NTNU
Det skapende universitet

Friezemønstre og triangulerte polygon

Sigurd Nybø Vagstad

Master i realfag
Innlevert: november 2015
Hovedveileder: Aslak Bakke Buan, MATH

Norges teknisk-naturvitenskapelige universitet
Institutt for matematiske fag

Abstract

Friezes were introduced by Conway and Coxeter. In what is practically a problem sheet, they imply that there exists a bijection between friezes and triangulated polygons.
SL_2-tilings with enough ones were introduced by Holm and Jørgensen, who showed that there exists a bijection between SL_2-tilings with *enough ones* and *good* triangulations of the strip. Their work builds on Conway and Coxeter's.
In this paper we expand on Conway and Coxeter's work by explaining the bijection in depth, in part by defining a sufficient and necessary condition to create a frieze pattern. We then explain the bijection between SL_2-tilings with enough ones and good triangulations of the strip, using the results on friezes and triangulated polygons. Most of the examples are new.

Sammendrag

Friezes ble introdusert av Conway og Coxeter. De antyder at det finnes en bijeksjon til triangulerte polygon i en oppgavesamling.
SL_2-tilings med nok enere ble introdusert av Holm og Jørgensen, som viste at det finnes en bijeksjon til *gode* trianguleringer av stripen. Deres arbeid bygger på Conway og Coxeters.
I denne artikkelen fortsetter vi det Conway og Coxeter begynte ved å utdype og forklare bijeksjonen mellom friezes og triangulerte polygon. Spesielt introduserer vi et nødvendig og tilstrekkelig krav for å generere friezes. Videre i oppgaven forklarer vi bijeksjonen, som vist av Holm og Jørgensen, ved bruke resultatene om friezes og triangulerte polygon. De fleste eksemplene i oppgaven er nye.

Acknowledgements

Takk til Aslak Buan for solid veiledning og motivasjon gjennom hele arbeidet. Jeg er imponert over hvor raskt du setter deg inn i nytt stoff.
Jeg har gjennom arbeidet hatt behov for distraksjoner, og vil i den sammenheng takke alle som har bidratt til distrahere meg: Takk til Camilla Næss for kjærlig støtte. Takk til Anna Vederhus, Marit Langfeldt og Anna Testaniere for godt vennskap og godt bordtennisspill. Takk til Martin Nybø Vagstad og Jesper Staahl Simonsen for en åpen invitasjon til prokrastinering. Til slutt, takk til Mamma og Pappa.

Contents

FRIEZE PATTERNS AND TRIANGULATED POLYGONS

1. Introduction

A frieze pattern consists of finitely many rows of integers with a specific set of restrictions. Triangulating a polygon is partitioning a polygon by non-intersecting lines so it consists entirely of triangles. In this paper we show a bijection between frieze patterns and triangulated polygons. We build on this result by looking at expansions of triangulated polygons and frieze patterns and showing relations between these objects as well.

SL_2-tilings are essentially frieze patterns with infinitely many rows. We show that a subset of SL_2-tilings are in bijection with a subset of what is known as triangulations of the strip. Triangulations of the strip can be viewed as an infinite number of triangulated polygons in a row. The bijection is realized by reducing the problem to the case of frieze patterns and polygons.

Conway and Coxeter introduced friezes and their relation to triangulated polygons in [2] and [3]. Holm and Jørgensen have shown the bijection between SL_2-tilings with enough ones and good triangulations of the strip.

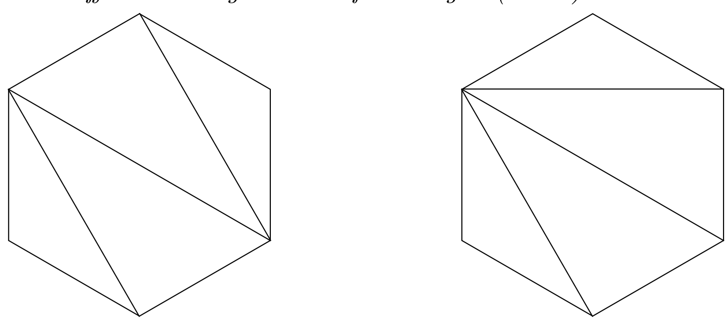
In the first few sections we will remind the reader what a triangulated polygon is and what a frieze pattern is, before explaining the bijection between the two. Our notation is also that used in [4]. Section 5 is dedicated to expand the reader's understanding of friezes. This section is inspired by [1]. As we proceed with SL_2-tilings and triangulations of the strip we will continuously use the preceding sections.

2. Triangulated polygons

Sections 2 through 4 are based on Conway and Coxeter's work on the subject, *Triangulated polygons and frieze patterns* ([2]) and *Triangulated polygons and frieze patterns (continued)* ([3]). Their paper is constructed as a problem sheet, and we have expanded a fair bit where much was left to the imagination. The primary difference is that in our paper we heavily rely on Theorem 3.15 which is previously glossed over. Our notation is generally that of [4] to keep the paper readable.

Definition 2.1. *A **triangulated polygon** is a convex polygon partitioned into non-intersecting triangles. For a polygon with n vertices, a triangulation will create $n-2$ triangles by $n-3$ non-crossing diagonals.*

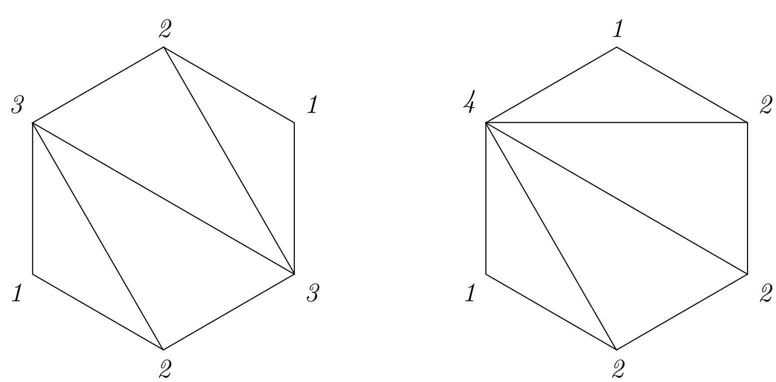
Example 2.2. *Two different triangulations of a hexagon ($n = 6$).*



We will present a way of telling different triangulations apart and present a few results that will come in handy when proving a bijection to friezes.

Definition 2.3. *Given any triangulated polygon with n vertices, label each vertex by how many triangles share that vertex. Going around the polygon counter clockwise once, the set of these values is called a **quiddity cycle**.*

Example 2.4. *For the two hexagons in Example 2.2 such a numbering would look like this:*



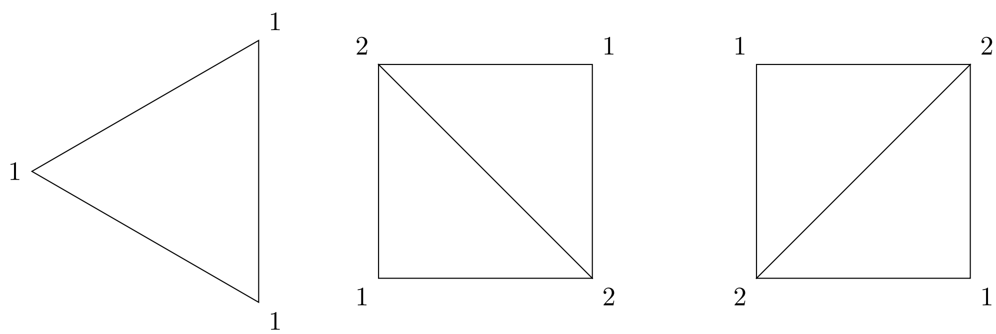
The quiddity cycle of the first hexagon is $1, 2, 3, 1, 2, 3$, or some cyclic shift of the same pattern. The second hexagon has quiddity cycle $1, 2, 2, 2, 1, 4$.

Definition 2.5. *We consider two triangulated polygons with the same number of vertices to be equal if and only if their quiddity cycle is equal up to a cyclic shift. This corresponds to the triangulations being identical if we rotate either polygon.*

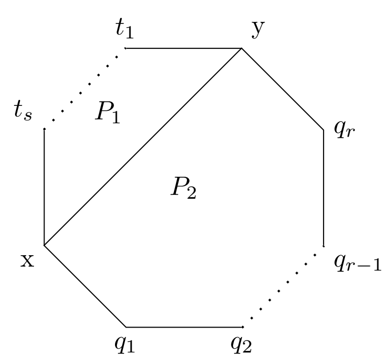
Definition 2.6. *A vertex contained in only one triangle is called a* ***special vertex****. All ones in a quiddity cycle correspond to special vertices.*

Proposition 2.7. *Any triangulated polygon must have at least 2 special vertices.*

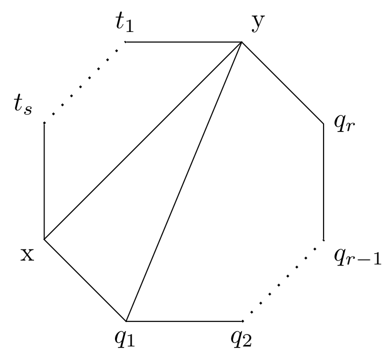
Proof. A polygon with 3 vertices, a triangle is triangulated in itself, and all vertices are special. For a polygon with 4 vertices there are two options for a triangulation, but the number of special vertices is the same.



Assume then that our statement holds for all polygon with k or less vertices, and consider a polygon P with $k+1$ vertices. Any diagonal between any two nodes separates the triangulated polygon into two smaller triangulated polygons, each with at least 3 and at most k vertices. Take a diagonal (x, y) and name the two smaller triangulated polygons P_1, P_2 as illustrated below. We let P_1 be the polygon consisting of vertices $x, y, t_1, ..., t_s$, where $1 \leq s \leq k-2$. Similarly P_2 has vertices $x, q_1, ..., q_r, y$ where $1 \leq r \leq k-2$.



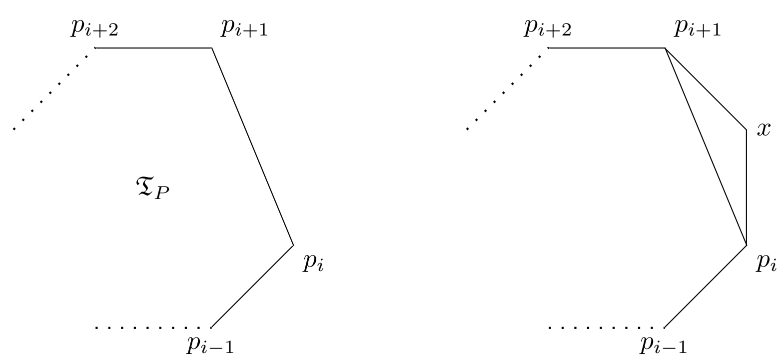
By our assumption P_1 has ≥ 2 special vertices, as does P_2. To show that P has at least two special vertices, we show that P_1, P_2 each have at least one special vertex other than x, y. Since the triangulation of P_1, P_2 is the same as for P, if a vertex in either smaller polygon is special, it is also special in P unless it is x or y. The argument for P_1 and P_2 is identical, but we show it for P_2 to make use of the figures. If P_2 has 3 vertices, it consists of x, q_1, y, all of which are special. Consider the case that P_2 has more than 3 vertices. If x is a special vertex in P_2 then y and q_1 must be connected. This makes vertex y not special in P_2. Similarly we could say that if y is special in P_2 then x is not.



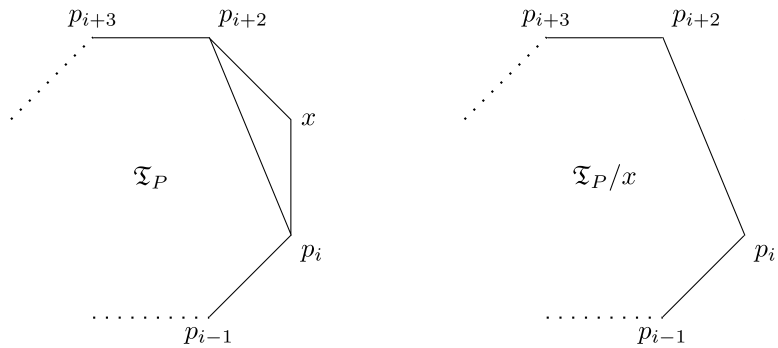
In other words, vertices x and y can not both be special vertices in P_2, but we know P_2 has at least 2 special vertices. Therefore q_i is special in P_2 for some i. This vertex will also be special in P since the triangulation of P_2 is in P. So P_2 has at least one special vertex other than x, y. We mirror the argument to say that so too must P_1. □

Note that for a polygon with more than 3 vertices, two adjacent vertices may not be special in a triangulation. If they were, the polygon would not be triangulated. We will now describe methods that constructs a new triangulated polygon with fewer or more vertices than a given triangulated polygon.

Construction 2.1. *Let $n \geq 3$. Given a triangulated polygon $\mathfrak{T}_P$ with n vertices we create a triangulated polygon with $n+1$ vertices in the following manner. We add one new triangle to $\mathfrak{T}_P$ by adding a vertex x between two adjacent vertices p_i, p_{i+1} and adding the edges $(p_i, x), (x, p_{i+1})$, as illustrated below. The diagonals of $\mathfrak{T}_P$ remains unchanged.*



Construction 2.2. *Let $n \geq 3$. Given a triangulated polygon $\mathfrak{T}_P$ with $n+1$ vertices, we create a triangulated polygon with n vertices in the following manner. Remove one triangle connected to a special vertex. This means removing the special vertex, and removing the two edges connecting it to the polygon. This is illustrated below. By Proposition 2.7 a special vertex is always present in $\mathfrak{T}_P$. This construction is therefore always applicable for polygons of $n+1$ vertices when $n \geq 3$. We let the triangulated polygon we get by removing vertex x from $\mathfrak{T}_P$ be named $\mathfrak{T}_P/x$.*



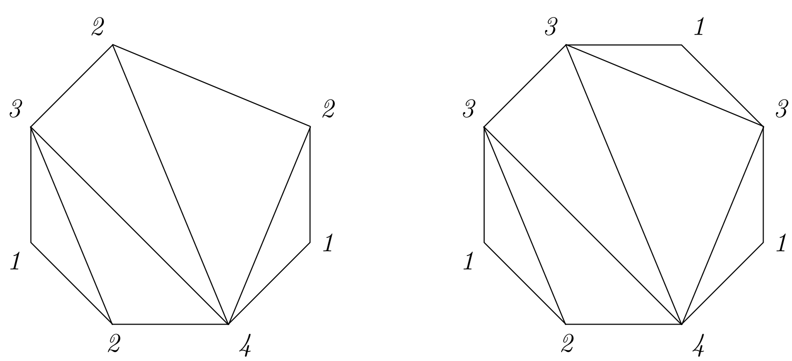
Theorem 2.8. *For $n \geq 3$:*
i) Any triangulated polygon with $n+1$ vertices can be created by applying Construction 2.1 to a triangulated polygon with n vertices.
ii) Any triangulated polygon with n vertices can be created by applying Construction 2.2 to a triangulated polygon with $n+1$ vertices.

Proof. i): Let $\mathfrak{T}_P$ be a triangulated polygon with $n+1$ vertices. We apply Construction 2.2 to $\mathfrak{T}_P$ and remove a special vertex p_i, adjacent to some vertices p_{i-1}, p_{i+1}. $\mathfrak{T}_P/p_i$ is a triangulated polygon with n vertices. We apply Construction 2.1 to $\mathfrak{T}_P/p_i$, inserting a vertex between the vertices p_{i-1}, p_{i+1} to create T_P. The proof for ii) is analogous. □

Let us now explore how our constructions affect the quiddity cycle of a triangulated polygon.

Remark 2.9. *Let $\mathfrak{T}_P$ be a triangulated polygon with $n \geq 3$ vertices. When we apply Construction 2.1 to $\mathfrak{T}_P$, the change to the quiddity cycle is the following. The elements p_i, p_{i+1} between which we insert a new vertex, have the values u, v in the quiddity cycle $a_0, a_1, ..., u, v, ..., a_{n-1}$. When we insert the new vertex x, the vertices p_i, p_{i+1} are now a part of one more triangle each, namely the triangle with vertices p_i, x, p_{i+1}. Their values in the quiddity cycle is therefore incremented by one. The quiddity cycle now reads $a_0, a_1, ..., u+1, 1, v+1, ..., a_{n-1}$ which consists of $n+1$ elements. The 1 between $u+1, v+1$ represents the special vertex x.*

Example 2.10. *The procedure of inserting a vertex between two adjacent vertices in a triangulated polygon*



Notice how the adjacent vertices have their value in the quiddity cycle increased by 1 while the rest of the vertices remain untouched. For the next Remark this illustration may be used from right to left for a visualization.

Remark 2.11. *Let $\mathfrak{T}_P$ be a triangulated polygon with $n+1$ vertices where $n \geq 3$. When we apply Construction 2.2 to $\mathfrak{T}_P$, the change to the quiddity cycle is the following. Let p_i be a special vertex in $\mathfrak{T}_P$. Then the value of p_i in the quiddity cycle is one. Furthermore the adjacent elements in the quiddity cycle must have value greater than one. This is because p_{i-1}, p_{i+1} can not be special when p_i is. This leaves the quiddity cycle reading $a_0, ..., u+1, 1, v+1, ..., a_n$ for $u, v > 0$. Construction 2.2 removes p_i, creating the triangulated polygon T_P/p_i with n vertices. Removing p_i decreases the number of triangles p_{i-1} and p_{i+1} are a part of by one. The quiddity cycle of T_P/p_i is $a_0, ..., u, v, ..., a_n$.*

3. Frieze patterns

In this section we will define frieze patterns. We will show several properties of frieze patterns. This section differs from [2] and [3] in particular because of Theorem 3.15.

Definition 3.1. *A* ***frieze pattern****, or a* ***frieze****, is a finite set of staggered infinite rows of positive integers, such that the top and bottom rows are all ones. In addition we require that each set of elements in a diamond shape*

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

have the property that $ad - bc = 1$. We will refer to this as ***the unimodular rule****. A frieze with $n-1$ rows is said to be of order n.*

Example 3.2. *A frieze pattern of order 6.*

$$\begin{array}{ccccccccccccccccc}
\dots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \dots & & \\
& \dots & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & \dots & \\
\dots & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & \dots & & \\
& \dots & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & \dots & \\
\dots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \dots & &
\end{array}$$

As fate would have it, a frieze pattern of order n also has the property that row i repeats itself with a period k_i such that $k_i|n$ for all $1 \leq i \leq n-1$. This is will be proven in Corollary 3.11. This in a way helps with the idea of linking frieze patterns to triangulated polygons, as going around a polygon more than one round would repeat the same pattern. Note however that not all rows in a single frieze pattern must share the same period. This is seen in Example 3.2, where the second row has period 6, while the third row has period 3.

Example 3.3. *A frieze of order 6, where the second, third and fourth rows have the same period, 3.*

$$\begin{array}{ccccccccccccccccc}
\dots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \dots & & \\
& \dots & 1 & & 2 & & 3 & & 1 & & 2 & & 3 & & 1 & \dots & \\
\dots & 2 & & 1 & & 5 & & 2 & & 1 & & 5 & & 2 & \dots & & \\
& \dots & 1 & & 2 & & 3 & & 1 & & 2 & & 3 & & 1 & \dots & \\
\dots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \dots & &
\end{array}$$

Definition 3.4. *Two friezes of order n are said to be equal if the are equal up to a cyclic shift.*

An interesting problem is characterizing all frieze patterns. To do this, we will give a sufficient requirement to create a frieze, and show a bijection between triangulated polygons and frieze patterns. This will give us a description of all valid frieze patterns, as well as several other interesting results. To show the bijection we will describe a way to relate any frieze pattern to a triangulated polygon and vice versa, starting with the smallest example and using induction. When determining the validity of a frieze we will show that looking at diagonals is enough. This is in part because one can calculate all elements in a frieze if given only a diagonal.

