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Optimal Capital Structure in Depository Financial Institutions

A Dynamic Programming Approach

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Problem Formulation

Purpose: Determine optimal capital structure behavior in depository financial institutions in the presence of uncertainty with dynamic programming methods.

1. Review and discuss literature related to a DFI's capital structure behavior and various solution approaches for the related mathematical optimization problem. Justify the choice of a dynamic programming-based solution approach.

2. Formulate the DFI's capital structure problem as a stochastic optimal control problem and solve with two distinct methods: by deriving and solving the Hamilton-Jacobi-Bellman partial differential equation and approximate dynamic programming using artificial neural networks.

3. Evaluate the results and their implications. Overall assessment of the strengths and weaknesses of the proposed model and solution methods.

Preface

We submit this thesis in fulfillment of the requirements for our Master of Science degrees in Industrial Economics and Technology Management at The Norwegian University of Science and Technology (NTNU).

Several people deserve special acknowledgement for their contribution towards its completion. First and foremost, we thank our supervisor, Associate Professor Einar Belsom at the Department of Industrial Economics and Technology Management at NTNU, for his patience, likeable humour and invaluable guidance. Special thanks are also due to Kjell Bjørn Nordal and Norges Bank for —ideas, inspiration and genuine interest in our work.

Moreover, debts of gratitude are owed to the following poeple: to Professor Warren B. Powell of Princeton University for helpful correspondence; to Professors Anne Kværnø and Anton Evgrafov of NTNU for steady mathematical guidance; to the MathWorks team for vital software, and to you, Siv, for the always dependable flow of coffee.

Abstract

We formulate a stochastic optimal control problem for the capital structure of depository financial institutions (DFIs), where the market value of equity is maximized in a dividend discount framework. The key objective is to study capital structure behavior of DFIs under uncertain market conditions when governmental capital regulations are imposed, in particular minimum requirements for the Core Tier 1 capital ratio given by the Basel III directive. We solve the problem by two distinct approaches based on dynamic programming: (1) by deriving and numerically solving the problem's Hamilton-Jacobi-Bellman partial differential equation, and (2) by an approximate dynamic programming approach using Q-learning with artificial neural networks.

The model is calibrated and tested for DNB ASA. The results obtained suggest that there is a significant cost for equity holders associated with governmental capital regulations, seen from the reduction in market value of equity when these are imposed. We find the impact of regulations to be stronger in times of high economic growth than during downturns. Moreover, given the current regulatory requirements, our model suggests that DNB ASA could increase shareholder value substantially by lowering its capital adequacy ratio.

Sammendrag

Vi formulerer et stokastisk kontrollproblem for innskuddsbaserte finansinstitusjoner, hvor målet er å bestemme kapitalstrukturen som maksimerer markedsverdien av egenkapitalen gitt dividendediskontering som underliggende verdsettelsesmodell. Hovedformålet med studien er å observere slike institusjoners oppførsel under usikre markedsforhold når regulatoriske kapitalkrav innføres, da spesifikt minimumskrav til kjernekapitaldekning gitt av Basel III-direktivet. Vi løser problemet med to ulike tilnærminger basert på dynamisk programmering: (1) ved å utlede og løse problemets Hamilton-Jacobi-Bellman-ligning, og (2) ved en approksimert dynamisk programmeringsmetode basert på Q-læring med kunstige nevrale nettverk.

Modellen kalibreres og testes for DNB ASA. Oppnådde resultater antyder en signifikant kostnad for egenkapitalholdere knyttet til regulatoriske kapitalkrav utover det implisitte markedskravet, representert ved en reduksjon i markedsverdien av egenkapitalen. Vi finner at effekten av myndighetskrav er større i tider med høy økonomisk vekst enn i nedgangstider. Resultatene våre antyder videre at DNB ASA, gitt dagens kapitalkrav, kunne skapt betydelig merverdi for investorer gjennom lavere kjernekapitaldekning.

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Introduction

This thesis aims to shed light on depository financial institutions (DFIs) and their fundamental behavior as value maximizing entities. These institutions play an important role in society as one of the key financiers of the real economy, being essential to secure growth both domestically and globally. Throughout history we have several times witnessed how damaging a crisis that arises within the financial system can be. This was last seen during the financial crisis of 2007-08 that was triggered by a complex interplay of several factors. A major factor was the real estate devaluation and financial asset depreciation associated with the US subprime mortgages in the aftermath of the burst of the US housing bubble (Shiller, 2008). The collapse of the bulge bracket investment bank Lehman Brothers and the major issues within the shadow banking system also contributed in turning the downturn into the ensuing credit crunch that led to the worst recession in 80 years (Hilsenrath et al., 2008; Romer, 2009).

The shadow banking system comprises non-bank financial institutions engaging in maturity transformation. When commercial banks fund long-term loans by the use of short-term deposits they engage in maturity transformation. Shadow banks did the equivalent by funding longer term assets with short term borrowing in money markets. The important distinction between commercial banks and shadow banks is that the latter are not subject to traditional bank regulation, which implies that they can neither borrow in an emergency from the central bank, nor have their short-term funding covered by insurance companies. Consequently, during the financial crisis, the too-big-to-fail (TBTF) problematics were brought to surface as the entire shadow banking system came close to a full collapse, and taxpayer-funded bailouts became a reality (Simkovic, 2009). As the credit markets tightened and financial institutions became increasingly risk averse in their lending activities, the problems moved from Wall Street to "Main Street".

Following the severe consequences of crises that arise within the financial system, we have seen that banks have been subject to massive regulation. The Basel Accords are a series of banking industry recommendations agreed upon by The Basel Committee on Banking Supervision (BCBS). The Committee has no authority to enforce the recommendations by law, so these must be approved and passed through national legislation. The main objective of the Basel Accords is to strengthen bank capital requirements by enforcing financial institutions to hold sufficient loss-absorbing capital, in addition to encourage banks to continue to finance economic activity and growth. Basel I and II were agreed upon in 1998 and 2004, respectively.

In retrospect of the financial crisis of 2007-08 and the following debt crisis in the Eurozone, a strengthened iteration of the Basel Accords (Basel III), has been proposed to remedy some of the fundamental shortcomings within the financial system that triggered the crisis. These were agreed upon in 2013, and the implementation act of Basel III is the new Capital Requirements Directive IV (CRD IV) in the European Union (EU) and The Market Risk Final Rules in the US. The new legislation concerns two main areas: capital requirements and asset and liability management. The new capital requirements are *"intended to help ensure that banks maintain strong capital positions, enabling them to continue lending to creditworthy households and businesses even after unforeseen losses, and during severe economic downturns"* (Federal Reserve, 2012).

The above snapshot of the last financial crisis and the regulatory changes agreed upon in the aftermath of the crisis, offers rudimentary insight into the scale and reach of such a crisis, and the large set of distinct stakeholders being affected. The fact that the crisis was centered around financial institutions, wherein DFIs are important entities, provides some rationale for why it is important to study and understand these types of firms.

The core business of DFIs is within depository lending, where they leverage a spread between lending and borrowing rates in order to generate revenues. Their primary functions are thus to accept deposits and to use the money collected for lending purposes. There exist three main types of DFIs: commercial banks, savings institutions and credit unions. We will primarily focus on commercial banks, and the terms DFI, bank and commercial bank will be used interchangeably depending on the context.

Given the central role DFIs play in respective economies, we seek to determine how these entities should channel resources most productively in an optimal stochastic control setting. We pay particular attention to the moral hazard problem arising between DFIs and regulatory forces, and how governmental institutions constrain the degree of freedom banks have when opting their capital structure in an attempt to maximize the value of shareholders' stake. In this regard, we use the Core Tier 1 capital ratio (CT1 ratio), defined as the strictest measure of a DFIs capital adequacy in the Basel Accords, to study how DFI's should adjust their behavior optimally.

We present a model and two solution methods based on dynamic programming in order to determine optimal DFI behavior under uncertainty and regulatory constraints in order to maximize the market value of equity. The uncertainty is represented by two important economic variables, the real GDP growth per capita (business cycle, economic cycle) and the money market interest rate (interbank rate, reference rate). The optimal behavior is assessed under different regulatory regimes and different economic states. One of our solution approaches follows the traditional line of DFI literature solving a stochastic optimal control problem through the Hamilton-Jacobi-Bellman equation. The other solution method is novel in a DFI setting in the sense that it is based on approximate dynamic programming with methods from artificial intelligence.

The results enable the assessment of optimal financing policies and corresponding capital structures under different economic conditions and regulatory regimes. The cost to shareholders of the regulatory capital requirement is also possible to quantify through differences in the implied fair value per share for various regimes and economic conditions.

This thesis differs from the existing research on optimal capital structure in banks in that our solution methods are based on numerical and simulation-based approximation methods rather than closed-form analytical expressions. In addition our model include multiple stochastic processes to model the uncertainty affecting these entities. We will now review some of the existing literature on the topic and position our work in greater detail.

1.1 Relation to Previous Literature

Modigliani and Miller (1958) argue that given a set of idealized market conditions, the value of any firm should be independent of how it is financed. This is not the case in real world markets, however; taxes, bankruptcy costs, governmental regulations, asymmetric information and other monetary frictions make the question of optimal capital structure essential to all firms, including DFIs. Since we know that these frictions are real and existing, the Modigliani-Miller theorem actually suggests that the value of a firm depends on how it is financed. Our approach, like the majority of the existing literature, take at least some of these frictions into account, meanwhile recognizing that the Modigliani-Miller theorem forms the basis for modern research on capital structure.

A subject that has been devoted considerable attention in modeling capital structure and banks' behavior, is the agency problem concerning moral hazard. The problem has been assessed in different papers, including Merton (1977), Dothan and Williams (1980), and Furlong and Keeley (1989). It comes from the fact that as regulators guarantee the value of bank deposits, bank shareholders are protected by limited liability. This limited liability, in combination with the bank's objective of maximizing shareholder value, creates a simple option interpretation, where the shareholder holds a deposit-insurance put option. The value of this option always increases in the volatility of returns, which can be obtained through leverage. Thus, the shareholders have an incentive to take higher risk, and in the case of liquidation can put the losses onto the regulator (Merton, 1977).

Berger et al. (1995) investigate how the role of market and regulatory capital requirements affect the behavior of financial institutions. They argue that the two main reasons why banks have a different capital structure than other firms, are (1) the presence of a regulatory safety net and (2) the regulatory capital requirements. They discuss the concept of the market's capital requirement. We understand this as the implied capital structure a firm should have such that the cost of capital provides fair compensation for the inherent risk of the assets. The market's capital requirements are possibly lowered by banks' access to a safety net, whereas regulatory capital requirements only matter if they affect the behavior of the bank beyond that of the market requirement, i.e. if the regulatory capital requirement is a binding constraint for the bank's behavior. Other findings include that banks may hold capital above the regulatory capital minimum in order to possibly exploit unexpected investment opportunities and withstand negative shocks, respectively. Our model enables us to assess the bank's behavior under different regulatory regimes. By altering the regulatory capital requirement, the bank's behavior and the consequences of the regulatory regime become apparent. The market capital requirement under a no-regulation regime, and the capital banks choose to hold above the regulatory minimum under different regulatory penalties, can both be assessed and evaluated against the theories of Berger et al. (1995).

Milne and Whalley (2001) present a continuous-time model of the capital decision problem in banks, as an extension of the Poisson audit model of Merton (1978). They combine Merton's model with standard corporate finance analysis, in which capital is chosen so as to balance the relatively lower cost of debt financing against the marginal increase in expected costs of liquidation. By assuming an arithmetic Brownian motion for the dynamics of the capital, and by applying the Hamilton-Jacobi-Bellman (HJB) partial differential equation with smooth-pasting, they arrive at an expression for the value function. The decision variables are dividends and a variable affecting cash flow uncertainty. The bank chooses these variables so as to maximize the sum of current dividends and the expected instantaneous capital gain as long as it continues in operation. This approach is in part similar to ours in that it uses stochastic dynamic programming through the HJB equation to arrive at an expression for the value function. However, we do not consider the uncertainty of cash flows a decision variable. This could be regarded as an allocation decision between risky and risk-free assets, which is an important aspect of the moral hazard problem described above.

Mukuddem-Petersen et al. (2007) argue that competition policy and decisions concerning capital structure constitute the key elements in describing DFI behavior. They model a macroeconomic factor stochastically to capture the cyclical dynamics of credit prices, risk-weights, provisions, profitability and capital. The dynamics of the factor is assumed to follow a geometric Brownian motion. Hackbarth et al. (2007) look at the impact of macroeconomic conditions on credit prices and dynamic capital structure. Their findings imply that firms should adjust their capital structure more frequently and by smaller amounts in booms than in recessions. They also discuss how the debt capacity is significantly larger in booms than in recessions. However, they do not look at industry-specific firms, as opposed to our work. We follow the line of Hackbarth et al. (2007) and model the stochastics of the macroeconomic conditions explicitly, through real GDP growth per capita and the interbank rate. Based on the exogenously determined economic variables in the state vector, the bank attempts to make optimal decisions.

Hugonnier and Morellec (2014) develop a dynamic decision model of banking incorporating taxation, floating costs of securities and default costs. In their model banks fund their operations with equity, insured deposits and risky debt. By applying Bellman's equation, the decision variables are jointly determined such that the maximization of shareholder value is achieved. As in our model the decision variables are dividend payments, presented as liquidity management by the authors, financing policies and default decision. Their findings suggest that capital requirements increase shareholders' willingness to absorb losses, thereby reducing default risk. However, they also describe how such requirements may substantially reduce bank value. They argue that introducing a capital requirement of 20% of total assets reduces the value of the bank by 6%. In our model setup we are also able to quantify the loss in value as a consequence of increased capital requirements.

Our model is inspired by the literature presented above. The bank faces a stochastic optimal control problem and applying methods from the field of dynamic programming such as the HJB equation and approximate methods makes sense. In line with parts of the existing literature, we model central economic measures in a stochastic fashion. In order to obtain a complete forward-looking model of DFIs' operations, we apply methods from stochastic calculus and Monte Carlo simulations to capture these uncertainties. Furthermore, we add to existing literature by applying numerical and approximate methods from the dynamic programming field of research to solve the optimal control problem that emerges.

Mukuddem-Petersen et al. (2007) and Hackbarth et al. (2007) model the economic cycle as stochastic and following GBM. In line with these articles we model the economic cycle stochastically, but we propose describing the dynamics with another process. Marcellino (2007) and Barnett et al. (2012) find that the Ornstein-Uhlenbeck (OU) process is well-suited for describing real GDP growth. Real GDP growth in turn is a good indicator of the economic cycle (Feld and Voigt, 2003; King and Levine, 1993). We therefore rely on these findings and model the real GDP growth as an OU process to represent the economic cycle.

In both Mukuddem-Petersen et al. (2007) and Hackbarth et al. (2007) the interest rate is calculated as a deterministic variable dependent on the economic cycle. However, knowing the importance of the interest rate on DFIs' operations we model it stochastic and in line with the findings of Cox et al. (1985). Consequently, we incorporate the economic cycle and interest rate effects explicitly. This enables us to dynamically model interest rates, risk-weighted assets and the impairment of loans and guarantees, which is crucial in order to provide a realistic model as both intuition, theory and empirical findings suggest that they are interrelated (see e.g. Borio et al., 2001).

To make inference on optimal DFI capital structure, we formulate a stochastic optimal control problem involving balance sheet items such as assets (loans and cash), liabilities (deposits and bonds) and DFI capital. Furthermore, we include key income statement items like net interest income and impairments of loans and guarantees, both of which are largely driven by their capitalized counterparts. This problem is in turn solved with different methods from dynamic programming.

To summarize, we argue that our work differs from the existing research in three main areas: (1) we apply other stochastic processes than GBMs in modeling the uncertainty, (2) we model both the economic cycle and the interest rate as stochastic variables and (3) we use numerical and in parts approximate solution methods to solve the optimal control problem DFIs face as analytic solution methods are intractable due to the complexity of the problem.

1.2 Outline of the Thesis

The rest of the paper is organized as follows. We start off with a specification of the problem in Chapter 2, including a more detailed description of DFIs and aspects surrounding these entities.

In Chapter 3 we study the anatomy of DFIs by defining important internal and external relationships. First we describe the valuation framework, as well as the balance sheet and income statement items used in our modelling approach. Subsequently we introduce the underlying stochastic processes assumed for the exogenous economic variables. Following, in Chapter 4 we present the dynamic programming framework and the specific methods applied to solve the optimal capital structure problem. In Chapter 5 discussions on model implementation and parameter estimation is presented.

Chapter 6 presents the results from a case study of DNB ASA, Norway's largest commercial bank, for which the model has been calibrated and tested. This forms the basis for an in-depth discussion of optimal capital structure decision-making under different regimes of capital regulation.

Concluding remarks are given in Chapter 7. We share our thoughts on the models' limitations, and propose our ideas for improvements and further research. Appendices with supporting concepts complete the paper.

2

Problem Description

In this chapter we offer a more detailed view on the key issues DFIs face when attempting to make optimal decisions on behalf of a selection of, or all, stakeholders. We pay particular attention to under what objective banks makes its decisions when constrained by regulatory requirements and the market's perception of the risk-return trade-off.

In traditional corporate finance literature, the objective of any for-profit firm is the maximization of market value. Jensen (2001), for example, writes that: *"Two hundred years of work in economics and finance implies that in the absence of externalities and monopoly, social welfare is maximized when each firm in an economy maximizes its total market value"*. However, in this objective lies assumptions that have been criticized by "managerial" literature, which apply other objective functions (Grossman and Stiglitz, 1977). The critics argue that the maximization of market value is possibly not the best measure of value creation, and that the interests of other stakeholders are not fairly served under this objective.

