Chapter 1

Lever arm

TO DO list:

- legg til eksempel p PE
- fullfr Observability proof
- gjr simulering med null bevegelse

1.1 Introduction

For ships an accurate position measurement is important, and for a vessel with several global navigation satellite system (GNSS) antennas, a pre-defined common reference point for these antennas is normally introduced. The length from an antenna to the reference point is called a lever arm. These lever arms needs to measured quite accurately to ensure good position measurements, and this is normally done done by surveying with laser equipment. This process is accurate, but time consuming and expensive. If the lever arms could be estimated numerically, this would save time and expenses.

For GPS and INS integration extensive research has been performed in [2], [3], and [4]. A single antenna GPS with accurate measurements is used in combination with a low-grade inertial measurement unit (IMU). In [2] the observability of the error states in the INS/GPS integrated system is studied, and in [3] experimental studies verify the lever arm estimation of the GPS antenna using a .

In the following a similar observability analysis will be performed, but only using the GPS measurements, and allowing for several antennas. Other measurements needed are assumed available.

1.2 Scope

The scope of the chapter is to investigate what manuevre is needed for a vessel in order to estimate the GNSS lever arms, of potentially several GNSS antennas. A Luenberger type observer, and an adaptive approach are investigated. The GNSS Real data from a sea trial perfomed with the NTNU owned vessel RV Gunnerus will be used to see how well the lever arms can be estimated with the proposed approaches. 6 degrees of freedom will be assumed, and measurements from all the global navigation satellite system (GNSS) positions are available, but measurements of the common and pre-defined reference point, P_0, is not.

1.3 Problem formulation

The GNSS positions in the North-East-Down (NED) frame can be presented as

$$\begin{aligned} P_{GNSS1}(t) &= P_0(t) + R(\Theta)l_1 \\ P_{GNSS1}(t) &= P_0(t) + R(\Theta)l_2 \\ &\vdots \\ P_{GNSSm}(t) &= P_0(t) + R(\Theta)l_m \end{aligned} \tag{1.1}$$

where $P_0(t)$ is some common and pre-defined reference point in the vessel body frame, $R(\Theta)$ is the rotation matrix between the body and the NED frame [1], and is given by

$$R(\Theta) = \begin{bmatrix} c\psi c\theta & -s\psi c\phi + c\psi s\theta s\phi & s\psi s\phi + c\psi c\phi s\theta \\ s\psi c\theta & c\psi c\phi + s\phi s\theta s\psi & -c\psi s\phi + s\theta s\psi c\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix} \in \mathbb{R}^{3\times 3} \tag{1.2}$$

where $c(\cdot) = cos(\cdot), s(\cdot) = sin(\cdot)$ and $\Theta = \begin{bmatrix}\phi & \theta & \psi\end{bmatrix}^T$ contains the Euler angles in roll, pitch and yaw, respectively. p, q and r are the angular body velocities in roll, pitch and yaw, respectively.
Also,

$$\dot{R}(\Theta) = R(\Theta)S(\omega) \tag{1.3}$$

where

$$S(\omega) = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}. \tag{1.4}$$

This can be written on a state space form as

$$x = \begin{bmatrix} P_0 & l_1 & l_2 & \cdots & l_m \end{bmatrix}^T \in \mathbb{R}^{3n}, \quad n = m + 1 \tag{1.5}$$

$$y = \begin{bmatrix} P_{GNSS1} & P_{GNSS2} & \cdots & P_{GNSSm} \end{bmatrix}^T \in \mathbb{R}^{3m} \tag{1.6}$$

where $\nu = \begin{bmatrix} u & v & w \end{bmatrix}^T$ is the linear velocity of the vessel in body coordinates, assumed measured. Thus, the dynamics of P_0 can be represented as [1]

$$\dot{P}_0 = R(\Theta, t)\nu(t) \tag{1.7}$$

The lever arms are constants and hence $\dot{l}_i = 0, i = 1 \cdots m$. The total system description can be written as

$$\dot{x} = B(t)\nu(t) \tag{1.8}$$

$$y(t) = C(t)x \tag{1.9}$$

The system can be thought of as a linear time varying (LTV) system, setting $R(\Theta) := R(t)$, since R $(\cdot)$ is not state dependent, but will vary in time. The matrices of (1.8) and (1.9) are given by

