NTNU
Norwegian University of
Science and Technology

Bratteli Diagrams

Modeling AF-algebras and Cantor Minimal
Systems Using Infinite Graphs

Petter Kjeverud Nyland

Master of Science in Mathematics (for international students)
Submission date: May 2016
Supervisor: Eduardo Ortega, MATH
Co-supervisor: Christian Skau, MATH

Norwegian University of Science and Technology
Department of Mathematical Sciences

Abstract

This thesis gives a thorough introduction to the infinite graphs known as Bratteli diagrams and their two most common uses — namely as a means of representing AF-algebras in a combinatorial way and as a way of modeling Cantor minimal systems. The thesis is therefore naturally divided into two parts.

In the first part, the machinery needed study AF-algebras is introduced, in particular the structure theorem for finite-dimensional C^*-algebras is proved from first principles, and direct limits of finite-dimensional C^*-algebras are constructed. We show how an AF-algebra can be represented by a Bratteli diagram and how information about the AF-algebra may be extracted from its Bratteli diagram. In particular we demonstrate how the ideals of an AF-algebra may be read off its Bratteli diagram and also how the Bratteli diagrams of isomorphic AF-algebras are related. Some classic examples of AF-algebras are given, and their Bratteli diagrams are computed and used to illustrate the general theory.

In the second part, ordered Bratteli diagrams are introduced and we construct the associated Cantor minimal systems, known as Bratteli-Vershik systems. The associated dimension groups are also briefly introduced. We give the full proof of the model theorem for Cantor minimal systems.

Sammendrag

Denne avhandlingen gir en grundig innføring i en type uendelige grafer som kalles Brattelidiagrammer, og deres to vanligste bruksområder — nemlig som en måte å representere AF-algebraer på en kombinatorisk måte og som en måte å modellere Cantor minimale systemer. Avhandlingen er naturlig delt opp i to deler.

I den første delen introduseres maskineriet som trengs for å studere AF-algebraer. Blant annet bevises struktursatsen for endeligdimensjonale C^*-algebraer kun ved bruk av elementær operatorteori, og direktegrenser av endeligdimensjonale C^*-algebraer konstrueres. Vi viser hvordan en AF-algebra kan representeres ved et Brattelidiagram og hvordan egenskapene til en AF-algebra kan leses ut fra dens tilhørende Brattelidiagram. Spesielt demonstrerer vi hvordan idealstrukturen til en AF-algebra kan leses av Brattelidiagrammet, og hvilken sammenheng det er mellom Brattelidiagrammer som tilhører isomorfe AF-algebraer. Det gis eksempler på klassiske AF-algebraer, og deres tilhørende Brattelidiagrammer beregnes og brukes til å illustrere den generelle teorien.

I den andre delen introduseres ordnede Brattelidiagrammer og det tilhørende Cantor minimale systemet, som kalles et Bratteli-Vershik system, konstrueres. Den tilhørende dimensjonsgruppen introduseres også kort. Modellteoremet for Cantor minimale systemer bevises i full detalj.

Acknowledgements

First of all, I would like to thank my main supervisor, Christian Skau, for his help and support during the last year. He's always taken the time to proofread my thesis and assist me when I had problems, no matter how trivial (or nontrivial) they turned out to be. I am also grateful for being encouraged to attend several master classes during the last two years. Christian also set up extra lectures to prepare me for the master classes.

I also want to thank my co-supervisor Eduard Ortega for proofreading and valuable advice during the write-up of the thesis. And also for funding traveling to master classes. Associate Professor Mark Tomforde of University of Houston deserves a mention for letting me read some chapters from his unpublished book. Typesetting Bratteli diagrams turned out to be a time consuming task, so I am very grateful for Siri-Malén Høynes' LaTeX templates.

I would also like to thank my family for their love and support during my years at the university. In particular my father, Håvard, my brothers, Amund and Lars, and my late mother, Venke, from whom I've learned so much. My many trips back home has helped me unwind and maintain a high motivation.

I'd like to thank all my classmates in room 395B for five great years. Special thanks to the "Decorating Committee" for keeping morale up. In particular, I would like to thank Ulrik Bo Rufus Enstad, who has been my traveling companion on every master class. I also have to thank Sigurd Seierstad for many good reading sessions in his makeshift office at home, and also for uncountable amounts of coffee.

Finally I have to thank my girlfriend, Lisa. Even though I have frightened you away from taking a major in mathematics, your steady support has been an important factor during the work with this thesis. Your ability to make me relax during the most hectic periods of writing has been invaluable. Thank you!

Petter Kjeverud Nyland
Trondheim, May 2016

Preface

This thesis was written during the academic year 2015-2016 (part-time during the fall semester and full-time during the spring semester), and it marks the end of my two-year international Master's programme in Mathematical Sciences at the Norwegian University of Science and Technology.

The original goal of the thesis was to prove the model theorem of Herman, Putnam and Skau [7, Theorem 4.5] in all its detail. The main tool in the proof of the model theorem is Bratteli diagrams. In fact, the statement of the theorem is, loosely speaking, that any Cantor minimal system can be realized on a Bratteli diagram. My supervisor suggested that I should introduce Bratteli diagrams in their original context. Ola Bratteli introduced Bratteli diagrams and AF-algebras (which are direct limits of finite-dimensional C^*-algebras) in seminal paper from 1972 [1]. The AF-algebras are one of the most extensively studied classes of C^*-algebras. A reason for this is that they are quite accessible, yet highly nontrivial.

When I started to work on my thesis in the fall semester, I did not have much knowledge of C^*-algebras. I followed an introductory course on C^*-algebras lectured by Eduard Ortega and Franz Luef. But in order to gain some experience on my own, me and my supervisor decided that I should include an elementary proof of the structure theorem of finite-dimensional C^*-algebras in the thesis.

As I continued studying AF-algebras I found them to be quite interesting and fun to work with. The books I read sometimes had too big gaps in the proofs for my taste, so I decided to fill them in with detail, for my own sake. I also thought that the notation and definitions were a bit cumbersome at times so I have tried to make it as "clean", yet detailed, as possible. The proofs I will present are to my liking and may not necessarily be the ones mostly found in the literature.

As I went on, the thesis became more centered around Bratteli diagrams, rather than just Cantor minimal systems. Strictly speaking I ended up considering two slightly different types of Bratteli diagrams in the two parts of the thesis (Chapter 1-3 and Chapter 4), but if one restricts to unital AF-algebras they coincide.

The model theorem for Cantor minimal systems (Theorem 4.6.2), which we prove in the final Section of this thesis, is a fundamental result when it comes to classifying Cantor minimal systems up to orbit equivalence. Since Bratteli diagrams are very combinatorial and in a sense "very concrete", they simplify many proofs and allow the construction of computable invariants. The main idea in the proof is the construction of so-called Kakutani-Rokhlin partitions, which we shall simply call towers, that give rise to a Bratteli diagram.

The thesis is pretty much self-contained. The only preliminaries needed to understand most of the text are some general topology and elementary theory on C^*-algebras. This may be found in [10], and [11] and [12], respectively.

An overview of each chapter is given below.

Notation and conventions Some of the notation and conventions used in the thesis, which is also used in other areas of mathematics, are spelled out, in order to avoid any confusion.

Chapter 1 In the first chapter we classify the finite-dimensional C^*-algebras. Given a finite-dimensional C^*-algebra, A, we construct several finite-dimensional Hilbert spaces inside A and let the elements of A act as operators on these Hilbert spaces. This yields an isomorphism onto a multimatrix algebra. We also make some observations needed for later chapters.

Chapter 2 This chapter is devoted to the construction of direct limits of finite-dimensional C^*-algebras, in order to make sense of AF-algebras. We show that any homomorphism between multimatrix algebras is inner equivalent to a canonical homomorphism. When we have a chain system of multimatrix algebras where the connecting homomorphisms are canonical, the chain system gives rise to a Bratteli diagram. We also make some remarks about unital and non-unital chain systems.

Chapter 3 This is the main chapter in the first part of the thesis. AF-algebras are defined, and several equivalent characterizations are given. We show how AF-algebras correspond to Bratteli diagrams and vice versa. This correspondence allows us to prove several theorems. We classify commutative AF-algebras, show that there is a very strong uniqueness condition on a chain system defining an AF-algebra, we classify the ideal structure of AF-algebras in terms of their Bratteli diagrams and we give a criterion for when AF-algebras are simple. Several examples are also introduced, and analyzed, in order to illustrate the general results.

Chapter 4 In the fourth and final chapter we prove the model theorem for Cantor minimal systems. We also give some motivation for studying these kinds of dynamical systems. Ordered Bratteli diagrams are introduced and the space of infinite paths in the diagram is shown to be a Cantor space. Also, a naturally defined homeomorphism on the path space yields a Cantor minimal system. Given a Cantor minimal system (X, T), we construct, using Kakutani-Rokhlin partitions, an ordered Bratteli diagram whose associated dynamical system is conjugate (in fact pointedly conjugate) to (X, T). We also include several figures to illustrate and illuminate the properties of such diagrams. The associated dimension group, which is an important invariant, is introduced, but not studied.

Contents

Notation and conventions

We shall denote the positive integers by $\mathbb{N}$, and the non-negative integers by $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$. When a collection of objects is indexed by n, as in "A_n", then unless otherwise stated, this means that n runs through $\mathbb{N}$. We use $\subseteq$ to indicate inclusion of sets and $\subsetneq$ to indicate proper inclusion. To indicate that a union is disjoint we will use $\sqcup$ instead of $\cup$. The symbol $\cong$ will be used to denote different types of isomorphisms (which type will be clear from the context), except for topological spaces, where $\simeq$ denotes homeomorphism. Unless otherwise stated, $\oplus$ denotes direct sum in the appropriate category. When i and j are integers, then δ_{ij} denotes the Kronecker delta.

If T is a linear map we use $\ker(T)$ to denote the kernel of T and $\mathrm{Ran}(T)$ to denote the Range (i.e. image) of T. Inner products in Hilbert spaces will be denoted by $\langle \cdot, \cdot \rangle$. When H is a Hilbert space, $\mathcal{B}(H)$ denotes the bounded linear operators on H and $\| \cdot \|_{op}$ denotes the operator norm. We let $M_n(\mathbb{C})$ denote the C^*-algebra of $n \times n$ matrices with complex entries. If A is a C^*-algebra, then $\tilde{A}$ denotes the minimal unitization of A. If A is unital we use 1_A to denote the unit in A, or just 1 when no confusion arises. For $a \in A$, $spec(a)$ denotes the spectrum of a. If $S \subseteq A$ is a subset of a C^*-algebra A, then $C^*(S)$ denotes the C^*-subalgebra generated by S. And if A is a pre-C^*-algebra, then $C^*(A)$ denotes the C^*-completion of A.

When X is a compact Hausdorff topological space, then $C(X)$ denotes the C^*-algebra of continuous functions from X to $\mathbb{C}$. More generally, if X is locally compact and Hausdorff, then $C_0(X)$ denotes the C^*-algebra of continuous complex-valued functions on X vanishing at infinity. Also, $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$ as a topological space.

If X is a topological space, then by "an open neighbourhood of a point x" we shall mean an open set containing x. A clopen set is a set which is both open and closed. And if X is a metric space, then $B_\epsilon(x)$ denotes the open ball of radius ϵ centered at x. We also use $d(x, Y)$ to denote the distance from a point $x \in X$ to a subset $Y \subseteq X$, that is $d(x, Y) = \inf_{y \in Y} d(x, y)$. More generally, $d(Y_1, Y_2) = \inf_{y_1 \in Y_1, y_2 \in Y_2} d(y_1, y_2)$.

Chapter 1

Finite-dimensional C^*-algebras

1.1 The structure theorem

The main goal of this chapter is to prove the structure theorem for finite-dimensional C^*-algebras. In the course of the proof we will also gain insight into some properties of finite-dimensional C^*-algebras needed for later chapters. The structure theorem says that any finite-dimensional C^*-algebra is isomorphic to a direct sum of full matrix algebras, a so called *multimatrix algebra*. The precise statement is as follows.

Theorem 1.1.1. *Let A be a finite-dimensional C^*-algebra. Then there exists positive integers K and $N_1, \dots, N_K$ such that*

$$A \cong M_{N_1}(\mathbb{C}) \oplus M_{N_2}(\mathbb{C}) \oplus \cdots \oplus M_{N_K}(\mathbb{C})$$

Furthermore, K is uniquely determined, and $N_1, \dots, N_K$ are unique up to permutation.

We shall first prove Theorem 1.1.1 under the additional hypothesis that A is unital. This is because having a unit will make our proof easier. The main idea of the proof is to use certain "small" projections to construct finitely many finite-dimensional Hilbert spaces inside A, and then let the elements of A act as operators on these Hilbert spaces. When we have established the result for unital finite-dimensional C^*-algebras, we will extend it without much effort to not necessarily unital C^*-algebras by looking at the unitization, $\tilde{A}$. However, all finite-dimensional C^*-algebras are unital, as a consequence of Theorem 1.1.1.

1.2 Rank-one operators on Hilbert spaces

We begin by examining a class of rank-one operators on Hilbert spaces. Given two vectors ξ and η in a Hilbert space H we define an operator which maps vectors ζ

along ξ by multiplying with the scalar $\langle \zeta, \eta \rangle$. Since $\langle \zeta, \eta \rangle \eta$ is a scalar multiple of the orthogonal projection of ζ onto η, our operator "projects" ζ onto ξ as if it was "projecting" onto η.

Definition 1.2.1. Let H be a Hilbert space and let ξ and η be vectors in H. Define an operator $\xi \otimes \eta^*$ on H by

$$\xi \otimes \eta^*(\zeta) \coloneqq \langle \zeta, \eta \rangle \xi \text{ for } \zeta \in H.$$

Let us pause for a moment and consider the special case when $H = \mathbb{C}^N$ and we view vectors as column vectors. Also, let $\xi^\dagger$ denote the conjugate transpose of ξ. Then $\xi \cdot \eta^\dagger$ is a $N \times N$ matrix and $\xi^\dagger \cdot \eta = \langle \eta, \xi \rangle$ is a 1×1 matrix, i.e. a scalar. The formula in the definition above states associativity for matrix multiplication because

$$\xi \otimes \eta^*(\zeta) = \langle \zeta, \eta \rangle \xi = (\eta^\dagger \cdot \zeta) \cdot \xi = \xi \cdot (\eta^\dagger \cdot \zeta) = (\xi \cdot \eta^\dagger) \cdot \zeta.$$

So the matrix $\xi \cdot \eta^\dagger$ (which corresponds to $\xi \otimes \eta^*$) applied to ζ equals ξ scaled by $\eta^\dagger \cdot \zeta$ (which corresponds to $\langle \zeta, \eta \rangle$).

Lemma 1.2.2. *Let H be a Hilbert space, let ξ, η, ζ, ω be vectors in H and let $T \in \mathcal{B}(H)$ be a bounded linear operator on H. Then we have*

(1) $\xi \otimes \eta^* \in \mathcal{B}(H)$ *and* $\|\xi \otimes \eta^*\|_{op} = \|\xi\| \|\eta\|$.

(2) $\operatorname{Ran}(\xi \otimes \eta^*) = \operatorname{span}\{\xi\}$ *when* $\eta \neq 0$. *In particular,* $\operatorname{rank}(\xi \otimes \eta^*) = 1$ *when* $\eta, \xi \neq 0$.

(3) $(\xi \otimes \eta^*)^* = \eta \otimes \xi^*$.

(4) $(\xi \otimes \eta^*) \circ (\zeta \otimes \omega^*) = \langle \zeta, \eta \rangle (\xi \otimes \omega^*)$.

(5) $T \circ (\xi \otimes \eta^*) = (T\xi) \otimes \eta^*$.

(6) $(\xi \otimes \eta^*) \circ T = \xi \otimes (T^* \eta)^*$.

(7) *If H is finite-dimensional and $\xi_1, \xi_2, \ldots, \xi_n$ is an orthonormal basis for H, then* $\mathcal{B}(H) = \operatorname{span}\{\xi_i \otimes \xi_j^* | 1 \leq i, j \leq n\}$ *and* $\sum_{i=1}^n \xi_i \otimes \xi_i^* = \operatorname{Id}_H$.

Proof. Parts (1) through (6) are routine calculations.

(1) Let $x, y \in H$ and $\lambda, \mu \in \mathbb{C}$. Then $\xi \otimes \eta^*(\lambda x + \mu y) = \langle \lambda x + \mu y, \eta \rangle \xi = \lambda \langle x, \eta \rangle \xi + \mu \langle y, \eta \rangle \xi = \lambda(\xi \otimes \eta^*(x)) + \mu(\xi \otimes \eta^*(y))$. By Cauchy-Schwarz $\|\xi \otimes \eta^*(x)\| = \|\langle x, \eta \rangle \xi\| = |\langle x, \eta \rangle| \|\xi\| \leq \|x\| \|\eta\| \|\xi\|$, so $\|\xi \otimes \eta^*\|_{op} \leq \|\xi\| \|\eta\|$. On the other hand, $\|\xi \otimes \eta^*(\eta)\| = \|\langle \eta, \eta \rangle \xi\| = \|\eta\|^2 \|\xi\|$, hence $\|\xi \otimes \eta^*\|_{op} = \|\xi\| \|\eta\|$.

(2) Clearly $\xi \otimes \eta^*(x) \in \operatorname{span}\{\xi\}$. And if $\eta, \xi \neq 0$, then we saw in part (1) that $\xi \otimes \eta^*(\eta) \neq 0$. Since $\operatorname{span}\{\xi\}$ is one-dimensional we have $\operatorname{Ran}(\xi \otimes \eta^*) = \operatorname{span}\{\xi\}$.

(3) $\langle \xi \otimes \eta^*(x), y \rangle = \langle \langle x, \eta \rangle \xi, y \rangle = \langle x, \eta \rangle \langle \xi, y \rangle = \langle x, \overline{\langle \xi, y \rangle} \eta \rangle = \langle x, \langle y, \xi \rangle \eta \rangle = \langle x, \eta \otimes \xi^*(y) \rangle$.

(4) $(\xi \otimes \eta^*) \circ (\zeta \otimes \omega^*)(x) = (\xi \otimes \eta^*)(\langle x, \omega\rangle \zeta) = \langle x, \omega\rangle \xi \otimes \eta^*(\zeta) = \langle x, \omega\rangle\langle \zeta, \eta\rangle \xi = \langle \zeta, \eta\rangle \xi \otimes \omega^*(x)$.

(5) Let $T \in \mathcal{B}(H)$. Then $T(\xi \otimes \eta^*(x)) = T(\langle x, \eta\rangle \xi) = \langle x, \eta\rangle T(\xi) = (T\xi) \otimes \eta^*(x)$.

(6) $\xi \otimes \eta^*(Tx) = \langle Tx, \eta\rangle \xi = \langle x, T^*\eta\rangle \xi = \xi \otimes (T^*\eta)^*(x)$.

(7) Suppose H is finite-dimensional and let $\xi_1, \xi_2, \ldots, \xi_n$ be an orthonormal basis for H. Then for any $x \in H$ we have the Fourier expansion

$$x = \sum_{i=1}^{n} \langle x, \xi_i\rangle \xi_i = \sum_{i=1}^{n} \xi_i \otimes \xi_i^*(x),$$

so we see that $\sum_{i=1}^{n} \xi_i \otimes \xi_i^*$ is the identity operator in $\mathcal{B}(H)$. Therefore, for any $T \in \mathcal{B}(H)$ we have

$$\begin{aligned} T = T \circ (\sum_{j=1}^{n} \xi_j \otimes \xi_j^*) &= \sum_{j=1}^{n} T \circ (\xi_j \otimes \xi_j^*) = \sum_{j=1}^{n} (T\xi_j) \otimes \xi_j^* \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \langle T\xi_j, \xi_i\rangle \xi_i \right) \otimes \xi_j^* = \sum_{j=1}^{n}\sum_{i=1}^{n} \langle T\xi_j, \xi_i\rangle \xi_i \otimes \xi_j^*, \end{aligned}$$

which clearly lies in the linear span of $\{\xi_i \otimes \xi_j^* | 1 \leq i, j \leq n\}$. The last equality above follows from the following computation:

$$\begin{aligned} &\left(\sum_{i=1}^{n} \lambda_i \xi_i \right) \otimes \xi_j^*(x) = \langle x, \xi_j\rangle \left(\sum_{i=1}^{n} \lambda_i \xi_i \right) = \sum_{i=1}^{n} \lambda_i \langle x, \xi_j\rangle \xi_i \\ &= \sum_{i=1}^{n} \lambda_i \xi_i \otimes \xi_j^*(x). \end{aligned}$$

□

1.3 Elementary properties of finite-dimensional C^*-algebras

We now give a proof of the very simple fact that $\mathbb{C}$ is the only one-dimensional C^*-algebra, up to isomorphism. We shall need this at one point.

Lemma 1.3.1. *Let A be a one-dimensional C^*-algebra. Then A is unital and $A \cong \mathbb{C}$ via $\lambda \cdot 1_A \mapsto \lambda$.*

Proof. Our first observation is that if B is any (non-zero) unital C^*-algebra, then $1_B^* = 1_B$ and $\|1_B\| = 1$. Indeed, $1_B^* = 1_B 1_B^* = (1_B^*)^* 1_B^* = (1_B 1_B^*)^* = (1_B^*)^* = 1_B$, and $\|1_B\|^2 = \|1_B^* 1_B\| = \|1_B 1_B\| = \|1_B\|$ and since $1_B \neq 0$, $\|1_B\| = 1$.

Now assume that A is a one-dimensional C^*-algebra. Then there is an $a \in A, a \neq 0$ such that $A = \{\lambda \cdot a | \lambda \in \mathbb{C}\}$. Then $a^2 = \mu a$ for some scalar μ. $\mu \neq 0$, because if $\mu = 0$, then $\|a\|^2 = \|a^* a\| = \|(\alpha a)a\| = \|\alpha \cdot 0\| = 0$, which is a contradiction since $a \neq 0$. We claim that $\mu^{-1}a$ is the unit in A. Indeed, for any $b = \lambda a \in A$, $b(\mu^{-1}a) = \lambda\mu^{-1}a^2 = \lambda a = b$. Similarly, $(\mu^{-1}a)b = b$. So $\mu^{-1}a = 1_A$. By noting that $A = \{\lambda \cdot 1_A | \lambda \in \mathbb{C}\}$, since $1_A \neq 0$ and A is one-dimensional, it is easy to verify that the map $\phi \colon A \to \mathbb{C}$ defined by $\phi(\lambda \cdot 1_A) = \lambda$ is a unital isometric $*$-isomorphism. □

The next result shows that $\oplus_{n=1}^{K}\mathbb{C}$ are the only unital commutative C^*-algebras.

Lemma 1.3.2. *Let X be a compact Hausdorff topological space. If $C(X)$ is finite-dimensional, then X is a finite set.*

Proof. We prove the contrapositive. Assume that X is infinite. We will show that $C(X)$ is infinite-dimensional by finding n linearly independent functions in $C(X)$ for an arbitrary $n \in \mathbb{N}$.

Since X is an infinite set we can pick n distinct points in X, say $x_1, \ldots, x_n$. As X is Hausdorff we can, by induction, find open neighbourhoods A_i of x_i such that A_i and A_j are disjoint whenever $i \neq j$.

Since X is compact Hausdorff, X is normal. For each $1 \leq i \leq n$, $\{x_i\}$ and $X \setminus A_i$ are disjoint closed sets. We now appeal to Urysohn's Lemma for the existence of Urysohn functions $f_i \in C(X)$ such that $f_i(x_i) = 1$ and $f_i \equiv 0$ on $X \setminus A_i$. If $0 \equiv a_i f_i + \ldots + a_n f_n$, then by evaluating the right hand side in x_i we get that $a_i = 0$. Hence the set $\{f_1, \ldots, f_n\}$ are linearly independent in $C(X)$. □

Proposition 1.3.3. *Let A be a unital finite-dimensional C^*-algebra. Then*

(1) *Every normal element in A has finite spectrum, i.e. spec(a) is a finite set.*

(2) *Every element in A can be written as a linear combination of projections.*

Proof. Let a be a normal element in A. By the Gelfand transform we know that $C(spec(a)) \cong C^*(a, 1_A)$. The latter is a C^*-subalgebra of A and is therefore finite-dimensional. But then $C(spec(a))$ is also finite-dimensional, and it follows from Lemma 1.3.2 that $spec(a)$ is finite.

Since $spec(a)$ is a finite subset of $\mathbb{C}$, it carries the discrete topology. For each $\lambda \in spec(a)$, let $\chi_{\{\lambda\}} \colon spec(a) \to \mathbb{C}$ denote the characteristic function of $\{\lambda\}$. Then $\chi_{\{\lambda\}} \in C(spec(a))$. Since $\chi_{\{\lambda\}}$ is a projection in $C(spec(a))$, we have that $\chi_{\{\lambda\}}(a)$ is a projection in A, by the functional calculus. We also have that for every $z \in spec(a)$

$$\sum_{\lambda \in spec(a)} \lambda \chi_{\{\lambda\}}(z) = z$$

Now, for the function $f(z) = z$ in $C(spec(a))$ we have that $f(a) = a$. So, again by the functional calculus, we obtain

$$\sum_{\lambda \in spec(a)} \lambda \chi_{\{\lambda\}}(a) = a.$$

So we can write any normal element as a linear combination of projections. If b is any element in A, we can write $b = c + id$ where c and d are self-adjoints. Then c and d are linear combinations of projections, and therefore b is as well. $\square$

This shows that in a unital finite-dimensional C^*-algebra we have, in a sense, lots of projections. For a general C^*-algebra it need not be so. The trivial projections are p = 0, and $p = 1$ (if the C^*-algebra is unital). There are C^*-algebras with no nontrivial projections at all. An example of this is $C(X)$ where X is a compact, Hausdorff and connected topological space, e.g. $C(\mathbb{T})$.

1.4 Minimal projections

Definition 1.4.1. Let A be a C^*-algebra. Define a relation on the projections in A by $p \geq q$ if $pq = q$.

Note that when p and q are projections, then $pq = q$ if and only if $qp = q$, by taking adjoints. Also, we write $p > q$ when $p \geq q$ and $p \neq q$.

Lemma 1.4.2. *The relation $\geq$ in Definition 1.4.1 is a partial ordering.*

Proof. Let A be a C^*-algebra, and let p, q and r be projections in A. Since $p^2 = p$ we have $p \geq p$, so $\geq$ is reflexive. If $p \geq q$ and $q \geq p$, then $q = pq = qp = p$, so $\geq$ is antisymmetric. And if $p \geq q$ and $q \geq r$, then $pr = p(qr) = (pq)r = qr = r$ which means that $p \geq r$, so $\geq$ is transitive. $\square$

Lemma 1.4.3. *Let A be a C^*-algebra.*

(1) *If p and q are projections in A, then $p \geq q$ if and only if $pAp \supseteq qAq$. In particular $pAp = qAq$ implies that $p = q$.*

(2) *pAp is a unital C^*-subalgebra of A with $1_{pAp} = p$.*

(3) *If $1 < \dim(pAp) < \infty$, then there exists a non-zero projection $q \in pAp$ such that $p > q$.*

(4) *If A is unital and finite-dimensional, then A has minimal non-zero projections with respect to the ordering $\geq$.*

Proof. (1) If $p \geq q$, then $pq = q = qp$, which implies that $qAq = p(qAq)p \subseteq pAp$. Conversely, if $qAq \subseteq pAp$, then $qqq \in pAp$, so $q = qqq = pap$ for some $a \in A$. But then $pq = p(pap) = p^2ap = pap = q$, so $p \geq q$. The latter statement now follows from the fact that $\geq$ is a partial ordering.

(2) Consider $pAp = \{pap | a \in A\}$. We first show that pAp is closed under addition, scalar multiplication, multiplication and adjoints. Let $a, b \in A$ and $\lambda \in \mathbb{C}$.

Then,

$$\begin{aligned}
pap + pbp &= p(ap + bp) = p(a + b)p \in ApA,\\
\lambda(pap) &= p(\lambda a)p \in pAp,\\
(pap)(pbp) &= pap^2bp = p(apb)p \in pAp,\\
(pap)^* &= p^*a^*p^* = pa^*p \in pAp.
\end{aligned}$$

So pAp is a $*$-subalgebra of A. Sub-multiplicativity, i.e. $\|ab\| \leq \|a\|\|b\|$, implies that left and right multiplication by a fixed element is a continuous operation. Therefore, if we have a sequence $(pa_np)_{n=1}^{\infty}$ in pAp such that $pa_np \longrightarrow b$ for some $b \in A$, then $p^2a_np^2 = pa_np \longrightarrow pbp$. By uniqueness of limits we get that $b = pbp \in pAp$. This shows that pAp is closed in A, which is complete, hence pAp is complete. So pAp is a C^*-subalgebra of A. Furthermore, we observe that $p = ppp \in pAp$ and that $p(pap) = p^2ap = pap = (pap)p$. So p is the unit in pAp.

(3) Consider the functions $f(z) = z$ and $g(z) \equiv \lambda$, where $\lambda \in \mathbb{C}$. Since f and g are polynomials in z (and $\overline{z}$) we know from the functional calculus that $f(a) = a$ and $g(a) = \lambda \cdot 1$ for a normal element a in a unital C^*-algebra. We claim that if a is a self-adjoint element in pAp whose spectrum consists of a single point, then a is a multiple of p. Indeed, if a is a self-adjoint element in pAp and $spec(a) = \{\lambda\}$, then $f = g$ as elements in $C(spec(a))$. And it follows that $a = f(a) = g(a) = \lambda \cdot 1_{pAp} = \lambda p$. Since any C^*-algebra is spanned by its self-adjoint elements, it follows that if all the self-adjoint elements in pAp has one point spectrum, then pAp is in fact one dimensional and spanned by p.

Now, assume $1 < \dim(pAp) < \infty$. Since pAp is not one dimensional, there exists a self-adjoint element $a \in pAp$ such that $|spec(a)| \geq 2$. Now let $f \colon spec(a) \to \{0, 1\}$ be any surjective function. By Proposition 1.3.3, $spec(a)$ is finite, and therefore f is continuous. Let $q = f(a)$. q is a projection in pAp since f is a projection in $C(spec(a))$. q is also non-zero because f is non-zero. Since p is the unit in pAp we get that $pq = q$, so $p \geq q$. Lastly, to see that $q \neq p$, define $g \colon spec(a) \to \mathbb{C}$ by $g(z) \equiv 1$. Of course, g is continuous as well, and $g(a) = 1 \cdot 1_{pAp} = p$. Since $f \neq g$ as functions on $spec(a)$, we get that $q = f(a) \neq g(a) = p$.

(4) Let A be a (non-zero) unital finite-dimensional C^*-algebra. Then A contains non-zero projections because the unit is a non-zero projection. Now let p be any projection in A. Since p is non-zero and A is finite-dimensional, $1 \leq \dim(pAp) < \infty$. And if $\dim(pAp) > 1$ we can use (3) to find a non-zero projection q such that $q < p$. Since $q \neq p$, but $p \geq q$ we get that qAq is a proper linear subspace of pAp, and thus $\dim(qAq) < \dim(pAp)$. By continuing this way, we will after a finite number of steps end up with a non-zero projection q such that $\dim(qAq) = 1$. We claim that q is a minimal non-zero projection. Indeed, if r is a non-zero projection with $q \geq r$, then $qAq \supseteq rAr$. Now, rAr is a linear subspace of qAq and $rAr \neq \{0\}$ because r is non-zero. But qAq is 1-dimensional, and therefore $rAr = qAq$ which means that $r = q$ by part (1). □

Observe that when A is a unital finite-dimensional C^*-algebra, then the minimal non-zero projections in A are exactly those that satisfy $\dim(pAp) = 1$. Since p is the unit in pAp, we have $pAp = \mathbb{C} \cdot p \cong \mathbb{C}$ by Lemma 1.3.1.

Lemma 1.4.4. *Let A be a unital finite-dimensional C^*-algebra. If $p_1, \ldots, p_K$ are minimal non-zero projections in A satisfying $p_k A p_l = 0$ whenever $k \neq l$, then $p_1, \ldots p_K$ are linearly independent.*

Proof. Assume, for the sake of contradiction, that $p_1, p_2, \ldots, p_K$ are linearly dependent. Then, by possibly reordering the p_k's, we have $p_1 = \sum_{k=2}^{K} \alpha_j p_j$. But then

$$\mathbb{C} \cong \mathbb{C} \cdot p_1 = p_1 A p_1 = p_1 A \left(\sum_{k=2}^{K} \alpha_k p_k \right) = \sum_{k=2}^{K} \alpha_k (p_1 A p_k) = 0.$$

And we have arrived at a contradiction. □

Henceforth we shall abbreviate *minimal non-zero projection* as MNP.

Definition 1.4.5. A finite, non-empty, set of MNPs satisfying the hypothesis of Lemma 1.4.4 is called *independent*.

Let us briefly summarize what we have learned from the last two results. As per usual, A is a unital finite-dimensional C^*-algebra. Lemma 1.4.3 guarantees the existence of MNPs in A and Lemma 1.4.4 shows that an independent set of MNPs cannot contain more elements than the dimension of A. The next results tells us that there always exists a (finite) *maximal* independent set of MNPs. By maximal we mean an independent set which is not properly contained in any other independent set.

Lemma 1.4.6. *Let A be a unital finite-dimensional C^*-algebra. Then there exists MNPs $p_1, p_2, \ldots, p_K$ in A such that $\{p_1, p_2, \ldots, p_K\}$ is a maximal independent set.*

Proof. Consider all independent sets of MNP's in A. Any such set has at least one element and at most $\dim(A)$ elements. Since A is finite-dimensional, we can pick an independent set, say $\{p_1, \ldots, p_K\}$, having largest cardinality. Clearly this set is maximal, because there are no independent sets with more than K elements. □

When proving the existence of a maximal element, one usually resorts to Zorn's Lemma. We could have done this here as well, but that would be to crack a nut with a sledgehammer (since we are in a finite-dimensional setting).

It will be apparent from the proof of Proposition 1.5.3 that K, the cardinality of a maximal independent set, always will be equal to $\dim(Z(A))$. It may very well happen that a singleton set $\{p\}$ is a maximal independent set, namely when $\dim(Z(A)) = 1$, where p is a MNP in A. Note that $\{p\}$ will always be independent because the condition $p_k A p_l = 0$ for $k \neq l$ is vacuously true. For example if $A = M_N(\mathbb{C})$, then the maximal independent sets consists of only one element. An example of a MNP in $M_2(\mathbb{C})$ is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

1.5 The proof of the structure theorem

The next lemma is the key result in proving Theorem 1.1.1.

Lemma 1.5.1. *Let A be a unital finite-dimensional C^*-algebra and let $\{p_1, p_2, \ldots, p_K\}$ be a maximal independent set of MNPs in A.*

(1) *For each $1 \leq k \leq K$, Ap_k is a finite-dimensional Hilbert space when equipped with the inner product implicitly defined by $\langle a, b\rangle p_k = b^*a$ for $a, b \in Ap_k$. Furthermore, the induced inner product norm coincides with the norm on A.*

(2) *For each $1 \leq k \leq K$, $Ap_kA = \mathrm{span}\{ap_ka' | a, a' \in A\}$ is a unital C^*-subalgebra of A.*

(3) *$A = \sum \oplus_{k=1}^{K} Ap_kA$, by which we mean that the sum is direct.*

(4) *For each $1 \leq k \leq K$, define $\pi_k \colon A \to \mathcal{B}(Ap_k)$ by $(\pi_k(a))(b) = ab$ for $a \in A$ and $b \in Ap_k$. Then*

   (i) *π_k is a unital $*$-homomorphism.*

   (ii) *${\pi_k}\big|_{Ap_lA} \equiv 0$ for $l \neq k$.*

   (iii) *${\pi_k}\big|_{Ap_kA}$ is an (isometric) $*$-isomorphism.*

Proof. (1) Fix some $1 \leq k \leq K$. Clearly $Ap_k = \{ap_k | a \in A\}$ is a (finite-dimensional) linear subspace of A. Let $a, b \in Ap_k$. Observe that $ap_k = a$ since p_k is idempotent. Using this we get $b^*a = (bp_k)^*(ap_k) = p_k^* b^* a p_k = p_k b^* a p_k \in p_k A p_k = \mathbb{C}p_k$ since p_k is a MNP. Hence there exists a unique scalar $\alpha_{a,b}$ such that $\alpha_{a,b}p_k = b^*a$. We define $\langle a, b\rangle = \alpha_{a,b}$. We now confirm that this is an inner product on Ap_k.

Linearity: $\langle \lambda a, b\rangle p_k = b^*(\lambda a) = \lambda(b^*a) = \lambda\langle a, b\rangle p_k$ and $\langle a + a', b\rangle p_k = b^*(a + a') = b^*a + b^*a' = \langle a, b\rangle p_k + \langle a', b\rangle p_k = (\langle a, b\rangle + \langle a', b\rangle)p_k$.

Conjugate symmetry: $\langle b, a\rangle p_k = a^*b = (b^*a)^* = (\langle a, b\rangle p_k)^* = \overline{\langle a, b\rangle} p_k^* = \overline{\langle a, b\rangle} p_k$.

Positive definiteness: $\langle a, a\rangle p_k = a^*a = (a^*a)^* = \overline{\langle a, a\rangle} p_k$, which implies that $\langle a, a\rangle \in \mathbb{R}$. Let $\alpha = \langle a, a\rangle$. Then $a^*a = \alpha p_k$ and $spec(a^*a) = spec(\alpha p_k) = \alpha spec(p_k)$ by the Spectral Mapping Theorem. Both a^*a and $p_k = p_k^* p_k$ are positive elements and thus $spec(a^*a), spec(p_k) \subseteq [0, \infty)$. If $\alpha < 0$, then the above implies that $spec(a^*a) = spec(p_k) = \{0\}$, but p_k is non-zero so this is a contradiction, hence $\alpha \geq 0$. Lastly, $\langle a, a\rangle = 0 \implies a^*a = 0 \cdot p_k = 0 \implies a = 0$ by the C^*-identity.

Next, observe that

$$\|a\|^2 = \|a^*a\| = \|\langle a, a\rangle p_k\| = |\langle a, a\rangle| \|p_k\| = \langle a, a\rangle \cdot 1 = \langle a, a\rangle.$$

So the induced inner product norm on Ap_k coincides with the norm induced from A. Also, Ap_k is finite-dimensional, and therefore complete, hence Ap_k is a finite-dimensional Hilbert space.

(2) $Ap_kA = \operatorname{span}\{ap_ka' | a, a' \in A\}$ is by definition a linear subspace of A. Ap_kA is also closed under multiplication and adjoints. Indeed, $(ap_ka')^* = (a')^*p_ka^*$, hence the adjoint of a linear combination of such elements is again in Ap_kA. Similarly, $(ap_ka')(bp_kb') = ap_k(a'bp_kb')$, hence the product of linear combinations of such elements is again in Ap_kA. Thus Ap_kA is a $*$-subalgebra of A. Since A is finite-dimensional, Ap_kA is closed and hence a C^*-subalgebra.

We proceed to construct the unit in Ap_kA. To this end, let B_k be an orthonormal basis for Ap_k. Then we have that

$$a = \sum_{b\in B_k} \langle a, b\rangle b = \sum_{b\in B_k} \langle a, b\rangle bp_k = \sum_{b\in B_k} b(\langle a, b\rangle p_k) = \sum_{b\in B_k} bb^*a = \left(\sum_{b\in B_k} bb^*\right) a$$

for any $a \in Ap_k$. Therefore it is natural to define

$$q_k := \sum_{b\in B_k} bb^* = \sum_{b\in B_k} bp_kb^* \in Ap_kA.$$

By the above calculation we have $q_ka = a$ for any $a \in Ap_k$. Let $a, a' \in A$. We compute

$$q_k(ap_ka') = (q_kap_k)a' = ap_ka'.$$

Observe that q_k is self-adjoint since it is a sum of self-adjoint elements. This gives

$$\begin{aligned}(ap_ka')q_k &= a(p_ka'q_k) = ((p_ka'q_k)^*a^*)^* = ((q_k^*(a')^*p_k^*)a^*)^* \\ &= ((q_k(a')^*p_k)a^*)^* = ((a')^*p_ka^*)^* = ap_ka'.\end{aligned}$$

Since Ap_kA is spanned by elements of the form ap_ka' we see that q_k is the unit in Ap_kA. In particular, $p_k \in Ap_kA$, so we have $q_kp_k = p_k$, i.e. $q_k \geq p_k$.

(3) Since the q_k's are units in C^*-subalgebras of A, they are projections. Let $k \neq l$. Since the p_k's are independent we have $p_kAp_l = 0$. Then $(ap_ka')(bp_lb') = a(p_k(a'b)p_l)b' = 0$, and this extends to products of such linear combinations. Thus $(Ap_kA)(Ap_lA) = 0$ as well. As $q_k \in Ap_kA$, we get $q_k(Ap_lA) = 0$. In particular $q_kq_l = 0$, i.e. the q_k's are pairwise orthogonal projections. Define $q := 1_A - \sum_{k=1}^K q_k$. Then $q^* = 1_A^* - \sum_{k=1}^K q_k^* = 1_A - \sum_{k=1}^K q_k = q$, and

$$\begin{aligned}q^2 &= \left(1_A - \sum_{k=1}^K q_k\right)^2 = \left(1_A - \sum_{k=1}^K q_k\right) - \sum_{k=1}^K q_k\left(1_A - \sum_{l=1}^K q_l\right) \\ &= \left(1_A - \sum_{k=1}^K q_k\right) - \sum_{k=1}^K \left(q_k - \sum_{l=1}^K q_kq_l\right) = \left(1_A - \sum_{k=1}^K q_k\right) - \sum_{k=1}^K (q_k - q_k) \\ &= 1_A - \sum_{k=1}^K q_k = q.\end{aligned}$$

So q is a projection. Additionally, $qq_k = 0$ for all k. We want to show that $q = 0$. Assume for the sake of contradiction that $q \neq 0$. By Lemma 1.4.3, $q \geq p$ for some MNP p. But then $pap_k = (pq)ap_k = (pq)(q_kap_k) = p(qq_k)ap_k = 0 \; \forall \; k$, i.e. $pAp_k = 0$. Similarly $p_kap = (pa^*p_k)^* = 0^* = 0$. So $p_kAp = 0$ as well. But this contradicts the maximality of $\{p_1, \ldots, p_K\}$. Thus we must have $q = 0$, which means that

$$\sum_{k=1}^{K} q_k = 1_A.$$

At this point, we should also note that $q_kA = Ap_kA$. Since $q_k \in Ap_kA$, we have $q_ka \in Ap_kA$. Hence $q_kA \subseteq Ap_kA$. On the other hand, $q_k(ap_ka') = ap_ka'$ and the same holds for linear combinations. Hence $Ap_kA \subseteq q_kA$.