Remark 3.5. *A single diagonal determines the rest of the frieze pattern.*

To convince yourself of Remark 3.5, look at the frieze below.

```
... 1       1       1       1   ...
    ...  3      x       z   ...
        ...  2      y       w   ...
    ... 1       1       1       1   ...
```

the unimodular rule gives us an expression for x:

```
      1
  3       x
      2
```

$$3x - 2 = 1 \implies x = 1$$

Now, knowing the value of x we calculate the value for y, then z and so on. Note that Remark 3.5 does not say that any diagonal determines a valid frieze, but rather that given any diagonal F in a valid frieze we can recreate the frieze from only F.

So, a frieze is determined by a single diagonal, yet we wish to find a relation to triangulated polygons. To do that we take a closer look at the second row of friezes. Our next course of action is to find a close relation between diagonals and the second row, before we show how the second row relates to triangulated polygons.

Assume that a frieze pattern of positive integers could continue beyond the top and bottom rows. The unimodular rule would then give us that the whole zeroth row would be all 0, since $1 \cdot 1 - x \cdot a_i = 1 \implies x = 0$ for $a_i > 0$. Furthermore the row two steps away from the first, the -1st row, would need to satisfy $0 \cdot 0 - 1 \cdot x = 1 \implies x = -1$ for all elements in the row. The same argument is mirrored for the two rows below the last row of the frieze as illustrated below.

Example 3.6. *A frieze pattern continued two rows above, and two rows below a normal frieze.*

```
... -1     -1     -1     -1     -1     -1     -1   ...
    ...  0      0      0      0      0      0      0   ...
... 1      1      1      1      1      1      1    ...
    ...  1      2      2      2      1      4      1   ...
... 3      1      3      3      1      3      3    ...
    ...  2      1      4      1      2      2      2   ...
... 1      1      1      1      1      1      1    ...
    ...  0      0      0      0      0      0      0   ...
... -1     -1     -1     -1     -1     -1     -1   ...
```

Definition 3.7. *Let $\{a_i\}_{i=0}^{n-1}$ be any n consecutive elements of the second row of a frieze pattern. Let $\{f_i\}_{i=-1}^{n-2}$ be the diagonal such that $f_{-1} = 0, f_0 = 1, f_1 = a_0, \dots , f_{n-2} = 1$ going from north west to south east. Let the neighbouring diagonal to the right be $g_{-1} = -1, g_0 = 0, g_1 = 1, g_2 = a_1, \dots , g_{n-1} = 1$, such that f_i and g_{i+1} is in line. This is illustrated below.*

$$
\begin{array}{ccccccccccccc}
\dots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \dots \\
& \dots & f_1 & & g_2 & & a_2 & & a_3 & & a_4 & \dots & \\
& & \dots & f_2 & & g_3 & \dots & & & & & & \\
& & & & \dots & & & & & & & & \\
& & & & & & g_{i-1} & & & & & & \\
& & & & & f_{i-1} & & g_i & & & & & \\
& & & & & & f_i & & & & & & \\
& & & & & & \dots & & & & & & \\
& & & & & & & f_{n-3} & & g_{n-2} & \dots & & \\
& \dots & 1 & & 1 & & 1 & & f_{n-2} & & 1 & & \dots
\end{array}
$$

Definition 3.8. *Let the notation (r,s) represent*

$$(r, s) = \begin{vmatrix} f_r & f_s \\ g_r & g_s \end{vmatrix} = f_r g_s - g_r f_s$$

where f_r, f_s, g_r, g_s is as in Definition 3.7

Proposition 3.9. *With the notation above the following holds.*
i) $(r, r) = 0$
ii) $(r, s) = -(s, r)$
iii) $(s - 1, s) = 1$
iv) $(x, y)(z, w) + (x, z)(w, y) + (x, w)(y, z) = 0$
v)

$$\begin{vmatrix} (r-1, s) & (r, s) \\ (r-1, s+1) & (r, s+1) \end{vmatrix} = 1$$

Proof. *i)*: $(r, r) = f_r g_r - g_r f_r = 0$

ii): $(r, s) = f_r g_s - g_r f_s = -1(f_s g_r - g_s f_r) = -1(s, r)$

iii): $(s - 1, s) = f_{i-1} g_i - g_{i-1} f_i$. For adjacent diagonals f, g in a frieze $f_{i-1} g_i - g_{i-1} f_i = 1$ by the unimodular rule $\forall i \; 0 \leq i < n - 2$.

iv): $(x, y)(z, w) + (x, z)(w, y) + (x, w)(y, z)$
$= (f_x g_y - f_y g_x)(f_z g_w - f_w g_z) + (f_x g_z - f_z g_x)(f_w g_y - f_y g_w) + (f_x g_w - f_w g_x)(f_y g_z - f_z g_y)$

We multiply this out and sort the terms alphabetically on the subscript.
$= f_x f_z g_w g_y - f_w f_x g_y g_z - f_y f_z g_w g_x + f_w f_y g_x g_z + f_w f_x g_y g_z - f_x f_y g_w g_z - f_w f_z g_x g_y + f_y f_z g_w g_x + f_x f_y g_w g_z - f_x f_z g_w g_y - f_w f_y g_x g_z + f_w f_z g_x g_y$
We color code this to make it readable.

$f_x f_z g_w g_y - f_w f_x g_y g_z - f_y f_z g_w g_x + f_w f_y g_x g_z + f_w f_x g_y g_z - f_x f_y g_w g_z - f_w f_z g_x g_y + f_y f_z g_w g_x + f_x f_y g_w g_z - f_x f_z g_w g_y - f_w f_y g_x g_z + f_w f_z g_x g_y = 0$

v): by inserting $x = r - 1,\; y = s,\; z = r,\; w = s + 1$ into *iv)* we get that

$$(r - 1, s)(r, s + 1) + (r - 1, r)(s + 1, s) + (r - 1, s + 1)(s, r) = 0$$

wherein $(r - 1, r) = 1$, $(s + 1, s) = -1$ by *iii)* and *ii)*. By *ii)* we also get $(s, r)(r - 1, s + 1) = -(r, s)(r - 1, s + 1)$ all of which we insert in our expression to get

$(r-1,s)(r,s+1)+(s,r)(r-1,s+1) = (r-1,r)(s,s+1) = 1$ However

$$1 = (r-1,s)(r,s+1)+(s,r)(r-1,s+1) = \begin{vmatrix} (r-1,s) & (r,s) \\ (r-1,s+1) & (r,s+1) \end{vmatrix} = 1$$

□

Having shown that any diagonal determines a frieze, we proceed with a way of expressing the second row of a frieze as a function of a diagonal.

Theorem 3.10. *As in Definition 3.7, let* $\{a_i\}_{i=0}^{n-1}$ *be any* n *consecutive elements of the second row of a frieze pattern. Let* $\{f_i\}_{i=-1}^{n-2}$ *be the diagonal such that* $f_{-1} = 0, f_0 = 1, f_1 = a_0, \dots, f_{n-2} = 1$. *Then*

$$a_s = \frac{f_{s-1}+f_{s+1}}{f_s}$$

$\forall s$ *such that* $0 \leq s \leq n-2$.

Proof. Consider the grid below, in the shape of a frieze, in which any touple (r,s) is as described in Definition 3.8.

(-1,0)		(0,1)		(1,2)		(2,3)		(3,4)	
	(-1,1)		(0,2)		(1,3)		(2,4)		...
		(-1,2)		(0,3)		(1,4)		(2,5)	
			...				...		
			(-1,n-3)		(0,n-2)		(1,n-1)		
				(-1,n-2)		(0,n-1)		(1,n)	
					(-1,n-1)		(0,n)		...

$(-1,s) = f_{-1}g_s - g_{-1}f_s = 0\cdot g_s - (-1)\cdot f_s = f_s$ so the first diagonal is indeed $f_0, f_1, ..., f_{n-2} = 1, f_{n-1} = 0$. $(0,s) = f_0 g_s - g_0 f_s = g_s - 0\cdot f_s = g_s$. Furthermore any diamond shape in the pattern is such that

	(r,s)	
(r-1,s)		(r,s+1)
	(r-1,s+1)	

for some integers r and s. We know from Proposition 3.9 v) however, that $(r,s)(r-1,s+1) - (r-1,s)(r,s+1) = 1$ which for the pattern above is the unimodular rule. Additionally all elements in the top row, $(i,i+1) = 1$. Now, as we know, a diagonal determines a frieze, and we have a pattern following the unimodular rule with the diagonal $\{f_i\}$. This means that the grid in the figure must indeed be the frieze we began with. In other words all friezes can be described in the manner of the grid with the notation from Definition 3.8. The second row is expressed $\{a_s\} = \{(s-1,s+1)\}$.
For $0 \leq s \leq n-2$, we get, by substituting $x = s-1,\ y = s+1,\ z = -1,\ w = s$ into Proposition 3.9 iv), that

$$a_s = (s-1,s+1) = \frac{(-1,s-1)(s,s+1)+(-1,s+1)(s-1,s)}{(-1,s)}$$

where $(-1,i) = f_i$ and $(i,i+1) = 1$ by Proposition 3.9 iii).

$$\frac{(-1,s-1)(s,s+1)+(-1,s+1)(s-1,s)}{(-1,s)} = \frac{f_{s-1}\cdot 1 + f_{s+1}\cdot 1}{f_s} = \frac{f_{s-1}+f_{s+1}}{f_s}$$

□

Several useful results follow directly from Theorem 3.10 and some are explicitly stated in the proof of the theorem. We rephrase and state some useful consequences for future use.

Corollary 3.11. *All rows in a frieze of order n have periods that divide n. Phrased differently $(r, s) = (r + n, s + n)$. Note however that all rows need not have the same period.*

Proof. By definition we have $(1, n) = (2, n + 1) = ... = (i, n + i) = 1$ as it is the bottom row of a frieze, and the row below that; $(0, n) = (1, n + 1) = ... = (j, n + j) = 0$. Inserting into Proposition 3.9 *iv*): $x = r,\ y = s,\ z = r + 1,\ w = r + n$, gives us

$$(r, s)(r + 1, r + n) + (r, r + 1)(r + n, s) + (r, r + n)(s, r + 1) = 0 \tag{1}$$

We use that $(r, r + 1) = 1$ by Proposition 3.8 *iii*). Additionally, as stated at the start of the proof, $(r + 1, r + n) = 1$ and $(r, r + n) = 0$. We insert into equation (1) to get

$$(r, s) + (r + n, s) + 0 = 0$$

or by Proposition 3.8 *ii*) $(r, s) - (s, r + n) = 0$. Adding $(s, r + n)$ to both sides of the equation gives us

$$(r, s) = (s, r + n)$$

Repeating the process above once more, starting with $(s, r+n)$, gives us $(s, r+n) = (r+n, s+n)$ which means $(r, s) = (r + n, s + n)$. □

Corollary 3.12. *Any element of a frieze pattern divides the sum of its diagonal neighbours.*

Proof. By Theorem 3.10 we have $a_s = \dfrac{f_{s-1} + f_{s+1}}{f_s}$, where a_s is an integer $\forall s$. Now any element is a part of a diagonal that intersects the second row, it is only a matter of shifting the index of the set $\{a_i\}$ to get $a_s = \dfrac{f_{s-1} + f_{s+1}}{f_s}$ for some s. □

Corollary 3.13. *The second row of any frieze pattern must have at least one 1.*

Proof. By Theorem 3.10 we have that

$$f_{s+1} = a_s f_s - f_{s-1}$$

Now to prove that at least one element is equal to 1 lets assume otherwise, that $a_s \geq 2\ \forall s$, which gives us

$$a_s f_s - f_{s-1} \geq 2f_s - f_{s-1}$$

by inserting this into the first expression we get the inequality

$$f_{s+1} \geq 2f_s - f_{s-1} \implies f_{s+1} - f_s \geq f_s - f_{s-1}$$

$$f_{s+1} - f_s \geq f_s - f_{s-1} \geq ... \geq f_1 - f_0 = a_0 - 1 \geq 1$$

However, this implies that the sequence $f_0, f_1, ..., f_{n-2}$ is strictly increasing, while by definition $f_{n-2} = 1$ which is a contradiction. □

Remark 3.14. *The neighbouring elements of a 1 in the second row are strictly greater than 1, in a frieze of order $n > 1$.*

Proof. Assume otherwise, that two adjacent elements in the second row, $a_i = a_{i+1} = 1$. Name the element below them b_i. Then by the unimodular rule $a_i a_{i+1} - 1b_i = 1 \implies b_i = 0$ which is a contradiction since the order of the frieze is greater than 1. □

We have now shown how we calculate the elements in the second row when given a diagonal. So far, however, we assume we are given a valid frieze. The next theorem states exactly when a diagonal generates a *valid* frieze. This result will help us show the continued validity of friezes when we apply certain maps to valid friezes.

Theorem 3.15. *Let F be a sequence of $n-1$ positive integers, $f_0, f_1, ..., f_{n-3}, f_{n-2}$ with $f_0 = 1 = f_{n-2}$, such that $\frac{f_{s-1}+f_{s+1}}{f_s}$ is a positive integer for $s \in 1, 2, ..., n-3$. Let F be a diagonal in an empty frieze of order n. Then F generates a valid frieze.*

Proof. Let $1 = f_0, f_1, ..., f_{n-3}, f_{n-2} = 1$ be positive integers such that $\frac{f_{s-1}+f_{s+1}}{f_s}$ is a positive integer for $s \in 1, 2, ..., n-3$. We let the f-elements be a diagonal in a potential frieze, as shown below. We compute the next diagonal by the unimodular rule, and name the elements $1 = g_1, g_2, ..., g_{n-1} = 1$. This setup is depicted below.

$$\begin{array}{ccccccccccccc}
... & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & ... \\
 & ... & f_1 & & g_2 & & & & & & & ... & \\
 & & ... & f_2 & & g_3 & ... & & & & & & \\
 & & & & ... & & & & & & & & \\
 & & & & & & g_{i-1} & & & & & & \\
 & & & & & f_{i-1} & & g_i & & & & & \\
 & & & & & & f_i & & ... & & & & \\
 & & & & & & & ... & & & & & \\
 & & & & & & & f_{n-3} & & g_{n-2} & ... & & \\
 & ... & 1 & & 1 & & 1 & & f_{n-2} & & 1 & & ...
\end{array}$$

We begin by showing $\frac{g_{i-1}+g_{i+1}}{g_i}$ is a positive integer before showing that all g_i are positive integers. By the unimodular rule the way we compute the elements g_i is the following.

$$f_{i-1}g_i - f_i g_{i-1} = 1 \implies g_i = \frac{1+f_i g_{i-1}}{f_{i-1}}. \tag{2}$$

We want to show $\frac{g_{i-1}+g_{i+1}}{g_i} \in \mathbb{Z}^+$ and in fact more specifically

$$\frac{g_{i-1}+g_{i+1}}{g_i} = \frac{f_{i-1}+f_{i+1}}{f_i} \quad \text{for } i \in 2, ..., n-3$$

Using equation (2) we have $g_i = \frac{1+f_i g_{i-1}}{f_{i-1}}$, $\quad g_{i+1} = \frac{1+f_{i+1}g_i}{f_i} = \frac{1+f_{i+1}(\frac{1+f_i g_{i-1}}{f_{i-1}})}{f_i}$
Inserting these expressions for g_i, g_{i+1} we get

$$\frac{g_{i-1}+g_{i+1}}{g_i} = \frac{g_{i-1} + \frac{1+f_{i+1}(\frac{1+f_i g_{i-1}}{f_{i-1}})}{f_i}}{\frac{1+f_i g_{i-1}}{f_{i-1}}}$$

We multiply the numerator and the denominator by $\frac{f_{i-1}}{1+f_ig_{i-1}}$ to get

$$(g_{i-1}+\frac{1+f_{i+1}(\frac{1+f_ig_{i-1}}{f_{i-1}})}{f_i})(\frac{f_{i-1}}{1+f_ig_{i-1}}) = \frac{f_{i-1}g_{i-1}}{1+f_ig_{i-1}} + \frac{f_{i-1}}{f_i(1+f_ig_{i-1})} + \frac{f_{i+1}}{f_i}$$

We multiply the first term by $\frac{f_i}{f_i}$ and the third term by $\frac{1+f_ig_{i-1}}{1+f_ig_{i-1}}$ to have a common denominator. We get

$$\frac{f_{i-1}f_ig_{i-1}+f_{i-1}+f_{i+1}(1+f_ig_{i-1})}{f_i(1+f_ig_{i-1})} = \frac{(f_{i-1}+f_{i+1})(1+f_ig_{i-1})}{f_i(1+f_ig_{i-1})} = \frac{f_{i-1}+f_{i+1}}{f_i} \in \mathbb{Z}^+$$

We now want to show g_i is a positive integer.
By definition $g_1 = 1 \in \mathbb{Z}^+$. By equation (2) $g_2 = \frac{1+f_2g_1}{f_1} = \frac{1+f_2\cdot 1}{f_1}$ which is in $\mathbb{Z}^+$ since $f_0 = 1$ and $\frac{f_0+f_2}{f_1} \in \mathbb{Z}^+$.
Furthermore, $g_i = \frac{1+f_ig_{i-1}}{f_{i-1}}$ gives us that if g_{i-1} is positive, so is g_i. It remains only to show g_i is an integer. We have shown g_1, g_2 are both integers. Assume $g_1, ..., g_i$ are integers. We have $\frac{g_{i-1}+g_{i+1}}{g_i} = \frac{f_{i-1}+f_{i+1}}{f_i} = r$ for some integer r. Then

$$g_{i+1} = r\cdot g_i - g_{i-1}$$

which is an integer. We have then shown all g_i are positive integers.

To sum up, if we start with a sequence beginning and ending with ones, with the property $f_s|(f_{s-1}+f_{s+1})$ we can create a valid frieze from that sequence. The sequence $g_1, g_2, ..., g_{n-1}$ has the same properties as $f_0, f_1, ..., f_{n-2}$, which means it also generates the next diagonal $\{h_i\}$, and so on.

□

Remark 3.16. *Corollary 3.12 is the converse of Theorem 3.15. The requirement that all elements in a diagonal divide its diagonal neighbours is therefore a necessary and sufficient requirement to generate a frieze.*

Having a sufficient requirement to create a frieze by Theorem 3.15, we can now introduce methods of creating friezes of greater or lesser rank when given an arbitrary frieze.