$$B(t) = \begin{bmatrix} R(t) & 0 & 0 & \cdots & 0 \end{bmatrix}^T \in \mathbb{R}^{3n \times 3} \tag{1.10}$$

$$C(t) = \begin{bmatrix} I_{3\times3} & R(t) & 0_{3\times3} & \cdots & 0_{3\times3} \\ I_{3\times3} & 0_{3\times3} & R(t) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0_{3\times3} \\ I_{3\times3} & 0_{3\times3} & \cdots & 0_{3\times3} & R(t) \end{bmatrix} \in \mathbb{R}^{3m \times 3n}. \tag{1.11}$$

x includes all the state parameters to be estimated. The only time-varying parameter is $P_0(t)$, and in itself it is not of interest - the constant lever arms are. Time indexes might be omitted for notational simplicity, when there is no ambiguity.

1.3.1 Observability assessment

In this section the observability criterion for the system given by Eq. (1.8) and (1.9) will be investigated. Because the $A(t)$ matrix of the system is zero, the state transition matrix is identity. This gives the following observability criterion for the system as [**?**],

$$W_0(t_0, t_1) = \int_{t_0}^{t_1} C(\tau)^T C(\tau) \mathrm{d}\tau \tag{1.12}$$

if $W_0(t_0, t_1)$ is nonsingular (if this theorem is to be included, fix it).

$$C(t)^T = \begin{bmatrix} I_{3\times3} & I_{3\times3} & \cdots & I_{3\times3} \\ R(t)^T & 0_{3\times3} & \cdots & 0_{3\times3} \\ 0_{3\times3} & R(t)^T & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_{3\times3} \\ 0_{3\times3} & \cdots & 0_{3\times3} & R(t)^T \end{bmatrix} \in \mathbb{R}^{3n \times 3m} \tag{1.13}$$

$$C(t)^T C(t) \quad = \begin{bmatrix} (n-1)I_{3\times3} & R(t) & \cdots & R(t) & R(t) \\ Rt)^T & I_{3\times3} & 0_{3\times3} & \cdots & 0_{3\times3} \\ \vdots & 0_{3\times3} & \ddots & \ddots & \vdots \\ R(t)^T & \vdots & \ddots & I_{3\times3} & 0_{3\times3} \\ R(t)^T & 0_{3\times3} & \cdots & 0_{3\times3} & I_{3\times3} \end{bmatrix} \in \mathbb{R}^{3n\times3n} \tag{1.14}$$

From (1.11) it can be found that if $R(t)$ has full rank, then $C(t)$ and $C^T(t)$ also have full rank, i.e. 3m. From [**?**] it is found that the rank of $C(t)^T C(t) \in \mathbb{R}^{3n\times3n}$ will at most be 3m, as

$$rank(AB) \leq min(\text{rank}(A), \text{rank}(B)) \tag{1.15}$$

where both A and B are matrices.

In other words the matrix is not full rank in the first place, and need to "add rank" as time increases [**?**]. To find the observability condition(s), Theorem 6.O12 from [**?**] is used, and the theorem is restated here for its convenience.

Theorem 1 (Thm 6.O12 [**?**]). *Let A(t) and C(t) be continously differentiable, then the n-dimensional pair (A(t),C(t)) is observable at t_0 if there exists a finite $t_1 > t_0$ such that*

$$rank \begin{bmatrix} N_0 \\ N_1 \\ \vdots \\ N_{n-1} \end{bmatrix} = n$$

where $N_{m+1} = N_m(t)A(t) + \frac{d}{dt}N_m(t) \quad m = 0, 1 \ldots n-1$
with $N_0 = C(t)$.

For the system at hand, both C(t) and A(t) are continuously differentiable, and C(t) has rank 3m given that R(t) has full rank.

