Chapter 1

Hybrid integral action in PID control: Application to dynamic positioning of marine vessels

1.1 Introduction

1.1.1 Motivation

The motivation behind the hybrid integral action proposed in the following chapter is a Dynamically Positioned (DP) marine vessel experiencing large unknown disturbances. DP vessels normally experience wave loads, wind loads, and current. The loads that the integral action part of the controller normally compensate for are slowly varying forces, almost constant for given periods of time. Because of this, the integral action is normally tuned very low, such that it does not induce unnecessary oscillation.

However, if the vessel experiences sudden load changes such as ice loads, or a mooring line that snaps, the integral action spends a long time (about 20 minutes) reaching the new steady state value. This chapter proposes a method that improves the convergence for the integral action when it is subject to large sudden disturbances that are constant some time after impact. The proposed method augments the standard PID controller with a hybrid integral action law.

1.1.2 Literature review

In **?**, hybrid resetting of integral action for a PID controller is discussed. Based on the sign of the position state and the integral state, the integral action value is reset if they are of opposite sign, thereby reducing transient behaviour of the closed loop plant. For **?** a hybrid high-gain observer is constructed to reduce the peaking behaviour of the observer on a second order planar nonlinear system. Trajectories with peaking are projected into areas without peaking behaviour.

A framework with several continuous controller and observer-pairs are proposed for hybrid control of DP vessels in **?**. The operational window of a DP vessel is extended by switching to different observer-controller pairs depending on the sea state. This approach could have been used to augment the plant with a controller with a high gain for the continuous integral action, and then switching back to a controller with lower integral action gains when the integral error is small.

The method of **?** is similar to that of the following chapter in that it uses the sign of the integral value and the states to determine when jumps can occur. However, the goal of the following chapter is to use information about the states to update the integral value, not only reset the integral value when the signs changes. This will be especially useful when large constant disturbances should quickly be compensated for by integral action.

1.1.3 Scope

The objective of the chapter is to improve performance of the PID controller when a system is subject to large disturbance changes that remain constant for some time (step disturbance). A PID-controller is augmented with a hybrid (**?**) integral action law that changes the integral action value at discrete instances (jumps). When the absolute value of the error in states are small, jumps are no longer allowed. This discrete change in integral action value allows higher convergene of the integral action, with no, or small overshoot. This will be developed for both a first order linear system, and a DP system.

Section (Section 1.2) consideres the mathematical modelling. Section 1.2.1 the preliminaries for hybrid control theory and the Lyapunov stability theory needed is summarized. The stability conditions is then derived for both the first order system (Section 1.2.3), and the DP system (Section 1.2.4). In these sections a theorem concludes the stability conditions for the hybrid system. In Section 1.3 there are case studies for both a linear system (Section 1.3.1), and a DP system (Section 1.3.2).

1.2 Mathematical modelling

1.2.1 Preliminaries

The theory for hybrid control theory is based on **?**. The benefit of the theory is that continuous and discrete dynamics can be combined, and stability can be proven. The aspects relevant for the approach of the chapter is mentioned below, but for a more in-depth analysis of hybrid control theory, it is referred to **?**.

Continous dynamics, here called flow, given generally by a differential inclusion $F(x)$ is allowed on the flow set C. The discrete dynamics, here called jumps, given by the difference inclusion $G(x)$ is allowed on the jump set D. In the following only a differential equation $f(x)$, and difference equation $g(x)$ will be used instead of $F(x)$, and $G(x)$ respectively.

In order to prove stability of the systems in this paper, Theorem 3.18 of **?**, using a Lyapunov function is applied. This theorem proves global uniform (pre-)asymptotic stability of the system, and requires the Lyapunov function to decrease in value for both flow and for a jump. Since the jump augmentation of the PID controller intends to improve performance, it makes sense to demand that $V(x)$ also decreases in jumps. To prove stability for a set A Theorem 3.18 in its most basic form is applied, and restated below for convenience.

Theorem 1 (**?** Theorem 3.18)**.** *(Sufficient Lyapunov conditions)*
Let $H = (C, F, D, G)$ be a hybrid system and let $A \subset \mathrm{R}^{\mathrm{n}}$ be closed. If V is a Lyapunov function candidate for H and there exists α_1, $\alpha_2 \in K_\infty$, and a continuous positive definite function ρ such that

$$\alpha_1(|x|_A) \leq V(x) \leq \alpha_2(|x|_A) \ \ \forall x \in C \cup D \cup G(D) \tag{1.1}$$

$$\langle V(x), f \rangle \leq -\rho(|x|_A) \ \ \forall x \in C, f \in F(x) \tag{1.2}$$

$$V(g) - V(x) \leq -\rho(|x|_A) \ \ \forall x \in D, g \in G(D) \tag{1.3}$$

then A is uniformly globally pre-asymptotically stable for H.

In the above theorem $|x|_A$ is the distance to the set A. In this paper the $\dot{V}$ notation will be used instead of $\langle V(x), f \rangle$, and for Eq. (**??**) $V(g)$ and $V(x)$ represents the value of $V(x)$ after and before a jump respectively. All stability proofs later in the chapter will use this theorem to prove stability.

About the approach

The key for the proposed hybrid integral action in Section 1.2.3, and 1.2.4 is the location of the jump set. It will be shown that when within the jump set, the proposed jump rule will guarantee decrease of the Lyapunov function. The clue is therefore to restrict the jump set to a set that will depend on the sign and size of the error in the integral value, and the sign and error in other states. By construction of the problem, the integral error (the difference between the unknown disturbance and the integral value) is unknown. Therefore, the dynamic equations will be used to find an expression for the integral error, and a new jump set will be formulated based on known states, and an estimate of a state derivative (that will be found by sampling). Flow can occur in the entire state space. The jump rule proposed is a linear jump rule, and jumps are proportional to the error in other state variables.

1.2.2 drawing of the jump set and jumping

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1.2.3 First order linear systems

In this approach a general first order system subject to an unknown constant disturbance is presented. A control law with proportional control, and integral action is proposed, and the closed loop system dynamics is derived. Then the flow set, flow map, and the jump map are defined, before Theorem 2 defines a jump set such that the jumps always decrease the Lyapunov function. The theorem also gives conditions for stability in flow, such that the combined system is stable.