Construction 3.1. *Given a frieze of order n we create a frieze of order $n+1$ in the following manner*

The frieze is determined by a diagonal $f_0, f_1, ..., T, U, V, W, ..., f_{n-2}$ such that each element in the sequence divides its neighbours. Expand the sequence $f_0, f_1, ..., T, U, V, W, ..., f_{n-2}$ to $f_0, f_1, ..., T, U, U+V, V, W, ..., f_{n-2}$ and let this sequence be a diagonal in the new frieze. Note that the element f_{n-2} is now the n-th element rather than the $n-1$-st

Proposition 3.17. *Construction 3.1 gives a valid frieze of order $n+1$ when applied to a frieze of order n*

Proof. That the order increases by one follows from the altered sequence being one term longer. The validity of the frieze is seen by direct computation. By Theorem 3.10 we have $f_s|(f_{s-1}+f_{s+1})$ for the diagonal in our frieze. This property is kept intact for all elements before and after the altered elements $...T, U, U+V, V, W, ...$ in the sequence. We need only prove that the property still holds here as well, and by Theorem 3.15 the new sequence then determines a valid frieze. Since T, U, V, W appears in the original diagonal we know that

$$U|(V+T),\ V|(W+U)$$

and that T and W divide their neighbours, which remain unaltered by the construction. In the new sequence, we need to show

$$U|((U+V)+T),\ (U+V)|(U+V),\ V|((U+V)+W)$$

First of all, $(U+V)|(U+V)$ is trivial.

$$\frac{U+V+T}{U} = \frac{V+T}{U} + \frac{U}{U} \text{ and } U|(V+T)$$
$$\frac{U+V+W}{V} = \frac{U+W}{V} + \frac{V}{V} \text{ and } V|(W+U)$$

We have shown that the sequence $f_0, f_1, ..., T, U, U+V, V, W, ..., f_{n-2}$ of n elements, retains the property that any element divides the sum of its neighbours, other than the elements at either end. □

Although Construction 3.1 focuses on diagonals, the expansion is in fact equal to inserting a one anywhere in the second row and incrementing the adjacent elements by one. We see this by letting $a_0, ..., t, u, v, w, ..., a_{n-1}$ be the second row of a frieze with diagonal $f_0, ..., T, U, V, W, ..., f_{n-2}$ such that $\frac{T+V}{U} = u$, $\frac{W+U}{V} = v$. When we apply Construction 3.1 we get the diagonal $f_0, ..., T, U, U+V, V, W, ..., f_{n-2}$ which generates the second row $a_0, ..., t, x, y, z, w, ..., a_{n-1}$ where

$$x = \frac{T+(U+V)}{U} = u+1, y = \frac{U+V}{(U+V)} = 1, z = \frac{(U+V)+W}{V} = v+1$$

So the second row of the new frieze is $a_0, ..., t, u+1, 1, v+1, w, ..., a_{n-1}$

Example 3.18. *Construction 3.1 in practice.*
Label the green elements in the frieze below as $a_0, a_1, a_2, a_3 ...$ with the corresponding diagonal f_0, f_1, f_2, f_3 starting in the row of ones and passing through the first green element.

$$\begin{array}{cccccccccccccc}
\dots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \dots & \\
& \dots & 1 & & \color{green}{2} & & \color{green}{2} & & \color{green}{1} & & \color{green}{3} & & 1 & \dots \\
\dots & 2 & & 1 & & \color{red}{3} & & 1 & & 2 & & 2 & \dots & \\
& \dots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \dots
\end{array}$$

by considering the 4 green elements as ...t, u, v, w... in Construction 3.1, let's expand this frieze of order 5 into one of order 6. The new frieze pattern can then be found by using the unimodular rule from the first two rows, but notice how the red number in the diagonal is equal to the sum of its diagonal neighbours. Additionally, the five green numbers below are the expansion of the green numbers in the frieze above.

$$\begin{array}{cccccccccccccccc}
\dots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \dots & \\
& \dots & 1 & & \color{green}{2} & & \color{green}{3} & & \color{green}{1} & & \color{green}{2} & & \color{green}{3} & & 1 & \dots \\
\dots & 2 & & 1 & & \color{red}{5} & & 2 & & 1 & & 5 & & 2 & \dots & \\
& \dots & 1 & & 2 & & 3 & & 1 & & 2 & & 3 & \dots & & \\
\dots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \dots & & &
\end{array}$$

We will now show a method for creating friezes of lower order than a given frieze. As with Construction 3.1, this construction has a choice of index as well. There exists choices for both constructions where the maps are inverses of each other.

Construction 3.2. *For $n \geq 3$, given a frieze of order $n+1$ we create a frieze of order n in the following manner:*
By Corollary 3.13 the second row has at least a one. Strictly speaking we know it has infinitely many ones, but the second row has a one within any set of $n+1$ consecutive elements. Additionally the two elements adjacent to the one must be greater than one. We name the elements $a_0, ..., t, u+1, 1, v+1, w..., a_n$ where $u, v \geq 1$. We construct the new frieze of order n by letting removing the one, and decrementing $u+1, v+1$ by one, so the second row is $a_0, ..., t, u, v, w..., a_{n-1}$, repeating. This is then expanded by the unimodular rule to find a diagonal and the rest of the frieze.

Proposition 3.19. *Construction 3.2 gives a valid frieze of order n when applied to a frieze of order $n+1$.*

Proof. As in the construction let $a_0, ..., t, u+1, 1, v+1, w..., a_n$ where $u, v \geq 1$ be $n+1$ consecutive elements in the second row of a frieze of order $n+1$. By Theorem 3.10 we have the relation $a_s = \dfrac{f_{s-1} + f_{s+1}}{f_s}$ for $s \leq n-1$, where $f_0 = 1, f_1 = a_0, ..., f_{n-1} = 1$ is the diagonal passing through a_0. Specifically this means that some consecutive elements in the diagonal $f_0 = 1, f_1 = a_0, f_2, ..., T, U, \alpha, V, W, ..., f_{n-1} = 1$ have the property

$$\frac{T+\alpha}{U} = u+1, \frac{U+V}{\alpha} = 1, \frac{\alpha+W}{V} = v+1$$

However, $\dfrac{U+V}{\alpha} = 1$ implies $\alpha = U+V$, so the diagonal is $f_0, f_1, f_2, ..., T, U, U+V, V, W, ..., f_{n-1}$. We wish to show that removing the one in the second row corresponds to a change in a diagonal that still gives a valid frieze, of one order less.
We construct a diagonal for the frieze of order n by removing the element $U+V$. The diagonal is then $f_0 = 1, f_1 = a_0, f_2, ..., T, U, V, W, ..., f_{n-1} = 1$, where f_{n-1} is the $n-1$-st element, as the sequence was shortened by one element. We show that this diagonal fulfils the requirements of determining a valid frieze as described in Theorem 3.15. $U|(T+(U+V))$ implies $U|(T+V)$ and $V|((U+V)+W)$ implies $V|(U+W)$. The elements in $f_0, f_1, ..., T \cup W, ..., f_{n-1}$ retain their values and thus their divisibility. The change to the second row in Construction 3.2 creates a diagonal of length $n-1$ which creates a valid frieze of order n. □

Construction 3.1 is applied to a diagonal while Construction 3.2 focuses on the second row. Construction 3.2 could strictly speaking also be applied to a diagonal but it is far harder to find elements in the diagonal such that the element is equal to the sum of its neighbours. Instead we find a one in the second row. Similarly, expanding a frieze could also be done by focusing on the second row.

Remark 3.20. *Construction 3.1 could be rewritten in the following manner:*

Let $\mathcal{F}_n$ be a frieze of order n with a diagonal $f_0, a_0, ..., T, U, V, W, ..., f_{n-2}$ for some values T, U, V, W that correspond to the elements $...t, u, v, w, ...$ in the second row of $\mathcal{F}_n$ such that $\dfrac{T+V}{U} = u, \dfrac{U+W}{V} = v$. We create a frieze F_{n+1} of order $n+1$ by letting the second row be $a_0, ..., t, u+1, 1, v+1, w, ..., a_{n-1}$ repeating. The sequence has $n+1$ elements, and corresponds exactly with inserting $U+V$ between elements U, V in the diagonal. The calculations to show this is similar to those in the proof of Proposition 3.19.

In Theorem 2.8 we showed that any triangulated polygon can be created by applying a construction to a polygon with a greater or smaller number of vertices. As our intentions are to relate triangulated polygons to friezes, it is in our best interest to prove a similar result for friezes.

Theorem 3.21. *For $n \geq 3$:*
i) Any frieze of order n can be created by applying Construction 3.2 to some frieze of order $n+1$.
ii) Any frieze of order $n+1$ can be created by applying Construction 3.1 to some frieze of order n.

Proof. i): Begin with any frieze $\mathcal{F}_n$ of order n. We apply Construction 3.1 to the frieze, expanding the diagonal $f_0, ..., T, U, V, W, ..., f_{n-2}$ to $f_0, ..., T, U, U+V, V, W, ..., f_{n-2}$ which has n elements, and generates a valid frieze, $\mathcal{F}_{n+1}$. $\mathcal{F}_{n+1}$ now has a 1 in the second row by Theorem 3.10, inserting U,U+V,V into the formula as $1 = \dfrac{U+V}{(U+V)}$. Apply Construction 3.2 to $\mathcal{F}_{n+1}$, removing the specific one corresponding to the diagonal elements $..., U, U+V, V, ...$ The diagonal is then reduced back to $f_0, ..., T, U, V, W, ..., f_{n-2}$, generating the frieze we started with, $\mathcal{F}_n$. Therefore, for any frieze $\mathcal{F}_n$ of order n, there exists a frieze of order $n+1$ such that Construction 3.2 generates $\mathcal{F}_n$. The proof for *ii*) is analogous. □

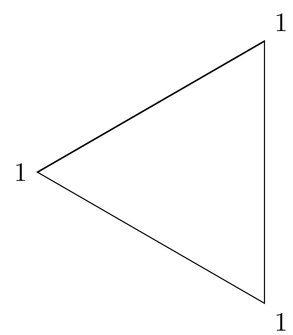
4. Maps between triangulated polygons and friezes

In this section we will present maps from triangulated polygons of n vertices to frieze patterns of order n and back. We will then show that the maps are inverse bijections. We begin with a map from any triangulated polygon to a frieze pattern.

Construction 4.1. *For a triangulated polygon with $n \geq 3$ vertices, let the quiddity cycle as in Definition 2.3 be n consecutive elements repeating in the second row of the frieze-to-be. The rest of the frieze is determined by the unimodular rule. Let ω_n denote Construction 4.1 for a polygon with n vertices.*

Proposition 4.1. *Construction 4.1 gives a valid frieze of order $n \geq 3$ when applied to a triangulated polygon with n vertices.*

Proof. We show the statement inductively. For $n = 3$ there is only one option, a triangle with quiddity cycle $1, 1, 1$. Construction 4.1 creates the frieze below with two rows, the smallest worthwhile frieze.



... 1 1 1 1 1 1 1 ...
... 1 1 1 1 1 1 1 ...

Assume ω_n gives a valid freeze for $3 \leq n \leq k$. Let $\mathfrak{T}_P$ be a triangulated polygon with $k+1$ vertices. $\mathfrak{T}_P$ has a quiddity cycle $a_0, ..., a_{i-1}, a_i, a_{i+1}, ..., a_k$. By Proposition 2.7 we know $a_i = 1$ for some i as $\mathfrak{T}_P$ has at least 2 special vertices. We apply Construction 2.2 to $\mathfrak{T}_P$, removing the special vertex corresponding to a_i to create $\mathfrak{T}_P/a_i$, a triangulated polygon with k vertices. By Remark 2.11 the quiddity cycle of $\mathfrak{T}_P/a_i$ is $a_0, ..., a_{i-1}-1, a_{i+1}-1, ..., a_k$ of length k. By our assumption ω_k applied to $\mathfrak{T}_P/a_i$ creates a valid frieze of order k with $a_0, ..., a_{i-1}-1, a_{i+1}-1, ..., a_k$ repeating as its second row.

By Theorem 3.10 we know $a_0, ..., a_{i-1}-1, a_{i+1}-1, ..., a_k$ have the relation to a diagonal $f_0, ..., T, U, V, W, ..., f_{k-1}$ such that $\frac{T+V}{U} = a_{i-1}-1, \frac{W+U}{V} = a_{i+1}-1$. We apply Construction 3.1 to the frieze, inserting $U+V$ into the diagonal making it $..., T, U, U+V, V, W, ...$ which we know by Proposition 3.17 gives a valid frieze. Moreover we know that such a change to the diagonal corresponds to altering the second row from $a_0, ..., a_{i-2}, a_{i-1}-1, a_{i+1}-1, a_{i+2}, ..., a_k$ of length k to $a_0, ..., a_{i-2}, a_{i-1}, 1, a_{i+1}, a_{i+2}, ..., a_k$ of length $k+1$. We have now created a valid frieze of order $k+1$ equal to $\omega_{k+1}(\mathfrak{T}_P)$. The commutative diagram below represents the way we built this proof where χ, γ represent Construction 2.2 and Construction 3.1 respectively, and $\mathcal{F}_k, \mathcal{F}_{k+1}$ represent the friezes of order k and $k+1$.

$$\begin{array}{ccc} \mathfrak{T}_P & \xrightarrow{\omega_{k+1}} & \mathcal{F}_{k+1} \\ \chi \downarrow & & \uparrow \gamma \\ \mathfrak{T}_P/a_i & \xrightarrow[\omega_k]{} & \mathcal{F}_k \end{array}$$

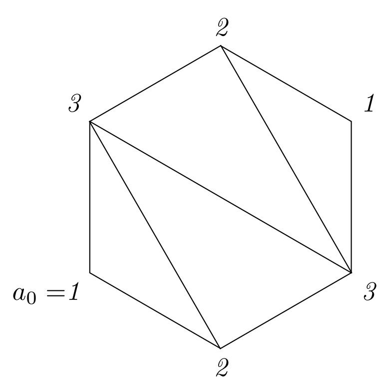
Keep in mind that Construction 3.1 and 2.2 leave room for a choice regarding which special vertices are removed, and where we insert extra elements. To make the maps χ, γ a bit more precise we may define a specific choice for the maps so that the diagram commutes. We know that triangulated polygons can be rotated without changing the quiddity cycle. Similarly we can change the index of the second row in a frieze without altering the frieze. Apply a cyclic shift to the quiddity cycle of $\mathfrak{T}_P$ such that a_1 is a special vertex, and let χ always remove a_1, the second element. The quiddity cycle of $\mathfrak{T}_P/a_i$ is then $a_0 - 1, a_2 - 1, a_3, ..., a_k$. ω_k sets the quiddity cycle of $\mathfrak{T}_P/a_i$ as the second row of $\mathcal{F}_k$. Let us temporarily let $a_0 - 1$ be defined as the first element of $\mathcal{F}_k$. We define γ to insert the 1 between the first two elements in the second row. The maps $\chi, \gamma \circ \omega_k$ are then well defined. Now $\omega_{k+1} = \gamma \circ \omega_k \circ \chi$.

□

Remark 4.2. *The quiddity cycle of a triangulated polygon $\mathfrak{T}_P$ with n vertices has period k such that $k|n$, and contains at least two ones. The period is straight forward to show since the quiddity cycle has n elements. It must contain at least two ones because $\mathfrak{T}_P$ has at least two special vertices by Proposition 2.7.*

Below are two short examples showing how we apply Construction 4.1. The two examples are both hexagons and yet yield different friezes.

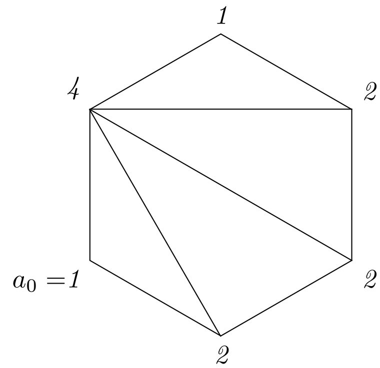
Example 4.3. *Construction 4.1 applied to a triangulated hexagon with quiddity cycle 1, 2, 3, 1, 2, 3.*



This polygon gives us the second row $...1, 2, 3, 1, 2, 3...$ *repeating. In full frieze form this expands by the unimodular rule to the order* 6 *frieze below:*

$$\begin{array}{ccccccccccccccccc}
\dots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \dots & & \\
& \dots & 1 & & 2 & & 3 & & 1 & & 2 & & 3 & & 1 & \dots \\
\dots & 2 & & 1 & & 5 & & 2 & & 1 & & 5 & & 2 & \dots & & \\
& \dots & 1 & & 2 & & 3 & & 1 & & 2 & & 3 & & 2 & \dots \\
\dots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \dots & &
\end{array}$$

Example 4.4. *Construction 4.1 applied to a triangulated hexagon with quiddity cycle 1,2,2,2,1,4.*



The resulting frieze pattern differs from that of Example 4.3, and notice in particular that although the orders are the same, the periods of the frieze patterns differ in the two examples.

... *1* *1* *1* *1* *1* *1* *1* ...

... *1* *2* *2* *2* *1* *4* *1* ...

... *3* *1* *3* *3* *1* *3* *3* ...

... *2* *1* *4* *1* *2* *2* *2* ...

... *1* *1* *1* *1* *1* *1* *1* ...

Which node you chose as a_0 will shift the pattern to the side, but not otherwise alter the frieze. By Definition 3.4 we know that a different choice of a_0 results in the same frieze. Next let us explain a method to go from any frieze pattern of order n to a triangulated polygon with n vertices.

Construction 4.2. *Take any frieze of order $n \geq 3$. Let n consecutive elements in the second row of the frieze be the quiddity cycle of a polygon P with n vertices. We work out the triangulation in accordance with the quiddity cycle, starting with the special vertices. When removing a special vertex x we repeat the process by finding a special vertex in P/x and so on. The whole process is reduced to only separating special vertices one at a time. This is illustrated in an example below. Let β_n denote Construction 4.2 for a frieze of order n.*

Proposition 4.5. *Let $\mathcal{F}_n$ be a frieze of order $n \geq 3$. Construction 4.2 gives a valid triangulation of a polygon with n vertices when applied to $\mathcal{F}_n$.*

Proof. We show the claim similarly to the proof of Proposition 4.1, by induction. As before the case $n = 3$ holds here as well. The second row in the order 3 frieze is all ones, so we label all vertices in a triangle with ones, satisfying the definition of a triangulation.

Assume β_i gives a valid triangulated polygon for $3 \leq i \leq k$. We wish to show β_{k+1} gives a valid triangulated polygon with $k+1$ vertices when applied to a frieze $\mathcal{F}_{k+1}$ of order $k+1$. We apply Construction 3.2 to $\mathcal{F}_{k+1}$ to obtain a valid frieze $\mathcal{F}_k$ of order k. In the process we remove a 1 from the second row of the frieze and decrease the neighbours by one. Name the elements in the second row of $\mathcal{F}_k$ that were decremented by the construction $u-1, v-1$. By our assumption the new frieze of order k corresponds do a valid triangulation $\mathfrak{T}_Q$ of a polygon Q with k vertices, such that $\mathfrak{T}_Q = \beta_k(\mathcal{F}_k)$. $\mathfrak{T}_Q$ has quiddity cycle $a_0, ..., u-1, v-1, ..., a_{k-1}$. We apply Construction 2.1 to $\mathfrak{T}_Q$, adding a vertex between the vertices corresponding to $u-1, v-1$

in the quiddity cycle to obtain a new polygon P with triangulation $\mathfrak{T}_P$. $\mathfrak{T}_P$ has quiddity cycle $a_0, ..., u, 1, v, ..., a_{k-1}$ of length $k+1$. Note that $Q = P/x$ where x is the special vertex inserted by Construction 2.1. $\mathfrak{T}_P$ is a valid triangulated polygon, and is equal to $\beta_{k+1}(\mathcal{F}_{k+1})$. Below is a small diagram showing the steps, where χ, γ correspond to Constructions 3.2 and 2.1 respectively.

$$\begin{array}{ccc} \mathcal{F}_{k+1} & \xrightarrow{\beta_{k+1}} & \mathfrak{T}_P \\ \chi \downarrow & & \uparrow \gamma \\ \mathcal{F}_k & \xrightarrow[\beta_k]{} & \mathfrak{T}_Q \end{array}$$

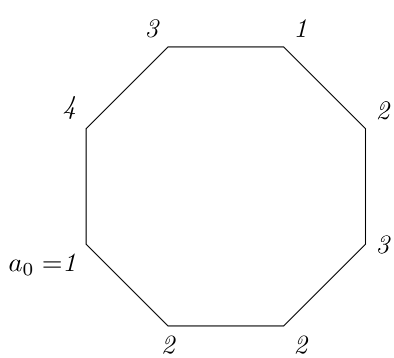
Note that for this diagram to commute the maps χ, γ require specific choices, but it is always possible. For example let χ always remove a_1 in $\mathcal{F}_{k+1}$. Then the second row of $\mathcal{F}_k$ is $a_0 - 1, a_2 - 1, a_3, ..., a_k$, which β_k sets as the quiddity cycle of $\mathfrak{T}_Q$. Let γ insert a vertex between the vertices corresponding to the first two elements of the quiddity cycle. Then $\gamma \circ \beta_k \circ \chi = \beta_{k+1}$. $\square$

Below follows a small yet lengthy example showing how Construction 4.2 is applied in practice.