In the modeling of DFIs, the latter argument becomes particularly apparent. Merton (1977), among others, formalized that when the deposits of the firm are guaranteed by the regulator, and the firm operates under the objective of maximizing shareholder value, a principal-agent problem arises which may lead to moral hazard. The moral hazard comes from the idea that the shareholder has an incentive to take higher risk, and in the case of liquidation, can put losses onto the regulator. To prevent this, the regulator may introduce legislation which forces the firm to operate in a certain manner to reduce the probability and severity of liquidation.

A second principal-agent problem could arise between management and shareholders of the firm. This problem arises if there is a divergence between the objective function of maximizing shareholder value and the utility function of management, who are ultimately responsible for the direction and decisions of the firm. Such a divergence could be due to a number of factors, both in the short and long run. However, in our model we assume that the managers are inaugurated in a capacity to ensure that the interests of the shareholders are realized, thus assuming that the DFI maximizes the market value of equity in coherence with traditional corporate finance theory. Therefore we only consider the asymmetry between regulatory forces and the firm.

Banks can first and foremost impact their capital structure by altering the dividend payout ratio, issuing new securities and repaying capital providers. Any disposable net cash flow that stems from operating activities, investing activities and a change in financing items may be allocated to a set of assets with a given level of risk. The constraints on the decision variables of the DFI deserves a thorough discussion and have a central role in this thesis. We distinguish between constraints imposed by the regulatory and competitive environment, and firm-specific restrictions such as the balance between sources and uses of funding. The former category will be given more attention in the remainder of this chapter.

As a consequence of the moral hazard problem existing between DFIs and regulators, the latter have introduced legislation. National laws based on the Basel framework requires banks to hold enough loss-absorbing capital in order to avoid default if the bank is forced to write down the value of its assets in times of crisis. In particular, the Core Tier 1 Capital ratio (CT1 ratio) is the strictest measure of financial strength in the Basel II and III Accords. The CT1 ratio is defined as:

$$\text{CT1 ratio} = \frac{\text{Core Tier 1 Capital}}{\text{Risk-Weighted Assets (RWA)}} \tag{2.1}$$

Core Tier 1 Capital is defined by the Basel Accords as common stock and non-cumulative perpetual preferred stock, as well as retained earnings. In our model, CT1 capital is equivalent to the book value of equity (BVE), as this consists of the fully issued and paid stocks, as well as retained earnings. The denominator in the CT1 ratio are RWAs. The standard approach to calculating these involves multiplying the exposure by a standardized risk weight associated with that type of exposure. The standardized risk weights are prescribed in the legislation, and reflect regulatory judgment. In our model the DFI's exposure to risk class ω at time t is defined as $A_{\omega,t}$. Further, we let each risk class have a corresponding risk-weight, v_ω. Cash and deposits to central banks are assumed to be risk-free. In general risk-weighted assets at time t, RWA_t, are given as:

$$RWA_t = \sum_{\omega \in \Omega} v_\omega A_{\omega,t} \tag{2.2}$$

The CT1 ratio in our model, at time t, is then given by:

$$CT1_t = \frac{BVE_t}{RWA_t} \tag{2.3}$$

A restriction on the bank's capital structure is therefore that the CT1 ratio of the bank at time t, $CT1_t$, should be above the CT1 ratio imposed by current legislation, Λ_t:

$$CT1_t \geq \Lambda_t \tag{2.4}$$

The capital adequacy requirements in Basel III are more strict than in its ancestor directives Basel I and II. This implies that a large number of banks are forced to build capital in order to satisfy the constraint given by inequality (2.4). Since we calibrate our model to fit DNB, we believe it is appropriate to present distinct characteristics in the Norwegian model. In brevity, the Norwegian banks need to adapt their capital structure to comply with several capital buffers proposed by the Ministry of Finance and legislated by the Norwegian Parliament. The requirements of Norwegian banks are explained in detail in appendix A.

Although capital adequacy and capital building undoubtedly receive and deserve substantial attention from the stakeholders of DFIs, other financial metrics may be of equal or higher importance for managers in these types of institutions. During the internal employee presentation of Morgan Stanley's 2014 second quarter figures in London, Colm Kelleher, Head of Institutional Securities Group, emphasized the overarching goal of the firm to increase return on equity (ROE). This confirms our view that maximization of ROE is traditionally acknowledged as best practice from management's standpoint in financial institutions, including DFIs. This behavior could very well be a direct consequence of how senior management compensations are structured. It is the board of directors that determine these compensations, and their main responsibility is to represent the interests of the shareholders.

The sentiment on The Street to unconditionally pursue accretive deals is not completely consistent with the framework of Koller et al. (2010) that point to three main value drivers for any firm: growth, return on invested capital (ROIC) and risk, measured by the cost of capital (WACC). In a flow-to-equity framework where you assess the market value of equity as opposed to the enterprise value, the return on invested capital is replaced with return on equity (ROE) and cost of capital is replaced with cost of equity (COE). The aforesaid inconsistency stems from the fact that shareholders are not made better off in an accretive deal if the increase in cash flows is offset by more risky cash flows on a discounted basis. The conclusion we draw from this is that creation of shareholder value is first and foremost achieved through an increase in economic profit, defined by:

$$\text{Economic Profit} = (ROE - COE) \cdot BVE \tag{2.5}$$

This becomes even more interesting when you look at a DuPont analysis, which shows mathematically that ROE is determined by net income margin, asset turnover and, most importantly, leverage. This yields an expanded version of the framework of Koller et al., depicted in Figure 2.1, which indicates that increasing leverage,

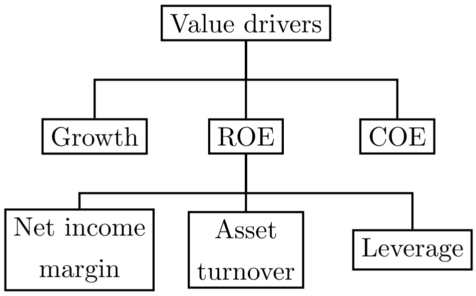

Figure 2.1: Key value drivers according to Koller et al. (2010)

increases ROE. The key question is the implication of increased gearing on levered cost of equity (COE). We will revert to the fundamentals within modern capital structure thinking, and employ Modigliani and Miller's proposition II (M&M II) with taxes. We argue that this way of structuring the risk premium for DFI's, being firms with high leverage, makes sense in order to make the risk for shareholders explicitly dependent on the degree of gearing:

$$COE = COE^U + \frac{D}{BVE}(COE^U - r^D)(1 - \tau) \tag{2.6}$$

where COE^U is the unlevered cost of equity, D is debt and r^D is the interest rate paid on debt. The tax benefit resulting from the interest expense deduction, will increase firm value by reducing the premium of the cost of equity capital related to leverage.

It can be seen directly from equation (2.5) that unconditional maximization of ROE is not optimal if the objective is to create shareholder value, since increasing leverage beyond a certain point should lead to greater marginal costs than benefits. The firm should under this objective rather maximize the spread between ROE and COE. Nevertheless, we do not question the behavioral pattern of management suggested by our empiricism. Instead we assume that they as utility maximizing agents observe how the board structures compensations and act accordingly. The role of the executive board, whether it carries out its duties and how it incentivizes management is beyond the scope of this thesis. Hence, when choosing the objective function of the firm we will adhere to the ideas of Koller et al. (2010). This allows us to test whether the underlying risk-return trade-off that is present will ensure the convergence of the CT1 ratio to a certain level in optimality.

3

Understanding Depository Financial Institutions

In this chapter we will offer an overview of the anatomy of banks' financial statements, key decision variables they face and the stochasticity that drive the value of these institutions. In this regard we will introduce decision variables and stochastic variables that we study in our modelling approach.

3.1 Internal Dynamics

The DFI solves a maximization problem for the market value of equity, which means an appropriate financial valuation method is needed. Our chosen method, and the rationale behind the approach, will be given next. Pertaining accounting items are subsequently explained.

3.1.1 Valuation of Financial Institutions

Based on the discussion in chapter 2, the DFI seeks to make optimal decisions such that the infinite series of cash flows to equity investors is maximized on a discounted basis. There exist several valuation frameworks that can be employed to model this objective, but one very intuitive approach to look at is the infinite flow of discounted dividends. The reason is that dividends paid out to investors is a direct cash flow they receive. Moreover, any net profit not paid out in dividends and hence reinvested in the business, constitutes internal (equity) funding, which will alter the capital structure and CT1 ratio. In order to explain the intuition behind the model, let us start with the mathematical formulation of the valuation expression. The market value of equity, MVE_t at time t is given by:

$$MVE_t = \sum_{j=t}^{\infty} \frac{\pi_j}{[1 + COE_j]^j} \tag{3.1}$$

where π_j is the nominal dividend flow to investors at time j, and COE_j is the cost of equity calculated by M&M's proposition II with taxes in equation (2.6). The functions presented in equation (3.1), for the market value of equity, is a perpetual sum, which stems from the common notion that the value of a company is a function of all future dividends shareholders receive. It is important to emphasize that the future dividends in the model are random variables, and thus the market value of equity depends on the probability space of the variables that determine these dividends. Therefore the market value of equity will be a discounted expectation in this probability space. Our treatment and assumptions concerning the stochastic value drivers for the DFI will be outlined in the remaining sections of this chapter.

As indicated earlier, company valuation can be done along multiple avenues. We believe it is important to cite competing approaches, and an alternative method of bank valuation is the excess return model. Although the various models should yield the same results as long as the input parameters match each other, we believe the dividend discount model is more intuitive in light of the discussion of capital structure and regulatory constraints in chapter 2. Under the model assumptions of excess return valuation, however, the market value of equity is given by:

$$MVE_t = BVE_t + PV_t(Excess\ Returns) \tag{3.2}$$
$$= BVE_t + \sum_{j=t}^{\infty} \frac{[ROE_j - COE_j]BVE_j}{[1 + COE_j]^j} \tag{3.3}$$

We can make use of the following relationship between net income, book value of equity and return on equity:

$$ROE_t = \frac{\Pi_t}{BVE_t} \tag{3.4}$$

in order to arrive at the following representation of the market value of equity:

$$MVE_t = BVE_t + \sum_{j=t}^{\infty} \frac{\Pi_j - COE_j BVE_j}{[1 + COE_j]^j} \tag{3.5}$$

where Π_t is the net income or net profit at time t, which will be discussed in further detail in section (3.1.3). Please note that in the above subsection the notational convention used with terms such as BVE, MVE, ROE and COE is merely in order to be very clear about the distinction between market values and book values, as well as market value of equity, enterprise value and share value. In the remainder of the thesis we use a more lean notation.

3.1.2 Sources and Uses of Funding

The core business model of DFIs is to generate income through a spread on interest rates between lending and borrowing activities. The individual DFI observes the aggregate market credit demand, and satisfies it in order to maintain its market share. The set Ω represents the loan classes that the bank can invest in. This set is assumed to contain loans with a risk weight strictly greater than zero. Consequently, the exposure to loan class ω at time $t + 1$ is given by:

$$L_{\omega,t+1} = L_{\omega,t}(1 + f^L_\omega(m_t, r_t)) \tag{3.6}$$

For our modeling purposes we will assume that the market credit demand, and hence the exposure the individual DFI has to loan class $\omega \in \Omega$, evolves as a function $f^L_\omega(m_t, r_t)$ of the economic cycle and the money market interest rate:

$$f^L_\omega(m_t, r_t) = i^L_\omega + j^L_\omega m_t + k^L_\omega r_t \tag{3.7}$$

The economic cycle variable, m_t, and the variable money market rate, r_t, are both modeled as exogenous stochastic variables. A more detailed discussion of the properties of these variables is offered in section 3.2. The linear form of the function is assumed to make the model simpler and more tractable. Consequently, assuming the only risky assets the DFI invests in are different loans then the total asset exposure at time t is given by:

$$A_t = R + \sum_{\omega \in \Omega} L_{\omega,t} \tag{3.8}$$

where R represents cash and cash equivalents and reserves that carry a risk-weight of zero.

In order to satisfy the loan demand the DFI must fund its lending activities. Deposits are the prime source of liability funding for a bank and a necessary ingredient needed to invest in a portfolio of loans that generate revenues. We will assume that deposits is a constant fraction, ϕ, of loans, and hence the evolution of deposits is given by:

$$D_{t+1} = \phi L_{\omega,t+1} = \phi L_{\omega,t}(1 + f^L_\omega(m_t, r_t)) \tag{3.9}$$

In addition to deposits the bank may fund its lending activities internally through its operating activities or through alternative external sources in the financial markets with issuance of debt and/or equity. Operating activities influence sources and uses of funding as the book value of equity increases when net income is retained in the business. Similarly, when dividends are paid out to equity investors the book value of equity is reduced by the same amount. The dividend payout amount introduces a decision variable in the model. We let $u_{t,1}$ be the dividend per share paid out to investors and n_t be the amount of fully diluted shares outstanding at time t and.

Issuance of equity also influence the book value of equity. When the DFI issues equity, the book value of equity increases by the same amount. The equity issuance amount introduces another decision variable in the model. We let $u_{t,3}$ be the nominal amount obtained through issuance of equity at time t. Based on this, the book value of equity at time $t+1$ is given as:

$$E_{t+1} = E_t + \Pi_{t+1} - u_{t+1,1}n + u_{t+1,3} \tag{3.10}$$

The issuance of additional equity increases the amount of fully diluted shares outstanding. Given the nominal amount obtained through equity issuance, $u_{t,3}$, the number of shares needed to obtain this amount is dependent on the issuance price. The new shares could be sold for the current share price or at a discount. Consequently, the new amount of fully diluted shares outstanding after an equity issuance is given as:

$$n_{t+1} = n_t + \frac{u_{t,3}}{\zeta MVE_t} n_t \tag{3.11}$$

where ζ is the premium or discount of issuance price relative to current share price.

Issuance of market debt or bonds introduces another decision variable, $u_{t,2}$, which represents the nominal amount of debt raised in fiscal period t. We will model the aggregate static exposure to debt capital markets at time t through the variable B_t. Consequently, total liabilities L_t^D at time t is given as:

$$L_t^D = D_t + B_t \tag{3.12}$$

Considering the intuitive requirement that uses of funding need to match sources of funding, we get the following constraint in our model:

$$A_t = E_t + L_t^D \tag{3.13}$$

Having this overview of sources and uses of funding in mind, we will move on to explain our assumptions on income statement items and how we arrive at an expression for net income, P_t.

3.1.3 Income Statement (P&L)

Banks generate revenue from numerous sources, including interest, commissions and fees, derivatives, and investment banking and financial advisory.

The main source of revenue in a bank is net interest income, defined as the difference between interest revenues and interest expenses. We let interest revenue R_t^{Int} be the sum of interest rate payments the bank receives on its assets. Assets are here defined as cash and cash equivalents or reserves and loans. Thus, interest revenue is given by:

$$\mathrm{R}_{t+1}^{Int} = r_{t+1}R_t + \sum_{\omega\in\Omega}(r_{t+1} + \Delta^{L_\omega})L_{\omega,t} \tag{3.14}$$

where R_t is cash or reserves, $L_{\omega,t}$ is the exposure to class ω, r_{t+1} is the appropriate interbank rate and Δ^{L_ω} is the spread or margin required for a loan with risk indicated by class ω. Note that in order to avoid a circular reference in the calculation of balance sheet and income statement items, we let the interest revenues be a function of the asset exposure at the beginning of the year. Similarly, we let interest cost, C_{t+1}^{Int}, be the interest rate payments on deposits and other debt-like instruments, given by:

$$\mathrm{C}_{t+1}^{Int} = (r_{t+1} + \Delta^D)D_t + (r_{t+1} + \Delta^B)(B_t + u_{t+1,2}) \tag{3.15}$$

where D_t is deposits, B_t is the beginning-of-the-year amount of bonds, $u_{t+1,2}$ is the amount of bonds issued in the interim, r_{t+1} is the appropriate interbank rate, Δ^D is the spread paid on deposits and Δ^B is the spread paid on bonds.

Furthermore, the bank generates revenues through other sources, such as commissions and fees, derivatives, and investment banking and financial advisory. It also incurs operating expenses such as personnel expenses (salaries and pension expenses), IT expenses, marketing expenses, and expenses associated with property and premises. These revenues and expenses may be modeled in a deterministic variable. However, in our dynamic model such a deterministic variable has a limited effect. Recognizing that banks primarily generate revenues through net interest income we leave out other revenues and cost from the model.

An important item that we include in the P&L model is the impairment of loans and guarantees. Impairment of loans and guarantees is only applicable to assets in Ω that carries a risk weight strictly greater than zero. We let $I_{\omega,t}$ be the dollar value of impaired loans in class ω at time t, and assume that the percentage impairment, $f^I_\omega(m_t, r_t)$, is a function of the two exogenous stochastic variables m_t and r_t such that:

$$f^I\omega(m_t, r_t) = \min\{0, f^L_\omega(m_t, r_t)\} \tag{3.16}$$

Hence total impairments is given by:

$$I_{\omega,t} = \sum_{\omega\in\Omega} f^I\omega(m_t, r_t)L_{\omega,t} \tag{3.17}$$

The bank's pre-tax profit, Π^τ_t, at time t is then given by:

$$\Pi^\tau_t = R^{Int}_t - C^{Int}_t + \sum_{\omega\in\Omega} I_{\omega,t} \tag{3.18}$$

By applying the corporate tax rate τ, the bank's net income at time t, P_t, is given by

$$\Pi_t = \Pi^\tau_t(1 - \tau) \tag{3.19}$$

3.2 External Dynamics

In sections 3.1.2 and 3.1.3 we briefly presented the two exogenous stochastic variables in our DFI model: the economic cycle variable, m_t, and the money market interest rate variable, r_t. It is the stochasticity in these two variables that determines the value of the DFI under our model assumptions. Without them the DFI would face zero risk and we would be able to calculate the infinite series of dividends deterministically based on optimal decisions. The problem of determining the true DFI value function is therefore essentially a problem of pricing a derivative on the economic cycle and the interest rate variables. We will defer the technicalities surrounding this problem to chapter 4, and will in this section rather focus on the stochastic processes we assume for the respective variables as well as the discretization of these.

