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Modelling of multimode and layer structure by transfermatrices

Helge Mjølhus Sveinsson

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Supervisor: Johannes Skaar, IET

Problem Description

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Modelling of multimode and layer structure by transfer matrices

Helge M. Sveinsson

helgemjo@stud.ntnu.no

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Faculty of Information Technology,
Mathematics and Electrical Engineering
Department of Physical Electronics

Preface

This report was written as my masters degree in electronics at the Norwegian University of Science and Technology (NTNU). I specialise in photonics, and I worked at the University Graduate Center (UniK), Kjeller. My thanks goes to Johannes Skaar and Guro Svendsen, for guidance and for supplying me with the vital Matlab code.



Summary

The intention of this report is twofold. The main goal is to find out if it is possible to increase the reflection at the end of a waveguide, by adding blocks of different refractive indices. This has been done by utilising a method called the Transfer Matrix Method. Most commonly, this method is used with a single mode, but this report will focus on a planar dielectric waveguides with several modes.

A code has been utilised to obtain matrices with reflection and transmission amplitudes, these have in turned been used with the Transfer Matrix method to obtain the reflection as a function of the number of blocks added to the waveguide. The results have been mixed. The transfer matrix method, as described in this report, seem to function perfectly for single modes, but with several modes, the results are less satisfactory, since it give impossible values for the reflection. These bad results may very well be tied to the inevitable use of pseudoinverse in the code.



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1 Introduction

With nanotechnology it is possible to manipulate matter down to the molecular level, in other words down to 10^{-9} m. It is then possible to be able to guide light on this level. This can be achieved by using something called nanowires[1]. A nanowire is basically a one dimensional nanostructure that can be synthesised by a chemical vapour process, and can be used as waveguides. Until now scientists and engineers have probably barely scratched the surface of what nanowires are capable of and how they can be applied. Among their potential applications are as components in optoelectronics electronics at the nano level. More specifically, this includes but is not limited to being components in lasers, optical waveguides and single photon sources [2]. It is worth noting that the wire can not only consist of many different materials, but through controlling the chemical precursors during growth one can also change the impurity doping profiles among other things. With such detailed control of the composition of the nanowire, one can produce specimens with a vast variety of properties.

As already mentioned, nanowires can be used as laser, called nanowire lasers. The length of this kind of laser is normally between $2 - 40\mu m$ and the width or diameter is between $20 - 400nm$ [3]. They are often made of materials with a big dielectric contrast between the surrounding materials, for example air. This results in the nanowire's ability to laterally confine electrons, holes and photons, and it is this ability that makes them very attractive to use in laser technology. In order to function properly as a laser, the end facets of the nanowire has to function as mirrors so that as much light as possible is reflected. For natural unaltered facets, however, the reflectivity is low, which means that there will be high losses during lasing. This can be prevented in several ways. One possible solution is by forming a distributed bragg reflector at the end facets of the wire. Another solution might be to add a media with a different refractive index at the end of the nanowire, and perhaps even add several layers of media of varying refractive index. To find out what kind of effect this will have on the reflective coefficient it is necessary to do simulations, using an appropriate program, for example Matlab. In this paper, one will use a premade matlab code to find the reflection and transmission coefficients of a transition from a medium with an arbitrary refraction index to vacuum[4]. The code in question will produce matrices of modal amplitudes, which will be used in something called the "Transfer Matrix Method" to find out how the reflection depends on the number of layers of media of varying refractivity. The "Transfer matrix method" will basically involve taking the reflection and transmission matrices produced in the code, and use them to make a so called "Transfer Matrix". One important characteristic of the Transfer Matrix is that by multiplying matrices of different refractions one can produce one single matrix that describes the reflection of the entire system. The fact

that one will work with large number of modes described by matrices, in stead of single digits, makes the calculations slightly different from the ones shown in some textbooks[5]. By using the code one can easily check the reflection coefficients as a function of the number of layers. However there are also other methods of finding the reflection. By using the Fresnel equations one can find the reflection between two media, and by using the Fabry-Perot one can find the reflection of a system of three media. These methods can therefore be used to test the Transfer Matrix Method.

2 Background theory

This is the theory behind the Matlab code and the Transfer Matrix method.

2.1 Simplifying the problem

Trying to do calculations on modes within a fiber can quickly become very complicated; luckily there are some methods of simplifying this. One has to keep in mind that modes through a fiber will change continuously, however this can be simplified by dividing the fiber into small sections, so small in fact that it is reasonable to assume that each section is uniform. It is then possible to apply multilayer optics to the modes, so that one can look at the reflectance and transmittance of the transitions between the layers and consider the sections between the transitions as homogenous. With the help of matrices it is possible to calculate a large number of modes at the same time. Using these assumptions one can use transfer matrices to do calculations on the modes(see fig 2.1). This is called the Transfer Matrix Method[5].

2.2 Transfer matrix method

2.2.1 Transfer matrix

One can call the forward moving vector a and the backward moving one b , and the vector in section 1 has the subscript 1 and the vector in section 2 has the subscript 2(see figure 2.1). The elements in the vectors are modal amplitudes. Writing it on matrix form, one will get the following situation[5]:

$$\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = M \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \quad (2.1)$$

where M is the transfer matrix, and the elements inside the matrix can be considered invertible matrices. As demonstrated on the picture, the vectors a_1 and b_2 go into the matrix, and a_2 b_1 go out. One important property of M , is that N different matrices can be multiplied together to form one transfer matrix representing the entire system:

$$M = M_N \dots M_2 M_1. \quad (2.2)$$

Much of this is material covered in text books like Saleh and Teich, but there is a key difference. In most books the elements inside the transfer matrix and the scattering matrix(which will be described shortly) are scalars. In this report however, one will work with matrices.

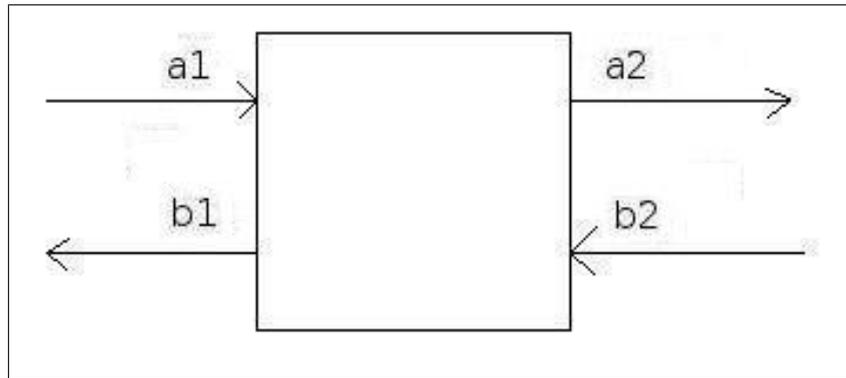


Figure 2.1:

2.2.2 Scattering matrix

There is also another important matrix, which is called scattering matrix S [5]:

$$\begin{bmatrix} b_1 \\ a_2 \end{bmatrix} = S \begin{bmatrix} a_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} R_1 & T^T \\ T & R_2 \end{bmatrix} \begin{bmatrix} a_1 \\ b_2 \end{bmatrix}. \quad (2.3)$$

The input of the transfer matrix is the backward and forward travelling waves of the left side, and the output is the corresponding waves on the right side. The input of the the scattering matrix on the other hand, is the incoming waves, and the output is the outgoing waves(see fig. 2.2). The big advantage with this method

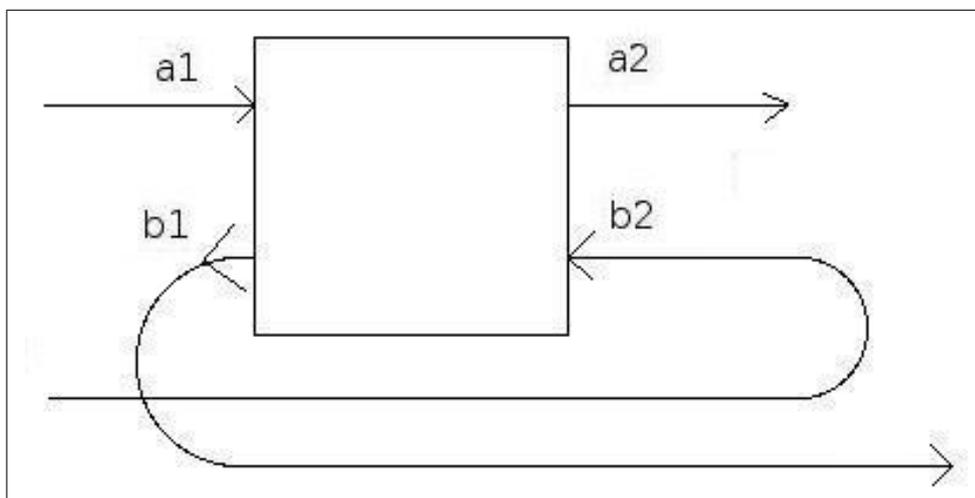


Figure 2.2:

is that the elements of the matrix have physical significance. R_1 and R_2 are the reflectances for waves incident from respectively the left and the right, while T and

T^T are the transmittances. The disadvantage with this matrix is that unlike the T matrix, one can not multiply a cascade of S-matrices to find a complete matrix of the system. It is therefore convenient to convert between S and T-matrices to exploit the advantages of each.