Consider the first order system with an unknown constant disturbance d as input to the system. The system is written as

$$\dot{z} = -az + d + u \tag{1.4}$$

$$\dot{d} = 0, \tag{1.5}$$

where $a > 0$. Let z_d be the desired z-value, and by selecting the control input u as

$$u = \dot{z}_d + az_d - k_p(z - z_d) - \hat{d}, \tag{1.6}$$

with $k_p > 0$, the closed loop error dynamics becomes

$$\dot{\tilde{z}} = -(a + k_p)\tilde{z} + \tilde{d} = -a'\tilde{z} + \tilde{d} \tag{1.7}$$

$$\dot{\tilde{d}} = \dot{d} - \dot{\hat{d}} = -\dot{\hat{d}} = -k_i z, \tag{1.8}$$

where $\tilde{d} = d - \hat{d}$, $a' = a + k_p > 0$.

Below the flow set, flow map, and the jump map are defined. The jump set is defined in Theorem 2. The states are defined as

$$\boldsymbol{x} = \begin{bmatrix} \tilde{z} \\ \tilde{d} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{1.9}$$

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Flow set

The flow set is the entire state space, so C is given as

$$C = \left\{ \boldsymbol{x} \in \mathrm{R}^2 \right\}. \tag{1.10}$$

Flow map

From (1.9), (1.7), and (1.8) the time derivative of the state $\dot{\boldsymbol{x}}$, or the flow map $f(\boldsymbol{x})$ is given as

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\tilde{z}} \\ \dot{\tilde{d}} \end{bmatrix} = \begin{bmatrix} -a'x_1 + x_2 \\ -k_i x_1 \end{bmatrix} = \begin{bmatrix} -a' & 1 \\ -k_i & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{A}\boldsymbol{x} = f(\boldsymbol{x}). \tag{1.11}$$

Jump map

In the proposed jump map x_1 remain the same, and x_2 is updated based on the x_1-value, such that

$$\boldsymbol{x}^+ = \begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - \lambda x_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\lambda & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{\Omega}\boldsymbol{x} = g(\boldsymbol{x}) \tag{1.12}$$

Theorem 2. *Given a linear system with $\boldsymbol{x} \in \mathrm{R}^2$ of (1.9), the flow set C given by (1.10), and the closed loop flow map $f(\boldsymbol{x})$ given by (1.11), and jump map $g(\boldsymbol{x})$ given by (1.12), where the constants a', k_i, $\lambda > 0$, let α_1, $\alpha_2 \in \mathrm{K}_\infty$, and a Lyapunov function be given as*

$$\alpha_1(|\boldsymbol{x}|_A) \leq V(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{P} \boldsymbol{x} \leq \alpha_2(|\boldsymbol{x}|_A), \tag{1.13}$$

where

$$\boldsymbol{P} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \boldsymbol{P}^\top > 0, \tag{1.14}$$

is set such that the Lyapunov function decreases in flow. That is,

$$\dot{V}(\boldsymbol{x}) \leq -\rho(|x|_A) \;\; \forall \boldsymbol{x} \in C. \tag{1.15}$$

Let

$$\beta := 2\lambda p_3 > 0, \tag{1.16}$$

$$\gamma := \lambda^2 p_3 - 2\lambda p_2 > 0, \tag{1.17}$$

$$\varepsilon > 0, \tag{1.18}$$

$$\sigma > 0, \tag{1.19}$$

where ε and σ are constants. For the jump set given by

$$D = \left\{ \boldsymbol{x} \in \mathrm{R}^2 : |x_1| \geq \varepsilon, x_1 x_2 \geq \frac{(\gamma + \sigma)x_1^2 + \sigma x_2^2)}{\beta} \right\}, \tag{1.20}$$

then

$$V(g(\boldsymbol{x})) - V(\boldsymbol{x}) \leq -\sigma |\boldsymbol{x}|^2, \tag{1.21}$$

and by Theorem 1 the set $A = \{0, 0\}$ is uniformly globally pre-asymptotically stable for $\mathcal{H} = (C, f, D, g)$.

Proof. Consider the Lyapunov function given by (1.13). The time derivative of the Lyapunov function gives

$$\begin{aligned} \dot{V}(\boldsymbol{x}) &= \left(\boldsymbol{x}^\top \boldsymbol{P} \dot{\boldsymbol{x}} + \dot{\boldsymbol{x}}^\top \boldsymbol{P} \boldsymbol{x}\right) &(1.22)\\ &= \left[\boldsymbol{x}^\top (\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^\top \boldsymbol{P}) \boldsymbol{x}\right], &(1.23) \end{aligned}$$

so for

$$\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^\top \boldsymbol{P} < 0, \tag{1.24}$$

$$\dot{V}(\boldsymbol{x}) < 0 \quad \forall \boldsymbol{x} \in C, \tag{1.25}$$

so if the matrix $\boldsymbol{P}$ is set such that $\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^\top \boldsymbol{P} < 0$, the Lyapunov function decrease in flow.

The value of the Lyapunov function after a jump, $V(g)$ is

$$V(g) = \boldsymbol{x}^\top \boldsymbol{\Omega}^\top \boldsymbol{P} \boldsymbol{\Omega} \boldsymbol{x}, \tag{1.26}$$

such that

$$\begin{aligned} V(g) - V(\boldsymbol{x}) &= \boldsymbol{x}^\top [\boldsymbol{\Omega}^\top \boldsymbol{P} \boldsymbol{\Omega} - \boldsymbol{P}] \boldsymbol{x}, \\ &= \boldsymbol{x}^\top \begin{bmatrix} \lambda^2 p_3 - 2\lambda p_2 & -\lambda p_3 \\ -\lambda p_3 & 0 \end{bmatrix} \boldsymbol{x}. \end{aligned} \tag{1.27}$$

Expanding (1.27) gives

$$\begin{aligned} V(g) - V(\boldsymbol{x}) &= (\lambda^2 p_3 - 2\lambda p_2) x_1^2 - 2\lambda p_3 x_1 x_2, \\ &= \gamma x_1^2 - \beta x_1 x_2, \end{aligned} \tag{1.28}$$

and by inserting for (1.20) the inequality of (1.28) becomes

$$V(g) - V(\boldsymbol{x}) \leq -\sigma |\boldsymbol{x}|^2 \quad \forall \boldsymbol{x} \in D, \tag{1.29}$$

so by Theorem (1.3), uniform global pre-asymptotic stability is guaranteed. $\square$

For Theorem 2 knowledge of the integral error is used to find the jump set. The integral error is not known, since the integral action is used to compensate for an unknown disturbance. Therefore, the results of Theorem (2) is not applicable for a practical implementation. In Remark 1, knowledge of the system dynamics is used to estimate the integral error, and to find a jump set based on x_1 and $\dot{x}_1$.