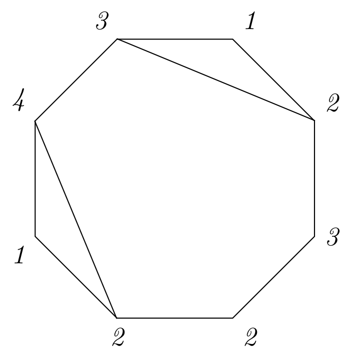
Example 4.6. *Construction 4.2 applied to a frieze of order 8.*

$$\begin{array}{ccccccccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \dots & & \\
& 1 & & 2 & & 2 & & 3 & & 2 & & 1 & & 3 & & 4 & & \dots & \\
3 & & 1 & & 3 & & 5 & & 5 & & 1 & & 2 & & 11 & & & & \\
& 2 & & 1 & & 7 & & 8 & & 2 & & 1 & & 7 & & 8 & & & \\
5 & & 1 & & 2 & & 11 & & 3 & & 1 & & 3 & & 5 & & \dots & & \\
& 2 & & 1 & & 3 & & 4 & & 1 & & 2 & & 2 & & 3 & & \dots & \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \dots & &
\end{array}$$

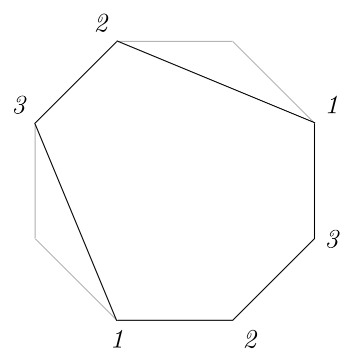
We focus on the second row, which gives us the quiddity cycle $1, 2, 2, 3, 2, 1, 3, 4$. *The frieze has 7 rows, so we hope to match this frieze to a triangulated octagon. Starting nowhere in particular on a convex octagon, number the vertexes in order, as seen below.*



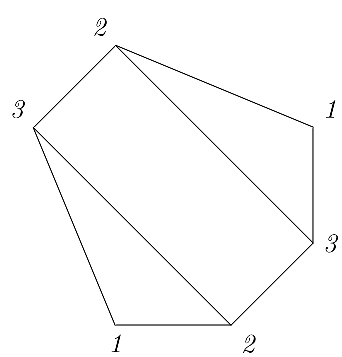
From here, we start with the two special vertices, cutting them off from the rest of the polygon. We do this by inserting a diagonal between the adjacent vertices to each special vertex.

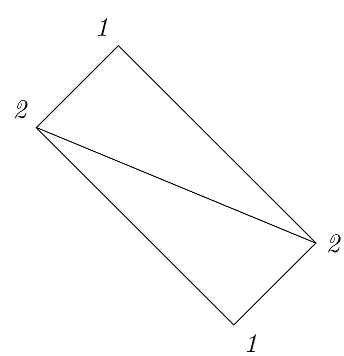


The problem of completing the triangulation can be reduced to triangulating the hexagon created by lopping off the triangles with special vertexes, and decreasing the number at each connected vertex by one. This is illustrated below.

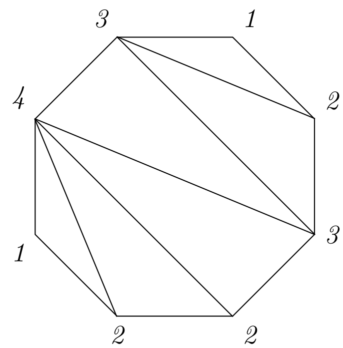


Notice how removing a pair of special vertices creates more special vertices. Recall from Proposition 2.7 that while the number of special vertices is not the same for all triangulated polygons, it is always ≥ 2*. Repeating this process another few steps yields:*





As such we can reduce the problem of creating a triangulation to exclusively creating triangles around special vertices. Now putting this triangulation back into the octagon yields the full triangulation:



Theorem 4.7. *The maps from Constructions 4.1 and 4.2 are inverse bijections between frieze patterns of order n and triangulated polygons with n vertices, for $n \geq 3$.*

Proof. We have shown that both maps are well defined. We let ω_k denote Construction 4.1 for k vertices and β_k denote Construction 4.2 for order k.

$$\mathfrak{T}_P \xrightarrow{\ \omega_k\ } \mathcal{F}_k$$

$$\mathfrak{T}_Q \xleftarrow{\ \beta_k\ } \mathcal{F}_k$$

We will show that $\beta_k \circ \omega_k = \mathfrak{T}_{id}$, where $\mathfrak{T}_{id}$ is the identity on triangulated polygons. We show this by letting $\omega_k(\mathfrak{T}_P) = \mathcal{F}_k$ and $\mathfrak{T}_Q = \beta_k(\mathcal{F}_k)$ and showing $\mathfrak{T}_P = \mathfrak{T}_Q$.
$\mathfrak{T}_P$ has quiddity cycle $a_0, a_1, ..., a_{k-1}$. ω_k constructs $\mathcal{F}_k$ by setting the repeating quiddity cycle of $\mathfrak{T}_P$ as the second row and expanding a frieze from there. The second row of $\mathcal{F}_k$ therefore contains the k consecutive elements $a_0, a_1, ..., a_{k-1}$. We construct a triangulated polygon $\mathfrak{T}_Q = \beta_k(\mathcal{F}_k)$ by first creating a polygon Q of k vertices, and setting $a_0, a_1, ..., a_{k-1}$ as the quiddity cycle before completing the triangulation. $\mathfrak{T}_Q$ and $\mathfrak{T}_P$ therefore have the same quiddity cycles which makes them equal by Definition 2.5. The proof that $\omega_k \circ \beta_k = \mathcal{F}_{id}$ is analogous, where $\mathcal{F}_{id}$ is the identity on friezes. □

Corollary 4.8. *We improve a previous result on friezes. For a frieze of order n, it is now clear that within n consecutive elements of a second row of a frieze, two or more elements must equal 1. This is because the second row corresponds to the quiddity cycle of a triangulated polygon. By Proposition 2.7, the corresponding polygon must have at least 2 special vertices.*

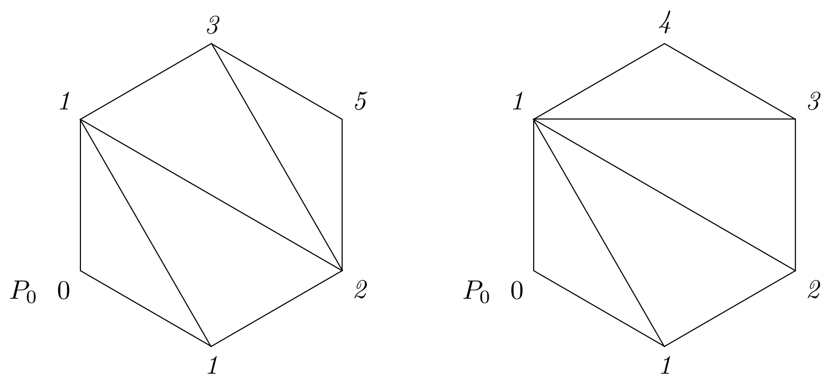
One thing we glossed over before, is that Theorem 3.10 only applies to $a_i, i \leq n-2$. While we know $a_n = a_0$, what then of a_{n-1}? We could use the same theorem and express the element by the next diagonal instead, but we now have another way of doing it. We have shown that the second row of a frieze is the same as the quiddity cycle of a triangulated polygon. Since a triangulated polygon of n vertices has $n-2$ triangles, it is clear that the sum of the quiddity cycle is $3(n-2)$, as each triangle contains 3 vertices. Expressed differently, $\sum_{i=0}^{n-1} a_i = 3(n-2)$. Our $n-1$-st term becomes $a_{n-1} = 3(n-2) - \sum_{i=0}^{n-2} a_i$.

5. Diagonals of friezes

The diagonals of friezes determine validity of frieze patterns. Diagonals also determine the second row of a frieze which we have linked to triangulated polygons. It is, however, interesting to explore what the diagonals themselves express. For a better understanding of the diagonals of friezes we introduce some new notation. This section is inspired by *The geometry of frieze patterns* by Broline, Crowe and Isaacs ([1]). In their paper they introduce the notation we will use. We prove that it relates to the preceding sections. In particular Theorem 5.5 is not previously shown.

Definition 5.1. *In a triangulated polygon $\mathfrak{T}_P$ with vertices $\{P_0, P_1, ..., P_{n-1}\}$, give vertex P_r the value 0. Next, label all vertices connected to P_r with 1, including P_{r-1} and P_{r+1}. Next, for any triangle where two vertices have been assigned a value but not the third, label the third vertex as the sum of the other two. Continue this way until all vertices have a value. We let* $(\mathbf{P_r}, \mathbf{P_s})$ *denote the value of vertex P_s when P_r is the initial vertex. We sometimes write* $\mathfrak{T}_\mathbf{P}(\mathbf{P_r}, \mathbf{P_s})$ *instead of (P_r, P_s) to specify which polygon the vertices are a part of.*

Example 5.2. *Our two favourite hexagons with (P_0, P_s) in each vertex P_s for $s \in 0, 1, ..., 5$.*

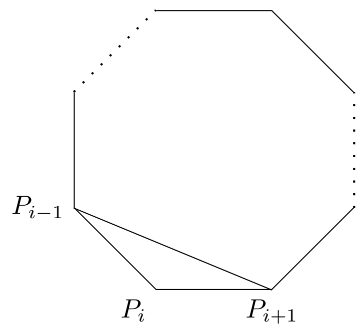


Proposition 5.3. *Let $\mathfrak{T}_P$ be a triangulated polygon, with P_i a special vertex in $\mathfrak{T}_P$. Let $\mathfrak{T}_P/P_i$ be the triangulated polygon obtained by removing P_i and its connected edges. Let P_r, P_s be vertices in $\mathfrak{T}_P/P_i$. Then*

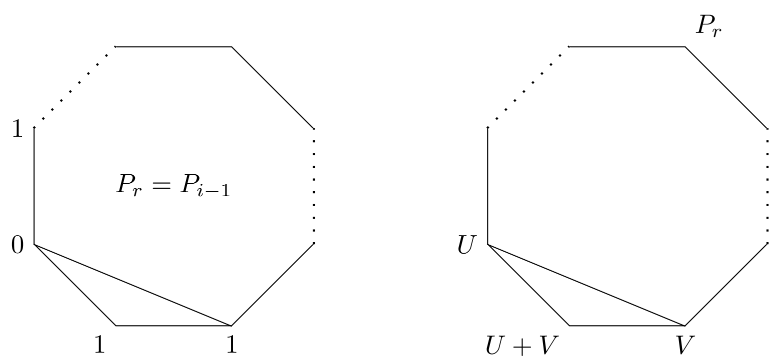
$$(P_r, P_s) = (P_r, P_s)'$$

where $(P_r, P_s)'$ denotes the value $(P_r, P_s) \in \mathfrak{T}_P/P_i$

Proof. We want to show that a special vertex P_i can never contribute to any value (P_r, P_k) as it is only a part of one triangle. P_i will get a value in one of three ways:
i) The two vertices adjacent to P_i already have a value.
ii) P_i is adjacent to the starting point, P_r.
iii) P_i is the starting point.



If $P_i = P_r$ as in *iii*) then the conditions of the proposition are not met. The other two cases are illustrated below.
Case *ii*): If P_i is adjacent to P_r we consider the left figure below. Since P_i is special there exists an edge (P_{i-1}, P_{i+1}). Now, P_{i-1} or P_{i+1} is the initial vertex with value 0, and the other has value 1 since they are connected. So although P_i is given a value, it is still cut off from the rest of the polygon. The remaining values in the triangulated polygon are then calculated using the values in verticesP_{i-1} and P_{i+1}.



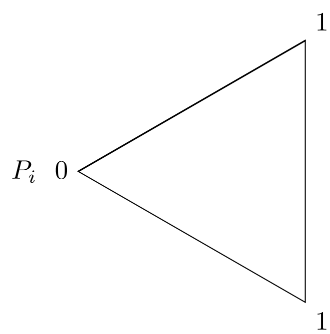
It remains to consider case *i*). Since we have shown the claim for case *iii*) we assume neither of the vertices adjacent to P_i is the starting point. This case is illustrated in the figure above to the right. Since P_i is not adjacent to P_r and its only edges are to P_{i-1} and P_{i+1} the value (P_r, P_i) will not be set until (P_r, P_{i-1}) and (P_r, P_{i+1}) are both determined. Let $(P_r, P_{i-1}) = U, (P_r, P_{i+1}) = V$. Then $(P_r, P_i) = U + V$. The special vertex does not affect the rest of the triangulated polygon, as it ever is determined by P_{i-1} and P_{i+1}. □

Remark 5.4. *As all triangulated polygons can be expanded and shortened one special vertex at a time, Proposition 5.3 can be generalized to removing any number of vertices. Let* $\mathfrak{T}_P$ *be a triangulated polygon with* P_i *special. Furthermore let* P_j *be special in* $\mathfrak{T}_P/P_i$*. Let* P_r *and* P_s *be vertices in* $\mathfrak{T}_P/P_i, P_j$*. Then* $(P_r, P_s) = (P_r, P_s)''$ *where* $(P_r, P_s)'' = (P_r, P_s) \in \mathfrak{T}_P/P_i, P_j$*. The argument is the same as for removing just one vertex. This argument is analogous for removing any number of vertices.*

In the proof above, we see the relation of consecutive elements $U, U+V, V$. We have previously seen this pattern in diagonals of friezes. This is no mere coincidence as the next theorem will show.

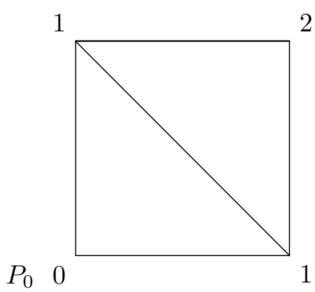
Theorem 5.5. *Let (r, s) be as in Definition 3.8. Let (r, s) describe the elements of a frieze pattern $\mathcal{F}_n$ of order $n \geq 3$ and let $\mathfrak{T}_P$ be the corresponding triangulated polygon. Then $\mathfrak{T}_P(P_r, P_s) = (r, s)$ for $r < s$ and $(P_r, P_s) = (s, r)$ for $s < r$.*

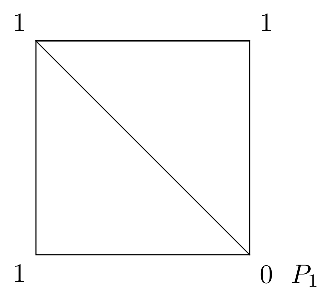
Proof. $\mathfrak{T}_P$ has vertices $P_0, P_1, ..., P_{n-1}$ we wish to show that any sequence $(P_i, P_j),\ j = i + 1, i + 2, ..., i + n - 1$ matches a diagonal $(i, j),\ j = i + 1, ..., i + n - 1$ in $\mathcal{F}_n$. We show this by induction. For $n = 3$ we show that any diagonal is equal to two ones, which matches the sequences $(P_i, P_i + 1), (P_i, P_i + 2)$ for all three choices of P_i.



... 1 1 1 1 1 1 1 ...
... 1 1 1 1 1 1 1 ...

For $n = 4$ we obtain a more interesting example, as shown by the figures below below.





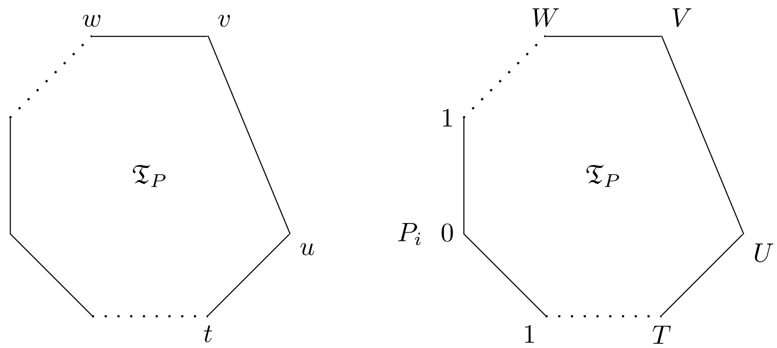
These are the only two truly different choices. Both quadrangles correspond to the frieze below, which is the only frieze pattern with 3 rows.

... 1 1 1 1 1 1 1 ...
... 2 1 2 1 2 1 2 ...
... 1 1 1 1 1 1 1 ...

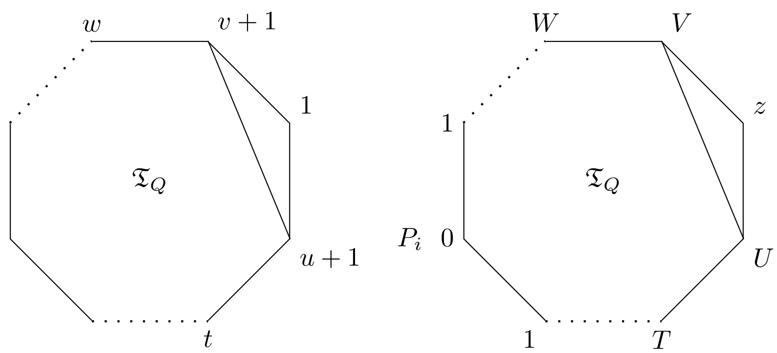
Notice the two different diagonals in the frieze are 1,2,1 and 1,1,1.

Assume that for all orders $\leq k$ we have $(i, j) = \mathfrak{T}_P(P_i, P_j),\ j = i + 1, ..., i + k - 1$ for a frieze $\mathcal{F}_k$ of order k with a corresponding triangulated polygon $\mathfrak{T}_P$.
This sequence is a diagonal in $\mathcal{F}_k$. The elements of this diagonal are $(i, i+1), ..., T, U, V, W, ..., i + k - 1$. The values T, U, V, W correspond to the elements $...t, u, v, w, ...$ in the second row of $\mathcal{F}_k$

such that $\dfrac{T+V}{U} = u, \dfrac{U+W}{V} = v$. The figures below illustrate the quiddity cycle and the values (P_i, P_j) for $\mathfrak{T}_P$.



We create a frieze of order $k+1$ by applying Construction 3.1 to $\mathcal{F}_k$. We insert $U+V$ between the values U, V in the diagonal of $\mathcal{F}_k$ and let the diagonal $f_0, ..., T, U, U+V, V, W, ..., f_{k-1}$ determine the new frieze $\mathcal{F}_{k+1}$. We know that the second row of $\mathcal{F}_{k+1}$ is $...t, u+1, 1, v+1, w, ...$ We can determine which triangulated polygon $\mathfrak{T}_Q$ this corresponds to by Construction 4.2.



The figure above to the right illustrates the sequence (P_i, P_j) for all vertices $P_i \neq P_j$ as before. We name the special vertex x such that $\mathfrak{T}_Q/x = \mathfrak{T}_P$. We let all vertices in both $\mathfrak{T}_P, \mathfrak{T}_Q$ retain their notations since we know by Remark 5.4 that $\mathfrak{T}_Q(P_i, P_j) = \mathfrak{T}_P(P_i, P_j)$ for $P_i, P_j \neq x$. We calculate $(P_i, x) = U + V$. The sequence $\mathfrak{T}_Q(P_i, y)$ for all vertices $y \in \mathfrak{T}_Q$ such that $y \neq P_i$ counter-clockwise then reads $(P_i, P_{i+1}), ..., U, U+V, V, ..., 1$. This is the diagonal used to determine the frieze that corresponds to $\mathfrak{T}_Q$. □

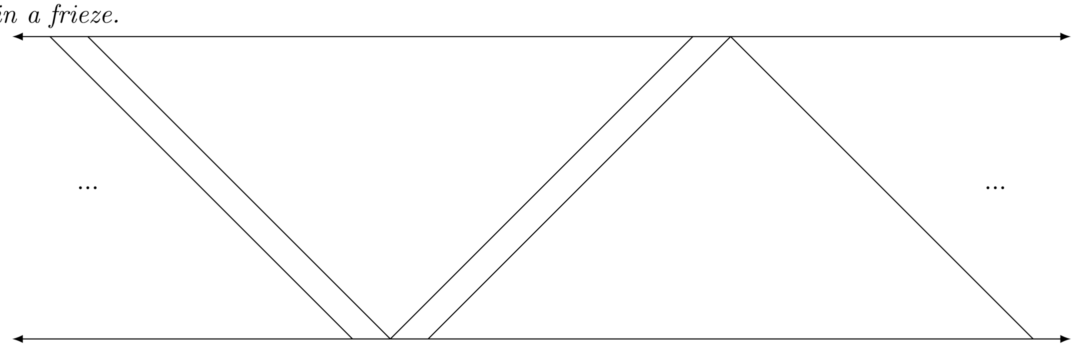
Remark 5.6. *This result gives us some understanding of the diagonals of the frieze. Any diagonal starting at P_r describes how well connected that vertex is. In particular, we have an edge between P_r and P_s wherever $(r, s) = 1$. The top and bottom row describe all non-diagonal edges.*

Our new understanding of the elements of a frieze make it easier to draw the triangulated polygon corresponding to a given frieze, as the diagonal edges need not be drawn by special vertices as shown before. We only have to find the ones in the diagonals of the frieze and draw the edges. In Definition 5.7 we describe the region where all elements of the form (r, s) are such that $r < s \leq n - 1$.