2.2.3 Properties of scattering matrix

In this case one assumes that the S-matrices are lossless and reciprocal, which makes it possible to obtain information about the elements in the matrix. One can first define

$$v = \begin{bmatrix} a_2 \\ b_1 \end{bmatrix}, u = \begin{bmatrix} a_1 \\ b_2 \end{bmatrix}. \quad (2.4)$$

And assuming that the S-matrix is lossless, one can then write

$$v = Su, \quad (2.5)$$

where u is the modes in, v is the modes out and S is the S-matrix. However since there is an unknown number of modes in and out, it is more useful to write the a and b as vectors as also mentioned above:

$$|u_1|^2 + |u_2|^2 + \dots = u^{*T}u, \quad (2.6a)$$

$$|v_1|^2 + |v_2|^2 + \dots = v^{*T}v. \quad (2.6b)$$

So using vectors in 2.4, one can write

$$v^{*T}v = Su^{*T}u. \quad (2.7)$$

So, combining 2.4 with 2.6, one can write

$$(Su)^{*T}Su = u^{*T}u \quad (2.8a)$$

$$u^{*T}S^{*T}Su = u^{*T}u \quad (2.8b)$$

$$u^{*T}(S^{*T} - I)u = 0 \quad (2.8c)$$

One have then shown that the S-matrix is unitary, by assuming that it is lossless:

$$S^{*T}S = I. \quad (2.9)$$

Secondly, because the matrix is reciprocal, one can assume that the matrix is symmetrical:

$$S = S^T. \quad (2.10)$$

The fact that the matrix is both unitary and symmetric gives it the useful property that it is possible calculate the numbers in the second column, from only the numbers in the first column. From the symmetry property one knows that

$$\begin{bmatrix} R_1 & T^T \\ T & R_2 \end{bmatrix} = \begin{bmatrix} R_1^T & T^T \\ T & R_2^T \end{bmatrix} \quad (2.11)$$

From the above definition of unitary matrices one can find out that

$$\begin{bmatrix} R_1^* & T^{*T} \\ T^* & R_2^* \end{bmatrix} \begin{bmatrix} R_1 & T^T \\ T & R_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (2.12)$$

which can also be written

$$\begin{bmatrix} (R_1^*)R_1 + (T^{*T})T & (R_1^{*T})T^T + (T^{*T})R_2 \\ (T^*)R_1 + (R_2^*)T & (T^*)T^T + (R_2^*)R_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (2.13)$$

From 2.13 one can see find R_2 in two different ways:

$$R_2^* = -(T^*)R_1(T)^{-1}, \quad (2.14a)$$

$$R_2 = (T^{*T})^{-1}R_1^{*T}T^T. \quad (2.14b)$$

where * signifies the complex conjugate. It is preferable to use 2.14a since this does not require the matrix to be transposed, which can create a problem because of the pseudoinverse.

2.2.4 Relation between the scattering and transfer matrix

In order to convert between S and M matrices one must first know the relation between them. This can be done by comparing 2.1 and 2.3. Firstly, equation 2.3 can be written out like this:

$$b_1 = R_1a_1 + T^Tb_2, \quad (2.15a)$$

$$a_2 = Ta_1 + R_2b_2. \quad (2.15b)$$

It can then be rewritten

$$a_2 = a_1(T - R_2(T^T)^{-1}R_1) + b_1R_2(T^T)^{-1}, \quad (2.16a)$$

$$b_2 = -a_1(T^T)^{-1}R_1 + b_1(T^T)^{-1}, \quad (2.16b)$$

and placed into an M-matrix.

$$M = \begin{bmatrix} T - R_2(T^T)^{-1}R_1 & R_2(T^T)^{-1} \\ -(T^T)^{-1}R_1 & (T^T)^{-1} \end{bmatrix} \quad (2.17)$$

This can also be done the other way. One begins by writing out the M-matrix:

$$a_1 = (M_{12} - M_{11}(M_{21})^{-1}M_{22})b_1 + M_{11}M_{21}^{-1}b_2 \quad (2.18)$$

$$a_2 = M_{21}^{-1}b_2 - (M_{21})^{-1}M_{22}b_1 \quad (2.19)$$

Finally one gets a matrix, that can convert from M-matrix to S-matrix.

$$S = \begin{bmatrix} -M_{22}^{-1}M_{21} & M_{22}^{-1} \\ (M_{22}^T)^{-1} & M_{12}M_{22}^{-1} \end{bmatrix} \quad (2.20)$$

2.2.5 Propagation matrix

In order to calculate to the complete matrix for the medium one has to calculate the matrices for the individual sections. Until now we have covered the matrices representing the transitions between the layers, however, one also has to find the matrices representing propagation in a constant medium. Continuing with the convention of using 2×2 matrices one has to imagine homogenous medium with the width d and the refraction index n . From [5] we know that $a_1 = e^{i\varphi}b_1$ and $a_2 = e^{i\varphi}b_2$. Using these equations one gets the matrices

$$M_p = \begin{bmatrix} \exp(i\varphi) & 0 \\ 0 & \exp(-i\varphi) \end{bmatrix}, S_p = \begin{bmatrix} \exp(i\varphi) & 0 \\ 0 & \exp(i\varphi) \end{bmatrix}, \varphi = nk_0d, \quad (2.21)$$

where d is the length of the uniform section and n is the refractive index. Like before, M is the transfer matrix and S is the scattering matrix. This can easily be transformed into matrices bigger than 2×2 matrices by using the relation $a_n = e^{i\varphi}b_n$ [6].

2.2.6 Finding the total reflection

Firstly one has to construct the S-matrices, which can easily be done as described in section 2.2.2, and in 2.2.5 in case of the propagation matrix. Secondly one has to convert these into M-matrices by using 2.17. As explained in the following section, the matrices produced in the code describes the modes going from a medium with refraction index n to a medium with refraction index 1. Since the matrix is reciprocal, the inverse of the matrix describes the modes going from a medium with refraction 1 and into one with refraction n . One then has to multiply these matrices like shown in 2.2. By taking all these things into considerations one can find the complete matrix for a wave guide with refraction index varying between n_1 and n_2 , where each section has a length d , given by the propagation matrix. This has been done in equation 2.22:

$$M = \dots M_{n_1} M_p (M_{n_1})^{-1} M_{n_2} M_p (M_{n_2})^{-1} M_{n_1}, \quad (2.22)$$

where M_{n_1} and M_{n_2} are the M-matrices representing the transitions from a medium with refraction indices n_1 and n_2 to vacuum. Like in 2.21 M_p is the propagation matrix.

