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Finite-temperature Casimir effect in Randall–Sundrum models

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Abstract. The finite-temperature Casimir effect for a scalar field in the bulk region of two Randall–Sundrum models, RSI and RSII, is studied. We calculate the Casimir energy and the Casimir force for two parallel plates with separation a on the visible brane in the RSI model. High-temperature and low-temperature cases are covered. Attraction versus repulsion of the temperature correction to the force is discussed in the special cases of Dirichlet–Dirichlet, Neumann–Neumann and Dirichlet–Neumann boundary conditions at low temperature. The Abel–Plana summation formula is used, as this is found to be the most convenient. Some comments are made on the related contemporary literature.

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Contents

1. Introduction

Inspired by the Randall–Sundrum models [1], there has recently been considerable interest in the Casimir effect in higher-dimensional space. We may recall the characteristic features of this model. In the first variant, called RSI, one assumes that we are living on a (3+1)-dimensional subspace called a 3-brane, separated from an additional hidden brane by a bulk region. Only gravity is assumed to propagate in the bulk. The extra dimension is a circle S^1 with radius r_c, represented by a coordinate ϕ in the range $-\pi \leqslant \phi \leqslant \pi$. The hidden and the visible branes are located at $\phi = 0$ and $\phi = \pi$, respectively. Imposition of Z_2 symmetry means that the points (x^μ, ϕ) and $(x^\mu, -\phi)$ are identified. In the second variant of the model, RSII, the hidden brane is sent to infinity. The major difference between the RS model and other higher-dimensional models lies in the warp factor $e^{-2kr_c|\phi|}$ in the metric

$$ds^2 = e^{-2kr_c|\phi|}\eta_{\mu\nu}\, dx^\mu\, dx^\nu - r_c^2\, d\phi^2, \tag{1}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric of flat spacetime and k is a constant of order M_{Pl}, the Planck mass. The warp factor plays an important role, helping to solve the hierarchy problem without introducing additional hierarchies.

Whereas in the original RS model the fields of the standard model (SM) were, as mentioned, confined to the visible brane only, the possibility of having additional fields in the bulk was soon investigated, beginning with scalar fields [2, 3] introduced to stabilize the inter-brane distance. Subsequently, the possibility of having other fields, such as fermion fields [4, 5], gauge fields [5]–[7], or even the full assembly of SM fields [8], was investigated. Tests of Newton's law at short distances may show aberrations at short distances (see for instance [9]).

Only recently have there appeared papers on the Casimir effect in the Randall–Sundrum models. To our knowledge the first group working on this was Frank *et al*, who published two papers [10, 11] on Casimir force in both the RSI/RSII and the RSI-q/RSII-q models. Here RSI-q and RSII-q refer to generalizing the 3-branes on RSI/RSII to $(3+q)$-branes (this kind of generalization will however not be dealt with in the present paper). There are also two papers from the group of Morales-Técotl *et al* [12], which focus on RSI-q/RSII-q. While Frank *et al* used zeta function regularization to calculate Casimir force, Morales-Técotl *et al* used a Green's function formalism. A delicate point is that their results are seemingly in conflict, except in the Minkowskian case. We shall comment on this point later.

A third class of papers are those of Cheng [13]–[15]. These papers, except for [15], assume zero temperature. Moreover, Elizalde *et al* have recently studied repulsive Casimir effects from extra dimensions for a massive scalar field with a general curvature parameter [16].

Among previous papers in this research field, the ones most similar to the present work are those of Teo [17]–[21]. She calculated the temperature Casimir force but not, however, the Casimir free energy.

As already mentioned, we will assume a scalar field Φ in the bulk. Both the RSI and the RSII models will be considered. Formally, expressions pertaining to the RSII case may be derived by letting $r_c \to \infty$ in the RSI expressions. The use of a scalar field makes the situation more unphysical than would be the case when assuming an electromagnetic field in the bulk. But we avoid the complications arising from photon spin in higher-dimensional spacetimes (for some recent papers in this direction, see for instance [22]–[27]). The conflict is not even resolved when we have only one extra spatial dimension and spacetime is *flat*. On the one hand, we have e.g. Poppenhaeger *et al* [27] and Pascoal *et al* [26], who find the electromagnetic Casimir force by multiplying by a factor p to account for the possible polarizations of the photon and subtract the mode polarized in the direction of the brane. On the other hand, we have Edery and Marachevsky, who start out with decomposition of the five-dimensional (5D) Maxwell action. This conflict in not the central issue in this paper and we avoid it by only considering a scalar field.

Our main purpose is to calculate the Casimir free energy and the Casimir force for the RSI model when there are two parallel plates with separation a on the visible brane. This is the piston model. Our main focus is on the following points:

1. The calculation is given for arbitrary temperature T, and the low- and high-temperature limits are considered thereafter. The attraction versus the repulsion of the temperature corrections for different boundary corrections at low temperature is of definite physical interest and is therefore pointed out. We regularize infinite expressions by using zeta functions and the Abel–Plana summation formula, as this formula turns out to be better suited to the problem than the more commonly used Euler–Maclaurin.
2. We assume Robin boundary conditions on the physical plates at $x=0$ and $x=a$. Usually, one considers the more simple Dirichlet conditions when working on this kind of problem, although very recently Robin considerations have begun to attract attention [21, 28].

As an introductory step, we consider in the next section the partition function and the free energy of a bulk scalar field. We discuss the distinction between even and odd fields, and also consider the mode localization problem. After a brief survey of the Abel–Plana formula in section 3, we consider in section 4 the Dirichlet–Dirichlet (DD), Neumann–Neumann (NN) and Dirichlet–Neumann (DN) boundary conditions in flat space, at finite temperature without extra dimensions. Our main topic, the temperature RSI case, is covered in section 5, where the

Casimir free energy and force are calculated for different boundary conditions. A brief treatment of the RSII case is given in section 6.

It should be recognized that the warp factor is an important element in the present problem. One might analyze instead the analogous higher-dimensional cases taking spacetime to be *flat*. Considerable interest has been devoted to this simpler variant of higher-dimensional Casimir theories in recent years. See for instance [19, 25, 26], [29]–[32], and further references therein.

2. Free energy of a bulk scalar field

To find the partition function for a non-minimally coupled scalar field Φ with mass m in the RSI model, we follow a Kaluza–Klein reduction approach [28, 33], starting from the Lagrangian density

$$\mathcal{L} = \sqrt{-G}\left(\tfrac{1}{2}\partial_M\Phi\partial^M\Phi - \tfrac{1}{2}(m^2 + \zeta R + c_{\text{hid}}\delta(z) + c_{\text{vis}}\delta(z - z_r))\Phi^2\right). \tag{2}$$

Here $G = \det G_{MN}$ (with $M, N = 0, 1, 2, 3, 5$) is the determinant of the 5D metric, R is the 5D Ricci scalar, ζ is the conformal coupling and $c_{\text{hid/vis}}$ are the boundary mass terms on the branes. Throughout the article we use $\hbar = c = k_{\text{B}} = 1$. We have introduced above a new position coordinate z so that $|z| = (\mathrm{e}^{k|r_c\phi|} - 1)/k$, implying that $z_{\text{r}} = (\mathrm{e}^{k\pi r_c} - 1)/k$. It is convenient to also introduce the quantity $A(z) = 1/(1 + k|z|)$. The partition function

$$Z = \int \mathcal{D}\Phi \exp\left(\mathrm{i}\int \mathrm{d}^4x\, \mathrm{d}z\, \mathcal{L}\right) \tag{3}$$

can now be calculated, making use of the Euclideanization $\tilde{x}^i = x^i, \tilde{x}^0 = \tau = \mathrm{i}x^0$. Partial integration yields

$$Z = \int \mathcal{D}\Phi \exp\left[-\int \mathrm{d}^4\tilde{x}\, \mathrm{d}z\, \frac{1}{2}\Phi A^3(z)(\hat{p}^2 + \hat{M}_z^2)\Phi\right]. \tag{4}$$

Here

$$\hat{p}^2 = \tilde{\eta}^{\mu\nu}\partial_\mu\partial_\nu, \tag{5}$$

where $\tilde{\eta}^{\mu\nu} = -\delta^{\mu\nu}$ is the metric in the coordinates $\tilde{x}^\mu$, and

$$\hat{M}_z^2 = A^{-3}(z)\left[-\partial_z A^3(z)\partial_z + A^5(z)(m^2 + \zeta R + c_{\text{hid}}\delta(z) + c_{\text{vis}}\delta(z - z_r))\right]. \tag{6}$$

We now expand Φ in the eigenfunctions $\chi_p(\tilde{x}^\mu)$ and $\psi_N(z)$ of $\hat{p}^2$ and $\hat{M}_z^2$, respectively,

$$\Phi(\tilde{x}, z) = \sum_{N,p} c_N(p)\chi_p(\tilde{x})\psi_N(z), \tag{7}$$

where ($\tau = \mathrm{i}t$)

$$\hat{p}^2\chi_p(\tilde{x}) = -(\partial_\tau^2 + \partial_x^2 + \partial_y^2 + \partial_z^2)\chi_p(\tilde{x}) = p^2\chi_p(\tilde{x}), \tag{8}$$

$$\hat{M}_z^2\psi_N(z) = M_N^2\psi_N(z). \tag{9}$$

The eigenfunctions are normalized as

$$\int \mathrm{d}^4\tilde{x}\, \chi_p(\tilde{x})\chi_{p'}(\tilde{x}) = \delta_{pp'}, \tag{10}$$

$$\int_{-z_r}^{z_r} \mathrm{d}z\, \psi_N(z)A^3(z)\psi_{N'}(z) = \delta_{NN'}. \tag{11}$$

The partition function now takes the form (with an unimportant factor omitted)

$$Z = \prod_{M_N, p} \left(M_N^2 + p^2\right)^{-1/2}, \tag{12}$$

where the sum goes over all eigenvalues of M_N and p. Our next step is to identify M_N and p.