3.2.1 Economic Cycle

The variable used to indicate the economic cycle is real GDP per capita. Feld and Voigt (2003) and King and Levine (1993) both describe real GDP as a good indicator of the economic cycle. Worth noting is that it is a procyclical indicator. The value of the economic cycle, m_t, is assumed to evolve according to an Ornstein-Uhlenbeck process (OU process). These dynamics are inspired by the work of Marcellino (2007) and Barnett et al. (2012), which assess different time series models for forecasting GDP growth. Their findings show that using an OU process is well-suited for such modeling. Even better models would be mean-reverting processes with time-varying long-term mean and volatility. However, for simplicity we apply the OU process which is given on the differential form:

$$\mathrm{d}m_t = \delta(\mu - m_t)\mathrm{d}t + \eta \mathrm{d}Z_t^m \tag{3.20}$$

where Z_t^m is a standard Brownian motion. μ represents the long-term average GDP growth, whereas δ is the reversion or attraction rate to the long-term average, that is, how strongly the variable tends to return to the long-term average. η represents the volatility of the economic cycle. We also note that the Ornstein-Uhlenbeck process is stationary, Gaussian and Markovian, all important properties exploited in the further work.

3.2.2 Money Market Rate

For the dynamics of the money market rate variable, r_t, we have assumed that the rate follows a Cox-Ingersoll-Ross model (Cox et al., 1985). The model is frequently used in describing the evolution of the interest rate, see Brigo and Mercurio (2007), among others. This process is a combination of a mean-reversion movement and a Brownian motion based diffusion part. Consequently, the dynamics of the interest rate is given by the following stochastic differential equation:

$$\mathrm{d}r_t = \alpha(\beta - r_t)\mathrm{d}t + \sigma\sqrt{r_t}\mathrm{d}Z_t^r \tag{3.21}$$

where Z_t^r is another standard Brownian motion. The process resembles the Ornstein-Uhlenbeck process used in modelling the economic cycle. α is the reversion or attraction rate to the long-term average interest rate level β. The volatility term, $\sigma\sqrt{r_t}$, implies that the CIR model assumes non-negative interest rates. σ is the volatility parameter of the interest rate variable.

Lastly, the correlation between the two processes governing the evolution of m_t and r_t is given by:

$$\mathrm{d}Z_t^m \mathrm{d}Z_t^r = \gamma \mathrm{d}t \tag{3.22}$$

3.2.3 Discretization and Simulation

In order to use simulations and numerical methods to solve the optimal control problem we first need to discretize the continuous differential equations 3.20 and 3.21. For the economic cycle dynamics the forward equations become:

$$m_{t+\Delta t} = m_t + \delta(\mu - m_t)\Delta t + \eta\Delta Z^1 \tag{3.23}$$

where ΔZ^1 is the Brownian motion increment from time t to time $t + \Delta t$, which is normally distributed with mean 0 and variance Δt. We note that the discrete-time version of the Ornstein-Uhlenbeck process is the auto-regressive(1) process, AR(1).

For the dynamics of the base rate we have the following forward equation:

$$r_{t+\Delta t} = r_t + \alpha(\beta - r_t)\Delta t + \sigma\sqrt{r_t}\Delta Z^2 \tag{3.24}$$

Similarly as in the economic cycle, ΔZ^2 denotes the Brownian motion increment from time t to time $t + \Delta t$.

As the values of all other right-hand side elements of the equations are known, all we need to do to simulate the values of the stochastic variables in the next increment, is to sample two independent random variables x_1 and x_2 from this distribution. We use a Cholesky Decomposition (Gentle, 1998) to create ΔZ^1 and ΔZ^2 with the correct correlation, ρ_{12}:

$$\Delta Z^1 = x_1$$

$$\Delta Z^2 = \rho_{12} x_1 + \sqrt{1 - {\rho_{12}}^2}\, x_2$$

$$x_1, x_2 \sim \mathcal{N}(0, \Delta t) \quad i.i.d.$$

This procedure is repeated for all samples in the simulation to create a complete set of samples. When a large number of samples are simulated, an approximate probability distribution for the transitions is derived. For a description on estimation methods to derive the nominal values of the parameters please see section 5.1.

4

Decision Model and Solution by Dynamic Programming

The problem description in chapter 2 and the overview of DFIs in chapter 3 provide the necessary foundation to formulate the stochastic optimal control problem that these institutions solve. The objective is to determine a control law or policy for some stochastic system such that a given set of optimality conditions are met. Problems of this kind can be solved by traditional mathematical programming methods or by dynamic programming. We have chosen a dynamic programming based approach. In the chapter to follow, we will justify our choice with an in depth discussion of the decision model and solution approach.

4.1 Dynamic Programming

Dynamic programming is a powerful algorithmic paradigm that is widely used for solving problems in mathematics, economics and computer science. This section presents the basic concepts of dynamic programming and shows how they can be applied to our DFI optimal capital structure problem.

4.1.1 Key Dynamic Programming Concepts

Dynamic programming offers a general optimization strategy in which a complex multistage decision problem is transformed into a sequence of simpler sub-problems that are solved individually and recombined to form a solution to the original problem. The sub-problems must be overlapping and exhibit what is known as optimal substructure for the dynamic programming approach to be applicable. By overlapping sub-problems we understand that the solution to each sub-problem is reused many times to solve other sub-problems, and by optimal sub-structure we mean that an optimal solution to a problem can be found by combining the optimal solutions to its sub-problems. Depending on the nature of the original problem, the term "simpler sub-problem" could both refer to problems of smaller size and to problems with shorter time-horizons.

Regardless of the interpretation, dynamic programming calls for a recursive mathematical formulation in which the optimal solution to the original problem is written in terms of the optimal solutions to its sub-problems. Typically, in a discrete-time model world, the complex problem can be written as a sum of contributions (costs or rewards), where the objective is to pick a sequence of decisions (a control law) so as to minimize or maximize this sum given the current state of the underlying system. In the case of a stochastic problem, where the state variables evolve according to some stochastic process, we would consider the expected value of the sum. In a continuous-time model world, the sum would become an integral.

Let us consider the discrete-time, finite time-horizon case. x_t represent the stochastic state variables of the system and u_t the control variables at time-step $t \in \{1, 2, ..., T\}$. A control law (policy) is denoted by $\psi = (u_1, u_2, ...u_T)$. The time-step t value of the immediate contribution from the system is given as a function of the state vector and the control policy by $\pi_t^{\psi}(x_t, u_t)$. ρ is the discount rate. We establish the risk premium such that it is increasing in the interbank rate and leverage. We refer to chapter 2 for the discussion of the appropriate method of pricing risk in DFIs. Please note that in earlier chapters we have used COE for the cost of equity to be clear about its meaning, while we will proceed with ρ which is more neat in some of our lengthy expressions. We can now formulate the original optimization problem as follows:

$$\max_{\psi} \mathbb{E}\bigg\{ \sum_{t=1}^{T} \frac{\pi_t^{\psi}(x_t, u_t)}{(1+\rho)^t} \bigg\} \tag{4.1}$$

Solving for the optimal control law directly is often computationally intractable, however, and we utilize the dynamic programming assumptions of overlapping sub-problems with optimal sub-structure to split the problem into two parts: the first part would be the immediate contribution received by applying some control given the current state of the system, and the second part would be the optimal solution to the next (or closest) sub-problem left to solve. In other words, dynamic programming models a trade-off between immediate rewards and the desirability of the context in which future decisions are to be made. Typically, the next sub-problem would be the problem one time-step closer to the time-horizon or one size "smaller" than the original problem. This sub-problem could be recursively formulated in a similar manner, and we could continue this process until some terminal value was reached. By recombining the optimal solutions to the sub-problems, we could construct an optimal solution to the original problem. Mathematically, the recursion can be written as:

$$V_t(x_t) = \max_{u_t} \left(\pi_t(x_t, u_t) + \frac{1}{1+\rho} \mathbb{E}\{V_{t+\Delta t}(x_{t+\Delta t}) \mid x_t, u_t\} \right) \tag{4.2}$$

Equation 4.2 is known by a range of different names. First introduced by Richard Bellman in 1957, the inventor of dynamic programming, it is commonly known as the Bellman equation or Bellman's optimality equation. Other names include the "functional equation", "optimality equation" or "recurrence equation" of dynamic programming. We will primarily refer to it as the Bellman equation in the remainder of this thesis.

4.1.2 DFI Decision Problem in a Dynamic Programming Context

Let us now consider the DFI optimal capital structure problem in a dynamic programming context. Following the dividend discount valuation model presented in chapter 3, the market value of a DFI's equity is given by the present value of all future dividends paid to shareholders. By dividing the equity value by the total number of shares issued, we get the stock price P_0. Mathematically, this can be seen as the infinite time-horizon version of 4.1, with $\pi_t^{\psi}(x_t, u_t)$ now being the dividends per share paid to shareholders at time t under policy ψ:

$$P_0 = \max_{\psi} \mathbb{E}\left\{ \sum_{t=1}^{\infty} \frac{\pi_t^{\psi}(x_t, u_t)}{(1+\rho)^t} \right\} \tag{4.3}$$

We saw in chapter 3 that the net profits available for dividend payouts depend on some exogenous stochastic processes and are consequently stochastic themselves. Hence we consider the expected present value. The corresponding Bellman equation becomes:

$$P_t(x_t) = \max_{u_t} \left(\pi_t(x_t, u_t) + \frac{1}{1+\rho} \mathbb{E}\{P_{t+\Delta t}(x_{t+\Delta t}) \mid x_t, u_t\} \right) \tag{4.4}$$

Due to the infinite time-horizon assumed, the notion of a terminal value does no longer make sense. There will always be an infinite amount of time left for the system to exist, so there is nothing that fundamentally separates one time-step from another anymore. In other words, while the current state x_t certainly matters, it is not important at which particular point on the time-line the system is considered.

Consequently, we can assume the system to be in a steady state, making the optimal controls time-independent. This allows us to restrict the search for an optimal sequence of controls to a search for an optimal stationary policy. When we omit the time index from the equation we get:

$$P(x) = \max_u \left(\pi(x,u) + \frac{1}{1+\rho} \mathbb{E}\{P(x') \mid \; x, u\} \right) \tag{4.5}$$

Here x' is used to represent a generalized successor state.

The rest of this chapter is devoted to presenting two distinct approaches for solving equation 4.5. In section 4.2, we derive a partial differential equation known as the Hamilton-Jacobi-Bellman equation and solve this numerically. In section 4.3, we combine dynamic programming concepts with elements from artificial intelligence in a solution framework called approximate or neuro-dynamic programming.

4.2 The Hamilton-Jacobi-Bellman Equation

Equation 4.5 is sometimes also referred to as the Hamilton-Jacobi-Bellman equation, but we reserve this term for the partial differential equation (HJB-PDE) you arrive at when you Taylor expand the value function on the right hand side of 4.5.

4.2.1 Formulating the HJB Equation

In the general formulation of the Bellman equation given by 4.5, x denotes the vector of state variables. In the context of the HJB-PDE, this vector of state variables corresponds to the stochastic state variables of the system. Our formulation of the DFI problem deals with two such variables: the economic cycle explained by the macroeconomic variable m and the money market interest rate r. In the derivation of the HJB-PDE for our problem, we will treat the dependence on both these variables explicitly, as opposed to the earlier vector notation in the general model formulation.

Due to the stationary nature of the DFI optimal capital structure problem, the corresponding HJB-PDE will only be a partial differential equation in spatial dimensions and not in time. Intuitively, since the value function is common to all periods, any partial derivative taken with respect to time will be zero. However, we should emphasize that this only works when the dividend flow function π, the transition probability distribution and the discount rate ρ are all independent of the label of the date, all of which are properties assumed in this modelling approach. The number of spatial dimensions is equivalent to the number of stochastic variables that the firm value depends on. We hence get a partial differential equation in the economic cycle and the interest rate.

In chapter 3 we introduced the decision variables we will study in our model. We let $u_{t,1}$ be the dividend per share paid out to the shareholders. The dividend per share will be a function of the state variables m and r. Taking the considerations in this subsection into account we can formulate the Bellman equation for our problem in the following manner:

$$P(m,r) = \max_{u \in \mathcal{U}} \left\{ u_1(m,r) + \frac{1}{1+\rho} \mathbb{E}[P(m',r')|m,r,u] \right\} \tag{4.6}$$

Here m' and r' denote any feasible state the system can transition to, and the expectation is conditioned on the information in the current values of m and r and the control that is applied u. The reason the optimization problem is over u as opposed to only u_1 is that the model has multiple decision variables. In the HJB approach we will, however, not consider equity issuance (captured by the variable $u_{t,3}$ defined in chapter 3) and incorporate market debt funding as the only external funding alternative. This is to make the solution approach more tractable. Later when we look at approximate dynamic programming we will include both market debt and equity funding. Furthermore we observe that the controls u are constrained by the set $\mathcal{U}$, which is intended to indicate the set of admissible controls. The space of admissible controls is spanned by the following relationships:

$$L(1 + f^L(m,r)) + R = D + B + u_2 + E + \Pi(m,r) - u_1 n \tag{4.7}$$

$$\Pi(m,r) = [(r+\Delta^L)L + rR - (r+\Delta^D)D - (r+\Delta^B)B - \left(r + \Delta^B + \Delta^{u_2}\frac{(u_2+B+D)}{E}\right)u_2 + f^I(m,r)][1-\tau] \quad (4.8)$$

$$E + \Pi(m,r) - u_1 n \geq \Lambda L(1 + f^L(m,r)) \quad (4.9)$$

$$u_1, u_2 \geq 0 \quad (4.10)$$

We see from equation 4.7 that any market debt issuance in any interim becomes just the residual funding requirement in order to support the growth the bank needs to maintain it's market share. Equation 4.7 is the balance sheet equation between sources and uses of funding. The reader should notice that in reality this residual could be a combination of bond and equity funding depending on what is optimal and the availability of capital in debt and equity capital markets. In the HJB approach, however, we force this residual to only represent fixed income instruments.

Furthermore we also need to emphasize that constraint 4.9, the Core Tier 1 Capital Constraint, is not imposed as a hard constraint in our implementation. Rather we use it as a condition that if not met has the immediate implication that the government takes control of the bank's operations and the value of the bank will be set to zero in state (m,r). This is supposed to model the bankruptcy option held by equity investors.

Inasmuch as our stochastic processes are continuous time processes we will derive the HJB-PDE based on the continuous time version of the Bellman equation. This version is obtained by letting $u_1(m,r)$ and ρ be rates that we multiply with Δt in order to obtain the respective dollar amounts. Then we multiply with $(1+\rho\Delta t)$ on both sides, rearrange and let Δt go to zero in order to obtain:

$$\rho P(m,r) = \max_{u \in \mathcal{U}} \left\{ u_1(m,r) + \frac{1}{dt}\mathbb{E}[dP] \right\} \quad (4.11)$$

We can Taylor expand dP around (m,r) and use the appropriate multiplication rules and that higher than second order terms captured by $o(dt)$ goes to zero faster than dt. This exercise yields the HJB-PDE given by:

$$\max_{u\in\mathcal{U}}\left\{\frac{1}{2}\eta^2 P_{mm}(m,r)+\gamma\eta\sigma\sqrt{r}P_{mr}(m,r)+\frac{1}{2}\sigma^2 rP_{rr}(m,r)+\right.$$
$$\left.\delta(\mu-m)P_m(m,r)+\alpha(\beta-r)P_r(m,r)-\rho P(m,r)+u_1(m,r)\right\}=0 \quad (4.12)$$

For a more detailed derivation of the HJB-PDE please see appendix B.

4.2.2 Numerical Solution of the HJB-PDE with an Upwind Finite Difference Scheme

The HJB-PDE is a partial differential equation with the additional characteristic, relative to a traditional PDE, that it also includes the optimization problem of the system control. A way to look at the HJB-PDE is to recognize that one seeks the DFI value $P(m,r)$, for every point (m,r) in the two-dimensional domain, which is a response to an optimal decision being made in the control space $\mathcal{U}$. Any optimal dividend action $\pi(m,r)$ substituted in the HJB-PDE for $u_1(m,r)$ produces a partial differential equation that can be solved using traditional PDE methods to find the value or reward. Let us first explain how we solve these PDEs, and subsequently outline the iterative implementation structure that the HJB-PDE requires due to the action-reward tuples. Our PDE can be expressed in the following more compact vector form:

$$\frac{1}{2}\mathrm{tr}\{C(r)\nabla^T\nabla P\}+\theta\cdot\nabla P-\rho P+\pi(m,r)=0 \quad (4.13)$$

where $\pi(m,r)$ denotes the optimal dividend decision in (m,r), i.e. the dividend yielding the largest value $P(m,r)$ for the given state. Using jargon from the world of differential equations, this term is called the source term or the source function.

In equation 4.13, tr$\{\cdot\}$ is the trace of the matrix $[C(r)\nabla^T\nabla P]$. ∇ is the nabla operator, which for the two-dimensional case is given by $[\frac{\partial}{\partial m}, \frac{\partial}{\partial r}]$. $C(r)$ is the symmetric diffusion coefficient matrix as a function of r given by:

$$C(r) = \begin{bmatrix} \eta^2 & \gamma\eta\sigma\sqrt{r} \\ \gamma\eta\sigma\sqrt{r} & \sigma^2 r \end{bmatrix}$$

θ is the advection coefficient vector given by $[\delta(\mu - m), \alpha(\beta - r)]$. ρ is the discount rate, which represents the absorption coefficient in the PDE. The PDE in equation 4.13 can be classified as an elliptic partial differential equation. Mathematically, the ellipticity of the equation is a result of the positive definiteness of the diffusion coefficient matrix $C(r)$, i.e.: $\det(C(r)) = \eta^2\sigma^2 r(1-\gamma^2) > 0$ inasmuch as $-1 < \gamma < 1$ and $\gamma \neq 0$.