Definition 5.7. *The general form of a **fundamental region***

(0,1)		(1,2)		(2,3)		(3,4)	...	(n-2,n-1)
	(0,2)		(1,3)		(2,4)	...	(n-3,n-1)	
		...		...		...		
			(0,n-2)		(1,n-1)			
				(0,n-1)				

Remark 5.8. *In a frieze pattern a fundamental region occurs repeatedly alternating being turned about the middle as illustrated below. The top and bottom borders are the rows of ones in a frieze.*



Remark 5.8 follows directly from $(r,s) = (s, r+n)$ (as seen in the proof of Corollary 3.11). Consider the bottom row. In a frieze the bottom row would continue $(0, n-1), (1, n), (2, n+1)$ and so on, where $(1, n) = (0, 1)$ and $(2, n+1) = (1, 2)$.

Remark 5.9. *As a direct result of Remark 5.8, a fundamental region contains all integers that occur in the frieze.*

6. SL_2-tilings

Sections 6 through 12 follow a similar process as Holm and Jørgensen in *$SL_2 -$ tilings and triangulations of the strip* ([4]) in terms of results and notation. The frieze patterns we have explored so far are infinite only in one dimension. In this section we will explore different, yet similar patterns which have infinitely many rows and columns. We intend to show a bijection between two new sets, both of which contain either triangulated polygons or frieze patterns.

Definition 6.1. *An **SL_2-tiling** is a matrix with infinitely many rows and columns, in which every 2×2-submatrix has determinant 1.*

Example 6.2. *An SL_2-tiling.*

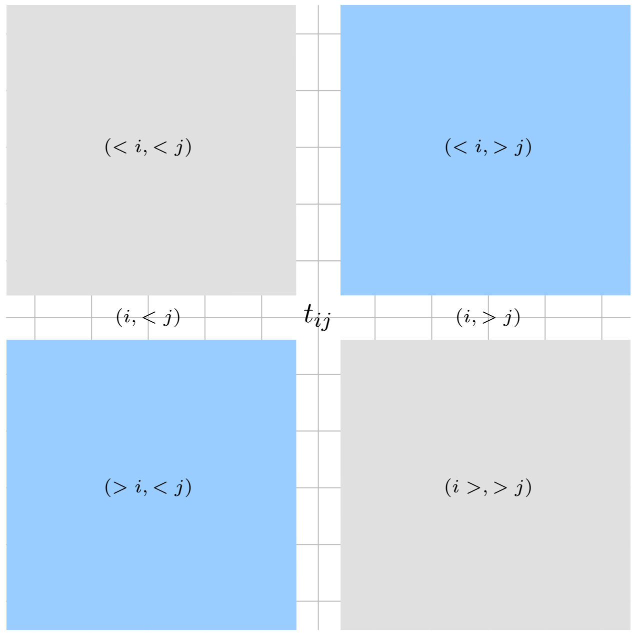
$$
\begin{array}{ccccccccccccc}
 & & & & & & \vdots & & & & & & \\
 & 61 & 50 & 39 & 28 & 17 & 6 & 7 & 8 & 9 & 10 & 11 & \\
 & 50 & 41 & 32 & 23 & 14 & 5 & 6 & 7 & 8 & 9 & 10 & \\
 & 39 & 32 & 25 & 18 & 11 & 4 & 5 & 6 & 7 & 8 & 9 & \\
 & 28 & 23 & 18 & 13 & 8 & 3 & 4 & 5 & 6 & 7 & 8 & \\
 & 17 & 14 & 11 & 8 & 5 & 2 & 3 & 4 & 5 & 6 & 7 & \\
\ldots & 6 & 5 & 4 & 3 & 2 & 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\
 & 7 & 6 & 5 & 4 & 3 & 2 & 5 & 8 & 11 & 14 & 17 & \\
 & 8 & 7 & 6 & 5 & 4 & 3 & 8 & 13 & 18 & 23 & 28 & \\
 & 9 & 8 & 7 & 6 & 5 & 4 & 11 & 18 & 25 & 32 & 39 & \\
 & 10 & 9 & 8 & 7 & 6 & 5 & 14 & 23 & 32 & 41 & 50 & \\
 & 11 & 10 & 9 & 8 & 7 & 6 & 17 & 28 & 39 & 50 & 61 & \\
 & & & & & & \vdots & & & & & &
\end{array}
$$

We number the elements (x, y) of tilings as we would in a matrix, so that x increases from top to bottom, and y increases from left to right. It is useful for us to introduce a way of describing whole quadrants of tilings.

Definition 6.3. *We will use the notation*

$$(< i, > j) = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} |\ x < i, y > j\}$$

to describe whole quadrants of SL_2-tilings. This notation applies for the other inequality signs making it possible to describe all infinite quadrants from a starting point. Additionally, we let $(< i, j) = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} |\ x < i, y = j\}$ and $(i, > j) = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} |\ x = i, y > j\}$ describe rays of rows and columns. The figure below illustrates a sample of these notations.



Definition 6.4. *An SL_2-tiling has* ***enough ones*** *if each quadrant $(> i, < j), (< i, > j)$ contains 1 for $i, j \in \mathbb{Z}$. Expressed differently, an SL_2-tiling has enough ones if the top right and bottom left quadrants contain the value 1 regardless of the starting point.*

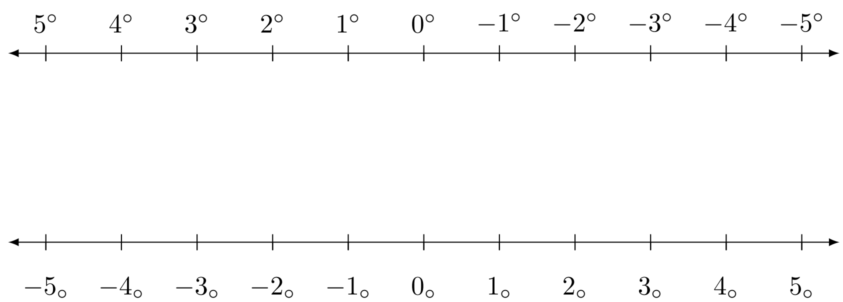
Example 6.2 does not have enough ones if we continue the middle column and middle row in the obvious way $(1, 2, 3, 4, ..., i, i+1, ...)$. We are primarily interested in SL_2-tilings with enough ones. Our main goal is to show a bijection between such tilings, and *good* triangulations of the strip.

There are similarities between tilings and friezes, but SL_2-tilings are drawn without staggering the rows. The restriction that 2×2-submatrices have determinant 1 is similar to the unimodular rule but not quite equal. If we look at friezes as diagonal bands of the SL_2-tilings the restrictions coincide. We will return to this point in Section 10.

7. Triangulations of the strip

In this section we introduce an expansion of triangulated polygons. This is similar to how SL_2-tilings are expanded friezes. Our intention is to eventually prove a bijection between these new expansions.

Definition 7.1. ***The strip*** *consists of two disjoint copies of* $\mathbb{Z}$*, denoted* $\mathbb{Z}^\circ$ *and* $\mathbb{Z}_\circ$*. Every element in either copy of* $\mathbb{Z}$ *is* ***a vertex of the strip****.* $\mathbb{Z}^\circ = \{..., -1^\circ, 0^\circ, 1^\circ, ...\}, \mathbb{Z}_\circ = \{..., -1_\circ, 0_\circ, 1_\circ, ...\}$*. A vertex* a *in the top half of the strip is represented by* a°*. A vertex* b *in the bottom half of the strip is represented by* $b_\circ$*. Notice in the figure below that the top half decreases from left to right, while the bottom half increases.*

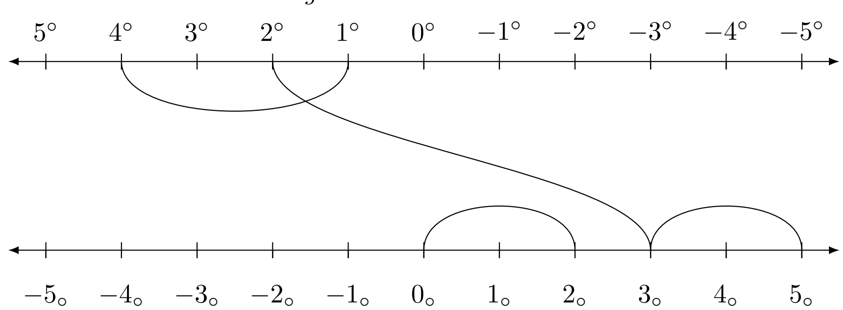


The vertices of the strip can be connected either to vertices on the same line, or by a line crossing between top and bottom. We will use the word *arc* for any connecting line between two vertices of the strip.

Definition 7.2. *An* ***arc*** *adjoins 2 vertices of the strip. An arc is either internal or connecting.* ***Internal arcs*** *are arcs* $(p_\circ, q_\circ) \in \mathbb{Z}_\circ \times \mathbb{Z}_\circ$ *or* $(p^\circ, q^\circ) \in \mathbb{Z}^\circ \times \mathbb{Z}^\circ$ *such that* $|p - q| \geq 2$*.* ***Connecting arcs*** *are arcs* $(p^\circ, q_\circ) \in \mathbb{Z}^\circ \times \mathbb{Z}_\circ$*.*

We do not allow internal arcs with $|p - q| = 1$. This is because adjacent vertices are already connected by the strip itself.

Example 7.3. *Below is a strip with 4 edges.* $(4^\circ, 1^\circ)$*,* $(0_\circ, 2_\circ)$*,* $(3_\circ, 5_\circ)$*,* $(2^\circ, 3_\circ)$*. The first three are internal and the last is connecting.*



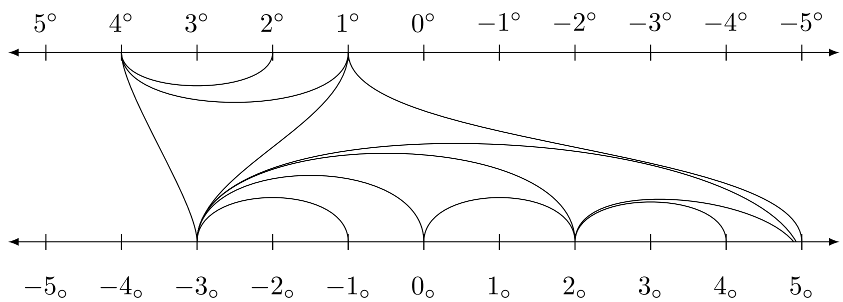
Definition 7.4. *Two arcs that intersect, but not at a vertex, are said to be* ***crossing****.*
Two internal arcs $(i^\circ, k^\circ), (j^\circ, l^\circ)$ *or* $(i_\circ, k_\circ), (j_\circ, l_\circ)$ *cross if* $i < j < k < l$ *or* $j < i < l < k$*.*
Two diagonal arcs $(i^\circ, p_\circ), (j^\circ, q_\circ)$ *cross if both* $i < j$ *and* $p < q$ *or both* $i > j$ *and* $p > q$*.*
Lastly a diagonal arc may cross an internal arc. If $i < j < k$*,* $(j^\circ, a_\circ)$ *crosses* (i°, k°) *and* $(a^\circ, j_\circ)$ *crosses* $(i_\circ, k_\circ)$ *for all* $a \in \mathbb{Z}$*.*

Note that two arcs that share a vertex do not cross one another. An example of this is the arcs $(3_\circ, 5_\circ)$, $(2^\circ, 3_\circ)$ in the example above.

Definition 7.5. *We have a way of expressing when arcs cross, so we can define when an arc is* ***between*** *two connecting arcs. For non-intersecting connecting arcs* $(i^\circ, j_\circ)$ *and* $(l^\circ, k_\circ)$ *with* $i > l, j < k$*:*
i) (p°, q°) *is between* $(i^\circ, j_\circ)$ *and* $(l^\circ, k_\circ)$ *if* $l < p, q < i$*.*
ii) $(p_\circ, q_\circ)$ *is between* $(i^\circ, j_\circ)$ *and* $(l^\circ, k_\circ)$ *if* $j < p, q < k$*.*
iii) $(p^\circ, q_\circ)$ *is between* $(i^\circ, j_\circ)$ *and* $(l^\circ, k_\circ)$ *if* $l < p < i$*,*$j < q < k$*.*

Definition 7.6. *A* ***triangulation of the strip*** *is a maximal set* $\mathfrak{T}$ *of non-crossing arcs in the strip.*

Example 7.7. *A triangulation of a small subset of the strip. The subset contains 9 vertices in* $\mathbb{Z}_\circ$ *and 4 vertices in* $\mathbb{Z}^\circ$*.*



Definition 7.8. *A triangulation* $\mathfrak{T}$ *of the strip is called* ***good*** *if for each connecting arc* $(p^\circ, q_\circ) \in \mathfrak{T}$ *there exists*

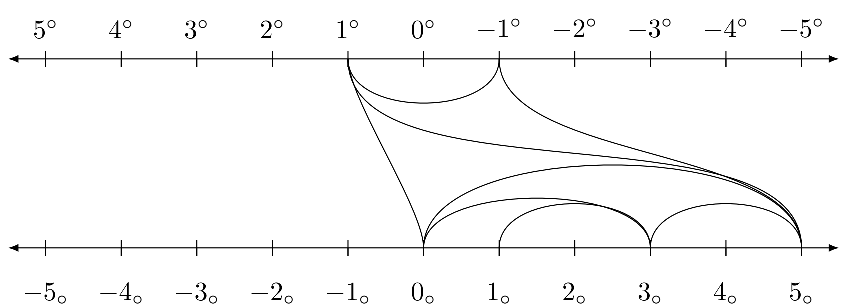
$$(p'^\circ, q'_\circ) \in \mathfrak{T}, (p', q') \in (< p, > q) \text{ and}$$

$$(p''^\circ, q''_\circ) \in \mathfrak{T}, (p'', q'') \in (> p, < q)$$

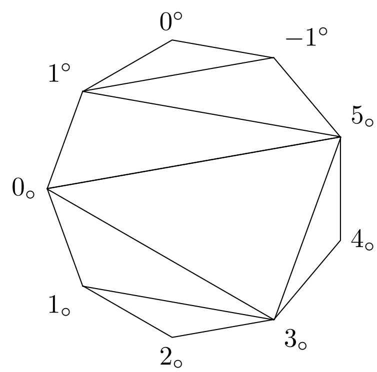
Expressed more simply, a triangulation is good if it has infinitely many connecting arcs in both directions. We notice how a good triangulation has infinitely many diagonal arcs, and an SL_2-tiling with enough ones has infinitely many ones. The notation $(> i, < j), (< i, > j)$ seems to occur both places as well.

Remark 7.9. *We are well acquainted with triangulated polygons. We may relate finite subsets of a triangulated strip to triangulated polygons. We illustrate how by a short example.*

Example 7.10. 9 *vertices bound together by the connecting arcs* $(1^\circ, 0_\circ)$ *and* $(-1^\circ, 5_\circ)$*.*



Within the confines of these two connecting arcs, we consider all other arcs to be diagonals of the polygon P *with vertices* $\{1^\circ, 0_\circ, 1_\circ, 2_\circ, 3_\circ, 4_\circ, 5_\circ, -1^\circ, 0^\circ\}$. *The diagonals of* P *are given by the arcs* $\{(0_\circ, 3_\circ), (0_\circ, 5_\circ), (1_\circ, 3_\circ), (3_\circ, 5_\circ), (1^\circ, -1^\circ), (1^\circ, 5_\circ)\}$. *We construct the 9-gon and draw the diagonals as listed.*



This 9-gon has quiddity cycle 3,2,1,4,1,4,2,1,3 which translates to the frieze below.

… *1* *1* *1* *1* *1* *1* *1* *1* *1* …
… *3* *2* *1* *4* *1* *4* *2* *1* *3* …
… *8* *5* *1* *3* *3* *3* *7* *1* *2* …
… *13* *2* *2* *2* *8* *5* *3* *1* *5* …
… *8* *5* *3* *1* *5* *13* *2* *2* *2* …
… *3* *7* *1* *2* *8* *5* *1* *3* *3* …
… *1* *4* *2* *1* *3* *3* *2* *1* *4* …
… *1* *1* *1* *1* *1* *1* *1* *1* *1* …

8. Constructing SL_2-tilings from triangulations of the strip

In this section we will describe a map that takes any good triangulation of the strip to an SL_2-tiling with enough ones. We show the validity of the map, and further explain the map through an example.

Construction 8.1. *An SL_2-tiling t with enough ones is constructed from a good triangulation $\mathfrak{T}$ such that $t = \Phi(\mathfrak{T})$ in the following manner: Consider a pair of vertices $(i^\circ, j_\circ)$ which is not necessarily an arc. Choose a pair of connecting arcs $(p^\circ, q_\circ)$, $(r^\circ, s_\circ) \in \mathfrak{T}$ such that $p < i < r$ and $s < j < q$. Since $\mathfrak{T}$ is a good triangulation this can be done for any pair $(i^\circ, j_\circ)$.*

We view $\{p^\circ, ..., r^\circ, s_\circ, ..., q_\circ\}$ as a polygon P of $|\{p^\circ, ..., r^\circ, s_\circ, ..., q_\circ\}|$ vertices. The arcs of $\mathfrak{T}$ between $(r^\circ, s_\circ)$ and $(p^\circ, q_\circ)$ are considered diagonals in the triangulation of the polyon (see Definition 7.5). We denote this triangulation $\mathfrak{T}_P$.

We define t by

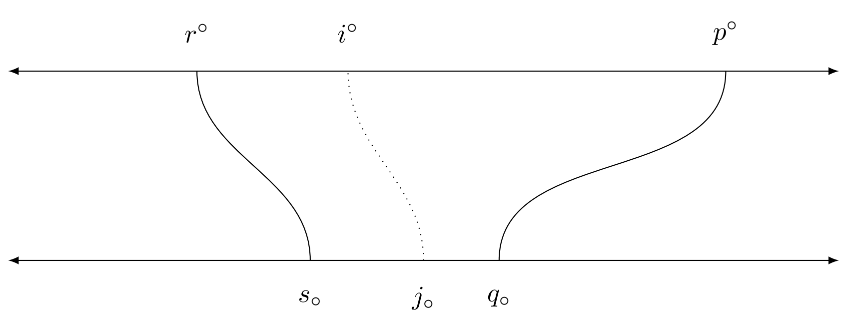
$$t_{ij} = \mathfrak{T}_P(i^\circ, j_\circ)$$

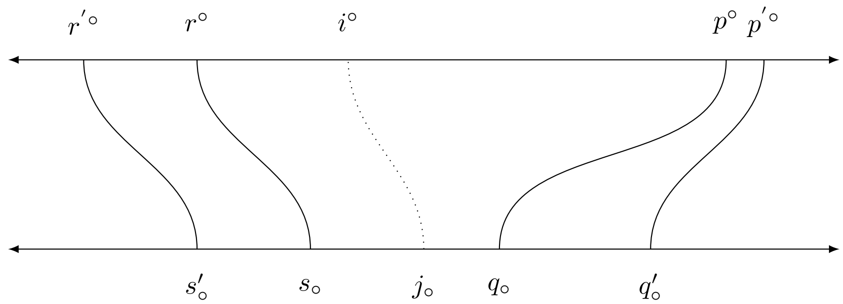
where $\mathfrak{T}_P(i^\circ, j_\circ)$ is the value $(P_{i^\circ}, P_{j_\circ})$ in the polygon P in the notation introduced in Definition 5.1.