2.3 Planar waveguide

The following is a description of how the Matlab code mentioned earlier, works. Normally the cross section of a waveguide is circular. This can sometimes be a bit harder to work with, compared to other kinds of cross sections, like that of a planar waveguides. One can therefore assume a planar waveguide with a thickness $2a$, with waves propagating in the z -axis, like shown in fig 2.3 [4]. The

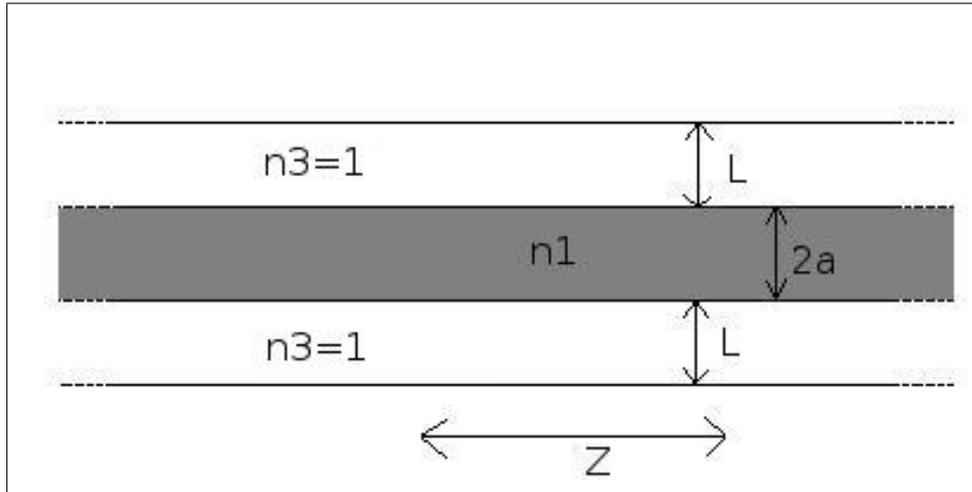


Figure 2.3:

core ($-a < x < a$) has a refractive index n , while outside the core there is vacuum. In order to discretize the modes, the waveguide has to be encapsulated by mirrors on either side, at a distance $\pm L$ where $L \gg a$. At $z = 0$ the waveguide ends, which means that in the region $z > 0$ everything is vacuum. The modal fields calculated with these conditions can be divided into two groups (TE and TM), depending on their polarisation, and these groups can again be divided into even and odd modes. These four groups of modes will not couple with each other through $z=0$ [x], so they can be studied separately.

The goal is now to find out how a mode couples through a $z=0$. As an example one can use the even modes with TE polarisation.

2.3.1 Modes in the $z > 0$ region

For $z > 0$ the electric field behaves according to

$$\mathbf{e}_{TE}^m = \mathbf{A}^m \cos(k_x^m x) \exp(ik_z^m z) \hat{\mathbf{y}}, \quad (2.23)$$

where

$$k_x^m = \pi/2L + m\pi/L, \quad (2.24)$$

$$k_z^{2m} = \omega^2/c^2 - k_x^{2m}, \quad (2.25)$$

and the normalisation constant is defined as

$$A^m = 1/\sqrt{|k_z^m|L}. \quad (2.26)$$

The corresponding transversal component of the magnetic field is given by

$$\mathbf{h}_{TE}^m \cdot \hat{\mathbf{x}} = -\frac{A^m k_z^m}{\omega\mu} \cos(k_x^m x) \exp(ik_z^m z) \hat{\mathbf{y}}. \quad (2.27)$$

It can then be shown that

$$\omega\mu \int_{-L}^L \mathbf{e}_{TE}^m \cdot \mathbf{h}_{TE}^{m'*} \cdot \hat{\mathbf{z}} dx = \frac{k_z^{m*}}{|k_z^{m*}|} \delta_{mm'}. \quad (2.28)$$

Then we will do the same for the $z < 0$.

2.3.2 Modes in the $z < 0$ region

Since $z < 0$ consists of different regions, the core and outside of the core, the solution will be a bit different. One noticeable difference is that the modes behave differently in $0 < x < a$ and $a < x < L$. Continuing working with even modes with TE polarisation, the electric field for the region $0 < x < a$ is $B \cos(k'_x x)$ and for $a < x < L$ it is $C \sin(k_x x + \varphi)$. Here

$$k_x'^2 = \frac{\omega n^2}{c^2} - \beta^2, \quad (2.29)$$

$$k_x^2 = \frac{\omega}{c^2} - \beta^2. \quad (2.30)$$

while B , C and φ are real constants.

One also has to take into account that the modes will vary in the z -direction with $\exp(i\beta z)$. For the modes to exist in the $z < 0$ region, certain requirements need to be met. The first is that the electric field at the L , which is the boundary between the vacuum and the mirror is 0, in other words $C \sin(k_x x + \varphi)$. Conditions number two and three state that both the electric field and the derivative of the electric field with respect to x , must be continuous at $x = a$. Condition number one can be condensed down to

$$\varphi = -k_x L, \quad (2.31)$$

while condition two can be written

$$C \sin(k_x a + \varphi) = B \cos(k'_x a), \quad (2.32a)$$

$$\frac{C}{B} = \frac{\cos(k'_x a)}{\sin(k_x a + \varphi)}, \quad (2.32b)$$

and number three can be written

$$Ck_x \cos(k_x a + \varphi) = -Bk'_x \sin(k'_x a) \quad (2.33a)$$

$$\frac{C}{B} = \frac{k'_x \sin(k'_x a)}{k_x \cos(k_x a + \varphi)} \quad (2.33b)$$

By combining these three conditions, one gets the following equation

$$\frac{C}{B} = \frac{\cos(k'_x a)}{\sin(k_x a + \varphi)} = -\frac{k'_x \sin(k'_x a)}{k_x \cos(k_x a + \varphi)}, \quad (2.34a)$$

$$\tan[k_x(L - a)]\tan(k'_x a) = \frac{k_x}{k'_x}. \quad (2.34b)$$

The electric field \mathbf{e} can then be expressed:

$$\mathbf{e}^j = \frac{1}{\sqrt{|\beta^j|}} \psi^j(x) \exp(i\beta^j z) \hat{\mathbf{y}}. \quad (2.35)$$

Here

$$\psi^j(x) = \begin{cases} D^j \cos(k'_x{}^j x) & \text{for } 0 < x < a \\ E^j \sin[k_x^j(L - x)] & \text{for } a < x < L \end{cases} \quad (2.36)$$

and

$$E^j = D^j \frac{\cos(k'_x{}^j a)}{\sin[k_x^j(L - a)]}. \quad (2.37)$$

The transversal component of the magnetic field can be written

$$\mathbf{h}^j \cdot \hat{\mathbf{x}} = -\frac{\beta^j}{\omega\mu\sqrt{|\beta^j|}} \psi^j(x) \exp(i\beta^j z) \hat{\mathbf{y}}. \quad (2.38)$$

D^j is a normalisation constant and is chosen so that

$$\omega\mu \int_{-L}^L \mathbf{e} \times \mathbf{h}^{h'*} \cdot \hat{\mathbf{z}} dx = \frac{\beta^{j*}}{|\beta^j|} \int_{-L}^L \phi^{j*}(x) \phi^{j'}(x) = \frac{\beta^{j*}}{|\beta^j|} \delta_{jj'} \quad (2.39)$$

2.3.3 Coupling through the $z = 0$ boundary

The next goal is to find out how the modes behave through the boundary $z = 0$. Assuming the incoming wave is e^i and the using the Fresnel equation we get:

$$\mathbf{e}^i + \sum_j r_{ij} \mathbf{e}^j = \sum_m t_{im} \mathbf{e}^m. \quad (2.40)$$

Since $\exp(i\beta z)$ is on both sides cancel each other out, one can write

$$\frac{1}{\sqrt{|\beta^i|}} + \sum_j r_{ij} \frac{1}{\sqrt{|\beta^i|}} = \sum_m t_{im} \frac{1}{\sqrt{|k_z^m|L}} \cos(k_x^m x). \quad (2.41)$$

The corresponding equations for the transversal magnetic fields are

$$\mathbf{h}^i + \sum_j r_{ij} \mathbf{h}^j = \sum_m t_{im} \mathbf{h}^m, \quad (2.42)$$

and

$$\frac{\beta^i}{\sqrt{|\beta^i|}} + \sum_j r_{ij} \frac{\beta^j}{\sqrt{|\beta^i|}} = \sum_m t_{im} \frac{k_z^m}{\sqrt{|k_z^m|L}} \cos(k_x^m x). \quad (2.43)$$

2.3.4 The reflection matrix

In order to find the reflection matrix, one can combine 2.39 with 2.43.