For any $a \in A$, we have that $a = 1_Aa = \left(\sum_{k=1}^{K} q_k\right) a = \sum_{k=1}^{K} q_ka$. So $A = \sum_{k=1}^{K} q_kA$. This sum is in fact direct. To see this, suppose $0 = \sum_{k=1}^{K} q_ka_k$, then multiplying by q_l on the left yields $q_la_l = 0$ for each l. Thus we have

$$A = \sum \oplus_{k=1}^{K} q_kA = \sum \oplus_{k=1}^{K} Ap_kA.$$

(4) Let $a, a' \in A$ and $b, c \in Ap_k$. We first verify that $\pi_k(a) \in \mathcal{B}(Ap_k)$. Since $b \in Ap_k$, so is $\pi_k(a)b = ab$. And $\pi_k(a)$ is linear because of distributivity. Finally, $\|\pi_k(a)b\| = \|ab\| \leq \|a\|\|b\|$. Hence $\pi_k(a)$ is bounded.

Next we show that π_k is a unital $*$-homomorphism. π_k is clearly linear, and $\pi_k(aa')b = aa'b = a(a'b) = a(\pi_k(a')b) = (\pi_k(a) \circ \pi_k(a')b)$. As for the adjoint, $\langle \pi_k(a)b, c\rangle p_k = \langle ab, c\rangle p_k = c^*ab = (a^*c)^*b = \langle b, a^*c\rangle p_k = \langle b, \pi_k(a^*)c\rangle p_k$. So $\pi_k(a^*) = \pi_k(a)^*$. And $\pi_k(1_A)b = 1_Ab = b \implies \pi_k(1_A) = Id_{Ap_k}$.

We saw in part (3) that Ap_lA acts trivially on $Ap_kA \supseteq Ap_k$ for $k \neq l$. Hence $\pi_{k|Ap_lA} \equiv 0$. Finally, we must show that π_k restricts to an isomorphism on Ap_kA. Recall that B_k is our chosen orthonormal basis for Ap_k. We have $p_kA = (Ap_k)^* \subseteq \operatorname{span} B_k^*$. Since $ap_ka' = (ap_k)(p_ka')$ we must have $Ap_kA \subseteq \operatorname{span}\{bc^* | b, c \in B_k\}$. Let $b_0, c_0 \in B_k$ and let $\alpha_{b,c}$ be any scalars. Then

$$\langle \pi_k \left(\sum_{b,c \in B_k} \alpha_{b,c} bc^* \right) c_0, b_0 \rangle p_k = \sum_{b,c \in B_k} \alpha_{b,c} \langle bc^*c_0, b_0 \rangle p_k = \sum_{b,c \in B_k} \alpha_{b,c} b_0^* bc^*c_0.$$

By orthonormality of B_k we have that $b_0^*b = \langle b, b_0\rangle p_k = 0$, unless $b = b_0$, in which case it equals p_k. Similarly for c^*c_0. From this we get

$$\langle \pi_k \left(\sum_{b,c \in B_k} \alpha_{b,c} bc^* \right) c_0, b_0 \rangle p_k = \alpha_{b_0,c_0} p_k.$$

Now if $a \in Ap_kA$, then we can write $a = \sum_{b,c \in B_k} \alpha_{b,c} bc^*$ for a suitable choice of scalars $\alpha_{b,c}$. And if $\pi_k(a) \equiv 0$, then the above calculation yields $\alpha_{b,c} = 0$ for each b and c. Which means that $a = 0$. Hence $\pi_{k|Ap_kA}$ is injective. (And since π_k is unital, it is also isometric).

Now let $a, b, c \in Ap_k$. Then $ab^* \in Ap_kA$. And $\pi_k(ab^*)c = ab^*c = a(b^*c) = a\langle c, b\rangle p_k = \langle c, b\rangle ap_k = \langle c, b\rangle a = a \otimes b^*(c)$. Hence $\pi_k(ab^*) = a \otimes b^*$. In particular $b \otimes c^* \in \pi_k(Ap_kA) \; \forall \; b, c \in B_k$. By part (7) of Lemma 1.2.2, these rank-one operators span $\mathcal{B}(Ap_k)$, hence $\pi_{k|Ap_kA}$ is surjective. $\square$

Lemma 1.5.2. *The center of $M_N(\mathbb{C})$ is the scalar multiples of the identity, i.e. $Z(M_N(\mathbb{C})) = \{\lambda I_N | \lambda \in \mathbb{C}\}$. The center of $\oplus_{k=1}^K M_{N_k}(\mathbb{C})$ is isomorphic to $\mathbb{C}^K$ and is spanned by the identity elements of the summands.*

Proof. For $N = 1$ we have $Z(M_1(\mathbb{C}) = Z(\mathbb{C}) = \mathbb{C}$. So assume $N \geq 2$. We identify $M_N(\mathbb{C})$ with $\mathcal{B}(\mathbb{C}^N)$ via $a \longleftrightarrow a(\cdot)$. Suppose $a \in Z(\mathcal{B}(\mathbb{C}^N))$, i.e. $ab = ba \ \forall \ b \in \mathcal{B}(\mathbb{C}^N)$. Let $\xi \in \mathbb{C}^N$. Then by Lemma 1.2.2

$$\begin{aligned}
&(a\xi) \otimes \xi^* = a(\xi \otimes \xi^*) = (\xi \otimes \xi^*)a = \xi \otimes (a^*\xi)^* \\
\implies &\langle \xi, \xi \rangle a\xi = (a\xi) \otimes \xi^*(\xi) = \xi \otimes (a^*\xi)^*(\xi) = \langle \xi, a^*\xi \rangle \xi \\
\implies &a\xi = \frac{\langle \xi, a^*\xi \rangle}{\langle \xi, \xi \rangle} \xi \text{ whenever } \xi \neq 0.
\end{aligned}$$

So for every $\xi \in \mathbb{C}^N$ there exists a scalar $r \in \mathbb{C}$ such that $a\xi = r\xi$. Now if $\xi, \eta \in \mathbb{C}^N$ are linearly independent. Then $\exists \, r, s, t \in \mathbb{C}$ such that $r\xi + s\eta = a\xi + a\eta = a(\xi + \eta) = t(\xi + \eta) = t\xi + t\eta$. But by linear independence, $r = s = t$. So actually, $a\xi = r\xi$, and the scalar r does not depend on ξ. Hence $a = r \cdot Id$, and $Z(M_N(\mathbb{C})) \cong \mathbb{C}$.

Now consider $\oplus_{k=1}^K M_{N_k}(\mathbb{C})$. It's clear that $(a_1, \ldots, a_K)$ lies in the center of the direct sum if and only if $a_k \in Z(M_{N_k}(\mathbb{C}))$ for all $1 \leq k \leq K$. Hence

$$Z\left(\oplus_{k=1}^K M_{N_k}(\mathbb{C})\right) = \{(\lambda_1 I_{N_1}, \ldots, \lambda_K I_{N_K}) | \lambda_1, \ldots, \lambda_K \in \mathbb{C}\} \cong \mathbb{C}^K. \qquad \square$$

We now have everything we need in order to prove the structure theorem under the unital assumption.

Proposition 1.5.3. *Let A be a unital finite-dimensional C^*-algebra. Then there exists positive integers K and $N_1, \ldots, N_K$ such that*

$$A \cong M_{N_1}(\mathbb{C}) \oplus M_{N_2}(\mathbb{C}) \oplus \cdots \oplus M_{N_K}(\mathbb{C})$$

Furthermore, K is uniquely determined, and $N_1, \ldots, N_K$ are unique up to permutation.

Proof. By Lemma 1.4.6 there exists a maximal independent set $\{p_1, \ldots, p_K\}$ of MNPs. Let π_k denote the maps defined in part (4) of Lemma 1.5.1. Define

$$\pi \colon A = \oplus_{k=1}^K A p_k A \to \oplus_{k=1}^K \mathcal{B}(A p_k)$$

by $\pi(a) = (\pi_1(a), \ldots, \pi_k(a))$. Then π is a unital $*$-homomorphism because each π_k is.

Suppose $\pi(a) = 0$. We know that $a = \sum_{k=1}^K a_k$ where $a_k \in A p_k A$. Since π_k is identically zero on $A p_l A$ when $k \neq l$ we get that

$$\begin{aligned}
0 = \pi(a) &= (\pi_1(a), \ldots, \pi_k(a)) = (\pi_1\left(\sum_{k=1}^K a_k\right), \ldots, \pi_k\left(\sum_{k=1}^K a_k\right)) \\
&= (\pi_1(a_1), \ldots, \pi_K(a_K)).
\end{aligned}$$

But the π_k's are injective when restricted to Ap_kA and therefore $a_k = 0$ for each k, hence $a = 0$. So π is injective.

Let $(T_1, \ldots, T_K) \in \oplus_{k=1}^K \mathcal{B}(Ap_k)$. Since $\pi_{k|Ap_kA}$ is an isomorphism there exists $a_k \in Ap_kA$ such that $\pi_k(a_k) = T_k$ for each k. Then, by a similar computation as above, $\pi(\sum_{k=1}^K a_k) = (T_1, \ldots, T_K)$. Thus π is an isomorphism.

By part (1) of Lemma 1.5.1, for each $1 \leq k \leq K$ Ap_k is a finite-dimensional Hilbert space. Therefore $\mathcal{B}(Ap_k) \cong \mathcal{B}(\mathbb{C}^{\dim(Ap_k)}) \cong M_{\dim(Ap_k)}(\mathbb{C})$. Thus $A \cong \oplus_{k=1}^K M_{N_k}(\mathbb{C})$.

It remains to show the uniqueness of K and $\{N_1, \ldots, N_K\}$. Since an isomorphism preserves the center we get that $Z(A) \cong Z(\oplus_{k=1}^K M_{N_k}(\mathbb{C})) \cong \mathbb{C}^K$. Hence $K = \dim(Z(A))$.

Now suppose that $\oplus_{k=1}^K M_{N_k}(\mathbb{C}) \cong A \cong \oplus_{i=1}^K M_{L_i}(\mathbb{C})$. Let

$$I_k = (0, \ldots, 0, I_{N_k}, 0, \ldots, 0) \in \oplus_{k=1}^K M_{N_k}(\mathbb{C}),$$

where I_{N_k} occurs in the k'th coordinate. Define J_i in $\oplus_{i=1}^K M_{L_i}(\mathbb{C})$ similarly. Note that the I_k's are exactly the MNPs in the C^*-algebra $Z(\oplus_{k=1}^K M_{N_k}(\mathbb{C}))$, and similarly for the J_i's. Therefore, if ϕ is an isomorphism between $\oplus_{k=1}^K M_{N_k}(\mathbb{C})$ and $\oplus_{i=1}^K M_{L_i}(\mathbb{C})$, we must have $\phi(\{I_1, \ldots, I_K\}) = \{J_1, \ldots, J_K\}$, since the MNPs are preserved under isomorphism. If now $\phi(I_{k_0}) = J_{i_0}$, then we can chop down on each side by these projections to get

$$M_{N_{k_0}}(\mathbb{C}) \cong I_{k_0}\left(\oplus_{k=1}^K M_{N_k}(\mathbb{C})\right) \cong J_{i_0}\left(\oplus_{i=1}^K M_{L_i}(\mathbb{C})\right) \cong M_{L_{i_0}}(\mathbb{C}),$$

which implies that $N_{k_0} = L_{i_0}$. Doing this for each k we get $\{I_1, \ldots, I_K\} = \{J_1, \ldots, J_K\}$. This finishes the proof. $\square$

We move straight on to the proof of the general case.

Proof of Theorem 1.1.1. Let A be a (not necessarily unital) finite-dimensional C^*-algebra. Let $\tilde{A}$ denote the unitization of A. Since $\tilde{A} = A \oplus \mathbb{C}$ as vector spaces, $\dim(\tilde{A}) = \dim(A) + 1$. In particular $\tilde{A}$ is unital and finite-dimensional, so by Proposition 1.5.3 there exists a unique positive integer K and a unique set of positive integers $\{N_1, \ldots, N_K\}$ such that $\tilde{A} \cong \oplus_{k=1}^K M_{N_k}(\mathbb{C})$. We know that $A \cong A' \subseteq \tilde{A}$, where A' is a closed two-sided ideal in $\tilde{A}$ such that $\tilde{A}/A' \cong \mathbb{C}$. It follows that $A \cong A'' \subseteq \oplus_{k=1}^K M_{N_k}(\mathbb{C})$ where A'' is a closed two-sided ideal of codimension 1. However, we know from ring theory, that $M_N(\mathbb{C})$ is simple. Therefore, the only two-sided ideals in $\oplus_{k=1}^K M_{N_k}(\mathbb{C})$ is ideals of the form $\oplus_{k=1}^K \Lambda_k$ where each Λ_k is either 0 or $M_{N_k}(\mathbb{C})$. Since A'' has codimension 1 we must have that $dim\left(M_{N_{k_0}}(\mathbb{C})\right) = 1$ for some k_0, i.e., the k_0'th summand is $\mathbb{C}$, and that $A'' = \oplus_{k=1}^K \Lambda_k$ where $\Lambda_{k_0} = 0$ and $\Lambda_k = M_{N_k}(\mathbb{C})$ for $k \neq k_0$.

Thus $A \cong A''$ which is a multimatrix algebra and the number of summands of A'' is $K - 1 = \dim(Z(\tilde{A})) - 1$, and since $\tilde{A}$ is unique, the number of summands is unique. Furthermore, the summands of A'' are unique up to permutation because these coincide with the ones of $\tilde{A}$, except that one (of possibly several) of the one-dimensional summands, $\mathbb{C}$, is removed. $\square$

The following are a few simple consequences of Theorem 1.1.1.

Corollary 1.5.4. *Every finite-dimensional C^*-algebra is unital.*

Corollary 1.5.5. *Two multimatrix algebras are isomorphic if and only if one is a "permutation" of the other, that is,*

$$\oplus_{k=1}^{K} M_{N_k}(\mathbb{C}) \cong \oplus_{i=1}^{I} M_{L_i}(\mathbb{C}) \iff K = I \text{ and } \{N_k\}_{k=1}^{K} = \{L_i\}_{i=1}^{I}.$$

Corollary 1.5.6. *There are only countably many finite-dimensional C^*-algebras, up to isomorphism.*

Proof. Let S_K be the set of all multimatrix algebras with K summands, i.e. $S_K = \{\oplus_{k=1}^{K} M_{N_k}(\mathbb{C}) | N_1, \ldots, N_K \in \mathbb{N}\}$. Clearly $|S_K| = |\mathbb{N}^K|$. The set $\cup_{K=1}^{\infty} S_K$ contains every finite-dimensional C^*-algebra, up to isomorphism. It is also a countable set since it is a countable union of countable sets. $\square$

Chapter 2

Direct limits and labeled Bratteli diagrams

In the previous chapter we saw that all finite-dimensional C^*-algebras are isomorphic to multimatrix algebras. Direct limits is a standard way to create new objects from old ones. We will be interested in the "simplest" of these, namely direct limits of finite-dimensional C^*-algebras. We begin by looking at sets, then we move on to $*$-algebras, and we end with defining the C^*-algebraic direct limit of finite-dimensional C^*-algebras. This leads us to the class of C^*-algebras called AF-algebras, and they will be properly introduced in the next chapter. Labeled Bratteli diagrams will emerge naturally as a convenient way to represent certain "canonical" types of direct limits.

2.1 Direct limits of sets

We begin by considering directed sequences of sets. Suppose we have a sequence of sets X_n and maps $\phi_n\colon X_n \to X_{n+1}$. We shall call $(X_n, \phi_n)_{n\in\mathbb{N}}$ a *chain system* and represent it as a diagram in the following way:

$$X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} X_3 \xrightarrow{\phi_3} \cdots \tag{2.1}$$

The maps ϕ_n is referred to as the *connecting homomorphisms* in the chain system. Our first goal is to define the set-theoretic direct limit of such a chain system. For $m < n$ we define the maps $\phi_{mn}\colon X_m \to X_n$ by $\phi_{mn} = \phi_{n-1} \circ \phi_{n-2} \circ \cdots \circ \phi_m$, and we define $\phi_{mm} = \mathrm{Id}_{X_m}$. So ϕ_{mn} takes us from X_m to X_n by sequentially applying the maps in the diagram (2.1). Now let

$$\bigsqcup_{n=1}^{\infty} X_n = \bigcup_{n=1}^{\infty} \{(x,n) | x \in X_n\}$$

be the "labeled" disjoint union of the sets X_n. Next, we define $\mathcal{R}$ to be the smallest equivalence relation on $\bigsqcup_{n=1}^{\infty} X_n$ such that $(x,n)\mathcal{R}(\phi_n(x), n+1)$ for all

$(x,n) \in \bigsqcup_{n=1}^{\infty} X_n$, i.e. each $x \in X_n$ is related to its image by ϕ_n in X_{n+1}. To get a firm grip of the equivalence relation $\mathcal{R}$ we will prove that $\mathcal{R}$ can be explicitly characterized as follows; If $x \in X_n$ and $x' \in X_m$ then $(x,n)\mathcal{R}(x',m)$ if and only if x and x' eventually coincide by following the diagram in (2.1). More precisely, we have the following lemma, which will turn out to be very useful in proofs.

Lemma 2.1.1. *Let $(X_n, \phi_n)_{n\in\mathbb{N}}$ be a chain system and let $\mathcal{R}$ be the equivalence relation on $\bigsqcup_{n=1}^{\infty} X_n$ defined above. If $x \in X_n$ and $x' \in X_m$ then $(x,n)\mathcal{R}(x',m)$ if and only if there exists an integer $k \geq \max\{m,n\}$ such that $\phi_{nk}(x) = \phi_{mk}(x')$.*

Proof. Define another relation, $\sim$, on $\bigsqcup_{n=1}^{\infty} X_n$ by $(x,n) \sim (x',m)$ if there exists an integer $k \geq \max\{m,n\}$ such that $\phi_{nk}(x) = \phi_{mk}(x')$. By definition, $\sim$ is reflexive and symmetric. To show transitivity, suppose that $(x,n) \sim (y,m)$ and $(y,m) \sim (z,l)$. Then there exists k_1, k_2 such that $\phi_{nk_1}(x) = \phi_{mk_1}(y)$ and $\phi_{mk_2}(y) = \phi_{lk_2}(z)$. If $k_1 = k_2$, then $x \sim z$ and we are done. So assume, without loss of generality, that $k_2 < k_1$. We wish to have x and z "coinciding" in X_{k_1}, see the following diagram:

$$\begin{tikzcd}
y \in X_m \arrow[r, "\phi_{mk_2}"] \arrow[rr, bend left, "\phi_{mk_1}"] & X_{k_2} \arrow[r, "\phi_{k_2k_1}"] & X_{k_1} \\
z \in X_l \arrow[ur, "\phi_{lk_2}"'] & x \in X_n \arrow[ur, "\phi_{nk_1}"'] &
\end{tikzcd}$$

We have $\phi_{nk_1}(x) = \phi_{mk_1}(y) = \phi_{k_2k_1} \circ \phi_{mk_2}(y) = \phi_{k_2k_1} \circ \phi_{lk_2}(z) = \phi_{lk_1}(z)$, which means that $x \sim z$. Thus $\sim$ is an equivalence relation.

Clearly $(x,n) \sim (\phi_n(x), n+1)$, and since $\mathcal{R}$ is the smallest equivalence relation satisfying this, we get $\mathcal{R} \subseteq \sim$. On the other hand, if $(x,n) \sim (x',m)$, then there is a $k \geq \max\{m,n\}$ such that $\phi_{nk}(x) = \phi_{mk}(x')$. But we also have that

$$\begin{aligned}
&(x,n)\mathcal{R}(\phi_n(x), n+1)\mathcal{R}\cdots\mathcal{R}(\phi_{nk}(x),k)\mathcal{R}(\phi_{mk}(x'),k)\mathcal{R} \\
&(\phi_{m(k-1)}(x'), k-1)\mathcal{R}\cdots\mathcal{R}(\phi_m(x'), m+1)\mathcal{R}(x',m)
\end{aligned}$$

And then $(x,n)\mathcal{R}(x',m)$ since $\mathcal{R}$ is an equivalence relation. Hence we get $\sim \subseteq \mathcal{R}$, so $\sim = \mathcal{R}$. ☐

We define $\varinjlim X_n = \left(\bigsqcup_{n=1}^{\infty} X_n\right)/\mathcal{R}$ to be the set of equivalence classes and denote the equivalence class of (x,n) under $\mathcal{R}$ by $[x,n]$. We say that $\varinjlim X_n$ is the *direct limit* of the chain system (2.1). To get more compact notation we often write $\varinjlim X_n = X_\infty$. If we wish to emphasize the maps ϕ_n, we shall write $\varinjlim(X_n, \phi_n)$ for the direct limit. For each $n \in \mathbb{N}$, define the map $\phi_{n\infty}\colon X_n \to X_\infty$ by $\phi_{n\infty}(x) = [x,n]$ for $x \in X_n$. I.e. $\phi_{n\infty}$ maps x to its equivalence class in X_∞.

A *chain system map* from a chain system (X_n, ϕ_n) to another chain system (Y_n, ψ_n) is a sequence of maps $\theta_n\colon X_n \to Y_n$ such that the following diagram commutes:

$$\begin{tikzcd}
X_1 \arrow[r, "\phi_1"] \arrow[d, "\theta_1"] & X_2 \arrow[r, "\phi_2"] \arrow[d, "\theta_2"] & X_3 \arrow[r, "\phi_3"] \arrow[d, "\theta_3"] & \cdots \\
Y_1 \arrow[r, "\psi_1"] & Y_2 \arrow[r, "\psi_2"] & Y_3 \arrow[r, "\psi_3"] & \cdots
\end{tikzcd}$$

Given such a chain system map, we get an induced map $\theta_\infty\colon X_\infty \to Y_\infty$ on the direct limits defined by $\theta_\infty([x,n]) = [\theta_n(x), n]$. The well-definedness of θ_∞ follow from the commutativity of the diagram above.

Suppose we have a chain system (X_n, ϕ_n) and subsets $W_n \subseteq X_n$ such that $\phi_n(W_n) \subseteq W_{n+1}$ for each $n \in \mathbb{N}$. Then we can identify $\bigcup_{n=1}^{\infty} \phi_{n\infty}(W_n)$ with $\varinjlim(W_n, \phi_{n|W_n})$ in a natural way, because the equivalence classes in either setting will be exactly the same.

It should also be noted that *telescoping*, i.e. passing to a subsequence, produces bijective limits. Namely, given positive numbers $n_1 < n_2 < n_3 < \ldots$, we telescope the diagram (2.1) by"collapsing" the sets and maps between X_{n_k} and $X_{n_{k+1}}$. This is illustrated in the following diagram:

$$X_{n_1} \xrightarrow{\phi_{n_1 n_2}} X_{n_2} \xrightarrow{\phi_{n_2 n_3}} X_{n_3} \xrightarrow{\phi_{n_3 n_4}} \cdots$$

There is a natural correspondence between $\varinjlim X_{n_k}$ and $\varinjlim X_n$ because each equivalence class in X_∞ correspond to the same one in $\varinjlim X_{n_k}$, where the elements not coming from the X_{n_k}'s are removed.

To simplify our notation we shall write $x \mapsto y$ if $y = \phi_{mn}(x)$, and $x \to y$ if $y = \phi_{m\infty}(x)$, where it is understood that $x \in X_m, y \in X_n, n \geq m$ and $x \in X_n, y \in X_\infty$ respectively.

2.2 Direct limits of $*$-algebras

Throughout the rest of this chapter an *algebra* will mean a complex $*$-algebra, a *subalgebra* will mean a $*$-subalgebra, and by a *homomorphism* will mean a $*$-homomorphism, i.e. a homomorphism which preserves the $*$-operation. If A is a unital algebra, then each unitary $u \in A$ defines an automorphism, $\operatorname{Ad} u\colon A \to A$ by $\operatorname{Ad} u(a) = uau^*$. If γ is an automorphism of A, then γ is called an *inner automorphism* if $\gamma = \operatorname{Ad} u$ for some unitary $u \in A$.

For matrix algebras, we will often use M_n as shorthand for $M_n(\mathbb{C})$. More generally for multimatrix algebras, if $\vec{p} = (p_1, p_2, \ldots, p_r) \in \mathbb{N}^r$, then we let $M(\vec{p}) = M_{p_1} \oplus M_{p_2} \oplus \cdots \oplus M_{p_r}$. We say that an algebra is *semisimple* if it is isomorphic to $M(\vec{p})$ for some $\vec{p} \in \mathbb{N}^r$.

We make a small remark on this choice of terminology. A ring is usually called semisimple if it is semisimple as a module over itself. And the Artin-Wedderburn Theorem implies that any finite-dimensional algebra (over $\mathbb{C}$) which is semisimple as a ring is isomorphic to $M(\vec{p})$ for some $\vec{p} \in \mathbb{N}^r$. Since we are ultimately interested in direct limits of finite-dimensonal C^*-algebras, which are isomorphic to multimatrix algebras by Theorem 1.1.1, these two notions of semisimplicity coincide for our purposes.

Consider a chain system (A_n, ϕ_n) of algebras and homomorphisms:

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \cdots \tag{2.2}$$

We wish to turn $A_\infty = \varinjlim A_n$ into an algebra. The operations in A_∞ are defined on the equivalence classes as follows. Let $a = [a_n, n]$ and $b = [b_m, m]$ be elements in A_∞ and let $\lambda \in \mathbb{C}$. We define λa by $\lambda a_n \to \lambda a$, i.e. $\lambda a = \phi_{n\infty}(\lambda a_n)$. Similarly $a_n^* \to a^*$. If $m \geq n$, let $a_n \mapsto a'_m \in A_m$. Define $a + b$ by $a'_m + b_m \to a + b$ and define ab by $a'_m b_m \to ab$. And if $m < n$, do the same with $b_m \to b'_n \in A_n$.

Lemma 2.2.1. *The operations defined above are well defined and they turn A_∞ into an algebra.*

Proof. First of all, it suffices to show that the operations are well defined. Once this is established the algebra axioms follow automatically because all operations are actually done in the A_n's, which are algebras. We show that the adjoint and multiplication are well defined. Scalar multiplication and addition are treated in exactly the same way.

Suppose $a = [a_n, n] = [a_m, m] \in A_\infty$. Then $\phi_{nk}(a_n) = \phi_{mk}(a_m)$ for some $k \in \mathbb{N}$ by Lemma 2.1.1. We compute:

$$\begin{aligned}\phi_{n\infty}(a_n^*) = [a_n^*, n] &= [\phi_{nk}(a_n^*), k] = [\phi_{nk}(a_n)^*, k] = [\phi_{mk}(a_m)^*, k] \\ &= [\phi_{mk}(a_m^*), k] = [a_m^*, m] = \phi_{m\infty}(a_m^*).\end{aligned}$$

Hence a^* is well defined. Now let $b = [b_l, l] \in A_\infty$. Assume first that $l \geq \max\{m, n\}$. k can be chosen larger than l, because if it isn't we just compose ϕ_{nk} with ϕ_{kl} and use l as our new k. Let $a_n \mapsto a_l \in A_l$ and $a_m \mapsto a'_l \in A_l$. Then

$$\begin{aligned}\phi_{l\infty}(a_l b_l) &= [a_l b_l, l] = [\phi_{lk}(a_l b_l), k] = [\phi_{lk}(a_l)\phi_{lk}(b_l), k] \\ &= [\phi_{lk}(a'_l)\phi_{lk}(b_l), k] = [\phi_{lk}(a'_l b_l), k] = [a'_l b_l, l] \\ &= \phi_{l\infty}(a'_l b_l).\end{aligned}$$

If $l < \max\{m, n\}$, then we use $b_l \mapsto b_L$ for a large enough L and the conclusion remains the same. Next, if we pick two different representatives for b, then we still get the same result by a computation similar to the above. Hence ab is well defined. $\square$

The proof above is somewhat messy, as is usual when one deals with equivalence classes. The idea however, is clear. Namely, "to do an operation in A_∞, pick representative(s), do the operation on the representative(s) and then the result is the resulting equivalence class". We say that $A_\infty = \varinjlim(A_n, \phi_n)$, when equipped with the operations above, is the *algebraic direct limit* of the chain system (2.2). We now define the algebraic precursor to the C^*-algebraic direct limits we are interested in.

Definition 2.2.2. An algebra A is called *locally semisimple* if it is isomorphic to a direct limit of semisimple algebras, i.e. $A \cong \varinjlim A_n$ where the A_n's are semisimple.

We shall see in Section 2.5 that each element in a locally semisimple algebra is contained in a semisimple subalgegra. We say that two algebraic chain systems (A_n, ϕ_n) and (B_n, ψ_n) are *isomorphic* (as chain systems) if there exists isomorphisms $\gamma_n \colon A_n \to B_n$ such that the following diagram commutes:

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\phi_1} & A_2 & \xrightarrow{\phi_2} & A_3 & \xrightarrow{\phi_3} & \cdots \\ \downarrow{\scriptstyle\gamma_1} & & \downarrow{\scriptstyle\gamma_2} & & \downarrow{\scriptstyle\gamma_3} & & \\ B_1 & \xrightarrow{\psi_1} & B_2 & \xrightarrow{\psi_2} & B_3 & \xrightarrow{\psi_3} & \cdots \end{array}$$

Note that $(\gamma_n)_{n\in\mathbb{N}}$ is then a chain system map consisting of isomorphisms. It is readily checked that isomorphic chain systems have isomorphic direct limits, and that the induced map $\gamma_\infty\colon A_\infty \to B_\infty$ is an isomorphism.

Nevertheless, one must be careful. Even if (A_n, ϕ_n) and (A_n, ψ_n) are two algebraic chain systems such that, for each n, ϕ_n and ψ_n are *automorphically equivalent*, i.e. there exists automorphisms α_n and β_n of A_n and A_{n+1} respectively, such that the following diagram commutes:

$$\begin{tikzcd} A_n \arrow[r, "\phi_n"] \arrow[d, "\alpha_n"] & A_{n+1} \arrow[d, "\beta_n"] \\ A_n \arrow[r, "\psi_n"] & A_{n+1} \end{tikzcd}$$

it does not necessarily follow that $\varinjlim(A_n, \phi_n) \cong \varinjlim(A_n, \psi_n)$. We shall soon see that if the automorphisms are inner, then we do actually get isomorphic direct limits.

Definition 2.2.3. Let A and B be unital algebras. We say that two homomorphisms $\phi, \psi\colon A \to B$ are *inner equivalent* if there exists inner automorphisms γ and δ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\gamma"] & B \arrow[d, "\delta"] \\ A \arrow[r, "\psi"] & B \end{tikzcd}$$

Note that inner equivalence is an equivalence relation on the set of homomorphisms between two algebras.

Lemma 2.2.4. *Let A and B be unital algebras and let $\psi\colon A \to B$ be a homomorphism. Then any inner automorphism γ of A can be "extended" to B in the sense that there exists an inner automorphism δ' of B such that the following diagram commutes:*

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\gamma"] & B \arrow[d, dashed, "\delta'"] \\ A \arrow[r, "\psi"] & B \end{tikzcd}$$

Proof. As γ is inner we have $\gamma = \operatorname{Ad} u$ for some unitary $u \in A$. Let $e = \psi(1_A)$. Then e is a projection and e acts like the identity on $\psi(A)$. We also have $\psi(u)\psi(u)^* = e = \psi(u)^*\psi(u)$. Now let $v = \psi(u) + 1_B - e$. Then v is a unitary in B since

$$\begin{aligned} vv^* &= (\psi(u) + (1_B - e))(\psi(u)^* + (1_B - e)) \\ &= \psi(u)\psi(u)^* + \psi(u)(1_B - e) + (1_B - e)\psi(u)^* + (1_B - e)^2 \\ &= e + \psi(u) - \psi(u)e + \psi(u)^* - e\psi(u)^* + 1_B - e \\ &= 1_B, \end{aligned}$$

and a similar computation shows that $v^*v = 1_B$. Define $\delta' = \operatorname{Ad} v$. For $a \in A$ we have

$$\begin{aligned}\delta' \circ \psi(a) &= \operatorname{Ad} v(\psi(a)) = v\psi(a)v^* = (\psi(u) + (1_B - e))\psi(a)(\psi(u^*) + (1_B - e)) \\ &= \psi(u)\psi(a)\psi(u^*) + \psi(u)\psi(a)(1_B - e) + (1_B - e)\psi(a)\psi(u^*) \\ &+ (1_B - e)\psi(a)(1_B - e) \\ &= \psi(uau^*) + 0 + 0 + 0 = \psi \circ \operatorname{Ad} u(a) \\ &= \psi \circ \gamma(a).\end{aligned}$$

□

The following lemma gives an alternative definition for inner equivalence, and is referred to as *unitary equivalence* in the general theory.

Lemma 2.2.5. *Let A and B be unital algebras. If $\phi, \psi\colon A \to B$ are inner equivalent homomorphisms, then there exists an inner automorphism θ of B such that $\theta \circ \phi = \psi$, i.e. the following diagram commutes:*

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[dr, "\psi"'] & B \arrow[d, dashed, "\theta"] \\ & B \end{tikzcd}$$

Proof. Let $\gamma\colon A \to A$ and $\delta\colon B \to B$ be inner automorphisms such that $\psi \circ \gamma = \delta \circ \phi$. By Lemma 2.2.4 there is an inner automorphism δ' of B such that $\psi \circ \gamma = \delta' \circ \psi$. Let $\theta = \delta'^{-1} \circ \delta$. Then θ is inner and we have

$$\theta \circ \phi = \delta'^{-1} \circ \delta \circ \phi = \delta'^{-1} \circ \psi \circ \gamma = \delta'^{-1} \circ \delta' \circ \psi = \psi.$$

□

It is worth emphasizing that in the following proposition, the connecting homomorphisms ϕ_n and ψ_n are not assumed to be unital.

Proposition 2.2.6. *Let (A_n, ϕ_n) and (A_n, ψ_n) be algebraic chain systems where the A_n's are unital. If, for each $n \in \mathbb{N}$, ϕ_n and ψ_n are inner equivalent, then $\varinjlim(A_n, \phi_n) \cong \varinjlim(A_n, \psi_n)$.*

Proof. We will construct a chain system isomorphism $(\theta_n)_{n\in\mathbb{N}}$ between the two chain systems as in the following diagram:

$$\begin{tikzcd} A_1 \arrow[r, "\phi_1"] \arrow[d, "\theta_1"] & A_2 \arrow[r, "\phi_2"] \arrow[d, "\theta_2"] & \dots \arrow[r] & A_n \arrow[r, "\phi_n"] \arrow[d, "\theta_n"] & A_{n+1} \arrow[r] \arrow[d, "\theta_{n+1}"] & \cdots \\ A_1 \arrow[r, "\psi_1"] & A_2 \arrow[r, "\psi_2"] & \dots \arrow[r] & A_n \arrow[r, "\psi_n"] & A_{n+1} \arrow[r] & \cdots \end{tikzcd}$$

by using Lemma 2.2.5 inductively. First, let $\theta_1 = \operatorname{Id}_{A_1}$. Since ϕ_1 and ψ_1 are inner equivalent there is an inner automorphism θ_2 of A_2 such that $\psi_1 = \theta_2 \circ \phi_1$, by Lemma 2.2.5. Then the first square in the diagram above commutes since $\psi_1 \circ \theta_1 = \psi_1 \circ \operatorname{Id}_{A_1} = \psi_1 = \theta_2 \circ \phi_1$.

Now assume that we have found inner automorphisms $\theta_1, \dots, \theta_n$ such that the first $n-1$ squares in the diagram above commutes. Since ϕ_n and ψ_n are inner equivalent, we have that $\psi_n \circ \gamma_n = \delta_n \circ \phi_n$ for inner automorphisms $\gamma_n \colon A_n \to A_n$ and $\delta_n \colon A_{n+1} \to A_{n+1}$. And then we have

$$\delta_n \circ \phi_n = \psi_n \circ \gamma_n = (\psi_n \circ \theta_n) \circ (\theta_n^{-1} \circ \gamma_n),$$

which means that the following diagram commutes:

$$\begin{tikzcd} A_n \arrow[r, "\phi_n"] \arrow[d, "\theta_n^{-1}\circ\gamma_n"'] & A_{n+1} \arrow[d, "\delta_n"] \\ A_n \arrow[r, "\psi_n\circ\theta_n"] & A_{n+1} \end{tikzcd}$$

Since $\theta_n^{-1} \circ \gamma_n$ is an inner automorphism we see that ϕ_n and $\psi_n \circ \theta_n$ are inner equivalent. We appeal to Lemma 2.2.5 again for the existence of an inner automorphism $\theta_{n+1} \colon A_{n+1} \to A_{n+1}$ such that $\psi_n \circ \theta_n = \theta_{n+1} \circ \phi_n$. Hence the first n squares in the diagram commutes.

By induction we get a chain system isomorphism $(\theta_n)_{n\in\mathbb{N}}$ between (A_n, ϕ_n) and (A_n, ψ_n), hence $\varinjlim(A_n, \phi_n) \cong \varinjlim(A_n, \psi_n)$. □

2.3 Canonical homomorphisms

Our next goal is to classify homomorphisms of multimatrix algebras, up to inner equivalence. It turns out that if we are allowed to "twist" by a unitary element (as in inner equivalence), then all homomorphisms are of a canonical type.

Let $q, p \in \mathbb{N}$ and $\kappa \in \mathbb{Z}^+$ be such that $\kappa q \leq p$. Then the triple (q, p, κ) gives rise to a homomorphism $\rho \colon M_q \to M_p$ as follows:

$$\rho(A) = \left[\begin{array}{c:c:c:c:c} A & 0 & 0 & 0 & 0 \\ \hdashline 0 & A & 0 & 0 & 0 \\ \hdashline 0 & 0 & \ddots & 0 & 0 \\ \hdashline 0 & 0 & 0 & A & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0_h \end{array}\right]_{p\times p} \quad \text{for } A \in M_q,$$

where there are κ copies of A along the diagonal and the last block of zeros has size $h = p - \kappa q$. This is clearly a homomorphism since doing an operation on $\rho(A)$ in M_p corresponds to doing the operation on A in each block on the diagonal. The condition $\kappa q \leq p$ ensures that there is enough space to place κ copies of $q \times q$ matrices on the diagonal.

Example 2.3.1. Let $q = 2,\ p = 6$ and $\kappa = 2$. Then $h = 2$ and the corresponding homomorphism $\rho \colon M_2 \to M_6$ is given by

$$\rho(A) = \left[\begin{array}{c:c:c} A & 0 & 0 \\ \hdashline 0 & A & 0 \\ \hdashline 0 & 0 & 0_2 \end{array}\right]_{6\times 6} \quad \text{for } A \in M_2.$$

This quickly becomes quite cumbersome to write. Therefore we introduce the following notation. Given (q, p, κ) we write the corresponding homomorphism $\rho\colon M_q \to M_p$ as

$$A \mapsto \overbrace{A \oplus A \oplus \cdots \oplus A}^{\kappa} \oplus 0_h.$$

This notation is sensible because having block matrices on the diagonal in a larger matrix correspond to direct sums, since each computation is done within each block on the diagonal. It is worth noting that if $q > p$, then there are no nonzero homomorphisms from M_q to M_p. This is because M_q is simple, so such a homomorphism is either injective or identically zero. And when $q > p$ the former cannot occur.

More generally, given $\vec{q} \in \mathbb{N}^s$, $\vec{p} \in \mathbb{N}^r$ and an $r \times s$ matrix $\kappa = [\kappa_{ij}]$, $\kappa_{ij} \in \mathbb{Z}^+$ such that $\kappa\vec{q} \leq \vec{p}$. Then this gives rise to a homomorphism $\rho\colon M(\vec{q}) \to M(\vec{p})$ as follows:

$$\begin{aligned}
\rho(A) &= \rho\left(A_1 \bigoplus \cdots \bigoplus A_s\right) \\
&= \left(\overbrace{A_1 \oplus \cdots \oplus A_1}^{\kappa_{11}} \oplus \overbrace{A_2 \oplus \cdots \oplus A_2}^{\kappa_{12}} \oplus \cdots \oplus \overbrace{A_s \oplus \cdots \oplus A_s}^{\kappa_{1s}} \oplus 0_{h_1}\right) \\
&\bigoplus \cdots \bigoplus \\
&\left(\overbrace{A_1 \oplus \cdots \oplus A_1}^{\kappa_{r1}} \oplus \overbrace{A_2 \oplus \cdots \oplus A_2}^{\kappa_{r2}} \oplus \cdots \oplus \overbrace{A_s \oplus \cdots \oplus A_s}^{\kappa_{rs}} \oplus 0_{h_r}\right) \\
&\text{for } A = A_1 \bigoplus \cdots \bigoplus A_s \in M(\vec{q}) = M_{q_1} \bigoplus \cdots \bigoplus M_{q_s},
\end{aligned}$$

where $\kappa\vec{q} + \vec{h} = \vec{p}$ and $\vec{h} = (h_1, \ldots, h_r)$. In the above equation $\bigoplus$ separates each summand (or coordinate) and $\oplus$ separates blocks on the diagonal. It is clear that ρ is a homomorphism. In words, ρ can be described as follows. The first summand of $\rho(A)$ consists of a $p_1 \times p_1$ matrix which has κ_{11} copies of A_1 along the diagonal, followed by κ_{12} copies of A_2 along the diagonal, all the way down to κ_{1s} copies of A_s, and then the rest are zeros. The last diagonal block (which is zero) has size $h_1 = p_1 - \sum_{j=1}^{s} \kappa_{1j} q_j$. The remaining summands of $\rho(A)$ are similar. Again, the notation is a little cumbersome, but the concept is quite clear. The nonnegative integer κ_{ij} denotes how many copies of A_j which are placed along the diagonal in M_{p_i}. The condition $\kappa\vec{q} \leq \vec{p}$, which really means that $\sum_{j=1}^{s} \kappa_{ij} q_j \leq p_i$ for each i, ensures that there is enough space in $M(\vec{p})$ to place the matrices on the diagonals. It's time to consider some examples.

Example 2.3.2. Let $\vec{q} = (2, 3, 7)$, $\vec{p} = (5, 11)$, and $\kappa = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{smallmatrix}\right)$, then the corresponding homomorphism $\rho\colon M_2 \oplus M_3 \oplus M_7 \longrightarrow M_5 \oplus M_{11}$ is given by

$$A \oplus B \oplus C \longmapsto \left[\begin{array}{c:c} A & 0 \\ \hdashline 0 & B \end{array}\right] \oplus \left[\begin{array}{c:c:c} A & 0 & 0 \\ \hdashline 0 & A & 0 \\ \hdashline 0 & 0 & C \end{array}\right]$$

Example 2.3.3. Let $\vec{q} = (1, 2, 3)$, $\vec{p} = (13)$, and $\kappa = (\,1\;3\;1\,)$, then the corresponding homomorphism $\rho\colon \mathbb{C} \oplus M_2 \oplus M_3 \longrightarrow M_{13}$ is given by

$$\lambda \oplus A \oplus B \longmapsto \left[\begin{array}{c:c:c:c:c:c} \lambda & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & A & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & A & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & A & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & B & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0_3 \end{array}\right]$$

Definition 2.3.4. A homomorphism $\rho\colon M(\vec{q}) \to M(\vec{p})$ between multimatrix algebras is called *canonical* if ρ is defined by a matrix $[\kappa_{ij}]$ as above.