Remark 8.1. *By Remark 5.6, we have that $(P_r, P_s) = 1$ if and only if there exists an edge between the two vertices. More specifically, Construction 8.1 gives us $t_{ij} = 1 \leftrightarrow (i^\circ, j_\circ) \in \mathfrak{T}$.*

Proposition 8.2. *Construction 8.1 gives a well defined SL_2-tiling, and the tiling $t = \Phi(\mathfrak{T})$ has enough ones.*

Proof. To show Φ is well defined, we need to show that the tiling is unaffected by our choice of connecting arcs $(p^\circ, q_\circ)$, $(r^\circ, s_\circ) \in \mathfrak{T}$. We show this by choosing two different pairs of diagonal arcs $(p^\circ, q_\circ)$, $(r^\circ, s_\circ)$, and $(p'^\circ, q'_\circ)$, $(r'^\circ, s'_\circ)$. Both pairs of diagonal arcs restrict a finite subset of the strip which contains $(i^\circ, j_\circ)$. The two choices give us two corresponding triangulated polygons $\mathfrak{T}_P$ and $\mathfrak{T}_Q$. We wish to show that $t_{ij} = \mathfrak{T}_P(i^\circ, j_\circ) = \mathfrak{T}_Q(i^\circ, j_\circ)$. In the following figures the choices of connecting arcs are drawn. The dotted line $(i^\circ, j_\circ)$ represents that there will not always be such an arc.





Let P be the smallest subset of the strip containing $(i^\circ, j_\circ)$. Then Q becomes an expansion of P by adding vertices on the outskirts of P by methods seen in Section 4. The triangulation of the inner polygon remains unchanged. Furthermore, by Remark 5.4 we see that the value of $(i^\circ, j_\circ)$ is the same for any choice of surrounding polygon. In other words, t_{ij} remains the same, regardless which pairs $(r^\circ, s_\circ), (p^\circ, q_\circ)$ we choose. Φ is then well defined.

Next, we show that t has enough ones. By Remark 8.1, we get that a good triangulation gives us infinitely many ones in the tiling, as each diagonal arc corresponds to a $1 \in t$.

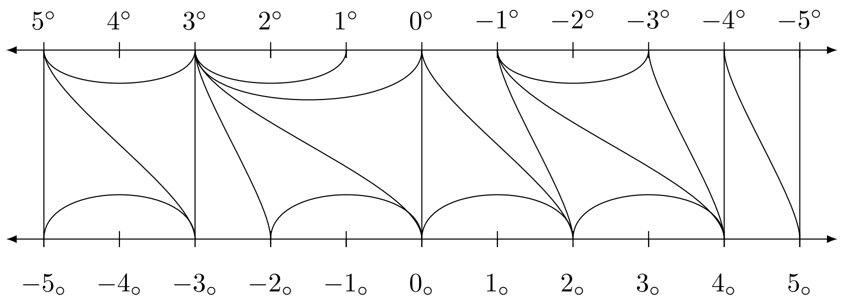
Additionally, we describe where these ones must occur. A diagonal arc $(p^\circ, q_\circ)$ has a neighbouring diagonal arc to the left, such that $(p'^\circ, q'_\circ) \in \mathfrak{T}$ and $p'^\circ > p^\circ,\ q'_\circ < q_\circ$. Because they correspond to diagonal arcs, $t_{pq} = 1$ and $t_{p'q'} = 1, (p', q') \in (> p, < q)$. The element $t_{p'q'}$ then is a 1 in the bottom left quadrant from t_{pq}. Similarly a diagonal to the right of $(p^\circ, q_\circ)$, say $(p''^\circ, q''_\circ)$ gives us $t_{p''q''} = 1$ with $p'' < p,\ q'' > q$, so $(p'', q'') \in (< p, > q)$. Then the element $t_{p''q''}$ is a 1 in the upper right quadrant from t_{pq}. This means that $t = \Phi(\mathfrak{T})$ has enough ones, as for every t_{ij} we have $1 \in (> i, < j), 1 \in (< i, > j)$.

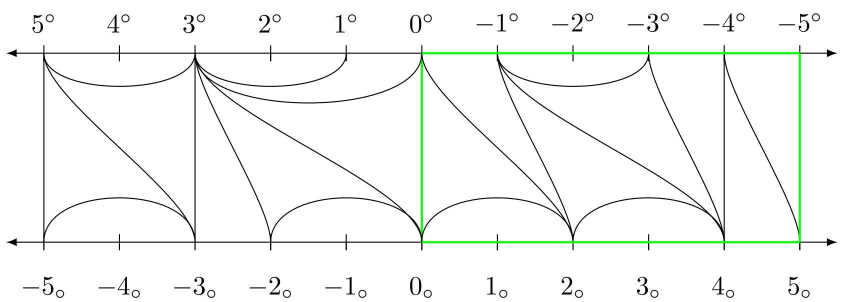
Lastly, we show that t is an SL_2-tiling. $t_{ij} > 0\ \forall\ i, j$ since $\mathfrak{T}_P(-,-)$ are all positive integers. We only need to show that all adjacent 2 x 2-submatrices have determinant 1.

$$\begin{vmatrix} \mathfrak{T}_P(i^\circ, j_\circ) & \mathfrak{T}_P(i^\circ, (j+1)_\circ) \\ \mathfrak{T}_P((i+1)^\circ, j_\circ) & \mathfrak{T}_P((i+1)^\circ, (j+1)_\circ) \end{vmatrix} = 1$$

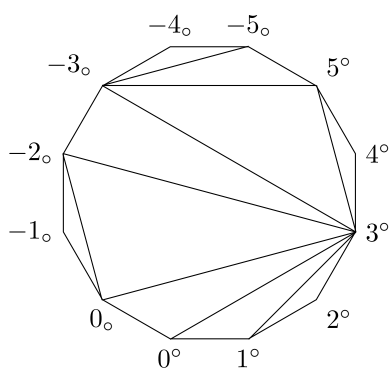
Here P is bounded by $(p^\circ, q_\circ), (r^\circ, s_\circ)$, where p, q, r, s are chosen such that $p < i < i+1 < r,\ s < j < j+1 < q$. This is always possible, as we can choose connecting arcs far enough from $(i^\circ, j_\circ)$ for the requirement to hold. The determinant above becomes a description of the unimodular rule for the frieze of P, as we defined $\mathfrak{T}_P(i^\circ, j_\circ)$ to be the value $(P_{i^\circ}, P_{j_\circ})$ in the polygon P. □

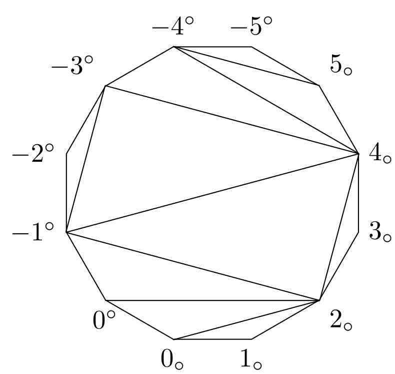
Example 8.3. *We illustrate how we can create the start of an SL_2-tiling from a triangulated strip, in practice. Since both figures will be infinite we will illustrate this for* $[-5_\circ, 5_\circ], [-5^\circ, 5^\circ]$ *which will yield an* 11×11*-submatrix of an* SL_2*-tiling.*





The nodes included in the green box are considered to be a 12-gon, where the triangulation is given by the arcs in the strip between those 12 vertices. This is illustrated in the figure below to the right. The non-green part of the triangulation is illustrated below to the left.





Using these two figures, we can calculate $(0^\circ, i_\circ), (i^\circ, 0_\circ)$ *for* $i \in \{-5, ..., 5\}$*:*

i	-5	-4	-3	-2	-1	0	1	2	3	4	5
$(i^\circ, 0_\circ)$	19	8	5	7	2	1	2	3	1	4	3
$(0^\circ, i_\circ)$	7	10	3	2	3	1	2	1	3	2	7

We then insert the middle row and middle column and calculate the remaining elements in the 11×11*-submatrix by using that the determinants of all* 2×2*-submatrices are 1. We start this process at the intersection, moving in either direction from* t_{00}.

$$
\begin{array}{ccccccccccccc}
 & & & & & & \vdots & & & & & & \\
 & 193 & 274 & 81 & 50 & 69 & 19 & 26 & 7 & 9 & 2 & 1 & \\
 & 81 & 115 & 34 & 21 & 29 & 8 & 11 & 3 & 4 & 1 & 1 & \\
 & 50 & 71 & 21 & 13 & 18 & 5 & 7 & 2 & 3 & 1 & 2 & \\
 & 69 & 98 & 29 & 18 & 25 & 7 & 10 & 3 & 5 & 2 & 5 & \\
 & 19 & 27 & 8 & 5 & 7 & 2 & 3 & 1 & 2 & 1 & 3 & \\
\cdots & 7 & 10 & 3 & 2 & 3 & 1 & 2 & 1 & 3 & 2 & 7 & \cdots \\
 & 9 & 13 & 4 & 3 & 5 & 2 & 5 & 3 & 10 & 7 & 25 & \\
 & 11 & 16 & 5 & 4 & 7 & 3 & 8 & 5 & 17 & 12 & 43 & \\
 & 2 & 3 & 1 & 1 & 2 & 1 & 3 & 2 & 7 & 5 & 18 & \\
 & 3 & 5 & 2 & 3 & 7 & 4 & 13 & 9 & 32 & 23 & 83 & \\
 & 1 & 2 & 1 & 2 & 5 & 3 & 10 & 7 & 25 & 18 & 115 & \\
 & & & & & & \vdots & & & & & &
\end{array}
$$

Should we want to expand the tiling beyond this 11×11-*submatrix we would need to continue the triangulation of the strip. It is also worth mentioning that although the set up we used now helps us compute a partial tiling, it has its limitations. For us to compute* t_{ij}*, we need a polygon large enough to include both* i° *and* $j_\circ$*. This may end up being very large polygons, and for this reason we will not show many full examples of this construction.*

Notice how even in such a small example, when we model a tiling after a subset of the triangulated strip, the value 1 seems to occur in the top right and bottom left quadrants, in a zig-zag pattern, and only in this zig-zag pattern. This could perhaps be special for the example we chose here, but we will show that it is not. By Remark 8.1 we get that in a tiling t created through Construction 8.1 all ones must follow a zig-zag path. We wish to show that all SL_2-tilings with enough ones have this property, but in order to do so we need to introduce some new computational notation.

9. Computational tools for SL_2-tilings

In this section we introduce a new computational notation used to show various results for SL_2-tilings. The notation has a geometrical interpretation which is explored more by Holm and Jørgensen in *SL_2 − tilings and triangulations of the strip* ([4], section 5). We will in this paper use it solely as a tool to be used in other results.

Definition 9.1. *For an SL_2-tiling t let $i < j$, $i, j \in \mathbb{Z}$. Choose $a \in \mathbb{Z}$. We define $\mathbf{c_{ij}}, \mathbf{d_{ij}}$ as follows.*

$$c_{ij} = \begin{vmatrix} t_{ia} & t_{i,a+1} \\ t_{ja} & t_{j,a+1} \end{vmatrix}, d_{ij} = \begin{vmatrix} t_{ai} & t_{aj} \\ t_{a+1,i} & t_{a+1,j} \end{vmatrix}$$

Remark 9.2. $c_{i,i+1} = d_{i,i+1} = 1$ *for* $i \in \mathbb{Z}$.

This follows from t being an SL_2-tiling. Below we show the insertion, and the resulting matrices which describe the demand for an SL_2-tiling that all 2×2-submatrices have determinant 1.

$$c_{i,i+1} = \begin{vmatrix} t_{ia} & t_{i,a+1} \\ t_{i+1,a} & t_{i+1,a+1} \end{vmatrix}, d_{i,i+1} = \begin{vmatrix} t_{ai} & t_{a,i+1} \\ t_{a+1,i} & t_{a+1,i+1} \end{vmatrix}$$

Proposition 9.3. *Let t be an SL_2-tiling, $i < j < k < l$ are integers. Then*

$$c_{ik}c_{jl} = c_{ij}c_{kl} + c_{il}c_{jk}, \; d_{ik}d_{jl} = d_{ij}d_{kl} + d_{il}d_{jk}$$

.

Proof. We will prove only $c_{ik}c_{jl} = c_{ij}c_{kl} + c_{il}c_{jk}$, as $d_{ik}d_{jl} = d_{ij}d_{kl} + d_{il}d_{jk}$ is done the exact same way, and this is horribly tedious to show.

$$\begin{vmatrix} t_{ia} & t_{i,a+1} \\ t_{k,a} & t_{k,a+1} \end{vmatrix} \begin{vmatrix} t_{ja} & t_{j,a+1} \\ t_{l,a} & t_{l,a+1} \end{vmatrix} = \begin{vmatrix} t_{ia} & t_{i,a+1} \\ t_{j,a} & t_{j,a+1} \end{vmatrix} \begin{vmatrix} t_{ka} & t_{k,a+1} \\ t_{l,a} & t_{l,a+1} \end{vmatrix} + \begin{vmatrix} t_{ia} & t_{i,a+1} \\ t_{l,a} & t_{l,a+1} \end{vmatrix} \begin{vmatrix} t_{ja} & t_{j,a+1} \\ t_{k,a} & t_{k,a+1} \end{vmatrix}$$

$(t_{ia}t_{k,a+1} - t_{k,a}t_{i,a+1})(t_{ja}t_{l,a+1} - t_{l,a}t_{j,a+1}) = (t_{ia}t_{j,a+1} - t_{j,a}t_{i,a+1})(t_{ka}t_{l,a+1} - t_{l,a}t_{k,a+1}) + (t_{ia}t_{l,a+1} - t_{l,a}t_{i,a+1})(t_{ja}t_{k,a+1} - t_{k,a}t_{j,a+1})$

$t_{ia}t_{k,a+1}t_{ja}t_{l,a+1} - t_{k,a}t_{i,a+1}t_{ja}t_{l,a+1} - t_{ia}t_{k,a+1}t_{l,a}t_{j,a+1} + t_{k,a}t_{i,a+1}t_{l,a}t_{j,a+1} = t_{ia}t_{j,a+1}t_{ka}t_{l,a+1} - t_{j,a}t_{i,a+1}t_{ka}t_{l,a+1} - t_{ia}t_{j,a+1}t_{l,a}t_{k,a+1} + t_{j,a}t_{i,a+1}t_{l,a}t_{k,a+1} + t_{ia}t_{l,a+1}t_{ja}t_{k,a+1} - t_{l,a}t_{i,a+1}t_{ja}t_{k,a+1} - t_{ia}t_{l,a+1}t_{k,a}t_{j,a+1} + t_{l,a}t_{i,a+1}t_{k,a}t_{j,a+1}$
This is quite the mess. We sort the terms alphabetically on the first term in the subscript, and add a splash of colour to more easily identify equal terms in the equation.

$t_{ia}t_{ja}t_{k,a+1}t_{l,a+1} - t_{i,a+1}t_{ja}t_{k,a}t_{l,a+1} - t_{ia}t_{j,a+1}t_{k,a+1}t_{l,a} + t_{i,a+1}t_{j,a+1}t_{k,a}t_{l,a} = t_{ia}t_{j,a+1}t_{ka}t_{l,a+1} - t_{i,a+1}t_{j,a}t_{ka}t_{l,a+1} - t_{ia}t_{j,a+1}t_{k,a+1}t_{l,a} + t_{i,a+1}t_{j,a}t_{k,a+1}t_{l,a} + t_{ia}t_{ja}t_{k,a+1}t_{l,a+1} - t_{i,a+1}t_{ja}t_{k,a+1}t_{l,a} - t_{ia}t_{j,a+1}t_{k,a}t_{l,a+1} + t_{i,a+1}t_{j,a+1}t_{k,a}t_{l,a}$

∎

Proposition 9.4. *Let t,i,j,k be as above, and choose an integer a. Then*

$$t_{ja}c_{ik} = t_{ia}c_{jk} + t_{ka}c_{ij}, \; t_{aj}d_{ik} = t_{ai}d_{jk} + t_{ak}d_{ij}$$

Proof. We again prove this only for the c-terms as the proof for the d-terms is identical.
$t_{ja}c_{ik} = t_{ia}c_{jk} + t_{ka}c_{ij}$

$$t_{j,a} \cdot \begin{vmatrix} t_{ia} & t_{i,a+1} \\ t_{k,a} & t_{k,a+1} \end{vmatrix} = t_{ia} \cdot \begin{vmatrix} t_{ja} & t_{j,a+1} \\ t_{k,a} & t_{k,a+1} \end{vmatrix} + t_{ka} \cdot \begin{vmatrix} t_{ia} & t_{i,a+1} \\ t_{j,a} & t_{j,a+1} \end{vmatrix}$$

$t_{ja}(t_{ia}t_{k,a+1} - t_{k,a}t_{i,a+1}) = t_{ia}(t_{ja}t_{k,a+1} - t_{k,a}t_{j,a+1}) + t_{ka}(t_{ia}t_{j,a+1} - t_{j,a}t_{i,a+1})$
Multiplied and with terms sorted alphabetically
$t_{ia}t_{ja}t_{k,a+1} - t_{i,a+1}t_{ja}t_{k,a} = t_{ia}t_{ja}t_{k,a+1} - t_{ia}t_{j,a+1}t_{k,a} + t_{ia}t_{j,a+1}t_{ka} - t_{i,a+1}t_{j,a}t_{ka}$
$t_{ia}t_{ja}t_{k,a+1} - t_{i,a+1}t_{ja}t_{k,a} = t_{ia}t_{ja}t_{k,a+1} - t_{ia}t_{j,a+1}t_{k,a} + t_{ia}t_{j,a+1}t_{ka} - t_{i,a+1}t_{j,a}t_{ka}$ □

Remark 9.5. *As a consequence of Proposition 9.4 we can show that $c_{ij}, d_{ij} \in \mathbb{Z}^+$ for $i < j$ in an SL_2-tiling t and for any $a \in \mathbb{Z}$.*

Proof. We prove $c_{ij} > 0$ by induction on j. $d_{ij} > 0$ is proven similarly.
For $j = i + 1$ we have shown $c_{i,i+1} = 1$. Assume $c_{ij} > 0$ for $i < j \leq r$. Then $c_{i,r+1} = \dfrac{t_{ia}c_{r,r+1} + t_{r+1,a}c_{ir}}{t_{ra}}$. Since $c_{r,r+1} = 1$ we reduce the expression.
$c_{i,r+1} = \dfrac{t_{ia} + t_{r+1,a}c_{ir}}{t_{ra}} > 0$, because we know $t_{ia} > 0$ $\forall i, a$ and $c_{ir} > 0$ by our assumption. We know c_{ij} is an integer by its definition because c_{ij} is multiplications and subtractions of integers as t_{ij} is always an integer. □

Proposition 9.6. *Let t be an SL_2-tiling and $i < j$ and $p < q$ integers. Then*

$$\begin{vmatrix} t_{ip} & t_{iq} \\ t_{jp} & t_{jq} \end{vmatrix} = c_{ij}d_{pq} \in \mathbb{Z}^+$$

Proof. The expression is clearly a positive integer as Remark 9.5 states both c_{ij} and d_{pq} are positive integers. The remainder of the proof is stated in 5.7 [4] as computational. □

From these results we can show more general results for SL_2-tilings. First we can show restrictions for where ones may occur in a tiling, before we describe accurately where they in fact must occur.

Proposition 9.7. *Let t be an SL_2-tiling, and $n \in \mathbb{Z}^+$. For a fixed i, $t_{ij} = n$ finitely many times. Similarly for a fixed j, $t_{ij} = n$ occurs finitely many times. In other words each row and each column contains any specific number at most a finite number of times.*

Proof. Fix j and let $k < l < ...$ be an increasing sequence such that $t_{kj} = t_{lj} = ... = n$. The first two terms give us

$$c_{kl} = \begin{vmatrix} t_{ka} & t_{k,a+1} \\ t_{la} & t_{l,a+1} \end{vmatrix} \tag{3}$$

By Remark 9.5, we know this to be positive for all $a \in \mathbb{Z}$, so we let $a = j$. Then

$$0 < \begin{vmatrix} t_{kj} & t_{k,j+1} \\ t_{lj} & t_{l,j+1} \end{vmatrix} = \begin{vmatrix} n & t_{k,j+1} \\ n & t_{l,j+1} \end{vmatrix} = n(t_{l,j+1} - t_{k,j+1}) \implies t_{l,j+1} > t_{k,j+1}$$

Continuing this process for the second and third terms of the sequence, and so forth, we get that $k < l < ...$ gives us $t_{l,j+1} > t_{k,j+1} > ...$, which must be finite since all $t_{pq} > 0$. Therefore $k < l < ...$ must also be finite.