Remark 1 (Practical implementation)**.** *For a practical implementation of Theorem 2, the jump set D can not depend on x_2. From the flow map of (1.11) $x_1 x_2$ can be written as*

$$x_1 x_2 = a' x_1^2 + x_1 \dot{x}_1, \tag{1.30}$$

such that the jump set of (1.20) can be rewritten as

$$D = \left\{ \boldsymbol{x} \in \mathrm{R}^2 : |x_1| \geq \varepsilon, x_1 \dot{x}_1 \geq \frac{(\gamma + \sigma - a'\beta) x_1^2 + \sigma x_2^2)}{\beta} \right\}. \tag{1.31}$$

The value of σ can be set arbitrarily small, such that Eq. (1.31) does in practice not depend on the value of x_2.

1.2.4 DP system

Similar to the first order system, a closed loop system with integral action in the controller will be derived. Then the flow set, flow map, and jump map are defined. Thereafter Theorem 3 will define stability of the system by specifying a jump set that depends on the states of the system.

Consider the linearized DP system of Section **??**, that has kinematics and kinetics given as

$$\dot{\boldsymbol{\eta}} = \boldsymbol{R}(\psi)\boldsymbol{\nu}, \tag{1.32}$$

$$\boldsymbol{M}\dot{\boldsymbol{v}} = -\boldsymbol{D}\boldsymbol{\nu} + \boldsymbol{R}(\psi)^\top \boldsymbol{b} + \boldsymbol{\tau}, \tag{1.33}$$

$$\dot{\boldsymbol{b}} = 0, \tag{1.34}$$

where $\boldsymbol{b}$ is considered a constant disturbance or bias force (from Eq. (1.34)), and the linear damping matrix satisfies $\boldsymbol{D} > 0$, and the mass matrix has the following properties $\boldsymbol{M} = \boldsymbol{M}^\top > 0$, and $\dot{\boldsymbol{M}} = \boldsymbol{0}$. The system now contain a rotation matrix, making it nonlinear. However, the jump map used will be linear, and similar to the approach used in Section 1.2.3.

By using a backstepping approach with integral action (**?**), the x-variables can be defined as

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{R}^\top \tilde{\boldsymbol{\eta}} \\ \boldsymbol{\nu} - \boldsymbol{\mu}(\boldsymbol{\eta}, t) \\ \tilde{\boldsymbol{b}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \end{bmatrix}, \tag{1.35}$$

where $\boldsymbol{\mu}(\boldsymbol{\eta}, t)$ is a virtual control law to be defined later. An integral state $\hat{\boldsymbol{b}}$ is augmented to the plant, and its dynamics are given as

$$\dot{\hat{\boldsymbol{b}}} = -\boldsymbol{K}_i \boldsymbol{R}(\psi)\boldsymbol{x}_2, \tag{1.36}$$

with $\boldsymbol{K}_i = \boldsymbol{K}_i^\top > 0$. The other variables are defined as $\tilde{\boldsymbol{\eta}} = \boldsymbol{\eta} - \boldsymbol{\eta}_d$, $\tilde{\boldsymbol{b}} = \boldsymbol{b} - \hat{\boldsymbol{b}}$, where $\boldsymbol{\eta}_d$ is the desired position, and the desired velocity is zero ($\boldsymbol{\nu}_d = 0$). By setting the virtual control law $\boldsymbol{\mu}(\eta, t)$ as

$$\boldsymbol{\mu}(\boldsymbol{\eta}, t) = -\boldsymbol{K}_p \boldsymbol{x}_1 + \boldsymbol{R}(\psi)^\top \dot{\boldsymbol{\eta}}_d, \tag{1.37}$$

with $\boldsymbol{K}_p = \boldsymbol{K}_p^\top > 0$, and the actual control input $\boldsymbol{\tau}$ as

$$\boldsymbol{\tau} = -\boldsymbol{x}_1 - \boldsymbol{K}_d \boldsymbol{x}_2 - \boldsymbol{R}(\psi)^\top \hat{\boldsymbol{b}} + \boldsymbol{D}\boldsymbol{\nu} + \boldsymbol{M}\dot{\boldsymbol{\mu}}, \tag{1.38}$$

where $\boldsymbol{K}_d = \boldsymbol{K}_d^\top > 0$. This results in the closed loop continuous dynamics

$$\dot{\boldsymbol{x}}_1 = -r\boldsymbol{S}\boldsymbol{x}_1 - \boldsymbol{K}_p \boldsymbol{x}_1 + \boldsymbol{x}_2 \tag{1.39}$$

$$\boldsymbol{M}\dot{\boldsymbol{x}}_2 = -\boldsymbol{x}_1 - \boldsymbol{K}_d \boldsymbol{x}_2 + \boldsymbol{R}(\psi)^\top \boldsymbol{x}_3 \tag{1.40}$$

$$\dot{\boldsymbol{x}}_3 = -\boldsymbol{K}_i \boldsymbol{R}(\psi)\boldsymbol{x}_2, \tag{1.41}$$

and this is shown in Appendix A.

From the states defined in (1.35), and the continuous closed loop dynamics of (1.39 - 1.41), the flow set, the flow map, and the jump map are defined below. The jump map is defined similarly to that of Section 1.2.3, and the jump set is defined in Theorem 3.

Flow set

Flow should be allowed in the entire state space, and the flow map C is given as

$$C = \left\{\boldsymbol{x} \in \mathrm{R}^9\right\}. \tag{1.42}$$

Flow map

Rewriting the equations (1.39 - 1.41), the flow map $f(\boldsymbol{x})$ is given as

$$\dot{\boldsymbol{x}}_1 = -r\boldsymbol{S}\boldsymbol{x}_1 - \boldsymbol{K}_p\boldsymbol{x}_1 + \boldsymbol{x}_2 \tag{1.43}$$
$$\boldsymbol{M}\dot{\boldsymbol{x}}_2 = -\boldsymbol{x}_1 - \boldsymbol{K}_d\boldsymbol{x}_2 + \boldsymbol{R}(\psi)^\top\boldsymbol{x}_3 \tag{1.44}$$
$$\dot{\boldsymbol{x}}_3 = -\boldsymbol{K}_i\boldsymbol{R}(\psi)\boldsymbol{x}_2. \tag{1.45}$$

Jump map

For the jump map chosen the jump in integral action is proportional to $\boldsymbol{x}_2$. That is, the same state variable used in the continuous integral action.