Partial differential equations can be solved with analytic and numerical methods. However, the feasible solution methods is largely dependent on the form of the partial differential equation. Generally it is difficult or potentially even impossible to find exact solutions of PDEs unless they are fairly simple and/or live on a simple geometry.

One way to pursue analytic solutions is to look at the characteristic curves of the second order PDE and the classification of the partial differential equation into one of three forms: hyperbolic, parabolic or elliptic, where our equation is elliptic. This implies that the characteristic equation (formed by the quadratic equation $ax^2 + 2bxy + cy^2 = constant$ representing the principal part of the equation) produces an ellipsis. We can introduce a variable transformation into a new system of coordinates (ξ, χ) such that the new variables are constant along the characteristic curves. This exercise should make the cross-partial term P_{mr} vanish, and we are left with an equation that is similar to the Poisson equation. However, unlike the Poisson equation we will still have the lower order terms. Additionally, the characteristic curves are in general imaginary for elliptic equations. Thus we would need a second change of variables as the variables (ξ, χ) will be complex conjugates. The diffusion coefficient matrix's dependency on r also makes the problem more difficult. Bearing this in mind it is not immediately clear that the method outlined above will make the PDE substantially easier to solve analytically. Hence we will turn to numerical methods when we attempt to solve equation 4.13.

Numerical methods for partial differential equations is a well-studied mathematical field with a wide range of plausible avenues to success. We solve our equation using finite differences. Our proposed solution scheme contains a mixture of forward, backward and centered finite differences. More precisely we implement an upwind finite difference scheme in order to avoid oscillations that we observe in the value function when limiting ourselves to centered finite differences. According to Binder and Aichinger (2013) upwind schemes generally use an adaptive finite difference stencil to numerically simulate the direction of propagation of information in a flow field . It is normally the advection (transport) terms that cause numerical instability. Consequently the discretization of these terms rely on their signs. The scheme employ a forward finite difference in the case of a negative sign and a backward finite difference whenever the sign is positive. This ensures that you never use information you do not have. The finite difference approximations we use for the partial derivatives are given by:

$$
\begin{aligned}
P_m(m,r) &= \frac{P(m+h,r) - P(m,r)}{h} \qquad \forall \delta(\mu - m) < 0 \\
P_m(m,r) &= \frac{P(m,r) - P(m-h,r)}{h} \qquad \forall \delta(\mu - m) > 0 \\
P_r(m,r) &= \frac{P(m,r+k) - P(m,r)}{k} \qquad \forall \alpha(\beta - r) < 0 \\
P_r(m,r) &= \frac{P(m,r) - P(m,r-k)}{k} \qquad \forall \alpha(\beta - r) > 0 \\
P_{mm}(m,r) &= \frac{P(m+h,r) - 2P(m,r) + P(m-h,r)}{h^2} \\
P_{rr}(m,r) &= \frac{P(m,r+k) - 2P(m,r) + P(m,r-k)}{k^2} \\
P_{mr}(m,r) &= \frac{P(m+h,r+k) - P(m+h,r-k) - P(m-h,r+k) + P(m-h,r-k)}{4hk}
\end{aligned}
$$

Together these finite differences include all points in the grid that lie on the perimeter of $P(m,r)$ in addition to $P(m,r)$ itself. Consequently this is a 9-point finite difference stencil. Now we can replace the partial derivative terms in the PDE with the finite differences above, which yields:

$$\begin{aligned}
&\frac{1}{2}\eta^2\left[\frac{P(m+h,r)-2P(m,r)+P(m-h,r)}{h^2}\right]+\\
&\gamma_{mr}\eta\sigma\sqrt{r}\left[\frac{P(m+h,r+k)-P(m+h,r-k)-P(m-h,r+k)+P(m-h,r-k)}{4hk}\right]+\\
&\frac{1}{2}\sigma^2 r\left[\frac{P(m,r+k)-2P(m,r)+P(m,r-k)}{k^2}\right]+\\
&\min\{\delta(\mu-m),0\}\left[\frac{P(m+h,r)-P(m,r)}{h}\right]+\max\{\delta(\mu-m),0\}\left[\frac{P(m,r)-P(m-h,r)}{h}\right]+\\
&\min\{\alpha(\beta-r),0\}\left[\frac{P(m,r+k)-P(m,r)}{k}\right]+\max\{\alpha(\beta-r),0\}\left[\frac{P(m,r)-P(m,r-k)}{k}\right]-\\
&\rho P(m,r)+\pi(m,r)=0 \qquad (4.14)
\end{aligned}$$

The objective is to find the value function $P(m,r)$ and hence we solve equation 4.14 for $P(m,r)$:

$$\begin{aligned}
&P(m,r)=\\
&\frac{1}{4}\cdot[k^2\eta^2+h^2\sigma^2 r+\\
&hk^2[\min\{\delta(\mu-m),0\}-\max\{\delta(\mu-m),0\}]+\\
&h^2k[\min\{\alpha(\beta-r),0\}-\max\{\alpha(\beta-r),0\}]]^{-1}\cdot\\
&[2k^2\eta^2[P(m+h,r)+P(m-h,r)]+hk\gamma\eta\sigma\sqrt{r}[P(m+h,r+k)-P(m+h,r-k)-\\
&P(m-h,r+k)+P(m-h,r-k)]+2h^2\sigma^2 r[P(m,r+k)+P(m,r-k)]+\\
&4hk^2[\min\{\delta(\mu-m),0\}P(m+h,r)-\max\{\delta(\mu-m),0\}P(m-h,r)]+\\
&4h^2k[\min\{\alpha(\beta-r),0\}P(m,r+k)-\max\{\alpha(\beta-r),0\}P(m,r-k)]+\\
&\pi(m,r)] \qquad (4.15)
\end{aligned}$$

Equation 4.15 allows us to compute the DFI value in (m, r) as a function of the grid values surrounding it.

The Hamilton-Jacobi-Bellman equation is a boundary value problem (BVP) since the partial differential equation, in addition to being constrained in the control space $\mathcal{U}$, also needs to satisfy boundary conditions in the independent variables m and r. Consequently, considerations of these boundary conditions and the geometry on which the PDE is valid, are in order.

When solving the HJB-PDE, we first need to construct the grid on which we would like to compute the value function. Not surprisingly it is the boundary conditions that constitute the perimeter of this grid. Our aim in this thesis is to determine optimal capital structure in DFIs, and the decision model we build chooses the controls u such that the value function is maximized for each state (m, r). Hence, computing the value function for all (m, r) is a vital part. However, in order to compute the value function for a given control, we need boundary data. This implies that we need a relationship between the value function and/or some derivative of the value function and the state variables on the boundary of the domain.

Characteristic for elliptic equations is that you need boundary conditions on all boundaries of the geometry where the equation is valid. These boundary conditions could be Dirichlet, Neumann or a combination. A relationship between the value function and the variables on the boundary of the domain is called a Dirichlet boundary condition. A relationship between the first derivative of the value function with respect to one of its state variables, and these variables themselves on the other hand, is termed a Neumann boundary condition.

We cannot be completely sure of any such relationships, because the value of the DFI is a random variable that depends on many additional variables besides the two we model in this thesis. Hence we approximate the boundary conditions to the best of our ability by leveraging empirical data and intuition on limiting cases. Let us briefly discuss how the boundary data is determined in our study of DNB.

The Norwegian economy entered a recession in 2009 as the financial crisis that hit financial markets in 2008 made its way into the real economy. The DNB share price plummeted to an all-time low of NOK 16.70 per share during the financial turmoil. If we recognize the market sentiment during the financial crisis of 2008 to be representative of a downside extrema in the business cycle, we are able to construct a boundary condition for the value of DNB when the macroeconomic variable takes on its most extreme value on the downside. Equivalently, historic share price data allows us to easily find DNB's all-time high. Consequently, the value of DNB for the most extreme value of the macroeconomic variable on the upside could be set to some number above the all-time high.

The arguments presented above on the mapping between a value of the macroeconomic variable and the value of DNB is more ambiguous when looking at the interest rate variable. There are multiple reasons for this ambiguity, but an important one is that the value of any DFI is perhaps even more dependent on interest rate margins or spreads than upon the level of the reference rate itself. Therefore it is difficult to envision how the gradient of the interest rate variable changes as you move along the domain of r. This in turn makes it difficult to establish a Dirichlet condition on the extrema of this variable. So instead of speculating on a Dirichlet condition we use a fine grid and apply the Neumann conditions that the derivative of the value function in the r-direction is zero for these two extreme values.

Since the validness of the specific boundary conditions we apply cannot be confirmed from exact science, but are rather heuristically determined, we construct our grid such that the domain of the grid extends significantly beyond our domain of interest. The boundary conditions become increasingly insignificant when the extreme values of the grid is set sufficiently far outside the domain of interest. This is possible to do on all sides of our rectangular grid except for the side in which r approaches its lowest feasible value, which has to be zero due to our choice of the CIR interest rate process, whose volatility term contains the square root of r.

Next to the boundary conditions defined on the perimeter of the rectangular grid there also exists another boundary condition, which traverses the (m, r) plane according to some function. This boundary divides our plane in two regions: the continuation region (going-concern region) and the stopping region (liquidation region). Let $\pi_{liquidation}$ be the value in the stopping region. Then by continuity we can impose the following condition:

$$P(m^*, r^*) = \pi_{liquidation} \tag{4.16}$$

The condition in 4.16 matches the values of the unknown function $P(m, r)$ with the known termination payoff function $\pi_{liquidation}$. Hence it is sometimes referred to as the "value-matching condition" (Dixit and Pindyck, 1994). The boundary is contingent on the unknown function $P(m, r)$ and so the boundary itself is an unknown. This implies that the region in the (m, r) space where the partial differential equation 4.13 is valid is endogenously determined. Boundary conditions of this kind are termed free boundary conditions. In our case the free boundary is represented with a curve $r^*(m)$.

Once the boundary data is defined it is possible to compute the value function of the DFI iteratively using equation 4.15, as the value of each point within the grid depends on the value of the eight points surrounding it. In order to solve the HJB-PDE we need to solve a PDE for all plausible dividend actions, and the resulting value in a given state (m, r) corresponds to the dividend decision that maximizes this value. The complete solution of the Hamilton-Jacobi-Bellman equation is summarized in algorithm 1.

Algorithm 1 Numerical solution of HJB-PDE with 9-point upwind finite difference stencil

Result: $P(m,r), \pi(m,r), CT1(m,r)$

Initialize grid parameters and define domain of interest

Initialize parameters $\Theta(m,r)\ \forall\ (m,r)$

Set initial conditions $P(m,r,u) = 0, Pn(m,r,u) = 0\ \forall\ (m,r,u)$

Set boundary conditions on perimeter of the rectangular grid

for $u = 0 \rightarrow U$ **do**

  for $n = 1 \rightarrow nIterations$ **do**

    Update $Pn(m,r,u)$ from last iteration: $Pn(m,r,u) = P(m,r,u)$

    if $CT1(m,r,u) \geq \Lambda$ **then**

      Compute: $P(m,r,u) = F(Pn(m,r,u), \Theta(m,r))$ with $F(\cdot)$ from (4.15)

    else

      $P(m,r,u) = 0$

    end

  end

end

Compute value function: $[P(m,r), u_1(m,r)] = \max_u(P(m,r,u))$

Determine optimal dividend policy: $\pi(m,r) = u_1(m,r)$

Compute Core Tier 1 Capital Ratio: $CT1(m,r,\pi(m,r))$

4.3 Approximate Dynamic Programming

In classical dynamic programming, an exact solution to the Bellman equation (4.2) is typically found by backward recursion. By assuming $V_{t+1}(x)$ is known, $V_t(x)$ and the corresponding policy $\psi_t(x)$ can be computed for all states x. When the state and action spaces can be described by a discrete, scalar variable, this is a manageable task. However, when the state and action spaces are represented by vectors, the number of possible states grows exponentially with the number of dimensions in these vectors and backward recursion quickly becomes intractable.

Moreover, in the extreme case of continuous state and action spaces, there is an infinite number of possible states, making this approach theoretically impossible. This is commonly known as "the curse of dimensionality in dynamic programming" (Powell, 2007).

In recent decades, several research communities have independently developed methods to cope with this limiting property of dynamic programming. As an alternative to backward recursion, the key idea of these methods is to step forward through time, replacing the true value function $V_t(x_t)$ with some statistical approximation $\bar{V}_t(x_t)$. Depending on its origin, this field of research is referred to with a range of different names. Iterative dynamic programming, forward dynamic programming, heuristic dynamic programming, adaptive dynamic programming, reinforcement learning and neuro-dynamic programming are some of the terms used. We will adhere to Powell (2007) and refer to it as "approximate dynamic programming", or ADP for short.

In the subsections to follow, we will present the building blocks of approximate dynamic programming. We start off explaining in detail how ADP overcomes the curse of dimensionality, before moving on to a general algorithmic description of the ADP concept. We then discuss a specific ADP algorithm known as Q-learning, before we turn to the subject of value function approximation and present a particular approximation architecture known as "artificial neural networks". The section is completed with a description of an ADP algorithm that combines Q-learning and value function approximation by artificial neural networks - the Neural Fitted Q-Iteration algorithm.

4.3.1 Overcoming the Curse of Dimensionality

The curse of dimensionality is in reality three curses, one for each of the three solution spaces in a stochastic optimal control problem. These are the state space, the action space and the probability space governing the state transition expectation, respectively. An ADP algorithm must overcome the curses associated with all three dimensions to be useful.

The curse of dimensionality associated with the state space comes from the fact that the true value function, $V(x)$, can only be found by enumerating all states x. As the number of possible states grows exponentially in the number of dimensions in the state vector, this becomes an increasingly difficult task. Let us illustrate this in the context of our DFI optimal capital structure model.

Our model assumes a conceptual DFI with five different balance sheet items: risky assets (loans), risk-free assets (cash), equity, deposits and bonds, respectively. The initial equity consists of share capital, additional paid in capital and retained earnings. The value of these balance sheet items are affected by three external factors: the current economic cycle, the interest rate level and a regulatory capital requirement. Counting the number of shares issued, this adds up to a total of nine state variables. Our implementation assumes all variables to be continuous, but let us for the sake of illustration assume discrete variables that can take a hundred different values each. This would give a total of $100^9 = 10^{18}$ possible states. Enumerating all states would clearly be an immense task; the problem is cursed.

The first step to overcome the state space curse is to realize that the discretization of the state variables is in fact causing the curse. A solution to the problem is thus to treat all state variables as continuous and rather find a continuous approximation to the value function given a state. Following the example from above, a very simple, linear approximation to the value function would be on the form:

$$\bar{V}(x) = \sum_{i=I} \theta_i x_i \tag{4.17}$$

Here I is used to denote the set of state variables and θ_i is a parameter that should capture the marginal value of state variable i. A linear approximation like this might not work very well if the true value function has considerable nonlinear properties (Powell, 2009), which is the case in our DFI model. In such cases, more sophisticated approximation architectures must be considered. Please see subsection 4.3.4 for a more detailed discussion of value function approximation methods.

Having an intuition of how to avoid the curse of dimensionality in the state space, we now turn to the action/decision space. As with the state space, if there is only one decision to be made, finding the optimal one is often fairly simple. However, in the case of multidimensional decision vectors, we are again faced with the exponentially growing space problem. If the decision variables are binary, or can take values from a small discrete set, the action space could still be manageable in size. But as soon as the discrete set of values grows slightly, or in the case of continuous decision variables, the size of the action space explodes.

Let us again consider the DFI model. In our implementation, we assume that a DFI can tune its capital structure in three different ways: by paying dividends to shareholders (or retaining earnings), by issuing new equity, and by issuing additional bonds. In other words, there are three decisions to be made. In reality, all of these are continuous decision variables that can take values ranging from 0 to several hundreds of billions. Even if they were discretized in millions, there would be a thousand possible values for each billion - for each of the three variables. The complete action space would be enormous, and the need for more sophisticated processing methods is evident.

This leads us to the field of mathematical programming. If we cannot naively enumerate all possible actions, we can use linear, mixed-integer or nonlinear programming methods to search for the optimal decisions in the action space. In particular, for the DFI optimal capital structure problem, we solve a constrained nonlinear program to find the optimal decision u^*, given the current state x and value function approximation $\bar{V}(x)$.

Lastly, we must overcome the curse of dimensionality in the probability space caused by the state transition expectation in equation 4.2. The established method for doing this is using so-called post-decision states. Described by both Powell (2007) and Van Roy et al. (1997), a post-decision state can be defined as the state the system transitions deterministically to given the current state and decision made. In other words, it is the abstract state of the system immediately after a decision has been done, but before any new stochastic information has been realized. This is supposed to capture the fact that in stochastic decision problems, decisions have to be made before certain outcomes are known. We denote the post-decision state by x^u, and let $T^u(x, u)$ be some general deterministic transition function. The post-decision state is then given by:

$$x^u = T^u(x, u) \tag{4.18}$$

After the system has transitioned to a post-decision state, some stochastic information ω is realized and the system transitions to a successor state $x^{'}$, also called a pre-decision state. Assuming some general stochastic transition function $T^s(x^u, \omega)$, and given a post-decision state x^u, the system evolves according to:

$$x' = T^s(x^u, \omega) \tag{4.19}$$

The most common treatment of post-decision states are given by equation 4.18. In a Q-learning context, however, the interpretation is somewhat different. Q-learning is based on the idea that a system's behavior is learnt by observing the consequences of doing random actions in given states. For this purpose, a post-decision state is just a concatenation of a pre-decision state vector and an action vector (Powell, 2009). This will be further elaborated in subsection 4.3.3 on Q-learning.

