S}}(t)] = 3
$$

Example 1. *For a constant yaw rate $r = 1$, $\dot{p} = 0.1$, and $\dot{r} = 0.1$, and p, q, $\dot{q}$, $\ddot{p}$, $\ddot{q}$, $\ddot{r} = 0$ at t_1, and , $\boldsymbol{G}(t)$ (Eq. (15) becomes*

$$
\boldsymbol{G}(t) = \begin{bmatrix} -0.3 & 1 & 0.2 \\ -1 & -0.3 & -0.1 \\ 0.1 & 0 & 0 \end{bmatrix}
$$

which has full rank, so the system is observable at t_0 for this excitation.

0.1.2 Observer equations and stability

Given the observer dynamics below

$$
\begin{aligned}
\dot{\hat{\boldsymbol{x}}} &= \boldsymbol{B}(t)\boldsymbol{u}(t) + \boldsymbol{W}\boldsymbol{C}(t)^\top \tilde{\boldsymbol{y}} && (18) \\
&= \boldsymbol{B}(t)\boldsymbol{u}(t) + \boldsymbol{W}\boldsymbol{C}(t)^\top \boldsymbol{C}(t)\tilde{\boldsymbol{x}}, && (19)
\end{aligned}
$$

the error dynamics is given as

$$
\tilde{\boldsymbol{x}} = -\boldsymbol{W}\boldsymbol{C}(t)^\top \boldsymbol{C}(t)\tilde{\boldsymbol{x}}, \tag{20}
$$

which is similar to the error dynamics of the observer of Section **??**. Therefore, following the same approach as Section **??**, if uniform observability can be shown for this system, and exponential stability is concluded if Proposition 1 is satisfied.