2.1. Eigenfunctions and eigenvalues for $\hat{p}^2$ and $\hat{M}_N^2$

We start from equation (8), assuming Robin boundary conditions on the physical walls,

$$(1 + \beta_0 \partial_x)\chi_p(\tilde{x})\big|_{x=0} = 0, \tag{13}$$

$$(1 - \beta_a \partial_x)\chi_p(\tilde{x})\big|_{x=a} = 0, \tag{14}$$

with constants β_0 and β_a referring to $x = 0$ and $x = a$. The forms above are as in [28]. Dirichlet and Neumann boundary conditions correspond to $\beta = 0$ and $\beta = \infty$, respectively. We assume eigenfunctions of the form

$$\chi_p(\tilde{x}) = N e^{i(\epsilon_l \tau + k_y y + k_z z)} \cos(k_x x + \alpha), \tag{15}$$

with eigenvalues

$$p^2 = \epsilon_l^2 + k_x^2 + k_y^2 + k_z^2. \tag{16}$$

For a bosonic field at temperature T, the Matsubara frequencies are

$$\epsilon_l = 2\pi T l, \quad l \in \mathbb{Z}. \tag{17}$$

Equation (13) leads to the following constraint on α,

$$\cos\alpha = \frac{\beta_0 k_x}{\sqrt{1 + \beta_0^2 k_x^2}}, \tag{18}$$

whereas equation (14) yields $F_x(k_x) = 0$, where

$$F_x(k_x) = (1 - k_x^2 \beta_0 \beta_a)\sin(k_x a) - k_x(\beta_0 + \beta_a)\cos(k_x a). \tag{19}$$

Consider next the eigenfunctions $\psi_N(z)$. We insert expression (6) into equation (9), taking into account that the Ricci scalar for the RS metric is

$$R = -20k^2 + 16k(\delta(z) - \delta(z - z_r)), \tag{20}$$

and change the position coordinate in the bulk back to y using $\mathrm{d}/\mathrm{d}z = A(y)\,\mathrm{d}/\mathrm{d}y$. Then,

$$\psi_N''(y) - 4k\psi_N'(y) + \left(M_N^2 e^{2ky} - (m^2 - 20\zeta k^2)\right)\psi_N(y) = 0. \tag{21}$$

The solution is (we consider the region $0 < y < \pi r_c$ only)

$$\psi_N(y) = \frac{e^{2ky}}{C_N}\left(J_\nu\left(\frac{M_N}{k}e^{ky}\right) + b_\nu(M_N) Y_\nu\left(\frac{M_N}{k}e^{ky}\right)\right), \tag{22}$$

where $\nu = \sqrt{4 + (m/k)^2 - 20\zeta}$ and C_N is a normalization constant. This is the same result as in [5], except that we include curvature ($\zeta \neq 0$) in our model.

One should now distinguish between even fields satisfying $\psi_N(-y) = \psi_N(y)$ and odd fields satisfying $\psi_N(-y) = -\psi_N(y)$. Their behavior may be summarized as follows:

- Even scalar fields obey the Robin BC on the branes. If the field is minimally coupled ($\zeta = 0$) and there is no mass boundary term ($c_{\rm brane} = 0$), the boundary condition reduces to the Neumann BC, $\psi_N'(y)|_{\rm brane} = 0$.
- Odd scalar fields obey the Dirichlet BC on the branes.

These two cases may be combined. We introduce the two functions

$$\begin{aligned} j_\nu^{\rm brane}(z) &= (2 - (k\beta_{\rm brane})^{-1}) J_\nu(z) + z J_\nu'(z), \\ y_\nu^{\rm brane}(z) &= (2 - (k\beta_{\rm brane})^{-1}) Y_\nu(z) + z Y_\nu'(z), \end{aligned} \tag{23}$$

and now let z mean $z = \mathrm{e}^{k\pi r_c} M_N/k$ (not to be confused with the coordinate z in section 2), and $d = \mathrm{e}^{-k\pi r_c}$. Then we can write the general BC as $F_N(z) = 0$, where

$$F_N(z) = j_\nu^{\rm hid}(zd) y_\nu^{\rm vis}(z) - j_\nu^{\rm vis}(z) y_\nu^{\rm hid}(zd). \tag{24}$$

This is in accordance with [35] in the case of minimal coupling, if we choose $c_{\rm hid} = -c_{\rm vis} = 2\alpha/k$.

Special attention ought to be given to the massless case, $M_N = 0$. For fields with $m^2 - 20\zeta k^2 \neq 0$, there is no solution of equation (21) with $M_N = 0$ satisfying the Robin BC on both branes. For an even field with $m^2 - 20\zeta k^2 \neq 0$ with no boundary mass term, the situation is different, as ψ_0 =const is a solution of equation (21) and also satisfies the boundary condition which in that case is the Neumann. The $M_N = 0$ case has important consequences for the Casimir force from a bulk scalar field. This is related to the localization problem for the Kaluza–Klein modes in general. In RSI, the massless mode is localized near the hidden brane at $y = 0$. In RSII, the situation is reversed, as the massless mode is localized near the visible brane at $y = 0$ and the massive modes are delocalized. The reader may consult [12, 36] for discussion of what weight is to be given to the massless modes in RSI due to the fact that it is localized near the hidden brane only.

2.2. Approximate expressions for the masses

We assume $d = \mathrm{e}^{-k\pi r_c} \ll 1$ but keep $z = \mathrm{e}^{k\pi r_c} M_N/k$ as arbitrary to find convenient approximative expressions for the Kaluza–Klein masses. As in this case $j_\nu^{\rm brane}(z) \ll y_\nu^{\rm brane}(z)$, the equation $F_N(z) = 0$ reduces to

$$j_\nu^{\rm vis}(z) = 0. \tag{25}$$

The situation can be divided into two classes.

1. For *Dirichlet* BC ($\beta_{\rm brane} = 0$) it follows that we need the zeros of $J_2(z)$. Making use of the large-z approximation $J_\nu(z) \sim (2/\pi)^{1/2} \cos(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)$, we find that the expression

$$M_N = k\pi \mathrm{e}^{-k\pi r_c} \left(N + \tfrac{1}{2}\nu - \tfrac{1}{4}\right), \quad N = 1, 2, \ldots. \tag{26}$$

is useful for practical purposes.

2. For *non-Dirichlet* BC ($\beta_{\rm brane} \neq 0$) we obtain from equation (23)

$$j_\nu(z) = (2 + \nu - (k\beta)^{-1}) J_\nu(z) - z J_{\nu+1}(z), \tag{27}$$

leading to approximately

$$M_N = k\pi \mathrm{e}^{-k\pi r_\mathrm{c}} \left(N + \tfrac{1}{2}\nu - \tfrac{3}{4}\right), \quad N = 1, 2, \ldots. \tag{28}$$

The formula works well for $k\beta_{\mathrm{brane}} > 1$ and better for higher N. As an example, choosing $k\beta_{\mathrm{brane}} = 10^3$, $d = 10^{-12}$, $\nu = 2$, we find the numerical error of the zeros to be about four per cent when $N = 3$ and around one per cent when $N = 5$. As we will see later, the first (i.e. smallest) values of M_N are the most significant for the Casimir force in RSI.

2.3. *Two expressions for the free energy*

From equation (12) we obtain for the free energy

$$F = -T \ln Z = \frac{1}{2} T V_\perp \int \frac{\mathrm{d}^2 k_\perp}{(2\pi)^2} \sum_{l=-\infty}^{\infty} \sum_{M_N, k_x} \int \ln(M_N^2 + \epsilon_l^2 + k_x^2 + k_\perp^2), \tag{29}$$

where $k_\perp^2 = k_y^2 + k_z^2$, $V_\perp$ is the transverse volume, $\epsilon_l = 2\pi T l$, and the summations over k_x and M_N go over all real zeros of the functions $F_x(k_x)$ (equation (19)) and $F_N(z)$ (equation (24)).

By making use of the zeta function

$$\zeta(s) = \sum_{l=-\infty}^{\infty} V_\perp \int \frac{\mathrm{d}^2 k_\perp}{(2\pi)^2} \sum_{M_N, k_x} \left(M_N^2 + \epsilon_l^2 + k_x^2 + k_\perp^2\right)^{-s}, \tag{30}$$

following [34], we can re-express F as

$$F = -\frac{1}{2} T \frac{\partial}{\partial s} \mu^{2s} \zeta(s)\Big|_{s=0}, \tag{31}$$

where μ is an arbitrary parameter with dimension mass.