This concludes the subsection on the curses of dimensionality. We have seen how the state space curse can be avoided by a continuous approximation to the value function, and how we can make the action space manageable by the help of mathematical programming. Introducing post-decision states helps circumvent the curse of the probability space when computing the state transition expectation of the Bellman equation. We will now move on to a general algorithmic description of approximate dynamic programming.

4.3.2 General Approximate Dynamic Programming Algorithm

The most central part of approximate dynamic programming is how to find a good approximation to the true value function. Value iteration, policy iteration, Q-learning and temporal-difference learning are some of the main strategies that exist. Each strategy has its own algorithmic setup, but they all share some common traits and are based on the same basic principle, which we call value function iteration. In the subsection to follow, we will present a generic value function iteration algorithm. It will form the basis for the more specialized Q-learning algorithm presented in subsection 4.3.3.

As the name implies, value function iteration is a procedure where the value function approximation is improved in an iterative manner. If we let $\bar{V}^n(x)$ be the approximated value function in iteration n, we can write the step of finding an improved estimate mathematically as the following optimization problem:

$$\hat{v}^n = \max_{u \in \mathcal{U}} \left\{ \pi(x,u) + \frac{1}{1+\rho}[\bar{V}^{n-1}(x'|x,u,\omega)] \right\} \tag{4.20}$$

Here, ω is a vector containing a realization of the stochastic variables governing the problem. It is sampled from Monte Carlo simulations of the underlying stochastic processes. The processes relevant to the DFI optimal capital structure problem are presented in section 3.2, and subsection 3.2.3 describes how they are simulated.

Solving maximization problem 4.20 gives an estimated value, $\hat{v}^n$, of being in state x and making the optimal decision, u. This estimate is used to update the value function approximation $\bar{V}^n(x)$. Such an updating scheme can take many forms and depends on the approximation architecture applied. The most basic method is to let the new value function approximation be a weighted sum of the previous approximation $\bar{V}^{n-1}(x)$ and the new estimate $\hat{v}^n$:

$$\bar{V}^n(x) = (1-\alpha_{n-1})\bar{V}^{n-1}(x) + \alpha_{n-1}\hat{v}^n \tag{4.21}$$

$\bar{V}^n(x)$ should now be viewed as the best estimate of being in state x after iteration n. The weighting coefficient α_n is also known as a smoothing factor or learning rate. It can either be constant, deterministic or adapt to the current estimate. Care should be taken when choosing the learning rate rule, because it has a strong impact on the convergence conditions and rate of convergence for the overall iteration procedure (Bertsekas, 1995). The most established learning rate rule is given by $\alpha_{n-1} = 1/n$. This is the one that we found to work best for our problem, because it neither made the algorithm converge too fast nor too slow. Please see Powell (2009) for an extensive overview of other available rules.

Having updated the value function approximation according to equation 4.21, the next step is to determine an appropriate successor state for the system. We are now faced with two choices: we can either use the current value function approximation to find the optimal state to transition to, or we can transition to some random state in the state space. There is a trade-off between exploiting the work we have done so far to make a better approximation in the current region and exploring new parts of the state space. Powell (2007) refers to this as the question of exploitation vs. exploration. The exploitation strategy alone will make the value function approximation more accurate in the region close to the current state, but not very good in other regions. Only using the exploration strategy will ensure a reasonably decent approximation in larger parts of the state space, but not with the same accuracy. The best results are obtained with a well-balanced combination of the two strategies. There are different ways to implement this algorithmically, and the combined strategy must be tailored to the problem being solved. Busoniu et al. (2010) and Tokic (2010) present a number of available heuristics. Our implementation alternates between exploiting and exploring with a fixed probability p. See subsection 4.3.6 for further details.

When appropriate approximation strategies and updating rules are determined, we almost have a complete ADP algorithm. The whole idea now is to transition through the state space in an iterative manner and repeat the updating step until a satisfactory approximator is obtained. As the number of iterations increases, the approximation should converge towards the true value function. For $n = N$ we have:

$$V(x) \approx \bar{V}^N(x) \tag{4.22}$$

Formal proofs of convergence as $n \to \infty$ are provided by Bertsekas (1995), Jaakkola et al. (1994) and Watkins and Dayan (1992). These assume that all reachable states in the state space are visited infinitely often, however, which does not help much in a real life implementation. ADP algorithms hence usually have explicit termination criteria, e.g. that the procedure is stopped when the absolute deviation between $V^n(x)$ and $V^{n-1}(x)$ is sufficiently small. The simplest termination criterion is just to stop the algorithm after a predefined number of iterations N.

Let us now formalise the generic ADP algorithm outlined so far:

Algorithm 2 General ADP algorithm

Result: $\bar{V}^N(x)$

Initialization:

Initialize $\bar{V}^0(x)$ for all states x

Set $n = 1$

Set initial state x_0^1

while $n < N$ **do**

  Choose a sample ω^n

  Solve: $\hat{v}^n = \max_{u \in \mathcal{U}} \left\{ \pi(x,u) + \frac{1}{1+\rho}[\bar{V}^{n-1}(x')|x, u, \omega] \right\}$

  Update value function: $\bar{V}^n = U^V(\bar{V}^{n-1}, x^n, \hat{v}^n)$

  Update next state: $x^{n+1} = U^x(x^n, u^n, w^n)$

  Increment n: $n = n + 1$

end

This generic algorithm forms the basis for more specialised ADP algorithms, among them Q-learning. The next subsection will present this technique and give it a formal algorithmic definition.

4.3.3 Q-learning

Q-learning is an extension of algorithm 2 that has its roots in the machine learning community. Central to Q-learning is the use of state-action pairs. In the general ADP algorithm, the objective was to learn a near-optimal value of being in a state. In Q-learning, the objective is instead to learn a near-optimal value of being in a state and taking some specified action which need not be optimal. The value function approximation thus represents the value of taking a specific action in a given state, and not just being in the state. In reality, no knowledge about the underlying system model is needed if enough state-action samples are available. Hence, Q-learning is classified as a model-free technique (Watkins and Dayan, 1992), which makes it a versatile tool.

Formally we let the value function approximation be on the form $\bar{Q}(x,u)$ instead of the traditional $\bar{V}(x)$. The Q-function now represents the value of being in state x and taking action u. The objective is therefore to find the action u that maximizes the function $Q(x,u)$. The relation between $Q(x,u)$ and $V(x)$ can be stated as follows:

$$V(x) = \max_{u \in \mathcal{U}} Q(x,u) \tag{4.23}$$

Q-learning has the same iterative nature as the generic ADP algorithm. The idea is to gradually learn the system or agent the consequences of taking a specific action in a specific state. By performing numerous iterations and sampling sufficient stochastic realizations, the agent should eventually learn the expected value of specific actions in given states, and thus know which actions causes the value to increase and which actions it should avoid. If all available actions in a particular state are value-destructing, the agent should learn to avoid this state. Similarly, if all actions in a particular state are value-increasing, the agent should learn to like that state, i.e. take actions in other states so that it would transition towards that state.

In the DFI optimal capital structure problem, the objective is to learn the manager how much dividends to pay out and how much bonds and equity to issue given a particular economic state of the DFI, so as to maximize the market value of equity (represented by the share price). This must be done without breaching any regulatory capital requirements imposed. A state where a capital requirement is breached, would be a bad state, and the share price would be set to zero. A good state would be one where the shareholders receive high returns at a small risk of breaching the requirements. The manager should therefore learn which actions lead the DFI towards states where the requirements are breached, and which actions that transition it towards good states.

The Q-learning equivalent to the improvement step in 4.21 is given by:

$$\hat{q}^n = \left\{ \pi(x,u) + \frac{1}{1+\rho} \max_{u' \in \mathcal{U}'} \bar{Q}^{n-1}(x',u') \right\} \tag{4.24}$$

$\hat{q}^n$ is here the current best estimated value of being in state x and taking action u. We update the approximate value function accordingly using the smoothing method presented in subsection 4.3.2:

$$\bar{Q}^n(x,u) = (1-\alpha_{n-1})\bar{Q}^{n-1}(x,u) + \alpha_{n-1}\hat{q}^n \tag{4.25}$$

We proceed in an iterative manner, improving our estimates and updating the value function approximation as we transition through the joint state-action space. Algorithm 3 formalises the procedure:

Algorithm 3 Q-learning algorithm

Result: $\bar{Q}^N(x,u)$

Initialization:

Initialize $\bar{Q}^0(x,u)$ for all states x and actions u

Set $n = 1$

Set initial state x_0

while $n < N$ **do**

  Choose a sample ω

  Choose an action u

  Find transition state: $x' = T(x,u,\omega)$

  Solve: $\hat{q}^n = \left\{ \pi(x,u) + \frac{1}{1+\rho} \max_{u' \in \mathcal{U}'} \bar{Q}^{n-1}(x',u') \right\}$

  Update value function: $\bar{Q}^n = U^Q(\bar{Q}^{n-1}, x, \hat{q}^n)$

  Update next state: $x = U^x(x,u,w)$

  Increment n: $n = n + 1$

end

Q-learning is the value function iteration technique we use to solve our DFI optimal capital structure problem. Deciding on an appropriate approximation architecture for the value function is the last building block we need for a complete ADP algorithm. This will be discussed in the subsection to follow.

4.3.4 Value Function Approximation

Finding an appropriate architecture for approximating the true value function is an essential part of ADP. Linear regression, multilevel aggregation and artificial neural networks are some of the classes of methods available. This subsection will describe these and how they are incorporated in a value function iteration procedure.

Due to the iterative nature of ADP, we need approximation methods that easily let us update our value function approximation as new states are visited and new estimations are learned. Moreover, as ADP is forward recursive, we need approximation methods that let us evaluate states without explicitly having to visit them. In other words, in order to evaluate a specific state, we need to be able to say something useful about the value of states we might transition to from the current state. Only then can we overcome the curse of dimensionality in the state space and make many problems manageable.

Multilevel aggregation is one of the first approximation architectures that was used in ADP, and it is still widely used. The idea is to partition the state space into subsets of states containing similar attributes. By assuming that the value function is constant over the subset, we can update all states in the subset efficiently as a new value estimate is found for a state in the subset. It is particularly useful in resource allocation problems where distinct discrete attributes are available, such as geographical locations, equipment characteristics or industrial segmentation. Near-optimal solutions have been found for many different industrial problems, including pilot scheduling, blood inventory management and fleet management (George et al., 2008).

Multilevel aggregation is not very suitable for our DFI problem, however, as it is very hard to find a reasonable partitioning of the state space; it is hard to find suitable attributes that describe the balance sheet items on different aggregation levels.

Linear regression with basis functions is another class of approximations methods that is widely used. The idea here is to choose a set of features $\mathcal{F}$ that we believe are important for explaining the value of a state, and then devise basis functions, $\phi_f(x), f \in \mathcal{F}$ that measure these features. The approximated value of a state is expressed as a linear combination of the basis functions:

$$\bar{V}^n(x|\theta^n) = \sum_{f \in \mathcal{F}} \theta_f^n \phi_f(x) \tag{4.26}$$

The approximation is linear as it is linear in the basis functions. The basis functions themselves may be nonlinear, however, e.g $\phi_1(x) = x_1^2$ or $\phi_1(x) = x_1 x_2$. The main challenge is estimating the parameters θ_f and tuning them as the algorithm iterates. A much used algorithm for doing this is the stochastic gradient algorithm:

$$\begin{aligned}\theta^n =&\theta^{n-1} - \alpha_{n-1}(\bar{V}^{n-1}(x|\theta^{n-1}) - \hat{v}^n)\nabla_\theta \bar{V}^{n-1}(x|\theta^{n-1}) \\ =&\theta^{n-1} - \alpha_{n-1}(\bar{V}^{n-1}(x|\theta^{n-1}) - \hat{v}^n)\begin{bmatrix}\phi_1(x)\\ \phi_2(x)\\ \vdots \\ \phi_F(x)\end{bmatrix}\end{aligned} \tag{4.27}$$

This method is popular due to its simplicity. Powell (2009) argues that it may be very unstable, however, and urges caution when applied. One of the reasons for this is that the learning rate α must be scaled to account for the fact that θ and the gradient have different units. Different features may also need different scaling factors, and small changes in the scaling factors may have considerable impacts on the regression coefficients θ_f. Besides, even finding a suitable number of features with enough explanatory power may be challenging. Using too many redundant basis functions may only add noise and distort the approximation. More sophisticated methods exist, such as recursive least squares and Kalman filters. Please see Powell (2009) for further details.

Linear regression with basis functions did not work very well for our DFI problem. First of all, finding suitable basis functions proved quite difficult. Secondly, we had a hard time scaling the learning rates appropriately; sometimes the algorithm converged, sometimes it converged to unreasonable values, and sometimes it did not converge at all.

This led us to the last class of approximation methods mentioned at the beginning of this subsection - artificial neural networks.

4.3.5 Artificial Neural Networks

Our model is based on continuous state and action spaces, which calls for a continuous value function approximation. Multilevel aggregation and linear regression with basis functions did not satisfy our needs and had us search for more powerful tools. This led us to a family of statistical models known as artificial neural networks (ANNs), which originally emerged from the fields of artificial intelligence and machine learning. ANNs are inspired by biological nervous systems and how these process information; more specifically, they try to mimic how the human brain learns from experience.

ANNs can be used to approximate any mathematical function, no matter how complicated the function is and how many inputs and outputs it has. This is formally known as the Universal approximation theorem of artificial neural networks, first proven by Gybenko (1989). Please see Nielsen (2015) for an informal visual proof of the theorem, or Hornik et al. (1989) if you feel the need for a bit more mathematical rigour. For a general, yet comprehensive, introduction to the theoretical foundations of ANNs, the reader is referred to Hassoun (1995).

Artificial neural networks are generally presented as graphs of interconnected processing elements, "neurons". The nodes of the graph represent the neurons and the edges represent the connections between the neurons, equivalent to biological synapses. We use a particular type of neural networks known as multilayer feedforward perceptrons - MLPs for short. An MLP can be thought of as a layered directed graph, where the neurons of one layer are fully connected to the neurons of the next layer. An MLP has three or more layers of neurons. The first layer is called the input layer and the final layer is called the output layer. Between these, the network can have one or more intermediate layers, often called hidden layers. One hidden layer is usually sufficient for most problems (Nielsen, 2015). The number of neurons in the input layer equals the number of input variables in the function to be approximated. If the network is used for classification purposes, the output layer has as many neurons as the number of possible output classes. If the network is used as a regression model, where a single numerical output value is assigned to every vector of input variables (as in our case), the output layer has one neuron. The number of neurons in the hidden layers is somewhat arbitrary; there is not a single correct answer here. A pragmatic, yet common, approach is simply to start with a small number of neurons and increase this until a sufficiently good regression is obtained. As a rule of thumb, the more complex the function to be approximated is, the more neurons are required in the hidden layers. However, if too many neurons are used, the input samples will in practice be memorized, instead of underlying relationships being learned. This is what statisticians call overfitting. If that happens, generalization of trends beyond the observed data will not happen and the network will be useless for out-of-sample predictions.

Each neuron has a nonlinear activation function that transforms its input signal to a desired output signal for the next layer. Typically, the activation function is used to dampen or amplify the input signal as it propagates through the network, consequently capturing the desired nonlinearities of the function to be approximated. The connections between the neurons have numeric weights that are tuned iteratively as new samples are given to the model. This procedure is called "training" the network, mimicking the biological procedure of learning from observations/experience. There are numerous methods available for training neural networks, most of which lend themselves to basic statistics and optimization theory. We use a specific algorithm called Levenberg-Marquardt backpropagation, which is a fast and robust training algorithm based on non-linear least squares estimation. We will go no further into the details, but point the curious reader to Hagan and Menhaj (1994) for an in-depth description.

When artificial neural networks are used for approximating the value function, approximate dynamic programming may be referred to as "neuro-dynamic programming" (NDP). This term was coined by Bertsekas and Tsitsiklis (1996) as an equivalent to "reinforcement learning" in the artificial intelligence and machine learning communities. Riedmiller (2005) proposes an NDP implementation with Q-learning as the surrounding ADP algorithm, which is named Neural Fitted Q Iteration. This is the implementation we use for our model. It will be further explained in the section to come.

4.3.6 Neural Fitted Q Iteration

This subsection completes section 4.3 on Approximate Dynamic Programming and will try to unify the elements presented so far in one algorithmic implementation: the Neural Fitted Q Iteration algorithm, or NFQ for short.

NFQ is a special realisation of the Fitted Q Iteration algorithm of Ernst et al. (2005), where a multilayer perceptron is used as approximation architecture. Pseudo code is given in algorithm 4:

Algorithm 4 Neural Fitted Q Iteration

Result: $Q^N(x,u)$
Initialization:
Input a set of transition samples $\mathcal{D}$
Initialize an approximation $\mathcal{Q}^0(x,u)$
$n = 1$
while $n < N$ **do**
  GeneratePatternSet($\mathcal{D}$) $\rightarrow \mathcal{P} = \{input^l, target^l\}, l = 1, ..., \#\mathcal{D}$ where
  $input^l = (x^l, u^l)$
  $target^l = \pi(x^l, u^l) + \max_{b \in \mathcal{B}} \frac{1}{1+\rho}[Q^{n-1}(x', b)]$
  TrainNetwork($\mathcal{P}$) $\rightarrow Q^n$
  Increment n: $n = n + 1$
end

The algorithm returns the best approximation to the true value function obtained after N iterations, $\mathcal{Q}^N(x,u)$. It consists of two major steps: The generation of the set of training data $\mathcal{P}$, and the training of a multilayer perceptron from these with an appropriate training algorithm.