For a constant $\lambda > 0$,

$$\boldsymbol{\Lambda} = diag\{\lambda, \lambda, \lambda\} > 0, \tag{1.46}$$

the jump map is given as

$$\boldsymbol{x}^+ = \begin{bmatrix} \boldsymbol{x}_1^+ \\ \boldsymbol{x}_2^+ \\ \boldsymbol{x}_3^+ \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 - \Lambda\boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{\Lambda} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \boldsymbol{x}_3 \end{bmatrix} = \boldsymbol{\Omega}\boldsymbol{e} = g(\boldsymbol{x}) \tag{1.47}$$

Theorem 3. *Given the closed loop DP system with $\boldsymbol{x} \in \mathrm{R}^9$ of (1.35), the flow set given by (1.42), and the closed loop flow map given by (1.43 - 1.45), and jump map given by (1.47), let $\boldsymbol{M} = \boldsymbol{M}^\top$, $\dot{\boldsymbol{M}} = \boldsymbol{0}$, $\boldsymbol{K}_i = \boldsymbol{K}_i^\top$, α_1, $\alpha_2 \in \mathrm{K}_\infty$, and a Lyapunov function be given as*

$$\alpha_1(|\boldsymbol{x}|_A) \leq V(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}_1^\top\boldsymbol{x}_1 + \frac{1}{2}\boldsymbol{x}_2^\top\boldsymbol{M}\boldsymbol{x}_2 + \frac{1}{2}\boldsymbol{x}_3^\top\boldsymbol{K}_i^{-1}\boldsymbol{x}_3 \leq \alpha_2(|\boldsymbol{x}|_A), \tag{1.48}$$

such that $\dot{V}(\boldsymbol{x}) < 0, \ \ \forall x \in C$ can be shown.

Let

$$\boldsymbol{\beta} := 2\boldsymbol{\Lambda}\boldsymbol{P}_3 > 0, \tag{1.49}$$
$$\boldsymbol{\Gamma} := \boldsymbol{\Lambda}^2\boldsymbol{P}_3 - 2\boldsymbol{\Lambda}\boldsymbol{P}_2 > 0, \tag{1.50}$$
$$\varepsilon_1 > 0, \tag{1.51}$$
$$\varepsilon_2 > 0, \tag{1.52}$$
$$\sigma > 0, \tag{1.53}$$

where ε_1, ε_2, and σ are constants. For the jump set given by

$$D = \Big\{\boldsymbol{x} \in \mathrm{R}^9 : |\boldsymbol{x}_1| \geq \varepsilon_1, |\boldsymbol{x}_2| \geq \varepsilon_2, \tag{1.54}$$
$$\boldsymbol{x}_2^\top\boldsymbol{K}_i^{-1}\boldsymbol{x}_3 \geq \frac{\lambda}{2}\boldsymbol{x}_2^\top\boldsymbol{K}_i^{-1}\boldsymbol{x}_2 + \frac{\sigma}{2\lambda}[\boldsymbol{x}_1^\top\boldsymbol{x}_1 + \boldsymbol{x}_2^\top\boldsymbol{x}_2 + \boldsymbol{x}_3^\top\boldsymbol{x}_3]\Big\}, \tag{1.55}$$

then

$$V(g(\boldsymbol{x})) - V(\boldsymbol{x}) \leq -\sigma|\boldsymbol{x}|^2, \tag{1.56}$$

and the set $A = \{\boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}\}$ is uniformly globally pre-asymptotically stable for $\mathcal{H} = (C, f, D, g)$.

Proof. Stability in flow:

The time derivative of the Lyapunov function given by (1.48) gives

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &= \boldsymbol{x}_1^\top \dot{\boldsymbol{x}}_1 + \boldsymbol{x}_2^\top \boldsymbol{M}\dot{\boldsymbol{x}}_2 + \boldsymbol{x}_3^\top \boldsymbol{K}_i^{-1}\dot{\boldsymbol{x}}_3 \\ &= \boldsymbol{x}_1^\top[-rS\boldsymbol{x}_1 - \boldsymbol{K}_p\boldsymbol{x}_1 + \boldsymbol{x}_2] + \boldsymbol{x}_2^\top[-\boldsymbol{x}_1 - \boldsymbol{K}_d\boldsymbol{x}_2 + \boldsymbol{R}(\psi)^\top \boldsymbol{x}_3] \\ &+ \boldsymbol{x}_3^\top \boldsymbol{K}_i^{-1}[-\boldsymbol{K}_i\boldsymbol{R}(\psi)\boldsymbol{x}_2] \\ &= -\boldsymbol{x}_1^\top \boldsymbol{K}_p\boldsymbol{x}_1 - \boldsymbol{x}_2^\top \boldsymbol{K}_d\boldsymbol{x}_2 \leq 0, \end{aligned} \tag{1.57}$$

and the continuous dynamics is uniformly globally stable (UGS) $\boldsymbol{referenceKHALIL}$, and to proove UGAS Barbalats Lemma ($\boldsymbol{REFpreviouschapter}$) needs to be applied, since the system is time varying. The double time derivative of $V(\boldsymbol{x})$, $\ddot{V}(\boldsymbol{x})$ is

$$\ddot{\boldsymbol{V}}(\boldsymbol{x}) = -2\boldsymbol{x}_1^\top \boldsymbol{K}_p\dot{\boldsymbol{x}}_1 - 2\boldsymbol{x}_2^\top \boldsymbol{K}_d\dot{\boldsymbol{x}}_2. \tag{1.58}$$

From Eq. (1.48) it is known that $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ are bounded. From Eq. (1.43) is can be seen that $\dot{\boldsymbol{x}}_1$ is bounded, and from Eq. (1.44) $\dot{\boldsymbol{x}}_2$ is bounded, and hence, $\ddot{V}(\boldsymbol{x})$ is bounded, and

$$\lim_{t\to\infty}\dot{V} = \lim_{t\to\infty}(-\boldsymbol{x}_1^\top \boldsymbol{K}_p\boldsymbol{x}_1 - \boldsymbol{x}_2^\top \boldsymbol{K}_d\boldsymbol{x}_2) = 0$$

Since $\boldsymbol{x}_2$ goes to zero as time goes to infinity, then

$$\lim_{t\to\infty}\dot{\boldsymbol{x}}_2 = 0,$$

and by Eq. (1.44)

$$\lim_{t\to\infty}\boldsymbol{x}_3 = 0,$$

and UGAS can be concluded for flow $\forall \boldsymbol{x} \in C$.