We now derive the classical expression for F using the fact that the Mellin transform of $b^{-z}\Gamma(z)$ is e^{-bt}, i.e.

$$b^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \mathrm{e}^{-bt}\, \mathrm{d}t. \tag{32}$$

Applying the Poisson summation formula (details omitted), we can then derive

$$F = \frac{1}{4\sqrt{\pi}} V_\perp \int \frac{\mathrm{d}^2 k_\perp}{(2\pi)^2} \sum_{M_N, k_x} \sum_{p=-\infty}^{\infty} \int_0^\infty \mathrm{d}t\; t^{-s-1} \mathrm{e}^{-(p^2/4T^2 t) - t\left(k_x^2 + k_\perp^2 + M_N^2\right)}. \tag{33}$$

Further manipulations lead us to the desired expression

$$F = T V_\perp \int \frac{\mathrm{d}^2 k_\perp}{(2\pi)^2} \sum_{M_N, k_x} \ln\left[2 \sinh\left(\frac{1}{2T}\sqrt{M_N^2 + k_x^2 + k_\perp^2}\right)\right]. \tag{34}$$

One may note here that a boson with energy E_p contributes with ($\beta = 1/T$)

$$Z_p = \sum_{n=1}^{\infty} \mathrm{e}^{-\beta E_p(n+1/2)} = \frac{1}{2\sinh(\beta E_p/2)} \tag{35}$$

to the total partition function [35]. Summing over all energies, we obtain the classical expression corresponding to equation (34). The expression of the free energy of a scalar bulk field is equal to that of bosons with energy $E_p^2 = M_N^2 + k_x^2 + k_\perp^2$, where M_N are the masses found in section 2.1.

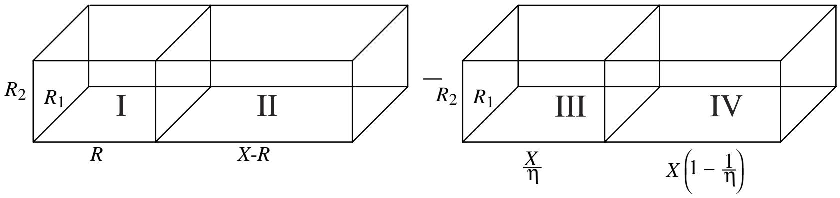

Figure 1. Illustration of the four cavities in the piston model.

For $M_N = 0$, this is the free energy of a scalar field in Minkowski (i.e. flat) spacetime without extra spatial dimensions. By letting $T \to 0$ we find the zero-point energy

$$E = V_\perp \int \frac{d^2 k_\perp}{(2\pi)^2} \sum_{M_N, k_x} \frac{1}{2}\sqrt{M_N^2 + k_x^2 + k_\perp^2}. \tag{36}$$

Again, we observe that the $M_N = 0$-term in the sum corresponds to the familiar expression for (3+1)-dimensional Minkowski spacetime.

Another expression for F can be derived, which is more useful in view of our application of the Abel–Plana summation formula later. We start from expression (31), introduce a generalized polar coordinate transformation along the same lines as in [37], and integrate over all angles. We then obtain (the limit $s \to 0$ is understood)

$$F = -\frac{T V_\perp}{4\pi}\frac{\partial}{\partial s}\mu^{2s} \sum_{l=-\infty}^{\infty} \sum_{M_N, k_x} \int_0^\infty dr\, r (C + r^2)^{-s}, \tag{37}$$

where C is defined as

$$C = M_N^2 + \epsilon_l^2 + k_x^2. \tag{38}$$

The integral is solved using the variable change $x = r^2/C$ and leads essentially to the Beta function $B(q, v) = \Gamma(q)\Gamma(v)/\Gamma(v+q)$. We obtain

$$F = -\frac{T V_\perp}{8\pi}\Gamma(-1) \sum_{l=-\infty}^{\infty} \sum_{M_N, k_x} (M_N^2 + \epsilon_l^2 + k_x^2). \tag{39}$$

This is the finite-temperature form that we will use below. The corresponding zero-temperature form is found by a limiting procedure to be

$$E = -\frac{V_\perp}{16\pi^{3/2}}\Gamma\left(-\frac{3}{2}\right) \sum_{M_N, k_x} (M_N^2 + k_x^2)^{3/2}. \tag{40}$$

From now on we will set $V_\perp = 1$. Hence E, F and P (force) refer to, respectively, energy, free energy and force per unit area of the physical plates.

2.4. *The piston model*

Before finding explicit expressions and specifying BCs we introduce the piston model. The model has received a great deal of attention [21], [29] and [38]–[41]. We introduce the piston (figure 1) with the same notation as in chapter 4.3 of [42]. Instead of only using the free energy $F_{\rm I}$ of cavity I as the Casimir free energy, we use

$$F_{\rm piston} = F^{\rm I}(a) + F^{\rm II}(X - a) - F^{\rm III}(X/\eta) - F^{\rm IV}(X(1 - 1/\eta)). \tag{41}$$

Initially the system is in an unstressed situation where the cavities have size X/η and $X(1-1/\eta)$. Then we shift the middle plate so that the lengths of the two cavities are a and $X-a$; the system is now in a stressed situation. The Casimir free energy is the sum of the free energies of two cavities in the stressed case (I and II) minus the free energies of the cavities in the unstressed case (III and IV). The constant η is ~ 2, characterizing the unstressed situation. In the end, we let $X \to \infty$ and effectively remove the rightmost plate from the setup.

The piston model introduces a term linear in a that may cancel an already existing term. Detailed analysis shows that all terms linear in a cancel in our final expression for the free energy in RSI and RSII. Terms independent of a do not contribute to the Casimir force. Hence from now on we discard all terms being linear in a, or independent of a.

3. Casimir free energy and force: initial remarks

3.1. The Abel–Plana formula

We want to find a more explicit expression for the Casimir free energy, one that can be evaluated numerically. Thus all the summations over k_x and M_N from equation (39) need to be taken care of. Instead of using equation (39), we look at the complex function

$$F(s) = -\frac{T}{8\pi}\Gamma(s)\sum_{l=-\infty}^{\infty}\sum_{M_N,k_x}(M_N^2+\epsilon_l^2+k_x^2)^{-s}, \tag{42}$$

which reduces to the free energy in equation (39) when $s=-1$. The function $F(s)$ is well defined for large, positive $\mathrm{Re}(s)$ and we analytically continue it to the whole complex plane. Together with $F(s)$, we will use a variant of the Abel–Plana formula [28, 43]

$$\sum_{n=1}^{\infty}\frac{\pi f(z_n)}{1+(1/z_n)\sin z_n\cos(z_n+2\alpha)} = \underbrace{-\frac{\pi}{2}\frac{f(0)}{1-\beta_0/a-\beta_a/a}}_{1} + \underbrace{\int_0^\infty \mathrm{d}z\, f(z)}_{2}$$
$$+\underbrace{\mathrm{i}\int_0^\infty \mathrm{d}z\frac{f\left(\mathrm{e}^{\mathrm{i}\pi/2}z\right)-f\left(\mathrm{e}^{-\mathrm{i}\pi/2}z\right)}{\frac{(\beta_0/a-1)(\beta_a/a-1)}{(\beta_0/a+1)(\beta_a/a+1)}\mathrm{e}^{2z}-1}}_{3}, \tag{43}$$

especially suited for plates with Robin BC. Here, z_n denotes the nth zero in the right half of the complex plane of the complex function $F_x(z=ak_x)$ in equation (19). From equations (18) and (19) we can find the relation

$$1+(1/z_n)\sin z_n\cos(z_n+2\alpha) = 1-\frac{\beta_0/a}{1+(\beta_0 z_n/a)^2}-\frac{\beta_a/a}{1+(\beta_a z_n/a)^2}. \tag{44}$$

By choosing

$$f(z) = \frac{1}{\pi}\left(M_N^2+\epsilon_l^2+z^2/a^2\right)^{-s}\left(1-\sum_{j=0,a}\frac{\beta_j/a}{1+(\beta_j z/a)^2}\right), \tag{45}$$

the left-hand side of equation (43) matches the sum over k_x in $F(s)$. The notation $\sum_{j=0,a}$ means there are contributions from both the left ($j=0$) and the right ($j=a$) plates.

3.2. Application of the Abel–Plana formula

We can divide the free energy F at arbitrary temperature T into two separate parts, $F = F(M_N = 0) + F(M_N > 0)$. For a massive scalar field, there is no massless mode ($M_N = 0$) at all. For a massless field, even and minimally coupled, there is an $M_N = 0$ mode. Recall that $F(M_N = 0)$ yields the same expression as the free energy of the massless scalar in Minkowski spacetime. To find the Casimir energy and force for such a field, one can simply add the massless mode term. It is natural, therefore, to analyze the $M_N = 0$ mode separately. The formal expressions are divergent and will be regularized by the use of zeta functions.