The training data $\mathcal{P}$ is generated based on the set of input transition observations $\mathcal{D}$. Each transition observation is a triplet consisting of a current state of the system, a vector of actions and the accompanying successor state. For each transition sample, an input-target pair - a "pattern" - is constructed. The input is essentially the current state and action vectors from the transition sample. The target value is given by Bellman's equation, using the current version of the neural network to get the approximate value of the successor state. The collection of these input-target pairs is called a "pattern set". For every time a pattern set is generated, a new transition sample is simulated, which means the approximator is always fed with new training data. The pattern set is given as input to the training algorithm, which outputs an incrementally improving approximator to the true value function. As mentioned in subsection 4.3.5, we use the Levenberg-Marquardt backpropagation algorithm for the training part. The two main steps are repeated in a loop until the predefined number of iterations N is reached or a satisfactory value function approximation is obtained.

The key idea behind the NFQ algorithm, as proposed by Riedmiller (2005), is to update the value function approximator "offline" after a batch of transition samples have been collected, rather than "online" for each new sample, as it is done in the general value function iteration (Algorithm 2) and Q-learning (Algorithm 3) algorithms presented in subsections 4.3.2 and 4.3.3, respectively. Multilayer perceptrons are global approximators, which means that changes to the network weights will apply to the whole state-action space. So if the network is updated based on a single new input-target pair, then this can cause unpredictable changes to the value function approximation in completely different regions of the state-action space. Riedmiller (2005) argues that this typically leads to longer learning times and unstable approximations, and suggests this batch learning approach as a reliable solution.

In principle, the input transitions can be arbitrarily sampled. However, when learning with a real system, collecting transitions that lie on actual trajectories usually makes most sense, as we primarily want the approximator to be accurate in regions the system is likely to touch upon. A straightforward way to do this is to use the exploitation strategy mentioned in subsection 4.3.2, exploiting the current value function approximation to determine the current optimal policy, and then greedily following this policy for a certain number of transitions or until a terminal condition is reached. The resulting sequences of states and actions are collected as transition sample.

Our implementation for the DFI problem is based on transition samples of 2000 observations each. We run the main loop for 300 iterations and use a multilayer perceptron with one hidden layer of 26 neurons. For the initial set of transition data we sample the state variables randomly and use a price-to-book multiple to find an appropriate target value. Please see appendix C for a derivation of the multiple.

5

Parameter Estimation and Implementation

We test our model on DNB ASA, Norway's largest commercial bank. This chapter will present details on model calibration and implementation. Section 5.1 presents the procedures used for estimating the model parameters, and section 5.2 elaborates on the computer implementation.

5.1 Parameter Estimation

Our model is calibrated using appropriate methods for the given parameters. Parameters associated with market data, DNB's financial statements and the growth function of loans are either taken directly from annual reports and statistical databases, or estimated according to fundamental corporate finance theory. The parameters in the stochastic processes are primarily estimated by means of statistical inference.

5.1.1 Parameters Based on Market Data and Financial Statements

A great portion of the parameters are taken directly from the financial statements of DNB. The initial levels of cash and cash equivalents, book value of equity and total loans, as well as fully diluted shares outstanding are all collected from the annual accounts of 2014. The interest rate margin received on loans (178 basis points (bps)) and paid on deposits (-79 bps) are taken as the weighted-average interest margins financial corporations has offered to the general public in Norway over the period from 2000 to 2014. These time series are provided by Statistics Norway (2015) and are updated quarterly. As DNB's average market share in the period has been 30.81% on loans and 40.63% on deposits (Finans Norge, 2015), these interest margins are very close to DNB's actual interest margins. We assume that the interest rate received on cash is identical to the money market rate, hence the margin is set to 0 bps. The interest rate margin paid on bond debt (61 bps) is taken as the yearly average of the interest rate margin on such liabilities in the period 2010 to 2014. These numbers are found in Note 20 in the annual accounts of DNB.

The deposits-to-loans ratio is taken as the average over the last five years for the ten largest Norwegian banks. These time series are provided by Finans Norge (2015) and are updated yearly. The deposits-to-loans ratio is calculated to 70.0%. Using the balance sheet items presented above we are able to compute deposits with the estimated deposits-to-loans ratio and the initial level of bonds under the assumption of equality between sources and uses of funding.

The unlevered cost of equity for DNB can be estimated in several ways. The standard approach is to first regress the stock price returns of DNB on the returns of OSEBX for a given time period, e.g. from 2009 up until today. The slope in this regression provides an estimate of the levered beta for DNB. The unlevered beta is then calculated using the current capital structure and the corporate tax rate of 27%. Finally the unlevered cost of equity can be found using CAPM.

CAPM only incorporates one risk-factor, which is the sensitivity in asset returns measured against the wider market return. The dynamic behaviour of asset prices might require a more exhaustive model. Therefore we propose to do a reversed excess return valuation of DNB by taking the current market capitalization as given, along with analyst estimates of future earnings, and then triangulate the implied unlevered cost of equity by using the solver function in Microsoft Excel. Consensus analysts estimates are available from DNB's website (DNB Investor Relations, 2015).

This analysis yields an unlevered cost of equity of 2.56%, implying a margin of 111 bps on the current money market rate of 1.45%. We believe that a good way to incorporate as many risk factors as possible is to look to the market price (of the firm in study) itself. Under the assumption of efficient markets the market price should capture all risk for that asset in the sense that you should be fairly compensated in terms of returns for the risk you undertake. For liquid stocks with high volumes each trading day, it is fair to assume that markets are at least approximately efficient. This should be the case for a blue-chip stock like DNB.

We assume a multivariate linear expression for the loan growth function, as given by equation 3.7. The functional form enables us to estimate the parameters based on least squares regression. Such a regression was performed where the predictor variable was the yearly growth of loans, and the explanatory variables were the economic cycle and the money market rate. However, such an estimation yielded that the constant parameter in the growth function was not statistically significant at a 5.0% p-level. Thus, calibration of these three parameters was based on intuition and the arguments of Jacobsen and Naug (2004) regarding debt in Norwegian households. The parameters were set to be 0.0231 for the constant i^L, 1.077 for the economic cycle j^L and -0.503 for the money market rate k^L, respectively. With such parameters the loan demand is grows with increasing GDP growth and falling with increasing money market rate.

5.1.2 Parameters for Stochstic Processes

The parameters of the stochastic processes are important to estimate as these processes are driving key variables that affect items on both the income statement and the balance sheet. For both the economic cycle and the money market rate, we have assumed processes that let us derive closed form expressions. This allows us to apply theory from statistical inference in the calibration. First we describe the derivations for the process governing the economic cycle before turning to the money market rate.

The dynamics of the economic cycle is given by the Ornstein-Uhlenbeck process presented in equation (3.2.1). We therefore need to estimate the mean reversion speed δ, the mean reversion level μ, and the volatility parameter η. The form of the process allows us to derive a closed form representation of the stochastic variable m_t, and hence find its distribution. This derivation can be found in Appendix D.1.

With these expressions, we can calculate numerical estimates of the parameters based on the annual growth in real GDP per capita in Norway in the time period 1970 - 2015, which are available from Statistics Norway. It is important to note that the growth in GDP in the first year of the data set (from 1970 to 1971) represents the initial observation u_0. The numerical values of the estimated parameters were found to be 0.763 for the reversion speed θ, 0.076 for the mean reversion level μ and 0.061 for the volatility parameter η. Note that the estimate for the mean reversion speed, δ, corresponds to a half-life of around 0.91 years. Half-life is the expected time it would take for the process to revert halfway back to its long-term mean.

Furthermore we believe it is necessary to offer a brief comment on the long-term average μ of 7.6%. We have used a time series extending from 1970-2015. The first major commercially viable discovery of petroleum on the Norwegian Continental Shelf was made late in 1969. Since the discovery of Ekofisk countless major discoveries have been made. Hence the time series we use for GDP corresponds to a period of time where the Norwegian economy has been transformed substantially due to the petroleum industry. Consequently a long-term average growth level of 7.6% might not be representative of what the mean real GDP growth per capita will be in the Norwegian economy going forward. One plausible implication of this parameter is that it might provide an optimistic picture of the value of DNB's interest generating activities.

The dynamics of the money market rate is given by the Cox-Ingersoll-Ross process presented in equation (3.2.2). We therefore need to estimate the mean reversion speed α, the mean reversion level β, and the volatility parameter σ. The derivation of the log-likelihood function can be found in appendix . We maximize the log-likelihood function with respect to the vector of parameters by applying a MATLAB implementation of the MLE estimation procedure developed by Kladıvko (2007). The estimation was performed on averaged yearly data of the 3-month NIBOR rate between 1985 and 2014 available from Statistics Norway.

The parameters were found to be 0.139 for the reversion speed α, equivalent to a half-life of 4.99 years. The long-term average, β, was estimated to 0.039, and the volatility parameter σ was found to be 0.068.The correlation between the two process, as presented in equation 3.22, is found to be 0.0897. This is based on yearly data from the period 1985 - 2014 provided by Statistics Norway.

We would like to emphasize the importance of critically evaluating the quality of the point estimators above. There are essentially two possible sources of error present: one due to the specific stochastic processes assumed, and one due to the uncertain nature of statistical estimation. It is particularly important to look at unbiasedness and the variance of the estimators. The variance can be compared to the Cramér-Rao lower bound for variance of unbiased estimators to check whether the estimators are efficient. If they are not efficient, then it is possible to look for improved estimators by using the Rao-Blackwell theorem (Casella and Berger, 2002). In this way it is possible to search for the uniformly minimum variance unbiased estimators (UMVUE).

We observe that the maximum likelihood estimates are in line with our expectations based on intuition. Since they make sense both from a qualitative and quantitative point of view, we will defer the improvement exercise of these estimators to future work.

A complete overview of model parameters is given by table 5.1.

Parameter	**Notation**	**Value**	**Source**
Initial book value of equity	BVE_0	159bn NOK	DNB AR 2014
Initial cash and cash equivalents	R_0	59bn NOK	DNB AR 2014
Initial total loans to customers	L_0	1439bn NOK	DNB AR 2014
Number of shares outstanding	n_0	1.628bn	DNB AR 2014
Interest margin loans	Δ^L	1.78%	SSB 2000-2014
Interest margin deposits	Δ^D	- 0.79%	SSB 2000-2014
Interest margin bonds	Δ^B	0.60%	SSB 2000-2014
Risk-weight cash	v_1	0	Norges Bank 2014
Risk-weight loans	v_2	1	Norges Bank 2014
Drift in loan growth	i^L	2.31%	DNB AR 2006-2014
Economic cycle parameter in loan growth	j^L	1.077	DNB AR 2006-2014
Money market rate parameter in loan growth	k^L	-0.503	DNB AR 2006-2014
Deposit-to-loan fraction	ϕ	0.70	Finans Norge 2010-2014
Unlevered cost-of-equity margin	COE^U	1.11%	DNB IR 2015
Tax rate	τ	27.00%	DNB AR 2014
Attraction rate OU process	δ	0.763	SSB 1970-2014
Long-term mean OU process	μ	0.076	SSB 1970-2014
Volatility OU process	η	0.061	SSB 1970-2014
Attraction rate CIR process	α	0.139	SSB 1985-2014
Long-term mean CIR process	β	0.039	SSB 1985-2014
Volatility CIR process	σ	0.068	SSB 1985-2014
Correlation between processes	γ	0.090	SSB 1985-2014

Table 5.1: Parameters in calibrated model

5.2 Implementation

Both solution approaches were implemented in MATLAB R2014a (Version 8.3)[1]. For the HJB-PDE approach presented in section 4.2, the solution framework was built from scratch. For the ADP approach (NFQ) presented in section 4.3, several of MATLAB's additional toolboxes were utilized. The Neural Network Toolbox was used to build and train the multilayer perceptron we use for our value function approximation. The intermediate constrained nonlinear optimization problem of the NFQ algorithm is solved using the Fmincon solver of the Optimization Toolbox. The NFQ algorithm is well-suited for parallel processing on multicore CPUs, which we implement with MATLAB's Parallel Computing Toolbox using parallel for-loops. Please refer to MATLAB's online documentation for further details on the toolboxes and their functionality.

All tests were run on 64-bit Windows 7 PCs with 3.40 GHz Intel Core i7-3770 CPUs (4 cores) and 16 GB RAM. As the HJB approach solves the DFI problem on a discrete joint action-state space, while the ADP approach solves the problem on a continuous action-state space, there are substantial differences in the computational complexity, and hence in the run time. The HJB model takes approximately 8 hours to solve the problem on a 100 by 100 grid in the stochastic variables with dividend values ranging from NOK 0 to NOK 40 billion with increments of NOK 125 million. The ADP model takes about 6 days to run 300 iterations ($N = 300$) of the NFQ algorithm with an input transition sample of 2000 observations.

[1] Code is available upon request. Please direct all inquiries to Fredrik Solbakk at fredriksol@gmail.com

6

Results and Discussion

The DFI model was tested on DNB ASA. In the following chapter we present the results from our two solution approaches. The results are visualized by graphs and tables in order to build intuition and provide support for our arguments and conclusions. Please note that in appendix E we provide a summary of key financial metrics for DNB. We encourage the reader to use DNB's actual key financial metrics as a reference whenever nominal values are presented, meanwhile keeping in mind that we only model the interest generating activities of the firm. Relative metrics are displayed whenever appropriate throughout the chapter. Section 6.1 presents the results from the Hamilton-Jacob-Bellman PDE approach and section 6.2 presents the results from the Neural Fitted Q-Iteration approach. The chapter is completed with a comparison of the two solutions in section 6.3.

6.1 Hamilton-Jacobi-Bellman Equation Results

In this section we present the results from the HJB-PDE approach. We first present the optimal financing policies our model suggests calibrated with DNB's figures. Next we describe the optimal capital structure that this financing policies imply. Lastly, we discuss the expected value function or share price in optimum, inasmuch as the share price is the criterion that quantifies the optimal control of the bank. Sensitivity analysis and discussion regarding changes in regulatory requirements accompany the corresponding subsections.

6.1.1 Optimal Financing Policies

In this section we discuss and illustrate optimal financing policies for a DFI when constrained by capital regulations and with underlying uncertainty in the economic cycle and the money market rate. In figure 6.1 the optimal dividend per share (DPS) in NOK is presented, which should be studied together with net profit or earnings per share (EPS) in figure 6.2. Earnings per share provide a reference to what a sound and healthy dividend could be. Moreover, we can triangulate retained earnings by looking at the difference between EPS and DPS, which constitute the internal source of funding. Figure 6.4 displays the optimal issuance of debt in NOK billion. Optimal DFI capital structure, which we discuss in the next section, is ultimately what we seek to determine in this thesis. Due to the tandem movement of optimal financing policies and corresponding CT1 ratio (see figure 6.5) we also refer to the CT1 ratio in this section whenever appropriate.

The surface displaying the optimal dividend policy point to three distinct regions in terms of how DNB should act in optimum. In a major fraction of the state space the dividends are stable in the range NOK 26-27 per share. We observe this behavior when the economy is in expansion, although high-interest rate environments require the economy to grow faster for this dividend level to be optimal. On the boundary where the change in demand for credit turns negative, we witness rapidly decreasing dividends in the direction of decreasing economic growth and increasing interbank rates. When our stochastic variables move toward the boundary of our domain in the direction described, DNB eventually enters the stopping region where a liquidating dividend is preferred. In this region the going-concern dividend is zero.

The optimal behavior described and suggested by the above paragraph is perhaps

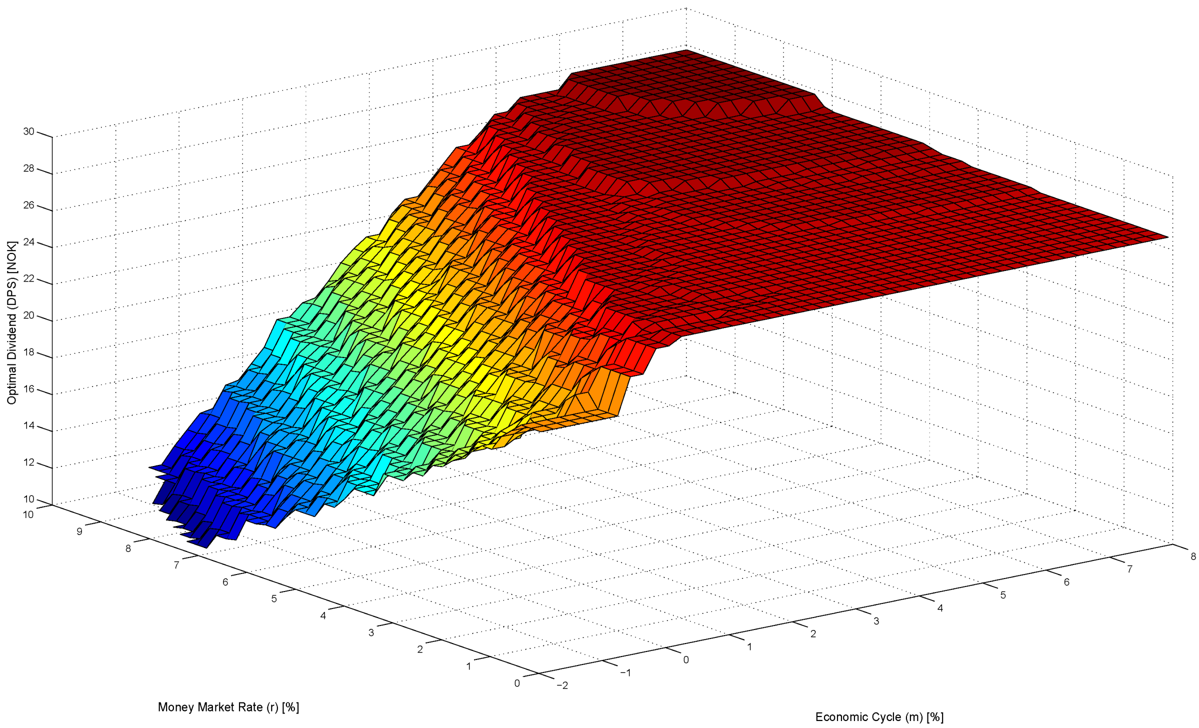

Figure 6.1: Optimal dividend policies per share (DPS) for DNB ASA

all the more clear if we compare the dividend policy to earnings per share. First of all we observe that the surfaces have the same form, indicating that DPS as a fraction of EPS is approximately constant throughout the state space. A more interesting detail is that the DFI should almost consistently utilize zero to negative fractions of its internal funding capacity, which is evident from the comparison of the magnitude between the two. However, the suggested optimal behavior for DNB in the next fiscal interim should be studied in the context of its beginning financial strength. For instance, the branches of DNB that we model has a CT1 ratio of 11.03% before the proposed optimal control is applied. Thus one of the key takeaways suggested by our model is that DNB should in the majority of the states adjust its CT1 ratio downward by paying out all the funds they generate internally and sometimes consume additional market debt to both fund the remaining part of the optimal dividends and any potential growth opportunities.