Hence, given C(t) full row rank, a rank of 3 is lacking to fulfil rank n of theorem 6.O12 (C(t) has three more columns than rows). Thus, N_1 must contribute with 3 more independent rows for the observability condition to be satisfied.
Finding N_1:

$$N_1 = N_0(t)A(t) + \frac{d}{dt}N_0(t) \tag{1.16}$$

$$= 0 + \frac{d}{dt}C(t) \tag{1.17}$$

$$\dot{C} = \begin{bmatrix} 0_{3\times3} & R(t)S(t) & 0_{3\times3} & \cdots & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & R(t)S(t) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & \cdots & 0_{3\times3} & R(t)S(t) \end{bmatrix}. \tag{1.18}$$

Finding N_2:

$$N_2 = N_1(t)A(t) + \frac{d}{dt}N_1(t) \tag{1.19}$$
$$= \ddot{C}(t) \tag{1.20}$$

$$\ddot{C} = \begin{bmatrix} 0_{3\times3} & R(t)S(t)^2 + R\dot{S} & 0_{3\times3} & \cdots & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & R(t)S(t)^2 + R\dot{S} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & \cdots & 0_{3\times3} & R(t)S(t)^2 + R\dot{S} \end{bmatrix}. \tag{1.21}$$

giving

$$\begin{bmatrix} N_0 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} I_{3\times3} & R(t) & 0_{3\times3} & \cdots & 0_{3\times3} \\ I_{3\times3} & 0_{3\times3} & R(t) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0_{3\times3} \\ I_{3\times3} & 0_{3\times3} & \cdots & 0_{3\times3} & R(t) \\ 0_{3\times3} & R(t)S(t) & 0_{3\times3} & \cdots & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & R(t)S(t) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & \cdots & 0_{3\times3} & R(t)S(t) \\ 0_{3\times3} & R(t)S(t)^2 + R\dot{S} & 0_{3\times3} & \cdots & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & R(t)S(t)^2 + R\dot{S} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & \cdots & 0_{3\times3} & R(t)S(t)^2 + R\dot{S} \end{bmatrix}, \tag{1.22}$$

where

$$\dot{S} = \begin{bmatrix} 0 & -\dot{r} & \dot{q} \\ \dot{r} & 0 & -\dot{p} \\ -\dot{q} & \dot{p} & 0 \end{bmatrix}. \tag{1.23}$$

For the system to be observable, the requirement is that rank($\begin{bmatrix} N_0 & N_1 & N_2 \end{bmatrix}^T$) $= 3n$. From analysis in MATLAB (see Appendix), this requirement is found to be fulfilled if either $p, q, r \neq 0$, and either $\dot{p}, \dot{q}, \dot{r} \neq 0$.

Proposition 1. *For the system of Eq. (1.8) and (1.9) to be observable, either $p, q, r \neq 0$, and either $\dot{p}, \dot{q}, \dot{r} \neq 0$.*

In practice, by trying to manoeuvre with a constant yaw rate r, there will be some nonzero $\dot{r}$, and observability will be assured. The will also be some motion in the other degrees of freedom. The fact that a constant yaw rate manoeuvre is sufficient will be investigated with sea trial results from RV Gunnerus.

1.4 Observer design

The following observer is proposed

$$
\begin{aligned}
\dot{\hat{x}} &= B(t)v(t) + C(t)^\top W\tilde{y} \\
&= B(t)v(t) + C(t)^\top WC\tilde{x}
\end{aligned}
\tag{1.24}
$$

$$
\tilde{y} = C(t)\tilde{x}, \tag{1.25}
$$

where $W = W^T > 0 \in \mathbb{R}^{3m\times 3m}$, $\tilde{x} = x - \hat{x}$, $\tilde{y} = y - \hat{y}$, and with closed loop error dynamics as

$$
\dot{\tilde{x}} = -C(t)^\top WC(t)\tilde{x}. \tag{1.26}
$$

In order to show exponential stability of the observer, the following lemma from [**?**] is applied.

Lemma 1. *Exponential stability of LTV system [**?**]*
For a system given by $\dot{x} = F(t)$, and the function $F(\cdot)$ is locally integrable. Suppose there exits a positive definite matrix $P = P^T > 0$ such that

$$
PF(t) + F^\top(t)P \leq -N(t)^T N(t) \tag{1.27}
$$

for some matrix function $N(\cdot)$ and all t. Then $\dot{x} = F(t)$ is uniformly stable in the sense of Lyapunov.