As a slight abuse of notation we will often write $\rho = [\kappa_{ij}]$ when ρ is canonical, but whether we mean the matrix itself or the homomorphism will always be clear from the context. The whole point of introducing these canonical homomorphisms is because of the fact that *all* homomorphisms are "essentially" canonical as the following theorem shows.

Theorem 2.3.5. *Let $\phi\colon M(\vec{q}) \to M(\vec{p})$ be a homomorphism, where $\vec{q} \in \mathbb{N}^s$ and $\vec{p} \in \mathbb{N}^r$. Then there exists a unique canonical homomorphism, ρ, which is inner equivalent to ϕ.*

Proof. We first consider the simplified case $\phi\colon M(\vec{q}) \to M_p$, i.e. a single summand in the codomain. Let E^k_{ij} for $1 \le k \le s$ and $1 \le i, j \le q_k$ denote the standard matrix units in $M(\vec{q})$. E.g. $E^2_{21} = 0 \oplus \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \oplus 0_3$ in Example 2.3.3. The matrix units satisfy the relation

$$E^k_{ij} E^{k'}_{i'j'} = \delta_{kk'} \delta_{ji'} E^k_{ij'}.$$

The projections $E^k = \sum_{i=1}^{q_k} E^k_{ii}$ are mutually orthogonal in $M(\vec{q})$, hence the same is true for $\phi(E^k)$ in M_p. From now on we will regard M_p as linear operators on $\mathbb{C}^p$.

For each $k = 1, \ldots, r$ let x^k_ν $(\nu = 1, \ldots, \kappa_k)$ be an orthonormal basis for $\phi(E^k_{11})\mathbb{C}^p$, where κ_k is the dimension of $\phi(E^k_{11})\mathbb{C}^p$. Later we will see that κ_k is the number of copies of A_k placed in M_p by the canonical homomorphism inner equivalent to ϕ. We claim that, for each k, $x^k_{j\nu} := \phi(E^k_{j1})x^k_\nu$ $(1 \le j \le q_k, 1 \le \nu \le \kappa_k)$ is an orthonormal basis for $\phi(E^k)\mathbb{C}^p$. Indeed, if $j \ne j'$, then

$$\begin{aligned} \langle x^k_{j\nu}, x^k_{j'\nu'} \rangle &= \langle \phi(E^k_{j1})x^k_\nu, \phi(E^k_{j'1})x^k_{\nu'} \rangle = \langle \phi(E^k_{j'1})^* \phi(E^k_{j1})x^k_\nu, x^k_{\nu'} \rangle \\ &= \langle \phi(E^k_{1j'})\phi(E^k_{j1})x^k_\nu, x^k_{\nu'} \rangle = \langle \phi(E^k_{1j'}E^k_{j1})x^k_\nu, x^k_{\nu'} \rangle \\ &= \langle \phi(0)x^k_\nu, x^k_{\nu'} \rangle = 0 \end{aligned}$$

since the E^k_{ij} are matrix units. And if $j = j'$, then

$$\langle x^k_{j\nu}, x^k_{j\nu'} \rangle = \langle \phi(E^k_{1j}E^k_{j1})x^k_\nu, x^k_{\nu'} \rangle = \langle \phi(E^k_{11})x^k_\nu, x^k_{\nu'} \rangle = \langle x^k_\nu, x^k_{\nu'} \rangle = \delta_{\nu\nu'}.$$

The second to last equality follows because $x^k_\nu \in \phi(E^k_{11})\mathbb{C}^p$ and $\phi(E^k_{11})$ is a projection. Because the E^k_{ij}'s are matrix units, we have $E^k_{i1}E^k_{11}E^k_{1i} = E^k_{ii}$ and vice versa. It follows that

$$\dim(\phi(E^k_{ii})\mathbb{C}^p) = \dim(\phi(E^k_{11})\mathbb{C}^p) = \kappa_k \text{ for each } i.$$

And since $\phi(E^k)\mathbb{C}^p = \oplus_{i=1}^{q_k}\phi(E^k_{ii})\mathbb{C}^p$ we get that $\dim(\phi(E^k)\mathbb{C}^p) = q_k\kappa_k$. As we have found $q_k\kappa_k$ linearly independent vectors, $x^k_{j\nu}$, in $\phi(E^k)\mathbb{C}^p$, these must span $\phi(E^k)\mathbb{C}^p$. This proves the claim.

Since the $\phi(E^k)$'s are mutually orthogonal projections, the subspaces $\phi(E^k)\mathbb{C}^p$ are orthogonal. This implies that the set $\{x^k_{j\nu} | 1 \leq k \leq s,\ 1 \leq \nu \leq \kappa_k,\ 1 \leq j \leq q_k\}$ is an orthonormal basis for $\phi(M(\vec{q}))\mathbb{C}^p = \oplus_{k=1}^{s}\phi(E^k)\mathbb{C}^p$. If this is a proper subspace of $\mathbb{C}^p$, i.e. the $x^k_{j\nu}$'s do not already span $\mathbb{C}^p$, then we complete it to an orthonormal basis for $\mathbb{C}^p$ by adding vectors $x_1, \dots, x_h$. Let $\mathcal{B}$ denote the ordered basis

$$
\begin{aligned}
&x^1_{11}, x^1_{21}, \dots, x^1_{q_1 1}, x^1_{1,2}, \dots, x^1_{q_1 2}, \dots, x^1_{1\kappa_1}, \dots, x^1_{q_1\kappa_1},\\
&x^2_{11}, x^2_{21}, \dots, x^2_{q_2\kappa_2}, x^3_{11}, \dots, x^s_{q_s\kappa_s}, x_1, \dots x_h.
\end{aligned}
$$

In other words, $\mathcal{B}$ is the basis described above, ordered by first ascending on j, then ν, then k, then h.

Let $A \in M(\vec{q})$. Using the matrix units we can write $A = \sum_{i,j,k} a^k_{ij}E^k_{ij}$, where the a^k_{ij}'s are scalars. As $\phi(A) \in M_p$, it may be viewed as a linear operator on $\mathbb{C}^p$. In order to find $\phi(A)$ relative to the basis $\mathcal{B}$ we compute

$$
\begin{aligned}
\phi(A)x^{k_0}_{j_0\nu} &= \left(\sum_{i,j,k} a^k_{ij}\phi(E^k_{ij})\right)\left(\phi(E^{k_0}_{j_0 1})x^{k_0}_{\nu}\right) = \sum_{i,j,k} a^k_{ij}\phi(E^k_{ij}E^{k_0}_{j_0 1})x^{k_0}_{\nu}\\
&= \sum_{i} a^{k_0}_{ij_0}\phi(E^{k_0}_{i1})x^{k_0}_{\nu} = \sum_{i=1}^{q_{k_0}} a^{k_0}_{ij_0}x^{k_0}_{i\nu}.
\end{aligned} \tag{2.3}
$$

As for the last basis vectors, x_j, we have $\phi(A)x_j = 0$. This is because $x_j \in (\phi(M(\vec{q}))\mathbb{C}^p)^{\perp} \supseteq (\mathrm{Ran}(\phi(A^*)))^{\perp} = \ker(\phi(A))$. We claim that $\phi(A)$ has the following representation with the respect to the basis $\mathcal{B}$:

$$
[\phi(A)]_{\mathcal{B}} = \overbrace{A_1 \oplus \cdots \oplus A_1}^{\kappa_1} \oplus \overbrace{A_2 \oplus \cdots \oplus A_2}^{\kappa_2} \oplus \cdots \oplus \overbrace{A_s \oplus \cdots \oplus A_s}^{\kappa_s} \oplus 0_h. \tag{2.4}
$$

To see that this is indeed the case we consider the columns of $[\phi(A)]_{\mathcal{B}}$. As for the first column we have $\phi(A)x^1_{11} = \sum_{i=1}^{q_1} a^1_{i1}x^1_{11}$ by (2.3), hence the first column of $[\phi(A)]_{\mathcal{B}}$ is $(a^1_{11}, a^1_{21}, \dots, a^1_{q_1 1}, 0 \dots, 0)^T$, which is just the first column of A_1, and the rest are zeros. Similarly, the second column equals the second column of A_1 followed by zeros. This pattern repeats for each value of ν, and the matrix is moved one block down the diagonal, hence we get κ_1 copies of A_1 along the diagonal. The same happens for A_2 up to A_s as well. Finally, since $\phi(A)x_j = 0$ for $j = 1, \dots, h$, the last block on the diagonal in $[\phi(A)]_{\mathcal{B}}$ is a $h \times h$ block of zeros.

Let $e_1, \dots, e_p$ denote the standard basis for $\mathbb{C}^p$ and let U be the change of basis matrix from the basis $\mathcal{B}$ to the basis e_i, i.e. $Ux^1_{11} = e_1$, $Ux^1_{21} = e_2$ etc. Then $U\phi(A)U^*$ has the form (2.4) relative to the standard basis. Also, U is unitary since the bases are orthonormal. By considering the following commutative diagram we see that ϕ is inner equivalent to the canonical homomorphism ρ defined by the triple $(\vec{q},\ p,\ [\kappa_1, \dots, \kappa_s])$:

$$\begin{array}{ccc} M(\vec{q}) & \xrightarrow{\phi} & M_p \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Ad}\,U} \\ M(\vec{q}) & \xrightarrow{\rho} & M_p \end{array}$$

As for the general case $\phi\colon M(\vec{q}) \to M(\vec{p})$, let π_i denote the projection on the ith summand in $M(\vec{p})$, i.e.

$$\pi_i(B) = B_i \text{ for } B = B_1 \oplus \cdots \oplus B_r \in M(\vec{p}) = M_{p_1} \oplus \cdots \oplus M_{p_r}.$$

Then we can write $\phi(A) = \phi_1(A) \oplus \cdots \oplus \phi_r(A)$, where $\phi_i = \pi_i \circ \phi\colon M(\vec{q}) \to M_{p_i}$ is a homomorphism. By the above we get, for each i, a canonical homomorphism ρ_i determined by the triple $(\vec{q},\ p_i,\ [\kappa_{i1}, \ldots, \kappa_{is}])$ and a unitary U_i such that $\rho_i = \mathrm{Ad}\,U_i \circ \phi_i$. And if we let $U = \mathrm{Ad}\,U_1 \oplus \cdots \oplus U_r$ and $\rho = [\kappa_{ij}]$, then $\rho = \mathrm{Ad}\,U \circ \phi$. Thus existence is established.

It remains to prove uniqueness. To this end, we first observe that inner automorphisms preserve trace. Indeed, if V is a unitary then by the trace property we get $\mathrm{Tr}(V(AV^*)) = \mathrm{Tr}((AV^*)V) = \mathrm{Tr}(A)$.

Suppose $\rho = [\kappa_{ij}]$ and $\rho' = \left[\kappa'_{ij}\right]$ define inner equivalent canonical homomorphisms from $M(\vec{q})$ to $M(\vec{p})$. We need to show that $\kappa_{ij} = \kappa'_{ij}$ for all i,j. By Lemma 2.2.5 there exists a unitary $V \in M(\vec{p})$ such that $\rho' = \rho \circ \mathrm{Ad}\,V$, i.e. $\rho'(A) = V\rho(A)V^*$ for $A \in M(\vec{q})$. Note that $V = V_1 \oplus \cdots \oplus V_r$ where each V_i is a unitary matrix in M_{p_i}. Using this we see that

$$\begin{aligned} V\rho(A)V^* &= V_1 \left(\overbrace{A_1 \oplus \cdots \oplus A_1}^{\kappa_{11}} \oplus \cdots \oplus \overbrace{A_s \oplus \cdots \oplus A_s}^{\kappa_{1s}} \oplus 0_{h_1} \right) V_1^* \bigoplus \\ &\quad \cdots \bigoplus V_r \left(\overbrace{A_1 \oplus \cdots \oplus A_1}^{\kappa_{r1}} \oplus \cdots \oplus \overbrace{A_s \oplus \cdots \oplus A_s}^{\kappa_{rs}} \oplus 0_{h_r} \right) V_r^* \\ &= \left(\overbrace{A_1 \oplus \cdots \oplus A_1}^{\kappa'_{11}} \oplus \cdots \oplus \overbrace{A_s \oplus \cdots \oplus A_s}^{\kappa'_{1s}} \oplus 0_{h'_1} \right) \bigoplus \\ &\quad \cdots \bigoplus \left(\overbrace{A_1 \oplus \cdots \oplus A_1}^{\kappa'_{r1}} \oplus \cdots \oplus \overbrace{A_s \oplus \cdots \oplus A_s}^{\kappa'_{rs}} \oplus 0_{h'_r} \right) \\ &= \rho'(A). \end{aligned} \tag{2.5}$$

Consider the multimatrix $A = 0 \oplus \cdots \oplus 0 \oplus I_{q_j} \oplus 0 \oplus \cdots \oplus 0 \in M(\vec{q})$, where the identity matrix I_{q_j} occurs in the j'th summand. Then the i'th summand in (2.5) reduces to

$$V_i \left(0 \oplus \cdots \oplus 0 \oplus \overbrace{I_{q_j} \oplus \cdots \oplus I_{q_j}}^{\kappa_{ij}} \oplus 0 \oplus \cdots \oplus 0 \right) V_i^*$$

$$= 0 \oplus \cdots \oplus 0 \oplus \overbrace{I_{q_j} \oplus \cdots \oplus I_{q_j}}^{\kappa'_{ij}} \oplus 0 \oplus \cdots \oplus 0.$$

And since inner automorphisms preserve trace, we can take the trace (in M_{p_i}) on both sides to get $\frac{\kappa_{ij} q_j}{p_i} = \frac{\kappa'_{ij} q_j}{p_i}$, which implies that $\kappa_{ij} = \kappa'_{ij}$. Thus $\rho = \rho'$. □

Now we are in good shape. For by combining Proposition 2.2.6 and Theorem 2.3.5 we see that any locally semisimple algebra can be obtained from a *canonical chain system*, i.e. a chain system where the algebras are multimatrix algebras and the connecting homomorphisms are canonical.

Proposition 2.3.6. *Let A be a locally semisimple algebra. Then A is isomorphic to the direct limit of a canonical chain system.*

Proof. As A is locally semisimple we have that $A \cong \varinjlim(A_n, \psi_n)$ where each A_n is semisimple. So there exists isomorphisms $\gamma_n \colon A_n \to M(\vec{p}(n))$. Defining $\phi_n = \gamma_{n+1} \circ \psi_n \circ \gamma_n^{-1}$ turns $(\gamma_n)_{n \in \mathbb{N}}$ into a chain system isomorphism, hence $\varinjlim(A_n, \psi_n) \cong \varinjlim(M(\vec{p}(n)), \phi_n)$. See the following diagram:

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{\psi_1} & A_2 & \xrightarrow{\psi_2} & A_3 & \xrightarrow{\psi_3} & \cdots \\
\cong \downarrow \gamma_1 & & \cong \downarrow \gamma_2 & & \cong \downarrow \gamma_3 & & \\
M(\vec{p}(1)) & \xrightarrow{\phi_1} & M(\vec{p}(2)) & \xrightarrow{\phi_2} & M(\vec{p}(3)) & \xrightarrow{\phi_3} & \cdots \\
M(\vec{p}(1)) & \xrightarrow{\rho_1} & M(\vec{p}(2)) & \xrightarrow{\rho_2} & M(\vec{p}(3)) & \xrightarrow{\rho_3} & \cdots
\end{array}$$

By Theorem 2.3.5 each ϕ_n is inner equivalent to a canonical homomorphism $\rho_n = [\kappa_{ij}(n)]$. And so, $\varinjlim(M(\vec{p}(n)), \phi_n) \cong \varinjlim(M(\vec{p}(n)), \rho_n)$ by Proposition 2.2.6. Hence $A \cong \varinjlim(M(\vec{p}(n)), \rho_n)$. □

2.4 Labeled Bratteli diagrams

Labeled Bratteli diagrams is a convenient way to represent canonical chain systems, and in turn AF-algebras (see Section 3.2). Let $\rho = [\kappa_{ij}] : M(\vec{q}) \to M(\vec{p})$ be a canonical homomorphism. To ρ we associate a (directed) graph whose "initial" vertices are $\{q_1, \ldots, q_s\}$ and whose "final" vertices are $\{p_1, \ldots, p_r\}$. There are κ_{ij} edges between q_j and p_i, and all these edges are directed from q_j to p_i. We draw such a graph as a bipartite graph as in Figure 2.1. When drawing the graph, we do not mark the direction of the edges because these are always directed downward. We call this graph simply the *graph of ρ*. The graph tells us how many copies of of M_{q_j} which maps inside M_{p_i} under ρ. For example, in Figure 2.1 there are two edges from q_1 to p_1, hence $\kappa_{11} = 2$. This means that ρ places two copies of the first summand, M_{q_1}, of $M(\vec{q})$ along the diagonal in the first summand, M_{p_1}, of $M(\vec{p})$.

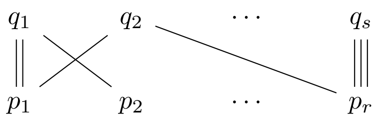

Figure 2.1: The graph of a canonical homomorphism.

Example 2.4.1. The graphs of the canonical homomorphisms in Example 2.3.2 and in Example 2.3.3 are depicted in Figure 2.2 and Figure 2.3, respectively.

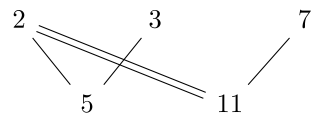

Figure 2.2: The graph of the canonical homomorphism in Example 2.3.2.

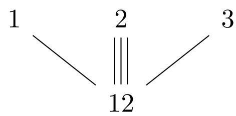

Figure 2.3: The graph of the canonical homomorphism in Example 2.3.3.

Given a canonical chain system $(M(\vec{p}(n)), \rho_n)$ we obtain an infinite graph by joining together the graphs associated to the ρ_n's. This infinite graph is the *labeled Bratteli diagram* of the canonical chain system. Notice that the graph has no sinks if and only if each connecting homomorphism in the canonical chain system is injective. The following two examples illustrate the concept. These examples will turn up again in the next chapter as labeled Bratteli diagrams representing classical examples of AF-algebras.

Example 2.4.2. Consider the following canonical chain system:

$$\mathbb{C} \xrightarrow{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} M(1,1) \xrightarrow{\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]} M(1,2) \xrightarrow{\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]} M(1,3) \xrightarrow{\left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]} \cdots \tag{2.6}$$

The labeled Bratteli diagram of this canonical chain system is depicted in Figure 2.4.

Example 2.4.3. Another canonical chain system is the following:

$$M(1,1) \xrightarrow{\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]} M(2,2) \xrightarrow{\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]} M(4,4) \xrightarrow{\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]} \cdots \tag{2.7}$$

And its labeled Bratteli diagram is depicted in Figure 2.5.

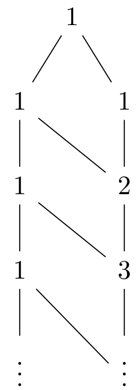

Figure 2.4: The labeled Bratteli diagram of the canonical chain system (2.6).

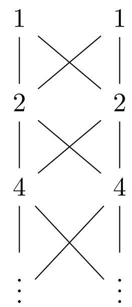

Figure 2.5: The labeled Bratteli diagram of the canonical chain system (2.7).

Bratteli diagrams will play a prominent role in this text. We will see how they can be used to represent both AF-algebras and certain dynamical systems on a Cantor space. Therefore we now define labeled Bratteli diagrams abstractly.

Definition 2.4.4. A *labeled Bratteli diagram* is a quintuple (V, E, r, s, d) satisfying the following properties:

(1) V, the vertex set, and E, the edge set, are both countable disjoint unions of non-empty finite sets; $V = \bigsqcup_{n=0}^{\infty} V_n$ and $E = \bigsqcup_{n=1}^{\infty} E_n$.

(2) r, the range map, is a map $r\colon E \to V$ such that $r(E_n) \subseteq V_n$. s, the source map, is a map $s\colon E \to V$ such that $s(E_n) \subseteq V_{n-1}$. Moreover, $s^{-1}(v) \neq \emptyset$ for all $v \in V$, i.e. the underlying graph has no sinks.

(3) d, the labeling of the Bratteli diagram, is a map $d\colon V \to \mathbb{N}$ such that $d(v) \geq \sum_{r(e)=v} d(s(e))$ for all $v \in V \setminus V_0$.

Clearly, the diagram obtained from an injective canonical chain system is a labeled Bratteli diagram, and any labeled Bratteli diagram gives rise to an injective canonical chain system - more on this in Section 3.2 We will also see that there is no loss of generality by working with injective chain systems.

When drawing a labeled Bratteli diagram, V_n are the vertices of level n, and these are drawn on the same horizontal line. E_n are the edges between level $n-1$ and n, i.e. between V_{n-1} and V_n. The edges are drawn without orientation because they are all directed downwards, i.e. from V_{n-1} to V_n. Also, a vertex $v \in V$ is drawn as the positive integer $d(v)$. The level of v is the unique n such that $v \in V_n$. One should think of the label $d(v)$ as the matrix size in the vth summand of a multimatrix algebra, and the sum on the right hand side in part (3) of Definition 2.4.4 as the sum of matrix sizes mapping into the vth summand. The maps r and s are the obvious range and source maps giving the range and source of an edge, respectively. We shall often write just (V, E) for a labeled Bratteli diagram, and supress r, s and d from our notation. These maps are easily deduced from a given diagram. In Chapter 4 we will drop the labeling, but for now it is needed to be able to represent non-unital embeddings and non-unital AF-algebras.

There is an obvious notion of *isomorphism* between two labeled Bratteli diagrams (V, E) and (V', E'); namely, there exists a pair of bijections between V and V' and between E and E' preserving the gradings and the labeling, and intertwining the respective source and range maps. One should think of isomorphism of labeled Bratteli diagrams as permuting the vertices within each level while keeping their labels and edges attached.

Example 2.4.5. The two labeled Bratteli diagrams depicted in Figure 2.6 are isomorphic. The correspondence of the vertices (i.e. the "permutations") within each level is indicated with colors.

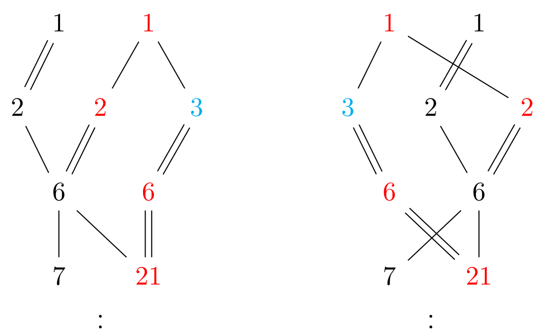

Figure 2.6: Two isomorphic labeled Bratteli diagrams.

The next result shows that a labeled Bratteli diagram uniquely determines a locally semisimple algebra.

Lemma 2.4.6. *Suppose $(M(\vec{q}(n)), \kappa_n)$ and $(M(\vec{p}(n)), \rho_n)$ are two canonical chain systems whose labeled Bratteli diagrams are isomorphic. Then* $\varinjlim(M(\vec{q}(n)), \kappa_n) \cong \varinjlim(M(\vec{p}(n)), \rho_n)$.

Proof. Since the labeled Bratteli diagrams are isomorphic, $\vec{q}(n)$ is a permutation of $\vec{p}(n)$ for each n. Thus there are isomorphisms $\phi_n \colon M(\vec{q}(n)) \to M(\vec{p}(n))$. However, the following diagram need *not* commute:

$$\begin{tikzcd}
M(\vec{q}(1)) \arrow[r, "\kappa_1"] \arrow[d, "\cong"', "\phi_1"] & M(\vec{q}(2)) \arrow[r, "\kappa_2"] \arrow[d, "\cong"', "\phi_2"] & M(\vec{q}(3)) \arrow[r, "\kappa_3"] \arrow[d, "\cong"', "\phi_3"] & \cdots \\
M(\vec{p}(1)) \arrow[r, "\rho_1"] & M(\vec{p}(2)) \arrow[r, "\rho_2"] & M(\vec{p}(3)) \arrow[r, "\rho_3"] & \cdots
\end{tikzcd}$$

But since the labeled Bratteli diagrams are isomorphic, we do have that $\phi_{n+1} \circ \kappa_n$ and $\rho_n \circ \phi_n$ are inner equivalent to the same canonical homomorphism for each n. In particular $\phi_{n+1} \circ \kappa_n$ and $\rho_n \circ \phi_n$ are inner equivalent. We will use this to construct a chain system isomorphism $(\psi_n)_{n\in\mathbb{N}}$. Let $\psi_1 = \phi_1$. By Lemma 2.2.5 there is an inner automorphism $\gamma_2 = \operatorname{Ad} u_2$ of $M(\vec{p}(2))$ such that $\rho_1 \circ \phi_1 = \gamma_2 \circ (\phi_2 \circ \kappa_1)$. Let $\psi_2 = \gamma_2 \circ \phi_2$. Then $\rho_1 \circ \psi_1 = \psi_2 \circ \kappa_1$.

To get to the next step, consider $\rho_2 \circ \psi_2$. Note that $\psi_2 = \operatorname{Ad} u_2 \circ \phi_2 = \phi_2 \circ \operatorname{Ad} \phi_2^{-1}(u_2)$. Now observe that $\rho_2 \circ \psi_2$ is inner equivalent to $\rho_2 \circ \phi_2$ since

$$(\rho_2 \circ \psi_2) \circ \operatorname{Ad} \phi_2^{-1}(u_2^*) = \rho_2 \circ \phi_2 \circ \left(\operatorname{Ad} \phi_2^{-1}(u_2) \circ \operatorname{Ad} \phi_2^{-1}(u_2^*)\right) = \rho_2 \circ \phi_2.$$

Thus $\rho_2 \circ \psi_2$ is inner equivalent to $\phi_3 \circ \kappa_2$. Again by Lemma 2.2.5 there is an inner automorphism $\gamma_3 = \operatorname{Ad} u_3$ of $M(\vec{p}(3))$ such that $\rho_2 \circ \psi_2 = \gamma_3 \circ (\phi_3 \circ \kappa_2)$. Let $\psi_3 = \gamma_3 \circ \phi_3$. Then $\rho_2 \circ \psi_2 = \psi_3 \circ \kappa_2$.

Continuing in this fashion we obtain a chain system isomorphism $(\psi_n)_{n\in\mathbb{N}}$, thus $\varinjlim(M(\vec{q}(n)), \kappa_n) \cong \varinjlim(M(\vec{p}(n)), \rho_n)$. □

When (V, E) is a labeled Bratteli diagram and $k, l \in \mathbb{Z}^+$ with $k < l$, we let $E_{k+1} \circ E_{k+2} \circ \cdots \circ E_l$ denote the set of all paths from V_k to V_l. That is,

$$E_{k+1} \circ E_{k+2} \circ \cdots \circ E_l = \{(e_{k+1}, e_{k+2}, \ldots, e_l) | e_i \in E_i, r(e_i) = s(e_{i+1}) \ \forall\ i\}.$$

We also define $r((e_{k+1}, \ldots, e_l)) = r(e_l)$ and $s((e_{k+1}, \ldots, e_l)) = s(e_{k+1})$ in this case.

Definition 2.4.7. Let (V, E, r, s, d) be a labeled Bratteli diagram and let $m_0 < m_1 < m_2 < \ldots$ be a sequence of non-negative integers. The *telescoping* of (V, E, r, s, d) with the respect to the sequence m_n is the labeled Bratteli diagram (V', E', r', s', d'), where $V'_n = V_{m_n}$, $E'_n = E_{m_{n-1}+1} \circ E_{m_{n-1}+2} \circ \cdots \circ E_{m_n}$, $d' = d_{|V'}$, and r' and s' are the extensions of r and s respectively, restricted to the paths E'_n as above.

It is easy to see that a telescoping of any labeled Bratteli diagram is again a labeled Bratteli diagram. Note that telescoping a labeled Bratteli diagram corresponds to telescoping the canonical chain system (as mentioned near the end of Section 2.1) it represents. The notion of telescoping is illustrated in Figure 2.7.

The inverse operation of telescoping is called *microscoping*. Microscoping is to fill in new levels and edges so that by telescoping to the old levels one gets the original diagram back again. We say that two labeled Bratteli diagrams are

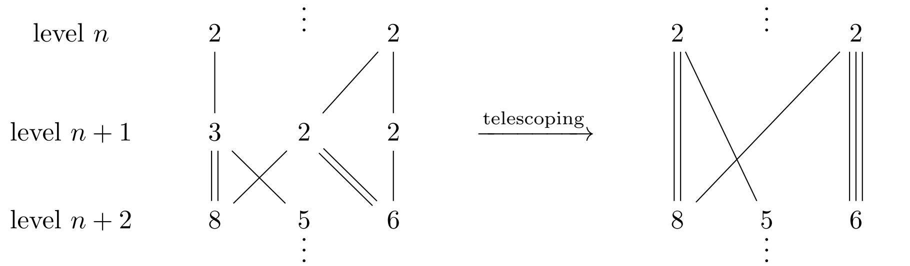

Figure 2.7: On the left, the levels n, $n+1$ and $n+2$ of a labeled Bratteli diagram, and on the right, the resulting telescoped diagram obtained by telescoping to levels n and $n+2$.

telescope equivalent if one diagram can be obtained from the other by a finite sequence of telescopings, microscopings and isomorphisms.

Sometimes it will be useful to describe the position of a vertex relative to another vertex in a labeled Bratteli diagram. In order to do this in a concise way we introduce a coordinate system of sorts for labeled Bratteli diagrams. When (V, E, r, s, d) is a labeled Bratteli diagram, we write $V_n = \{(n, 1), (n, 2), \ldots, (n, |V_n|)\}$ where we have associated each vertex, v, with a pair (n, p) where n is the level of v and p is an integer between 1 and $|V_n|$ such that each vertex in V_n is assigned a unique number, not to be confused with the label of the vertex. When drawing the labeled Bratteli diagram (V, E, r, s, d) we always draw the vertices on level n from left to right beginning with $(n, 1)$, then $(n, 2)$, and so forth. In this way, (n, p) refers to the pth vertex (from the left) on the nth level, and $d((n, p))$ is its label.

Example 2.4.8. Consider the labeled Bratteli diagram in Figure 2.8. The coloring in this example is only present to ease the reading, it carries no mathematical meaning. Here, $(0, 1)$ refers to the vertex labeled 1 at the top (0th) level. $(1, 1)$ refers to the first vertex on the first level and $(1, 2)$ refers to the second vertex on the first level, both wich are labeled 3. And $(2, 2)$ refers to the second vertex on the second level, which is labeled 2.

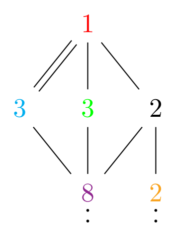

Figure 2.8: A labeled Bratteli diagram with colored vertices to ease the reading.

If (V, E, r, s, d) is a labeled Bratteli diagram and (n, p) and $(n+1, q)$ are vertices in V_n and V_{n+1} respectively, then we write $(n, p) \mapsto (n + 1, q)$ if there is an edge between them, i.e. if there exists an $e \in E_{n+1}$ with $s(e) = (n,p)$ and $r(e) = (n+1,q)$. More generally, when $n < m$, we write $(n,p) \to (m,q)$ if there is a path from (n,p) to (m,q), i.e. if there exists an $\alpha \in E_{n+1} \circ \cdots \circ E_m$ with $s(\alpha) = (n,p)$ and $r(\alpha) = (m,q)$.

For instance, in the diagram in Figure 2.8, $(0,1) \mapsto (1,1)$ and $(0,1) \to (2,1)$, while $(1,1) \nrightarrow (2,2)$ since there is no (downwards directed) path from the vertex $(1,1)$ to the vertex $(2,2)$.

Next we consider subdiagrams generated by subsets of vertices. Let (V,E,r,s,d) be a labeled Bratteli diagram and let $W \subseteq V$ be a non-empty subset of vertices. Let m be the smallest integer such that $W \cap V_m \neq \emptyset$. Then the *subdiagram corresponding to W* is $(V^{(W)}, E^{(W)})$ where $V_n^{(W)} = V_{m+n} \cap W$ for $n \in \mathbb{Z}^+$, and $E_n^{(W)} = \{e \in E_{m+n} | s(e) \in W \wedge r(e) \in W\}$ for $n \in \mathbb{N}$. The subdiagram corresponding to a subset of vertices will not always be a labeled Bratteli diagram, but for the cases we shall consider (e.g. ideals in Section 3.5), it will be. To obtain the subdiagram associated to W from (V,E) when $W \subseteq V$, simply remove every vertex not in W and remove every edge whose source or range does not belong to W.

Example 2.4.9. Let (V,E) be the labeled Bratteli diagram in Figure 2.8 and let

$$W = \{(0,1),\ (1,1),\ (2,1),\ (2,2), \ldots\} \subseteq V.$$

Then the subdiagram corresponding to W is depicted in Figure 2.9.

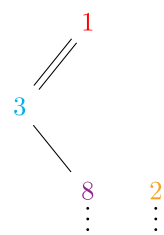

Figure 2.9: The subdiagram corresponding to W in Example 2.4.9.

2.5 Direct limits of finite-dimensional C^*-algebras

Having defined locally semisimple algebras, the next step is to equip these with a norm turning them into C^*-algebras, and thus obtaining direct limits of finite-dimensional C^*-algebras. Consider a *finite-dimensional chain system*, i.e. the A_n's are finite dimensional C^*-algebras and the ϕ_n's are homomorphisms:

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} \cdots \tag{2.8}$$

By the structure theorem for finite-dimensional C^*-algebras (Theorem 1.1.1), each A_n is semisimple as an algebra. Let A_∞ denote the algebraic direct limit of this chain system. Then A_∞ is locally semisimple. Recall the maps $\phi_{n\infty}\colon A_n \to A_\infty$ which maps an element to its equivalence class in A_∞. Define $B_n := \phi_{n\infty}(A_n) \subseteq A_\infty$. As $\phi_{n\infty}$ is a homomorphism we have that $B_n \cong A_n/\ker(\phi_{n\infty})$. And since A_n is semisimple, B_n is a direct summand of A_n, hence B_n is semisimple as well.

This can also be seen directly as follows. For a fixed n, $A_n \cong M_{p_1} \oplus \cdots \oplus M_{p_{r_n}}$. Each summand of A_n is simple, so each summand is either embedded into A_m or collapsed to 0 by each ϕ_{nm}. As there are finitely many summands, this process will eventually stabilize, i.e. for some N, $\phi_{nN}(A_n) \cong \phi_{n,N+1}(A_n) \cong \cdots$. At this point, a certain number of the summands of A_n may have been collapsed to 0, but the remaining summands which are then embedded ad infinitum are isomorphic, as a direct sum, to B_n.

Observe that $B_1 \subseteq B_2 \subset B_3 \subseteq \cdots$ and that $\bigcup_{n=1}^\infty B_n = A_\infty$. From this we see that each element in A_∞ is contained in a semisimple subalgebra, hence the name locally semisimple. We know that each multimatrix algebra $M(\vec{p})$ has a unique norm which turns it into a C^*-algebra. Each B_n is isomorphic to some multimatrix algebra $M(\vec{p}(n))$, so we endow B_n with the norm from the latter (which is then the unique norm turning B_n into a C^*-algebra). As $B_n \subseteq B_{n+1}$ the norm on B_n coincide with the norm on B_{n+1}. In this way we obtain a norm on A_∞ which satisfies all the axioms of a C^*-algebra, except possibly completeness. (Following the ideas in the previous paragraph one can see that $\|[a,n]\| = \|\phi_{n\infty}(a)\| = \lim_{m\to\infty} \|\phi_{nm}(a)\|_{A_m}$.) We may then form the C^*-completion of A_∞.

Definition 2.5.1. Let (A_n, ϕ_n) be a finite-dimensional (C^*-algebraic) chain system, and let A_∞ denote the algebraic direct limit of the chain system. The *C^*-algebraic direct limit* of the chain system (A_n, ϕ_n), denoted $\mathfrak{A}_\infty$, is the closure $C^*(A_\infty)$ with respect to the norm described in the preceding paragraph.

An important remark is that $\mathfrak{A}_\infty$ depends only on the limit algebra A_∞, and not on the particular approximating chain system. This is a trivial observation, but it will be used several times in the next chapter.

Lemma 2.5.2. *If (A_n, ϕ_n) and (A'_n, ϕ'_n) are finite-dimensional chain systems such that $A_\infty \cong A'_\infty$ as algebras, then $\mathfrak{A}_\infty \cong \mathfrak{A}'_\infty$ as C^*-algebras.*

Proof. Let $\|\cdot\|_{A_\infty}$ denote the norm on A_∞ induced by the semisimple subalgebras $B_n = \phi_{n\infty}(A_n)$, and similarly for $\|\cdot\|_{A'_\infty}$. Let $\psi\colon A_\infty \to A'_\infty$ be an isomorphism. Then $\cup_{n=1}^\infty \psi(B_n) = A'_\infty = \cup_{n=1}^\infty B'_n$. Since the unions are increasing and the algebras are semisimple, we have that for each n, $\psi(B_n) \subseteq B'_{m_n}$. This means that the $\psi(B_n)$'s and the B'_n's define the same norm on A'_∞ (since C^*-norms are unique). And since ψ restricts to an isomorphism between the C^*-algebras B_n and $\psi(B_n)$, we get that $\|\psi(a)\|_{A'_\infty} = \|a\|_{A_\infty}$. Hence A_∞ and A'_∞ are isometrically isomorphic as normed algebras, and it follows that their completions are isomorphic as C^*-algebras. $\square$

General direct limits exists in the category of C^*-algebras and one can show that $\mathfrak{A}_\infty$ is (isomorphic to) the categorical direct limit of the chain system in Equation (2.8) (see Section 6.1 in [11]). In a general direct limit one may use a directed set, instead of a directed sequence which we have used. General direct limits of finite-dimensional C^*-algebras exhibit certain pathological behaviour which the direct limits of directed sequences do not (see [8]). Such as inseparability and not being determined by a labeled Bratteli diagram (specifically Lemma 3.2.4 ceases to hold). AF-algebras are defined in terms of directed sequences, so therefore by a *direct limit* we shall always mean the direct limit of a directed sequence, and not a general directed set.

2.6 Unital homomorphisms and unital chain systems

In the preceding sections we have not assumed that all our algebras and/or homomorphisms are unital. We will now investigate when this happens and how to "unitize" a chain system.

For an algebra A, we let A^1 denote it's algebraic unitization, that is, the vector space $\{(a, \lambda) | a \in A,\ \lambda \in \mathbb{C}\}$ with adjoint $(a, \lambda)^* = (a^*, \overline{\lambda})$ and product

$$(a, \lambda)(b, \mu) = (ab + a\mu + b\lambda,\ \lambda\mu).$$

The unit in A^1 is $1_{A_1} = (0, 1)$. If A is a C^*-algebra then the C^*-algebraic unitization, $\tilde{A}$, is A^1 equipped with the unitization norm. If A is already unital, then $A^1 \cong A \oplus \mathbb{C}$, where the latter is the algebraic direct sum, i.e. coordinatewise multiplication and unit $(1_A, 1)$. The map $\psi_A \colon A^1 \to A \oplus \mathbb{C}$ given by $\psi_A(a, \lambda) = (a + \lambda 1_A) \oplus \lambda$ is a unital isomorphism. Note that $\psi_A \colon \tilde{A} \to A \oplus \mathbb{C}$ also defines a unital isomorphism if A is a unital C^*-algebra.

Let A and B be algebras. Then any homomorphism $\phi \colon A \to B$ extends uniquely to a unital homomorphism $\phi^1 \colon A^1 \to B^1$ by defining $\phi^1(a, \lambda) = (\phi(a), \lambda)$. If A and B are both unital, then we shall instead consider the unique extension $\tilde{\phi} \colon A \oplus \mathbb{C} \to B \oplus \mathbb{C}$ defined by

$$\tilde{\phi}(a \oplus \lambda) = (\phi(a) + \lambda(1_B - \phi(1_A))) \oplus \lambda.$$

Note that $\tilde{\phi} = \psi_B \circ \phi^1 \circ \psi_A^{-1}$.

We now consider canonical homomorphisms $\rho = [\kappa_{ij}] \colon M(\vec{q}) \to M(\vec{p})$. It's easily seen that ρ is unital precisely when $\rho\vec{q} = \vec{p}$, i.e. when ρ "fills up" every summand of $M(\vec{p})$. For instance, the canonical homomorphism in Example 2.3.2 is unital while the one in Example 2.3.3 is not. The multimatrix algebras are unital and the unital extension of ρ, $\tilde{\rho} \colon M(\vec{q}) \oplus \mathbb{C} \to M(\vec{p}) \oplus \mathbb{C}$, is given by the following matrix:

$$\tilde{\rho} = \left[\begin{array}{ccc:c} \kappa_{11} & \cdots & \kappa_{1s} & h_1 \\ \vdots & \ddots & \vdots & \vdots \\ \kappa_{r1} & \cdots & \kappa_{rs} & h_r \\ \hdashline 0 & \cdots & 0 & 1 \end{array}\right]$$

where $h_i = p_i - \sum_{j=1}^{s} q_j \kappa_{ij}$. This is indeed $\tilde{\rho}$ since it is unital and extends ρ. We see that $\tilde{\rho}$ simply puts enough copies of $\mathbb{C}$ into each summand of $M(\vec{p})$ to fill it up.

By looking at the graph of a canonical homomorphism $\rho\colon M(\vec{q}) \to M(\vec{p})$, like in Figure 2.1, one sees that ρ is unital exactly if, for each i, the integers leading into p_i, by counting multiplicities, add up to p_i. Also, the graph of $\tilde{\rho}$ is obtained by adding one vertex labeled 1 in the upper row, and one vertex labeled 1 in the lower row, each corresponding to the one-dimensional direct summand in the unitization. We also add h_i edges from the upper 1 to p_i for each i, and we add a single edge from the upper 1 to the lower 1. The additional edges introduced are depicted in Figure 2.10.

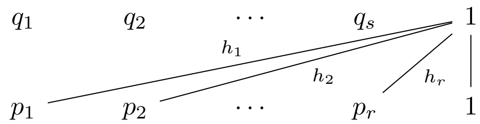

Figure 2.10: The additional edges added to the graph of ρ to obtain $\tilde{\rho}$.