Stability in jumps:

The value of the Lyapunov function after a jump $V(g)$ is given as

$$\begin{aligned} V(g) &= \frac{1}{2}\boldsymbol{x}_1^\top \boldsymbol{x}_1 + \frac{1}{2}\boldsymbol{x}_2^\top \boldsymbol{M}\boldsymbol{x}_2 + \frac{1}{2}(\boldsymbol{x}_3 - \boldsymbol{\Lambda}\boldsymbol{x}_2)^\top \boldsymbol{K}_i^{-1}(\boldsymbol{x}_3 - \boldsymbol{\Lambda}\boldsymbol{x}_2) \\ &= \frac{1}{2}\boldsymbol{x}_1^\top \boldsymbol{x}_1 + \frac{1}{2}\boldsymbol{x}_2^\top \boldsymbol{M}\boldsymbol{x}_2 \\ &+ \frac{1}{2}[\boldsymbol{x}_3^\top \boldsymbol{K}_i^{-1}\boldsymbol{x}_3 - \boldsymbol{x}_3^\top \boldsymbol{K}_i^{-1}\boldsymbol{\Lambda}\boldsymbol{x}_2 - \boldsymbol{x}_2^\top \boldsymbol{\Lambda}^\top \boldsymbol{K}_i^{-1}\boldsymbol{x}_3 + \boldsymbol{x}_2^\top \boldsymbol{\Lambda}^\top \boldsymbol{K}_i^{-1}\boldsymbol{\Lambda}\boldsymbol{x_2}], \end{aligned} \tag{1.59}$$

such that

$$V(g) - V(\boldsymbol{x}) = -\boldsymbol{x}_3^\top \boldsymbol{K}_i^{-1}\boldsymbol{\Lambda}\boldsymbol{x}_2 - \boldsymbol{x}_2^\top \boldsymbol{\Lambda}^\top \boldsymbol{K}_i^{-1}\boldsymbol{x}_3 + \boldsymbol{x}_2^\top \boldsymbol{\Lambda}^\top \boldsymbol{K}_i^{-1}\boldsymbol{\Lambda}\boldsymbol{x_2}, \tag{1.60}$$

and since $\boldsymbol{\Lambda} = \lambda \boldsymbol{I}$,

$$V(g) - V(\boldsymbol{x}) = -2\lambda \boldsymbol{x}_2^\top \boldsymbol{K}_i^{-1} \boldsymbol{x}_3 + \lambda^2 \boldsymbol{x}_2^\top \boldsymbol{K}_i^{-1} \boldsymbol{x}_2. \tag{1.61}$$

For

$$\boldsymbol{x}_2^\top \boldsymbol{K}_i^{-1} \boldsymbol{x}_3 \geq \frac{\lambda}{2} \boldsymbol{x}_2^\top \boldsymbol{K}_i^{-1} \boldsymbol{x}_2 + \frac{\sigma}{2\lambda}[\boldsymbol{x}_1^\top \boldsymbol{x}_1 + \boldsymbol{x}_2^\top \boldsymbol{x}_2 + \boldsymbol{x}_3^\top \boldsymbol{x}_3], \tag{1.62}$$

$$V(g) - V(\boldsymbol{x}) \leq -\sigma |\boldsymbol{x}|^2 < 0. \tag{1.63}$$

As with the approach of the first order system, σ can be arbitrary small, so it does not matter that $\boldsymbol{x}_3$ is unknown.

Since UGAS can be proved for both flow and jumps, UGAS (UGpAS) for the hybrid system is concluded. □

Remark 2 (Practical implementation)**.** *For a practical implementation of Theorem 3, the jump set D can not depend on $\boldsymbol{x}_3$. From the flow map equation of (1.44) $\boldsymbol{x}_2^\top \boldsymbol{K}_i^{-1} \boldsymbol{x}_3$ can be written as*

$$\boldsymbol{x}_2^\top \boldsymbol{K}_i^{-1} \boldsymbol{x}_3 = \boldsymbol{x}_2^\top \boldsymbol{K}_i^{-1} \boldsymbol{R}(\psi)[\boldsymbol{M}\dot{\boldsymbol{x}}_2 + \boldsymbol{x}_1 + \boldsymbol{K}_d \boldsymbol{x}_2] \tag{1.64}$$

such that the jump set of (1.55) can be rewritten as

$$D = \big\{\boldsymbol{x} \in \mathrm{R}^9 : |\boldsymbol{x}_1| \geq \varepsilon_1, |\boldsymbol{x}_2| \geq \varepsilon_2, \tag{1.65}$$

$$\boldsymbol{x}_2^\top \boldsymbol{K}_i^{-1} \boldsymbol{R}(\psi) \boldsymbol{M} \dot{x}_2 \geq \frac{\lambda}{2} \boldsymbol{x}_2^\top \boldsymbol{K}_I \boldsymbol{x}_2 + \frac{\sigma}{2\lambda}[\boldsymbol{x}_1^\top \boldsymbol{x}_1 + \boldsymbol{x}_2^\top \boldsymbol{x}_2 + \boldsymbol{x}_3^\top \boldsymbol{x}_3] \tag{1.66}$$

$$-\boldsymbol{x}_2^\top \boldsymbol{K}_i^{-1} \boldsymbol{R}(\psi)[\boldsymbol{x}_1 + \boldsymbol{K}_d \boldsymbol{x}_2]\big\} \tag{1.67}$$

As in Remark 1, the value of σ can be set arbitrarily small, such that Eq. (1.67) does in practice not depend on the value of the integral state error x_3.

1.3 Case studies

1.3.1 First order linear system

For this example both the cases when knowledge of x_2 is assumed known, and when x_2 is estimated from sampling of $\dot{x}_1$ are simulated. Consider the first order system

$$\dot{z} = -z + d + u \tag{1.68}$$

$$\dot{d} = 0, \tag{1.69}$$

and control input u as

$$u = \dot{z}_d + z_d - 9(z - z_d) - \hat{d}, \tag{1.70}$$

so that the closed loop continuous dynamics becomes

$$\dot{\tilde{z}} = -10\tilde{z} + \tilde{d} \tag{1.71}$$

$$\dot{\tilde{d}} = -k_i x_1. \tag{1.72}$$

Let the state vector be given as

$$\boldsymbol{x} = \begin{bmatrix} \tilde{z} \\ \tilde{d} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{1.73}$$

Typically the integral action would have a low tuning, so set $k_i = 1$, and a' is given as 10 by (1.71), so $\dot{\boldsymbol{x}}$ is written as

$$\dot{\boldsymbol{x}} = \begin{bmatrix} -10 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{A}\boldsymbol{x}, \tag{1.74}$$

and for

$$\boldsymbol{P} = \begin{bmatrix} 1 & -0.1 \\ -0.1 & 1 \end{bmatrix} \tag{1.75}$$

the condition given by (1.24) is satisfied, and the Lyapunov function decrease in flow.

For this example, the values chosen for λ, and ε are

$$\lambda = 3, \tag{1.76}$$

$$\varepsilon = 0.3, \tag{1.77}$$

such that Eq. (1.16), and (1.17) gives $\beta = 6$, and $\gamma = 9.6$, and the jump set given by (1.20) becomes

$$D = \left\{ \boldsymbol{x} \in \mathrm{R}^2 : |x_1| \geq 0.3, x_1 x_2 \geq 1.6 x_1^2 + \frac{\alpha_3}{\beta}(x_1^2 + x_2^2) \right\}. \tag{1.78}$$

Note that α_3 can be arbitrary small, so it is not considered in the simulation.