If, further the pair $[F(t), N(t)]$ is uniformly completely observable, that is, writing $\phi(t,\tau)$ as the transition function of $\dot{x} = F(t)$, there exits $T > 0,\ \beta > 0,\ \alpha > 0$ such that $\forall$

$$
\beta I \geq \int_t^{t+T} \phi(t,\tau)^T C(\tau)^T C(\tau)\phi(t,\tau)\mathrm{d}\tau \geq \alpha I
$$

then $\dot{x} = F(t)$ is exponentially stable.

From equations (1.25) and (1.26), $F(t) = -C(t)^\top WC(t)$ and
$P = I_{3n\times 3n}$ gives

$$
\begin{aligned}
PF(t) - F(t)^\top P &= -2C(t)^\top WC(t) \\
&= -N(t)^T N(t),
\end{aligned}
\tag{1.28}
$$

where

$$
N(t) = \sqrt{2W}C(t) \tag{1.29}
$$

W can be adjusted, and as long as (1.28) is satisfied, stability is assured. The system will be assumed uniformly completely observable for ”rich” input signal. Then (1.28) assures exponential stability of the observer. By ”rich” input signals, it is meant signals such as sinusoids. Observability will be confirmed by simulations of section 1.6.1. It must be noted that the error dynamics of the observer is different from the observability analysis done in section 1.3.1, where the transition matrix was identity, due to A(t) zero. In the observer design, the transition matrix will be different from identity, due to nonzero A(t) of (1.26).

1.5 analysis results

1.6 Adaptive design

Another way to solve the lever arm estimation problem is an adaptive solution. This set up will remove P_0 as a variable to be estimated, and thus have full state measurements available. This is reasonable as P_0 can be found from (1.1) once all the lever arms are estimated.

By taking the time derivative of (1.1), using (1.7) and $R(\Theta) = R(t)$, $S(\omega) = S(t)$, the lever arm problem can be formulated as

$$\begin{aligned}
\dot{P}_{GNSS1}(t) &= R(t)\upsilon(t) + R(t)S(t)l_1 \\
\dot{P}_{GNSS2}(t) &= R(t)\upsilon(t) + R(t)S(t)l_2 \\
&\vdots \\
\dot{P}_{GNSSn}(t) &= R(t)\upsilon(t) + R(t)S(t)l_n
\end{aligned}$$

Where $P_{GNSSi} = \begin{bmatrix} P_{GNSSi_N} & P_{GNSSi_E} & P_{GNSSi_D} \end{bmatrix}^T \in \mathbb{R}^3$ and contains the coordinates of the GNSS measurement expressed in the NED frame [1].

$$x(t) = \begin{bmatrix} P_{GNSS1} & P_{GNSS2} & \cdots & P_{GNSSn} \end{bmatrix}^T \in \mathbb{R}^{3n} \tag{1.30}$$

$$y(t) = I_{3n\times 3n}x, \tag{1.31}$$

where $\upsilon = \begin{bmatrix} u & v & w \end{bmatrix}^T \in \mathbb{R}^3$ is the measured velocity of the vessel in body coordinates. Written in state space form

$$\dot{x}(t) = A(t)x(t) + B(t)\upsilon(t) + \Omega\varphi \tag{1.32}$$

$$y(t) = C(t)x(t), \tag{1.33}$$

where

$$A(t) = 0_{3n\times 3n} \tag{1.34}$$

$$B(t) = \begin{bmatrix} R(t) & R(t) & \cdots & R(t) \end{bmatrix}^T \in \mathbb{R}^{3n\times 3} \tag{1.35}$$

$$C = I_{3n\times 3n} \tag{1.36}$$

$$\varphi = \begin{bmatrix} l_1 & l_2 & \cdots & l_n \end{bmatrix}^T \in \mathbb{R}^{3n} \tag{1.37}$$

$$\Omega = \begin{bmatrix} R(t)S(t) & 0_{3\times 3} & \cdots & 0_{3\times 3} \\ 0_{3\times 3} & R(t)S(t) & \ddots & \vdots \\ \vdots & & \ddots & 0_{3\times 3} \\ 0_{3\times 3} & \cdots & 0_{3\times 3} & R(t)S(t) \end{bmatrix} \in \mathbb{R}^{3n\times 3n}. \tag{1.38}$$