Now let M_N be arbitrary. Insert expression (45) into (43), and divide the sum into three separate parts as indicated by the underlines 1, 2 and 3. We do not give the details here, as the formalism is analogous to that of [28], pertaining to the zero-temperature case. The free energy can be written as the sum of three parts: one part F_{NP} as the contribution when no plates are present, one part F_j as the vacuum free energy along the transverse directions induced by the plates at $x_0 = 0$ and $x_a = a$, respectively, and the remaining part ΔF. Thus

$$F = F_{\mathrm{NP}} + \sum_{j=0,a} F_j + \Delta F. \tag{46}$$

The first two terms do not refer to the gap width a, or are linearly dependent on a, so we discard them. The last term ΔF, henceforth called simply F, is the term of physical importance. It is precisely the term corresponding to underline 3 in equation (43). We give this expression explicitly:

$$F(s) = -\frac{T}{(2\pi)^2}\Gamma(s)\sin\pi s \sum_{l=-\infty}^{\infty}\sum_{M_N}\int_{a\sqrt{M_N^2+\epsilon_l^2}}^{\infty} \mathrm{d}z \, \frac{[z^2/a^2 - (M_N^2+\epsilon_l^2)]^{-s}}{\frac{(\beta_0/a-1)(\beta_a/a-1)}{(\beta_0/a+1)(\beta_a/a+1)}\mathrm{e}^{2z} - 1} \times \left(1 - \sum_{j=0,a}\frac{\beta_j/a}{1-(\beta_j z/a)^2}\right). \tag{47}$$

We can now use this expression as a basis for discussing special cases, namely DD, NN and DN boundary conditions. We first consider flat space with no additional spatial dimensions.

4. Dirichlet–Dirichlet (DD), Neumann–Neumann (NN) and Dirichlet–Neumann (DN) boundary conditions in flat space with no extra dimensions

To demonstrate the procedure used for finding the Casimir free energy and force, we look at a well-known case; a massless, scalar field in flat spacetime (Minkowski metric) and no extra spatial dimensions. An additional motivation for including this section is that $F(M_N = 0) = F_{\mathrm{Mink}}$, as mentioned earlier.

Consider first the general formalism. With DD or NN boundary conditions we obtain, when making use of the substitution $z = xa\sqrt{M_N^2+\epsilon_l^2}$,

$$F^{\mathrm{DD,NN}} = -\frac{aT}{(2\pi)^2}\Gamma(s)\sin\pi s \sum_{l=-\infty}^{\infty}\sum_{M_N}(M_N^2+\epsilon_l^2)^{-s+1/2}\int_1^{\infty}\mathrm{d}x \, \frac{(x^2-1)^{-s}}{\mathrm{e}^{2a\sqrt{M_N^2+\epsilon_l^2}} - 1}. \tag{48}$$

We expand the denominator and use the relation

$$\int_1^\infty (x^2-1)^{\nu-1}\mathrm{e}^{-\mu x}\mathrm{d}x = \frac{1}{\sqrt{\pi}}\left(\frac{2}{\mu}\right)^{\nu-(1/2)}\Gamma(\nu)K_{\nu-(1/2)}(\mu). \tag{49}$$

In the limit $s\to -1$, we use the property $\Gamma(x)\sin\pi x = \pi/\Gamma(1-x)$ to obtain the free energy for arbitrary T:

$$F^{\mathrm{DD,NN}} = -\frac{\sqrt{\pi}aT}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{M_N}\sum_{n=1}^{\infty}\left(\frac{M_N^2+\epsilon_l^2}{n^2a^2}\right)^{3/4}K_{3/2}\left(2na\sqrt{M_N^2+\epsilon_l^2}\right). \tag{50}$$

The same expression follows if one makes use of zeta regularization. The Abel–Plana formula is effective in the present context, as it is easily adjustable to different choices for the boundary conditions.

Consider next flat space. With $M_N=0$ we obtain from equation (50)

$$F^{\mathrm{DD,NN}}_{\mathrm{Mink}} = -\frac{\zeta_R(3)T}{16\pi a^2} - \frac{1}{2}aT\left(\frac{2T}{a}\right)^{3/2}\sum_{l,n=1}^{\infty}\left(\frac{l}{n}\right)^{3/2}K_{3/2}(4\pi anlT). \tag{51}$$

The first term corresponds to $l=0$, and is derivable, for instance, by taking into account the properties of $K_\nu(z)$ for small arguments.

For high temperatures, $aT\gg 1$, expression (51) is suitable. The first term is the dominant one, as the K_ν terms decrease for increasing temperatures.

For low temperatures, $aT\ll 1$, some rewriting is, however, necessary. We go back to the complex function

$$F(s) = -\frac{T}{8\pi}\Gamma(s)\sum_{l=-\infty}^{\infty}\sum_{k_x}(\epsilon_l^2+k_x^2)^{-s}, \tag{52}$$

which corresponds to equation (39) when $M_N=0$, $s=-1$. Splitting off the $l=0$ term and using again Mellin transform (32), we can write $F(s)$ as

$$F(s) = F_{l=0} - \frac{T}{4\pi}\sum_{k_x}\int_0^\infty \mathrm{d}t\, t^{s-1}S_2(4\pi^2T^2t)\mathrm{e}^{-k_x^2t}. \tag{53}$$

Here

$$F_{l=0} = -\frac{T}{8\pi}\Gamma(s)\sum_{k_x}(k_x^2)^{-s} \tag{54}$$

and $S_2(t)$ is the function

$$S_2(t) = \sum_{m=1}^{\infty}\mathrm{e}^{-m^2t}, \tag{55}$$

possessing the property [42]

$$S_2(t) = -\frac{1}{2}+\frac{1}{2}\sqrt{\frac{\pi}{t}}+\sqrt{\frac{\pi}{t}}S_2\left(\frac{\pi^2}{t}\right). \tag{56}$$

The first of the three terms from the rhs of equation (56) cancels out $F_{l=0}$, leaving

$$F(s) = -\frac{1}{16\pi^{3/2}}\Gamma\left(s-\frac{1}{2}\right)\sum_{k_x} k_x^{-2(s-1/2)} - \frac{1}{8\pi^{3/2}}\sum_{k_x}\sum_{l=1}^{\infty}\int_0^{\infty} \mathrm{d}t\, t^{(s-1/2)-1}\exp\left(-tk_x^2 - \frac{l^2}{4T^2 t}\right). \tag{57}$$

The first term here is recognized as the zero-temperature energy, $F(T=0)=E$. With $s=-1$, we find

$$F_{\text{Mink}} = E_{\text{Mink}} - \frac{1}{4\pi^{3/2}}(2T)^{3/2}\sum_{k_x}\sum_{l=1}^{\infty}\left(\frac{k_x}{l}\right)^{3/2} K_{3/2}\left(\frac{k_x l}{T}\right). \tag{58}$$

We can now make use of the Abel–Plana formula (43), choosing for the function $f(z)$ the form

$$f(z) = \frac{1}{\pi}\left(\frac{z}{a}\right)^{3/2} K_{3/2}\left(\frac{lz}{aT}\right)\left(1 - \sum_{j=0,a}\frac{\beta_j/a}{1+(\beta_j z/a)^2}\right). \tag{59}$$

This leads to, when omitting terms not contributing to the piston model,

$$F_{\text{Mink}} = E_{\text{Mink}} - \frac{2a}{\pi^2}\sum_{n,l=1}^{\infty}\frac{1}{(4a^2n^2 + l^2/T^2)^2}. \tag{60}$$

We once more use the Mellin transform, but this time choosing $S_2(4a^2t)$, together with equation (56). Some calculation leads to the final expression

$$F_{\text{Mink}}^{\text{DD,NN}} = E_{\text{Mink}}^{\text{DD,NN}} - \frac{2T^{3/2}}{(2a)^{3/2}}\sum_{n,l=1}^{\infty}\left(\frac{n}{l}\right)^{3/2} K_{3/2}\left(\frac{\pi l n}{aT}\right), \tag{61}$$

where $E_{\text{Mink}}^{\text{DD,NN}} = -\pi^2/(1440a^3)$. The corresponding expression for the pressure is[2]

$$P_{\text{Mink}}^{\text{DD,NN}} = P_{\text{Mink}}^{\text{DD,NN}}(T=0) - \frac{3T^{3/2}}{\sqrt{2}\,a^{5/2}}\sum_{n,l=1}^{\infty}\left(\frac{n}{l}\right)^{3/2} K_{3/2}\left(\frac{\pi l n}{aT}\right) + \frac{\pi\sqrt{T/2}}{a^{7/2}}\sum_{n,l=1}^{\infty}\frac{n^{5/2}}{\sqrt{l}} K_{5/2}\left(\frac{\pi l n}{aT}\right), \tag{62}$$

where $P_{\text{Mink}}^{\text{DD,NN}}(T=0) = -\pi^2/(480a^4)$. The Casimir energy and force are equal to the zero-temperature expressions plus correction terms, the latter decaying exponentially as $T\to 0$.

In equation (62) we may insert the asymptotic expansion for large arguments, $K_\nu(z) = (\pi/2z)^{1/2}\,\mathrm{e}^{-z}[1+(4\nu^2-1)/8z]$. It is of interest to extract the dominant term in the correction, corresponding to $n=l=1$. Approximatively, we then obtain

$$P_{\text{Mink}}^{\text{DD,NN}} = P_{\text{Mink}}^{\text{DD,NN}}(T=0) + \frac{\pi}{2a^3}\exp\left(-\frac{\pi}{aT}\right). \tag{63}$$

[2] We are missing the Stefan-Boltzmann term in both the free energy and force since we have removed all terms linear in a.

The physically important point here is that the finite-temperature term is *positive*, corresponding to a repulsive force correction (recall that we are considering $aT \ll 1$). The situation is in some sense similar to that encountered in earlier studies when calculating the Casimir force between two parallel metallic slabs in physical space, assuming the Drude dispersion relation for the material; in that case the finite-temperature effect was also found to *weaken* the attractive $T = 0$ force [44].