The following results are some useful facts.

Lemma 2.6.1. *Let (A_n, ϕ_n) be an algebraic chain system. If the A_n's and ϕ_n's are all unital, then the limit algebra A_∞ is unital. Consequently, if $\mathfrak{A}_\infty$ is the C^*-algebraic direct limit of a finite-dimensional chain system where the connecting homomorphisms are unital, then $\mathfrak{A}_\infty$ is unital.*

Proof. As the connecting homomorphisms are unital, we have $[1_{A_1}, 1] = [1_{A_n}, n] = \phi_{n\infty}(1_{A_n})$ for each n. And then $[a, n][1_{A_1}, 1] = [a, n][1_{A_n}, n] = [a1_{A_n}, n] = [a, n]$, and similarly the other way around. Hence $[1_{A_1}, 1]$ is the unit in A_∞. And if the A_n's are finite-dimensional C^*-algebras, then $[1_{A_1}, 1]$ is the unit in $C^*(A_\infty) = \mathfrak{A}_\infty$ as well. $\square$

The following proposition shows that the unitization of a direct limit of a chain system is the direct limit of the unitized chain system.

Proposition 2.6.2. *Let $(A_n, \phi_n)_{n\in\mathbb{N}}$ be an algebraic chain system. Then*

(1) $\left(\varinjlim(A_n, \phi_n)\right)^1 \cong \varinjlim(A_n^1, \phi_n^1)$.

(2) *Moreover, if each A_n is unital, then* $\left(\varinjlim(A_n, \phi_n)\right)^1 \cong \varinjlim(A_n \oplus \mathbb{C}, \tilde{\phi_n})$.

Proof. Let $B_\infty = \varinjlim(A_n^1, \phi_n^1)$. Note that B_∞ is unital by Lemma 2.6.1. Define a map $\psi\colon A_\infty^1 \to B_\infty$ by

$$([a_n, n], \lambda) \longmapsto [(a_n, \lambda), n].$$

To see that ψ is well defined suppose $[a_n, n] = [a_m, m]$ in A_∞. Then there is a k such that $\phi_{nk}(a_n) = \phi_{mk}(a_m)$. Which implies that

$$[(a_n, \lambda), n] = [\phi_{nk}^1(a_n, \lambda), k] = [(\phi_{nk}(a_n), \lambda), k] = [(\phi_{mk}(a_m), \lambda), k] = [(a_m, \lambda), m]$$

in B_∞. Clearly, ψ is a bijection and a homomorphism, thus $A_\infty^1 \cong B_\infty$.

If each A_n is unital, then $(\psi_{A_n})_{n\in\mathbb{N}}$ defines a chain system isomorphism from (A_n^1, ϕ_n^1) to $(A_n \oplus \mathbb{C}, \tilde{\phi_n})$ because $\tilde{\phi_n} = \psi_{A_{n+1}} \circ \phi_n^1 \circ \psi_{A_n}^{-1}$. Thus $A_\infty^1 \cong B_\infty \cong \varinjlim(A_n \oplus \mathbb{C}, \tilde{\phi_n})$. □

An immediate consequence of this result is that the unitization of a locally semisimple algebra is also locally semisimple (since the unitization of a semisimple algebra is also semisimple). This extends to C^*-algebraic direct limits as follows.

Proposition 2.6.3. *If $\mathfrak{A}_\infty$ is the C^*-algebraic direct limit of a finite-dimensional chain system (A_n, ϕ_n), then the unitization $\widetilde{\mathfrak{A}_\infty}$ is isomorphic to the C^*-algebraic direct limit of the corresponding unitized chain system (A_n, ϕ_n^1).*

Proof. Let B_∞ denote the algebraic direct limit of (A_n^1, ϕ_n^1) and $\mathfrak{B}_\infty$ the C^*-algebraic limit. Then $A_\infty^1 \cong B_\infty$ by the previous proposition. Therefore $C^*(A_\infty^1) \cong \mathfrak{B}_\infty$ as C^*-algebras by Lemma 2.5.2. Since A_∞ is dense in $\mathfrak{A}_\infty$, it follows that A_∞^1 is dense in $\widetilde{\mathfrak{A}_\infty}$. Hence $\widetilde{\mathfrak{A}_\infty} \cong C^*(A_\infty^1) \cong \mathfrak{B}_\infty$ as C^*-algebras. □

Chapter 3

AF-algebras

It is time to utilize our constructions from the previous chapter. We start by defining AF-algebras and prove some of the basic properties of this class of C^*-algebras. Then we describe the correspondence between AF-algebras and labeled Bratteli diagrams. We will also look at some examples which illustrate various properties of AF-algebras before we move on to a local characterization and an investigation of the ideal structure of AF-algebras. We also prove that any separable commutative C^*-algebra can be embedded into a certain fixed AF-algebra.

3.1 Basic properties of AF-algebras

Definition 3.1.1. A C^*-algebra is called an *AF-algebra*, or simply *AF*, if it is isomorphic to the direct limit of (a directed sequence of) finite-dimensional C^*-algebras.

AF is an abbreviation for "approximately finite-dimensional". This name comes from the fact that in an AF-algebra one can make "finite-dimensional approximations" - more on this later. When we defined the direct limit of a finite-dimensional chain system in the previous chapter, we positively used the fact that the underlying locally semisimple algebra (which is dense by definition) could be written as an increasing union of finite-dimensional subalgebras. It is therefore not so surprising that we have the following characterization of AF-algebras.

Proposition 3.1.2. *Let A be a C^*-algebra. Then A is an AF-algebra if and only if there exists a sequence of finite-dimensional C^*-subalgebras $A_1 \subseteq A_2 \subseteq \ldots \subseteq A$ such that $\cup_{n=1}^{\infty} A_n$ is dense in A.*

Before moving on to the proof we make the following nice observation.

Lemma 3.1.3. *Suppose A is a C^*-algebra which contains a sequence of finite-dimensional C^*-subalgebras, A_n, such that $A_1 \subseteq A_2 \subset \ldots$ and $\overline{\cup_{n=1}^{\infty} A_n} = A$. Let i_n denote the inclusion map $i_n \colon A_n \hookrightarrow A_{n+1}$. Then A is isomorphic to the direct limit, $\mathfrak{A}_\infty$, of the chain system $(A_n, i_n)_{n \in \mathbb{N}}$.*

Proof. Let A_∞ be the $*$-algebraic limit of the following chain system:

$$A_1 \xhookrightarrow{i_1} A_2 \xhookrightarrow{i_2} A_3 \xhookrightarrow{i_3} \cdots$$

As the connecting homomorphisms i_n are inclusions we have that for each $n \in \mathbb{N}$,

$$[a, n] = [a, n+1] = [a, n+2] = \ldots \in A_\infty$$

for any $a \in A_n$. Therefore $\cup_{n=1}^\infty A_n \cong A_\infty$ as $*$-algebras. They are also isomorphic as normed $*$-algebras because

$$\|[a, n]\|_{A_\infty} = \lim_{m\to\infty} \|i_{nm}(a)\|_{A_m} = \lim_{m\to\infty} \|a\|_A = \|a\|_A.$$

Thus $A \cong \mathfrak{A}_\infty$ as C^*-algebras. $\square$

Proof of Proposition 3.1.2. Since A is AF, $A \cong \mathfrak{A}_\infty$ where $\mathfrak{A}_\infty$ is the direct limit of a finite-dimensional chain system. By definition $\mathfrak{A}_\infty$ contains an increasing sequence of finite-dimensional C^*-subalgebras whose union is dense. Hence so does A.

Conversely, if A contains a sequence of finite-dimensional C^*-subalgebras such that $A_1 \subseteq A_2 \subseteq \ldots$ and $\overline{\cup_{n=1}^\infty A_n} = A$, then A is AF by Lemma 3.1.3. $\square$

Taking direct limits of canonical chain systems is a nice way to construct AF-algebras, while Proposition 3.1.2 is a nice way to check whether a given C^*-algebra is AF. It is also often easier, and less notation-heavy, to work with an increasing union of subalgebras, since one does not have to deal with equivalence classes. The following results are some useful observations.

Proposition 3.1.4. *Every AF algebra is separable.*

Proof. Let A be an AF-algebra. Then $A = \overline{\cup_{n=1}^\infty A_n}$ for some increasing union of finite-dimensional C^*-subalgebras A_n. In particular each A_n is separable, so there is a countable dense subset $D_n \subset A_n$ for each n. We claim that $\cup_{n=1}^\infty D_n$ is a countable dense subset of A. Indeed for any $a \in A$, given $\epsilon > 0$, we can find $a_n \in A_n$ for some n such that $\|a - a_n\| < \frac{\epsilon}{2}$. We can also find $d_n \in D_n$ with $\|d_n - a_n\| < \frac{\epsilon}{2}$. And then $\|a - d_n\| < \epsilon$. $\square$

Proposition 3.1.5. *If A is an AF-algebra, then the unitization $\tilde{A}$ is also AF.*

Proof. This is immediate from Proposition 2.6.3. We also give a proof using Proposition 3.1.2 to demonstrate the usefulness of this result in a theoretical context.

Since A is AF, $A = \overline{\cup_{n=1}^\infty A_n}$ for some increasing union of finite-dimensional C^*-subalgebras A_n. Then $\tilde{A}_1 \subseteq \tilde{A}_2 \subseteq \ldots$ is an increasing family of finite-dimensional C^*-subalgebras of $\tilde{A}$, and clearly $\overline{\cup_{n=1}^\infty \tilde{A}_n} = \tilde{A}$. Thus $\tilde{A}$ is AF. $\square$

Lemma 3.1.6. *Let A be a ring with unit 1. Assume that B is a subring of A with (local) unit e. If B contains an element which is invertible in A, then $e = 1$.*

Proof. Suppose $b \in B$ is invertible in A. Then $\exists\ a \in A$ with $ab = 1$. As e is the unit in B we have $be = b$. This gives $1 = ab = abe = 1e = e$. $\square$

Lemma 3.1.7. *If $A = \overline{\cup_{n=1}^{\infty} A_n}$ is a unital AF-algebra, then there exists an $N \in \mathbb{N}$ such that $1_A \in A_N \subseteq A_{N+1} \subseteq \dots$*

Proof. As $\cup_{n=1}^{\infty} A_n$ is dense in A, there is an $N \in \mathbb{N}$ and an element $a \in A_N$ such that $\|a - 1_A\| < 1$. Then a is invertible in A, and since A_N is finite-dimensional, it has a local unit. By the previous lemma, $1_A \in A_N$. $\square$

It is also worth noting that there is no loss of generality by assuming that the connecting homomorphisms in a chain system are injective.

Lemma 3.1.8. *Suppose $(A_n, \phi_n)_{n\in\mathbb{N}}$ is a finite-dimensional chain system. Let $A'_n = A_n/\ker(\phi_{n\infty})$. The induced homomorphisms $\phi'_n \colon A'_n \to A'_{n+1}$ are given by $a_n + \ker(\phi_{n\infty}) \longmapsto \phi_n(a_n) + \ker(\phi_{(n+1)\infty})$. Then ϕ'_n is injective for each n and $\varinjlim(A_n, \phi_n) \cong \varinjlim(A'_n, \phi'_n)$ as algebras.*

Consequently, if (A_n, ϕ_n) is a finite-dimensional chain system, then the C^*-algebraic direct limit $\mathfrak{A}_\infty$ is isomorphic to the C^*-algebraic direct limit of the injective chain system (A'_n, ϕ'_n).

Proof. We start by noting that ϕ'_n is injective. Indeed, if $\phi'_n(a_n + \ker(\phi_{n\infty})) = 0$, then $\phi_n(a_n) \in \ker(\phi_{(n+1)\infty})$, but then $\phi_{(n+1)\infty}(\phi_n(a_n)) = \phi_{n\infty}(a_n) = 0$, so $a_n \in \ker(\phi_{n\infty})$. Let $B_n = \phi_{n\infty}(A_n) \subseteq A_\infty$ and let i_n denoted the inclusion of B_n into B_{n+1}. Let $\phi'_{n\infty} \colon A'_n \to B_n$ denote the induced isomorphism. Then the following diagram commutes for each n:

$$\begin{tikzcd} A'_n \arrow[r, "\phi'_n"] \arrow[d, "\cong"', "\phi'_{n\infty}"] & A'_{n+1} \arrow[d, "\cong"', "\phi'_{(n+1)\infty}"] \\ B_n \arrow[r, hook, "i_n"] & B_{n+1} \end{tikzcd}$$

Thus the family $(\phi'_{n\infty})_{n\in\mathbb{N}}$ is a chain system isomorphism from (A'_n, ϕ'_n) to (B_n, i_n). Hence $\varinjlim(A'_n, \phi'_n) \cong \varinjlim(B_n, i_n) \cong \varinjlim(A_n, \phi_n)$, where the last isomorphism follows from Lemma 3.1.3. The last statement follows from Lemma 2.5.2. $\square$

3.2 The labeled Bratteli diagram of an AF-algebra

To a given labeled Bratteli diagram (V, E, r, s, d), there corresponds a unique AF-algebra (up to isomorphism). Namely the direct limit of any canonical chain system whose labeled Bratteli diagram is (V, E, r, s, d). That is, the direct limit of the canonical chain system $(A_n, \rho_n)_{n\in\mathbb{Z}^+}$ where $A_n = \oplus_{v\in V_n} M_{d(v)}$ and ρ_n is the canonical homomorphism whose graph is $(V_n \sqcup V_{n+1}, E_{n+1})$. By Lemma 2.4.6 the order in which the vertices are listed does not matter, hence a labeled Bratteli diagram determines a unique AF-algebra.

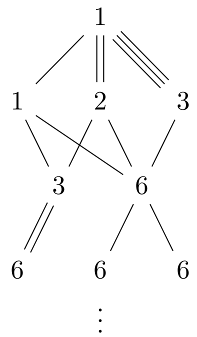

Figure 3.1: The labeled Bratteli diagram considered in Example 3.2.1.

Example 3.2.1. Let (V, E) be the labeled Bratteli diagram in Figure 3.1. Then the AF-algebra corresponding to (V, E) is the direct limit of the following chain system:

$$\mathbb{C} \xrightarrow{\left[\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right]} M(1,2,3) \xrightarrow{\left[\begin{smallmatrix}1&1&0\\1&1&1\end{smallmatrix}\right]} M(3,6) \xrightarrow{\left[\begin{smallmatrix}2&0\\0&1\\0&1\end{smallmatrix}\right]} M(6,6,6) \xrightarrow{\cdots} \cdots$$

Associating a labeled Bratteli diagram to a given AF-algebra is a bit more tricky. This is because many different labeled Bratteli diagrams yield the same AF-algebra. In particular, telescope-equivalent labeled Bratteli diagrams yield isomorphic AF-algebras. As this is an important fact on its own, we list it as a proposition. The proof consists mostly of “notation juggling”.

Proposition 3.2.2. *If (V, E) and (V', E') are labeled Bratteli diagrams such that (V', E') is a telescoping of (V, E), then the AF-algebras corresponding to the two diagrams are isomorphic.*

Proof. Let (V, E) be a labeled Bratteli diagram and suppose (V', E') is the labeled Bratteli diagram obtained when telescoping (V, E) with respect to a sequence $m_0 < m_1 < \ldots$ of positive integers. Let $(A_n, \phi_n)_{n\in\mathbb{Z}^+}$ be a canonical chain system coming from (V, E). As usual let A_∞ and $\mathfrak{A}_\infty$ denote the $*$-algebraic and C^*-algebraic direct limits. Also let $A'_n = A_{m_n}$ and $\phi'_n = \phi_{m_n m_{n+1}} = \phi_{m_{n+1}-1} \circ \phi_{m_{n+1}-2} \circ \cdots \circ \phi_{m_n}$. Then ϕ'_n is inner equivalent to the canonical homomorphism determined by E'_n. Therefore A'_∞ is the locally semisimple $*$-algebra corresponding to (V', E'), and $\mathfrak{A}'_\infty$ is the AF-algebra corresponding to (V', E'). Now define the homomorphisms $\theta_n \colon A_n \to A'_n$ by $\theta_n = \phi_{n m_n}$. Then the following diagram commutes:

$$\begin{tikzcd}
A_0 \arrow[r, "\phi_0"] \arrow[d, "\theta_0=\phi_{0m_0}"] & A_1 \arrow[r, "\phi_1"] \arrow[d, "\theta_1=\phi_{1m_1}"] & A_2 \arrow[r, "\phi_2"] \arrow[d, "\theta_2=\phi_{2m_2}"] & \cdots \\
A'_0 = A_{m_0} \arrow[r, "\phi'_0=\phi_{m_0m_1}"] & A'_1 = A_{m_1} \arrow[r, "\phi'_1=\phi_{m_1m_2}"] & A'_2 = A_{m_2} \arrow[r, "\phi'_2=\phi_{m_2m_3}"] & \cdots
\end{tikzcd}$$

Denote the equivalence classes in A'_∞ by $[a_{m_n}, n]'$ for $a_{m_n} \in A'_n$. We claim that the induced map $\theta_\infty \colon A_\infty \to A'_\infty$, i.e. $\theta_\infty([a_n, n]) = [\theta_n(a_n), n]'$, is an isomorphism. By construction, θ_∞ is a homomorphism.

For any $a_{m_n} \in A'_n$ we have

$$[a_{m_n}, n]' = [\theta_{m_n}(a_{m_n}), m_n]' = \theta_\infty([a_{m_n}, m_n])$$

since

$$\theta_{m_n}(a_{m_n}) = \phi_{m_n m_{m_n}}(a_{m_n}) = \phi'_{n m_n}(a_{m_n}),$$

so θ_∞ is surjective.

And if $a_n \in A_n$ and $a_l \in A_l$ with $\theta_\infty([a_n, n]) = \theta_\infty([a_l, l])$, then

$$[\phi_{n m_n}(a_n), n]' = [\phi_{l m_l}(a_l), l]',$$

which means that there is a k such that

$$\phi'_{nk}(\phi_{n m_n}(a_n)) = \phi'_{lk}(\phi_{l m_l}(a_l)).$$

But then $[a_n, n] = [a_l, l]$, hence θ_∞ is injective.

We have shown that $A_\infty \cong A'_\infty$ as $*$-algebras, and it follows that the C^*-algebraic limits $\mathfrak{A}_\infty$ and $\mathfrak{A}'_\infty$ are isomorphic by Lemma 2.5.2. Thus (V, E) and (V', E') determine isomorphic AF-algebras. $\square$

Let A be an AF-algebra. We would like to associate a labeled Bratteli diagram (V, E) to A such that the AF-algebra corresponding to (V, E) is (isomorphic to) A. Proposition 2.3.6 suggests how we might go at it.

Suppose $(A_n, \phi_n)_{n \in \mathbb{N}}$ is a finite-dimensional chain system with $\mathfrak{A}_\infty \cong A$. As the A_n's are finite-dimensional, $A_n \cong M(\vec{p}(n))$ for each n. Note that here we have made a choice regarding the order of the coordinates in $\vec{p}(n)$. Next we can choose isomorphisms $\psi_n \colon A_n \to M(\vec{p}(n))$ for each n. Defining $\theta_n = \psi_{n+1} \circ \phi_n \circ \psi_n^{-1}$ turns (ψ_n) into a chain system isomorphism, hence $\varinjlim(A_n, \phi_n) \cong \varinjlim(M(\vec{p}(n)), \theta_n)$. By Theorem 2.3.5 each θ_n is inner equivalent to a unique canonical homomorphism $\rho_n = [\kappa_{ij}(n)]$. And so $\varinjlim(M(\vec{p}(n)), \theta_n) \cong \varinjlim(M(\vec{p}(n)), \rho_n)$ by Proposition 2.2.6. Thus the C^*-algebraic direct limits of the following chain systems are all isomorphic:

$$\begin{array}{ccccccc}
A_1 & \xrightarrow{\phi_1} & A_2 & \xrightarrow{\phi_2} & A_3 & \xrightarrow{\phi_3} & \cdots \\
\cong\big\downarrow \psi_1 & & \cong\big\downarrow \psi_2 & & \cong\big\downarrow \psi_3 & & \\
M(\vec{p}(1)) & \xrightarrow{\theta_1} & M(\vec{p}(2)) & \xrightarrow{\theta_2} & M(\vec{p}(3)) & \xrightarrow{\theta_3} & \cdots
\end{array}$$

$$M(\vec{p}(1)) \xrightarrow{\rho_1} M(\vec{p}(2)) \xrightarrow{\rho_2} M(\vec{p}(3)) \xrightarrow{\rho_3} \cdots$$

In particular $A \cong C^*\left(\varinjlim(M(\vec{p}(n)), \rho_n)\right)$ and since this chain system is canonical, we obtain a labeled Bratteli diagram as in Section 2.4. By construction, A is the AF-algebra corresponding to this diagram.

Note that this also includes the case when A is given as $A = \overline{\cup_{n=1}^{\infty} A_n}$ for some increasing union of finite dimensional C^*-subalgebras A_n. For then $(A_n, i_n)_{n\in\mathbb{N}}$ is a finite-dimensional chain system whose limit is A.

In the construction above, one has to make some choices. First one must decide on a permutation of the coordinates of the vector $\vec{p}(n)$ for each n. Then one must choose isomorphisms $\psi_n \colon A_n \to M(\vec{p}(n))$ for each n. However, different choices only lead to a permutation of the vertices V_n at each level of the labeled Bratteli diagram with an associated permutation of the edges E_n. Hence the resulting labeled Bratteli diagrams are all isomorphic.

Definition 3.2.3. Given a finite-dimensional chain system $(A_n, \phi_n)_{n\in\mathbb{N}}$, then any labeled Bratteli diagram coming from the construction above will be called a *labeled Bratteli diagram associated to* (A_n, ϕ_n).

The next result encapsulates what we mean when we say that an AF-algebra is determined by its labeled Bratteli diagram. The "practical use" of the result is that if one has two AF-algebras, and manages to write down labeled Bratteli diagrams associated to each of them, and the labeled Bratteli diagrams turn out to be the same (i.e. isomorphic), then one can conclude that the AF-algebras are isomorphic.

Lemma 3.2.4. *If $A = \overline{\cup_{n=1}^{\infty} A_n}$ and $B = \overline{\cup_{n=1}^{\infty} B_n}$ are AF-algebras having isomorphic labeled Bratteli diagrams (associated to the two chain systems of inclusions), then $A \cong B$.*

Proof. Immediate from Definition 3.2.3, Lemma 2.4.6 and Lemma 2.5.2. □

Before moving on to some concrete examples of AF-algebras we are going to make some general assumptions for what follows. We would like the labeled Bratteli diagram to be "the same" whether we give an AF-algebra as a chain system of finite-dimensional C^*-algebras or as an increasing sequence of finite-dimensional C^*-subalgebras. The chain system associated to the latter is always injective, since the connecting homomorphisms are inclusions, while the former need not be injective. Thanks to Lemma 3.1.8 we can assume that any chain system is injective. And in that case we get the following result.

Lemma 3.2.5. *Let $(A_n, \phi_n)_{n\in\mathbb{N}}$ be a finite-dimensional chain system. Let $B_n = \phi_{n\infty}(A_n) \subseteq A_\infty$ and let i_n denote the inclusion of B_n into B_{n+1}. If each ϕ_n is injective, then any labeled Bratteli diagram associated to $(A_n, \phi_n)_{n\in\mathbb{N}}$ is also associated to $(B_n, i_n)_{n\in\mathbb{N}}$ and vice versa.*

Proof. By our hypothesis

$$B_n = \phi_{n\infty}(A_n) = \phi_{(n+1)\infty}(\phi_n(A_n)) \subseteq \phi_{(n+1)\infty}(A_{n+1}) = B_{n+1}, \tag{3.1}$$

and $\phi_{n\infty}$ and $\phi_{(n+1)\infty}$ are isomorphisms onto B_n and B_{n+1} respectively. Consider the diagram in Figure 3.2. Equation (3.1) states that the middle square commutes. The other two squares correspond to induced canonical homomorphisms ρ, ρ' which come from isomorphisms onto $M(\vec{p}(n))$ and $M(\vec{p}(n+1))$ respectively, as in the

$$\begin{tikzcd}
M(\vec{p}(n)) \arrow[r, dashed, "\rho"] \arrow[d, "\cong"'] & M(\vec{p}(n+1)) \\
A_n \arrow[r, "\phi_n"] \arrow[r, bend left] \arrow[r, bend right] \arrow[d, shift right, "\cong"', "\phi_{n\infty}"'] & A_{n+1} \arrow[u, "\cong"'] \arrow[d, shift left] \arrow[d, shift right, "\cong", "\phi_{(n+1)\infty}"] \\
B_n \arrow[r, hook, "i_n"] \arrow[r, shift right] \arrow[u, shift right] & B_{n+1} \arrow[d, "\cong"] \\
M(\vec{p}(n)) \arrow[r, dashed, "\rho'"] \arrow[u, "\cong"] & M(\vec{p}(n+1))
\end{tikzcd}$$

Figure 3.2: The commutative diagram in the proof of Lemma 3.2.5.

construction preceding Definition 3.2.3. By the commutativity of the diagram, the two blue paths both equal ρ'. Thus ρ and ρ' are both canonical homomorphisms induced by ϕ_n. Since i_n induces ρ', the graphs of ϕ_n and i_n are isomorphic. $\square$

General assumptions: Together with Lemma 3.1.3 the preceding lemma shows that for a given AF-algebra, an injective chain system of finite-dimensional C^*-algebras defining the AF-algebra corresponds to a dense increasing union of finite-dimensional C^*-subalgebras and vice versa. Therefore we shall always assume that the connecting homomorphisms in our chain systems are injective. This means that the associated labeled Bratteli diagrams will have no sinks. When we say that "$A = \overline{\cup_{n=0}^{\infty} A_n}$ is an AF-algebra" we shall always assume that $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ and that these are finite-dimensional C^*-subalgebras of A. Furthermore, if A is unital, we shall assume that any such sequence $A_0, A_1, \dots \subseteq A$ of C^*-subalgebras begins with $A_0 = \{\lambda \cdot 1_A | \lambda \in \mathbb{C}\}$. This can be assumed without loss of generality because of Lemma 3.1.7. Finally, when we say that "$A = \overline{\cup_{n=0}^{\infty} A_n}$ is an AF-algebra with an associated labeled Bratteli diagram (V, E)" we assume that (V, E) is associated to the chain system (A_n, i_n) where i_n denotes the inclusion map.

3.3 Examples of AF-algebras

Now it is about time we consider some concrete examples. We shall show that two familiar C^*-algebras are AF, and find their associated labeled Bratteli diagrams. We shall also consider an AF-algebra defined by a direct limit.

The construction preceding Definition 3.2.3 can always be done in theory. In practice however, i.e. for a given AF-algebra, it is not necessarily easy to write down an associated labeled Bratteli diagram. In the examples that follows it can be done very explicitly, but it still requires some thought and work.

Example 3.3.1. Let $A = \mathbb{C}I + \mathcal{K}$. Here $\mathcal{K}$ denotes the C^*-algebra of compact operators on some separable infinite-dimensional Hilbert space H, and I is the identity operator on H, so that $A \subseteq \mathcal{B}(H)$. (For instance $H = L^2([0,1])$.) Let

$\xi_1, \xi_2, \ldots$ be an orthonormal basis for H. Let P_n be the orthogonal projection onto the span of $\xi_1, \ldots, \xi_n$. Also let $P_n^\perp$ denote the orthogonal projection onto $\operatorname{span}\{\xi_1, \ldots, \xi_n\}^\perp = \overline{\operatorname{span}\{\xi_{n+1}, \xi_{n+2}, \ldots\}}$.

We wish to show that A is AF. To this end, define $A_n := \mathbb{C}P_n^\perp + P_n \mathcal{K} P_n$ for $n = 0, 1, 2, \ldots$. By interpreting P_0 as the zero-operator we get $A_0 = \mathbb{C} \cdot I$. To see that $A_n \subseteq A$ note that $P_n + P_n^\perp = I$, and then

$$\begin{aligned} \lambda P_n^\perp + P_n K P_n &= \lambda P_n^\perp + \lambda P_n - \lambda P_n + P_n K P_n \\ &= \lambda I + (P_n K P_n - \lambda P_n) \in \mathbb{C}I + \mathcal{K} \end{aligned}$$

for $\lambda \in \mathbb{C}$ and $K \in \mathcal{K}$. We also have that

$$P_n \mathcal{K} P_n = P_{n+1}(P_n \mathcal{K} P_n) P_{n+1} \subseteq P_{n+1} \mathcal{K} P_{n+1}.$$

Let E_{n+1} denote the orthogonal projection onto ξ_{n+1}. Then $P_n^\perp = P_{n+1}^\perp + E_{n+1}$ which implies that

$$\lambda P_n^\perp + P_n K P_n = \lambda P_{n+1}^\perp + (P_n K P_n + \lambda E_{n+1}) \in \mathbb{C}P_{n+1}^\perp + P_{n+1} \mathcal{K} P_{n+1}. \tag{3.2}$$

From this we see that $A_n \subseteq A_{n+1}$. Next, observe that

$$\begin{aligned} P_n \mathcal{K} P_n &= \{K \in \mathcal{K} \mid \operatorname{Ran}(K) \subseteq \operatorname{Ran}(P_n) \wedge \ker(K) \subseteq \ker(P_n)\} \\ &\cong \mathcal{B}(\operatorname{span}\{\xi_1, \ldots, \xi_n\}) \cong M_n. \end{aligned}$$

Hence $A_n \cong \mathbb{C} \oplus M_n$, and A_n is in particular finite-dimensional. To see that $\cup_{n=0}^\infty A_n$ is dense in A we first observe that $P_n \to I$ pointwise. It follows that $P_n K P_n \to K$ in the operator norm when K is compact. Therefore

$$\lambda P_n^\perp + P_n(K + \lambda P_n) P_n = \lambda I + P_n K P_n \to \lambda I + K$$

in the operator norm. Since the former operator is in A_n, the union is dense. Thus A is AF.

The computation in (3.2) shows that the canonical homomorphism corresponding to the inclusion $\mathbb{C} \oplus M_n \cong A_n \subseteq A_{n+1} \cong \mathbb{C} \oplus M_{n+1}$ embeds M_n into M_{n+1} and $\mathbb{C}$ into each of $\mathbb{C}$ and M_{n+1}, respectively. Hence the graph of this canonical homomorphism is the graph depicted in Figure 3.3. By joining together these graphs we obtain a labeled Bratteli diagram associated to $\mathbb{C}I + \mathcal{K} = A = \overline{\cup_{n=0}^\infty A_n}$, and this labeled Bratteli diagram is depicted in Figure 3.4.

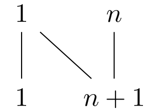

Figure 3.3: The graph of the canonical homomorphism corresponding to the inclusion $A_n \subseteq A_{n+1}$.

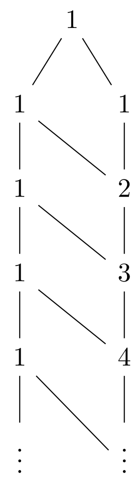

Figure 3.4: A labeled Bratteli diagram associated to the AF-algebra $A = \mathbb{C}I + \mathcal{K}$.

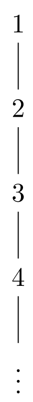

Figure 3.5: A labeled Bratteli diagram associated to $\mathcal{K}$.

One last thing to notice is that $\mathcal{K}$ is a (closed two-sided) ideal in A and $\mathcal{K} = \overline{\cup_{n=1}^{\infty}\mathcal{K}_n}$ where $\mathcal{K}_n = P_n\mathcal{K}P_n = \mathcal{K} \cap A_n$. Moreover, the subdiagram of the labeled Bratteli diagram of A (Figure 3.4) depicted in Figure 3.5 represents the ideal $\mathcal{K} = \overline{\cup_{n=1}^{\infty}\mathcal{K}_n}$. This is no coincidence. We shall see in Section 3.5 that ideals in AF-algebras are always coming from ideals of the defining sequence A_n, and that they can be read straight off a labeled Bratteli Diagram.

Example 3.3.2. Our next example is an AF-algebra defined (a priori) by a canonical chain system. Consider the following chain system:

$$\mathbb{C} \xrightarrow{[2]} M_2 \xrightarrow{[2]} M_4 \xrightarrow{[2]} M_8 \xrightarrow{[2]} M_{16} \xrightarrow{[2]} \dots \tag{3.3}$$

Here the canonical homomorphisms $\rho_n\colon M_{2^n} \to M_{2^{n+1}}$ are given by

$$\rho_n(A) = \left[\begin{array}{c:c} A & 0 \\ \hdashline 0 & A \end{array}\right]$$

The C^*-algebraic direct limit of this system is called the *CAR algebra*. CAR is an abbreviation for "Canonical Anticommutation Relations". We observe that the labeled Bratteli diagram corresponding to the canonical chain system (3.3) is the diagram depicted in Figure 3.6.

$$\begin{array}{c} 1 \\ \| \\ 2 \\ \| \\ 4 \\ \| \\ 8 \\ \| \\ 16 \\ \| \\ \vdots \end{array}$$

Figure 3.6: The labeled Bratteli diagram corresponding to the canonical chain system (3.3).

We shall now show that the CAR algebra is also the AF-algebra associated to the labeled Bratteli diagram in Figure 2.5. To this end, let

$$B_n = \left\{ \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \middle| A_1, A_2 \in M_{2^n} \right\} \cong M_{2^n} \oplus M_{2^n}.$$

Then $\rho_n(M_{2^n}) \subseteq B_n \subseteq M_{2^{n+1}}$. This implies that

$$\rho_{n\infty}(M_{2^n}) \subseteq \rho_{(n+1)\infty}(B_n) \subseteq \rho_{(n+1)\infty}(M_{2^{n+1}}).$$

Thus the algebraic limit of the chain system is

$$A_\infty = \cup_{n=0}^{\infty} \rho_{n\infty}(M_{2^n}) = \cup_{n=0}^{\infty} \rho_{(n+1)\infty}(B_n).$$

Hence the CAR algebra is also the C^*-algebraic direct limit of the chain system (B_n, ρ_{n+1}). To find a labeled Bratteli diagram associated to this system we consider the embedding $\rho_{n+1} \colon B_n \to B_{n+1}$, which is given by

$$\rho_{n+1}\left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\right) = \left[\begin{array}{cc:cc} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ \hdashline 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & A_2 \end{array}\right]$$

By identifying B_n with $M_{2^n} \oplus M_{2^n}$ we see that the canonical homomorphism from $M_{2^n} \oplus M_{2^n}$ to $M_{2^{n+1}} \oplus M_{2^{n+1}}$ corresponding to ρ_{n+1} is given by the matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. The corresponding labeled Bratteli diagram is therefore the diagram depicted in Figure 3.7.

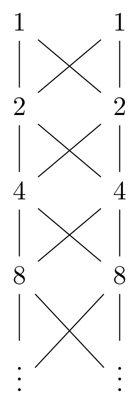

Figure 3.7: A labeled Bratteli diagram associated to the chain system (B_n, ρ_{n+1}).

Finally, consider the labeled Bratteli diagram in Figure 3.8. The thing to notice about this diagram is that telescoping to even levels (recall that the top level is level 0) gives the diagram of $(M_{2^n}, [2])$ and telescoping to odd levels gives the diagram of (B_n, ρ_{n+1}). This means that the two first diagrams are telescope equivalent. As we shall see in the next section, this is always the case when two diagrams represent the same AF-algebra.

Before we move on to the next example we shall make a few quick observations. By Proposition 1.3.3 every element in a finite-dimensional C^*-algebra can be written as a linear combination of projections. Therefore any AF-algebra is spanned

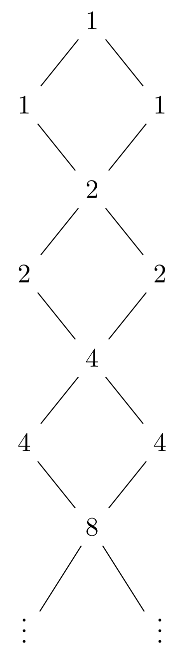

Figure 3.8: A third labeled Bratteli diagram representing the CAR algebra.

by its projections. By which we mean that the C^*-algebra equals the closed linear span of its projections.

Let X be a locally compact Hausdorff space. The projections in $C_0(X)$ are characteristic functions χ_Z where Z is a clopen compact subset of X. $C_0(X)$ is spanned by its projections precisely when X is totally disconnected and $C_0(X)$ is separable precisely when X is second countable. In toto, we get the following characterization of commutative AF-algebras.

Theorem 3.3.3. *A separable commutative C^*-algebra is AF if and only if it is spanned by its projections. Consequently, a commutative C^*-algebra $C_0(X)$ is AF if and only if X is a locally compact Hausdorff second countable and totally disconnected space.*

On the other hand, if A is an AF-algebra with an associated labeled Bratteli diagram (V, E, r, s, d), then A is commutative if and only if $d(v) = 1$ for all $v \in V$.

Proof. We have already argued for the "only if" direction of the first part. To prove the "if" direction, suppose A is a separable and commutative C^*-algebra which is spanned by its projections. This means that $A = \overline{\operatorname{span}\{p \in A | \, p \text{ projection}\}}$. By separability, we can find a countable set of projections $\{p_1, p_2, p_3, \ldots\}$ whose linear span is dense in A. Let A_n be the linear span of all finite products of $p_1, \ldots, p_n$, i.e.

$$A_n = \operatorname{span}\{p_1, \dots,\ p_n,\ p_1p_2, \dots,\ p_1p_n,\ p_2p_3, \dots, p_2p_n, \dots,\\ p_{n-1}p_n, \dots,\ p_1p_2\cdots p_{n-1}, \dots,\ p_2p_3\cdots p_n,\ p_1p_2\cdots p_n\}.$$

Clearly $A_n \subseteq A_{n+1}$. Since A is commutative, and the p_k's are projections, A_n is closed under multiplication and adjoints, hence A_n is a finite-dimensional C^*-subalgebra of A. As $\operatorname{span}\{p_1,\ p_2, \dots\} \subseteq \cup_{n=1}^{\infty} A_n$, we have $\overline{\cup_{n=1}^{\infty} A_n} = A$. Thus A is AF.

For the second part, assume that A is an AF-algebra with labeled Bratteli diagram (V, E, r, s, d). If $d(v) = 1$ for all $v \in V$, then each multimatrix algebra in the associated canonical chain system is commutative, hence so is the direct limit A. Conversely, if $d(w) \geq 2$ for some $w \in V$, then A contains a noncommutative C^*-subalgebra, hence A is noncommutative. □

Example 3.3.4. We now look at such an example of a commutative AF-algebra. Let X be the classical Cantor ternary set. Then $X = \cap_{n=0}^{\infty} X_n$ where X_n is the disjoint union of 2^n closed intervals of length 3^{-n} contained in $[0,1]$ which are left after deleting the "middle thirds" n times. More explicitly,

$$X_0 = [0,1],\ X_1 = \left[0, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, 1\right],\ X_2 = \left[0, \tfrac{1}{9}\right] \cup \left[\tfrac{2}{9}, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, \tfrac{7}{9}\right] \cup \left[\tfrac{8}{9}, 1\right] \text{ etc.}$$

For $n \in \mathbb{Z}^+$, let $A_n \subseteq C(X)$ be the C^*-subalgebra of functions that are constant on each of the intervals of X_n. By letting $I_n^{(k)}$ for $1 \leq k \leq 2^n$ denote the 2^n disjoint intervals which make up X_n, we get an explicit characterization of A_n as

$$A_n = \operatorname{span}\{\chi_{X \cap I_n^{(1)}}, \chi_{X \cap I_n^{(2)}}, \dots, \chi_{X \cap I_n^{(2^n)}}\} \cong \mathbb{C}^{2^n}.$$

As each interval of X_n contains exactly two disjoint intervals of X_{n+1} we get that

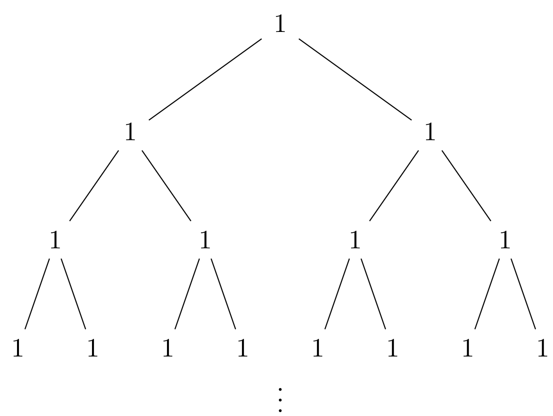

Figure 3.9: A labeled Bratteli diagram associated to $C(X)$, where X is the Cantor set.

$A_n \subseteq A_{n+1}$ and the inclusion is given explicitly as

$$\lambda_1 \chi_{X\cap I_n^{(1)}} + \cdots + \lambda_{2^n} \chi_{X\cap I_n^{(2^n)}} = \lambda_1 \left(\chi_{X\cap I_{n+1}^{(1)}} + \chi_{X\cap I_{n+1}^{(2)}} \right) + \cdots$$
$$+ \lambda_{2^n} \left(\chi_{X\cap I_{n+1}^{(2^{n+1}-1)}} + \chi_{X\cap I_{n+1}^{(2^{n+1})}} \right) \in A_{n+1}.$$

So the canonical homomorphism corresponding to the inclusion $A_n \subseteq A_{n+1}$ is the embedding of $\mathbb{C}^{2^n}$ into $\mathbb{C}^{2^{n+1}}$ given by splitting each copy of $\mathbb{C}$ into two copies. This yields the labeled Bratteli diagram in Figure 3.9. As $\cup_{n=0}^{\infty} A_n$ is a unital $*$-subalgebra of $C(X)$ which separates points in X, an appropriate version of the Stone-Weierstrass Theorem implies that $\cup_{n=0}^{\infty} A_n$ is dense.

If $A = \overline{\cup_{n=1}^{\infty} A_n}$ and $B = \overline{\cup_{n=1}^{\infty} B_n}$ are AF-algebras such that $A_n \cong B_n$ for each n, then at first glance it may seem like A and B should be isomorphic. After all, we are taking the closure of an increasing union of isomorphic C^*-subalgebras. However, this is far from true in general. And the next example is a nice illustration of this. The fact to remember is that for an AF-algebra, the nature of the embeddings of the defining sequence is crucial.

