Chapter 1

Modelling

1.1 Vessel models

The modelling of following section is to a high degree based on material from **?**.

Vessel kinetics is normally expressed in BODY-frame, and the kinematics in North-East-Down (NED)-frame. There exists numerous models with varying complexity, dependent on operation. The main difference is for high- and low-speed applications, where different forms of hydrodynamic damping dominate for different speed regimes, and the Coriolis and centripetal terms become negligible for low speed.

1.1.1 High fidelity model

Kinematics

Vessel kinematics is given as (**?**)

$$\dot{\boldsymbol{\eta}} = \boldsymbol{R}(\boldsymbol{\Theta})\boldsymbol{\nu}, \tag{1.1}$$

where

$$\boldsymbol{\eta} = \begin{bmatrix} N & E & D & \phi & \theta & \psi \end{bmatrix}^\top \in \mathbb{R}^{6\times 1}, \tag{1.2}$$

contain the North, East, and down positions, and the angular orientation (Euler angles) in roll (ϕ), pitch (θ), and yaw (ψ). The velocity vector $\boldsymbol{\nu}$ is given as

$$\boldsymbol{\nu} = \begin{bmatrix} u & v & w & p & q & r \end{bmatrix}^\top \in \mathbb{R}^{6\times 1}, \tag{1.3}$$

where u, v, and w are the velocities in surge (x-direction), sway (y-direction), and heave (down-direction), respectively. The other half of $\boldsymbol{\nu}$ is given by the angular velocities in roll (p), pitch (q), and yaw (r). The $\boldsymbol{\eta}$ vector is given in NED-coordinates, and $\boldsymbol{\nu}$ in BODY-coordinates. The rotation matrix $\boldsymbol{R}(\boldsymbol{\Theta})$ transforms the coordinate frame between BODY to NED and is given as

$$\boldsymbol{R}(\boldsymbol{\Theta}) = \begin{bmatrix} c\psi c\theta & -s\psi c\phi + c\psi s\theta s\phi & s\psi s\phi + c\psi c\phi s\theta \\ s\psi c\theta & c\psi c\phi + s\phi s\theta s\psi & -c\psi s\phi + s\theta s\psi c\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix} \in \mathbb{R}^{3\times 3}, \tag{1.4}$$

where $c(\cdot) = \cos(\cdot), s(\cdot) = \sin(\cdot)$.

Also, the time derivative of the rotation matrix is given by

$$\dot{\boldsymbol{R}}(\boldsymbol{\Theta}) = \boldsymbol{R}(\boldsymbol{\Theta})\boldsymbol{S}(\boldsymbol{\omega}) \tag{1.5}$$

where $\boldsymbol{S}(\boldsymbol{\omega})$ is a skew-symmetric matrix given by

$$\boldsymbol{S}(\boldsymbol{\omega}) = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}. \tag{1.6}$$

Kinetics

The vessel kinetics is generally given as (**?**)

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{M}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{C}_A(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{D}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r + \boldsymbol{g} = \boldsymbol{\tau} + \boldsymbol{\tau}_{wind} + \boldsymbol{\tau}_{wave}. \tag{1.7}$$

The terms in the equations are

- relative velocity $\boldsymbol{\nu}_r = \boldsymbol{\nu} - \boldsymbol{\nu}_c$, where $\boldsymbol{\nu}_c$ is the velocity of the current,
- interia terms: $\boldsymbol{M}_{RB}$ (rigid body) and $\boldsymbol{M}_A$ (added mass)
- Coriolis and centripetal terms: $\boldsymbol{C}_{RB}$ (rigid body), and $\boldsymbol{C}_A$ (due to added mass),
- damping forces: $\boldsymbol{D}(\boldsymbol{\nu}_r) = (\boldsymbol{D}_p + \boldsymbol{D}_v)$, where $\boldsymbol{D}_p$ is the linear potential damping, and $\boldsymbol{D}_v$ contains the viscous damping (vortex shedding, skin friction),
- restoring forces: $\boldsymbol{g}$ (hydrostatics),
- environmental forces:
  - $\boldsymbol{\tau}_{wind}$: wind forces,
  - $\boldsymbol{\tau}_{wave}$: first order oscillatory waves forces and second order waves forces (mean drift, slowly varying drift forces, sum-frequency forces) (**?**).
  - current: the forces from the current are included in $\boldsymbol{D}(\boldsymbol{\nu}_r)$ due to the relative velocity $\boldsymbol{\nu}_r$,
- propulsion forces, $\boldsymbol{\tau}$: the forces generated by the thrusters of the vessel

The Coriolis terms are present in the equation because the kinetics is expressed in BODY frame, which is a rotating moving reference frame with respect to an inertial reference frame (**?**).

1.1.2 Uncoupled surge dynamics

From **?** the uncoupled surge dynamics for a longitudinally symmetric ship, where m is the mass, $X_{\dot{u}}$ is the added mass in surge, X_u and $X_{|u|u}|$ are linear and nonlinear damping, respectively, can be written

$$(m - X_{\dot{u}})\dot{u} - X_u u_r - X_{|u|u}|u_r|u_r = \chi, \tag{1.8}$$

where χ comprise of the external forces, and the control input. The nonlinear damping will dominate for higher vessel speeds, and the linear damping dominates for low speeds.