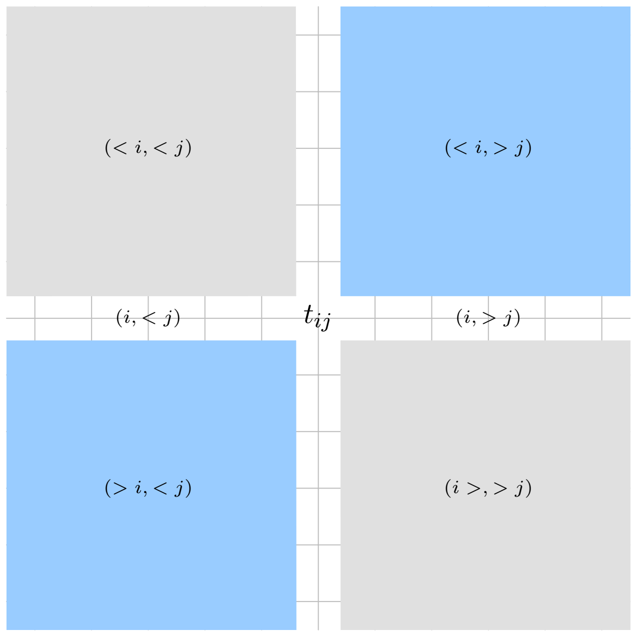
Next let $k > l > ... $ be a decreasing sequence, still such that $t_{kj} = t_{lj} = ... = n$. Then

$$c_{lk} = \begin{vmatrix} t_{la} & t_{l,a+1} \\ t_{ka} & t_{k,a+1} \end{vmatrix} \implies \begin{vmatrix} t_{lj} & t_{l,j+1} \\ t_{kj} & t_{k,j+1} \end{vmatrix} = \begin{vmatrix} n & t_{l,j+1} \\ n & t_{k,j+1} \end{vmatrix} > 0 \implies t_{k,j+1} > t_{l,j+1}$$

As before by continuing this step for the next few terms we get $k > l > ...$ which implies that $t_{k,j+1} > t_{l,j+1} > ... > 0$ so $k > l > ...$ must be finite.

So far we have shown that $t_{ij} = n$ finitely many times in each column, if there exist any at all. The proof that $t_{ij} = n$ finitely many times in each row is analogous using d_{kl}, d_{lk}. □

Recall the notation $(< i, > j)$ from Definition 6.3 to describe quadrants of SL_2-tilings.



Proposition 9.8. *Let t be an SL_2-tiling, and $i, j, p, q \in \mathbb{Z}$. If $t_{ij} = 1$ then $t_{pq} \neq 1, \forall\ (p, q) \in (< i, < j) \cup (> i, > j)$. This means the bottom right and top left quadrants may not contain the value 1, from the starting point $t_{ij} = 1$.*

Proof. To prove this we assume otherwise. Let $t_{xy} = t_{zw} = 1$, with $x < z, y < w$. Then by Proposition 9.6 we have by inserting $x = i, y = j, z = p, w = q$

$$c_{xz} d_{yw} = \begin{vmatrix} t_{xy} & t_{xw} \\ t_{zy} & t_{zw} \end{vmatrix} = t_{xy} t_{zw} - t_{zy} t_{xw} = 1 - t_{zy} t_{xw} \leq 0$$

This is a contradiction to Proposition 9.6 stating that $c_{xz} d_{yw} > 0$. Therefore we cannot have $t_{xy} = t_{zw} = 1$, with $x < z, y < w$. We need not check $x > z, y > w$, as that is the same case as above with different names. □

10. Revisiting friezes

Let us consider friezes as partial tilings, placed as diagonal bands. This way the unimodular rule is covered by the normal submatrix determinant requirement of a tiling. By considering diagonal bands friezes, vertical lines in the partial tiling become what we are used to referring to as diagonals of the frieze. This change in perspective is illustrated below.

$$
\begin{array}{ccccccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
 & & & & & & & & & \vdots & & & & & & & \\
 & & & & & & & & & b & & & & & & & \\
 & & & & & & & & a & & d & & & & & & \\
 & & & & & & & & & c & & & & & & & \\
 & & & & & & & & & \vdots & & & & & & & \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots
\end{array}
$$

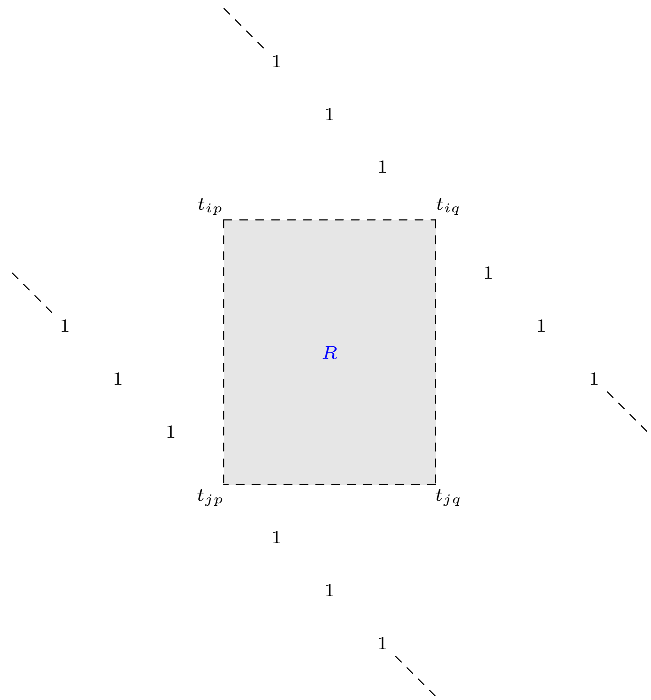
Here, $ad - bc = 1$ by the unimodular rule. Rotating the frieze 45 degrees clockwise, we obtain a pattern without staggered rows, like a diagonal band in a matrix.

$$
\begin{array}{cccccccccccc}
 & & & \ddots & & & & & & & & \\
 & & & & 1 & & & & & & & \\
 & & & & & 1 & & & & & & \\
\ddots & & & & & & 1 & & & & & \\
 & 1 & & & & & & 1 & & & & \\
 & & 1 & & & & & \iddots & 1 & & & \\
 & & & & & a & b & & & & & \\
 & & & 1 & & & & & & 1 & & \\
 & & & & & c & d & & & & & \\
 & & & & 1 & \iddots & & & & & 1 & \\
 & & & & & 1 & & & & & & \ddots \\
 & & & & & & 1 & & & & & \\
 & & & & & & & 1 & & & & \\
 & & & & & & & & \ddots & & &
\end{array}
$$

$ad - bc = 1$ is the determinant of a 2×2 submatrix in the diagonal band above. In this section we intend to relate diagonal band friezes to SL_2-tilings.

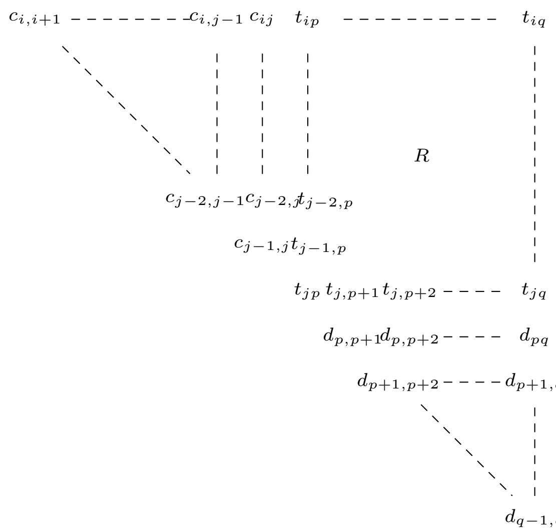
Theorem 10.1. *Let t be and SL_2-tiling. Let $i \leq j, p \leq q, (i,j) \neq (p,q)$ be integers with $t_{ij} = t_{pq} = 1$. Then there exists a frieze which matches t in the rectangle $R = \{t_{xy} \mid i \leq x \leq j, p \leq y \leq q\}$.*

Proof. We begin with restricting t to R, before extending it to a fundamental region within the potential borders of a frieze. Let us begin with a figure showing the rectangle R.



Note that a frieze defined on a diagonal band such as this may never occur in an SL_2-tiling with enough ones. The placement of the ones in the border would be contradictory to several results. The theorem merely states that between two ones moving diagonally from south west to north east, there is a whole rectangle which is identical to that of a diagonal band frieze.

Knowing a single diagonal of a frieze allows us to complete it. We wish to extend the rectangle R to a fundamental region, stretching from t_{iq} straight west and straight south until it reaches the border. This will give us a full diagonal. If we can prove that this region is in accordance with the rules of a frieze we know that the frieze is then made up of reflections of that fundamental region, and so we are done. Consider the figure below, in which we have filled out the fundamental region by adding triangular regions of c_{xy}, d_{xy} in the top left and bottom corners, respectively.



Here we continue the rectangle $(i...j, p...q) = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid i \leq x \leq j, p \leq y \leq q\}$ by adding small triangles to complete a fundamental region. Note here that $c_{i,i+1}, ..., c_{j-1,j}, t_{jp}, d_{p,p+1}, ...,$ $d_{q-1,q}$ consists only of ones. Note also that this may never be the case in an SL_2-tiling with enough ones, as this will breach several results regarding where ones may be positioned. However should we consider a partial tiling with a diagonal of ones, the rectangle which agrees with t may be extended in the manner shown. We will show why the triangles added to the rectangle are the d and c elements. We will show this only for the bottom triangle as both sides have similar proofs. Using d-notation we can do the following.

$$\begin{vmatrix} t_{j,p+1} & t_{j,p+2} \\ d_{p,p+1} & d_{p,p+2} \end{vmatrix} = \begin{vmatrix} t_{p-1,p+1} & t_{p-1,p+2} \\ d_{p,p+1} & d_{p,p+2} \end{vmatrix} = t_{p-1,p+1}d_{p,p+2} - t_{p-1,p+2}d_{p,p+1}$$

Next we apply Proposition 9.4 stating $t_{ik}d_{ik} = t_{ai}d_{jk} + t_{ak}d_{ij}$, $t_{ik}d_{ik} = t_{ai}d_{jk} + t_{ak}d_{ij}$ rewritten $t_{ik}d_{ik} - t_{ak}d_{ij} = t_{ai}d_{jk}$. We insert $a = p-1, i = p, j = p+1, k = p+2$, and get $t_{p-1,p+1}d_{p,p+2} - t_{p-1,p+2}d_{p,p+1} = t_{p-1,p}d_{p+1,p+2} = 1$ since $t_{p-1,p} = t_{j,p} = 1$. In other words $d_{p,p+2}$ must occupy the space below $t_{j,p+2}$ for the determinant of the 2×2-submatrix to be 1. In a similar fashion it is straight forward to show the whole row $d_{p,p+i}, ..., d_{pq}$ must be as in the figure. The rows further down we compute using Proposition 9.3.
We insert $i = p, j = p+1, k = p+2, l = p+3$ into $d_{ik}d_{jl} = d_{ij}d_{kl} + d_{il}d_{jk}$ to get $d_{p,p+2}d_{p+1,p+3} - d_{p,p+3}d_{p+1,p+2} = d_{p,p+1}d_{p+2,p+3} = 1$ which gives the next term in the second row of d-terms. We continue this way till the end of the triangle. We continue using the bottom row currently calculated to calculate the next row, starting at the left most element after the border. For the very last term in the bottom corner there is no full 2×2-submatrix we can use to determine the final term. However, we already know the value of $d_{q-1,q}$ as all $d_{s,s+1} = 1$. Similarly we use the same propositions to show the top left triangle of c-terms.

□

Incidentally, Theorem 10.1 also implies that in an SL_2-tiling, elements will divide the sum of their horizontal neighbours, and the sum of their vertical neighbours. This will not be needed for the proofs to come, but it is inarguably amusing.

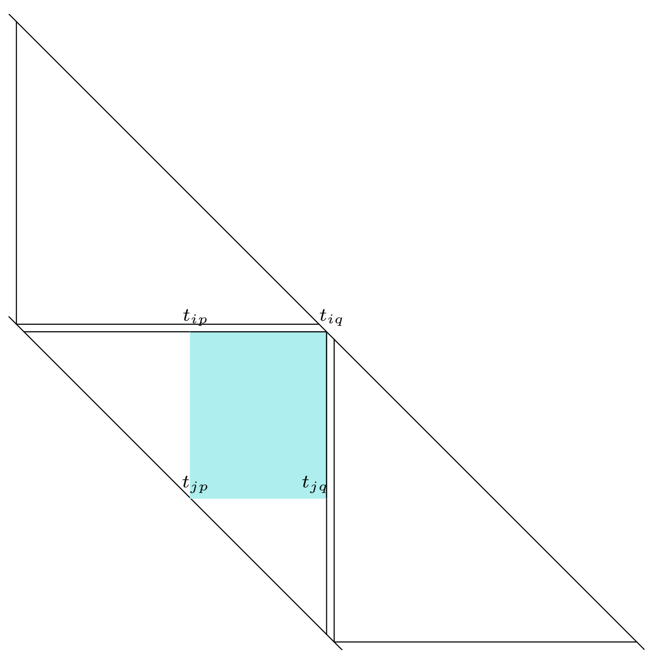
11. Zigzag path of ones

In this section we determine how all ones can and must be distributed in an SL_2-tiling. This is necessary before we can find an inverse map of Construction 8.1.

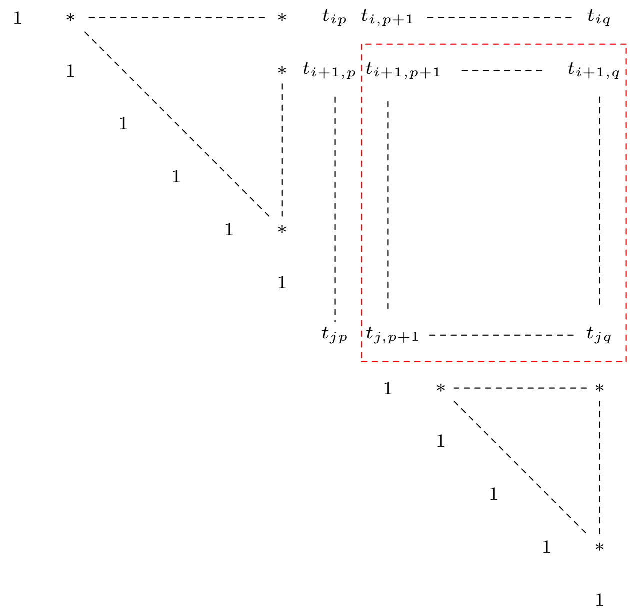
Proposition 11.1. *Let t be an SL_2-tiling with $t_{jp} = 1$*
i) $t_{rs} = 1$ for $(r,s) \in (< j, > p) \implies$ either $t_{qs} = 1, q < r$ or $t_{rw} = 1, w > s$ but not both. In other words if a 1 occurs in the top right quadrant it also occurs on one of the half lines restricting the quadrant but not both.
ii) $t_{rs} = 1$ for $(r,s) \in (> j, < p) \implies$ either $t_{qs} = 1, q > r$ or $t_{rw} = 1, w < s$ but not both. In other words if a 1 occurs in the bottom left quadrant it also occurs on one of the half lines restricting the quadrant but not both.

Proof. The proofs for i) and ii) are similar so we show only i).
If a 1 occurs on both half lines it contradicts Proposition 9.8, as one term then is in the top left quadrant in relation to the other. It then remains to show that if $t_{jp} = 1$ and a 1 occurs in the top right quadrant, $t_{iq} = 1, (i,q) \in (< j, > p)$, it also occurs on one of the half lines $(< j, p)$ or $(j, > p)$.

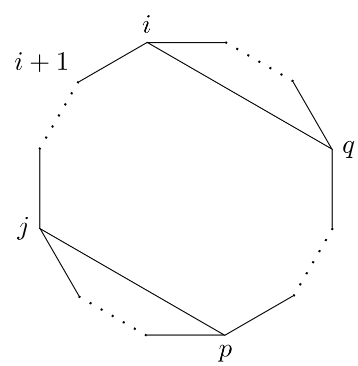
We wish to prove this by contradiction so assume that $t_{jp} = 1 = t_{iq}, (i,q) \in (< j, > p)$ and that $t_{xp} \neq 1 \neq t_{jy}, \forall x < j, y > p$. Furthermore let (i,q) be the closest term in this quadrant such that $t_{iq} = 1$ so that the rectangle $R = \{t_{xy} \mid i \leq x \leq j, p \leq y \leq q\}$ has only two ones, namely t_{jp}, t_{iq}. By Theorem 10.1 the SL_2-tiling agrees with a frieze on the rectangle R.



The figure above represents in the blue box, the rectangle R, placed within a frieze, here represented as a diagonal band. The diagonal borders are all ones, and the large triangles represent the alternating fundamental regions of the frieze. For $i < i+1 < j$ we can draw the following illustration.



We draw a polygon corresponding to the rectangle R (not the smaller red box, mind you).



The elements in the red box in the figure above correspond to the diagonals between the diagonals $(i+1,q),(j,p)$. For R to be a part of a frieze pattern the correlating triangulated polygon $\mathfrak{T}_P$ must be a maximal triangulation. We also know from Section 5 that in a frieze $(x,y)=1 \leftrightarrow (x,y) \in \mathfrak{T}_P$. However the whole column of elements (y,q) for $i+1 \leq y \leq j$ contains no ones by our assumption, and similarly (x,p) for $i+1 \leq x \leq j$ we have no diagonals crossing a potential diagonal from i to p. In other words for such a triangulation to be maximal, $(i,p) \in \mathfrak{T}$ which implies $t_{ip}=1$. This contradicts our assumption, and we are done. For the special case where $i+1=j$ we get that the rectangle R is a 2×2 matrix with

$$\begin{vmatrix} t_{ip} & t_{iq} \\ t_{jp} & t_{jq} \end{vmatrix} = \begin{vmatrix} t_{ip} & 1 \\ 1 & t_{jq} \end{vmatrix} = t_{ip}t_{jq} - 1 = 1 \implies t_{ip}=1, t_{jq}=2 \text{ or } t_{ip}=2, t_{jq}=1$$

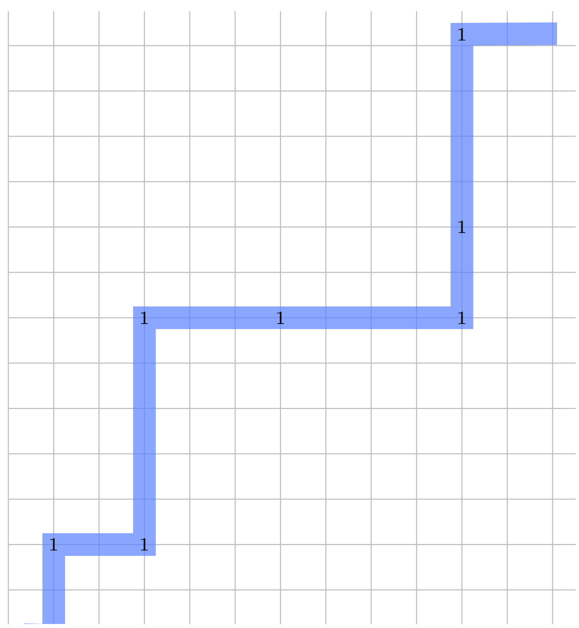
which again contradicts our assumption.