$$\omega\mu \int_{-L}^L \mathbf{e}^i \times \mathbf{h}^{m'*} \cdot \hat{\mathbf{z}} dx + \omega\mu \int_{-L}^L \sum_j r_{ij} \mathbf{e}^j \times \mathbf{h}^{m'*} \cdot \hat{\mathbf{z}} dx = \omega\mu \int_{-L}^L \sum_m r_{im} \mathbf{e}^m \times \mathbf{h}^{m'*} \cdot \hat{\mathbf{z}} dx \quad (2.44)$$

Written fully out the equation will be:

$$\begin{aligned} & \omega\mu \int_L^L \frac{1}{\sqrt{|k_z^i|L}} \cos(k_x^i x) \times -\frac{\beta^{m*}}{\omega\mu\sqrt{|\beta^m|}} \psi^{m*} dx \\ & + \omega\mu \int_L^L \sum_j r_{ij} \frac{1}{\sqrt{|k_z^j|L}} \cos(k_x^j x) \times -\frac{\beta^{m*}}{\omega\mu\sqrt{|\beta^m|}} \psi^{m*} dx \\ & = \omega\mu \int_L^L \sum_m t_{im} \frac{1}{\sqrt{|\beta^m|L}} \psi^m \times -\frac{\beta^{m*}}{\omega\mu\sqrt{|\beta^{m*}|}} \psi^{m*} dx \end{aligned} \quad (2.45)$$

This can also be written:

$$\begin{aligned} & \frac{\omega\mu}{\omega\mu} \cdot -\beta^{m*} \int_L^L \frac{1}{\sqrt{|k_z^i||\beta^m|L}} \cos(k_x^i x) \psi^{m*} dx \frac{\omega\mu}{\omega\mu} \\ & - \beta^{m*} \sum_j r_{ij} \int_L^L \frac{1}{\sqrt{|k_z^j||\beta^{m*}|L}} \cos(k_x^j x) \psi^{m*} dx \\ & = -\frac{\beta^m}{|\beta^m|} t_{im} \int_L^L \psi^m \psi^{m*} dx. \end{aligned} \quad (2.46)$$

By using the definition

$$\Psi_{jm} = \frac{1}{\sqrt{|\beta^m| |k_z^j| L}} \int_{-L}^L \psi^m(x) \cos(k_x^j x) dx \quad (2.47)$$

one can rewrite 2.47 as

$$\beta^{m*} \Psi_{im} + \sum_j r_{jm} \beta^{m*} \Psi_{jm} = t_{im} \frac{\beta^{m*}}{|\beta^m|}. \quad (2.48)$$

From 2.48, it is easy to find the reflection matrix.

2.4 Using the Matlab code

2.4.1 Obtaining the solution

As mentioned in the introduction it is interesting to find out what exactly happens to the reflection in the end of the waveguide if one adds media of different refractive index at the end. This will in practice mean that one adds several mirrors to the end of the waveguide. One can therefore use the information about the multilayer optics above to calculate the the reflectance and end transmittance of the entire system. By using the reflection coefficient between the individual layers one can construct scattering matrices. After converting them into transfer matrices one can multiply them together as explained earlier. However due to the functioning of the Matlab program used, in stead of finding the reflection and transmission coefficients between the two media, we will find the coefficients between a medium 1 and vacuum, and between medium 2 and vacuum(see fig 2.4). Converting the two resulting matrices into transfer matrices and then simply multiply them, we assume that the distance between the two mirrors are zero. We will then get the desired transfer matrix.

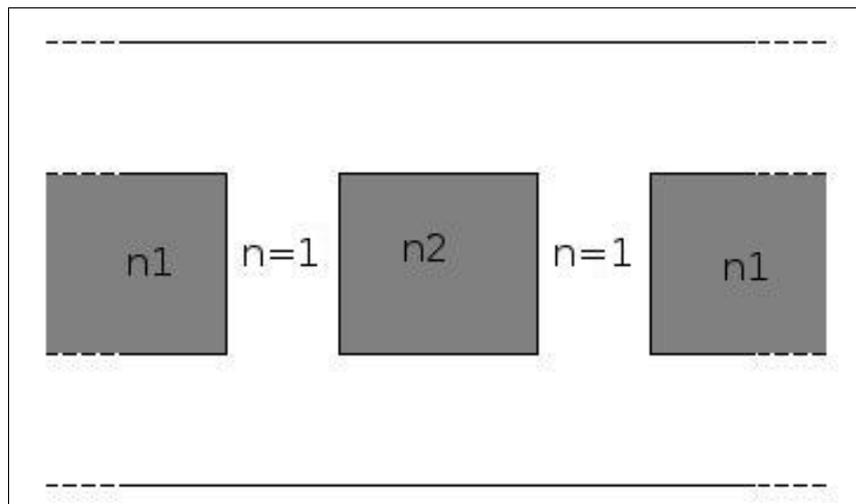


Figure 2.4:

2.4.2 The output data

It would then be interesting to see what kind of output data the code produces.

Firstly one gets the reflection matrix R_1 . This represents the reflection of the transition from the material with refraction index n_1 , to vacuum with refraction 1. This will be an $m \times m$ matrix, where m is the number of modes in the waveguide,

and includes both bound and radiation modes. Secondly there is the transmission matrix T which is a $k \times m$ matrix[4]. It describes the transition from the waveguide material to vacuum, and k represents the free modes found in vacuum where the precise number can be adjusted in the code. The transition matrix T^T represents the modes going the opposite way, from vacuum to the waveguide. The matrix R_2 describes the reflection from the vacuum to the waveguide material, and therefore is of the size $k \times k$.

2.5 The Moore-Penrose pseudoinverse

In order to do some of the calculations one needs to inverse certain matrices. In the majority of cases, it is impossible to use a normal inverse function on the matrices one is supposed to inverse. This is due to that the matrices are not square, which is a prerequisite for the inverse function[7]. In that case one can use the Moore-Penrose pseudoinverse, which has some of the same properties as the inverse function, and can be used on non square matrices. For a matrix B to be the pseudoinverse of matrix A , the following conditions has to be met:

$$A * B * A = A, \tag{2.49a}$$

$$B * A * B = B, \tag{2.49b}$$

$$A * B \text{ is Hermitian}, \tag{2.49c}$$

$$B * A \text{ is Hermitian}. \tag{2.49d}$$

Here B has the same dimensions as A' .

In many cases the solution of $Ax = b$ is not unique, but have an infinite amount of solutions. This happens when A has more rows than columns and is rank deficient. Otherwise, the pseudoinverse is the pseudoinverse is the unique solution. This can prove to be a problem when one transposes the matrices, because by transposing the matrix, it will no longer have full rank and there will be problems with the pseudoinverse. This is a problem which will have to be solved, by using the pseudoinverse as little as possible.

2.6 Redefining the the Amplitude parameters

The Matlab code which is used to find the reflection and transmission coefficients, produces coefficients for the E-field, and have the subscripts E_1 and E_2 . These coefficients however, cannot be used directly in the Transfer Matrix Method, since the modes used in the method, are a set of field amplitudes called a and b . By using the following method, one can translate the coefficients R_{E_1} , R_{E_2} , T_{E_1} and T_{E_2} into R_1 , R_2 , T_1 and T_2 . Firstly, one can define the scattering matrix for the E-field:

$$\begin{bmatrix} E_{b_1} \\ E_{a_2} \end{bmatrix} = \begin{bmatrix} R_{E_1} & T_{E_2} \\ T_{E_1} & R_{E_2} \end{bmatrix} \begin{bmatrix} E_{a_1} \\ E_{b_2} \end{bmatrix} \quad (2.50)$$

One possible way to translate the coefficients from the E-field into coefficients we can use, is to use the method shown in [6]. Here a and b can be redefined as

$$b_1 = E_{b_1}/\sqrt{Z_1}, \quad (2.51a)$$

$$a_1 = E_{a_1}/\sqrt{Z_1}, \quad (2.51b)$$

$$a_2 = E_{a_2}/\sqrt{Z_2}, \quad (2.51c)$$

$$b_2 = E_{b_2}/\sqrt{Z_2}, \quad (2.51d)$$

where Z_m is the characteristic impedance of the material. This can be inserted in 2.50:

$$\begin{bmatrix} \sqrt{Z_1}b_1 \\ \sqrt{Z_2}a_2 \end{bmatrix} = \begin{bmatrix} R_{E_1} & T_{E_2} \\ T_{E_1} & R_{E_2} \end{bmatrix} \begin{bmatrix} \sqrt{Z_1}a_1 \\ \sqrt{Z_2}b_2 \end{bmatrix}. \quad (2.52)$$