Let a state observer be given as

$$\begin{aligned}\dot{\hat{x}} &= Bv + \Omega\hat{\varphi} + Ly - LC\hat{x} \\ &= Bv + \Omega\hat{\varphi} + LC\tilde{x} \qquad (1.39)\end{aligned}$$

where $L \in \mathbb{R}^{3n\times 3n}$ and $\tilde{x} = x - \hat{x}$. Let $\tilde{\varphi} = \varphi - \hat{\varphi}$, and

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = -LC\tilde{x} + \Omega\tilde{\varphi}, \qquad (1.40)$$

$$\dot{\tilde{\varphi}} = \dot{\varphi} - \dot{\hat{\varphi}} = -\dot{\hat{\varphi}}. \qquad (1.41)$$

Define the following Control Lyapunov Function (CLF) [**?**]

$$V = \frac{1}{2}\tilde{x}^T\tilde{x} + \frac{1}{2}\tilde{\varphi}^T\Gamma^{-1}\tilde{\varphi}, \qquad (1.42)$$

where the constant matrix $\Gamma^{-1} > 0$, and hence also invertible.

$$\dot{V} = \tilde{x}^T[-LC\tilde{x} + \Omega\tilde{\varphi}] + \tilde{\varphi}^T\Gamma^{-1}(-\dot{\hat{\varphi}}) \qquad (1.43)$$

$$= -\tilde{x}^T LC\tilde{x} + \tilde{x}^T\Omega\tilde{\varphi} - \tilde{\varphi}^T\Gamma^{-1}\dot{\hat{\varphi}} \qquad (1.44)$$

For the following update law for $\hat{\varphi}$

$$\dot{\hat{\varphi}} := \Gamma\Omega^T\tilde{x}, \qquad (1.45)$$

Sigma-modification

$$\dot{\hat{\varphi}} := \Gamma\Omega^T\tilde{x} + \sigma\Gamma\hat{\phi}, \qquad (1.46)$$

$\dot{V}$ becomes

$$\dot{V} = -\tilde{x}^T LC\tilde{x} \leq 0. \qquad (1.47)$$

Let $LC > 0$, then $\dot{V}$ is negative semidefinite. Barbalat's lemma [**?**] is applied, and restated here for convenience.

Lemma 2. *(Barbalat's lemma) If the differentiable function f(t) has a finite limit as $t \to \infty$, and if $\dot{f}$ is uniformly continuous, then $f(t) \to 0$ as $t \to \infty$.*

To assess uniform continuity, [**?**] states that a sufficient condition for a differentiable function is that its derivative is bounded. For the above CLF, this implies that $\ddot{V}$ should be bounded. $\ddot{V}$ is given as

$$\ddot{V} = -2\tilde{x}^T LC\dot{\tilde{x}} \qquad (1.48)$$

From the fact that $\dot{V} \leq 0$ it is shown that V is bounded, and from (1.42) it can be concluded that both $\tilde{x}$ and $\tilde{\varphi}$ are bounded. From (1.40), since both $\tilde{x}$ and $\tilde{\varphi}$ are bounded, $\dot{\tilde{x}}$ is bounded, and hence is $\ddot{V}$ bounded. So $\dot{V}$ is uniformly continous, and by Barbalat's lemma

$\dot{V} \to 0$ as $t \to \infty$, and hence $\tilde{x} \to 0$ as $t \to \infty$. This implies that $\dot{\tilde{x}} \to 0$ as $t \to \infty$, and from (1.40) it follows that $\Omega\tilde{\varphi} \to 0$ as $t \to \infty$.