The jump set for when x_2 is not assumed known, given by (1.31) becomes

$$D = \left\{ \boldsymbol{x} \in \mathrm{R}^2 : |x_1| \geq 0.3, x_1 \dot{x}_1 \geq -8.4 x_1^2 + \frac{\alpha_3}{\beta}(x_1^2 + x_2^2) \right\}. \tag{1.79}$$

For both simulations a step of magnitude 100 at time $t = 5s$, and a step of -50 at time $t = 30$ is applied.

Even though it is not necessary in terms of stability, a dwell-time (**?**) (minimum amount of time between jumps) of $T = 0.1s$ is demanded between each jump in the simulations. For the simulation run with x_2 assumed known, it would work fine without a dwell time, but if ε is set small, then all jumps necessary would be performed with a magnitude $\lambda\varepsilon_2$, which could require a lot of jumps. A method without dwell-time could be computationally unfeasible. Also, a method without a dwell-time constraint could be less robust, since it allows for an infinite number of jumps in no time, which means that is could respond too quickly to noise. However, if the dwell-time is slower than the noise, this would work as a noise filter. For the sampled version a dwell-time is needed in order to sample values of x_1.

For the second simulation run where $\dot{x}_1$ is found by sampling, a sampling period of $T_s = 0.03s$ is applied. Let x_2' be the value of x_2 at previous sampling instant. Then $\dot{x}_2$ is found from

$$\dot{x_1} = \frac{x_1 - x_1'}{T_s}. \tag{1.80}$$

White noise of power 0.1 is added to make the sampling process more realistic. It is not crucial that the value for $\dot{x}_1$ is correct, but it should be close in magnitude to the true value, at least when close to the boundary of the jump set. This is because $\dot{x}_1$ is just used as a jump condition (defines the jump set), and the value itself is not used in any feedback.

Results first order system - with knowledge of x_2

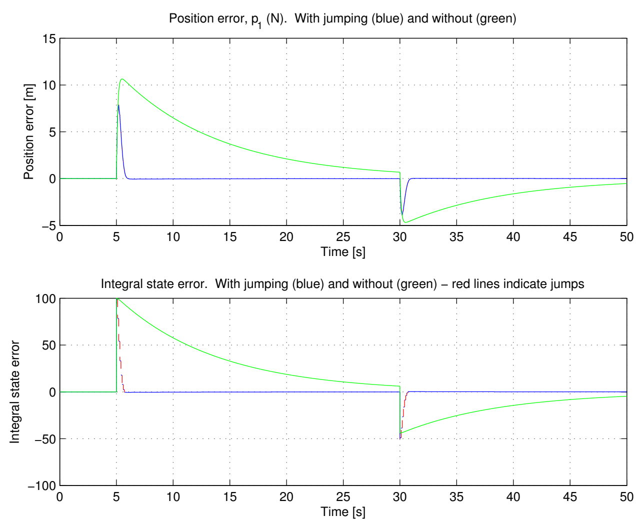

Figure 1.1: Position and integral state error first order system, no sampling

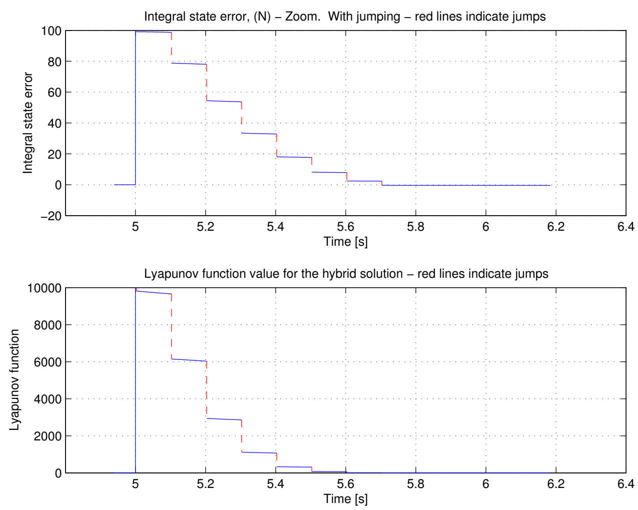

Figure 1.2: Lyapunov function and integral state error zoom at time $t = 5s$, no sampling

Results first order system - without knowledge of x_2

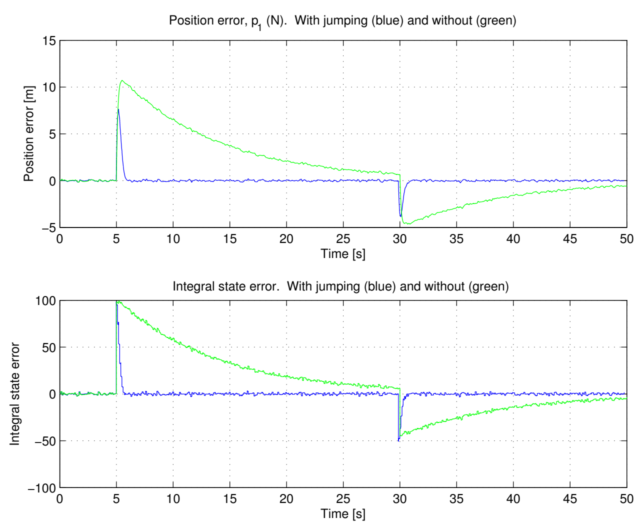

Figure 1.3: Position and integral state error first order system, with sampling, and white noise.

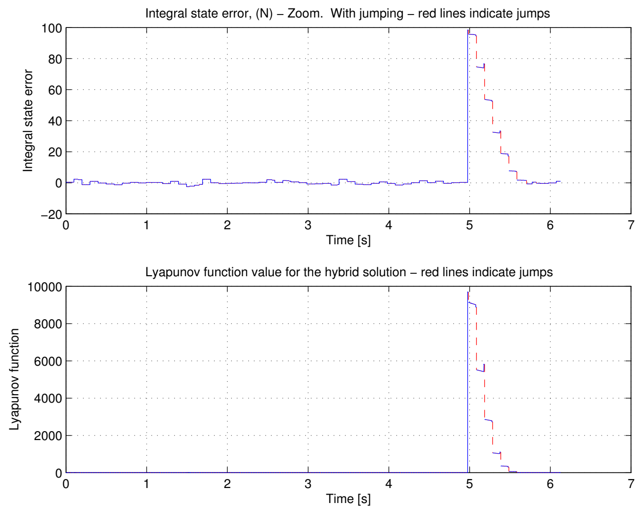

Figure 1.4: Lyapunov function and integral state error zoom at time $t = 5s$, with sampling, and white noise.