This can be multiplied into the S-matrix by

$$\begin{bmatrix} b_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} R_{E_1}/\sqrt{Z_1} & T_{E_2}/\sqrt{Z_1} \\ T_{E_1}/\sqrt{Z_2} & R_{E_2}/\sqrt{Z_2} \end{bmatrix} \begin{bmatrix} \sqrt{Z_1}a_1 \\ \sqrt{Z_2}b_2 \end{bmatrix} \quad (2.53)$$

and

$$\begin{bmatrix} b_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} (\sqrt{Z_1}/\sqrt{Z_1})R_{E_1} & (\sqrt{Z_2}/\sqrt{Z_1})T_{E_2} \\ (\sqrt{Z_1}/\sqrt{Z_2})T_{E_1} & (\sqrt{Z_2}/\sqrt{Z_2})R_{E_2} \end{bmatrix} \begin{bmatrix} a_1 \\ b_2 \end{bmatrix}. \quad (2.54)$$

And then be rewritten as

$$\begin{bmatrix} b_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} R_{E_1} & (\sqrt{Z_2}/\sqrt{Z_1})T_{E_2} \\ (\sqrt{Z_1}/\sqrt{Z_2})T_{E_1} & R_{E_2} \end{bmatrix} \begin{bmatrix} a_1 \\ b_2 \end{bmatrix}. \quad (2.55)$$

The characteristic impedance can be set equal to the wave impedance and it therefore depends on refractive index[6]:

$$Z_m = \frac{\eta_0}{\eta_m} = \frac{n_m}{n_0}. \quad (2.56)$$

Lastly, 2.54 can be inserted into 2.53 and one gets

$$\begin{bmatrix} R_1 & T_2 \\ T_1 & R_2 \end{bmatrix} = \begin{bmatrix} R_{E_1} & \sqrt{n_1/n_2}T_{E_2} \\ \sqrt{n_2/n_1}T_{E_1} & R_{E_2} \end{bmatrix}, \quad (2.57)$$

and one can see that T_1 and T_2 depend on the refractive index of both sides.

2.7 Testing the calculations

2.7.1 Fresnel

To find the reflection between two surface, it is possible to use the Fresnel equations [5]

$$r_x = \frac{n_1 \cos \varphi_1 - n_2 \cos \varphi_2}{n_1 \cos \varphi_1 + n_2 \cos \varphi_2}. \quad (2.58a)$$

$$r_y = \frac{n_1 \sec \varphi_1 - n_2 \sec \varphi_2}{n_1 \sec \varphi_1 + n_2 \sec \varphi_2}, \quad (2.58b)$$

where n_1 and n_2 are the refractive indices of the two media, while φ is the angle of incidence. The coefficients r_x is the complex amplitude reflectance for TE polarisation, while r_y is the complex reflectance for TM polarisation. Another important value is the power reflectance. It is defined as the ratio of power flow between the reflected wave and the incident wave. One can then define the reflectance to be:

$$R = |r|^2, \quad (2.59)$$

In the calculations it is assumed that the incident angle is normal to the surface, which means that the power reflectance in all cases can be written as

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2. \quad (2.60)$$

2.7.2 Fabry-perot

A Fabry-perot etalon is a kind of interferometer which consists of two mirrors, parallel to each other and highly reflective[5]. The light transmitted from it depends on the optical pathlength between the two mirrors, so that it can among other things be used as a filter. It is also assumed that the system is reciprocal and lossless. The entire Fabry-perot system can be described with matrices like this:

$$M = \begin{bmatrix} t_1 - \frac{(-r_1)r_1}{t_1} & \frac{-r_1}{t_1} \\ \frac{-r_1}{t_1} & \frac{1}{t_1} \end{bmatrix} \begin{bmatrix} \exp(-j\theta) & 0 \\ 0 & \exp(j\theta) \end{bmatrix} \begin{bmatrix} t_2 - \frac{(-r_2)r_2}{t_2} & \frac{-r_2}{t_2} \\ \frac{-r_2}{t_2} & \frac{1}{t_2} \end{bmatrix}. \quad (2.61)$$

The reflection r_1 is defined as the reflection between n_1 and n_2 from the right, and r_2 is the reflection between n_2 and n_3 from the left:

$$r_1 = \frac{n_1 - n_2}{n_2 + n_1}, \quad (2.62a)$$

$$r_2 = \frac{n_2 - n_3}{n_3 + n_2}. \quad (2.62b)$$

These reflections are illustrated in fig 2.5. In order to find the total reflection one has to add all the reflections. The first reflection, is the reflection between n_1 and n_2 from the left, which is simply r_1 . The other reflections are a bit more

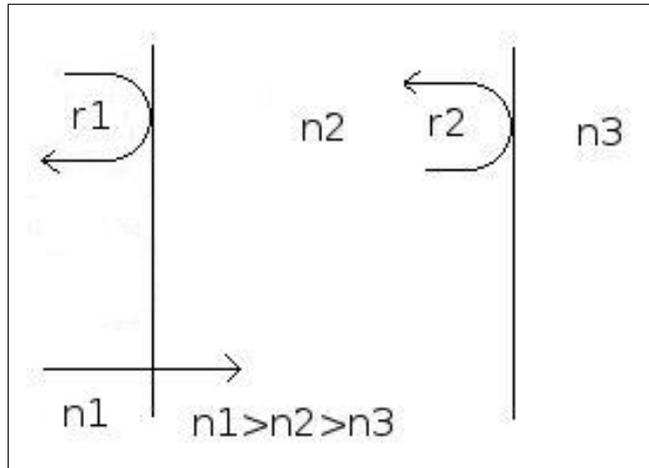


Figure 2.5:

complicated to explain. The second reflection can be described by the following equation where $\varphi = nk_0d$:

$$r_{01} = t_1 t_2 r^2 \exp(2j\phi). \quad (2.63)$$

The wave first has to cross the border between n_1 and n_2 , then it must traverse the medium and reflect at the other side. Then it must traverse the medium again and cross the $n_1 - n_2$ border a second time, but from the other side. One can call the sum of all the reflections that has traversed the medium r_t . Then one can write:

$$r_t = r_{01}(1 + f + f^2 \dots). \quad (2.64)$$

Here $f = r_1 r_2 \exp(-2i\phi)$, where the two reflections represent the internal reflections of the n_2 borders. By using the identity $1 + f + f^2 + f^3 \dots = 1/(1 - f)$, one can obtain the equation:

$$r = r_1 + \frac{t_1 t_2 r_2 \exp(-2i\phi)}{1 + r_1 r_2 \exp(-2i\phi)}. \quad (2.65)$$

Here t_1 and t_2 is the transmission to the left and right. They can be written $t_1 = 1 - r$ and $t_2 = 1 + r$. One can then use the equation $1 - r^2 = (1 - r)(1 + r)$ and write

$$r = r_1 + \frac{(1 - r_1^2) r_2 \exp(-2i\phi)}{1 + r_1 r_2 \exp(-2i\phi)}. \quad (2.66)$$

To find the reflection, one can use $R = |r|^2$.