Consider $\Omega\tilde{\varphi} = 0$. In order to have $\tilde{\varphi} = 0$ as only solution Ω need to be persistently excited. For a update law of the form (1.45), then persistence of excitation (PE) can be formulated as [**?**]

Theorem 2. *(Persistence of excitation) The matrix Ω is persistently excited if there exists $\alpha, T > 0$ such that $\forall t$*

$$\int_t^{t+T} \Omega(\tau)^T \Omega(\tau) \mathrm{d}\tau > \alpha I \tag{1.49}$$

Seeing that $\Omega(t)$ is diagonal, condition (1.49) will only depend on $R(t)S(t)$, and PE criterion can be evaluated based on $R(t)S(t)$ as the integrand (with $I := I_{3\times 3}$ in (1.49)). Looking at the integrand

$$\begin{aligned} (R(t)S(t))^T R(t)S(t) &= S(t)^T R(t)^T R(t)S(t) \qquad (1.50)\\ &= S(t)^T S(t). \qquad (1.51) \end{aligned}$$

This result is valid due to $R^T R = I$, which is a fundamental property of the rotation matrix [1]. The expression for $S(t)^T S(t)$ is given as

$$S(t)^T S(t) = \begin{bmatrix} r(t)^2 + q(t)^2 & -p(t)q(t) & -p(t)r(t) \\ -p(t)q(t) & r(t)^2 + p(t)^2 & -q(t)r(t) \\ -p(t)r(t) & -q(t)r(t) & p(t)^2 + q(t)^2 \end{bmatrix}, \tag{1.52}$$

and the PE criteria can be written as

$$\begin{aligned} &\int_t^{t+T} S(\tau)^T S(\tau) \mathrm{d}\tau = \\ &\begin{bmatrix} \int_t^{t+T} [q(\tau)^2 + r(\tau)^2] \mathrm{d}\tau & -\int_t^{t+T} p(\tau)q(\tau)\mathrm{d}\tau & -\int_t^{t+T} p(\tau)r(\tau)\mathrm{d}\tau \\ -\int_t^{t+T} p(\tau)q(\tau)\mathrm{d}\tau & \int_t^{t+T} [p(\tau)^2 + r(\tau)^2] \mathrm{d}\tau & -\int_t^{t+T} q(\tau)r(\tau)\mathrm{d}\tau \\ -\int_t^{t+T} p(\tau)r(\tau)\mathrm{d}\tau & -\int_t^{t+T} q(\tau)r(\tau)\mathrm{d}\tau & \int_t^{t+T} [p(\tau)^2 + q(\tau)^2] \mathrm{d}\tau \end{bmatrix} \\ &> I_{3\times 3}\alpha, \end{aligned} \tag{1.53}$$

A matrix is positive definite if the leading principal minors are positive [**?**]. This gives the following three conditions for Eq. (1.53) to be satisfied

1. $$\int_t^{t+T} [q(\tau)^2 + r(\tau)^2] \mathrm{d}\tau > 0 \tag{1.54}$$

2.

$$\int_t^{t+T}[q(\tau)^2+r(\tau)^2]\mathrm{d}\tau\int_t^{t+T}[p(\tau)^2+r(\tau)^2]\mathrm{d}\tau \\ -\left[\int_t^{t+T}p(\tau)q(\tau)\mathrm{d}\tau\right]^2>0 \tag{1.55}$$

3.

$$\int_t^{t+T}[q(\tau)^2+r(\tau)^2]\mathrm{d}\tau\left\{\int_t^{t+T}[p(\tau)^2+r(\tau)^2]\mathrm{d}\tau\int_t^{t+T}[p(\tau)^2+q(\tau)^2]\mathrm{d}\tau- \right. \\ \left.\left[\int_t^{t+T}q(\tau)r(\tau)\mathrm{d}\tau\right]^2\right\}- \\ \int_t^{t+T}p(\tau)q(\tau)\mathrm{d}\tau\left\{\int_t^{t+T}p(\tau)q(\tau)\mathrm{d}\tau\int_t^{t+T}[p(\tau)^2+q(\tau)^2]\mathrm{d}\tau+\right. \\ \left.\int_t^{t+T}q(\tau)r(\tau)\mathrm{d}\tau\int_t^{t+T}p(\tau)r(\tau)\mathrm{d}\tau\right\}- \\ \int_t^{t+T}p(\tau)r(\tau)\mathrm{d}\tau\left\{\int_t^{t+T}p(\tau)q(\tau)\mathrm{d}\tau\int_t^{t+T}q(\tau)r(\tau)\mathrm{d}\tau+\right. \\ \left.\int_t^{t+T}p(\tau)r(\tau)\mathrm{d}\tau\int_t^{t+T}[p(\tau)^2+r(\tau)^2]\mathrm{d}\tau\right\} \tag{1.56}$$