1.3.2 Example - DP

In this example, knowledge of $\boldsymbol{x}_3$ will be assumed known for the first simulation run. For the second simulation, $\boldsymbol{x}_2$ will be sampled to find an estimate of $\dot{\boldsymbol{x}}_2$. Some noise will be added to this simulation, to make the sampling process more realistic.

Consider the closed loop DP system of Eq. (1.43 - 1.45) as

$$\dot{\boldsymbol{x}}_1 = -r\boldsymbol{S}\boldsymbol{x}_1 - \boldsymbol{K}_p\boldsymbol{x}_1 + \boldsymbol{x}_2 \tag{1.81}$$

$$\boldsymbol{M}\dot{\boldsymbol{x}}_2 = -\boldsymbol{x}_1 - \boldsymbol{K}_d\boldsymbol{x}_2 + \boldsymbol{R}(\psi)^\top\boldsymbol{x}_3 \tag{1.82}$$

$$\dot{\boldsymbol{x}}_3 = -\boldsymbol{K}_i\boldsymbol{R}(\psi)\boldsymbol{x}_2 \tag{1.83}$$

with

$$\boldsymbol{M} = diag\{450, 450, 100\} \tag{1.84}$$

$$\boldsymbol{K}_p = diag\{100, 100, 50\} \tag{1.85}$$

$$\boldsymbol{K}_i = diag\{10, 10, 5\} \tag{1.86}$$

$$\boldsymbol{K}_d = diag\{500, 500, 200\}, \tag{1.87}$$

and λ, ε_1, and ε_2 chosen as

$$\lambda = 25, \tag{1.88}$$

$$\varepsilon_1 = 0.0001, \tag{1.89}$$

$$\varepsilon_2 = 0.01. \tag{1.90}$$

For both simulations a step of magnitude

$$\boldsymbol{b} = \begin{bmatrix} 3000 \\ 2000 \\ 1000 \end{bmatrix}, \tag{1.91}$$

is applied at time $t = 10s$.

By the same reasoning as for the example for the first order system of Section 1.3.1 a dwell-time is added between jumps here as well. The dwell chosen is $T = 0.1s$.

Similar to the example for the first order system of Section 1.3.1 a dwell-time is needed in order to sample values of x_2. For the second simulation run where $\dot{\boldsymbol{x}}_2$ is found by sampling, a sampling period of $T_s = 0.03s$ is applied. Let $\boldsymbol{x}_2'$ be the value of $\boldsymbol{x}_2$ at previous sampling instant then $\dot{\boldsymbol{x}}_2$ is found from

$$\dot{\boldsymbol{x}}_{\boldsymbol{2}} = \frac{\boldsymbol{x}_2 - \boldsymbol{x}_2'}{T_s}. \tag{1.92}$$

In the second simulation noise of power 20 has been added in all DOF.

Results DP - with knowledge of x_3

For the results below is says (N), (E), $(Heading)$, on the plots, but this is not entirely the case. The $\boldsymbol{x}_1$ state is $\tilde{\boldsymbol{\eta}}$ transformed to body coordinates, and $\boldsymbol{x}_2$ is not equal to $\boldsymbol{\nu}$, but in the plots $\boldsymbol{x}_1$, and $\boldsymbol{x}_2$ are called "position" and "velocity".