It can also be interesting to note that if that area traversed is one wavelength, or in other words if $\varphi = \pi$, the resulting reflection coefficient will always be equal to $(n_1 - n_3)/(n_1 + n_3)$. This can easily be proven by substituting 2.62 into 2.66:

$$\frac{n_1 - n_2}{n_1 + n_2} + \frac{\frac{n_3 - n_2}{n_3 + n_2} - \left(\frac{n_1 - n_2}{n_2 + n_1}\right)^2 \frac{n_2 - n_3}{n_2 + n_3}}{1 + \frac{n_2 - n_3}{n_2 + n_3} \frac{n_1 - n_2}{n_1 + n_2}}. \quad (2.67)$$

This can be reduced to

$$\frac{n_1 - n_2}{n_1 + n_2} + \frac{4n_2^2 n_1 - 4n_3 n_2 n_1}{2n_1^2 n_2 + 2n_3 n_2 n_1 + 2n_1 n_2^2 + 2n_2^2 n_3}, \quad (2.68)$$

which can again be reduced into:

$$\frac{n_1^2 - 2n_1 n_3 + n_2 + n_3}{n_1^2 + n_1 n_3 + n_2 n_1 + n_2 n_3}. \quad (2.69)$$

This can lastly be written as

$$\frac{(n_1 - n_3)(n_1 + n_2)}{(n_1 + n_3)(n_1 + n_2)}, \quad (2.70)$$

which can be reduced to $(n_1 - n_3)/(n_1 + n_3)$

3 Testing with one mode

In order to verify that the code is correct, it is also useful test it with just one mode. In stead of using a matrix as input, the reflection is given as scalars, where $r_1 = (n_3 - n_1)/(n_3 + n_1)$ and $r_2 = (n_3 - n_2)/(n_3 + n_2)$. The left going and right going transmission coefficients for r_1 are given by $\sqrt{n_1/n_3}(1 - r_1)$ and $\sqrt{n_3/n_1}(1 + r_1)$ and the corresponding coefficients for r_2 is $\sqrt{n_3/n_2}(1 - r_1)$ and $\sqrt{n_2/n_3}(1 + r_1)$. In all the following simulations, the reflectance is on the y axis, and the x axis represents the number of blocks with different refractive index, as demonstrated in fig 3.1. In other words fig 3.1 shows how the blocks are added to the waveguide, with the number to the right being the value of the x-axis. The thickness of these blocks are all half a wavelength, except when n_2 is at the end, then the last block is an entire wavelength in thickness. This prevents the reflection of fluctuating. The results from these tests, conform perfectly to the testing with Fabry Perot and Fresnel.

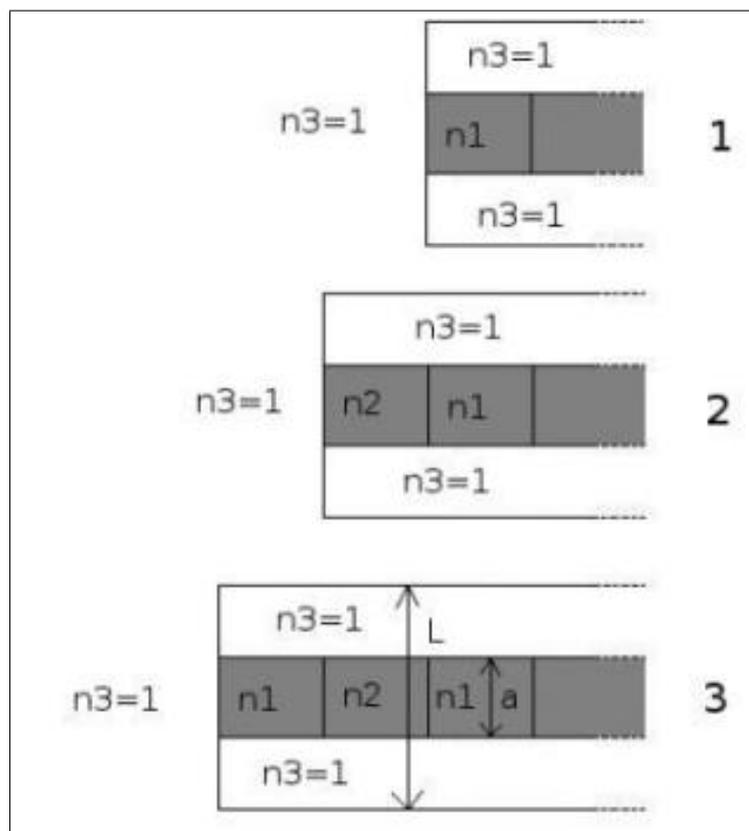


Figure 3.1:

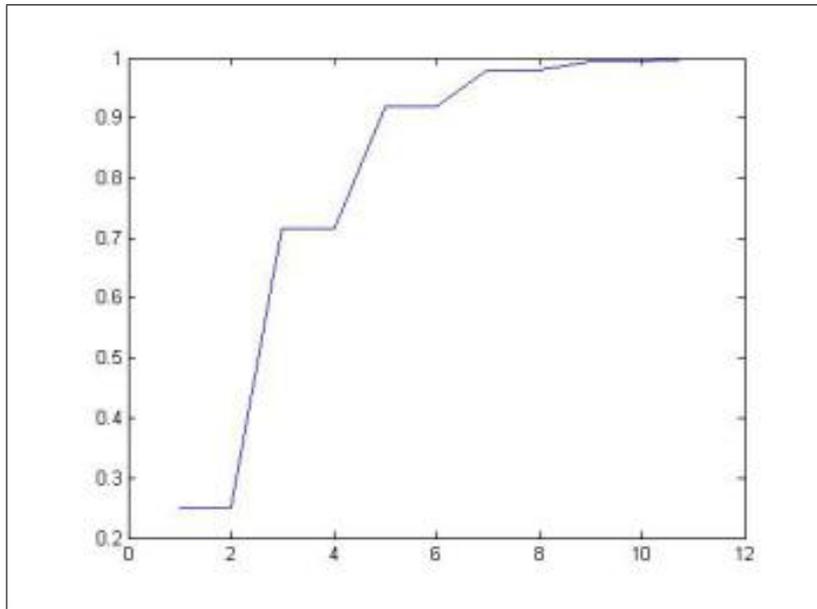


Figure 3.2: $n_1 = 3$ and $n_2 = 1.5$

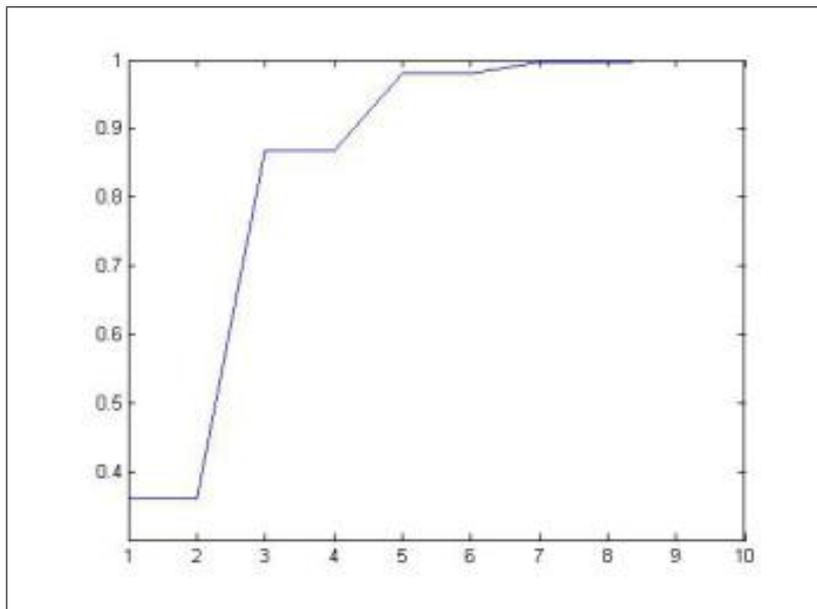
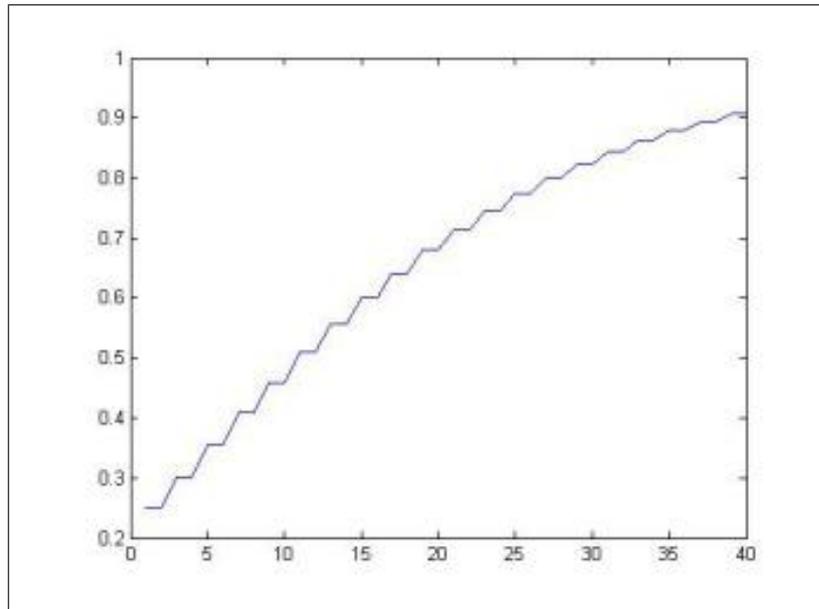
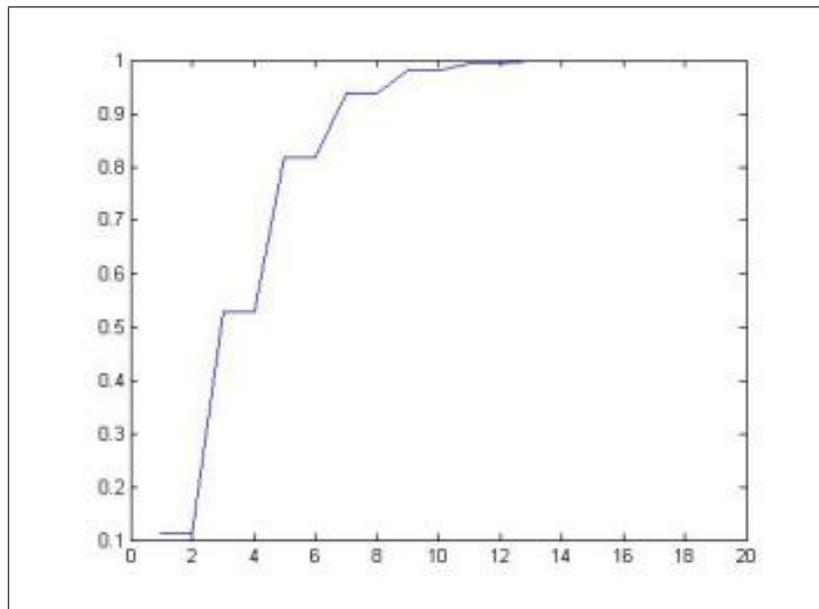