Equation (1.54) requires either $q, r \neq 0$, and considering Equation (1.55) with $p = 0$, gives that r needs to be nonzero. Looking at (1.56) with $p = 0$, and $r \neq 0$, gives that q needs to be nonzero, and that

$$\int_t^{t+T}r(\tau)^2\mathrm{d}\tau\int_t^{t+T}q(\tau)^2\mathrm{d}\tau>\left[\int_t^{t+T}q(\tau)r(\tau)\mathrm{d}\tau\right]^2 \tag{1.57}$$

Proposition 2 (PE of adaptive observer)**.** *The adaptive observer of Eq. (1.40 and (1.41) is persistently excited if the conditions of (1.54) - (1.56) are satisfied. The minimum requirement is that* $r \neq 0$ *and either* $q \neq 0$ *and*

$$\int_t^{t+T}r(\tau)^2\mathrm{d}\tau\int_t^{t+T}q(\tau)^2\mathrm{d}\tau>\left[\int_t^{t+T}q(\tau)r(\tau)\mathrm{d}\tau\right]^2 \tag{1.58}$$

or $p \neq 0$ *and*

$$\int_t^{t+T}r(\tau)^2\mathrm{d}\tau>\left[\int_t^{t+T}p(\tau)r(\tau)\mathrm{d}\tau\right]^2 \tag{1.59}$$

Both Equation (1.58) and (1.59) are satisfied for sufficiently large T for r constant, and p and q oscillating with zero mean. Simulation on real data from a sea trial with RV Gunnerus will later show that for a manoeuvre of a constant yawing rate r, the motion in pitch and roll will be small (but present), such that PE is satisfied.

1.6.1 Case study - Observer

For the sea trials with Gunnerus, three manuevers were performed.

Lever arm manuevers

- 3032: 1 turn of the vessel, with $100°/min$
- 3033: 1 turn of the vessel, with $200°/min$
- 3034: rotation of the vessel, with varying rotation velocity

Onboard Gunnerus there were two GPS antennas installed. One that was integrated with the DP system, and one that MARINTEK installed. Unfortunately, for these three manuevers, only the GPS antenna of the DP system gives results with sufficient resolution. The case study will therefore only be run with one GPS antenna. This should not affect the results, since the way the algorithm is implemented, does not require more than one antenna.

Data from GPS

The data from the GPS antenna is given as data from the centre of origin, P_0. Therefore, running the observer on the data should yield a lever arm of zero length. This will be verified. Also, in order to find a lever arm a artificial lever arm will be added to the GPS signal, such that the GPS input to simulation will be

$$y_{GPS_1} = y_{GPS,measured} + R(t)l_1, \tag{1.60}$$

where y_{GPS_1} is the measurement fed to the observer, $y_{GPS,measured}$ is the signal actually measured by the GPS, and $R(t)l_1$ is the lever arm coordinate.

Synching of data

A problem for the case study is that the GPS data comes from the DP system, and the other data needed for the observer, such as the linear velocities of the vessel, the Euler angles, the angular rates, they all come from MARINTEK data. The DP data needs to be synched with the MARINTEK data. The time difference between these data sets is not constant (VERIFY THIS), and could cause problems.