We now consider the third class of BCs mentioned above; assuming Dirichlet boundary conditions on one plate and Neumann on the other, we find

$$F^{\rm DN} = \frac{aT}{(2\pi)^2}\Gamma(s)\sin\pi s \sum_{l=-\infty}^{\infty}\sum_{M_N}(M_N^2+\epsilon_l^2)^{-s+1/2}\int_1^\infty {\rm d}x\,\frac{(x^2-1)^{-s}}{{\rm e}^{2a\sqrt{M_N^2+\epsilon_l^2}}+1}. \tag{64}$$

The steps are similar to those of the DD and NN calculations, only with a factor $(-1)^n$ due to the positive sign in the denominator and accordingly $E^{\rm DN}_{\rm Mink} = -7/8E^{\rm DD,NN}_{\rm Mink}$. The free energy density with DN boundary conditions becomes

$$F^{\rm DN} = -\frac{a\sqrt{\pi}T}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{M_N}\sum_{n=1}^{\infty}(-1)^n\left(\frac{M_N^2+\epsilon_l^2}{n^2a^2}\right)^{3/4}K_{3/2}\left(2na\sqrt{M_N^2+\epsilon_l^2}\right). \tag{65}$$

With $M_N = 0$, we obtain

$$F^{\rm DN}_{\rm Mink} = \frac{3\zeta_R(3)T}{64\pi a^2} - \frac{1}{2}aT\left(\frac{2T}{a}\right)^{3/2}\sum_{l,n=1}^{\infty}(-1)^n\left(\frac{l}{n}\right)^{3/2}K_{3/2}(4\pi anlT), \tag{66}$$

which is a convenient form for the case of high temperatures.

For low temperatures, we obtain, by reasoning similar to that given above,

$$F^{\rm DN}_{\rm Mink} = E^{\rm DN}_{\rm Mink} - \frac{2T^{3/2}}{(2a)^{3/2}}\sum_{n,l=1}^{\infty}(-1)^n\left(\frac{n}{l}\right)^{3/2}K_{3/2}\left(\frac{\pi ln}{aT}\right) \tag{67}$$

with corresponding force

$$P^{\rm DN}_{\rm Mink} = P^{\rm DN}_{\rm Mink}(T=0) - \frac{3\,T^{3/2}}{\sqrt{2}\,a^{5/2}}\sum_{n,l=1}^{\infty}(-1)^n\left(\frac{n}{l}\right)^{3/2}K_{3/2}\left(\frac{\pi ln}{aT}\right)$$
$$+\frac{\pi\sqrt{T/2}}{a^{7/2}}\sum_{n,l=1}^{\infty}(-1)^n\frac{n^{5/2}}{\sqrt{l}}K_{5/2}\left(\frac{\pi ln}{aT}\right). \tag{68}$$

Again extracting the dominant term by including only $n = l = 1$, we obtain approximately

$$P^{\rm DN}_{\rm Mink} = P^{\rm DN}_{\rm Mink}(T=0) - \frac{\pi T}{2a^3}\exp\left(-\frac{\pi}{aT}\right). \tag{69}$$

The correction term is the same as in equation (63), but with the opposite sign. The thermal correction is attractive.

5. DD, NN and DN boundary conditions in Randall–Sundrum Model I

Consider first the high-temperature regime. Whereas in flat space this corresponds to $aT \gg 1$, in RSI the natural choice for high temperatures is $T \gg k{\rm e}^{-k\pi r_c}$. Recall that the lowest values of M_N are $\sim k{\rm e}^{-k\pi r_c}$; this implies $aT \gg 1$ since $ak{\rm e}^{-k\pi r_c} \gg 1$ for all relevant distances in physical space. In this limit, equation (50) is a suitable expression for the free energy and the Casimir force is

$$P_{\rm RSI}^{\rm DD,NN} = \frac{\sqrt{\pi}\,T}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{M_N}\sum_{n=1}^{\infty}\left(\frac{M_N^2+(2\pi T l)^2}{n^2a^2}\right)^{3/4} K_{3/2}\left(2na\sqrt{M_N^2+(2\pi T l)^2}\right)$$
$$-\frac{2\sqrt{\pi}\,T}{(2\pi)^2}\sum_{l=-\infty}^{\infty}\sum_{M_N}\sum_{n=1}^{\infty}\frac{\left(M_N^2+(2\pi T l)^2\right)^{5/4}}{\sqrt{na}}K_{5/2}\left(2na\sqrt{M_N^2+(2\pi T l)^2}\right). \quad (70)$$

After some rewriting we find that this is in accordance with equation (23) in [20]. We need only to include the $E(M_N=0)$ term to get the Casimir force for a massless scalar instead of a massive scalar.

We find that the high-temperature limit is valid for $T \gg 10^{16}\,$K. Only temperatures much less than these are expected to be of physical importance. It is very natural therefore to find the Casimir energy and force for $T \ll k\mathrm{e}^{-k\pi r_{\rm c}}$. Note that the brane low-temperature condition does not fix the magnitude of the product aT relative to unity. With $k \sim 10^{19}\,$GeV, $\mathrm{e}^{-k\pi r_{\rm c}} \sim 10^{-16}$, we only obtain the weak condition $T \ll 10^3\,$GeV. As an example, choose $T = 300\,$K ($2.6\times 10^{-11}\,$GeV), $a = 1\,\mu$m, from which it follows that $aT = 0.15$. In most cases of practical interest we will have $aT \ll 1$, although one can easily consider cases where $aT \gg 1$, still compatible with the condition $T \ll k\mathrm{e}^{-k\pi r_{\rm c}}$.

Using the same procedure as in flat space, we find the RSI equivalent to equation (58),

$$F_{\rm RSI} = E_{\rm RSI} - \frac{1}{4\pi^{3/2}}(2T)^{3/2}\sum_{M_N}\sum_{k_x}\sum_{l=1}^{\infty}\left(\frac{k_x^2+M_N^2}{l^2}\right)^{3/4} K_{3/2}\left(\frac{l}{T}\sqrt{k_x^2+M_N^2}\right). \quad (71)$$

We can differentiate this expression to find the Casimir force. By assuming $\partial k_x/\partial a = -k_x/a$, we obtain equation (17) in [21], only missing the first term[3]. The assumption holds for all k_x proportional to $1/a$, which is the case for DD, NN and DN BC. Although the metric in [21] does not include the warp factor $\mathrm{e}^{-2kr_{\rm c}\phi}$, the expressions are the same, since the warp factor only affects the values of the M_Ns. In equation (71) $E_{\rm RSI}$ is the zero-temperature energy in RSI and is found from equation (40) using the Abel–Plana formula (43) with

$$f(z) = \frac{1}{\pi}\left(M_N^2 + z^2/a^2\right)^{3/2}\left(1 - \sum_{j=0,a}\frac{\beta_j/a}{1+(\beta_j z/a)^2}\right). \quad (72)$$

After some variable changes, the energy reads

$$E_{\rm RSI} = -\frac{1}{6\pi^2}\sum_{M_N}\int_{aM_N}^{\infty}\mathrm{d}z\,(z^2/a^2 - M_N^2)^{3/2}\frac{1-\sum_{j=0,a}\frac{1-\beta_j/a}{(\beta_j z/a)^2}}{\frac{(\beta_0/a-1)(\beta_a/a-1)}{(\beta_0/a+1)(\beta_a/a+1)}\mathrm{e}^{2z}-1}, \quad (73)$$

and for DD and NN boundary conditions it simplifies to

$$E_{\rm RSI}^{\rm DD,NN} = -\frac{1}{8\pi^2 a}\sum_{n=1}^{\infty}\sum_{M_N}\frac{M_N^2}{n^2}K_2(2aM_N n). \quad (74)$$

[3] This is the additional term from cavity II-IV in the piston model and is independent of a (the free energy is linear in a). With the free energy of this form, we do not get cancelation of the terms linear in a.

The Casimir force at zero temperature is

$$P_{\mathrm{RSI}}^{\mathrm{DD,NN}}(T=0) = -\frac{3}{8\pi^2 a^2}\sum_{n=1}^{\infty}\sum_{M_N}\frac{M_N^2}{n^2}K_2(2aM_N n) - \frac{1}{4\pi^2 a}\sum_{n=1}\sum_{M_N}\frac{M_N^3}{n}K_1(2aM_N n). \tag{75}$$

This is in accordance with [28], although that paper does not consider the Casimir effect arising from a bulk scalar in the RS model in particular. Inserting the approximation equation (28) for M_N, we see that the energy is essentially the same as in [10]. There are three minor differences. First of all the energy in [10] has some extra terms that are linear and independent of a since the piston model is not used. Secondly, factors p are included to make the expression hold for electromagnetic fields, where p is the polarization of the photon. The last difference is a factor of 2 included to account for 'the volume of the orbifold'. Since we cannot see how this factor occurs, it is not included. This is also equal to equation (26) in [20], except that it contains the $E(M_N=0)$ term since a massless field is considered.