Example 3.3.5. Let X be the Cantor ternary set and let $A_n \subseteq C(X)$ be as in Example 3.3.4. Let $Y = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \subseteq [0,1]$ and let $B_n \subseteq C(Y)$ be the C^*-subalgebra of functions that are constant on $[0, \frac{1}{2^n}] \cap Y$. Then

$$B_n = \operatorname{span}\{\chi_{\{1\}}, \chi_{\{\frac{1}{2}\}}, \ldots, \chi_{\{\frac{1}{2^n-1}\}}, \chi_{[0,\frac{1}{2^n}]\cap Y}\} \subseteq C(Y).$$

It is clear that $B_n \subseteq B_{n+1}$, and $\cup_{n=0}^{\infty} B_n$ is dense in $C(Y)$ by Stone-Weierstrass. So we have $C(X) = \overline{\cup_{n=0}^{\infty} A_n}$, $C(Y) = \overline{\cup_{n=0}^{\infty} B_n}$ and $A_n \cong \mathbb{C}^{2^n} \cong B_n$ for each n. Nevertheless, $C(X) \ncong C(Y)$ because $X \ncong Y$ as topological spaces. In particular, Y is countable while X is not.

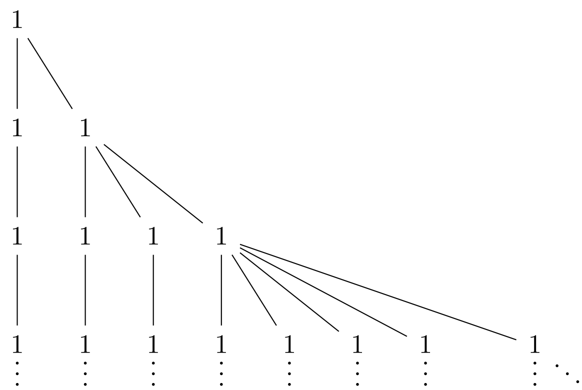

Figure 3.10: A labeled Bratteli diagram associated to $C(Y)$ from Example 3.3.5.

If we consider the inclusion $i_n \colon B_n \hookrightarrow B_{n+1}$, then we see that

$$\chi_{\{\frac{1}{m}\}} \longmapsto \chi_{\{\frac{1}{m}\}}\} \text{ for } 1 \leq m \leq 2^n - 1,$$
$$\chi_{[0,\frac{1}{2^n}]\cap Y} \longmapsto \chi_{\{\frac{1}{2^n}\}} + \chi_{\{\frac{1}{2^n+1}\}} + \cdots + \chi_{\{\frac{1}{2^{n+1}-1}\}} + \chi_{[0,\frac{1}{2^{n+1}}]\cap Y}$$

in terms of the generators. Hence the last summand, $\mathbb{C}$, of B_n maps into $2^n + 1$ copies of $\mathbb{C}$ in B_{n+1}. This yields the labeled Bratteli diagram in Figure 3.10 representing $C(Y)$. Comparing with the labeled Bratteli diagram in Figure 3.9 from Example 3.3.4 we see that they are quite different.

3.4 The local characterization of AF-algebras

In this section we shall prove the local characterization of AF-algebras, originally due to Bratteli [1]. We will also prove that there is a strong uniqueness condition on a chain of subalgebras defining an AF-algebra. In fact, any such dense increasing union is unique up to isomorphism.

We begin by stating a technical lemma due to Glimm which show how finite-dimensional C^*-subalgebras can be moved inside other subalgebras. A proof of the following result may be found in Section III.3 of [2].

Lemma 3.4.1. *Let D be a unital C^*-algebra and suppose A and B are C^*-subalgebras of D. Given $\epsilon > 0$ and $n \in \mathbb{N}$, there exists a $\delta = \delta(\epsilon, n) > 0$ such that whenever A is finite-dimensional with $\dim(A) \leq n$ and A has a system of matrix units $\{e_{ij}^k\}$ satisfying $d(e_{ij}^k, B) < \delta$, then there exists a unitary $u \in C^*(A, B) \subseteq D$ with $\|u - 1_D\| < \epsilon$ such that $uAu^* \subseteq B$.*

Furthermore, the unitary u can be chosen so that it commutes with $A \cap B$.

This lemma shows that if a finite-dimensional C^*-algebra is "almost contained" in another C^*-algebra, then it can be moved inside by twisting with a unitary close to the identity. By using this result inductively we shall prove the following theorem, which classifies AF-algebras by a local property.

Theorem 3.4.2. *Let A be a C^*-algebra. Then A is an AF-algebra if and only if A is separable and has the following property:*

- *For every finite subset $\{a_1, \ldots, a_n\} \subseteq A$ and for every $\epsilon > 0$, there exists a finite-dimensional C^*-subalgebra $B \subseteq A$ such that $d(a_i, B) < \epsilon$ for $1 \leq i \leq n$.*

Moreover, if A_0 is any finite-dimensional C^*-subalgebra of A, then B can be chosen such that $A_0 \subseteq B$.

Proof. We prove the easy direction first. Assume that A is an AF-algebra. Then A is separable by Proposition 3.1.4. By Proposition 3.1.2 there is an increasing union $\cup_{m=1}^{\infty} A_m$ of finite-dimensional C^*-subalgebras of A which is dense. Given $a_1, \ldots, a_n \in A$ and $\epsilon > 0$, pick m_i such that $d(a_i, A_{m_i}) < \epsilon$. Then $B = A_M$, where $M = \max\{m_1, \ldots, m_n\}$ does the trick.

If we are also given a finite-dimensional C^*-subalgebra $A_0 \subseteq A$, choose $\delta = \delta(\epsilon, \dim(A_0))$ as in Lemma 3.4.1. Then we can find an N such that A_N is δ-close to the matrix units in A_0. In this case, let $B = A_{\max\{M,N\}}$. By Lemma 3.4.1 there is a unitary $u \in \tilde{A}$ with $\|u - 1\| < \epsilon$ such that $uA_0u^* \subseteq B$. Then $A_0 \subseteq u^*Bu \subseteq A$, where the latter inclusion follows since A sits inside $\tilde{A}$ as an ideal. Also, since u is within ϵ of the unit and B is ϵ-close to $\{a_1, \ldots, a_n\}$ it follows that u^*Bu is $K\epsilon$-close to $\{a_1, \ldots, a_n\}$ for some constant $K > 0$ depending on the norms of $a_1, \ldots, a_n$.

Conversely, assume that A is separable and satisfies the property in the theorem. Let $\{a_1, a_2, \ldots\}$ be a countable dense subset of the unit ball in A and let $1 > \epsilon_1 > \epsilon_2 > \ldots > 0$ be a sequence of numbers such that $\epsilon_k \to 0$. We claim that there are finite-dimensional C^*-subalgebras A_n of A such that $A_1 \subseteq A_2 \subseteq \ldots$ and $d(a_i, A_n) < \epsilon_n$ for $1 \leq i \leq n$.

By the property in the theorem, there is a finite-dimensional C^*-subalgebra $A_1 \subseteq A$ such that $d(a_1, A_1) < \epsilon_1$. Suppose we have found finite-dimensional subalgebras $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_k$ of A such that $d(a_i, A_n) < \epsilon_n$ for $1 \leq i \leq n$, for each $n \leq k$. Choose $\delta = \delta(\frac{\epsilon_{k+1}}{4}, \dim(A_k))$ as in Lemma 3.4.1. Let $\{e_{ij}^l\}$ be a system of matrix units for A_k. Using $\epsilon = \min\{\delta, \frac{\epsilon_{k+1}}{4}\}$ there is, by the property in the theorem, a finite-dimensional subalgebra $B \subseteq A$ such that $d(e_{ij}^l, B) < \delta$ for the matrix units and such that $d(a_i, B) < \frac{\epsilon_{k+1}}{4}$ for $1 \leq i \leq k+1$. Again by Lemma 3.4.1 there is a unitary $u \in \tilde{A}$ with $\|u - 1\| < \frac{\epsilon_{k+1}}{4}$ such that $uA_ku^* \subseteq B$. Now let $A_{k+1} = u^*Bu \subseteq A$. Then A_{k+1} is finite-dimensional and contains A_k since $u^*Bu \supseteq u^*(uA_ku^*)u = A_k$. For any $b \in B$ and any a_i, where $1 \leq i \leq k+1$, we have

$$
\begin{aligned}
\|ua_iu^* - b\| &= \|((u-1)+1)a_i((u^*-1)+1) - b\| \\
&\leq \|(u-1)a_i\| + \|a_i(u^*-1)\| + \|(u-1)a_i(u^*-1)\| + \|a-b\| \\
&< \frac{\epsilon_{k+1}}{4} + \frac{\epsilon_{k+1}}{4} + \frac{(\epsilon_{k+1})^2}{4} + \|a-b\| \\
&\leq \frac{3\epsilon_{k+1}}{4} + \|a-b\|.
\end{aligned}
$$

It follows that $d(a_i, A_{k+1}) = d(ua_iu^*, uA_{k+1}u^*) = d(ua_iu^*, B) < \epsilon_{k+1}$ for $1 \leq i \leq k+1$. This proves the claim.

All that remains is to note that $\cup_{n=1}^{\infty} A_n$ is dense in A. To that end, let $0 \neq x \in A$ be arbitrary and let $\epsilon > 0$ be given. Choose a_i such that $\|a_i - \frac{x}{\|x\|}\| < \frac{\epsilon}{2\|x\|}$. Next, choose $a' \in A_N$ with $\|a' - a_i\| < \frac{\epsilon}{2\|x\|}$. Then $\|x\|a' \in A_N$ and $\|x - \|x\|a'\| < \epsilon$. Thus $\cup_{n=1}^{\infty} A_n$ is dense, so A is AF. □

Before continuing we make a quick remark. At first glance it might seem like the property stated in Theorem 3.4.2 will always hold. One's first impulse is perhaps to let $B = \operatorname{span}\{a_1, \ldots, a_n\}$. While this will be a closed subspace, it will generally not be closed under multiplication and hence not a subalgebra. A finite number of elements in a C^*-algebra need not be contained in a finite-dimensional C^*-subalgebra in general. As a case in point, the single element $f(z) = z$ generate the infinite-dimensional C^*-algebra $C(\mathbb{T})$.

Lemma 2.5.2 together with Lemma 3.1.3 implies that if $A = \overline{\cup_{n=1}^{\infty} A_n}$ and $B = \overline{\cup_{n=1}^{\infty} B_n}$ are two AF-algebras such that $\cup_{n=1}^{\infty} A_n \cong \cup_{n=1}^{\infty} B_n$ as $*$-algebras, then $A \cong B$ as C^*-algebras. We will now prove that the converse is also true, namely that if $A \cong B$, then we have $\cup_{n=1}^{\infty} A_n \cong \cup_{n=1}^{\infty} B_n$. This is a remarkable result which is far from trivial! It means that the union of C^*-subalgebras which define an AF-algebra is actually unique up to isomorphism. This is remarkable, because in general one does not have such uniqueness results for objects one takes the C^*-completion of. We shall accomplish this by finding a unitary which "twists" one union into the other.

Proposition 3.4.3. *Let A be an AF-algebra. Suppose $\overline{\cup_{n=1}^{\infty} A_n} = A = \overline{\cup_{n=1}^{\infty} B_n}$. Then for any $\epsilon > 0$, there exists a unitary $w \in \tilde{A}$ with $\|w - 1_{\tilde{A}}\| < \epsilon$ such that $\cup_{n=1}^{\infty} A_n = w(\cup_{n=1}^{\infty} B_n)w^*$. In fact, there are subsequences m_i and n_i of $\mathbb{N}$ such that $A_{m_i} \subseteq wB_{n_i}w^* \subseteq A_{m_{i+1}}$ for each i.*

Proof. Let $\epsilon > 0$ be given. First, pick positive numbers $\epsilon_i > 0$ such that $2\sum_{i=1}^{\infty} \epsilon_i < \epsilon$. Let $m_1 = 1$ and $\delta_1 = \delta(\epsilon_1, \dim(A_1))$ as in Lemma 3.4.1. As $\cup_{n=1}^{\infty} B_n$ is dense in A there is an n_1 such that B_{n_1} is within δ_1 of the matrix units of A_1. Hence there exists a unitary $u_1 \in \tilde{A}$ such that

$$u_1 A_1 u_1^* \subseteq B_{n_1} \text{ and } \|u_1 - 1\| < \epsilon_1,$$

where 1 denotes the unit in $\tilde{A}$. Next, let $\mu_1 = \delta(\epsilon_1, \dim(B_{n_1}))$. As above, there is an $m_2 > 1$ such that A_{m_2} is within distance μ_1 of the matrix units of $u_1^* B_{n_1} u_1$. We also have that $A_1 \subseteq A_{m_2} \cap u_1^* B_{n_1} u_1$, so by Lemma 3.4.1 there exists a unitary $v_1 \in \tilde{A}$ which commutes with A_1 such that

$$v_1 u_1^* B_{n_1} u_1 v_1^* \subseteq A_{m_2} \text{ and } \|v_1 - 1\| < \epsilon_1.$$

Since v_1 commutes with A_1 we get $A_1 = v_1 A_1 v_1^* \subseteq v_1 u_1^* B_{n_1} u_1 v_1^* \subseteq A_{m_2}$.

We now argue inductively. Assume that we have found integers $1 = m_1 < \cdots < m_{k+1}$ and $n_1 < \cdots < n_k$, and unitaries u_i and v_i for $i = 1, \ldots, k$ satisfying the following properties for each i:

(1) $\|u_i - 1\| < \epsilon_i$ and $\|v_i - 1\| < \epsilon_i$.

(2) $A_{m_i} \subseteq B'_{n_i} \subseteq A_{m_{i+1}}$, where $B'_{n_i} = v_i u_i^* \cdots v_1 u_1^* B_{n_i} u_1 v_1^* \cdots u_i v_i^*$.

(3) v_i commutes with A_{m_i} and u_{i+1} commutes with B'_{n_i}.

To get to the $(k+1)$th step we proceed as follows. As $\cup_{n=1}^{\infty} B_n$ is dense in A, so is $\cup_{n=1}^{\infty}(v_k u_k^* \cdots v_1 u_1^* B_n u_1 v_1^* \cdots u_k v_k^*)$. So there is an $n_{k+1} > n_k$ such that $v_k u_k^* \cdots v_1 u_1^* B_{n_{k+1}} u_1 v_1^* \cdots u_k v_k^*$ is close enough to the matrix units of $A_{m_{k+1}}$ to guarantee by Lemma 3.4.1 the existence of a unitary u_{k+1} within distance ϵ_{k+1} of 1 and

$$u_{k+1} A_{m_{k+1}} u_{k+1}^* \subseteq v_k u_k^* \cdots v_1 u_1^* B_{n_{k+1}} u_1 v_1^* \cdots u_k v_k^*.$$

Also, since $B'_{n_k} \subseteq A_{m_{k+1}} \cap (v_k u_k^* \cdots v_1 u_1^* B_{n_{k+1}} u_1 v_1^* \cdots u_k v_k^*)$, we may assume by Lemma 3.4.1 that u_{k+1} commutes with B'_{n_k}. Next, we choose $m_{k+2} > m_{k+1}$ so that $A_{m_{k+2}}$ is close enough to the matrix units of

$$u_{k+1}^* v_k u_k^* \cdots v_1 u_1^* B_{n_{k+1}} u_1 v_1^* \cdots u_k v_k^* u_{k+1}$$

to guarantee, again by Lemma 3.4.1, the existence of a unitary v_{k+1} within distance ϵ_{k+1} of 1 and

$$v_{k+1} u_{k+1}^* v_k u_k^* \cdots v_1 u_1^* B_{n_{k+1}} u_1 v_1^* \cdots u_k v_k^* u_{k+1} v_{k+1}^* = B'_{n_{k+1}} \subseteq A_{m_{k+2}}.$$

Then $A_{m_{k+1}} \subseteq A_{m_{k+2}} \cap (u_{k+1}^* v_k u_k^* \cdots v_1 u_1^* B_{n_{k+1}} u_1 v_1^* \cdots u_k v_k^* u_{k+1})$, so therefore we may assume by Lemma 3.4.1 that v_{k+1} commutes with $A_{m_{k+1}}$. It follows that $A_{m_{k+1}} \subseteq B'_{n_{k+1}}$. We have now established the $(k+1)th$ step.

Using the recursion steps above, we form the sequence $w_k = v_k u_k^* \cdots v_1 u_1^*$. We claim that $\|w_k - 1\| < 2\sum_{i=1}^k \epsilon_i$. For any $i \in \mathbb{N}$ we have

$$\|v_i u_i^* - 1\| = \|v_i(u_i^* - 1) + v_i - 1\| \leq \|u_i^* - 1\| + \|v_i - 1\| < 2\epsilon_i.$$

When $i = 1$, this is the formula above for $k = 1$. Assuming the formula holds for k, we have

$$\begin{aligned} &\|w_{k+1} - 1\| = \|v_{k+1}u_{k+1}^*(w_k - 1) + v_{k+1}u_{k+1}^* - 1\| \\ &\leq \|w_k - 1\| + \|v_{k+1}u_{k+1}^* - 1\| < 2\sum_{i=1}^{k} \epsilon_i + 2\epsilon_{k+1} = 2\sum_{i=1}^{k+1} \epsilon_i. \end{aligned}$$

Thus the formula is valid for all k. Had we done the exact same computation for $v_n u_n^* \cdots v_m u_n^*$, where $n > m$, would we have gotten

$$\|v_n u_n^* \cdots v_m u_m^* - 1\| < 2\sum_{i=m}^{n} \epsilon_i.$$

From this we can deduce that w_k is Cauchy. Indeed, if $n > m$ then

$$\begin{aligned} &\|w_n - w_m\| = \|(v_n u_n^* \cdots v_{m+1}u_{m+1}^* - 1)w_m\| \\ &\leq \|v_n u_n^* \cdots v_{m+1}u_{m+1}^* - 1\| < 2\sum_{i=m+1}^{n} \epsilon_i \longrightarrow 0 \text{ as } m, n \to \infty \end{aligned}$$

since this is the tail of a convergent series. Let $w := \lim_{k\to\infty} w_k$. Since w is a limit of unitaries it is itself unitary. It follows from our computations above that $\|w - 1\| < \epsilon$.

If we now let $w^{(k)} := \lim_{i\to\infty} v_i u_i^* \cdots v_{k+1}u_{k+1}^*$. Then $w = w^{(k)} w_k$ and $B'_{n_k} = w_k B_{n_k} w_k^*$. From property (1) and (2) in the induction part above, we see that $v_i u_i^* \cdots v_{k+1}u_{k+1}^*$ commutes with B'_{n_k} for each $i > k$. Hence $w^{(k)}$ commutes with B'_{n_k}. Using this, we compute

$$wB_{n_k}w^* = w^{(k)} w_k B_{n_k} w_k^* w^{(k)*} = w^{(k)} B'_{n_k} w^{(k)*} = B'_{n_k} w^{(k)} w^{(k)*} = B'_{n_k}.$$

Now we are done, since this means that

$$A_{m_i} \subseteq wB_{n_i}w^* \subseteq A_{m_{i+1}} \text{ for each } i.$$

In particular,

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{i=1}^{\infty} A_{m_i} = \bigcup_{i=1}^{\infty} wB_{n_i}w^* = w\left(\bigcup_{i=1}^{\infty} B_{n_i}\right) w^* = w\left(\bigcup_{n=1}^{\infty} B_n\right) w^*. \qquad \square$$

Theorem 3.4.4. *Let $A = \overline{\cup_{n=1}^{\infty} A_n}$ and $B = \overline{\cup_{n=1}^{\infty} B_n}$ be AF-algebras. Then $A \cong B$ as C^*-algebras if and only if $\cup_{n=1}^{\infty} A_n \cong \cup_{n=1}^{\infty} B_n$ as complex $*$-algebras.*

Similarly, if $\mathfrak{A}_\infty$ and $\mathfrak{B}_\infty$ are C^*-algebraic direct limits of finite-dimensional C^*-algebras, then $\mathfrak{A}_\infty \cong \mathfrak{B}_\infty$ if and only if the underlying $*$-algebraic direct limits A_∞ and B_∞ are isomorphic.

Proof. We have already argued for the "if" direction in the paragraph preceding Proposition 3.4.3. For the converse, let $\phi\colon A \to B$ be an isomorphism. Then $\overline{\cup_{n=1}^{\infty} \phi(A_n)} = B = \overline{\cup_{n=1}^{\infty} B_n}$. So by the Proposition 3.4.3 there is a unitary $w \in \tilde{B}$ such that

$$\bigcup_{n=1}^{\infty} A_n \cong \bigcup_{n=1}^{\infty} \phi(A_n) = w(\bigcup_{n=1}^{\infty} B_n)w^* \cong \bigcup_{n=1}^{\infty} B_n.$$

For the second part, apply the preceding argument to the union of the images of the finite-dimensional C^*-algebras in the chain system. $\square$

We are now able to prove the converse of Proposition 3.2.2 and thereby establishing the fact that isomorphic AF-algebras have telescope equivalent labeled Bratteli diagrams.

Theorem 3.4.5. *Let A and B be AF-algebras with associated labeled Bratteli diagrams (V,E) and (W,F), respectively. Then A is isomorphic to B if and only if (V,E) is telescope equivalent to (W,F).*

Proof. The "if" direction follows from Proposition 3.2.2, so assume that A is isomorphic to B and let $\phi\colon A \to B$ be an isomorphism. Let $A_1 \subseteq A_2 \subseteq \dots$ be finite-dimensional C^*-subalgebras of A whose union is dense, and such that (V,E) is associated to the chain system (A_n, i_n), where $i_n\colon A_n \hookrightarrow A_{n+1}$ denotes the inclusion map. Let $B_1, B_2, \dots$ be similar for B such that (W,F) is associated to (B_n, j_n). Then $\overline{\cup_{n=1}^{\infty} \phi(A_n)} = B = \overline{\cup_{n=1}^{\infty} B_n}$ so by Proposition 3.4.3 there is a unitary w and subsequences m_k, n_k of $\mathbb{N}$ such that

$$\phi(A_{m_1}) \subseteq wB_{n_1}w^* \subseteq \phi(A_{m_2}) \subseteq wB_{n_2}w^* \subseteq \phi(A_{m_3}) \subseteq wB_{n_3}w^* \subseteq \dots$$

Consider the following chains of subalgebras and their associated labeled Bratteli diagrams:

(1) $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ inducing (V,E).

(2) $\phi(A_1) \subseteq \phi(A_2) \subseteq \phi(A_3) \subseteq \dots$ inducing (V',E').

(3) $\phi(A_{m_1}) \subseteq \phi(A_{m_2}) \subseteq \phi(A_{m_3}) \subseteq \dots$ inducing (V'',E'').

(4) $B_1 \subseteq B_2 \subseteq B_3 \subseteq \dots$ inducing (W,F).

(5) $wB_1w^* \subseteq wB_2w^* \subseteq wB_3w^* \subseteq \dots$ inducing (W',F').

(6) $wB_{n_1}w^* \subseteq wB_{n_2}w^* \subseteq wB_{n_3}w^* \subseteq \dots$ inducing (W'',F'').

(7) $\phi(A_{m_1}) \subseteq wB_{n_1}w^* \subseteq \phi(A_{m_2}) \subseteq wB_{n_2}w^* \subseteq \dots$ inducing (U,G).

Since ϕ and $\operatorname{Ad} w$ are isomorphisms, (V, E) is isomorphic to (V', E') and (W, F) is isomorphic to (W', F'). We also have that (V'', E'') is the telescoping of (V', E') with respect to the sequence m_k, and similarly (W'', F'') is the telescoping of (W', F') with respect to the sequence n_k. Finally, telescoping (U, G) to even levels yields (V'', E'') and telescoping (U, G) to odd levels yields (W'', F''). This shows that (V, E) and (W, F) are telescope equivalent. $\square$

The following criterion for telescope equivalence can be extracted from the proof of the previous theorem; Two labeled Bratteli diagrams (V, E) and (W, F) are telescope equivalent if and only if there exists a third labeled Bratteli diagram (U, G) such that the telescoping of (U, G) to even levels is isomorphic to (V, E) and the telescoping of (U, G) to odd levels is isomorphic to (W, F). A very basic illustration of this criterion and Theorem 3.4.5 is the three labeled Bratteli diagrams representing the CAR-algebra in Example 3.3.2.

3.5 Ideals of AF-algebras

In this section we will see that ideals and quotients of AF-algebras are also AF. The main result shows how the ideal structure of an AF-algebra can be read off its labeled Bratteli diagram. Using this we shall comment on the ideal structure of the examples introduced thus far.

By an ideal we shall mean a closed two-sided ideal. We denote by $I \lhd A$ when I is an ideal of A. We begin by noting that ideals in an AF algebra $A = \overline{\cup_{n=1}^{\infty} A_n}$ are *inductive*, i.e. that they stem from ideals of the defining sequence A_n.

Lemma 3.5.1. *Let $A = \overline{\cup_{n=1}^{\infty} A_n}$ be an AF-algebra. If $I \lhd A$ is an ideal, then $I = \overline{\cup_{n=1}^{\infty} I \cap A_n}$ and $(I \cap A_n) \lhd A_n$.*

Proof. The Second isomorphism theorem, to wit,

$$\frac{A_n}{I \cap A_n} \cong \frac{A_n + I}{I}$$

is obtained via the map $a_n + I \cap A_n \longmapsto a_n + I$ for $a_n \in A_n$. As this is in particular an isometry, we obtain

$$d(a_n, I \cap A_n) = \|a_n\|_{\frac{A_n}{I \cap A_n}} = \|a_n\|_{\frac{A_n + I}{I}} = d(a_n, I)$$

by considering the quotient norms. Our aim is to show that $\cup_{n=1}^{\infty} I \cap A_n$ is dense in I. To this end let $x \in I$ and $\epsilon > 0$ be given. Choose $a_n \in A_n$ with $\|a_n - x\| < \frac{\epsilon}{2}$. This means that $d(a_n, I) < \frac{\epsilon}{2}$, so by the equation above there exists $a_n' \in I \cap A_n$ with $\|a_n - a_n'\| < \frac{\epsilon}{2}$. But then $\|a_n' - x\| < \epsilon$. Hence $I = \overline{\cup_{n=1}^{\infty} I \cap A_n}$.

Since $I \lhd A$ and A_n is a C^*-subalgebra of A, it follows that $I \cap A_n \lhd A_n$. $\square$

Proposition 3.5.2. *Let A be an AF-algebra. If $I \lhd A$ is an ideal, then I and A/I are both AF-algebras.*

Proof. Let $A = \overline{\cup_{n=1}^{\infty} A_n}$. By the preceding lemma, $I = \overline{\cup_{n=1}^{\infty} I \cap A_n}$. As $I \cap A_1 \subseteq I \cap A_2 \subseteq \dots$ are finite-dimensional C^*-subalgebras of I we see that I is AF.

As for the quotient, we have $A/I = \frac{A+I}{I}$. Therefore

$$\frac{A_1 + I}{I} \subseteq \frac{A_2 + I}{I} \subseteq \frac{A_3 + I}{I} \dots\dots \subseteq A/I.$$

By the Second isomorphism theorem,

$$\frac{A_n + I}{I} \cong \frac{A_n}{I \cap A_n}$$

and it follows that $\frac{A_n+I}{I}$ is finite-dimensional. It remains to show that $\cup_{n=1}^{\infty} \frac{A_n+I}{I}$ is dense in A/I. To this end, let $a + I \in A/I$ and $\epsilon > 0$ be given. Choose $a_n \in A_n$ with $\|a - a_n\| < \epsilon$. Then

$$\|(a + I) - (a_n + I)\|_{A/I} = \|(a - a_n) + I\|_{A/I} \leq \|a - a_n\| < \epsilon.$$

It follows that the increasing union is dense, hence A/I is AF. □

It can be shown that the converse of Proposition 3.5.2 is also true. Namely that if A is a C^*-algebra which has an ideal I such that both I and A/I are AF, then A is AF [3, Theorem 9.9]. In other words, extensions of AF-algebras by AF-algebras are also AF.

Being an AF-algebra can at first seem like being a "size restriction" and therefore it might be tempting to think that all C^*-subalgebras (and not just ideals) of AF-algebras are also AF. This fails miserably in general. Consider the following example.

Example 3.5.3. Let X be the Cantor ternary set as in Example 3.3.4. We will show that $C([0, 1])$ can be embedded into $C(X)$. Let ψ denote the Cantor function $\psi\colon X \to [0, 1]$. Since ψ is continuous, the map $\phi\colon C([0, 1]) \to C(X)$ defined by $f \mapsto f \circ \psi$ for $f \in C([0, 1])$ is a well defined $*$-homomorphism. ψ is also surjective and therefore ϕ is injective. And then $\phi(C([0, 1]))$ is a C^*-subalgebra of $C(X)$ isomomorphic to $C([0, 1])$. As $[0, 1]$ is connected, $C([0, 1])$ is not AF by Theorem 3.3.3. Consequently, C^*-subalgebras of AF-algebras need not be AF.

It turns out that the previous example can be generalized vastly. A well known result due to Alexandroff and Urysohn states that any compact metric space is a continuous image of the Cantor ternary set, see e.g. [13, Theorem 30.7]. We will use this to prove the following remarkable result.

Theorem 3.5.4. *Let X be the Cantor ternary set. Every separable commutative C^*-algebra can be embedded into $C(X)$.*

Proof. Let A be a separable commutative C^*-algebra. Then $A \cong C_0(Y)$ for some locally compact second countable Hausdorff space Y. First consider the case where Y is compact. Then Y is metrizable and compact, and therefore there exists a continuous surjective function $F\colon X \to Y$. By doing exactly the same as in Example 3.5.3 we obtain an injective $*$-homomorphism $\phi\colon C_0(Y) = C(Y) \to C(X)$.

Now, if Y is not necessarily compact, then let $\widetilde{C_0(Y)}$ denote the unitization of Y. Since $C_0(Y)$ is commutative, so is $\widetilde{C_0(Y)}$. The latter is also unital, and therefore $\widetilde{C_0(Y)} \cong C(Z)$ for some compact Hausdorff space Z. Since $C_0(Y)$ is separable, so is the unitization, and therefore Z is also second countable. In fact, it is not hard to show that $\widetilde{C_0(Y)} \cong C(Y^\dagger)$, where $Y^\dagger$ denotes the one-point compactification of Y ($Y^\dagger$ is Hausdorff since Y is Hausdorff and locally compact). An explicit isomorphism is given by $f \longmapsto (f(\infty), f_{|Y} - f(\infty))$ for $f \in C(Y^\dagger)$ where ∞ denotes the "point at infinity" in $Y^\dagger$. From the first part of the proof, we get an isometry $\phi\colon C(Y^\dagger) \to C(X)$. But then A embeds into $C(X)$ since

$$A \cong C_0(Y) \hookrightarrow \widetilde{C_0(Y)} \cong C(Y^\dagger) \xrightarrow{\phi} C(X). \qquad \square$$

The previous result demonstrates one of the ways in which the AF-algebras are a "rich class". Namely that they contain a lot of interesting non-AF-algebras as subalgebras.

Our next objective is to use Lemma 3.5.1 to describe the ideals of an AF-algebra in terms of its labeled Bratteli diagram. At this point it might be useful for the reader to brush up on the terminology introduced in Section 2.4.

Definition 3.5.5. Let (V, E, r, s, d) be a labeled Bratteli diagram and let $W \subseteq V$ be a subset of vertices. W is called *directed* if whenever $(n,p) \in W$ and $(n,p) \mapsto (n+1,q)$, then $(n+1,q) \in W$ also.

W is called *hereditary* if whenever $(n,p) = v \in V$ and $r(s^{-1}(v)) \subseteq W$, then $v \in W$ also.

In words, a subset W of vertices of a labeled Bratteli diagram (V, E) is directed if every vertex pointed to by a vertex belonging to W also belongs to W. And W is hereditary if every vertex which points only to vertices in W is itself in W. Note that if W is directed and hereditary, then a vertex (n,p) belongs to W if and only if all vertices on level $n+1$ connected to (n,p) by an edge lies in W. Another important observation is that if W is directed hereditary and contains every vertex on some level, then $W = V$.

Example 3.5.6. Let (V, E) be the labeled Bratteli diagram in Figure 3.11.

(1) The subset $W_1 = \{(2,1),\ (2,2),\ (3,1),\ (3,2),\ (3,3), \ldots\}$ is directed, but not hereditary.

(2) The subset $W_2 = \{(1,2),\ (1,3),\ (2,2),\ (3,2),\ (3,3), \ldots\}$ is hereditary, but not directed.

(3) The subset $W_3 = \{(1,3),\ (2,2),\ (3,2),\ (3,3), \ldots\}$ is both directed and hereditary.

We are going to show that the directed hereditary subsets of a labeled Bratteli diagram correspond exactly to the ideals of the associated AF-algebra in a natural way. The proof will be broken down into several lemmas. The first lemma is a technical one.

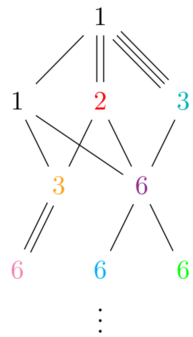

Figure 3.11: The labeled Bratteli diagram considered in Example 3.5.6.

Lemma 3.5.7. *Let* $A = \overline{\cup_{n=1}^{\infty} A_n}$ *be an AF-algebra. Suppose* J *and* K *are not necessarily closed ideals in* $\bigcup_{n=1}^{\infty} A_n$*. If* $\overline{J} = \overline{K}$ *(where the closure is taken in* A*), then* $J = K$*.*

Proof. Assume that $J \neq K$. This means that there is an $n \in \mathbb{N}$ with $J \cap A_n \neq K \cap A_n$. Assume, without loss of generality, that $J \cap A_n \nsubseteq K \cap A_n$. As these are both ideals in the semisimple algebra A_n there is a matrix algebra

$$0 \oplus \cdots 0 \oplus M_k \oplus 0 \cdots \oplus 0 \cong M \subseteq A_n$$

such that $M \subseteq J \cap A_n$ and $M \cap (K \cap A_n) = 0$. From our work in Chapter 1 we know that $M = eA_n$ where e is a central projection in A_n. Then $e^2 = e \in M \subseteq J \cap A_n$, so $e \in J$. On the other hand, $e \notin K \cap A_m$ for every m. To see this, assume to the contrary that $e \in K \cap A_m$ for some $m \geq n$. Then $eM = M \subseteq K$ since K is an ideal, but then $M \subseteq K \cap A_n$ which is a contradiction.

Consider the quotient maps

$$\pi_m \colon A_m \longrightarrow \frac{A_m}{K \cap A_m} \text{ for } m \geq n.$$

Since $e \notin K \cap A_m$, $\pi_m(e)$ is a nonzero projection and consequently has norm 1. Thus

$$d(e, K \cap A_m) = \|\pi_m(e)\| = 1.$$

We also have that $K = \bigcup_{m=1}^{\infty}(K \cap A_m)$ and it follows that $d(e, K) = 1$, hence $e \notin \overline{K}$. As $e \in J \subseteq \overline{J}$, we have $\overline{J} \neq \overline{K}$. □

Now we describe how an ideal determines a directed hereditary subset of vertices and vice versa. Let $A = \overline{\cup_{n=0}^{\infty} A_n}$ be an AF-algebra with an associated labeled Bratteli diagram (V, E). Then $A_n \cong M_{d(n,1)} \oplus \cdots \oplus M_{d(n,k)}$. So if $J \lhd A_n$, we have $J \cong M_{d(n,m_1)} \oplus \cdots \oplus M_{d(n,m_l)}$, where $\{m_1, m_2, \ldots m_l\}$ is a subsequence of $\{1, 2, \ldots, k\}$ since A_n is semisimple. We will identify A_n with $M_{d(n,1)} \oplus \cdots \oplus M_{d(n,k)}$, and we will identify $M_{d(n,p)}$ with $0 \oplus \cdots \oplus 0 \oplus M_{d(n,p)} \oplus 0 \oplus \cdots \oplus 0$ when $1 \leq p \leq k$.

If $I \lhd A$ is an ideal, let $I_n := I \cap A_n$ for $n \in \mathbb{Z}^+$. Then $I_n \lhd A_n$, so by the argument above, I_n corresponds to a subset of vertices in V_n, denote these by W_n. *Define*

$$W_I := \cup_{n=0}^{\infty} W_n$$

If $W \subseteq V$ is a directed hereditary subset, then let $J_n \lhd A_n$ be the ideal corresponding to the vertices in $W \cap V_n$, i.e. $J_n = \oplus_{(n,p)\in W\cap V_n} M_{d(n,p)}$. *Define*

$$J_W = \overline{\cup_{n=0}^{\infty} J_n}$$

Lemma 3.5.8. *Let $A = \overline{\cup_{n=0}^{\infty} A_n}$ be an AF-algebra with an associated labeled Bratteli diagram (V, E).*

(1) *If $I \lhd A$ is an ideal, then the subset $W_I \subseteq V$ associated to I, as described in the preceding paragraph, is directed and hereditary.*

(2) *If $W \subseteq V$ is a directed hereditary subset, then $J_W \subseteq A$, as described in the preceding paragraph, is an ideal in A.*

Proof. Let $I \lhd A$. Let $i_n \colon A_n \hookrightarrow A_{n+1}$ denote the inclusion maps. Suppose $(n,p) \in W_I$ and $(n,p) \mapsto (n+1,q)$. This means that $M_{d(n,p)} \subseteq I_n$ and $i_n(M_{d(n,p)}) \subseteq M_{d(n+1,q)}$. Thus $I \cap M_{d(n+1,q)} \supseteq i_n(M_{d(n,p)}) \cap M_{d(n+1,q)} \neq \{0\}$, so then $M_{d(n+1,q)} \subseteq I_{n+1}$ which means that $(n+1,q) \in W_I$. Thus W_I is directed.

Now suppose $(n,p) \in V$ and $r(s^{-1}(n,p)) \subseteq W_I$. Let $\Gamma = \{q|(n,p) \mapsto (n+1,q)\}$. Then $i_n(M_{d(n,p)}) \subseteq \oplus_{q\in\Gamma} M_{d(n+1,q)} \subseteq I$, hence $M_{d(n,p)} \subseteq I$, i.e. $(n,p) \in W_I$. This shows that W_I is hereditary.

Let W be a directed hereditary subset of V. W being directed implies that $J_0 \subseteq J_1 \subseteq J_2 \subseteq \ldots$. Therefore $\cup_{n=0}^{\infty} J_n$ is a (not necessarily closed) ideal in $\cup_{n=0}^{\infty} A_n$. It follows that $J_W = \overline{\cup_{n=0}^{\infty} J_n}$ is an ideal in A. □

Lemma 3.5.9. *Let $A = \overline{\cup_{n=0}^{\infty} A_n}$ be an AF-algebra with an associated labeled Bratteli diagram (V, E). The mappings $I \mapsto W_I$ and $W \mapsto J_W$ between ideals and directed hereditary subsets are inverses of one another.*

Proof. Let $I \lhd A$ be an ideal and let W_I be its associated directed hereditary subset. Then

$$J_{W_I} = \overline{\cup_{n=0}^{\infty} J'_n} = \overline{\cup_{n=0}^{\infty} I \cap A_n} = I,$$

where the last equality follows from Lemma 3.5.1.

Next, let W be a directed hereditary subset of V and let $J_W = \overline{\cup_{n=0}^{\infty} J_n}$ be its associated ideal. Let $i_n \colon A_n \hookrightarrow A_{n+1}$ denote the inclusion map. We claim that $J_n = A_n \cap J_{n+1}$ for every n. To see this, suppose $M_{d(n,p)} \subseteq A_n \cap J_{n+1}$. Then $i_n(M_{d(n,p)}) \subseteq J_{n+1}$, which means that $(n+1,q) \in W$ whenever $(n,p) \mapsto (n+1,q)$, but then $(n,p) \in W$ since W is hereditary. Hence $M_{d(n,p)} \subseteq J_n$, which means that $A_n \cap J_{n+1} \subseteq J_n$. Since $J_n \subseteq A_n \cap J_{n+1}$ trivially, the claim is proved. Using this, we see that

$$J_n = A_n \cap J_{n+1} = A_n \cap (A_{n+1} \cap J_{n+2}) = A_n \cap J_{n+2}.$$

By induction we obtain $J_n = A_n \cap J_m$ for every $m \geq n$.

Observe that $\cup_{n=0}^{\infty} J_n$ and $\cup_{n=0}^{\infty}(A_n \cap J_W)$ are both (not necessarily closed) ideals in $\cup_{n=0}^{\infty} A_n$. Moreover, $\overline{\cup_{n=0}^{\infty} J_n} = J_W = \overline{\cup_{n=0}^{\infty}(A_n \cap J_W)}$, again by Lemma 3.5.1. Appealing to Lemma 3.5.7 yields $\cup_{n=0}^{\infty} J_n = \cup_{n=0}^{\infty} A_n \cap J_W$. For every m, J_m and $A_m \cap J_W$ are finite-dimensional ideals in A_m. Therefore, for every $n \in \mathbb{Z}^+$ there is an $m \geq n$ such that $A_n \cap J_W \subseteq J_m$. And then $A_n \cap J_W = A_n \cap (A_n \cap J_W) \subseteq A_n \cap J_m = J_n$. As $J_n \subseteq A_n \cap J_W$ trivially, we get that $J_n = A_n \cap J_W$. This means that $W_{J_W} = W$. $\square$

By putting together the preceding results we obtain the following characterization of ideals in AF-algebras in terms of their labeled Bratteli diagrams.

Theorem 3.5.10. *Let $A = \overline{\cup_{n=0}^{\infty} A_n}$ be an AF-algebra with an associated labeled Bratteli diagram (V, E). The ideals in A are in one-to-one correspondence with the directed hereditary subsets of V. The correspondence is given by the maps $I \mapsto W_I$ and $W \mapsto J_W$ described in the paragraph preceding Lemma 3.5.8.*

Note that V and $\emptyset$ are always directed hereditary subsets of V, and they correspond to the ideals A and $\{0\}$ respectively. We make the following observation.

Corollary 3.5.11. *Let A be an AF-algebra with an associated labeled Bratteli diagram (V, E). The ideal lattice of A corresponds exactly to the lattice of directed hereditary subsets of V, i.e. $I \subseteq J \iff W_I \subseteq W_J$ whenever $I, J \lhd A$.*

Our next proposition describes the labeled Bratteli diagrams of ideals and quotients of AF-algebras.

Proposition 3.5.12. *Let A be an AF-algebra with an associated labeled Bratteli diagram (V, E). If $I \lhd A$ is a proper ideal then*

(1) *The subdiagram of (V, E) corresponding to W_I is a labeled Bratteli diagram and its associated AF-algebra is I.*

(2) *The subdiagram of (V, E) corresponding to $V \backslash W_I$ is a labeled Bratteli diagram and its associated AF-algebra is A/I.*

Proof. Let $A_0 \subseteq A_1 \subseteq \ldots$ be the finite-dimensional C^*-subalgebras of A, whose union is dense, associated to (V, E). The fact that $\emptyset \subsetneq W_I \subsetneq V$ and that W_I is directed hereditary implies that the subdiagram corresponding W_I satisfies Definition 2.4.4. It also implies that $V \setminus W_I$ satisfies Definition 2.4.4. Lemma 3.5.1 and the proof of Lemma 3.5.9 implies that I is the AF-algebra associated to the subdiagram of W_I.