show plot of GPS data

Observer design for the case study

$$x(t) = \begin{bmatrix} P_0 & l_1 \end{bmatrix}^T \in \mathbb{R}^6 \tag{1.61}$$
$$y(t) = Cx, \tag{1.62}$$

where

$$C = \begin{bmatrix} I_{3\times3} & R(t) \end{bmatrix} \in \mathbb{R}^{3\times6}, \tag{1.63}$$

with closed loop error dynamics as

$$\dot{\tilde{x}} = -C(t)^\top W C(t)\tilde{x} \tag{1.64}$$

where W is chosen as $W = 10000 I_{3\times3}$. The observer is then given as

$$\dot{\hat{x}} = B(t)v(t) + C(t)^\top W C(t)\tilde{x} \tag{1.65}$$

Added lever arm

For all the cases below, a lever arm of body coordinates

$$l_1 = \begin{bmatrix} l_{1x} \\ l_{1y} \\ l_{1z} \end{bmatrix} = \begin{bmatrix} 12m \\ 0.56m \\ 13m \end{bmatrix} \tag{1.66}$$

is added. These are the coordinates of the MARINTEK GPS antenna, so these are realistic numbers. The observer is also run without this lever arm, such that the P_0 estimate can be verified to be good.

1.6.2 Results - observer

The results in Table (1.1) are for all the cases, and the average value and standard deviation of the lever arm coordinates are given. The values are obtained from an interval where the measurement variance is low, and the selected interval in given in the table. The results obtained are for the plots shown in Figure (1.3, (1.4), and (1.5) - all with the same lever arm added. The lever arm that supposed to be estimated has coordinates of $\begin{bmatrix} 12.0m & 0.56m & 13m \end{bmatrix}^T$.

1.6.3 Case study 3032

For this case study the simulation results for when no lever arms are added to the GPS measurement will be shown. This will show that how well P_0 is estimated, and also show that the lever arm is in fact estimated to be zero when no arm is added.

Table 1.1: Results, lever arm - observer

Case		**Lever arm coordinates**						**Collected from interval**
		x [m]	y	y [m]		z [m]		
		Avg	*Std.*	*Avg*	*Std.*	*Avg*	*Std.*	
3032		12.003	0.032	0.569	0.0446	13.005	0.035	60 - 140 [s]
3033		12.042	0-049	0.522	0.086	13.009	0.080	40 - 140 [s]
3034		11.989	0.0657	0.540	0.084	12.984	0.117	100 - 150 [s]

Estimation of P_0

For this simulation no lever arm is added, and GPS measurement coincide with the P_0 measurement. Figure (1.1 show the GPS measurement and the estimated position of P_0.

Leverarmplots/GPSLeverarmsNEP03032-eps-converted-to.pdf

Figure 1.1: P0 3032

Estimation of lever arm - no added lever arm

Since no lever arm is added, the position in body frame should be zero in x-, y- and z-direction. Figure (1.2) show the results. The initial condition of the estimated arm coordinate is $\begin{bmatrix}-0.5m & 1m & 2m\end{bmatrix}^T$

Leverarmplots/GPSLeverarmsZero3032-eps-converted-to.pdf

Figure 1.2: Lever arm coordinates, 3032 - no lever arm added

Estimation of lever arm - added lever arm

A lever arm of coordinates $\begin{bmatrix}12.0m & 0.56m & 13m\end{bmatrix}^T$ is added. The initial condition of the estimated arm coordinate is $\begin{bmatrix}8m & 0.3m & 16m\end{bmatrix}^T$

Figure 1.3: Lever arm coordinates, 3032

1.6.4 Case study 3033

Estimation of lever arm - added lever arm

A lever arm of coordinates $\begin{bmatrix}12.0m & 0.56m & 13m\end{bmatrix}^T$ is added. The initial condition of the estimated arm coordinate is $\begin{bmatrix}8m & 0.3m & 16m\end{bmatrix}^T$

Leverarmplots/GPSLeverarmsAdded3033-eps-converted-to.pdf

Figure 1.4: Lever arm coordinates, 3033

1.6.5 Case study 3034

Estimation of lever arm - added lever arm

A lever arm of coordinates $\begin{bmatrix} 12.0m & 0.56m & 13m \end{bmatrix}^T$ is added. The initial condition of the estimated arm coordinate is $\begin{bmatrix} 8m & 0.3m & 16m \end{bmatrix}^T$

Leverarmplots/GPSLeverarmsAdded3034-eps-converted-to.pdf

Figure 1.5: Lever arm coordinates, 3034

1.6.6 Case study - Adaptive solution

The data does not seems to sufficiently accurate or sampled often enough for the adaptive solution. The estimates of the lever arms diverges.

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