The summation over k_x in equation (71) is still left and can be done using the Abel–Plana formula with

$$f(z) = \frac{1}{\pi}(M_N^2 + z^2/a^2)^{3/4} K_{3/2}\left(\frac{l}{T}\sqrt{z^2/a^2 + M_N^2}\right)\left(1 - \sum_{j=0,a}\frac{\beta_j/a}{1+(\beta_j z/a)^2}\right). \tag{76}$$

After inserting $K_3/2(z) = (\pi/2z)^{1/2}\,\mathrm{e}^{-z}(1+1/z)$, the free energy reads

$$F_{\mathrm{RSI}} = E_{\mathrm{RSI}} - \frac{T^3}{\pi^2}\sum_{M_N}\sum_{l=1}^{\infty} l^{-3}\int_{M_N a}^{\infty}\mathrm{d}z\,\frac{1-\sum_{j=0,a}\frac{\beta_j/a}{1-(\beta_j z/a)^2}}{\frac{(\beta_0/a-1)(\beta_a/a-1)}{(\beta_0/a+1)(\beta_a/a+1)}\mathrm{e}^{2z}-1}\left[\sin\left(\frac{l}{T}\sqrt{(z/a)^2 - M_N^2}\right)\right.$$
$$\left. -\frac{l}{T}\sqrt{(z/a)^2 - M_N^2}\cos\left(\frac{l}{T}\sqrt{(z/a)^2 - M_N^2}\right)\right]. \tag{77}$$

We continue by inserting the βs for DD and NN boundary conditions, expansion of the denominator and the variable exchange $x = z/a$ to obtain

$$F_{\mathrm{RSI}}^{\mathrm{DD,NN}} = E_{\mathrm{RSI}}^{\mathrm{DD,NN}} - \frac{T^3 a}{\pi^2}\sum_{M_N}\sum_{l=1}^{\infty} l^{-3}\sum_{n=1}^{\infty}\int_{M_N}^{\infty}\mathrm{d}x\,\mathrm{e}^{-2nax}\left[\sin\left(\frac{l}{T}\sqrt{x^2 - M_N^2}\right)\right.$$
$$\left. -\frac{l}{T}\sqrt{x^2 - M_N^2}\cos\left(\frac{l}{T}\sqrt{x^2 - M_N^2}\right)\right]. \tag{78}$$

Integrals of this form are solved in the appendix with the result

$$\int_C^{\infty}\mathrm{d}x\left(\sin(A\sqrt{x^2-C^2}) - A\sqrt{x^2-C^2}\cos(A\sqrt{x^2-C^2})\right)\mathrm{e}^{-Bx}$$
$$= \frac{C^2 A^3}{A^2+B^2}K_2(C\sqrt{A^2+B^2}), \tag{79}$$

giving the free energy

$$F_{\mathrm{RSI}}^{\mathrm{DD,NN}} = E_{\mathrm{RSI}}^{\mathrm{DD,NN}} - \frac{a}{\pi^2}\sum_{M_N}\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}\frac{M_N^2}{(2na)^2 + (l/T)^2}K_2\left(M_N\sqrt{(2na)^2 + (l/T)^2}\right) \tag{80}$$

and force

$$P_{\rm RSI}^{\rm DD,NN} = P_{\rm RSI}^{\rm DD,NN}(T=0) - \frac{1}{\pi^2}\sum_{M_N}\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}\frac{M_N^2(3(2na)^2-(l/T)^2)}{((2na)^2+(l/T)^2)^2}K_2\left(M_N\sqrt{(2na)^2+(l/T)^2}\right) - \frac{1}{\pi^2}\sum_{M_N}\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}\frac{M_N^3(2na)^2}{((2na)^2+(l/T)^2)^{3/2}}K_1\left(M_N\sqrt{(2na)^2+(l/T)^2}\right). \quad (81)$$

This expression, belonging to the low-temperature regime $T \ll k\mathrm{e}^{-k\pi r_c}$, can be used both for $aT \ll 1$ and for $aT \gg 1$. The argument of the Bessel functions will always be large since $ak\mathrm{e}^{-k\pi r_c} \gg 1$ for all relevant distances. The correction terms to the zero-temperature energy and force expressions are small. The expression, to our knowledge, has not been given before. The leading term for the force in terms of T/M_N is

$$P_{\rm RSI}^{\rm DD,NN} \sim P_{\rm RSI}^{\rm DD,NN}(T=0) - \frac{1}{\sqrt{2\pi}^3}\sum_{M_N}\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}M_N^4\mathrm{e}^{-\frac{M_N}{T}\sqrt{(2anT)^2+l^2}} \times\left(\frac{T}{M_N}\right)^{3/2}\frac{(2naT)^2}{\left((2naT)^2+l^2\right)^{7/4}}. \quad (82)$$

In contrast to flat space, *both* zero-temperature Casimir force and thermal correction a negative. Hence, the Casimir effect in RSI is stronger in the low-temperature limit ($T \ll k\mathrm{e}^{-k\pi r_c}$) both for $aT \ll 1$ and for $aT \gg 1$.

The DN expressions deviate from the DD and NN expressions in RSI in the same way as in flat space. The factor of $(-1)^n$ must be included in sum over n, where the sum over n originates from the expansion of the denominator in equations (73) and (77).

5.1. Comparison to flat space

The point of calculating the Casimir force in RSI is to find out where there are deviations from the Casimir force in flat spacetime without extra spatial dimensions. For an easier comparison, we give the full expression for the Casimir force of a *massless bulk scalar* in RSI with DD/NN BCs.

$$P_{\rm RSI}^{\rm DD,NN} = -\frac{\pi^2}{480a^4} - \frac{1}{a^4}\sum_{M_N}\sum_{n=1}^{\infty}\left[\frac{3}{8\pi^2}\frac{(aM_N)^2}{n^2}K_2(2aM_Nn) + \frac{1}{4\pi^2}\frac{(aM_N)^3}{n}K_1(2aM_Nn)\right] + \frac{1}{a^4}\sum_{n,l=1}^{\infty}\left[-\frac{3(aT)^{3/2}}{\sqrt{2}}\left(\frac{n}{l}\right)^{3/2}K_{3/2}\left(\frac{\pi ln}{aT}\right) + \pi\sqrt{Ta/2}\frac{n^{5/2}}{\sqrt{l}}K_{5/2}\left(\frac{\pi ln}{aT}\right)\right] - \frac{1}{a^4\pi^2}\sum_{M_N}\sum_{n,l=1}^{\infty}\left[\frac{(aM_N)^3(2n)^2}{\left((2n)^2+(l/aT)^2\right)^{3/2}}K_1\left(aM_N\sqrt{(2n)^2+(l/Ta)^2}\right) - \frac{(aM_N)^2\left(3(2n)^2-(l/aT)^2\right)}{\left((2n)^2+(l/aT)^2\right)^2}K_2\left(aM_N\sqrt{(2n)^2+(l/aT)^2}\right)\right]. \quad (83)$$

This expression is good at low temperatures because the Bessel function decreases exponentially at high arguments. Hence, we only need to sum over the first couple of values from M_N, n and l if the other factors (M_N, a and T) ensure that the argument is much greater than one. On the other hand, if the argument of the Bessel function is not large we need to be careful that the sum has converged. Now the essential question is, for what values (of M_N, a and T) are the sums of Bessel functions of the same magnitude as the flat spacetime at zero temperature (i.e. the first term)? Or, simply, when can we see a deviation from the ordinary Casimir force?

Looking at equation (83), we see that at zero temperature we need $aM_N \sim 1$ for a noticeable difference. We know that $M_N \approx k\mathrm{e}^{-k\pi r_c}$ for low N and k is usually set to $\sim M_{pl} \approx 10^{19}$ GeV in RSI. In the original paper of Randall and Sundrum, they propose choosing $kr_c \sim 10$ in order to solve the hierarchy problem. With these values, we find that a is $\sim 10^{-21}$ m. There is no point in looking at distances smaller than the size of an atom. Only distances of physical relevance (>1 nm) are of interest. In figure 2, we keep $k = 10^{19}$ GeV, but choose $\mathrm{e}^{-k\pi r_c} = 10^{-26}$. The difference from RSI to ordinary Casimir force F_{Mink} at zero temperature is given in figure 2. By a choice of parameters, the magnitude of the correction in RSI is of the same order of magnitude as F_{Mink}, given by the red line in figure 2(a). With a smaller value of kr_c we will not see any difference at separations larger than 1 nm. The corresponding size of the extra dimension is $r_c \approx 10^{-35}$ m. In figure 2(b), we see the ratio of this difference to the Casimir force at zero temperature. We observe that this extra term we obtain in RSI goes more quickly to zero than F_{Mink}. Now we turn to relevant values of the temperature. The choices of r_c, k and a are still the same, and after some testing it turns out that $a_{\max}T \sim 3$ is suitable. To be sure that the sums have converged, we let both n and l run to 30. The result is presented in figure 2. The green line is the $P_{\mathrm{RSI}} - P_{\mathrm{Mink}}$ for $T = 3 \times 10^6$ K and we see that this gives a stronger Casimir force than at zero temperature.