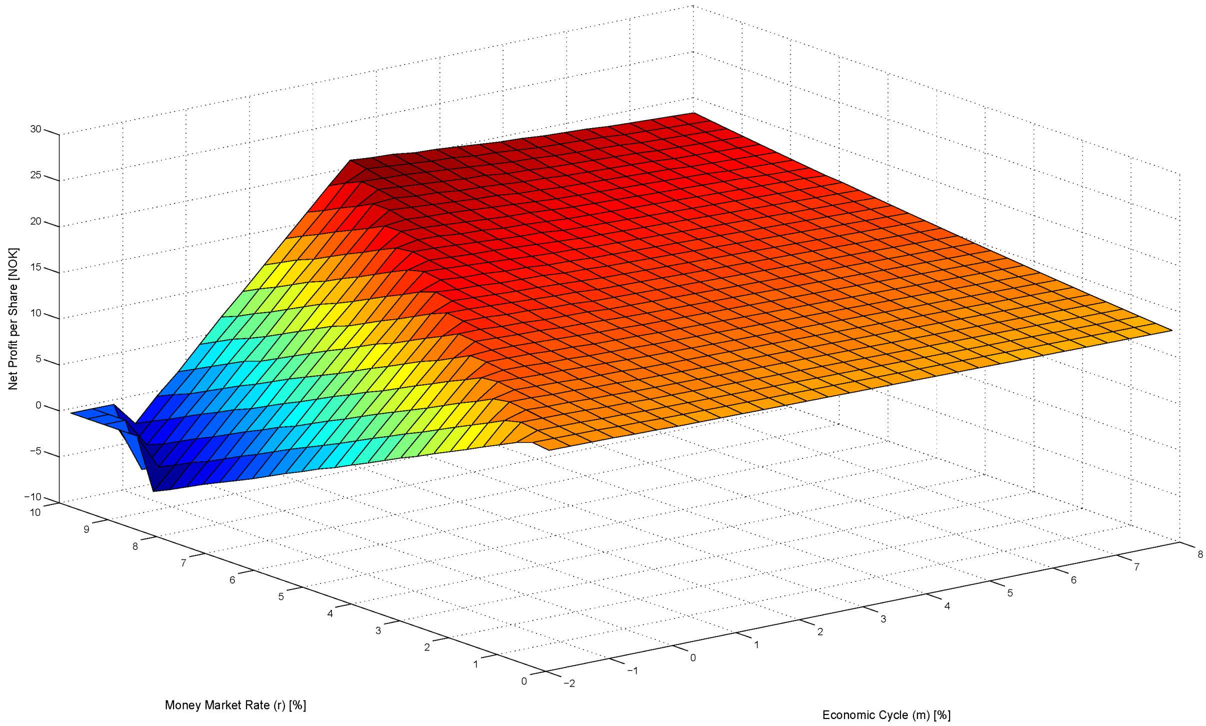

Figure 6.2: Expected earnings per share (EPS) for DNB ASA

Let us try to provide economic intuition behind the aforementioned observations. First of all DNB is able to sustain a vast nominal flow of dividends in states with larger operating cash flows and higher growth. In this region the likelihood of violating the regulatory capital requirement is also lower in the immediate future, which allows management to promise stable cash flows to shareholders going forward. Additionally, an increase in the leverage of the firm by consumption of debt to fund growth opportunities and hence adjust the CT1 ratio downward, is admissible for the same reason. However, it is interesting to notice that the dividend yield (see figure 6.3) is lower in this region. An effect of this kind can be explained by analyzing the optimal allocation of funds. We know that when the economy is in rapid growth, the growth DNB needs to support is accordingly high, and it needs to be funded with internal equity or external debt. Hence management needs to evaluate how it can maximize economic profit by weighing the trade-off between how much to distribute in dividends and how much to reinvest in the business. Management may argue that reinvesting a greater portion of its internal funds is the rational economic choice when the growth opportunities in the business will provide a higher return on equity.

Our results mostly indicate that all the internal funds (and sometimes even more) should be distributed in dividends. In the states where the growth is at its highest, interest rates are at its lowest, which allows DNB to fund growth by cheap debt meanwhile making a sound dividend payment. The implication is that leverage increases causing a surge in return on equity. At the same time the cost of equity is contained due to the low interest rate environment. Consequently, if we revert to the framework of Koller et al. (2010), we can conclude that since growth is higher, ROE is higher and COE is fairly low, the share price should be high. This is exactly what we observe. Hence the lower dividend yield in this region is a direct result of favorable value drivers and corresponding high share price.

We observe a rapid decrease in nominal dividends when the activity in the economy deteriorates, signalled by a worsening of the real GDP growth per capita and higher interbank rates. In this region credit demand is in contraction, and thus the degree of the downward adjustment on the initial capital adequacy is lower in response to weaker economic outlook. Furthermore, we observe that in the state with the largest CT1 ratio DNB should leverage some of its internal funding capacity triggering an increase in the CT1 ratio from 11.03% to 11.18%. Consequently, dividends per share as a fraction of earnings per share are decreasing in this region. The dividend payouts continue to fall until the free boundary between liquidation and going-concern is reached, where DNB pays out a dividend of NOK 9.00 per share while still being in a state of continuation. On the contrary to what we saw before, the states where the nominal dividend flow declines correspond to states where the dividend yield is quite high. Consequently, the value of DNB is here to a greater extent a function of the dividend generating component of the firm rather than its future growth opportunities. When the GDP growth declines and the interbank rate increases, it is therefore more attractive for shareholders to be compensated in the form of a dividend on a relative basis. The higher dividend yield provides some evidence of management having the shareholders best interest at heart.

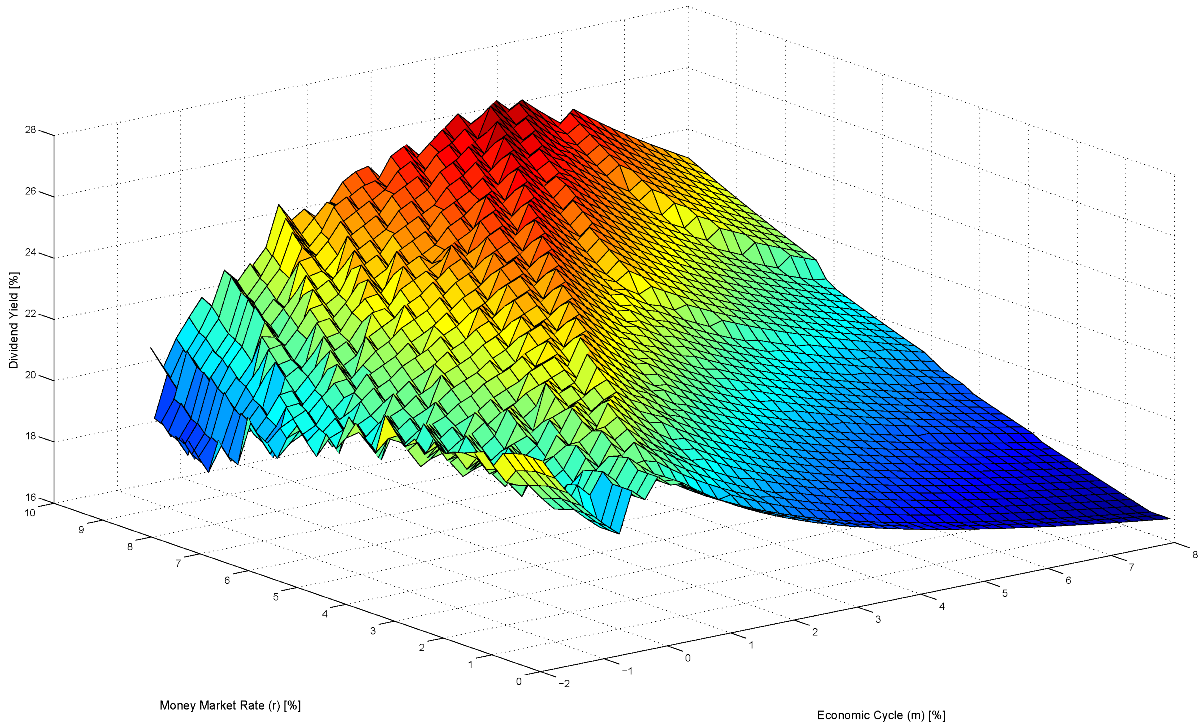

Figure 6.3: Dividend yield of DNB ASA

Figure 6.4 shows that issuance of debt is optimal in most states of the economy. In the majority of the debt-issuing states the DFI face an expanding credit market, which puts it in a position of having to fund the growth in loans through a combination of retained earnings and market debt. Since retained earnings for the most part is nonexistent due to significant dividend payouts, it is necessary to take on more debt in order to meet the increasing loan demand. DNB faces the largest demand for credit in times of high real GDP growth and low money market rates. In the figure we observe that this is the point where it is optimal to issue the largest amount of market debt, corresponding to NOK 69.1 billion.

The dark blue region in figure 6.4 is an area where DNB issues small to negative amounts of debt. In these states we also observe the largest CT1 ratio. When moving towards the stopping region we see that debt issuance increases again, and the CT1 ratio decreases. The intuition here is that we also have an expanding fraction of loans that become non-performing. Thus net profit is significantly reduced, triggering small debt offerings so that there is equality between sources and uses of funding. Furthermore, it could be an effect where shareholders try to maximize their piece of the pie in a state where the economic outlook is weak. Since the CT1 ratio is high in this region governmental intervention is not likely, and thus DNB is able to fund a dividend by issuing bonds.

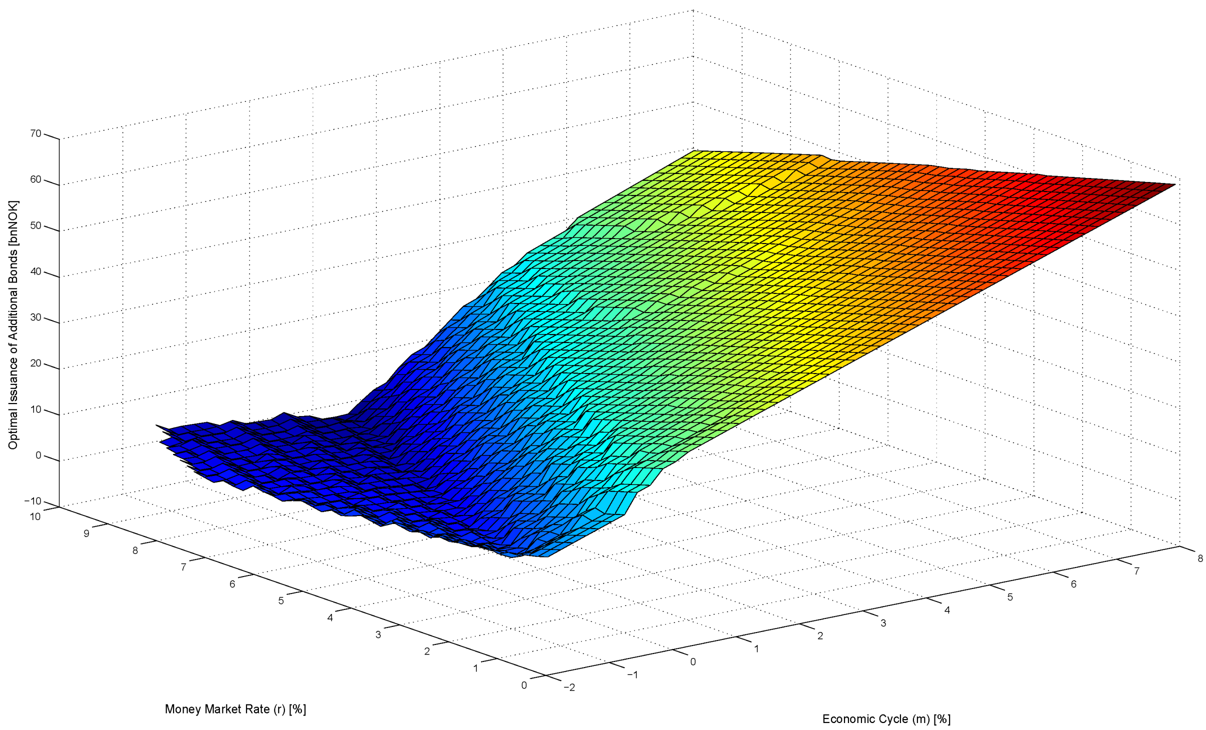

Figure 6.4: Optimal bond issuance for DNB ASA

Having presented the results of the optimal financing policy for the different economic states under the current capital requirement, we now turn to how changes in the capital requirement alters the optimal policy. These results are presented in table 6.1. Note that the table only includes the values for DNB under the going-concern assumption, as the numerical values presented in the table would be skewed by the outlier values obtained under liquidation.

	CT1 Requirement [%]			
Optimal Dividend per Share [NOK]	0.0	4.0	**8.0**	10.0
Mean	82.66	53.88	**23.83**	7.19
Median	83.00	54.00	**27.00**	7.00
Standard Deviation	1.02	0.90	**1.77**	0.24
Maximum	99.00	67.00	**28.00**	9.00
Minimum	61.00	33.00	**9.00**	5.00
Optimal Bond Issuance [bn NOK]	0.0	4.0	**8.0**	10.0
Mean	130.3	80.4	**28.3**	0.05
Median	127.1	76.8	**30.0**	-0.08
Standard Deviation	3.84	3.98	**3.22**	2.64
Maximum	189.5	135.9	**69.1**	34.1
Minimum	89.3	40.8	**-2.3**	-25.4

Table 6.1: Key statistical properties for the optimal financing policies of DNB ASA

Considering the optimal dividend payout first, we observe that the mean dividend in all states is falling when the regulatory requirement increases. The intuition here is merely that increasing capital requirements force the bank to hold more loss-absorbing capital by increasing its retained earnings. The suggested optimal behavior should also be viewed in relation with the diminishing issuance of debt to fund growth opportunities when the capital requirement increases. Thus we can conclude that when government impose stricter capital requirements it is optimal to shift from debt financing to equity financing, which is effectively what regulatory forces seek to achieve through these banking regulations.

6.1.2 Optimal Capital Structure

Figure 6.5 illustrates optimal DNB capital structure, measured by the CT1 capital ratio, when constrained by capital regulations and with underlying uncertainty in the business cycle and the interest rate. The horizontal plane beneath our surface represents the current governmental requirement that force going-concern banks to hold at least 8% equity.

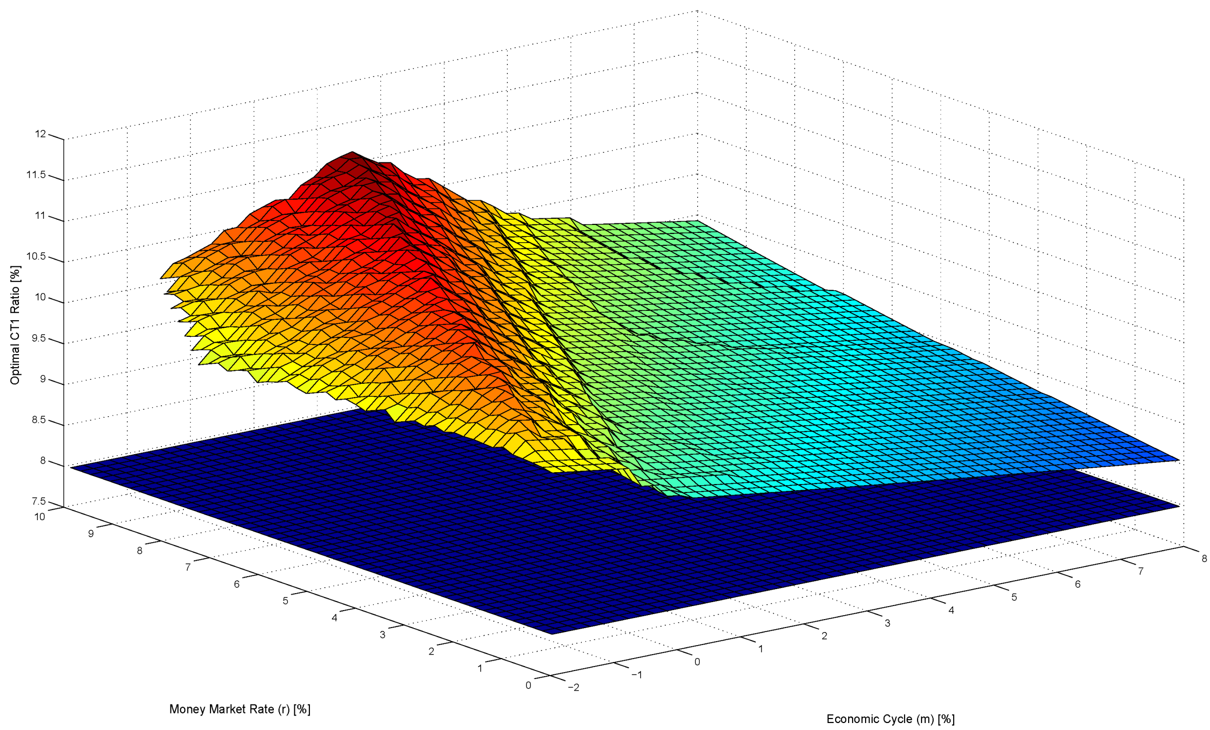

Figure 6.5: Optimal capital ratio for DNB ASA

The most obvious observation is that the bank chooses to lie well above the minimum requirement in the majority of the plausible states. The increase and convergence of the CT1 ratio is in line with the results of Berger et al. (1995). They argue that banks may hold capital above the regulatory minimum in order to possibly exploit unexpected investment opportunities in the future, as well as to withstand negative shocks. Our model only captures the latter, as the asset exposure and hence investments are governed by the exogenous growth function. However, the punishment for violating the regulatory capital requirement is very strict in our model, in the sense that the book value of equity and all future excess returns are immediately reduced to zero. Thus, in order to avoid this from happening in the case of an unexpected negative shock, the bank tends to hold capital well above the minimum requirement. However, we observe that when the GDP growth moves towards its minimum and the interbank rate towards its maximum there is a small region that the surface does not extend to. This represents the stopping region where realized values of the DFI assets are paid back to capital providers in order of seniority.