Let $I_n = I \cap A_n$. In the proof of Proposition 3.5.2 we saw that

$$A/I = \overline{\bigcup_{n=0}^{\infty} \frac{A_n + I}{I}}$$

and since

$$A_n \cong \bigoplus_{(n,i) \in V_n} M_{d(n,i)}, \quad I_n \cong \bigoplus_{(n,i) \in W_I \cap V_n} M_{d(n,i)}$$

we get that

$$\frac{A_n + I}{I} \cong \frac{A_n}{A_n \cap I} = A_n/I_n \cong \bigoplus_{(n,i)\notin W_I} M_{d(n,i)}$$

which corresponds to the vertices on level n of the subdiagram of $V \setminus W_I$. Furthermore, the partial embeddings of $M_{d(n,i)}$ into $M_{d(n+1,j)}$ stays the same if (n,i) and $(n+1,j)$ both belong to $V \setminus W_I$. Since these edges are exactly the ones in the subdiagram corresponding to $V \setminus W_I$, we get that the AF-algebra associated to the subdiagram is A/I. □

Now that we have established the connection between the ideals of an AF-algebra and the structure of its labeled Bratteli diagram, we shall try to find some criterion for determining whether an AF-algebra is simple. It turns out that this may be characterized on the level of labeled Bratteli diagrams by looking at the "asymptotic connectedness" of the diagram.

Definition 3.5.13. Let (V,E) be a labeled Bratteli diagram. We say that (V,E) is *eventually fully connected* if, for every vertex $(n,p) \in V$ there exists an integer $m > n$ such that $(n,p) \to (m,j)$ for every $(m,j) \in V_m$. That is, there is a path from (n,p) to every vertex on level m.

Eventually fully connectedness is sufficient for an AF-algebra to be simple, but it is not quite necessary. The following example illustrates what we are missing for necessity.

Example 3.5.14. Let (V,E) be the labeled Bratteli diagram in Figure 3.12. For every $v \in V$, the smallest directed hereditary subset of V containing v is V itself. So

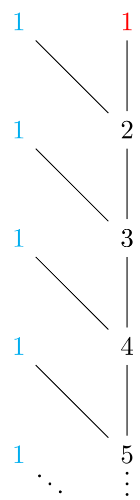

Figure 3.12: A labeled Bratteli diagram which is not eventually fully connected, yet its corresponding AF-algebra is simple.

by Theorem 3.5.10 the AF-algebra associated to this diagram is simple. However, (V, E) is not eventually fully connected since there is no path from the vertex $(0, 2)$ to any vertex $(n, 1)$.

The reason why the diagram in Figure 3.12 is not eventually fully connected is because there are lots of sources in the graph. We will obtain a necessary condition by assuming that the underlying graph has no sources, except at level 0. This is often the case, e.g. when the AF-algebra is unital, for then the embeddings are unital homomorphisms.

Theorem 3.5.15. *Let A be an AF-algebra with an associated labeled Bratteli diagram (V, E).*

(1) *If (V, E) is eventually fully connected then A is simple.*

(2) *If (V, E) has no sources except on level 0, then (V, E) is eventually fully connected if and only if A is simple.*

Proof. Assume that (V, E) is eventually fully connected. Let $I \lhd A$ be a nonzero ideal. Let $A_0 \subseteq A_1 \subseteq \dots$ be finite-dimensional C^*-subalgebras of A, whose union is dense, associated to (V, E). Then $I = \overline{\cup_{n=0}^{\infty} I \cap A_n}$ by Lemma 3.5.1. So for some n_0, $I \cap A_{n_0} \neq 0$, i.e. $(n_0, p) \in W_I$ for some p. Since (V, E) is eventually fully connected there exists $m > n_0$ such that $(n_0, p) \to (m, j)$ for all j. By Theorem 3.5.10, W_I is directed, hence $(m, j) \in W_I$ for all j. As every vertex on level m belongs to W_I, it follows that $W_I = V$ since W_I is directed hereditary. But then $I = A$.

Conversely, assume that (V, E) is not eventually fully connected. Also assume that (V, E) has no sources except on level 0. Then there is a vertex $(n_0, p_0) \in V$ which does not satisfy the eventually fully connected property. Let W' denote the directed subset of V generated by (n_0, p_0). Then W' consists of the vertices which can be reached from (n_0, p_0) and W' is a proper subset at each level above n_0, i.e. $W' \cap V_n \subsetneq V_n$ for $n > n_0$. Now let W be the directed hereditary subset generated by W'. W is obtained by adding to W' any vertex which maps completely into W' at some higher level. Suppose, for the sake of contradiction, that $W = V$. Then $(n_0, p) \in W$ for all p. By the construction of W and the fact that V_{n_0} is finite, there is some $N > n_0$ such that for every p, every path from (n_0, p) to level N terminates at a vertex in W'. Since (V, E) has no sources, except on level 0, every vertex on level N can be reached by some vertex on level n_0. Thus every vertex on level n_0 belongs to W'. This is a contradiction since W' is a proper subset at every level. We conclude that $\emptyset \subsetneq W \subsetneq V$, hence J_W is a proper ideal in A. $\square$

With the tools we have developed in this section we can analyze the ideal structure of the examples introduced in Section 3.3 using only the combinatorial data of their associated labeled Bratteli diagrams.

Example 3.5.16. Recall the AF-algebra $\mathbb{C}I + \mathcal{K}$ of Example 3.3.1. Let (V, E) be the associated labeled Bratteli diagram depicted in Figure 3.13. If $v = (n, 1)$ for $n \geq 0$, then the directed hereditary subset generated by v is all of V. And if $v = (n, 2)$ for $n \geq 1$, then the directed hereditary subset generated by v is the

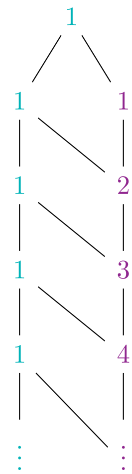

Figure 3.13: The labeled Bratteli diagram associated to $\mathbb{C}I+\mathcal{K}$ from Example 3.3.1.

subset $\{(1,2),(2,2),(3,2),\ldots\}$ whose corresponding subdiagram is the diagram in Figure 3.5. This subset corresponds to the ideal $\mathcal{K}$. As this exhausts our possibilities we find that the ideal lattice of $\mathbb{C}I+\mathcal{K}$ is given by

$$0 \lhd \ \mathcal{K} \lhd \ \mathbb{C}I+\mathcal{K}.$$

Example 3.5.17. Next we consider the CAR algebra of Example 3.3.2. Since the associated labeled Bratteli diagram in Figure 3.6 has only one vertex on every level, the diagram is eventually fully connected, hence the CAR algebra is simple by Theorem 3.5.15.

Example 3.5.18. Finally we take a look at the continuous complex-valued functions on the Cantor set from Example 3.3.4. $C(X)$, where X is the Cantor set, has uncountably many ideals so we will not try to list them in any way. Instead we will deduce this fact from the labeled Bratteli diagram of $C(X)$, (V,E), depicted in Figure 3.14. If v is any vertex in V, then the directed hereditary subset generated by v is just the directed subset generated by v, and moreover, the subdiagram corresponding to this subset is isomorphic to (V,E). This is illustrated in Figure 3.14 with $v=(1,1)$. The corresponding subdiagram is colored in red.

Now let

$$S = \{(n, 2^n-1) | n \in \mathbb{N}\} \subseteq V.$$

The subset S is depicted in magenta in Figure 3.15. Since the directed subsets generated by any two distinct vertices in S are disjoint, it follows that any two distinct subsets of S generates distinct directed hereditary subsets. Hence the power set of S defines uncountably many ideals in $C(X)$. Note that these are far from all ideals in $C(X)$. In fact the ideals in $C(X)$ are in 1-1 correspondence with the closed subsets of X.

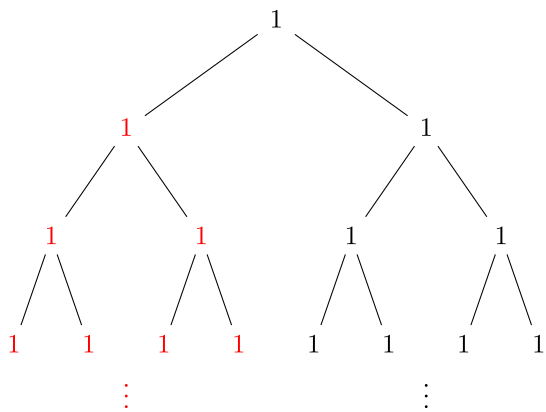

Figure 3.14: The directed hereditary subset generated by the vertex $(1, 1)$ marked in red.

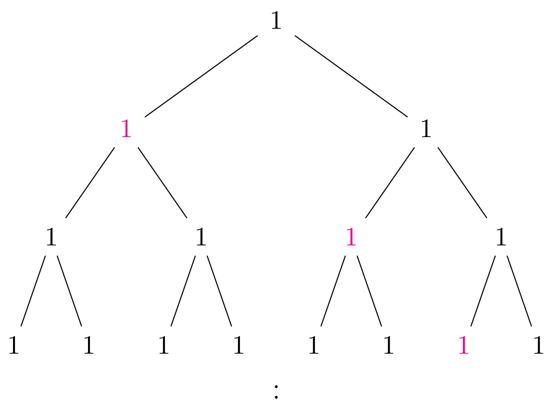

Figure 3.15: The subset S marked in magenta.

Chapter 4

Bratteli diagrams and dynamical systems

The goal of this chapter is to prove the Bratteli-Vershik model theorem of Herman, Putnam and Skau [7, Theorem 4.5] for Cantor minimal systems. Namely, that any Cantor minimal system is conjugate to a Bratteli-Vershik system. We will introduce properly ordered Bratteli diagrams and show how they can be viewed as Cantor minimal systems, when endowed with a particular action. These are the Bratteli-Vershik systems. Since they are Bratteli diagrams, they are very combinatorial by nature. In some sense, this makes them "easy to work with". We are only going to prove the model theorem itself (although in full detail), but there are a lot of important dynamical consequences of the model theorem.

4.1 Cantor minimal systems

Let X be a topological space. Recall the following from general topology: X is called *perfect* if X has no isolated points, *totally disconnected* if the connected components of X are the singleton sets, and *0-dimensional* if X admits a basis of clopen sets. If X is 0-dimensional and T_1, then X is totally disconnected. Also, if X is compact Hausdorff, then X is totally disconnected if and only if X is 0-dimensional.

Definition 4.1.1. A topological space X is called a *Cantor space* if X is non-empty, totally disconnected, perfect, compact and metrizable.

The canonical example of a Cantor space is the Cantor ternary set (See Example 3.3.4). Another example is $2^{\mathbb{N}} = \prod_{n\in\mathbb{N}}\{0,1\}$ (see Example 4.1.8). In fact, Definition 4.1.1 is a complete topological characterization of "being homeomorphic to the Cantor ternary set". Therefore we shall also refer to a Cantor space as simply a *Cantor set*. For a proof of the following result see [13, Theorem 30.3].

Theorem 4.1.2. *Any two Cantor spaces are homeomorphic.*

In the literature, a topological dynamical system can be quite general, like a compact Hausdorff space together with a continuous map from the space to itself, or more generally a semigroup of continuous maps. We are interested in the following type of systems.

Definition 4.1.3. A *topological dynamical system* is a pair (X, T) where X is a compact metrizable space and $T: X \to X$ is a homeomorphism.

As T is a homeomorphism, this is naturally a $\mathbb{Z}$-action on X by defining $kx = T^k(x)$ for $k \in \mathbb{Z}$ and $x \in X$. One can think of the powers of T as "observations" of the space X at discrete time intervals. And you can move both forward and backward in time. For a point $x \in X$, the *orbit* of x is the subset $\mathrm{orbit}_T(x) = \{T^k(x) | k \in \mathbb{Z}\}$, where $T^0 = \mathrm{Id}_X$. Also, the *positive orbit* of x is $\mathrm{orbit}_T(x)^+ = \{T^k(x) | k \in \mathbb{Z}^+\}$.

The notion of "sameness" or isomorphism of topological dynamical systems is that of conjugacy, which we now define.

Definition 4.1.4. Let (X_1, T_1) and (X_2, T_2) be topological dynamical systems. Then (X_1, T_1) is *conjugate* to (X_2, T_2) if there exists a homeomorphism $h: X_1 \to X_2$ such that $h \circ T_1 = T_2 \circ h$, i.e. the following diagram commutes:

$$\begin{array}{ccc} X_1 & \xrightarrow{T_1} & X_1 \\ \big\downarrow h & & \big\downarrow h \\ X_2 & \xrightarrow{T_2} & X_2 \end{array}$$

In that case we call h a *conjugacy*.

Note that conjugacy of topological dynamical systems is an equivalence relation. And if h is a conjugacy between (X_1, T_1) and (X_2, T_2), then $T_2 = h^{-1} \circ T_1 \circ h$, which implies that $T_2^k = h^{-1} \circ T_1^k \circ h$. So h also conjugates the powers of T_1 and T_2.

We shall also restrict to topological dynamical systems which have no (nontrivial) closed subsystems, so that they are minimal in this sense. This concept is made precise in the following definition.

Definition 4.1.5. A topological dynamical system (X, T) is *minimal* if X has no nontrivial closed T-invariant subsets, i.e. $T(F) \subseteq F \implies F = \emptyset$ or $F = X$ when F is a closed subset of X. We then call T a *minimal homeomorphism.*

When dealing with minimal topological dynamical systems it turns out that connected spaces cause difficulties. This is because when the systems are minimal, orbit equivalence implies conjugacy for connected spaces (This result, due to Sierpinski, may be found in §47.III of [9]). And classifying spaces up to conjugacy is virtually beyond hope. Therefore it is not so unnatural to move on to the other side of the spectrum, namely totally disconnected spaces. And by demanding minimality you wind up with the Cantor spaces. (We shall see shortly that minimality imply perfectness, except in trivial cases.)

There are several equivalent characterizations of minimality. Some of them are given in the following proposition.

Proposition 4.1.6. *Let (X,T) be a topological dynamical system. Then the following are equivalent:*

(1) *(X,T) is minimal.*

(2) *If $U \subseteq X$ is open and $U \subseteq T(U)$, then $U = \emptyset$ or $U = X$.*

(3) *If $F \subseteq X$ is closed and $T(F) = F$, then $F = \emptyset$ or $F = X$.*

(4) *If $U \subseteq X$ is open and $T(U) = U$, then $U = \emptyset$ or $U = X$.*

(5) *The orbit of every point is dense in X.*

(6) *The positive orbit of every point is dense in X.*

(7) *If $\emptyset \neq U \subseteq X$ is open, then $\bigcup_{k=-\infty}^{\infty} T^k(U) = X$.*

Proof. Observe that (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4) by taking complements. Also, (6) $\Rightarrow$ (5) trivially. We proceed to show the implications (1) $\Rightarrow$ (6), (5) $\Rightarrow$ (3), (4) $\Rightarrow$ (7) and (7) $\Rightarrow$ (2).

If (1) holds and $x \in X$, then $T(\mathrm{orbit}_T(x)^+) \subseteq \mathrm{orbit}_T(x)^+$, which implies that

$$T\left(\overline{\mathrm{orbit}_T(x)^+}\right) \subseteq \overline{T(\mathrm{orbit}_T(x)^+)} \subseteq \overline{\mathrm{orbit}_T(x)^+}.$$

So $\overline{\mathrm{orbit}_T(x)^+}$ is a non-empty closed invariant subset of X, and must therefore equal X.

Suppose (5) holds and assume that $T(F) = F$ where F is a non-empty closed subset of X. Let $x \in F$ and note that $T(F) = F = T^{-1}(F)$ implies that $\mathrm{orbit}_T(x) \subseteq F$. Since the former is dense and F is closed, we must have $F = X$.

Now suppose (4) holds and let U be a non-empty open subset of X. Let $O = \bigcup_{k=-\infty}^{\infty} T^k(U)$. Then $T(O) = O$ and $O \neq \emptyset$, hence $O = X$.

Finally, suppose (7) holds and assume that U is a non-empty open subset of X such that $U \subseteq T(U)$. Then $\ldots \subseteq T^{-1}(U) \subseteq U \subseteq T(U) \subseteq T^2(U) \subseteq \ldots$ and these sets form an open covering of X. Since X is compact there is a $K \in \mathbb{N}$ such that

$$X = \bigcup_{k=-\infty}^{\infty} T^k(U) = \bigcup_{k=-K}^{K} T^k(U) = T^K(U).$$

By applying T^{-K} to both sides we obtain $U = X$. □

Suppose (X,T) is a minimal topological dynamical system and that X has an isolated point, say x_0. Then $\{x_0\}$ is open, so by property (7) in the previous proposition we have $X = \mathrm{orbit}_T(x_0)$. And since T is a homeomorphism, it follows that all singleton sets are open in X. Thus X is discrete, and hence finite. This shows that when X is an infinite set, minimality implies that X is perfect.

We now define the topological dynamical systems we are going to study.

Definition 4.1.7. A *Cantor minimal system* is a topological dynamical system (X,T) where X is a Cantor space and T is minimal.

At this point it is worth noting that even though all Cantor spaces are homeomorphic, it does not by any means follow that all Cantor minimal systems are conjugate. The Cantor minimal systems are a rich class of dynamical systems.

We end this section with a classic example of a Cantor minimal system - the so called *dyadic odometer*. At the end of this chapter we will see how the proof of the model theorem applies to this particular example.

Example 4.1.8. Let $X = 2^{\mathbb{N}} = \prod_{n=1}^{\infty}\{0,1\}$ equipped with the product topology, where each $\{0,1\}$ is discrete. A compatible metric is $d(x,y) = 1/n$, where n is the first index where x and y differ. As $\{0,1\}$ is totally disconnected and compact, so is X. It is also easy to see that X has no isolated points since any point can be approximated arbitrarily well by other points. Hence X is a Cantor space.

Informally, the transformation T of X is addition of $(1,0,0,\ldots)$ mod 2, with carry over to the right. More formally, if $(x_1,x_2,x_3,\ldots) \neq (1,1,1,\ldots)$ let $N = \max\{n \in \mathbb{N} | x_i = 1 \text{ for } i \leq n\}$. Then we define

$$T(x_1,x_2,x_3,\ldots) = (\overbrace{0,0,\ldots,0}^{N},1,x_{N+2},x_{N+3},\ldots).$$

Also, we define

$$T(1,1,1,\ldots) = (0,0,0,\ldots).$$

For instance,

$$\begin{aligned} T(1,1,1,0,0,1,\ldots) &= (0,0,0,1,0,1,\ldots) \\ T(0,0,1,1,0,0,\ldots) &= (1,0,1,1,0,0,\ldots). \end{aligned}$$

Given $(x_1,x_2,x_3,\ldots) \in X$, let $M = \min\{n \in \mathbb{N} | x_n = 1\}$. Then

$$T(\overbrace{1,1,\ldots,1}^{M-1},0,x_{M+1},x_{M+2},\ldots) = (x_1,x_2,x_3,\ldots),$$

hence T is surjective. Now let $x = (x_1,x_2,x_3,\ldots), y = (y_1,y_2,y_3,\ldots) \in X$. Suppose $d(x,y) = 1/n$. That is, $x_i = y_i$ for $1 \leq i < n$ and $x_n \neq y_n$. Assume without loss of generality that $x_n = 0$. As above, let $N+1$ denote the first index in x where a 0 appears. Then $N+1 \leq n$. If $N+1 = n$, then $T(x)_i = T(y)_i = 0$ for $1 \leq i < n$ and $T(x)_n = 1 \neq 0 = T(y)_n$, hence $d(T(x),T(y)) = 1/n$. On the other hand, if $N+1 < n$, then we still have $T(x)_i = T(y)_i$ for $1 \leq i < n$ and $T(x)_n = x_n \neq y_n = T(y)_n$, so that $d(T(x),T(y)) = 1/n$. This shows that T is an isometry. In particular then, T is a homeomorphism.

To see that (X,T) is minimal, let $x \in X$ and let $n \in \mathbb{N}$. By looking at the iterates $x, T(x), T^2(x), \ldots$ one observes that every possible sequence of 0's and 1's of length $n-1$ appears in the $n-1$ first coordinates of the points $x, T(x), \ldots, T^{2^{n-1}}(x)$. This means that $d(\text{orbit}_T(x)^+, y) \leq 1/n$ for any $y \in X$. Since this is true for every $n \in \mathbb{N}$ it follows that $\text{orbit}_T(x)^+$ is dense in X. By Proposition 4.1.6, (X,T) is a Cantor minimal system.

4.2 Bratteli diagrams

We are now going to define (unlabeled) Bratteli diagrams, and we will in the sequel denote these simply by the term Bratteli diagrams. Broadly speaking, we are simply deleting the labeling we had when we introduced labeled Bratteli diagrams. Bratteli diagrams with the associated dimension groups are on the one hand crucial tools in determining the isomorphism type of AF-algebras, while on the other hand they yield invariants for the orbit structure of Cantor minimal systems.

Definition 4.2.1. A *Bratteli diagram* is a quadruple (V, E, r, s) satisfying the following properties:

(1) V, the vertex set, and E, the edge set, are both countable disjoint unions of non-empty finite sets; $V = \bigsqcup_{n=0}^{\infty} V_n$ and $E = \bigsqcup_{n=1}^{\infty} E_n$.

(2) $V_0 = \{v_0\}$ is a one point set.

(3) r, the range map, is a map $r\colon E \to V$ such that $r(E_n) \subseteq V_n$. s, the source map, is a map $s\colon E \to V$ such that $s(E_n) \subseteq V_{n-1}$. Moreover, $s^{-1}(v) \neq \emptyset$ for all $v \in V$ and $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus \{v_0\}$, i.e. the graph has no sinks, and no sources except for v_0.

It might be useful for the reader to compare the preceding definition with Definition 2.4.4. We shall often use just (V, E) to denote a Bratteli diagram. We draw Bratteli diagrams in the same way as we draw labeled Bratteli diagrams. The vertices in V_n are drawn on the same horizontal level and the edges in E_n are represented by a line segment between the source in V_{n-1} and the range in V_n. The maps r and s give the range vertex and source vertex of an edge, respectively. The vertices are represented by dots, instead of integers as in the labeled case.

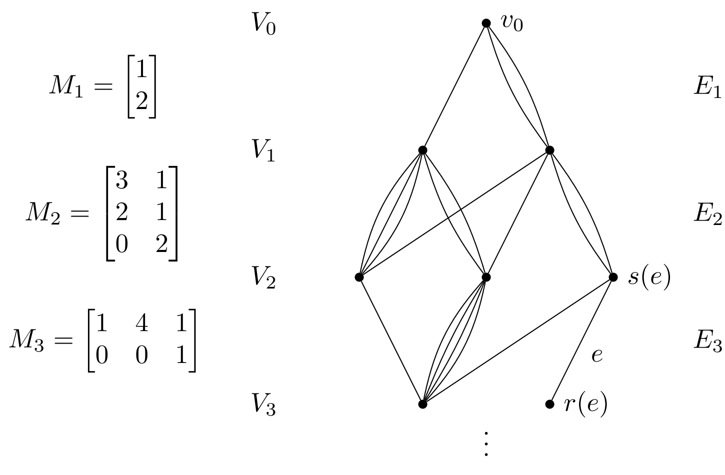

Figure 4.1: A diagrammatic presentation of a Bratteli diagram (V, E, r, s) with some of the terminology illustrated.

The notation introduced in Section 2.4 carries straight over to Bratteli diagrams. When (V, E) is a Bratteli diagram the edges in E_n can be represented by a $|V_n| \times |V_{n-1}|$ *incidence matrix* $M_n = (m_{ij}^{(n)})$, where $m_{ij}^{(n)}$ denotes the number of edges from the vertex $(n-1, j)$ to the vertex (n, i). See Figure 4.1.

Just as for labeled Bratteli diagrams, there is an obvious notion of *isomorphism* between Bratteli diagrams (V, E, r, s) and (V', E', r', s'). Namely, a pair of bijections between V and V' and between E and E' preserving the gradings (i.e. levels) and intertwining the respective source and range maps.

We define paths and telescoping of Bratteli diagrams entirely analogues to Section 2.4, but for clarity we include it here as well. Let (V, E, r, s) be a Bratteli diagram. For $k, l \in \mathbb{Z}^+$ with $k < l$, let $E_{k+1} \circ E_{k+2} \circ \cdots \circ E_l$ denote the set of all *paths* from V_k to V_l. That is,

$$E_{k+1} \circ E_{k+2} \circ \cdots \circ E_l = \{(e_{k+1}, e_{k+2}, \ldots, e_l) | e_i \in E_i, r(e_i) = s(e_{i+1}) \ \forall \ i\}.$$

When $l = k+1$ we identify the paths from V_k to V_{k+1} with E_{k+1}. We also extend the range and source maps to paths in the following way. Define $r(e_{k+1}, \ldots, e_l) = r(e_l)$ and $s(e_{k+1}, \ldots, e_l) = s(e_{k+1})$ for $(e_{k+1}, \ldots, e_l) \in E_{k+1} \circ \cdots \circ E_l$.

Definition 4.2.2. Let (V, E, r, s) be a Bratteli diagram and $m_0 = 0 < m_1 < m_2 < \ldots$ a sequence of integers. The *telescoping* of (V, E, r, s) with respect to the sequence m_n is the Bratteli diagram (V', E', r', s'), where $V'_n = V_{m_n}$, $E'_n = E_{m_{n-1}+1} \circ E_{m_{n-1}+2} \circ \cdots \circ E_{m_n}$, and r' and s' are the extensions of r and s respectively, restricted to the paths E'_n as above.

We need $m_0 = 0$ to ensure that the telescoping of a Bratteli diagram is again a Bratteli diagram. Note that in a telescoping as described in Definition 4.2.2 above, the incidence matrix of $E'_n = E_{m_{n-1}+1} \circ E_{m_{n-1}+2} \circ \cdots \circ E_{m_n}$ is simply the product $M'_n = M_{m_n} M_{m_n - 1} \cdots M_{m_{n-1}+1}$ of the incidence matrices involved.

Example 4.2.3. Let (V, E) be the Bratteli diagram to the left in Figure 4.2. The diagram to the right, (V', E'), is obtained by telescoping to levels $m_0 = 0, m_1 = 2, m_2 = 5, \ldots$. The edges in (V', E') correspond to paths in (V, E). For instance, the red path in (V, E) from V_0 to V_2 becomes an edge in (V', E') from V'_0 to V'_1.

We are now going to look at how a Bratteli diagram can represent a unital AF-algebra, as a labeled Bratteli diagram. Let (V, E, r, s) be a Bratteli diagram. Then (V, E, r, s, d) is a labeled Bratteli diagram where $d \colon V \to \mathbb{N}$ is defined recursively as follows:

$$d(v_0) = 1,$$
$$d(v) = \sum_{r(e)=v} d(s(e)) \text{ for } v \in V_n, \ n \geq 1.$$

See Figure 4.3 for an example of this. This definition of the labeling ensures equality in part 3 of Definition 2.4.4. Which in turn means that the connecting homomorphisms in the associated canonical chain system are unital. We say that the AF-algebra associated to (V, E, r, s, d) is also the AF-algebra associated to

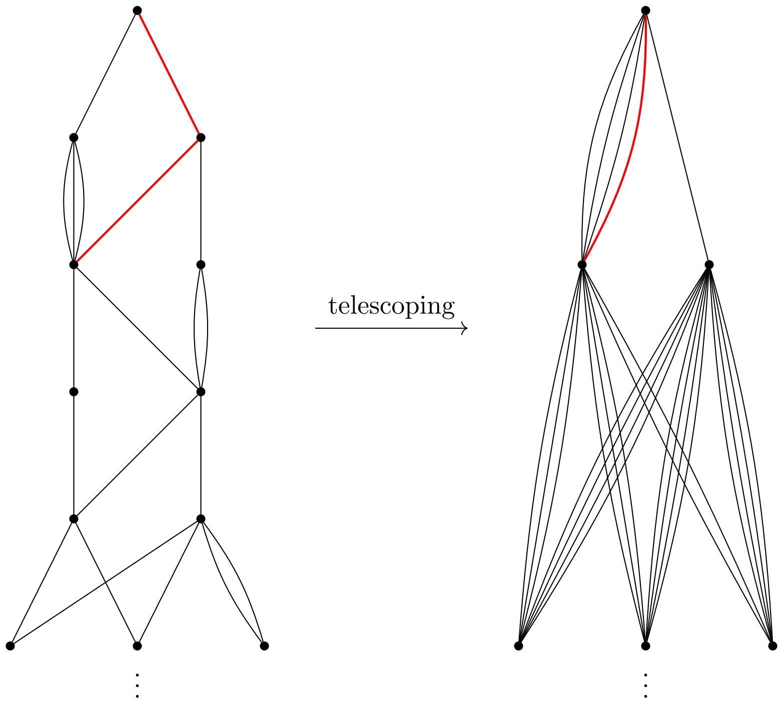

Figure 4.2: Telescoping a Bratteli diagram.

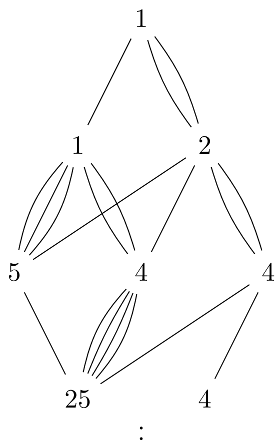

Figure 4.3: The Bratteli diagram in Figure 4.1 as a labeled Bratteli diagram.

(V, E, r, s) and we denote this AF-algebra by $\mathrm{AF}(V, E)$. The following proposition shows that Bratteli diagrams are adequate to represent unital AF-algebras.

Proposition 4.2.4. *If (V, E) is a Bratteli diagram, then $\mathrm{AF}(V, E)$ is unital. Conversely, if A is a unital AF-algebra, then there exists a Bratteli diagram (W, F) with $A \cong \mathrm{AF}(W, F)$.*

Proof. Let (V, E, r, s) be a Bratteli diagram, and let (V, E, r, s, d) be the corresponding labeled Bratteli diagram. Since the connecting homomorphisms in the associated canonical chain system are unital, it follows that the associated AF-algebra $\mathrm{AF}(V, E)$ is unital by Lemma 2.6.1.

Conversely, let $A = \overline{\cup_{n=0}^{\infty} A_n}$ be a unital AF-algebra. As discussed near the end of Section 3.2, we may assume that $A_0 = \mathbb{C}1_A$. Then the connecting homomorphisms in the chain system $(A_n, i_n)_{n \in \mathbb{Z}^+}$, where i_n is the inclusion map, are all unital. This implies that if (W, F, r, s, d) is a labeled Bratteli diagram associated to this canonical chain system, then (W, F) has no sources except v_0 and the sum of the labels leading into a vertex equals the label of that vertex. Hence (W, F, r, s) is a Bratteli diagram and its associated AF-algebra is, by construction, A. $\square$

Example 4.2.5. Let $A = C(Y)$ be the unital AF-algebra in Example 3.3.5. By simply removing the labels from the labeled Bratteli diagram in Figure 3.10 we obtain a Bratteli diagram which represents A. This diagram is depicted in Figure 4.4.

4.3 Associated dimension groups

In this section we briefly introduce the dimension group $K_0(V, E)$ associated to a Bratteli diagram (V, E). A lot more can be said about these.

Definition 4.3.1. By an *ordered group* we shall mean a pair (G, G^+) where G is a countable abelian group and $G^+ \subseteq G$ is a subset, which we will refer to as the *positive cone*, satisfying the following properties:

(1) $G^+ + G^+ \subseteq G^+$

(2) $G^+ - G^+ = G$

(3) $G^+ \cap (-G^+) = \{0\}$

The morphisms in the category of ordered groups are the positive homomorphisms. As the name would suggest, a map $\gamma \colon G_1 \to G_2$ between two ordered groups is a *positive homomorphism* if γ is a group homomorphism and $\gamma(G_1^+) \subseteq G_2^+$.

Let (G, G^+) be an ordered group. We write $a \leq b$ if $b - a \in G^+$, and we write $a < b$ if $a \leq b$ and $a \neq b$. Note that $\leq$ is a translation invariant partial order on G. We say that (G, G^+) is *unperforated* if $a \in G$ and $na \in G^+$ for some $n \in \mathbb{N}$ implies that $a \in G^+$. In a sense this means that the positive cone G^+ has no "gaps". Note that unperforated implies torsion-free. An *order unit* for (G, G^+) is an element $u \in G^+$ such that for every $a \in g$, $a \leq nu$ for some $n \in \mathbb{N}$.

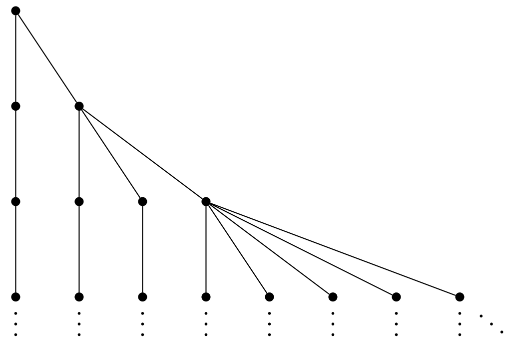

Figure 4.4: A Bratteli diagram representing the unital AF-algebra $C(Y)$ in Example 3.3.5.

An ordered group (G, G^+) is said to satisfy the *Riesz interpolation property* if whenever $a_1, a_2, b_1, b_2 \in G$ with $a_i \leq b_j$ for $i, j = 1, 2$ there exists a $c \in G$ with $a_i \leq c \leq b_j$ for $i, j = 1, 2$. We say that (G, G^+) satisfies the *strict Riesz interpolation property* if $\leq$ can be replaced with $<$ in the above.

Definition 4.3.2. A *dimension group* is an ordered group (G, G^+) which is unperforated and satisfies the Riesz interpolation property.

Example 4.3.3. One of the simplest examples of a dimension group is $\mathbb{Z}^n$ with the positive cone $(\mathbb{Z}^n)^+ = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n | a_i \geq 0 \text{ for } 1 \leq i \leq n\}$. Then $(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n)$ if and only if $a_i \leq b_i$ for each i. It is easily verified that $(\mathbb{Z}^n, (\mathbb{Z}^n)^+)$ is a dimension group. The "canonical" order unit is $(1, 1, \ldots, 1) \in (\mathbb{Z}^n)^+$.

However, this dimension group does not satisfy the strict Riesz interpolation property. To see this, let $a_1 = a_2 = (1, 1, \ldots, 1, 0)$ and $b_1 = b_2 = (1, 1, \ldots, 1)$. Then $a_i < b_j$, but clearly there is no $c \in \mathbb{Z}^n$ with $a_1 < c < b_1$.

If M is an $m \times n$ matrix with non-negative integer entries, then $M\colon \mathbb{Z}^n \to \mathbb{Z}^m$ given by matrix multiplication (on the left) of column vectors is a positive homomorphism. In fact, it is not hard to show that every positive homomorphism $\gamma\colon \mathbb{Z}^n \to \mathbb{Z}^m$ is given by such a matrix.

Example 4.3.4. Let G be a countable subgroup of $\mathbb{R}$. The induced (linear) ordering from $\mathbb{R}$ corresponds to $G^+ = G \cap \mathbb{R}^+$, and (G, G^+) is then a dimension group. If G is dense in $\mathbb{R}$, then (G, G^+) will satisfy the strict Riesz interpolation property. A basic concrete example is $G = \mathbb{Q}$.

In fact, any countable, torsion-free abelian group is isomorphic to a subgroup of $\mathbb{R}$ (see [6]), and can thus be equipped with a (linear) order turning it into a dimension group.

We are now going to adress direct limits of ordered groups. Suppose we have a chain system

$$G_1 \xrightarrow{\gamma_1} G_2 \xrightarrow{\gamma_2} G_3 \xrightarrow{\gamma_3} \cdots \tag{4.1}$$

of ordered groups (G_n, G_n^+) and positive homomorphisms γ_n. We construct the group theoretic direct limit G_∞ of the chain system (4.1) via equivalence classes exactly as we did for $*$-algebras in Section 2.2. The positive cone of G_∞ should come from the positive cones of the G_n's, so we let $G_\infty^+ = \{[a,n] \in G_\infty \mid a \in G_n^+,\ n \in \mathbb{N}\}$. It is easy to check that (G_∞, G_∞^+) is an ordered group, and it can also be verified that this is indeed the direct limit of the chain system (4.1) in the categorical sense. As we did in Section 2.2, we also denote the direct limit (G_∞, G_∞^+) by $\varinjlim(G_n, \gamma_n)$.

Neither is it hard to show that unperforatedness and the Riesz interpolation property are preserved under direct limits, hence a direct limit of dimension groups is again a dimension group.

Example 4.3.5. Continuing Example 4.3.3 we may consider a chain system

$$\mathbb{Z}^{k_1} \xrightarrow{M_1} \mathbb{Z}^{k_2} \xrightarrow{M_2} \mathbb{Z}^{k_3} \xrightarrow{M_3} \cdots$$

where each M_n is a $k_{n+1} \times k_n$ matrix with non-negative integer entries. Then $\varinjlim(\mathbb{Z}^{k_n}, M_n)$ is a dimension group. Furthermore, if no M_n has a row of zeroes, then $[(1,1,\ldots,1),1]$, i.e. the equivalence class of $(1,1,\ldots,1) \in \mathbb{Z}^{k_1}$, is an order unit for $\varinjlim(\mathbb{Z}^{k_n}, M_n)$. This is because $(M_n M_{n-1} \cdots M_1)(1,1,\ldots,1)^T$ is then a vector with strictly positive entries for each n.

Dimension groups were originally introduced by Elliott [5] as direct limits of $\mathbb{Z}^{k_n}$ as in Example 4.3.5. Effros, Handelman and Shen [4] gave the abstract definition in Definition 4.3.1 and proved that any dimension group can be realized as a direct limit as in Example 4.3.5. So that these are in fact all dimension groups.

We are now ready to define the dimension group associated to a Bratteli diagram. Let (V,E) be a Bratteli diagram and let M_n denote the incidence matrix representing E_n for $n = 1, 2, \ldots$. Since each M_n is a non-negative integer matrix we may form the following chain system of ordered groups:

$$\mathbb{Z} \xrightarrow{M_1} \mathbb{Z}^{|V_1|} \xrightarrow{M_2} \mathbb{Z}^{|V_2|} \xrightarrow{M_3} \mathbb{Z}^{|V_3|} \xrightarrow{M_4} \cdots$$

Since a Bratteli diagram has no sources, except for v_0, none of the incidence matrices M_n has a row of zeroes. By Example 4.3.5 the equivalence class of $1 \in \mathbb{Z} = \mathbb{Z}^{|V_0|}$ is an order unit in $\varinjlim(\mathbb{Z}^{|V_{n-1}|}, M_n)$.

Definition 4.3.6. Let (V,E) be a Bratteli diagram and let M_n for $n \in \mathbb{N}$ denote the incidence matrices. The *dimension group associated to* (V,E), denoted $K_0(V,E)$, is $\varinjlim(\mathbb{Z}^{|V_{n-1}|}, M_n)$.

The associated dimension group, $K_0(V,E)$, of a Bratteli diagram (V,E) turns out to be a crucial link between the following three mathematical structures:

- The C^*-algebraic structure of $\mathrm{AF}(V,E)$.
- The combinatorial structure of (V,E).
- The dynamical structure of $(X_{(V,E)}, T_{(V,E)})$ (see Section 4.5).

Among other things, it allows the construction of several highly computable algebraic invariants for unital AF-algebras and Cantor minimal systems, as opposed to general C^*-algebraic or dynamical invariants, respectively.

4.4 Bratteli diagrams as Cantor spaces

We are now going to change focus and instead think of Bratteli diagrams as topological spaces. The underlying set will be the set of infinite paths which begin at the top vertex, v_0.

Definition 4.4.1. Let (V, E, r, s) be a Bratteli diagram. The *infinite path space* of (V, E, r, s) is the set

$$X_{(V,E)} = \{(e_1, e_2, \ldots)|e_n \in E_n \wedge r(e_n) = s(e_{n+1}) \text{ for } n \geq 1\}.$$

By Definition 4.2.1 we have $X_{(V,E)} \neq \emptyset$. We are now going to topologize $X_{(V,E)}$ so that it becomes a Cantor space (under some mild assumptions). A natural approach is to note that $X_{(V,E)} \subseteq \prod_{n=1}^{\infty} E_n$. As each E_n is finite, we regard it as a discrete space. It's not hard to show that $X_{(V,E)}$ is a closed subset in the product topology, and from this most properties of Definition 4.1.1 follow. This is indeed the topology we desire, but in order to get some "hands on experience" with Bratteli diagrams in this new context we are going to prove those properties more directly, rather than only appealing to abstract theorems of general topology.

Let $\alpha = (e_1, \ldots, e_n) \in E_1 \circ \cdots \circ E_n$ be a finite path (starting at v_0) in (V, E). We define the *cylinder set* of α to be

$$U(\alpha) = \{(f_1, f_2, \ldots) \in X_{(V,E)}|f_i = e_i, i = 1, 2, \ldots n\}.$$

That is, all infinite paths in $X_{(V,E)}$ which "starts with" α. Let $B_n = \{U(\alpha)|\alpha \in E_1 \circ \cdots \circ E_n\}$ for $n \in \mathbb{N}$, and let $\mathcal{B} = \cup_{n=1}^{\infty} B_n$. Since each E_n is discrete, and since the product $\prod_{n=1}^{\infty} E_n$ is countable, a basis for the product topology on $\prod_{n=1}^{\infty} E_n$ is

$$\mathcal{B}' = \{\{f_1\} \times \cdots \times \{f_n\} \times E_{n+1} \times E_{n+2} \times \cdots |f_i \in E_i, 1 \leq i \leq n, n \in \mathbb{N}\}.$$

Since $\mathcal{B} = \mathcal{B}' \cap X_{(V,E)}$, $\mathcal{B}$ is a basis for the subspace topology on $X_{(V,E)}$.

Lemma 4.4.2. *Let (V, E) be a Bratteli diagram. Then $X_{(V,E)}$ when endowed with the topology generated by $\mathcal{B}$, the basis of cylinder sets, is a totally disconnected, compact metrizable space.*

Proof. Our first observation is that for every $n \in \mathbb{N}$, the set $B_n = \{U(\alpha)|\alpha \in E_1 \circ \cdots \circ E_n\}$ forms a (finite) partition of $X_{(V,E)}$. Hence, for any $\alpha \in E_1 \circ \cdots \circ E_n$

$$U(\alpha) = X_{(V,E)} \setminus \left(\bigcup_{\beta \in E_1 \circ \cdots \circ E_n \setminus \{\alpha\}} U(\beta) \right).$$

So $U(\alpha)$ is closed. As each cylinder set is clopen, $X_{(V,E)}$ is 0-dimensional.