□

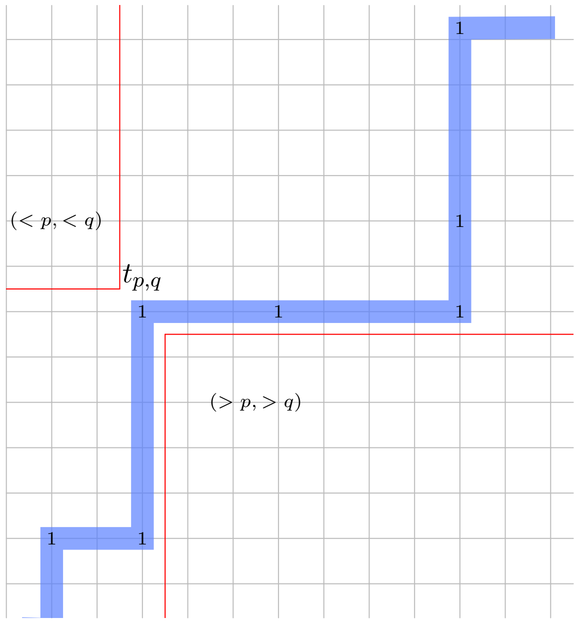
Proposition 11.2. *For any SL_2-tiling t with enough ones there exists a zigzag pattern of coordinates $(x_\alpha, y_\alpha) \in \mathbb{Z} \times \mathbb{Z}, \alpha \in \mathbb{Z}$ with the following properties.*
i) $t_{xy} = 1 \leftrightarrow (x, y) = (x_\alpha, y_\alpha)$ for some α
ii) For each α either $x_{\alpha+1} < x_\alpha, y_{\alpha+1} = y_\alpha$ or $x_{\alpha+1} = x_\alpha, y_{\alpha+1} > y_\alpha$
iii) There are infinitely many twists and turns in the zigzag pattern in both directions.

Proof. *iii*) is a direct consequence of Proposition 9.7, as otherwise there would be infinitely many ones in either a row or column since t has enough ones.

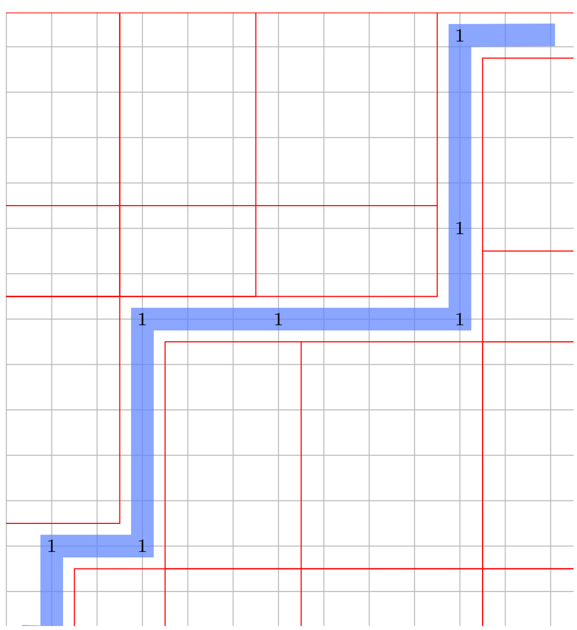
ii) follows from the definition, it simply numbers (x_α, y_α) so that α increases from south west to north east in t. We need to show $\rightarrow$ in *i*), and that such a system exists. Below we show a figure of what such a zigzag pattern looks like.



To show that the ones on the path are all possible ones, consider the figure below. Here we choose an arbitrary element in $\{(p, q) \in \mathbb{Z} \times \mathbb{Z} \,|t_{pq} = 1\}$. We mark in red the areas that may not contain a 1 by Proposition 9.8, in both the top left and bottom right quadrants from t_{pq}.



We repeat this process for all ones on the path to reveal a mesh pattern that blots out all of the tiling that is not on the path.



We know now that should such a path exist, all ones must be on it. SL_2-tiling with enough ones has infinitely many ones in the top right and bottom left quadrants. By Proposition 11.1 this also guarantees a zigzag path of ones as depicted and described above.

We choose the numbers on the path $\{(x_\alpha, y_\alpha)\}$ such that if more than one 1 occurs on the same half line we choose the closest one as the next element of the sequence. This means that for $t(x_\alpha, y_\alpha) = 1$, if t has the value 1 on the half line $(< x_\alpha, y_\alpha)$ let $x_{\alpha+1}$ be maximal on the half line such that $t(x_{\alpha+1}, y_\alpha) = 1$ while $y_\alpha = y_{\alpha+1}$.

If a 1 occurs on the half line $(x_\alpha, > y_\alpha)$ let $y_{\alpha+1}$ be minimal on the half line, such that $t(x_\alpha, y_{\alpha+1}) = 1$ and $x_\alpha = x_{\alpha+1}$. With this construction all ones in the tiling must be in the set $\{(x_\alpha, y_\alpha)\}$. □

Note that since all ones in the tiling must be in this zigzag path it is also unique, although one may add a constant to α to shift the names of the elements.

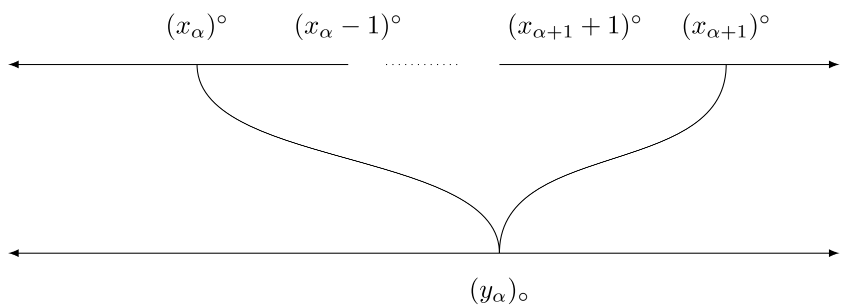
Remark 11.3. *Filling out the elements in the segments between ones in the zigzag path in Proposition 11.2 creates a path that determines the whole SL_2-tiling. This is realized by starting at any corner in the path and expanding by the determinant rule for the tiling.*

12. Constructing triangulations of the strip from SL_2-tilings

We will now present a way of constructing good triangulations of the strip from SL_2-tilings with enough ones. We can then show that this construction and Construction 8.1 are inverse bijections.

Construction 12.1. *Starting with an SL_2-tiling with enough ones, we construct a good triangulation of the strip $\mathfrak{T} = \Psi(t)$ in the following manner.*

We start with drawing all the connecting arcs $((x_\alpha)^\circ, (y_\alpha)_\circ)$ in $\mathfrak{T}$, where x_α, y_α are from the set of ones described in Proposition 11.2. This guarantees that the resulting triangulation will be good, should it be a triangulation at all. Additionally by ii) in Proposition 11.2 these diagonal arcs must be pairwise non-crossing, illustrated in the figure below. For this figure $y_{\alpha+1} = y_\alpha$.



We do this for every pair along our zigzag path to create a series of the segments seen above. Each of these is then viewed as a polygon in the manner we are accustomed to. Since we have $t(x_\alpha, y_\alpha) = 1$ and $x_\alpha = x_{\alpha+1}$ Theorem 10.1 states there exists a freeze which agreed with t on the area R. However R has width 1 in this case, an it is simply the vertical line depicted below.

$$
\begin{array}{ccccccc}
 & & & \ddots & & & \\
 & & & 1 & & & \\
 & & & & t(x_{\alpha+1}, y_\alpha) & & \\
\ddots & & & & & & \\
1 & & & & t(x_{\alpha+1}+1, y_\alpha) & 1 & \\
 & & & & \vdots & & \\
 & 1 & & & & & 1 \ \ddots \\
 & & & 1 & t(x_\alpha - 1, y_\alpha) & & \\
 & & & & t(x_\alpha, y_\alpha) & & \\
 & & & & & 1 \ \ddots &
\end{array}
$$

This vertical line defines the whole frieze, as a vertical (or horizontal) line in a diagonal band corresponds to diagonals in the friezes from Section 3. We construct $\mathfrak{T}_P$ by filling out the friezes and finding the triangulation of the polygon $P = (x_{\alpha+1}, x_{\alpha+1}+1, ..., x_\alpha, y_\alpha)$. We add the diagonals of $\mathfrak{T}_P$ to $\mathfrak{T}$. This completes the triangulation of a subset restricted by the diagonals $((x_\alpha)^\circ, (y_\alpha)_\circ), ((x_{\alpha+1})^\circ, (y_{\alpha+1})_\circ)$, for any choice of α.

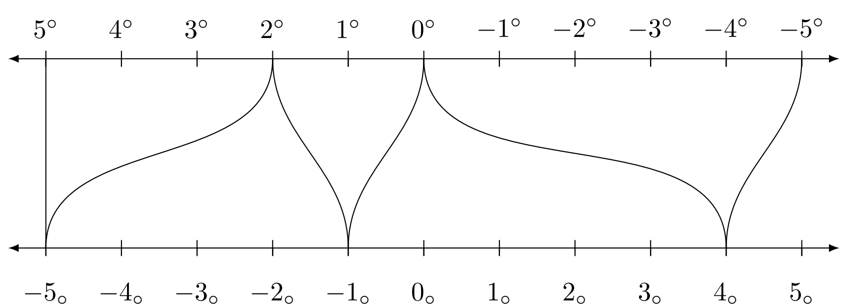
Example 12.1. *Creating a subset of a triangulation of the strip by applying Construction 12.1 to a 11×11-submatrix of an SL_2-tiling with enough ones.*

$$
\begin{array}{ccccccccccccc}
 & & & & & & \vdots & & & & & & \\
 & 5 & 12 & 7 & 9 & 2 & 3 & 7 & 11 & 4 & 1 & 2 & \\
 & 12 & 29 & 17 & 22 & 5 & 8 & 19 & 30 & 11 & 3 & 7 & \\
 & 19 & 46 & 27 & 35 & 8 & 13 & 31 & 49 & 18 & 5 & 12 & \\
 & 7 & 17 & 10 & 13 & 3 & 5 & 12 & 19 & 7 & 2 & 5 & \\
 & 9 & 22 & 13 & 17 & 4 & 7 & 17 & 27 & 10 & 3 & 8 & \\
\dots & 2 & 5 & 3 & 4 & 1 & 2 & 5 & 8 & 3 & 1 & 3 & \dots \\
 & 3 & 8 & 5 & 7 & 2 & 5 & 13 & 21 & 8 & 3 & 10 & \\
 & 1 & 3 & 2 & 3 & 1 & 3 & 8 & 13 & 5 & 2 & 7 & \\
 & 2 & 7 & 5 & 8 & 3 & 10 & 27 & 44 & 17 & 7 & 25 & \\
 & 3 & 11 & 8 & 13 & 5 & 17 & 46 & 75 & 29 & 12 & 43 & \\
 & 1 & 4 & 3 & 5 & 2 & 7 & 19 & 31 & 12 & 5 & 18 & \\
 & & & & & & \vdots & & & & & &
\end{array}
$$

We are most interested in the zigzag path of ones. Let the red number below be $t_{0,0}$.

$$
\begin{array}{ccccccccccccc}
 & & & & & & \vdots & & & & & & \\
 & & & & & & & & & & 1 & 2 & \\
 & & & & & & & & & & 3 & & \\
 & & & & & & & & & & 5 & & \\
 & & & & & & & & & & 2 & & \\
 & & & & & & & & & & 3 & & \\
\dots & & & & & 1 & \color{red}{2} & 5 & 8 & 3 & 1 & & \dots \\
 & & & & & 2 & & & & & & & \\
 & 1 & 3 & 2 & 3 & 1 & & & & & & & \\
 & 2 & & & & & & & & & & & \\
 & 3 & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & \\
 & & & & & & \vdots & & & & & &
\end{array}
$$

We identify that there are ones in the coordinates $(5,-5), (2,-5), (2,-1), (0,-1), (0,4), (-5,4)$. We add the corresponding diagonals to the strip.



This gives us a nice pattern of non-crossing arcs, which separate the strip into finite polygons which we can triangulate with ease.

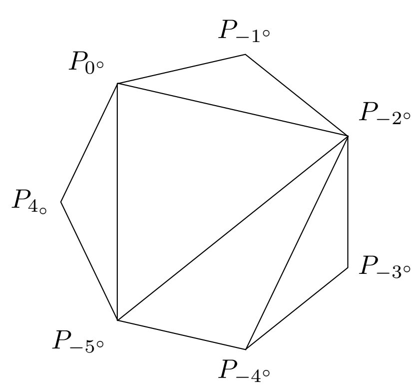
Consider first the vertical line in the top right corner, $1, 3, 5, 2, 3, 1$. *By Theorem 10.1 this line matches a diagonal in a frieze.*

...	1		1		1		1		1		1		1	...	
	...	3													...
...			5											...	
	...			2											...
...					3									...	
	...	1		1		1		1		1		1		1	...

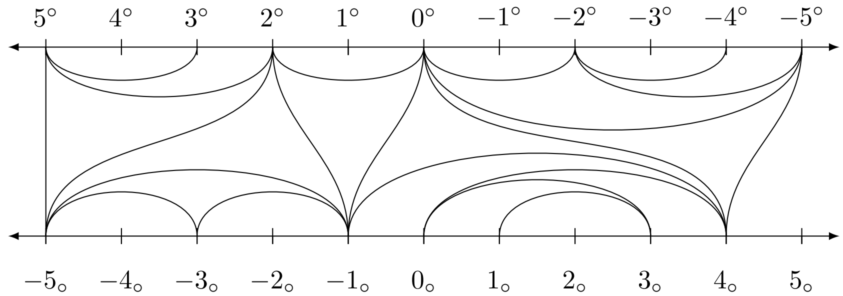
We fill out the frieze by the unimodular rule. Keep in mind here that the elements in this diagonal represent $(P_{-5^\circ}, P_{4_\circ}) = 1, (P_{-4^\circ}, P_{4_\circ}) = 3, ..., (P_{0^\circ}, P_{4_\circ}) = 1$ *for a polygon* P *with vertices* $P_{-5^\circ}, P_{-4^\circ}, P_{-3^\circ}, P_{-2^\circ}, P_{-1^\circ}, P_{0^\circ}, P_{4_\circ}$.

...	1		1		1		1		1		1		1	...	
	...	3		2		1		4		1		3		1	...
...			5		1		3		3		2		2	...	
	...			2		2		2		5		1		3	...
...					3		1		3		2		1	...	
	...	1		1		1		1		1		1		1	...

We need not fill out the frieze past 7 elements in the second row since the quiddity cycle of a frieze with 6 rows has order 7. The frieze above has quiddity cycle $3, 2, 1, 4, 1, 3, 1$. *To find the triangulation corresponding to this we apply Construction 4.2 to get the polygon.*



To find out how to name the vertices we use the diagonal we started with. Since the diagonal has only two ones, we know the node $4_\circ$ *is a special vertex. We can find the correct vertex by trial and error, filling out the polygon with the method from Definition 5.1. All the diagonals in this heptagon are a part of the triangulation of the strip, namely they are all arcs between* $(0^\circ, 4_\circ), (-5^\circ, 4_\circ) \in \mathfrak{T}$. *We fill out similar diagrams for all other segments on the zigzag path to obtain the triangulation below.*



Note that for horizontal lines in the zigzag path in the SL_2-tiling you have a bit of a choice. You may choose to read a horizontal line as a diagonal from south west to north east in a frieze pattern (as it would appear as such in a diagonal band) or you may treat it exactly the same way as you would a vertical line. This is because of the properties that frieze patterns are repeated and mirrored fundamental regions. In other words, a diagonal $(a, b, c, ..., r)$ going from north west to south east starting with a, will also appear going from south west to north east with a as the bottom element. We also know that a diagonal going either way in a frieze determines the whole frieze so it matters not which option we choose.

It then remains only to show this map is the inverse of the map in Construction 8.1 and vice versa.

Theorem 12.2. *The maps Φ and Ψ from Construction 8.1 and Construction 12.1 respectively are inverse bijections between good triangulations of the strip and SL_2-tilings with enough ones.*

Proof. We intend to show $\Psi \circ \Phi$ is the identity on the triangulated strip. Let $\mathfrak{T}$ be a good triangulation of the strip. Let $t = \Phi(\mathfrak{T}), \mathcal{U} = \Phi(t)$. We want to show then, that $\mathcal{U} = \mathfrak{T}$. Note that t is an SL_2-tiling with enough ones by Proposition 8.2. Let $((x_\alpha)^\circ, (y_\alpha)_\circ)$ be the connecting arcs in $\mathfrak{T}$. Since t was created by Construction 8.1 we have that $t_{xy} = 1 \leftrightarrow (x, y) = (x_\alpha, y_\alpha)$ for some α by Remark 8.1. By Proposition 11.2 we see that the set of ones must be the zigzag pattern described. Note that the connecting arcs $((x_\alpha)^\circ, (y_\alpha)_\circ)$ must also be in $\mathcal{U}$ as $\mathcal{U} = \Psi(t)$, and Construction 12.1 takes the ones of t to diagonals in $\mathcal{U}$.

We now know that the maps move ones to diagonal arcs and back, and we need only consider the stuffing in between ones, namely the triangulation of finite polygons. Consider the subset of the strip restricted by $((x_\alpha)^\circ, (y_\alpha)_\circ), ((x_{\alpha+1})^\circ, (y_{\alpha+1})_\circ) \in \mathfrak{T}$. We know that either $x_\alpha = x_{\alpha+1}$ or $y_\alpha = y_{\alpha+1}$. We show the following only for the case that $x_\alpha = x_{\alpha+1}$ as both cases are handled similarly.

Now $y_\alpha < y_{\alpha+1}$. Let P be the polygon with r vertices, $x_\alpha, y_\alpha, y_\alpha + 1, ..., y_{\alpha+1}$. Construction 8.1 defines t by $t_{x,y} = \mathfrak{T}_Q(x, y)$ for some surrounding polygon Q in the strip. More specifically

$$t_{x_\alpha,y} = \mathfrak{T}_P(x_\alpha^\circ, y_\circ),\ y_\alpha \le y \le y_{\alpha+1}$$

Next we apply Ψ to $t_{x_\alpha,y}$ to obtain a triangulation $\mathcal{U}_P$ for the same polygon P as the vertices are $x_\alpha, y_\alpha, y_\alpha + 1, ..., y_{\alpha+1}$. We wish to show that the triangulation is equal, that is, to show the quiddity cycle is the same. Now

$$\mathcal{U}_P(x_\alpha^\circ, y_\circ),\ y_\alpha \le y \le y_{\alpha+1} = t_{x_\alpha,y}$$

so

$$\mathcal{U}_P(x_\alpha^\circ, y_\circ) = \mathfrak{T}_P(x_\alpha^\circ, y_\circ),\ y_\alpha \le y \le y_{\alpha+1}$$

which states the friezes corresponding to $\mathcal{U}_P$ and $\mathfrak{T}_P$ have one equal diagonal. Now by Remark 3.5 a diagonal determines the whole frieze, so the friezes corresponding to $\mathcal{U}_P$ and $\mathfrak{T}_P$ are equal. More specifically the second rows of the friezes are equal. Construction 4.2 creates the triangulations $\mathcal{U}_P = \mathfrak{T}_P$ by setting the second row as the quiddity cycle of P. $\square$

13. Appendix

In the first 5 sections we explored the connection between triangulated polygons and friezes. It would be interesting as further work to look for different patterns in triangulated polygons and see if they also appear in friezes, and vice versa. A possible next step for further study is to alter the rules that bind friezes. Friezes follow the unimodular rule. Expanding them to have infinitely many rows is essentially what SL_2-tilings are. This change gave rise to a plethora of new questions and answers. Should we find another way to change the restrictions on these patterns, they too might relate to a geometrical or combinatoric object in a similar fashion. This might also include looking at patterns similar to friezes that do not follow the unimodular rule.

References

[1] C. Broline, D.W. Crowe, and I.M. Isaacs. The geometry of frieze patterns. *Geometriae Dedicata*, 3:p.171–176, 1974.

[2] J. H. Conway and H. S. M. Coxeter. Triangulated polygons and frieze patterns. *The Mathematical Gazette*, 57(400):p. 87–94, 1973.

[3] J. H. Conway and H. S. M. Coxeter. Triangulated polygons and frieze patterns (continued). *The Mathematical Gazette*, 57(401):p. 175–183, 1973.

[4] T. Holm and P. Jørgensen. SL_2-tilings and triangulations of the strip. *Journal Of Combinatorial Theory Series A*, 120:p.1817–1834, 2013.