5.2. Comparison to flat space with one extra dimension

In the previous section, we compared the Casimir force in RSI with the Casimir force for a massless scalar in ordinary flat (3+1)-spacetime. In this section, we will look at a higher-dimensional spacetime that is flat, i.e. with no warp factor in the metric. As mentioned in the introduction, this topic has received much interest lately. The extra dimension is a torus with circumference $2\pi L$. In this case $M_N = N/L$, with $N = 0, \pm 1, \pm 2, \ldots$. With these values for the M_Ns instead of those we have in RSI, we see that equation (70) is equal to the high-temperature expression in [19]. However, we have not found an expression corresponding to the Casimir force in equation (83). Since the new values for the M_Ns are equally separated, we can use the Chowla–Selberg formula found in [45]. The function $F(s)$ now reads for DD BCs

$$F(s) = -\frac{T}{8\pi}\Gamma(s)\sum_{l=-\infty}^{\infty}\sum_{N=-\infty}^{\infty}\sum_{n=1}^{\infty}((2\pi T)^2 + (n\pi/a)^2 + (N/L)^2)^{-s}. \tag{84}$$

Rewriting this to homogeneous Epstein zeta functions

$$Z_{\mathrm{E},p}(s; a_1, \ldots, a_p) = \sum_{k_1,\ldots,k_p=-\infty}^{\infty}{}' ((a_1k_1)^2 + (a_2k_2)^2 + \cdots + (a_pk_p)^2)^{-s}, \tag{85}$$

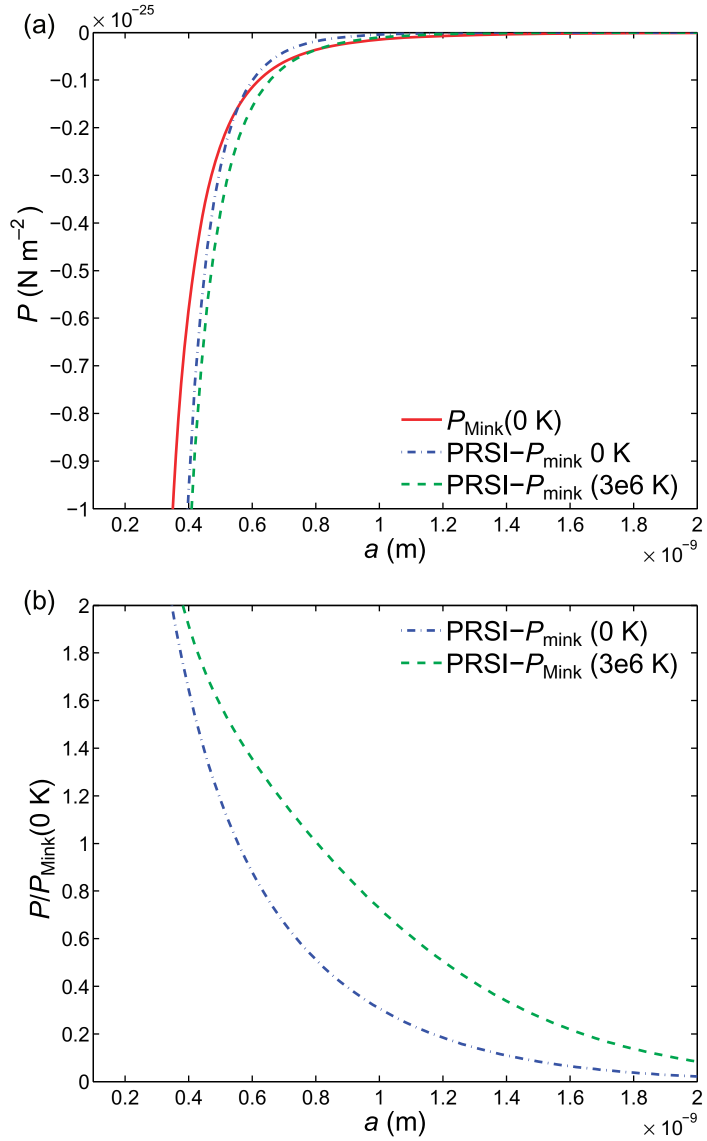

Figure 2. $P_{\mathrm{RSI}}^{\mathrm{DD,NN}} - P_{\mathrm{Mink}}^{\mathrm{DD,NN}}$ for a massless scalar bulk field with $k = 10^{19}$ GeV and $d = \mathrm{e}^{-k\pi r_c} = 10^{-26}$. (a) The difference between the Casimir force for RSI and flat (3+1)-spacetime at $T = 0$ and $T = 3 \times 10^6$ K. The Casimir force for a massless scalar field in flat spacetime at zero temperature is included for comparison. (b) The ratio of the difference between the Casimir force in RSI and flat spacetime at $T = 0$ and $T = 3 \times 10^6$ K to the force at zero temperature in flat spacetime.

we find

$$F(s) = -\frac{T}{16\pi}\Gamma(s) Z_{E,2}(s; 2\pi T, 1/L) - \frac{T}{16\pi}\Gamma(s) Z_{E,3}(s; 2\pi T, \pi/a, 1/L). \tag{86}$$

The notation $\sum_{k_1,k_2,\ldots k_p=-\infty}^{\infty}$ means that for k_1 to k_p we sum from $-\infty$ to ∞. The prime '′' behind the sum means that the term $k_1 = k_2 = \cdots = k_p = 0$ is omitted. The Chowla–Selberg formula is

$$Z_{\mathrm{E},p}(s;a_1,\ldots,a_p)=Z_{\mathrm{E},m}(s;a_1,\ldots,a_m)+\frac{\pi^{m/2}\Gamma\left(s-\frac{m}{2}\right)}{\left(\prod_{i=1}^m a_i\right)\Gamma(s)}Z_{\mathrm{E},p-m}\left(s-\frac{m}{2};a_{m+1},\ldots,a_p\right)$$
$$+\frac{2\pi^s}{\Gamma(s)\left(\prod_{i=1}^m a_i\right)}\sum_{k_1,\ldots,k_m=-\infty}^{\infty}{}'\sum_{k_{m+1},\ldots,k_p=-\infty}^{\infty}{}'\left(\frac{\sum_{i=1}^m (k_i/a_i)^2}{\sum_{j=m+1}^p (k_j a_j)^2}\right)^{(2s-m)/4}$$
$$\times K_{s-m/2}\left(2\pi\sqrt{\sum_{i=1}^m k_i/a_i}\sqrt{\sum_{j=m+1}^p (k_j a_j)^2}\right). \tag{87}$$

After using the Chowla–Selberg formula with $m=2$ we put $s=-1$ and remove all terms linear and independent of a. Then we see that the free energy is

$$F=-\frac{T}{16\pi}\Gamma(-1)Z_{E,2}(-1;2\pi T,\pi/a)-\frac{a}{4\pi^2}\sum_{N=-\infty}^{\infty}{}'\sum_{n,l=-\infty}^{\infty}{}'\frac{(N/L)^2}{(2na)^2+(l/T)^2}K_2$$
$$\times\left(N/L\sqrt{(2na)^2+(l/T)^2}\right). \tag{88}$$

The first term can be identified as $F(M_N=0)$ and the second term is equal to equation (80). From $\sum_{n,l=-\infty}^{\infty}{}'$, we obtain the factor 4 when we rewrite so that n and l run from 1 to ∞. The term $l=0$ but $n\neq 0$ gives the zero-temperature expression and $n=0$ with $l\neq 0$ is independent of a and is removed. Since equation (80) leads to equation (83), we can conclude that we obtain the same answer with the Chowla–Selberg formula as with the Abel–Plana formula in flat spacetime with one extra spatial dimension. However, the Abel–Plana formula can be used regardless of the values of M_N, while the Chowla–Selberg formula is only useful if we can rewrite our expressions to homogeneous Epstein zeta functions. The second advantage of the Abel–Plana formula is that different boundary condititions can easily be obtained.

6. DD, NN and DN boundary conditions in Randall–Sundrum Model II

In RSII, the Kaluza–Klein modes are continuous and we must replace the sum over M_N with an integral,

$$\sum_{M_N}\to\int_0^\infty\frac{\mathrm{d}M}{k}. \tag{89}$$

In equation (50), we obtain an integral of the form

$$\int_0^\infty \mathrm{d}z(z^2+A^2)^{3/4}K_{3/2}\left(B\sqrt{z^2+A^2}\right)=\sqrt{\frac{\pi}{2B}}A^2K_2(AB). \tag{90}$$

The derivation of this formula is given in the appendix. We find that the free energy and force in RSII are

$$F_{\mathrm{RSII}}^{\mathrm{DD,NN}}=-\frac{T\pi^3}{1440ka^3}-\frac{\pi T^3}{ak}\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}\frac{l^2}{n^2}K_2\left(4\pi Taln\right) \tag{91}$$

and

$$P_{\rm RSII}^{\rm DD,NN} = -\frac{3T\pi^3}{1440ka^4} - \frac{3\pi T^3}{a^2k}\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}\frac{l^2}{n^2}K_2(4\pi aTln) - \frac{4\pi^2T^4}{ak}\sum_{l=1}^{\infty}\sum_{n=1}^{\infty}\frac{l^3}{n}K_1(4\pi aTln). \quad (92)$$

These expressions are convenient for the high-temperature limit $aT \gg 1$. In RSII, there are only two temperature regimes, $aT \gg 1$ and $aT \ll 1$, since M_N is continuous. To find the low-temperature limit, we insert equation (89) into equation (80) and use the integral

$$\int_0^\infty {\rm d}z\, z^2 K_2(Az) = \frac{3\pi}{2A^3}. \quad (93)$$

The free energy reads

$$F_{\rm RSII}^{\rm DD,NN} = E_{\rm RSII}^{\rm DD,NN} - \frac{3a}{2\pi k}\sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{((2an)^2+(l/T)^2)^{5/2}}, \quad (94)$$

where $E_{\rm RSII}^{\rm DD,NN} = -\frac{3\zeta_R(5)}{128\pi ka^4}$ is the zero-temperature energy in RSII. We can find this energy from e.g. equation (74) by making use of equation (93). We use the Mellin transform as in low-temperature, flat spacetime (with $S_2(4a^2t)$) and find that

$$F_{\rm RSII}^{\rm DD,NN} = E_{\rm RSII}^{\rm DD,NN} - \frac{\pi T^2}{2ka^2}\sum_{n,l=1}^{\infty}\left(\frac{n}{l}\right)^2 K_2\left(\frac{\pi nl}{aT}\right). \quad (95)$$

The Casimir force is

$$P_{\rm RSII}^{\rm DD,NN} = P_{\rm RSII}^{\rm DD,NN}(T=0) + \frac{\pi^2 T}{2ka^4}\sum_{n,l=1}^{\infty}\frac{n^3}{l}K_1\left(\frac{\pi nl}{aT}\right), \quad (96)$$

with $P_{\rm RSII}^{\rm DD,NN}(T=0) = -\frac{3\zeta_R(5)}{32\pi ka^5}$.