Next we are going to define a compatible metric. For $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots) \in X_{(V,E)}$ define $d(x, y) = 0$ if $x = y$, and $d(x, y) = 1/n$ if $x \neq y$, where $n = \min\{m \in \mathbb{N}|x_m \neq y_m\}$. Clearly, d is non-negative, symmetric and non-degenerate. To see that d satisfies the triangle inequality, let $x, y, z \in X_{(V,E)}$. If $x = z$ there is nothing to prove. So suppose $d(x, z) = 1/n$. Then $x_i = z_i$ for

$1 \leq i \leq n-1$ and $x_n \neq z_n$. Therefore we must have $y_j \neq x_j$ or $y_j \neq z_j$ for some $j \leq n$. Consequently, $d(x,y) \geq 1/n$ or $d(y,z) \geq 1/n$ which gives

$$d(x,z) = 1/n \leq d(x,y) + d(y,z).$$

So d is a metric. Now let $x \in X_{(V,E)}$ and let $0 < \epsilon < 1$. Pick $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \epsilon \leq \frac{1}{n}$. Then

$$\begin{aligned} B_\epsilon(x) &= \{y \in X_{(V,E)} | d(x,y) < \epsilon\} \\ &= \{(y_1, y_2, \ldots) \in X_{(V,E)} | y_i = x_i, i = 1, 2, \ldots n\} \\ &= U((x_1, \ldots, x_n)). \end{aligned}$$

Since the open balls with respect to d coincide with the cylinder sets, d generates the same topology. Then $X_{(V,E)}$ is Hausdorff and 0-dimensional, hence $X_{(V,E)}$ is totally disconnected.

As $X_{(V,E)}$ is metric, it suffices to prove sequential compactness. Let $x^{(j)} = (x_1^{(j)}, x_2^{(j)}, \ldots)$ be a sequence in $X_{(V,E)}$. Since E_1 is finite there is an $e_1 \in E_1$ such that $x_1^{(j)} = e_1$ for infinitely many j. Let j_1 be one of these j's. Since E_2 is finite, and $x_1^{(j)} = e_1$ for infinitely many j, there is an $e_2 \in E_2$ such that $r(e_1) = s(e_2)$ and $x_1^{(j)} = e_1, x_2^{(j)} = e_2$ for infinitely many j. Let $j_2 > j_1$ be one of these j's. Continuing in this manner we obtain an infinite path $e = (e_1, e_2, \ldots) \in X_{(V,E)}$ and a subsequence $x^{(j_k)}$ of $x^{(j)}$ such that $x_i^{(j_k)} = e_i$ for $1 \leq i \leq k$. This means that $d(x^{(j_k)}, e) < 1/k$, hence $x^{(j_k)}$ converges to e as $k \to \infty$. □

Since telescoping essentially groups finite paths into single "edges", it is not very surprising that the infinite path space is preserved under telescoping. This is the content of the next result.

Lemma 4.4.3. *Let (V,E) be a Bratteli diagram and (V',E') the telescoping of (V,E) with respect to a sequence $0 < m_1 < m_2 < \ldots$. Then $X_{(V,E)} \simeq X_{(V',E')}$.*

Proof. Let $F \colon X_{(V,E)} \to X_{(V',E')}$ denote the natural map defined by

$$(e_1, e_2, \ldots) \mapsto ((e_1, \ldots, e_{m_1}), (e_{m_1+1}, \ldots, e_{m_2}), \ldots) \text{ for } (e_1, e_2, \ldots) \in X_{(V,E)}.$$

It is clear that F is a bijection. Now suppose $x, y \in X_{(V,E)}$ with $d(x,y) = 1/N$. Choose n so that $m_{n-1} < N \leq m_n$. Then $F(x)$ and $F(y)$ will agree up to level $n-1$, since x and y agree up to level m_{n-1}, but differ in the path from level m_{n-1} to level m_n. Thus $d(F(x), F(y)) = 1/n$. And since $n \leq m_{n-1} + 1$ we have $1/N \leq 1/n$, i.e. $d(x,y) \leq d(F(x), F(y))$. This shows that F^{-1} is Lipschitz, and in particular continuous. As the infinite path spaces are compact Hausdorff, F^{-1} is a homeomorphism, and consequently so is F. □

In what follows, we want to avoid the trivial cases, that is we want $X_{(V,E)}$ to be infinite. Therefore we make the following definition.

Definition 4.4.4. A Bratteli diagram (V,E) is called *nontrivial* if $|E_n| \geq 2$ for infinitely many $n \in \mathbb{N}$.

Clearly, $X_{(V,E)}$ is infinite if and only if (V, E) is nontrivial. In order for $X_{(V,E)}$ to be a perfect space we need the following property.

Definition 4.4.5. A Bratteli diagram (V, E) is called *simple* if

(1) (V, E) is nontrivial.

(2) There is a telescoping (V', E') of (V, E) such that (V', E') has full connectivity between any two consecutive levels. I.e. if $v \in V'_n$ and $w \in V'_{n+1}$, then $v \mapsto w$.

Example 4.4.6. Consider the Bratteli diagram depicted on the left in Figure 4.5. This diagram is simple since it is nontrivial and by telescoping to even levels we obtain the telescoped diagram on the right, which has full connectivity between consecutive levels.

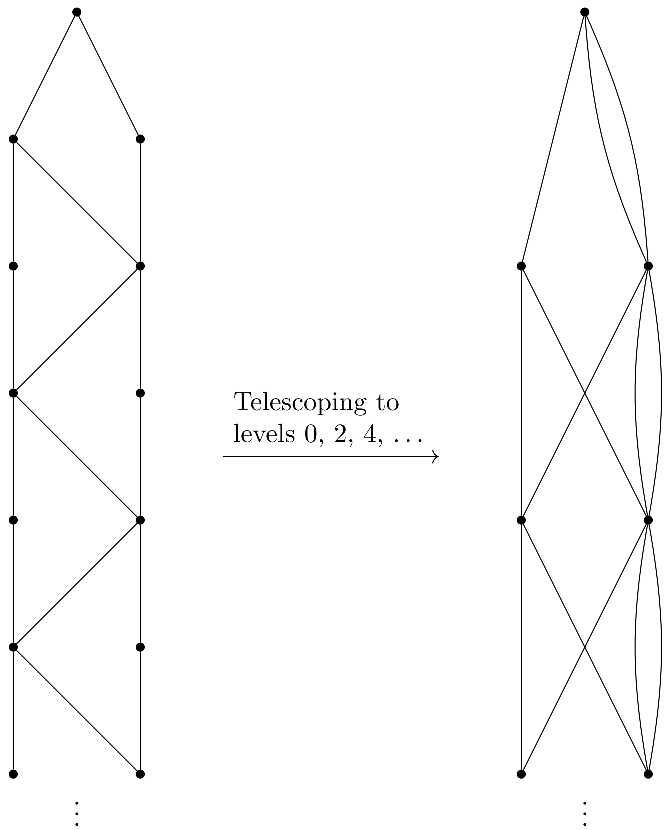

Figure 4.5: On the left, a simple Bratteli diagram, and on the right, a telescoping of this diagram with full connectivity between consecutive levels.

The following proposition illustrates why "simple" is a reasonable name for this property.

Proposition 4.4.7. *Let (V, E) be a nontrivial Bratteli diagram. Then the following are equivalent:*

(1) *(V, E) is simple.*

(2) *$\mathrm{AF}(V, E)$ is simple.*

(3) *(V, E) is eventually fully connected.*

Proof. Since (V, E) has no sources except for v_0 by definition, it suffices to show that (1) $\Leftrightarrow$ (3), by Theorem 3.5.15. Assume that (V, E) is simple. Let $m_0 = 0 < m_1 < m_2 < \dots$ be a sequence such that the telescoping (V', E') with respect to m_n has full connectivity. Let $v \in V_n$. Choose $m_k \geq n$ and let $\alpha \in E_{n+1} \circ \dots \circ E_{m_k}$ be a path with $s(\alpha) = v$. Let $w = r(\alpha) \in V_{m_k}$. Since (V', E') is fully connected we have that for every $u \in V_{m_{k+1}}$ there exists a path $\beta_u \in E_{m_k+1} \circ \dots \circ E_{m_{k+1}}$ with $s(\beta_u) = w$ and $r(\beta_u) = u$, see Figure 4.6. The concatenation of α and β_u is then a path from v to u. This shows that (V, E) is eventually fully connected.

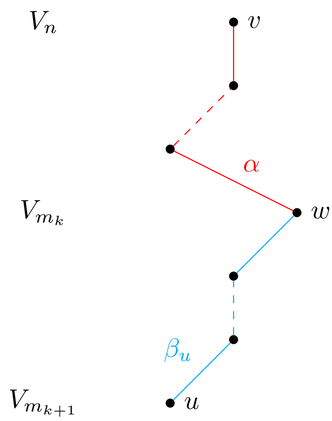

Figure 4.6: Constructing a path from $v \in V_n$ to every vertex u on level m_{k+1}.

Conversely, assume that (V, E) is eventually fully connected. Let $m_0 = 0$ and let m_1 be such that $v_0 \to w$ for all $w \in V_{m_1}$. For each $v \in V_{m_1}$ choose k_v such that $v \to w$ for all $w \in V_{k_v}$. Let $m_2 = \max\{k_v | v \in V_{m_1}\}$. Then there is path from every $v \in V_{m_1}$ to every $w \in V_{m_2}$. Continuing in this manner we obtain a sequence $m_0 = 0 < m_1 < m_2 < \dots$ such that the telescoping of (V, E) with respect to this sequence has full connectivity between consecutive levels. $\square$

It should be noted that when (V, E) is simple, then the associated dimension group $K_0(V, E)$ is also simple. A simple dimension group is one that has no non-trivial order ideals, but we shall not delve any further into this. The following criterion will simplify some proofs.

Lemma 4.4.8. *Let (V, E) be a Bratteli diagram. Then (V, E) is simple if and only if there exists a telescoping (V', E') of (V, E) such that there are at least two edges between any two vertices on consecutive levels in (V', E').*

Proof. One direction is trivial. Assume that (V, E) is simple. Let (V', E') be the telescoping of (V, E) with respect to a sequence m_n having full connectivity between consecutive levels. Since (V, E) is nontrivial, so is (V', E'). Therefore there is a sequence $n_1 < n_2 < \ldots$ such that $|E'_{n_k}| \geq 2$. Let (V'', E'') be the telescoping of (V', E') with respect to the sequence $0 < n_2 < n_4 < n_6 < \ldots$. If $v \in V'_{n_{2k}}$, then since there are at least two edges in $E'_{n_{2k+1}}$ and (V', E') has full connectivity between consecutive levels, there are at least two paths from v to any vertex in $V'_{n_{2k+2}}$. This means that there are at least two edges between any two vertices on consecutive levels in (V'', E''). Now we are done since (V'', E'') is (isomorphic to) the telescoping of (V, E) with respect to the sequence $m_{n_{2k}}$. □

We are now in a position to prove that, under suitable conditions on (V, E), $X_{(V,E)}$ is a Cantor space.

Proposition 4.4.9. *If (V, E) is a simple Bratteli diagram, then $X_{(V,E)}$ is a Cantor space.*

Proof. By Lemma 4.4.2 it suffices to show that when (V, E) is simple, then $X_{(V,E)}$ has no isolated points. To this end let $x = (x_1, x_2, \ldots) \in X_{(V,E)}$ and let $\epsilon > 0$ be given. We need to find a point $y \neq x$ within distance ϵ of x. Pick $n \in \mathbb{N}$ such that $\frac{1}{n} \leq \epsilon$. By Lemma 4.4.8 there is a sequence $0 < m_1 < m_2 < \ldots$ such that the telescoping with respect to this sequence has at least two edges between any two vertices on consecutive levels. Now pick k such that $m_k \geq n$. By virtue of this telescoping there are at least two distinct paths from $r(x_{m_k})$ to $r(x_{m_{k+1}})$, so let $(f_{m_k+1}, \ldots, f_{m_{k+1}})$ be a path different from the subpath of x between these vertices. If we set $y = (x_1, \ldots, x_{m_k}, f_{m_k+1}, \ldots, f_{m_{k+1}}, x_{m_{k+1}+1}, \ldots)$, then $y \neq x$ and $d(x, y) < \frac{1}{n} \leq \epsilon$. Hence $\{x\}$ is not open. Thus $X_{(V,E)}$ is perfect. □

4.5 Ordered Bratteli diagrams

In this section we are going to introduce ordered Bratteli diagrams. Under suitable conditions, the ordering will induce a minimal homeomorphism on the infinite path space $X_{(V,E)}$ and yield a Cantor minimal system.

Definition 4.5.1. An *ordered Bratteli diagram* is a quintuple $(V, E, r, s, \leq)$ where (V, E, r, s) is a Bratteli diagram and $\leq$ is a partial ordering on E such that two edges $e, e' \in E$ are comparable if and only if $r(e) = r(e')$. In other words, we have a linear ordering on $r^{-1}(v)$ for every $v \in V \setminus \{v_0\}$.

Since each set $r^{-1}(v)$ is finite (see Figure 4.7), each edge in $r^{-1}(v)$ can be uniquely assigned a natural number denoting its place in the linear ordering on $r^{-1}(v)$. When drawing ordered Bratteli diagrams we represent the ordering on each set $r^{-1}(v)$ by putting the ordinal number of an edge next to it. If all edges in $r^{-1}(v)$ have the same source we usually omit drawing the ordering.

Example 4.5.2. An example of an ordered Bratteli diagram is depicted in Figure 4.8. The underlying Bratteli diagram is the one from Figure 4.1.

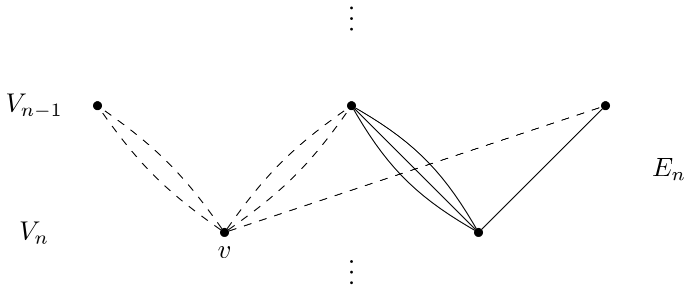

Figure 4.7: For a vertex $v \in V_n$, the edges in $r^{-1}(v) \subseteq E_n$ are the dashed edges.

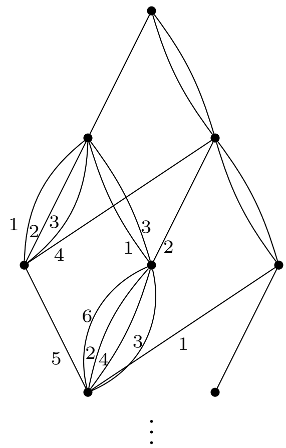

Figure 4.8: An ordered Bratteli diagram.

If $(V, E, \leq)$ is an ordered Bratteli diagram we get an induced "backwards lexicographic" ordering on the set of finite paths as follows. Just as with edges, two paths are comparable if and only if they are paths between the same levels and have the same range. If $\alpha = (e_{k+1}, \ldots, e_l), \beta = (f_{k+1}, \ldots, f_l) \in E_{k+1} \circ \cdots \circ E_l$ are paths from level k to level l with $r(\alpha) = r(\beta)$ (i.e. $r(e_l) = r(f_l)$), then $\alpha \lneq \beta$ if and only if there is an $i \in \mathbb{N}$ such that $k + 1 \leq i \leq l$, $e_j = f_j$ for $i < j \leq l$ and $e_i \lneq f_i$. It is easy to see that this is a partial order on the set of finite paths. Moreover, the paths from a certain level to certain vertex are linearly ordered.

If (V', E') is a telescoping of an ordered Bratteli diagram $(V, E, \leq)$ then it is easy to see that $(V', E', \leq')$ is again an ordered Bratteli diagram when $\leq'$ is the induced partial order on E' described above.

It is clear that a path $\alpha = (e_{k+1}, \ldots, e_l)$ is maximal, respectively minimal, (among the paths) if and only if e_j is maximal, respectively minimal, for each j. Note that for any vertex $v \in V$, there is a unique maximal, respectively minimal, path from v_0 to v. Motivated by this, if $x = (e_1, e_2, \ldots) \in X_{(V,E)}$ is an infinite path we say that x is *maximal*, respectively *minimal*, if e_j is maximal, respectively minimal, for all $j \geq 1$. When $(V, E, \leq)$ is an ordered Bratteli diagram we denote the set of maximal infinite paths in $X_{(V,E)}$ by $X_{\max}$ and the set of minimal infinite paths by $X_{\min}$.

Example 4.5.3. Consider the Bratteli diagram in Figure 4.9. This is the Bratteli diagram from Figure 4.5 with an ordering such that there are two minimal infinite paths and one maximal infinite path. The two minimal paths are colored green and the maximal path is colored red.

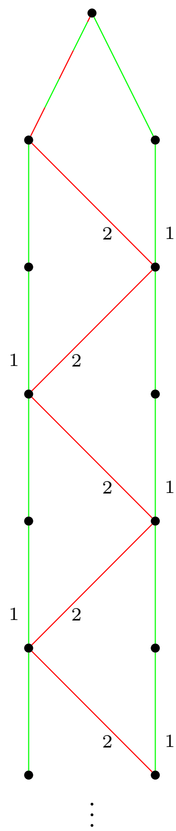

Figure 4.9: An ordered Bratteli diagram with two infinite minimal paths.

As the following proposition shows, there always exists a maximal and a minimal infinite path, but as we just saw, these need not be unique.

Proposition 4.5.4. *Let $(V, E, \leq)$ be an ordered Bratteli diagram. Then $X_{\max}$ and $X_{\min}$ are closed, non-empty subsets of $X_{(V,E)}$. And if (V, E) is simple, then $X_{\max} \cap X_{\min} = \emptyset$.*

Proof. For each $n \in \mathbb{N}$, let $K_n = \{(e_1, e_2, \dots) \in X_{(V,E)} | (e_1, \dots, e_n) \text{ maximal}\}$. For each $v \in V_n$ let α_v denote the unique maximal path from v_0 to v. Then

$K_n = \cup_{v \in V_n} U(\alpha_v)$ which is a finite union of clopen sets, hence K_n is clopen. Moreover, $K_1 \supseteq K_2 \supseteq \dots$ and $X_{\max} = \cap_{n=1}^{\infty} K_n$. Since $X_{(V,E)}$ is compact and $X_{\max}$ is a decreasing intersection of non-empty closed sets, we have that $X_{\max}$ is non-empty and closed itself. The same argument applies to $X_{\min}$.

If (V, E) is simple, then by Lemma 4.4.8 there is a sequence $m_0 = 0 < m_1 < m_2 < \dots$ such that the telescoping (V', E') with respect to m_n has at least two edges between any two vertices on consecutive levels. In particular, this means that for any vertex $v \in V_{m_1}$ there are at least two distinct paths from v_0 to v. And since the paths from v_0 to v are linearly ordered, the maximal path and the minimal path from v_0 to v are distinct. It follows that an infinite path cannot be both maximal and minimal. $\square$

Example 4.5.5. This next example is a bit more extreme than the previous one. The ordered simple Bratteli diagram depicted in Figure 4.10 has infinitely many minimal and maximal infinite paths. The "diagonal" minimal and maximal paths are colored green and red, respectively.

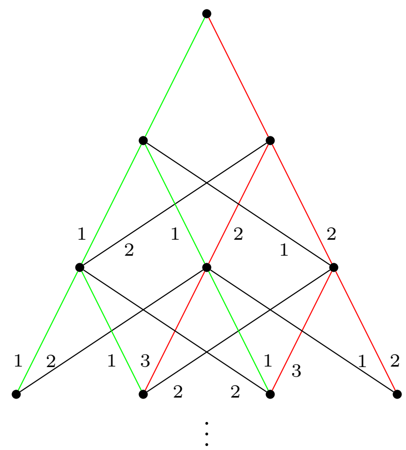

Figure 4.10: An ordered Bratteli diagram with infinitely many infinite minimal paths and infinitely many infinite maximal paths.

In order to get a homeomorphism of $X_{(V,E)}$ we will assume that we have exactly one minimal and one maximal path in $X_{(V,E)}$. We therefore make the following definition.

Definition 4.5.6. A *properly ordered Bratteli diagram* is an ordered Bratteli diagram $(V, E, \leq)$ so that:

(1) (V, E) is simple.

(2) $|X_{\max}| = |X_{\min}| = 1$. I.e. we have a unique maximal infinite path $x_{\max} \in X_{(V,E)}$ and a unique minimal infinite path $x_{\min} \in X_{(V,E)}$.

We are now going to introduce the homeomorphism $T_{(V,E)}$ on $X_{(V,E)}$, where $(V, E, \leq)$ is a properly ordered Bratteli diagram. If $x = (e_1, e_2, \ldots) \in X_{(V,E)}$ and $x \neq x_{\max}$, then let k be the smallest integer such that e_k is not maximal. Let f_k be the successor of e_k (in particular $r(e_k) = r(f_k)$) and let $(f_1, \ldots, f_{k-1})$ be the unique minimal path from v_0 to $s(f_k)$. Define

$$T_{(V,E)}(x) = (f_1, \ldots, f_{k-1}, f_k, e_{k+1}, e_{k+2}, \ldots).$$

Also, define $T_{(V,E)}(x_{\max}) = x_{\min}$. The map $T_{(V,E)} \colon X_{(V,E)} \to X_{(V,E)}$ is called the *Vershik map*. Note that the Vershik map simply gives the successor in the ordering of finite paths in $(V, E, \leq)$ if you disregard the tail of the infinite path.

When $x, y \in X_{(V,E)}$ we say that x and y are *cofinal* if there is an $N \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geq N$, in other words the tails agree from a certain point on. Observe that if $x \neq x_{\max}$, then x and $T_{(V,E)}(x)$ are cofinal (see Figure 4.11).

Example 4.5.7. In Figure 4.11 is a properly ordered Bratteli diagram with the action of the Vershik map $T_{(V,E)}$ applied to the minimal infinite path $x = x_{\min}$. The iterates are depicted in blue.

We now prove that $(X_{(V,E)}, T_{(V,E)})$ is indeed a Cantor minimal system. We call this system the *Bratteli-Vershik system* associated to the properly ordered Bratteli diagram $(V, E, \leq)$.

Proposition 4.5.8. *If $(V, E, \leq)$ is a properly ordered Bratteli diagram, then the associated Bratteli-Vershik system $(X_{(V,E)}, T_{(V,E)})$ is a Cantor minimal system.*

Proof. We need to prove that the Vershik map $T_{(V,E)}$ is a minimal homeomorphism. We proceed to construct the inverse of $T_{(V,E)}$. Define T' on $X_{(V,E)}$ as follows. For $x = (x_1, x_2, \ldots) \in X_{(V,E)}$ with $x \neq x_{\min}$ let l be the smallest integer such that x_l is not minimal. Let y_l be the predecessor of x_l and let $(y_1, \ldots, y_{l-1})$ be the unique maximal path from v_0 to $s(y_l)$. Define $T'(x) = (y_1, \ldots, y_{l-1}, y_l, x_{l+1}, x_{l+2}, \ldots)$. Also, define $T'(x_{\min}) = x_{\max}$. It is easily verified that T' and $T_{(V,E)}$ are inverses of one another. Thus $T_{(V,E)}$ is a bijection. By reversing the order $\leq$, we obtain another properly ordered Bratteli diagram $(V, E, \geq)$. Observe that T' is the Vershik map on $(V, E, \geq)$.

Recall the compatible metric d from Lemma 4.4.2. Let $x = (e_1, e_2, \ldots) \neq x_{\max}$ be an infinite path and let $\epsilon > 0$ be given. Let k be the smallest index such that e_k is not maximal. Pick $N \in \mathbb{N}$ such that $N \geq k$ and $N \geq 1/\epsilon$. If now $d(x, y) < 1/N$, then y agree with x up to level N, which means that the k first initial edges in $T_{(V,E)}(x)$ and $T_{(V,E)}(y)$ agree. And since the tails from level k remain unchanged, we have that $T_{(V,E)}(x)$ and $T_{(V,E)}(y)$ agree up to level N. And this in turn implies that $d\left(T_{(V,E)}(x), T_{(V,E)}(y)\right) < 1/N \leq \epsilon$. Thus $T_{(V,E)}$ is continuous at x.

It remains to verify continuity at $x_{\max}$. Let $N \in \mathbb{N}$ be such that $N \geq 1/\epsilon$. Let $w \in V_N$ be the vertex on level N which $x_{\min}$ passes through. For $n \geq N$, let

$$K_n = \{(e_1, e_2, \ldots) \in X_{(V,E)} |\ (e_1, \ldots, e_n) \text{ minimal } \wedge r(e_N) \neq w\}.$$

Then $K_N \supseteq K_{n+1} \supseteq \ldots$ are closed (finite unions of cylinder sets) and $\cap_{n=N}^{\infty} K_n = \emptyset$, since $x_{\min}$ is the only minimal infinite path. By the finite intersection property we

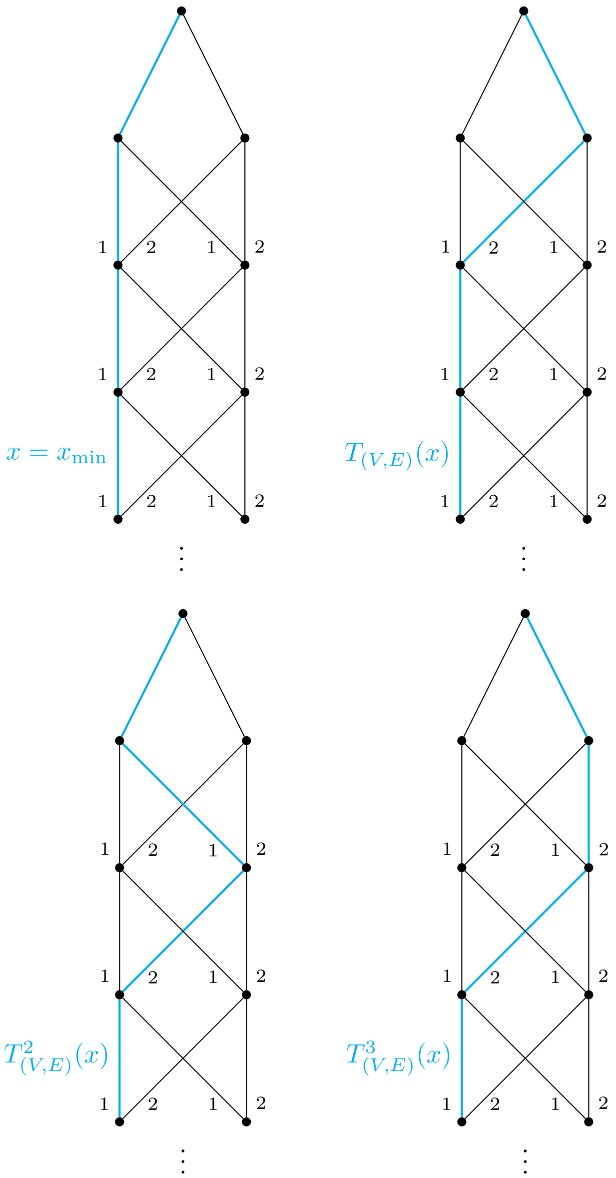

Figure 4.11: The Vershik map applied to $x_{\min}$.

have $K_M = \emptyset$ for some $M \geq N$. This means that any minimal path from v_0 to level M must agree with $x_{\min}$ up to level N. So if $d(x_{\max}, y) < 1/M$, then the first M edges in $T_{(V,E)}(y)$ is a minimal path from v_0 to level M, and therefore $d\left(x_{\min}, T_{(V,E)}(y)\right) < 1/N \leq \epsilon$. Since $T_{(V,E)}(x_{\max}) = x_{\min}$ we see that $T_{(V,E)}$ is continuous at $x_{\max}$. As noted above, the inverse T' is also a Vershik map, and is therefore continuous as well. Hence $T_{(V,E)}$ is a homeomorphism.

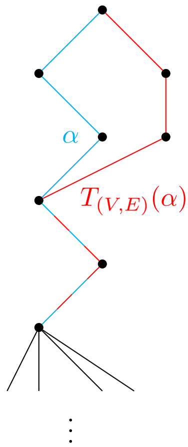

Figure 4.12: $T_{(V,E)}(U(\alpha)) = U(T_{(V,E)}(\alpha))$ when α is not maximal.

Let $\alpha \in E_1 \circ \cdots \circ E_n$ be a path which is not maximal. Let $T_{(V,E)}(\alpha)$ denote the successor of α (in the ordering of finite paths from v_0 to $r(\alpha)$). Note that this is consistent with viewing α as the beginning of an infinite path. Now observe that $T_{(V,E)}(U(\alpha)) = U(T_{(V,E)}(\alpha))$, see Figure 4.12. Similarly, if α is a non-minimal path, then $T^{-1}_{(V,E)}(U(\alpha)) = U(T^{-1}_{(V,E)}(\alpha))$, where $T^{-1}_{(V,E)}(\alpha)$ denotes the predecessor of α. Using this observation we shall show that the orbit of every cylinder set is all of $X_{(V,E)}$. Let $\alpha \in E_1 \circ \cdots \circ E_n$ be any finite path and let $x = (e_1, e_2, \ldots) \in X_{(V,E)}$ be any infinite path. Since (V, E) is simple there is a sequence $0 < m_1 < m_2 < \ldots$ such that the telescoping to these levels has full connectivity between consecutive levels. Pick k so that $m_k \geq n$. Let $q_1 = (e_1, \ldots, e_{m_{k+1}})$. Let q_2 be a superpath of α from v_0 to $r(q_1)$. Such a path exists because of the full connectivity. Since q_1 and q_2 are finite paths between the same vertices there is a $j \in \mathbb{Z}$ such that $T^j_{(V,E)}(q_2) = q_1$ (i.e. q_1 is the jth successor or predecessor of q_2). We therefore have

$$x \in U(q_1) = U(T^j_{(V,E)}(q_2)) = T^j_{(V,E)}(U(q_2)) \subseteq T^j_{(V,E)}(U(\alpha)).$$

Hence $\bigcup_{i=-\infty}^{\infty} T^i_{(V,E)}(U(\alpha)) = X_{(V,E)}$. Since every non-empty open set contains a cylinder set, it follows by Proposition 4.1.6 that $T_{(V,E)}$ is minimal. $\square$

The model theorem states that every Cantor minimal system is conjugate to a Bratteli-Vershik system. In the next example we demonstrate how the dyadic odometer of Example 4.1.8 is realized as a Bratteli-Vershik system. The Bratteli diagram in question has a very basic structure.

Example 4.5.9. Let (X, T) be the dyadic odometer of Example 4.1.8 and let $(V, E, \leq)$ be the properly ordered Bratteli diagram in Figure 4.13. Define the map $\delta\colon E \to \{0, 1\}$ by $\delta(e) = 0$ if e is a minimal edge and $\delta(e) = 1$ if e is a maximal edge. It is clear that the mapping $h\colon X_{(V,E)} \to X$ defined by

$$h((e_1, e_2, \ldots)) = (\delta(e_1), \delta(e_2), \ldots).$$

is a conjugacy.

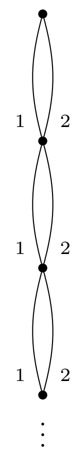

Figure 4.13: The dyadic odometer as a Bratteli-Vershik system.

We saw in Section 3.2 that the AF-algebra associated to a labeled Bratteli diagram is unchanged by telescoping. The next result shows that the same is true for the associated Bratteli-Vershik system. And just as it was in the AF case, the proof is mostly "notation juggling".

Proposition 4.5.10. *Let $(V, E, \leq)$ be a properly ordered Bratteli diagram. If $(V', E', \leq')$ is the telescoping of $(V, E, \leq)$ with respect to a sequence $0 < m_1 < m_2 < \ldots$. Then the Bratteli-Vershik systems $(X_{(V,E)}, T_{(V,E)})$ and $(X_{(V',E')}, T_{(V',E')})$ are conjugate.*

Proof. Note that being properly ordered is preserved by telescoping. We will show that the natural homeomorphism $F\colon X_{(V,E)} \to X_{(V',E')}$ from the proof of Lemma 4.4.3 is a conjugacy. Let $x_{\max}$ and $x'_{\max}$ denote the unique maximal paths in $(V, E, \leq)$ and $(V', E', \leq')$, respectively. Similarly, let $x_{\min}$ and $x'_{\min}$ denote the unique minimal paths. Since $F(x_{\max}) = x'_{\max}$ and $F(x_{\min}) = x'_{\min}$ we obtain

$$F(T_{(V,E)}(x_{\max})) = F(x_{\min}) = x'_{min} = T_{(V',E')}(x'_{\max}) = T_{(V',E')}(F(x_{\max})).$$

Now suppose $x = (e_1, e_2, \ldots) \in X_{(V,E)}$ with $x \neq x_{\max}$. Let k be the smallest integer such that e_k is not maximal. Then

$$T_{(V,E)}(x) = (f_1, \ldots, f_{k-1}, f_k, e_{k+1}, e_{k+2}, \ldots),$$

where f_k is the successor of e_k and $(f_1, \ldots, f_{k-1})$ is the minimal path from v_0 to $s(f_k)$. Now choose j so that $m_j < k \leq m_{j+1}$. By definition of F we have

$$\begin{aligned} F(T_{(V,E)}(x)) =&((f_1, \ldots, f_{m_1}), \ldots, (f_{m_j+1}, \ldots, f_k, e_{k+1}, \ldots, e_{m_{j+1}}), \\ &(e_{m_{j+1}+1}, \ldots, e_{m_{j+2}}), \ldots). \end{aligned}$$

On the other hand, we have that

$$\begin{aligned} F(x) =&((e_1, \ldots, e_{m_1}), \ldots, (e_{m_j+1}, \ldots, e_k, e_{k+1}, \ldots, e_{m_{j+1}}), \\ &(e_{m_{j+1}+1}, \ldots, e_{m_{j+2}}), \ldots). \end{aligned}$$

Since the edges e_i are all maximal for $i < k$, so are the paths in $F(x)$ up to level m_j. The first non-maximal path in $F(x)$ is $(e_{m_j+1}, \ldots, e_k, e_{k+1}, \ldots, e_{m_{j+1}})$ and its successor in the ordering of paths is $(f_{m_j+1}, \ldots, f_k, e_{k+1}, \ldots, e_{m_{j+1}})$. Therefore we have

$$\begin{aligned} T_{(V',E')}(F(x)) =&((f_1, \ldots, f_{m_1}), \ldots, (f_{m_j+1}, \ldots, f_k, e_{k+1}, \ldots, e_{m_{j+1}}), \\ &(e_{m_{j+1}+1}, \ldots, e_{m_{j+2}}), \ldots) = F(T_{(V,E)}(x)). \end{aligned}$$

This shows that F is a conjugacy. □

We end this section with a combinatorial description of the orbits in a Bratteli-Vershik system. The orbit of an infinite path consists of all cofinal infinite paths, with a slight exception for those paths which have maximal or minimal tails.

Proposition 4.5.11. *Let $(V, E, \leq)$ be a properly ordered Bratteli diagram and let $x \in X_{(V,E)}$. Then*

(1) $\mathrm{orbit}_{T_{(V,E)}}(x) = \{y \in X_{(V,E)} |\ y, x$ *cofinal*$\}$ *when x is not cofinal with $x_{\max}$ or $x_{\min}$.*

(2) $\mathrm{orbit}_{T_{(V,E)}}(x) = \{y \in X_{(V,E)} |\ y, x_{\max}$ *cofinal* $\vee$ $y, x_{\min}$ *cofinal*$\}$ *when x is cofinal with $x_{\max}$ or $x_{\min}$.*

Proof. Suppose x is not cofinal with $x_{\max}$ or $x_{\min}$. Then x and $T^j_{(V,E)}(x)$ are cofinal for each $j \in \mathbb{Z}$ (since y and $T_{(V,E)}(y)$ are cofinal when $y \neq x_{\max}$). And if y is cofinal with x, then for some $N \in \mathbb{N}$ we have $x_n = y_n$ for $n \geq N$. Then $(x_1, \ldots, x_{N-1})$ and $(y_1, \ldots, y_{N-1})$ are finite paths between the same vertices. As we observed in the proof of Proposition 4.5.8 we have $T^j_{(V,E)}((x_1, \ldots, x_{N-1})) = (y_1, \ldots, y_{N-1})$ for some $j \in \mathbb{Z}$. And then $T^j_{(V,E)}(x) = y$.

The second statement follows from the previous argument together with the fact that $T_{(V,E)}(x_{\max}) = x_{\min}$. □

4.6 The Bratteli-Vershik model theorem

Having introduced the Bratteli-Vershik systems, we are now ready to prove the model theorem for Cantor minimal systems. The main ingredient in the proof is a sequence of "tower constructions" which are partitions of the space X that behave nicely under the action of T. These "towers" will allow us to construct a properly ordered Bratteli diagram. The full proof becomes quite long since there are many details to check. Before we state the model theorem we introduce some relevant terminology.

Definition 4.6.1. A *pointed topological dynamical system* is a triple (X, T, x_0) where (X, T) is a topological dynamical system and x_0 is a point in X. We call x_0 the *base point*. Two pointed topological dynamical systems (X, T, x_0) and (Y, S, y_0) are *pointedly conjugate* if there exists a conjugacy h between (X, T) and (Y, S) such that $h(x_0) = y_0$.

An example of a pointed conjugacy is the map F from Proposition 4.5.10, when the base points are the respective minimal infinite paths. The exact statement of the model theorem is as follows.

Theorem 4.6.2 (Herman, Putnam, Skau)**.** *Let (X, T) be a Cantor minimal system and let $x_0 \in X$. Then there exists a properly ordered Bratteli diagram $(V, E, \leq)$ such that (X, T, x_0) is pointedly conjugate to $(X_{(V,E)}, T_{(V,E)}, x_{\min})$.*

Note that this theorem is actually a bit stronger that what have been claiming so far. Not only is every Cantor minimal system conjugate to a Bratteli-Vershik system, but for any point $x_0 \in X$ there is a conjugate Bratteli-Vershik system in which the unique minimal path $x_{\min}$ corresponds to x_0.

For the remainder of this section (X, T) is a Cantor minimal system. Suppose Y is a clopen and non-empty subset of X. Since all positive T-orbits are dense (Proposition 4.1.6) and Y is open, the iterated images of every $y \in Y$ must eventually return to Y. Therefore we may define a map $\lambda\colon Y \to \mathbb{N}$ by

$$\lambda(y) := \min\{n \in \mathbb{N} | T^n(y) \in Y\}.$$

We call this map the *first return map* of Y. The positive integer $\lambda(y)$ is how many times we must apply T to y before we wind up back in Y. Henceforth we assume that Y is non-empty.

Lemma 4.6.3. *Let $Y \subseteq X$ be clopen. Then the first return map $\lambda\colon Y \to \mathbb{N}$ is continuous (when $\mathbb{N}$ carries the discrete topology).*

Proof. For $n \in \mathbb{N}$, let $\delta_n = \{1, 2, \ldots, n\}$ and $\Delta_n = \{n+1, n+2, n+3, \ldots\}$. It suffices to show that $\lambda^{-1}(\delta_n)$ and $\lambda^{-1}(\Delta_n)$ are open in X for all $n \in \mathbb{N}$, since $\{n\} = \delta_n \cap \Delta_{n-1}$.

Suppose $y \in \lambda^{-1}(\delta_n)$. This means that $T^k(y) \in Y$ for some $1 \leq k \leq n$. Since Y is open and T^k is continuous, there exists an open neighbourhood U of y such that $T^k(U) \subseteq Y$. And then $U \subseteq \lambda^{-1}(\delta_n)$. This shows that $\lambda^{-1}(\delta_n)$ is open.

Showing that $\lambda^{-1}(\Delta_n)$ is open is equivalent to showing that $\lambda^{-1}(\delta_n)$ is closed. To this end, assume that $y_1, y_2, \ldots$ is a sequence in $\lambda^{-1}(\delta_n)$ which converges to y in Y. Then $T^{k_m}(y_m) \in Y$ where $1 \leq k_m \leq n$. Clearly, there is a $1 \leq k \leq n$ such that $T^k(y_m) \in Y$ for infinitely many m. Denote this subsequence by y_{m_j}. Then $y_{m_j} \to y$ as well and therefore $T^k(y_{m_j}) \to T^k(y)$. As $T^k(y_{m_j}) \in Y$ for each j and Y is closed we have $T^k(y) \in Y$. In other words $y \in \lambda^{-1}(\delta_n)$. Thus $\lambda^{-1}(\delta_n)$ is closed. □

Corollary 4.6.4. *Let $Y \subseteq X$ be clopen and let $\lambda\colon Y \to \mathbb{N}$ be the first return map. Then $\lambda(Y)$ is a finite subset of $\mathbb{N}$; $\lambda(Y) = \{m_1, m_2, \ldots, m_K\}$ where $m_k < m_{k+1}$. Moreover, $\{\lambda^{-1}(m_1), \lambda^{-1}(m_2), \ldots, \lambda^{-1}(m_K)\}$ is a clopen partition of Y.*

Proof. As X is compact and Y is closed, Y is also compact. Then $\lambda(Y)$ is a compact subset of $\mathbb{N}$, hence finite. Each subset $\lambda^{-1}(m_k)$ of Y is clopen since $\{m_k\}$ is clopen in $\mathbb{N}$. Also, these sets are disjoint since the integers m_k are distinct. □

For a given clopen set $Y \subseteq X$, we are now going to use the partition from Corollary 4.6.4 to "build finitely many towers" over Y. What this actually means will soon be clear. Define

$$Y(k, j) := T^j(\lambda^{-1}(m_k))$$

for $k = 1, 2, \ldots, K$, and $j = 0, 1, \ldots, m_k - 1$. We refer to $\{Y(k, 0), Y(k, 1), \ldots, Y(K, m_k - 1)\}$ as a *tower of height m_k*. We also say that $Y(k, j)$ is the *j'th floor of the k'th tower.* $Y(k, 0) = \lambda^{-1}(m_k)$ is a ground floor and $Y(k, m_k - 1)$ is a top floor. The reason for this terminology is explained by Figure 4.14.

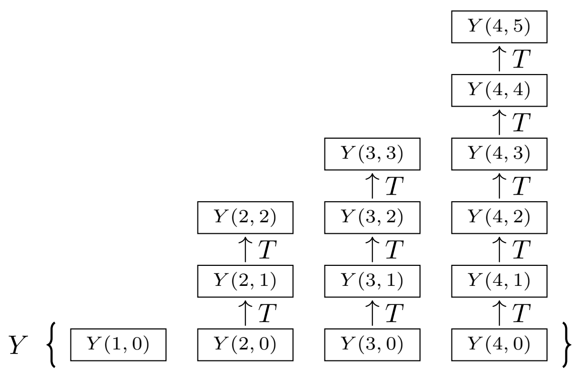

Figure 4.14: The four towers built over Y when $\lambda(Y) = \{1, 3, 4, 6\}$.