Figure 3.3: $n_1 = 4$ and $n_2 = 1.5$

Figure 3.4: $n_1 = 3$ and $n_2 = 2.8$ Figure 3.5: $n_1 = 2$ and $n_2 = 1.5$

4 Simulations

The following graphs show the reflection as a function the number of blocks of varying refraction index as described in section 2.4.1. Like mentioned in the theory, the waveguide is assumed to be planar. The length of the section will be a half wavelength, or $\varphi = \pi/2$. Like in section 3.3, the last section of a measurement with an even number(n) of blocks is one wavelength long. This causes the the reflection coefficient to be identical to the reflection of a waveguide $n - 1$ sections, and all lines from odd to even sections will be flat. In the following simulations the width of the waveguide, and the number of modes is shown for each figure. The number of modes in the waveguide depend on the width of the guide, while the number of free modes can be adjusted in the code. If the number of free modes and the number of modes in the waveguide are the same, the matrices will be quadratical. Fig 4.3 and 4.4 have quadratical matrices, but fig 4.3 was made only utilising the bound modes. The values L and a correspond the the values in fig 3.1.

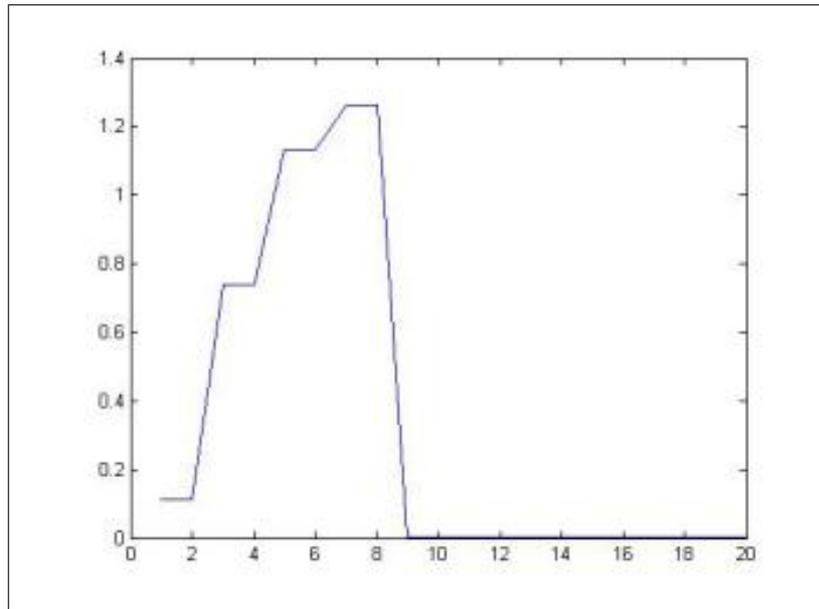


Figure 4.1:

$$a = 20nm$$

$$L = 100nm$$

Number of free modes: 200

$$n_1 = 2$$

Number of modes in n_1 : 104

Number of bound modes in n_1 : 12

$$n_2 = 1.5$$

Number of modes in n_2 : 102

Number of bound modes in n_2 : 8

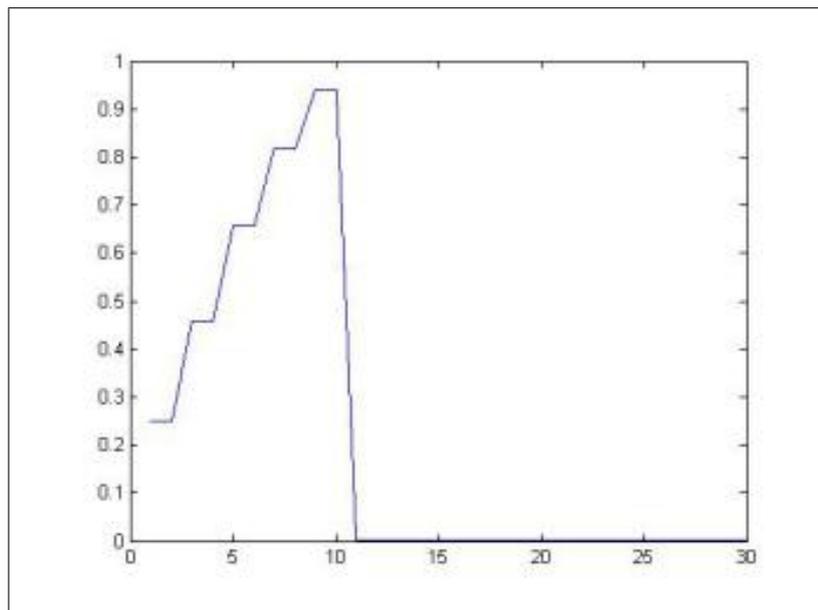


Figure 4.2:

$$a = 20nm$$

$$L = 100nm$$

Number of free modes: 200

$$n_1 = 3$$

Number of modes in n_1 : 108

Number of bound modes in n_1 : 19

$$n_2 = 2.8$$

Number of modes in n_2 : 107

Number of bound modes in n_2 : 17

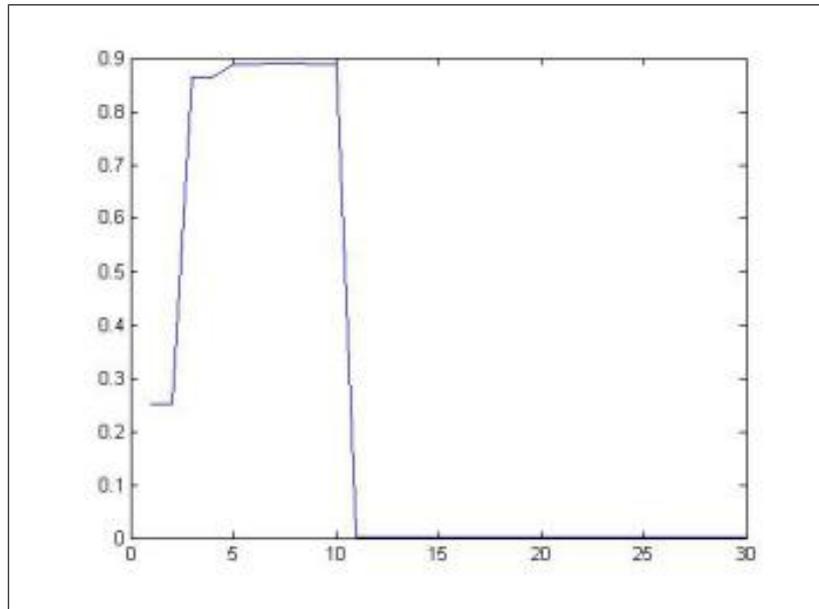


Figure 4.3:

$$a = 20nm$$

$$L = 100nm$$

Number of free modes: 200

$$n_1 = 3$$

Number of modes in n_1 : 108

Number of bound modes in n_1 : 19

$$n_2 = 1.5$$

Number of modes in n_2 : 102

Number of bound modes in n_2 : 8

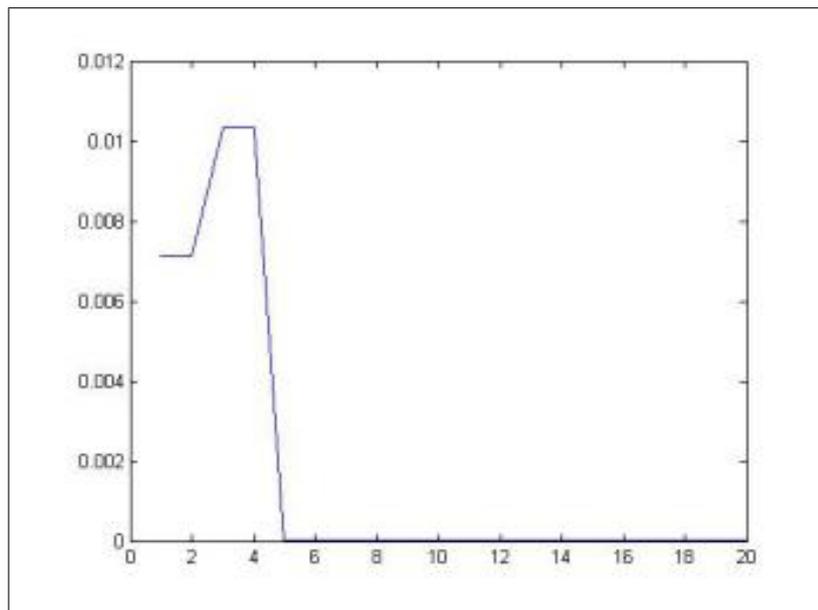


Figure 4.4:

$$a = 8nm$$

$$L = 40nm$$

Number of free modes: 8

$$n_1 = 5$$

Number of modes in n_1 : 8

Number of bound modes in n_1 : 8

$$n_2 = 4.9$$

Number of modes in n_2 : 8

Number of bound modes in n_2 : 8

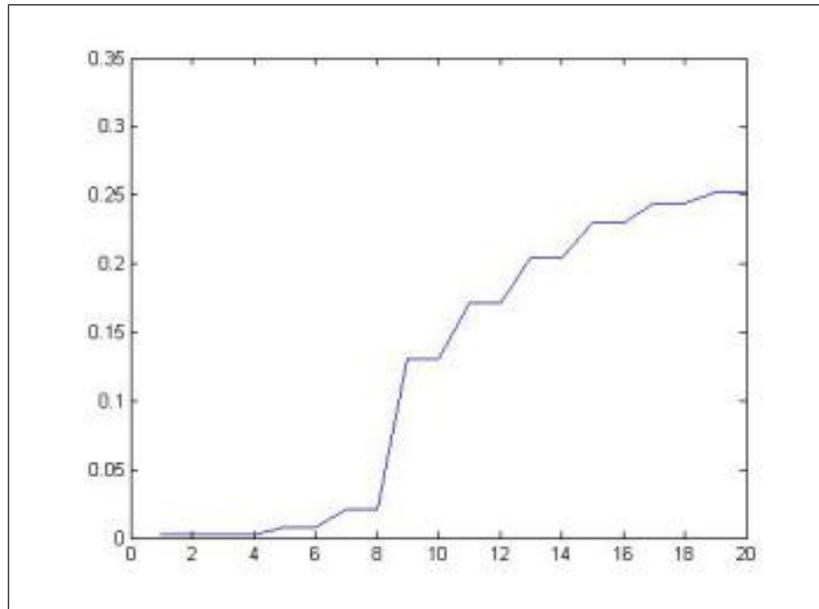


Figure 4.5:

$$a = 8nm$$

$$L = 40nm$$

Number of free modes: 30

$$n_1 = 5$$

Number of modes in n_1 : 8

Number of bound modes in n_1 : 30

$$n_2 = 4.9$$

Number of modes in n_2 : 8

Number of bound modes in n_2 : 30

5 Discussion

The major source of error in the code is probably the pseudoinverse. As described in section 2.5, the pseudoinverse doesn't necessarily have a unique solution, depending on whether or not it is rank deficient. This creates a problem in the code, when the matrices used are non quadratical, because the matrices invariably needs to be transposed which in turn makes them rank deficient. Since this means one can only find one solution among an infinite number, this can clearly become a problem. One possible remedy to this problem is to use quadratical matrices, in other words letting the number of free modes be equal to the number of modes in the waveguide. This has been simulated for several values, and the result is shown in fig. 4.4 and 4.5. 4.5 shows a nice increase in reflection and levels out at 0.25, which is a reasonable value. However, due to problems with the other simulations, it is difficult to say whether or not the simulation is reliable. 4.4 falls quickly to zero, due to a problem that will be explained below. One can then compare the single mode simulations in part 3, with the corresponding multi mode simulations in part 4.

Fig 4.1 shows the reflection as a function of the number of blocks, which have a refraction that varies between 2 and 1.5. One can clearly see it flattens out at a value higher than 1, which is physically impossible. This can possibly be attributed to the use of pseudoinverse in the code. Since there are instances in the code where the pseudoinverse is just one out of many solutions, it is very possible this can lead to problematic results. Another noteworthy feature of this figure, is that at block number nine, the reflectance dips towards zero. This is directly due to the fact that matrix M_{22}^{-1} from 2.20, suddenly drops sharply in value after adding a certain number of blocks. This can be tied to a decrease of the rank of this matrix. The fall in the reflectance value, correlate with a dramatic decrease in the rank of this matrix, indicating that these two things are connected to each other. The first value, matches perfectly with the single mode test in 3.5, while the second value is not quite as similar.

Fig 4.2 has more or less the same situation. The reflectance sharply falls after eleven blocks, but it is clear from the picture that if it had been able to continue, it would level out above one. The first value is correct, but it then increases more than it's supposed to.

The third figure, fig 4.3, flattens out before it reaches 1, however the second value is higher than it's single mode counterpart. But like the fig 3.2, it levels out after only a couple of blocks. This may indicate, that basic curvature of the graphs are correct, but each step is for some reason far too high.

One can then go back to fig 4.4 and 4.5. They show the exact same waveguide, but in 4.4 only the bound modes are included, as opposed to 4.5, where all modes are included. There is a clear difference between these two graphs, and this is

mainly due to the aforementioned problem; that the value of the pseudoinverse of M_{22} suddenly falls. In the system that produced fig 4.4, the rank of M_{22} falls almost immediatly, while the corresponding M_{22} for fig 4.5 retains full rank much longer.

6 Conclusion

The Transfer Matrix Method utilised as described in this report, seems to produce promising results, but it is doubtful that they are correct. The method gives what might be the correct shape of the reflectance as a function of the number of blocks, but the values are probably wrong. It is difficult to say with certainty why this is, but it is possible this is due to the pseudoinverse and the non quadratical matrices, since the pseudoinverse only gives one out of many possible solutions. The problem with inverting a non quadratical matrix, was avoided in fig 4.5 where a quadratical matrix was used, and the result looked promising. However without further testing, it is difficult to say whether or not the simulation is correct.

There is also another problem. The reflectance suddenly falls to zero after adding a certain number of blocks. The exact number of blocks, depend on the number of modes. This problem is most likely tied to the use of pseudoinverse in the code. That's because the fall in reflectance correlelates with a fall in the rank of the matrix, on which the pseudoinverse is used.

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