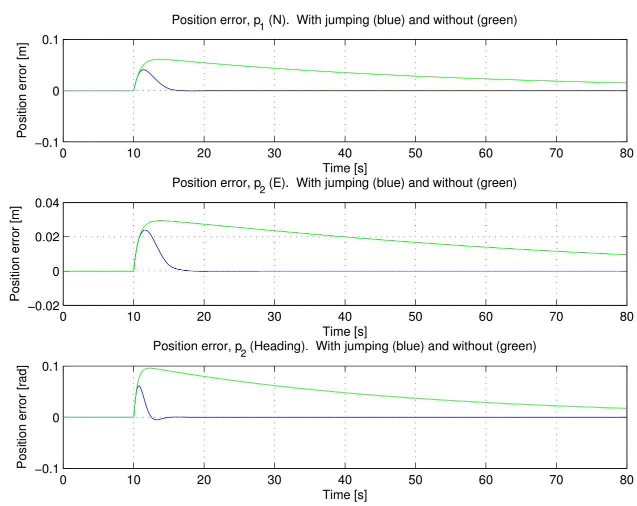

Figure 1.5: Position error DP, no sampling

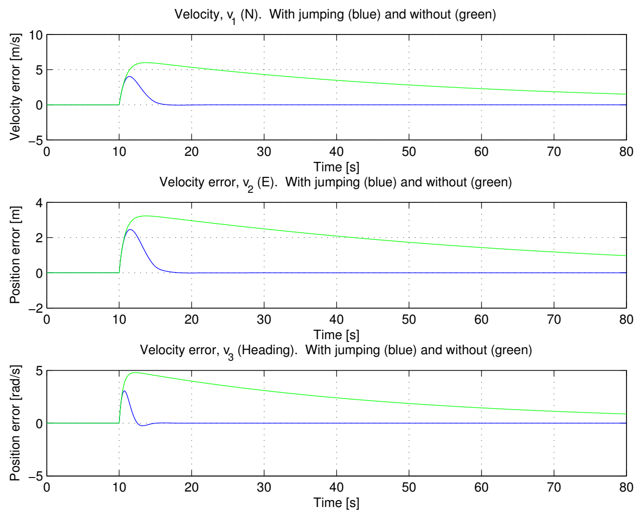

Figure 1.6: Velocity error DP, no sampling

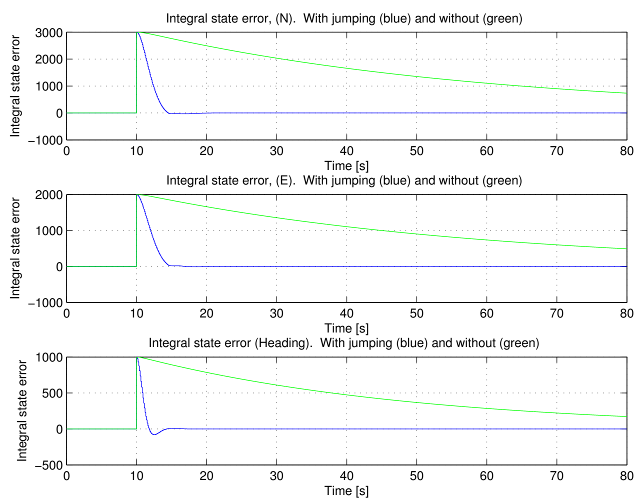

Figure 1.7: Integral error DP, no sampling

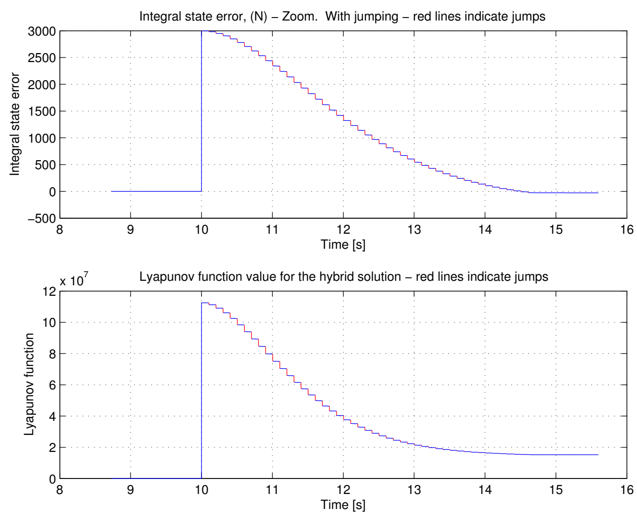

Figure 1.8: Lyapunov function and zoom on integral error, no sampling

Results DP - without knowledge of x_3 - sampling of x_2

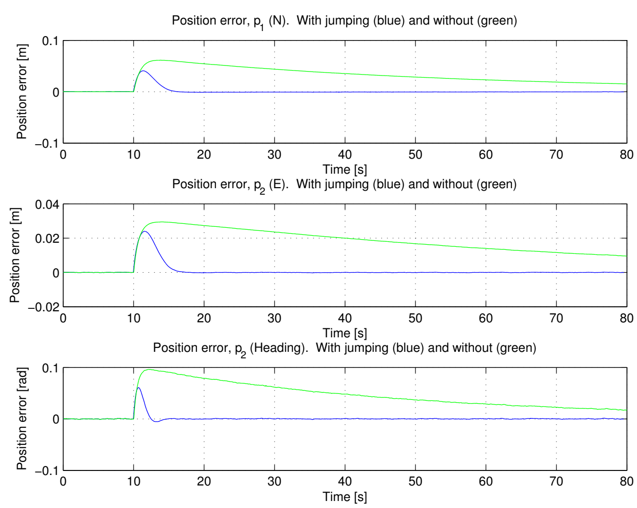

Figure 1.9: Position error DP, with sampling

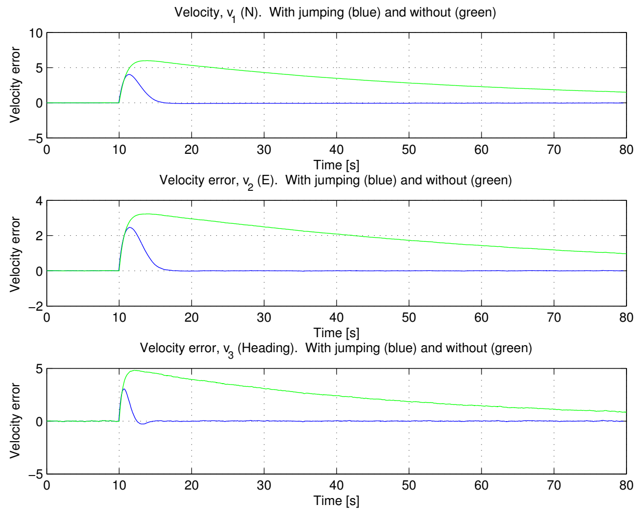

Figure 1.10: Velocity error DP, with sampling

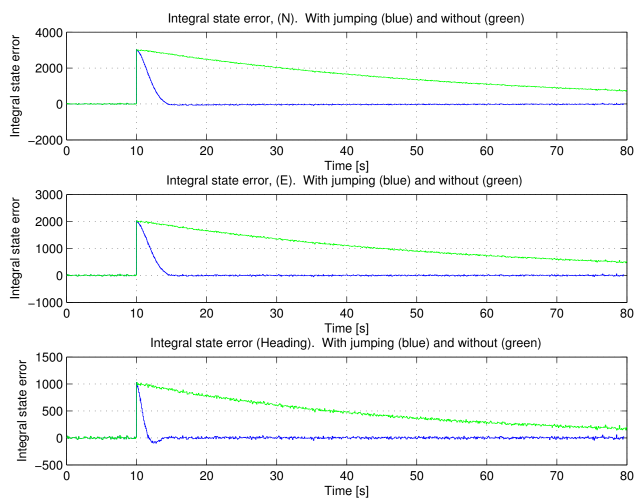

Figure 1.11: Integral error DP, with sampling

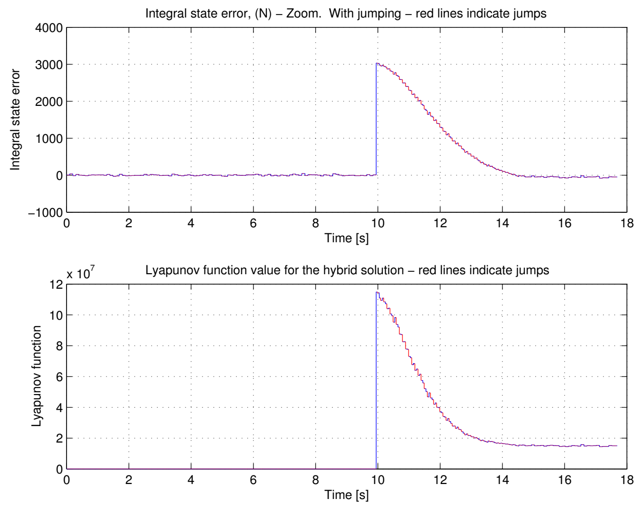

Figure 1.12: Lyapunov function and zoom on integral error, with sampling

1.4 Conclusions and further work

For both the first order linear system, and the DP system the hybrid integral action improves performance for large constant disturbances. The state estimates, and the integral action error converge much faster than in the case without jumps. After a step the integral action converge to the new value quickly, and (almost) without overshoot. When sampling is used to find the jump set, the performance decrease is negligible compared to the ideal case, and the noise does not induce problems.

For further work the algorithm should be tested on more realistic data, either a comprehensive numerical ice simulation, or a model test for a DP vessel subject to large step changes in disturbances. It would be interesting to see how this approach could improve performance (offset from desired position), thrust, and power consumption. One other interesting aspect for a model test is how performance is affected by sensor noise. For the DP system an alternative to sampling the velocity is to use accelerometers to find the accelerations.

Chapter 2

Summary and conclusions

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