Figure 4.14 makes it clear how to draw the general case, that is, when $\lambda(Y) = \{m_1, m_2, \ldots, m_K\}$. By drawing the sets $Y(k, j)$ in this manner T maps each floor onto the floor above. Each top floor is mapped back into Y, i.e. the ground floors, by T, but may be mapped into several ground floors. The K ground floors also form a partition of Y. As the next lemma shows, the collection of all the tower floors actually form a clopen partition of the whole space X. Such a partition is referred to as a *Kakutani-Rokhlin partition* in the literature.

Lemma 4.6.5. *Let $Y \subseteq X$ be clopen. Then the tower floors $Y(k,j)$ satisfy the following properties:*

(1) *Each $Y(k,j)$ is clopen.*

(2) $\bigsqcup_{k=1}^{K} Y(k,0) = Y$.

(3) $T(Y(k,j)) = Y(k,j+1)$ *for* $0 \leq j \leq m_k - 2$.

(4) $T(Y(k,m_k-1)) \subseteq Y$.

(5) *If $(k,j) \neq (k',j')$, then $Y(k,j) \cap Y(k',j') = \emptyset$.*

In particular, the collection $\{Y(k,j) | 1 \leq k \leq K,\ 0 \leq j \leq m_k - 1\}$ is a clopen partition of X.

Proof. The first four properties are trivial consequences of the definition and Corollary 4.6.4. We illustrate (4) for clarity. If $x \in Y(k, m_k - 1)$, then $x = T^{m_k-1}(y)$ where $y \in \lambda^{-1}(m_k)$, but then $T(x) = T^{m_k}(y) \in Y$.

To see that the different floors are disjoint, we first observe that $Y(k,j)$ is disjoint from Y when $1 \leq j \leq m_k - 1$. For if $x \in Y(k,j)$, then $x = T^j(y)$ where $y \in \lambda^{-1}(m_k)$. This means that $T^i(y) \notin Y$ for $1 \leq i \leq m_k - 1$. So $x \notin Y$. Now, consider floors of the same height in different towers, that is $j = j'$ and $k \neq k'$. Then

$$\begin{aligned} Y(k,j) \cap Y(k',j) &= T^j(\lambda^{-1}(m_k)) \cap T^j(\lambda^{-1}(m_{k'})) \\ &= T^j(\lambda^{-1}(m_k) \cap \lambda^{-1}(m_{k'})) = T^j(\emptyset) = \emptyset. \end{aligned}$$

On the other hand, if the floors have different heights, i.e. $j < j'$, assume that $x \in Y(k,j)$ and $x' \in Y(k',j')$. Then $T^{-j}(x) \in Y$, while $T^{-j}(x') \in Y(k', j'-j)$ which is disjoint from Y, hence $x \neq x'$.

Let

$$E = \bigcup_{k=1}^{K} \bigcup_{j=0}^{m_k-1} Y(k,j).$$

Then E is closed by (1), and properties (2), (3) and (4) imply that $T(E) \subseteq E$. Since Y is assumed non-empty, so is E. By minimality we have $E = X$. This completes the proof. $\square$

Now that we have established the tower construction for clopen subsets the strategy is as follows. Given a point $x_0 \in X$, we will choose a sequence of clopen sets shrinking to x_0 and build towers over each of them. Then we will observe how the tower floors on a given level is contained in the floors of the previous level. This will enable us to build an ordered Bratteli diagram encapsulating the action of T. By choosing the sequence of clopen sets in a a clever way, we will obtain a conjugacy. We begin with some preliminary lemmas.

Lemma 4.6.6. *Let X be a Cantor space and let $x_0 \in X$. Then there exists a decreasing sequence $Y_0 = X \supseteq Y_1 \supseteq Y_2 \supseteq \ldots$ of clopen sets containing x_0 such that $\cap_{n=0}^{\infty} Y_n = \{x_0\}$.*

Proof. For every $n \in \mathbb{N}$, $B_{2^{-n}}(x_0)$ is an open set containing x_0 having diameter less than (or equal to) 2^{-n+1}. Since X is 0-dimensional, there is a clopen set $Y'_n \subseteq B_{2^{-n}}(x_0)$ containing x_0. By recursively defining $Y_0 = X$ and $Y_n = Y'_n \cap Y_{n-1}$ for $n \geq 1$, we get $Y_n \supseteq Y_{n+1}$, $x_0 \in Y_n$ and $\operatorname{diam}(Y_n) \leq 2^{-n+1}$ for every n. It follows that $\cap_{n=0}^{\infty} Y_n = \{x_0\}$. □

If Y_n are clopen sets satisfying the conclusion of Lemma 4.6.6, then we say that Y_n *shrink to* x_0. The following lemma shows what shrinking means in technical terms.

Lemma 4.6.7. *Let Y_n be a sequence of clopen sets shrinking to $x_0 \in X$. Then for every open neighbourhood U of x_0, there is an $N \in \mathbb{N}$ such that $Y_n \subseteq U$ for $n \geq N$.*

Proof. Given the hypothesis we have that

$$\bigcap_{n=0}^{\infty} ((X \setminus U) \cap Y_n) = (X \setminus U) \cap \left(\bigcap_{n=0}^{\infty} Y_n \right) = (X \setminus U) \cap \{x_0\} = \emptyset.$$

Since $(X \setminus U) \cap Y_n$ is a decreasing sequence of closed sets we must have $(X \setminus U) \cap Y_N = \emptyset$ for some N by the finite intersection property for compact spaces. That is, $Y_N \subseteq U$ as desired. □

The next lemma shows that as the clopen sets gets "smaller", the towers built over them get taller.

Lemma 4.6.8. *Let Y_n be a sequence of clopen sets shrinking to $x_0 \in X$. Let $\lambda_n : Y_n \to \mathbb{N}$ be the first return map of Y_n. Then $\lambda_n(x_0) \to \infty$ as $n \to \infty$.*

Proof. Fix $L \in \mathbb{N}$. A consequence of minimality (and X being perfect) is that T has no periodic points. So $x_0, T(x_0), \ldots, T^L(x_0)$ are distinct points. As X is Hausdorff there are pairwise disjoint open neighbourhoods U_i of $T^i(x_0)$ for $0 \leq i \leq L$. And by continuity there are open neighbourhoods V_i of x_0 such that $T^i(V_i) \subseteq U_i$ for $1 \leq i \leq L$. Let $O_i = V_i \cap U_0$. Then $x_0 \in O_i \subseteq U_0$ and $T^i(O_i) \subseteq U_i$. Now let $O = \cap_{i=1}^{L} O_i$. Then O is an open neighbourhood of x_0 with $T^i(O) \subseteq U_i$. Also, $O \cap U_i = \emptyset$ for $1 \leq i \leq L$. This means that $T^i(x_0) \notin O$ for $1 \leq i \leq L$. By Lemma 4.6.7 there is an $N \in \mathbb{N}$ such that $Y_n \subseteq O$ for $n \geq N$. But then $\lambda_n(x_0) > L$ for $n \geq N$. As L was arbitrary, we have $\lim_{n\to\infty} \lambda_n(x_0) = \infty$. □

Suppose $\mathcal{P}$ and $\mathcal{Q}$ are partitions of X. We say that $\mathcal{P}$ is *finer* than $\mathcal{Q}$ if for every $A \in \mathcal{P}$ there is a $B \in \mathcal{Q}$ such that $A \subseteq B$. We denote this by $\mathcal{Q} \prec \mathcal{P}$.

Lemma 4.6.9. *Let X be a Cantor space. Then there exists a sequence $\mathcal{P}_n$ of finite clopen partitions of X getting increasingly fine, i.e. $\mathcal{P}_0 = \{X\} \prec \mathcal{P}_1 \prec \mathcal{P}_2 \prec \ldots$, such that $\cup_{n=0}^{\infty} \mathcal{P}_n$ is a basis for X.*

Proof. Since X is 0-dimensional there exists a clopen covering of X with sets of diameter less than $\frac{1}{2}$. By compactness, this covering can be chosen finite, say $X = C_1 \cup C_2 \cup \ldots \cup C_J$, where each set is non-empty and neither is completely contained in any of the other. Then $\mathcal{P}_1 = \{C_1, C_2 \setminus C_1, \ldots, C_J \setminus (\cup_{j=1}^{J-1} C_j)\}$ is a finite clopen partition of X consisting of sets having diameter less than $\frac{1}{2}$.

Assume that we have found finite clopen partitions $\mathcal{P}_1 \prec \mathcal{P}_2 \prec \ldots \prec \mathcal{P}_{n-1}$ such that every set in $\mathcal{P}_j$ has diameter less than 2^{-j}. Exactly as we did above for $\frac{1}{2}$ we can find a finite clopen partition $\{D_1, \ldots, D_{J_n}\}$ of X where the sets have diameter less than 2^{-n}. By letting $\mathcal{P}_n$ be the coarsest partition finer than both $\mathcal{P}_{n-1}$ and $\{D_1, \ldots, D_{J_n}\}$ we have that $\mathcal{P}_n$ is a finite clopen partition of X consisting of sets of diameter less than 2^{-n} such that $\mathcal{P}_{n-1} \prec \mathcal{P}_n$.

By induction there exists a sequence of finite clopen partitions $\mathcal{P}_0 = \{X\} \prec \mathcal{P}_1 \prec \mathcal{P}_2 \prec \ldots$ of X such that every set in $\mathcal{P}_n$ has diameter less than 2^{-n}. To see that $\cup_{n=0}^{\infty} \mathcal{P}_n$ is a basis for X we simply note that for every open set $U \subseteq X$ and every $x \in U$ there is an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. By choosing n such that $2^{-n} < \epsilon$, and choosing $V \in \mathcal{P}_n$ with $x \in V$, we get $V \subseteq B_\epsilon(x) \subseteq U$. $\square$

Lemma 4.6.10. *Let $Y \subseteq X$ be clopen. If $\mathcal{P}$ is a finite clopen partition of X, then by vertically subdividing each of the towers built over Y finitely many times, we obtain a tower partition which is finer than $\mathcal{P}$ and still satisfies the properties in Lemma* 4.6.5 *with the possible exception that there may be several towers of the same height.*

Proof. Suppose $C \in \mathcal{P}$ and that $\emptyset \subsetneq C \cap Y(k,j) \subsetneq Y(k,j)$. Then $Y(k,j)$ splits into two non-empty clopen sets $Y(k,j) \cap C$ and $Y(k,j) \cap (X \setminus C)$, the former contained in C and the latter disjoint from C. Now form the sets

$$\begin{aligned} Y(k,i)' &= T^{i-j}(Y(k,j) \cap C) \\ Y(k,i)'' &= T^{i-j}(Y(k,j) \cap (X \setminus C)) \end{aligned}$$

for $i = 0, 1, \ldots, m_k - 1$. This is a splitting of the kth tower into two separate towers (both of height m_k) such that the jth floor of the first tower is contained in C, while the jth floor of the second is disjoint from C.

By iterating this procedure for every $C \in \mathcal{P}$ and for every tower floor in the current partition which meets C, but is not completely contained in C, (which will be a finite process) we obtain a new $K' \geq K$ and new sets $Y'(k', j')$ such that possibly $m_{k'} = m_{k'+1} = m_k$ for some k''s. Since this new partition is simply the result of vertically subdividing each original tower finitely many times, it is clear that the subdivided tower floors $Y'(k', j')$ also satisfy the properties listed in Lemma 4.6.5. $\square$

Constructing a Bratteli-Vershik model: Given a Cantor minimal system (X, T) and a base point $x_0 \in X$, we use the preceding lemmas to construct a sequence of towers built over clopen sets as follows. Let $Y_0 = X \supseteq Y_1 \supseteq Y_2 \supseteq \ldots$ be a sequence of clopen sets shrinking to $\{x_0\}$ (Lemma 4.6.6) and let $\mathcal{P}_0 = \{X\} \prec \mathcal{P}_1 \prec \mathcal{P}_2 \prec \ldots$ be finite clopen partitions of X such that $\cup_{n=0}^{\infty} \mathcal{P}_n$ is a basis for X (Lemma 4.6.9).

For $n = 0$ the towers built over $Y_0 = X$ is just a single ground floor which is all of X and the associated tower partition then coincides with $\mathcal{P}_0 = \{X\}$. Inductively construct towers over Y_n for each $n \in \mathbb{N}$ such that the associated tower partition is finer than $\mathcal{P}_n$ and finer than the tower partition associated to the towers built over Y_{n-1} (Lemma 4.6.10). This implies that every tower floor over Y_n is contained in some tower floor over Y_{n-1}. If now $x \in Y_n \subseteq Y_{n-1}$, then as the iterates $x, T(x), T^2(x), \dots$ traverse the tower over Y_n which has x in the ground floor, the iterates also traverse some of the towers over Y_{n-1}. By looking at which of the towers built over Y_{n-1} are traversed by $x, T(x), T^2(x), \dots, T^{\lambda_n(x)-1}(x)$, and in which order, we can construct an ordered Bratteli diagram as follows.

For each $n \in \mathbb{Z}^+$ we associate each tower built over Y_n with a vertex in V_n, so that $|V_n| = K_n$, where K_n is the number of towers built over Y_n. See Figure 4.15. If A is a tower built over Y_n, then A_j refers to the jth floor of A. So that in Figure 4.15 for instance, we have $A_0 \sqcup B_0 \sqcup C_0 \sqcup D_0 \sqcup E_0 = Y_n$.

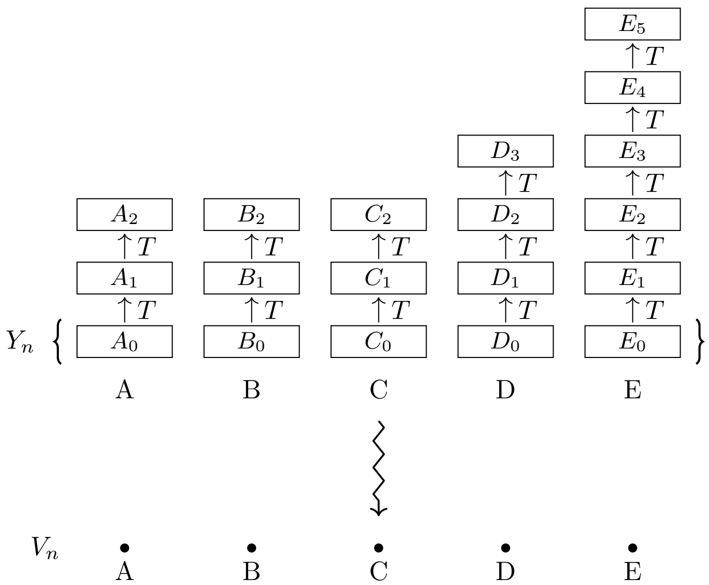

Figure 4.15: For each tower over Y_n we have a corresponding vertex in V_n.

Suppose R is a tower over Y_n ($n \geq 1$) of height m. Since the tower partition of Y_n is finer the tower partition of Y_{n-1}, we must have that the first m_{k_1} floors of R is contained in a tower of height m_{k_1} over Y_{n-1}, the next m_{k_2} floors is contained in a tower of height m_{k_2} over Y_{n-1}, et cetera, and the last m_{k_J} floors of R is contained in a tower of height m_{k_J} over Y_{n-1}. (We then have $m = \sum_{j=1}^{J} m_{k_j}$.) For each of these towers over Y_{n-1} containing some floors of R we have an edge, in E_n, between the vertex corresponding to that tower on level $n-1$ and the vertex corresponding to R on level n. The ordering on the edges going into the vertex corresponding to R is determined by the order in which the towers over Y_{n-1} are reached from the ground floor of R. E.g. for the tower over Y_{n-1} containing the first m_{k_1} floors of R we get the minimal edge, and for the tower over Y_{n-1} containing the last m_{k_J} floors of R we get the maximal edge. The figures Figure 4.17, Figure 4.18 and

Figure 4.19 illustrate this construction in a particular case.

Lemma 4.6.11. *The diagram* $(V = \bigsqcup_{n=0}^{\infty} V_n, E = \bigsqcup_{n=1}^{\infty} E_n, \leq)$ *obtained by the construction above is an ordered Bratteli diagram.*

Proof. Since there are finitely many towers built over each Y_n, and also a finite number of tower floors, we have that V and E are disjoint unions of finite sets. Also, the tower over $Y_0 = X$ is just a single tower of height 1 and therefore $|V_0| = 1$, see Figure 4.16.

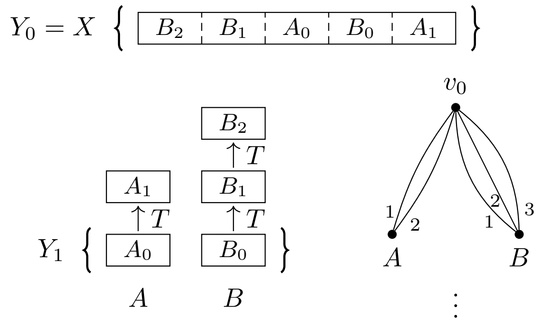

Figure 4.16: The towers over $Y_0 = X$ and Y_1, and the corresponding part of the ordered Bratteli diagram having a single vertex on level 0.

To see that (V, E) has no sinks, and no sources except v_0, it suffices to note that every tower over Y_n contains some floors of at least one tower over Y_{n+1}(resulting in an edge going out of the corresponding vertex), and every tower over Y_{n+1} has its ground floor contained in a tower over Y_n (resulting in an edge coming into the corresponding vertex). Since the induced ordering on the edges is a linear ordering on each set $r^{-1}(v)$, $(V, E, \leq)$ is an ordered Bratteli diagram. $\square$

An ordered Bratteli diagram obtained by the above procedure is called a *Bratteli-Vershik model for* (X, T).

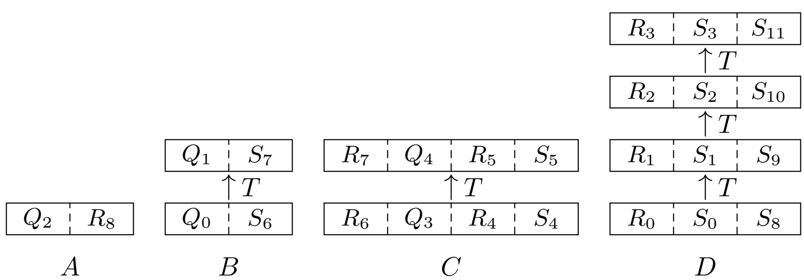

Figure 4.17: The towers built over Y_{n-1} and how they contain the floors of the towers built over Y_n in Figure 4.18.

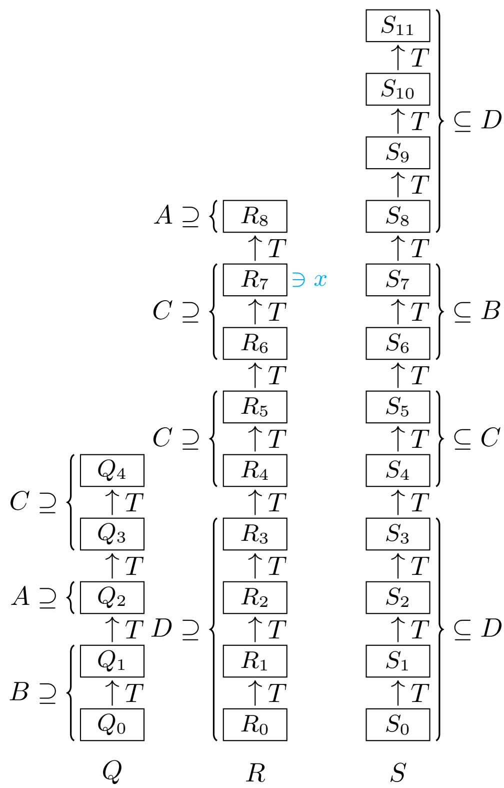

Figure 4.18: The towers built over Y_n and which of the towers over Y_{n-1} in Figure 4.17 they are contained in. Also, $x \in R_7$.

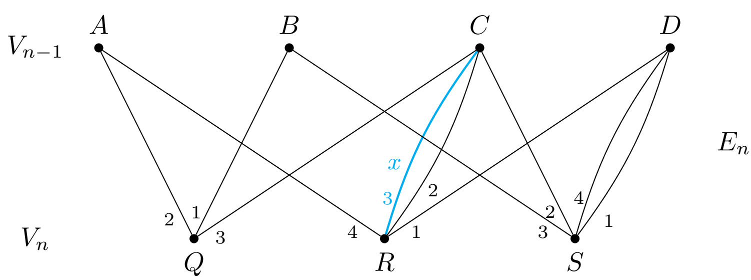

Figure 4.19: The edges between level $n-1$ and level n, and their ordering when Y_{n-1} and Y_n are as in Figure 4.17 and Figure 4.18, respectively. Also, the edge in E_n associated to x, when $x \in R_7$, is marked in blue.

Lemma 4.6.12. *If $(V, E, \leq)$ is a Bratteli-Vershik model for (X, T), then (V, E) is a simple Bratteli diagram.*

Proof. We first show that (V, E) is nontrivial and then that (V, E) is eventually fully connected, which suffices by Proposition 4.4.7. Since X is perfect, no finite set is open. So Y_n is infinite for each n. Since Y_n shrink to x_0, we have that $Y_{n-1} \supsetneq Y_n$ for infinitely many n. We show that $|E_n| \geq 2$ when this happens. If Y_n is properly contained in Y_{n-1}, then the ground floors over Y_n does not fill up all the ground floors over Y_{n-1}. Consequently, there is a tower over Y_{n-1} whose ground floor contains a floor R_j, where $j > 0$ and R is a tower over Y_n. This means that as you traverse R you traverse towers over Y_{n-1} at least twice before returning to Y_n, and thus there are at least two edges going into the vertex corresponding to R.

Suppose A is a tower over Y_n. We need to find an $N > n$ such that every tower over Y_N has a floor contained in A. Since T is minimal there is an $L \in \mathbb{N}$ such that $\{T^k(x_0) | 0 \leq k \leq L\}$ meets every floor of A. And by Lemma 4.6.8 there is an M so that $\lambda_M(x_0) > L$. This means that $T^k(x_0)$ does not return to Y_M before $k > L$. Let R be the tower over Y_M with $x_0 \in R_0$. Then R has some floors contained in A. By Lemma 4.6.7 there is an N with $Y_N \subseteq R_0$. Then every tower over Y_N has its first couple of floors contained in R and consequently has some floors contained in A. So if S is a tower over Y_N, then $S_j \subseteq A$ for some j. And since the tower partitions are increasingly fine we have $S_j \subseteq R^{(N-1)}_{j_{N-1}} \subseteq \ldots \subseteq R^{(n+1)}_{j_{n+1}} \subseteq A$, where $R^{(m)}$ is a tower over Y_m. Each inclusion corresponds to an edge in E and these edges form a path from the vertex corresponding to A on level n to the vertex corresponding to S on level N. This shows that (V, E) is eventually fully connected. □

Before we show that any Bratteli-Vershik model is properly ordered we are going to introduce the identification between X and $X_{(V,E)}$, which is going to serve as our conjugacy. For any $x \in X$ and $n \in \mathbb{N}$ we associate an edge in E_n to x as follows. Let R be the tower over Y_n containing x, say $x \in R_j$. Then $R_j \subseteq A_i$ for some tower A over Y_{n-1}. Then x shall be associated to an edge between the vertices corresponding to A and R. Before reaching the $(j-i)$th floor of R, $M - 1$ towers over Y_{n-1} have already been traversed, and then x is associated to the Mth edge in ordering of edges going into the vertex corresponding to R. And if S is the tower over Y_{n+1} containing x, then x shall be associated to an edge in E_{n+1} between the vertices corresponding R and S (since $x \in R$). This shows that the source of the next edge corresponding to x is the range of the previous one.

This correspondence is better explained by a concrete example. Consider Figure 4.18 and suppose $x \in R_7$. Since $R_7 \subseteq C$, x shall be associated to an edge between the vertices corresponding to C and R. Since R_7 lies in the third traverse of a tower over Y_{n-1} (D and C have been traversed once each), x is associated to the third edge going into R (see Figure 4.19).

Doing this for every $n \in \mathbb{N}$ associates a (unique) infinite path in $X_{(V,E)}$ to each $x \in X$. We denote this mapping simply by $h\colon X \to X_{(V,E)}$.

Lemma 4.6.13. *If $(V, E, \leq)$ is a Bratteli-Vershik model for (X, T), then the induced map $h\colon X \to X_{(V,E)}$ is a homeomorphism.*

Proof. We begin by noting that for every $n \in \mathbb{N}$, the paths in $(V, E, \leq)$ from level 0 to level n, i.e. $E_1 \circ \cdots \circ E_n$, are in one-to-one correspondence with the tower floors over Y_n. In fact, if R is a tower over Y_n, then the floors of R are in one-to-one correspondence with the paths from v_0 to the vertex corresponding to R. If R_j is a floor of R, then the path in $E_1 \circ \cdots \circ E_n$ associated to R_j is the subpath of $h(x)$ from v_0 to the vertex corresponding to R for any $x \in R_j$. More explicitly, it is the path whose edges correspond to the sequence of inclusions $R_j \subseteq A^{(n-1)}_{j_{n-1}} \subseteq \ldots \subseteq A^{(1)}_{j_1} \subseteq X$ of tower floors, where $A^{(l)}$ is a tower over Y_l.

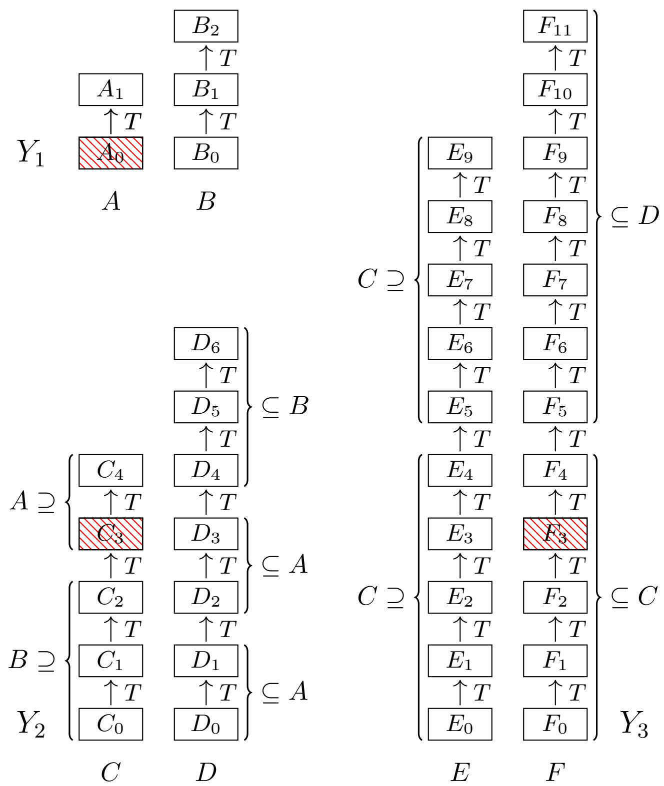

Figure 4.20: The towers A and B, C and D, E and F built over Y_1, Y_2, Y_3 respectively, with the tower inclusions indicated. In particular $F_3 \subseteq C_3 \subseteq A_0 \subseteq X$. So F_3 corresponds to the path α in Figure 4.21.

If $\alpha = (e_1, \ldots, e_n)$ is a path from v_0 to the vertex corresponding to a tower R over Y_n, then the tower floor of R associated to α is obtained inductively as follows. e_1 is the m_1'th edge going into $r(e_1)$, which corresponds to a tower $A^{(1)}$ over Y_1. This corresponds to the m_1'th floor of $A^{(1)}$. So e_1 corresponds to $A^{(1)}_{m_1}$. e_2 is the m_2'th edge going into $r(e_2)$, which corresponds to a tower $A^{(2)}$ over Y_2. Then e_2 corresponds to the inclusions $A^{(2)}_j \subseteq A^{(1)}_0, A^{(2)}_{j+1} \subseteq A^{(1)}_1, \ldots, A^{(2)}_{j+M-1} \subseteq A^{(1)}_{M-1}$, where M is the height of $A^{(1)}$. Since e_1 corresponds to the tower floor $A^{(1)}_{m_1}$ over Y_1, the path (e_1, e_2) is associated to the tower floor $A^{(2)}_{j+m_1}$ over Y_2 (since $A^{(2)}_{j+m_1} \subseteq A^{(1)}_{m_1}$). By continuing in this fashion all the way down to level n we obtain the tower floor over Y_n which is associated to α. Since this is just the "reverse" of the preceding paragraph one easily sees that we have a one-to-one correspondence between the tower floors of a tower R and the paths to the corresponding vertex. This correspondence is better illustrated by a concrete example as in Figure 4.20 and Figure 4.21.

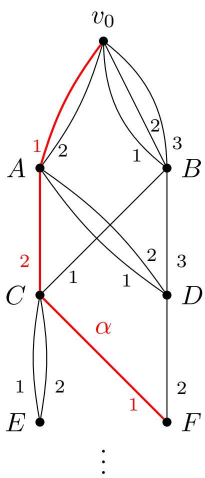

Figure 4.21: The first four levels of the Bratteli-Vershik model obtained from the tower construction in Figure 4.20. The edges making up the path α (marked red) correspond to the inclusions $X \supseteq A_0 \supseteq C_3 \supseteq F_3$ in Figure 4.20, hence α corresponds to F_3.

The correspondence above allows us to construct the inverse of h. If $e = (e_1, e_2, \ldots) \in X_{(V,E)}$ is an infinite path, then by the correspondence of finite paths we have that $(e_1, \ldots, e_n)$ corresponds to a tower floor $A^{(n)}_{j_n}$ for each n. In addition, these tower floors are nested, i.e. $A^{(n)}_{j_n} \supseteq A^{(n+1)}_{j_{n+1}}$. We claim that $\cap_{n=1}^{\infty} A^{(n)}_{j_n} = \{x_e\}$ for some point $x_e \in X_{(V,E)}$. The intersection is non-empty by compactness, and since the partition of tower floors are finer at each step than $\mathcal{P}_n$ (which in particular must have diameter shrinking to 0 as it constitutes a basis), the intersection can contain at most one point. We define $h^{-1}(e) = x_e$. Clearly this is the inverse of h.

The collection of all tower floors form a basis for X (since they are finer than the $\mathcal{P}_n$'s) and the cylinder sets of finite paths form a basis for $X_{(V,E)}$. Let $\alpha \in E_1 \circ \cdots \circ E_n$ be the path corresponding to the tower floor R_j over Y_n. If $x \in R_j$ then, as we saw above, α is the initial subpath of $h(x)$, i.e. $h(x) \in U(\alpha)$. And if $h(x') \in U(\alpha)$, then $x' \in R_j$, for otherwise the initial subpath of $h(x')$ would not be α. This shows that $h^{-1}(U(\alpha)) = R_j$, thus h is continuous. And $h(R_j) = U(\alpha)$, so h^{-1} is continuous. Hence h is a homeomorphism. $\square$

Lemma 4.6.14. *If $(V, E, \leq)$ is a Bratteli-Vershik model for (X, T, x_0), then $(V, E, \leq)$ is properly ordered.*

Proof. We show that $h(x_0)$ is the only minimal infinite path in $(V, E, \leq)$ and that $h(T^{-1}(x_0))$ is the only maximal path. Since $x_0 \in Y_n$ for every n, x_0 lies in a ground floor of a tower over Y_n for every n. This means that x_0 is always in the the first tower traversed on the next level, and therefore $h(x_0)$ is minimal. Now suppose $h(x) = (e_1, e_2, \ldots) \in X_{(V,E)}$ is minimal. Since e_1 is minimal, x lies in one of the ground floors over Y_1, say $x \in A_0$ where A is a tower over Y_1 of height m. Now if R is the tower over Y_2 containing x, then A contains the first m floors of R, which also contains x, since e_2 is minimal. But then $x \in R_0$ since $x \in A_0$ and R_0 is the only floor contained in A_0 of the first m floors. Continuing in this manner one sees that x is contained in a ground floor at each step, i.e. $x \in Y_n$ for all n. But then $x = x_0$, so $h(x_0)$ is the only minimal infinite path.

Since x_0 lies in a ground floor for every n, $T^{-1}(x_0)$ lies in a top floor for every n. This means that $T^{-1}(x_0)$ is always in the last tower traversed on the next level, and therefore $h(T^{-1}(x_0))$ is maximal. And if $h(x) = (e_1, e_2, \ldots) \in X_{(V,E)}$ is maximal, then by analogous reasoning as above, x lies in a top floor over each Y_n. But then $T(x) \in Y_n$ for each n. And then $T(x) = x_0$, so $x = T^{-1}(x_0)$. Thus $h(T^{-1}(x_0))$ is the only maximal infinite path. □

Proof of Theorem 4.6.2. All that remains to prove Theorem 4.6.2 is to check that the induced map h conjugates T and the Vershik map $T_{(V,E)}$. We saw in the proof of the previous lemma that $h(x_0) = x_{\min}$ and $h(T^{-1}(x_0)) = x_{\max}$, where $x_{\min}$ and $x_{\max}$ are the respective unique minimal and maximal paths in $X_{(V,E)}$. It is therefore a trivial calculation to check that

$$T_{(V,E)}(h(T^{-1}(x_0))) = T_{(V,E)}(x_{\max}) = x_{\min} = h(x_0) = h(T(T^{-1}(x_0))).$$

Now consider $x \neq T^{-1}(x_0)$ in X. Then $h(x) = (e_1, e_2, \ldots)$ is not maximal. As usual, let k be the smallest index so that e_k is not a maximal edge. Then x lies in a top floor over each Y_j for $j < k$, but x is not in a top floor over Y_k. Since x is in a top floor over Y_{k-1} and not in a top floor over Y_k, $T(x)$ lies in the next tower traversed over Y_{k-1} (See Figure 4.18, where $T(x) \in R_8 \subseteq A$). This means that the k'th edge in $h(T(x))$ is the successor, f_k, of e_k. Also, $T(x)$ lies in a ground floor over Y_j for $j < k$, which means that the $k-1$ first edges of $h(T(x))$ are all minimal. So the first $k-1$ edges of $h(T(x))$ must be $(f_1, \ldots, f_{k-1})$, which is the unique minimal path from v_0 to $s(f_k)$. And since x was not in a top floor over Y_k, $T(x)$ is still in the same tower over Y_k, and therefore $T(x)$ is in the same traversal over Y_{k+1} and so forth. Therefore $h(T(x))$ has the same tail as $h(x)$ from level k and onwards. In toto we get $h(T(x)) = (f_1, \ldots, f_{k-1}, f_k, e_{k+1}, e_{k+2}, \ldots)$ where f_k is the successor of e_k and $(f_1, \ldots, f_{k-1})$ is the unique minimal path from v_0 to $s(f_k)$. We recognize this as being the Vershik map on $X_{(V,E)}$, i.e. $h(T(x)) = T_{(V,E)}(h(x))$. We also have $h(x_0) = x_{\min}$ so h is a pointed conjugacy. □

Now that we have proved the model theorem a few remarks are in order. First of all, a Bratteli-Vershik model $(V, E, \leq)$ for a given Cantor minimal system (X, T)

obviously depends on the choice of base point $x_0 \in X$, the choice of sets Y_n shrinking to x_0 and also on the choice of the clopen partitions $\mathcal{P}_n$. In our proof we proved that such shrinking sets and clopen partitions exists, and then simply assumed that such a choice had been made. And regardless of the choice made, one ends up with a Bratteli-Vershik model whose Bratteli-Vershik system with base point $x_{\min}$ is pointedly conjugate to (X, T, x_0). Making different choices will yield different ordered Bratteli diagrams. This is similar to the situation we had in Chapter 3 with AF-algebras and their associated labeled Bratteli diagrams. In Theorem 3.4.5 we saw that isomorphic AF-algebras have telescope equivalent labeled Bratteli diagrams. It turns out that the same is true for pointed Cantor minimal systems (i.e. when fixing a base point). In [7] it is shown that two pointed Cantor minimal systems with associated Bratteli-Vershik models are pointedly conjugate if and only if the Bratteli-Vershik models are telescope equivalent (as ordered Bratteli diagrams).

However, changing the base point will generally not yield telescope equivalent ordered Bratteli diagrams. But the diagrams will be telescope equivalent as Bratteli diagrams (i.e. by removing the ordering).

In the proof of the model theorem we constructed a conjugate Bratteli-Vershik system for any Cantor minimal system. So if one started with a properly ordered Bratteli diagram $(V, E, \leq)$, what does a Bratteli-Vershik model for the Bratteli-Vershik system $(X_{(V,E)}, T_{(V,E)})$ look like? A priori, it seems like it should be possible to get $(V, E, \leq)$ back again when constructing a Bratteli-Vershik model for $(X_{(V,E)}, T_{(V,E)}, x_{\min})$. This is indeed the case, and it is explained in more detail in the following example.

Example 4.6.15. Let $(V, E, \leq)$ be a properly ordered Bratteli diagram. We can recover $(V, E, \leq)$ from $(X_{(V,E)}, T_{(V,E)})$ as follows. First, let $x_0 = x_{\min} \in X_{(V,E)}$ be the base point. For $n \in \mathbb{N}$ let

$$Y_n = \bigsqcup_{i=1}^{|V_n|} U\left(\alpha_i^{(n)}\right)$$

where $\alpha_i^{(n)}$ is the unique minimal path from v_0 to the i'th vertex on level n. Let $Y_0 = X$. Since Y_n is a finite union of cylinder sets (which are clopen), Y_n is clopen. A more explicit description of Y_n is

$$Y_n = \{(e_1, e_2, \ldots) \in X_{(V,E)} |\ (e_1, e_2, \ldots, e_n) \text{ is minimal}\}.$$

From this it is clear that $Y_n \supseteq Y_{n+1}$ and $\cap_{n=0}^{\infty} Y_n = \{x_0\}$. Let

$$\mathcal{P}_n = \{U(\alpha) |\ \alpha \in E_1 \circ \cdots \circ E_n\}.$$

Then, by definition, $\mathcal{P}_n$ is a sequence of increasingly fine, finite, clopen partitions of $X_{(V,E)}$ whose union is a basis for $X_{(V,E)}$. When building the towers over Y_n it will be necessary to build $|V_n|$ towers, where each ground floor is the cylinder set of a minimal path between v_0 and level n. Then the tower partition will coincide with $\mathcal{P}_n$ (and will also be finer than the tower partition over Y_{n-1}). Let $v \in V_n$ and let R_v be the tower whose ground floor is the cylinder set of the minimal path from v_0 to v. Then the first few floors of R_v are contained in the tower A_{w_1} built over Y_{n-1}, where w_1 is the source of the minimal edge coming into v. The next few floors of R_v are contained in the tower A_{w_2} built over Y_{n-1}, where w_2 is the source of the second smallest edge coming into v, et cetera. From this we see that the Bratteli-Vershik model obtained from this tower construction coincides with $(V, E, \leq)$.

We end this section with our favourite basic example. Namely the dyadic odometer (see Example 4.1.8). We will see that the ordered Bratteli diagram in Figure 4.13 is the Bratteli-Vershik model obtained from a very natural choice of base point, shrinking sets and partitions.

Example 4.6.16. Let (X, T) be the dyadic odometer. Let $\overline{0} = (0, 0, \ldots) \in X$ be the base point. If we let

$$Y_n = \{(0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots) |\ x_i \in \{0, 1\}\},$$

then Y_n are clopen sets shrinking to $\overline{0}$. The partitions $\mathcal{P}_n$ are just the collection of cylinder sets corresponding to binary sequences of length n. Observe that $\lambda_n(Y_n) = \{2^n\}$ and by building a single tower over each Y_n the tower partitions coincide with $\mathcal{P}_n$, see Figure 4.22. Since there is a single tower on each level, and two tower inclusions on the previous level, it follows that the Bratteli-Vershik model obtained is the ordered Bratteli diagram in Figure 4.13.

$$\begin{array}{ccc} \boxed{(\ldots)} & \begin{array}{r} A \supseteq \left\{ \boxed{(1,\ldots)} \right. \\ \uparrow T \\ A \supseteq \left\{ \boxed{(0,\ldots)} \right. \end{array} & \begin{array}{r} B \supseteq \left\{ \begin{array}{c} \boxed{(1,1,\ldots)} \\ \uparrow T \\ \boxed{(0,1,\ldots)} \end{array} \right. \\ \uparrow T \\ B \supseteq \left\{ \begin{array}{c} \boxed{(1,0,\ldots)} \\ \uparrow T \\ \boxed{(0,0,\ldots)} \end{array} \right. \end{array} \\ A & B & C \end{array}$$

Figure 4.22: The single towers A, B and C built over X, Y_1 and Y_2, respectively, for the dyadic odometer.

Bibliography

[1] Ola Bratteli. "Inductive limits of finite dimensional C*-algebras". In: *Transactions of the American Mathematical Society* 171 (1972), pp. 195–234.

[2] Kenneth R. Davidson. *C*-algebras by example.* Fields Institute monographs 6. Providence, R.I: American Mathematical Society, 1996.

[3] Edward G. Effros. *Dimensions and C*-algebras.* Regional conference series in mathematics 46. Providence, R.I: American Mathematical Society, 1981.

[4] Edward G. Effros, David E. Handelman, and Chao-Liang Shen. "Dimension groups and their affine representations". In: *American Journal of Mathematics* 102.2 (1980), pp. 385–407.

[5] George A. Elliott. "On the classification of inductive limits of sequences of semisimple finite-dimensional algebras". In: *Journal of Algebra* 38 (1976), pp. 29–44.

[6] László Fuchs. *Infinite Abelian groups.* Vol. 1. New York: Academic Press, 1970.

[7] Richard H. Herman, Ian F. Putnam, and Christian F. Skau. "Ordered Bratteli diagrams, dimension groups and topological dynamics". In: *International Journal of Mathematics* 3.6 (1992), pp. 827–864.

[8] Takeshi Katsura. "Non-separable AF-algebras". In: *Operator Algebras: The Abel Symposium 2004.* Vol. 1. Abel Symp. Springer, Berlin, 2006, pp. 165–173.

[9] Kazimierz Kuratowski. *Topology.* Vol. 2. New York: Academic Press, 1968.

[10] James R. Munkres. *Topology.* 2nd ed. Upper Saddle River: Prentice Hall, 2000.

[11] Gerard J. Murphy. *C*-algebras and operator theory.* Boston: Academic Press, 1990.

[12] Ian F. Putnam. *Lecture Notes on C^*-algebras.* University of Victoria. Sept. 23, 2015. URL: `http://www.math.uvic.ca/faculty/putnam/ln/C*-algebras.pdf`.

[13] Stephen Willard. *General topology.* Addison-Wesley series in mathematics. Reading, Mass: Addison-Wesley, 1970.