In the low-temperature limit $aT \ll 1$, we obtain from equation (96) the dominant term corresponding to $n = l = 1$,

$$P_{\rm RSII}^{\rm DD,NN} = P_{\rm RSII}^{\rm DD,NN}(T=0) + \frac{\pi^2 T}{2ka^4}\sqrt{\frac{aT}{2}}\exp\left(-\frac{\pi}{aT}\right). \quad (97)$$

The temperature correction term is repulsive.

As in RSI the only difference between DD/NN BC and DN is a factor $(-1)^n$ in the sum over n.

7. Concluding remarks

Our main objective has been to calculate the finite-temperature Casimir effect for a scalar field residing in the bulk in the two Randall–Sundrum models, RSI and RSII. Two parallel plates are envisaged, with gap a located on one of the RS branes. We have given most attention to the RSI model. Robin boundary conditions, see equations (13) and (14), are assumed on the two plates. The geometrical picture is the piston model, as illustrated in figure 1. We have made use of the Abel–Plana summation formula throughout, as this is found to be the most convenient choice in the present context.

In the case of flat space the basic expressions for the Casimir free energy and force (per unit surface area) are worked out in the form of the series in section 4, for both high and low temperatures. A characteristic feature for DD and NN boundary conditions on the two plates is that the dominant part of the finite-temperature correction term for low temperatures ($aT \ll 1$) is repulsive. That is, the force decreases slightly when the temperature increases from zero. In this sense the behavior is analogous to that encountered in the case of conventional Casimir theory for metallic slabs in physical space when the dispersive relation for the material is taken to have the Drude form [44].

The RSI model is covered in section 5 in an analogous way. The dominant term in the Casimir force shows that the characteristic property strengthens the zero-temperature effect instead of weakening it as in RSII. From equation (81) we can evaluate the Casimir force for both $aT \ll 1$ and $aT \gg 1$, provided $T \ll k\mathrm{e}^{-k\pi r_c}$. This section also covers a comparison to flat space with and without a compactified extra dimension.

In section 6, the RSII model is considered. We have $\int_0^\infty \mathrm{d}M/k = \pi/k \int_{-\infty}^{\infty} \mathrm{d}M/(2\pi)$ and thus the Casimir force has a characteristic π/k times the Casimir force of a (4+1)-dimensional Minkowski spacetime. This is pointed out by Morales-Técotl *et al* [12], but as an argument against using zeta functions in the regularization in favour of Green's functions. It is not only the regularization method that is different; the physical picture also differs from this article and the work by Frank *et al* [10, 11]. While Frank *et al* calculate the free energy of a slice of the bulk, Morales-Técotl *et al.* try to restrict the system to the brane by evaluating the Green's function at the visible brane ($y = 0$ for RSII). In this way they claim to incorporate the localization properties of the modes of the scalar bulk field ($\psi_N(y)$). Note that the Green's function method includes an integral over y, and by setting $y = 0$, Morales-Técotl *et al* thus remove the y-dependence of a part of the integrand before integrating. However, keeping the y-dependence before the integration makes for results different to those of Frank *et al*. The delicate point is how to include the localization properties of the modes. In [36], it is proposed to resolve the issue by changing the boundary conditions. The problem with the localization properties of the modes needs to be resolved before the Casimir effect from an electromagnetic field can be considered.

Appendix. Integrals and Bessel functions

We need to calculate the nontrivial integral

$$\int_C^\infty \mathrm{d}x \left(A\sqrt{x^2-C^2}\cos\left(A\sqrt{x^2-C^2}\right) - \sin\left(A\sqrt{x^2-C^2}\right)\right)\mathrm{e}^{-Bx}. \qquad \text{(A.1)}$$

First use the substitution $u = \sqrt{x^2-C^2}$ to find

$$\begin{aligned}
&\int_0^\infty \frac{\mathrm{d}u}{\sqrt{u^2+C^2}}(Au^2\cos(Au) - u\sin(Au))\mathrm{e}^{-B\sqrt{u^2+C^2}} \\
&= A\frac{\partial^2}{\partial B^2}\int_0^\infty \mathrm{d}u\, \frac{1}{\sqrt{u^2+C^2}}\cos(Au)\mathrm{e}^{-B\sqrt{u^2+C^2}} \\
&-A\int_0^\infty \mathrm{d}u\, \frac{1}{\sqrt{u^2+C^2}}\cos(Au)\mathrm{e}^{-B\sqrt{u^2+C^2}} \\
&-\int_0^\infty \frac{\mathrm{d}u}{\sqrt{u^2+C^2}}u\sin(Au))\mathrm{e}^{-B\sqrt{u^2+C^2}}.
\end{aligned} \qquad \text{(A.2)}$$

Using equation (3.961) in [46],

$$\int_0^\infty \frac{x\mathrm{d}x}{\sqrt{\gamma^2+x^2}}\mathrm{e}^{-\beta\sqrt{\gamma^2+x^2}}\sin(ax) = \frac{a\gamma}{\sqrt{a^2+\beta^2}}K_1\left(\gamma\sqrt{a^2+\beta^2}\right), \tag{A.3}$$

as well as

$$\int_0^\infty \frac{\mathrm{d}x}{\sqrt{\gamma^2+x^2}}\mathrm{e}^{-\beta\sqrt{\gamma^2+x^2}}\cos(ax) = K_0\left(\gamma\sqrt{a^2+\beta^2}\right), \tag{A.4}$$

we obtain

$$\begin{aligned}&\int_C^\infty \mathrm{d}x\left(A\sqrt{x^2-C^2}\cos\left(A\sqrt{x^2-C^2}\right)-\sin\left(A\sqrt{x^2-C^2}\right)\right)\mathrm{e}^{-Bx}\\ &= A\frac{\partial^2}{\partial B^2}K_0\left(C\sqrt{A^2+B^2}\right)-\frac{AM}{\sqrt{A^2+B^2}}K_1\left(C\sqrt{A^2+B^2}\right)\\ &\quad -AM^2K_0\left(C\sqrt{A^2+B^2}\right).\end{aligned} \tag{A.5}$$

After differentiating and using the relationships between K_0, K_1 and K_2, we find the solution

$$\begin{aligned}&\int_C^\infty \mathrm{d}x\left(A\sqrt{x^2-C^2}\cos\left(A\sqrt{x^2-C^2}\right)-\sin\left(A\sqrt{x^2-C^2}\right)\right)\mathrm{e}^{-Bx}\\ &= -\frac{C^2A^3}{A^2+B^2}K_2\left(C\sqrt{A^2+B^2}\right).\end{aligned} \tag{A.6}$$

Next consider the integral

$$\int_0^\infty \mathrm{d}z(z^2+A^2)^{3/4}K_{3/2}\left(B\sqrt{z^2+A^2}\right). \tag{A.7}$$

With the substitution $u=\sqrt{(z/A)^2+1}$, we obtain

$$A^{5/2}\int_1^\infty \frac{\mathrm{d}u\; u^{5/2}}{\sqrt{u^2-1}}K_{3/2}(BAu) = \sqrt{\frac{\pi}{2B}}A^2\int_1^\infty \frac{\mathrm{d}u\; u^2}{\sqrt{u^2-1}}\left(1+\frac{1}{BAu}\right)\mathrm{e}^{-BAu}, \tag{A.8}$$

again using $K_3/2(z)=(\pi/2z)^{1/2}\,\mathrm{e}^{-z}(1+1/z)$. As the integral representation of $K_2(BA)$ is [47]

$$K_2(ax) = \frac{\pi^{1/2}\left(\frac{1}{2}ax\right)^2}{\Gamma\left(\frac{5}{2}\right)}\int_1^\infty \mathrm{d}u\;\mathrm{e}^{-axu}(u^2-1)^{3/2}, \tag{A.9}$$

we find after several partial integrations

$$K_2(ax) = \frac{\pi^{1/2}\left(\frac{1}{2}ax\right)^2}{\Gamma\left(\frac{5}{2}\right)}\frac{3}{(ax)^2}\int_1^\infty \frac{\mathrm{d}u\; u^2}{\sqrt{u^2-1}}\left(1+\frac{1}{axu}\right)\mathrm{e}^{-axu}. \tag{A.10}$$

Thus, the integral becomes

$$\int_0^\infty \mathrm{d}z\left(z^2+A^2\right)^{3/4}K_{3/2}\left(B\sqrt{z^2+A^2}\right) = \sqrt{\frac{\pi}{2B}}A^2K_2(BA). \tag{A.11}$$

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