1.1.3 Sway-yaw subsystem

For a constant surge velocity $u \approx u_0$, a linear sway-yaw subsystem, known as the second order Nomoto Model, can be written (**?**)

$$\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{N}(u_0)\boldsymbol{v}_r = \boldsymbol{b}\alpha \tag{1.9}$$

with

$$\boldsymbol{M} = \begin{bmatrix} m - Y_{\dot{v}} & mx_g - Y_{\dot{r}} \\ mx_g - Y_{\dot{r}} & I_z - N_{\dot{r}} \end{bmatrix} \in \mathbb{R}^{2\times 2}, \tag{1.10}$$

and $\boldsymbol{N} \in \mathbb{R}^{2\times 2}$ contain the speed dependent terms from $\boldsymbol{C}(\boldsymbol{v})$ and the linear damping matrix, $\boldsymbol{D}$ from Eq. (1.7) (**?**). The rudder angles are collected in the vector $\boldsymbol{\alpha}$. Let the α in Eq. (1.9) be a scalar. To relate this to the tests with R/V Gunnerus, that has two rudders, assume that the rudders have an equal angle, and thus modelled as a single rudder. Looking at the transfer function from the rudder angle to the yaw-rate, it can be written as (**?**)

$$\frac{r}{\alpha}(s) = \frac{K_y(1 + T_3 s)}{(1 + T_1 s)(1 + T_2 s)}. \tag{1.11}$$

Also, for the sway motion a similar relationship is found as

$$\frac{v}{\alpha}(s) = \frac{K_v(1 + T_v s)}{(1 + T_1 s)(1 + T_2 s)}, \tag{1.12}$$

where r is the yaw rate, and v is the sway velocity, $K_y,\ K_v$ are the steady state gains, and $T_1,\ T_2$, and T_3 are the time constants.

First order approximation

? defines a equivalent time constant so that from (1.11) the equivalent time constant T_r is defined as

$$T_r := T_1 + T_2 - T_3, \tag{1.13}$$

and from (1.12) the equivalent time constant $T_{\bar{v}}$ is defines as

$$T_{\bar{v}} := T_1 + T_2 - T_v, \tag{1.14}$$

and hence, the models given by (1.11) and (1.12) can be approximated by the first order models as

$$\frac{r}{\delta}(s) = \frac{K_y}{1 + T_r s}, \tag{1.15}$$

and

$$\frac{v}{\delta}(s) = \frac{K_v}{1 + T_{\bar{v}} s}. \tag{1.16}$$

1.1.4 Low speed models

For low speed applications, like dynamic positioning, the linear damping will dominate the nonlinear part (**?**), so $\boldsymbol{D}(\boldsymbol{\nu}_r) \approx \boldsymbol{D}\boldsymbol{\nu}$, where $\boldsymbol{D}$ is the linear damping matrix. The Coriolis and centripetal terms can be neglected, also due to low speed.

DP (3 DOF) - Linearized model

For dynamic positioning of a vessel it is common to restrict the workspace to 3 degrees of freedom (DOF). Only the horizontal plane (surge, sway and yaw) is considered, and this gives $\boldsymbol{\eta}$ and $\boldsymbol{\nu}$ as

$$\boldsymbol{\eta} = \begin{bmatrix} N & E & \psi \end{bmatrix}^\top \tag{1.17}$$

$$\boldsymbol{\nu} = \begin{bmatrix} u & v & r \end{bmatrix}^\top, \tag{1.18}$$

and the rotation matrix reduces to

$$\boldsymbol{R}(\boldsymbol{\Theta}) = \boldsymbol{R}(\psi) = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.19}$$

In the linearized DP model the slowly varying drift forces, mean drift forces, and the current are all collected into a bias $\boldsymbol{b}$. Since current is captured in the bias, the relative velocity vector is not included in the model anymore (it becomes superfluous) (**?**).

The kinematics and kinetics for the linearized 3 DOF DP model is written as (**?**)

$$\dot{\boldsymbol{\eta}} = \boldsymbol{R}(\psi)\boldsymbol{\nu} \tag{1.20}$$

$$\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{D}\boldsymbol{\nu} = \boldsymbol{R}^T(\psi)\boldsymbol{b} + \boldsymbol{\tau} + \boldsymbol{\tau}_{wind} + \boldsymbol{\tau}_{wave1} \tag{1.21}$$

$$\dot{\boldsymbol{b}} = -\boldsymbol{T}^{-1}\boldsymbol{b} + \boldsymbol{w}_b \quad (\text{ or } \dot{\boldsymbol{b}} = \boldsymbol{w}_b), \tag{1.22}$$

where $\boldsymbol{w}_b$ is white Gaussian noise (**?**), and $\boldsymbol{\tau}_{wave1}$ comprise only of the first order wave forces. The bias force is modeled by a Gauss-Markov process above, and could also be modeled as a white noise process (**?**), which is given in parenthesis in Eq. (1.22). Here $\boldsymbol{b}$ is a slowly varying disturbance or bias force, and the linear damping matrix satisfies $\boldsymbol{D} > 0$.