The main trend we observe is that DNB should ideally hold more equity when the economy moves towards a recession and when the interbank rate grows. The largest CT1 ratio in our state space is 11.18%, 318 basis points above the requirement. The smallest CT1 ratio is 8.54%, 54 basis points above the requirement. On a risk-weighted asset base of NOK 1,439 billion, this difference corresponds to NOK 38.0 billion, about two times the size of DNB's net profit in a normal year.

There are several effects that explain the sign of the partial derivative of the CT1 ratio with respect to the state variables. For instance, the ratio is at its lowest when the real GDP growth per capita approaches its maximum and the interbank rate approaches zero. These are states where the exogenous growth, which we assume the bank needs to fund, is at its maximum. In order to both fund the growth and distribute a significant dividend (see figure 6.1) the bank raises a substantial amount of bonds evident from figure 6.4. This effect causes the leverage to increase. We have previously discussed that an increase in leverage implies an increase in economic profit and hence excess returns provided the increase in return on equity outweighs the increase in cost of equity. This provides rationale to why the bank adapts its capital structure in this manner. The states in discussion correspond to states where the prospects of DNB are good and thus the likelihood of violating the CT1 ratio going forward is low, which allows DNB to exploit the effect of increasing leverage.

The maximum CT1 ratio is chosen when the real GDP growth per capita is around 2.5% and interbank rate is 10%. It is interesting to observe that the comparative static derivative of the CT1 ratio with respect to the macroeconomic variable changes from positive to negative at this point, because it implies that DNB should hold less equity behind its risky assets as it approaches the stopping region from this point. Apparent from the surface there exists a line in the (m, r) plane where this comparative static changes from positive to negative. To explain why the CT1 ratio decreases when moving towards the stopping region from this line, we decompose the ratio into its component parts and study each one. There are three main factors that contribute to the change in the CT1 ratio from one fiscal period to another: net profit, dividends and the change in loans. The CT1 ratio is increasing in net profit and decreasing in the latter two. First and foremost in the area between the aforementioned line and the stopping region, the credit demand is shrinking, which implies that the DFI loan exposure is decreasing. On an isolated basis this should trigger an increase in the CT1 ratio. The dividend is also lower in this direction with the same influence as the decrease in loans. However, net profit is significantly reduced and outweighs the two other effects on a relative basis. The main contribution to the significant reduction in net profit in this region, is the combined effect of reduced demand and that a greater fraction of the loans becomes non-performing.

In addition to the optimal capital structure chosen at the current CT1 requirement, it is important to study the implications different capital requirements has on DFI capital behavior. Key statistical properties for various requirements are presented in table 6.2. The trend we observe is that when the minimum capital requirement increases the amount of capital the DNB chooses to hold on top of this requirement generally decreases. The intuition here is that the risk premium equity investors demand is increasing in leverage. Hence there seems to exist a trade-off function in the (m, r) plane where the marginal benefit of increasing leverage is lower than the marginal cost measured by the cost of equity. A plausible theory is therefore that optimal capital structure in DFIs is governed by a market capital requirement in absence of the regulatory requirement, and the market requirement is lower than the current regulatory.

Table 6.2 illustrates that the area of the stopping region is increasing in the capital requirement. This suggests that management is more likely to exercise their bankruptcy option when forced to hold more loss-absorbing capital.

	CT1 Requirement [%]			
Optimal CT1 [%]	0.0	4.0	**8.0**	10.0
Mean	2.82	6.19	**9.71**	11.75
Median	2.89	6.35	**9.56**	11.79
Standard Deviation	0.24	0.09	**0.18**	0.13
Maximum	4.80	8.16	**11.18**	12.80
Minimum	0.99	4.35	**8.54**	10.74
Area of Stopping Region [%]	0.00	0.00	**1.00**	16.44

Table 6.2: Key statistical properties for optimal capital structure under current regulatory requirement

6.1.3 Value Function Representation Based on HJB

Figure 6.6 shows the expected share price for DNB as a function of the business cycle and the money market interest rate. These prices assume the corresponding optimal financing policies are followed (see figures 6.1 and 6.4).

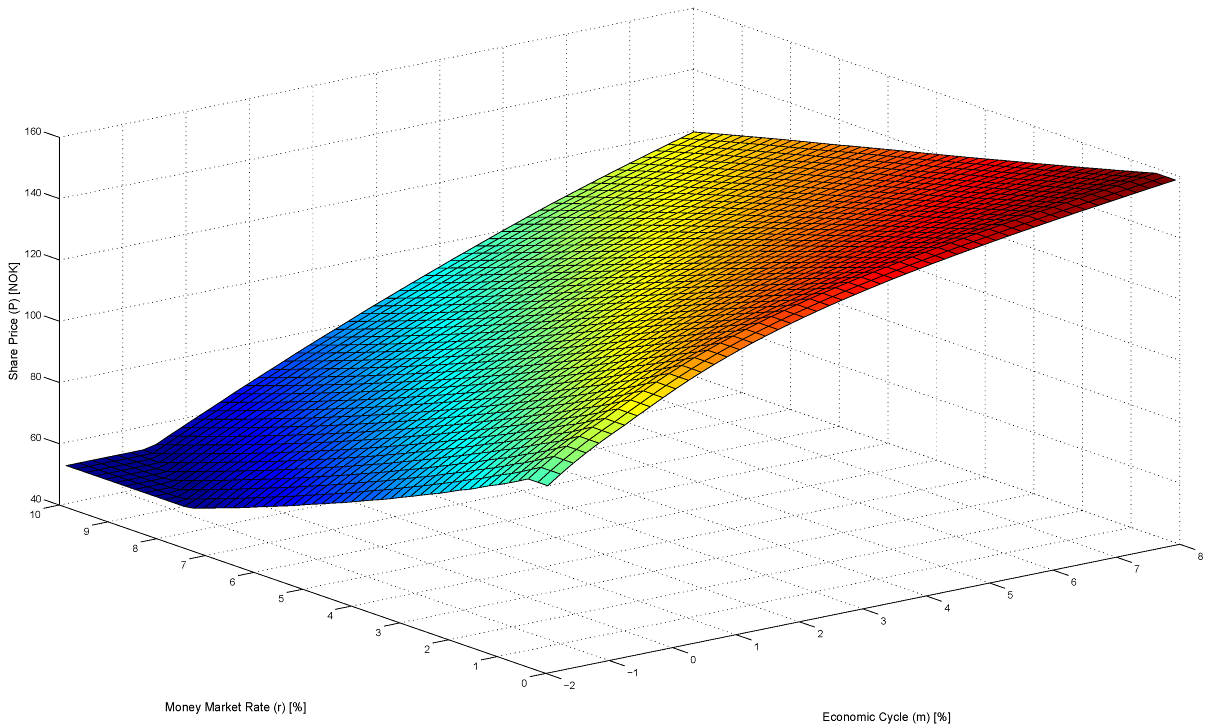

Figure 6.6: Expected share price for DNB ASA given optimal financing policies

We see that the share price is increasing in the business cycle variable for all money market rate levels. We believe this makes sense as the demand for loans, and hence revenues, increase when the economy is in a good state, while at the same time the expected impairment of loans decreases. The net effect is that net profits increase. The increase in the share price is marginally diminishing in the business cycle, the degree of which decreases in the money market rate (the second derivative is negative). In other words, in high interest rate scenarios, the business cycle seems to have a greater impact on the share price than in low interest rate scenarios.

Whether or not the share price should be increasing in the interbank rate, is perhaps a more ambiguous and interesting aspect. From the plot we see that the directional derivative is negative for all business cycle levels (excluding the liquidation region indicated by the free boundary). The second derivative is also negative, but it is monotonically increasing (it gets less negative) in the business cycle; for high business cycle levels, the decrease in the share price for increasing interest rate levels is less severe than for low business cycle scenarios.

We find these observations interesting, and will try to explain them from the plot of net profits or earnings per share in figure 6.2. Dividends paid to shareholders are the fundamental value driving mechanism in our model, and the magnitude of these is directly given by net profits available. We can see a distinct partitioning of the domain in two regions. The borderline between the two regions indicates the points where the growth in demand for loans goes from positive to negative. In the positive growth region, net profits are increasing in the interest rate for all business cycle levels. In the negative growth region, net profits are decreasing in the interest rate. This can explain why the derivative of the share price gets more negative the lower the business cycle level is. It does not explain why the derivative is negative in the first place, however.

To explain this, we need to have a look at how we model interest income. One can argue that a bank's net interest income is more dependent on the spread between lending and borrowing activities than on the level of the reference rate itself, because it to some extent can have its customers take the bill by increasing its loan rates when the money market rate increases. We have thus chosen to keep the spread on loans constant in our model. At the same time, we discount future dividend payments by shareholder's cost of equity. The cost of equity reflects shareholder's opportunity cost from investing their money in shares of the bank stock. As the money market rate increases, this opportunity cost also increases, and future dividend payments are discounted at a higher rate. Holding the spread on loans constant, the net effect is a decreasing share price in the interbank rate.

In the case of an economic downturn, corresponding to the flat region below GDP growth rates of approximately -0.8%, DNB would not be able to comply with the CT1 capital requirement imposed. The consequence would be governmental intervention, in practice a bankruptcy; we could say that equity holders exercise their bankruptcy option. Managers would liquidate the business, pay back realized values of the assets to creditors and distribute any remaining funds to shareholders through a dividend. The dividends per share amount would give the share price in this region - approximately NOK 57 per share in the DNB case.

Let us now have a look at how the dynamics of the expected share price change as the CT1 capital requirement is changed. Table 6.3 presents key statistical properties. As should be expected, increasing the requirement impacts the share price negatively. This is evident from both the mean, median, maximum and minimum values, which are strictly decreasing in the capital requirement. This is in accordance with basic optimization theory; if any binding constraints are made stricter, the new optimal solution will be worse than or, at best, equal to the old optimal solution.

	CT1 Requirement [%]			
Share Price [NOK]	0.0	4.0	**8.0**	10.0
Mean	287.7	202.1	**118.1**	75.8
Median	295.5	206.6	**122.4**	76.7
Standard Deviation	1.81	0.09	**2.77**	1.90
Maximum	362.1	266.5	**165.4**	107.4
Minimum	171.0	106.8	**53.3**	53.3

Table 6.3: Key statistical properties for the expected share price under different regulatory regimes

6.2 Results from Neural Fitted Q Iteration

This section presents the results from the DNB test case for the Neural Fitted Q Iteration approach when a CT1 capital ratio requirement of 8.0% is imposed. We start off with a discussion of the optimal financing policies in subsection 6.2.1, before we in subsection 6.2.1 take a look at the CT1 ratios these policies imply.

Subsection 6.2.3 presents the expected share prices given that the optimal financing policies are followed and the corresponding CT1 ratios are reached. The possibility of issuing equity enables a more comprehensive discussion of optimal financing policies and possible changes in the optimal capital structure.

6.2.1 Optimal Financing Policies

The NFQ solution incorporates three potential sources of funding: retained earnings, issuance of bonds and issuance of equity. This differs from the HJB model, which only incorporates retained earnings and issuance of bonds as financing sources. Figures 6.7, 6.9 and 6.10 illustrate optimal financing policies for DNB when constrained by a CT1 capital requirement of 8.0%. To better emphasize the dynamics of our solutions, we also include a plot of optimal dividends for the case when a minimum 6.0% CT1 capital ratio is required. We start off with a discussion of the dividends, before turning to the two other sources of funding.

Figure 6.7 displays optimal dividend payouts under the 8.0% regime. The surface suggests that it is only optimal to pay out dividends in economic states characterized by negative or slightly positive GDP growth and with low money market rates. In all other states the model suggests that DNB should retain all earnings in order to increase capital.

The shape of the surface is different from the one suggested by the HJB approach, where dividends are paid in all going-concern states. We found this result quite surprising, but recognize that there are some fundamental differences in how the two methods operate which may have them find different optimal solutions. We will elaborate further on this in section 6.3, and we confine ourselves to searching for some plausible economic explanation for this behavior for now.

Management must asses whether shareholders are better off getting paid dividends today or if they believe excess returns can be created in the future by retaining earnings. The plot suggests that the NFQ algorithm usually finds the latter to be the case, except for in the particular region mentioned earlier. Here dividends are

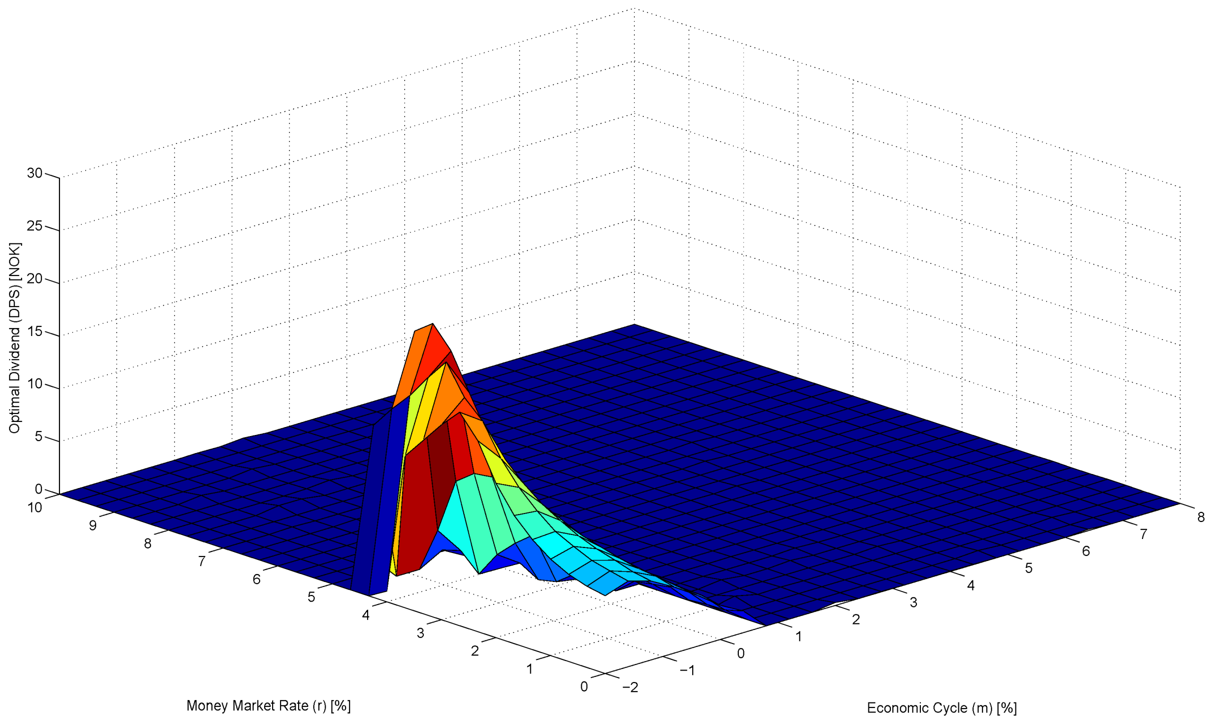

Figure 6.7: Optimal dividend policies for DNB ASA with $\Lambda_t = 8.0\%$

paid at an increasing rate as the GDP growth gets more negative and the money market rate increases. Due to the difficult economic conditions, we believe this can be interpreted as the model expecting a liquidation to be imminent, and hence that the prospects of future excess returns are considered poor (the bank is drained for values before the bankruptcy is a fact). This trend comes to a sudden stop as the money market rate approaches 5%. No dividends are paid out from this point on, because this is where expected net profits turn negative. If that was not the case, we believe this trend would have continued for all money market rates.

Although there are discrepancies between the proposed dividend surfaces for the two solution approaches at the 8.0% minimum requirement, we find the solutions at the 6.0% requirement to be more coherent.

Figure 6.8 presents the optimal dividend policies for the 6.0% regime. It bears more resemblance to the HJB solution, and is in line with the economic intuition drawn from the latter, as described in section 6.1.2. It is clear that the capital requirement has a great impact on the optimal dividend policies, and in the case of lower capital requirements it is possible to support higher dividend payouts in most regions. This also applies to the most extreme point of the dividend policies for the 8% regime. In addition, the curvature in the flat region of figure 6.8 indicates that there are some effects related to dividend payments that are not captured in the HJB solution.

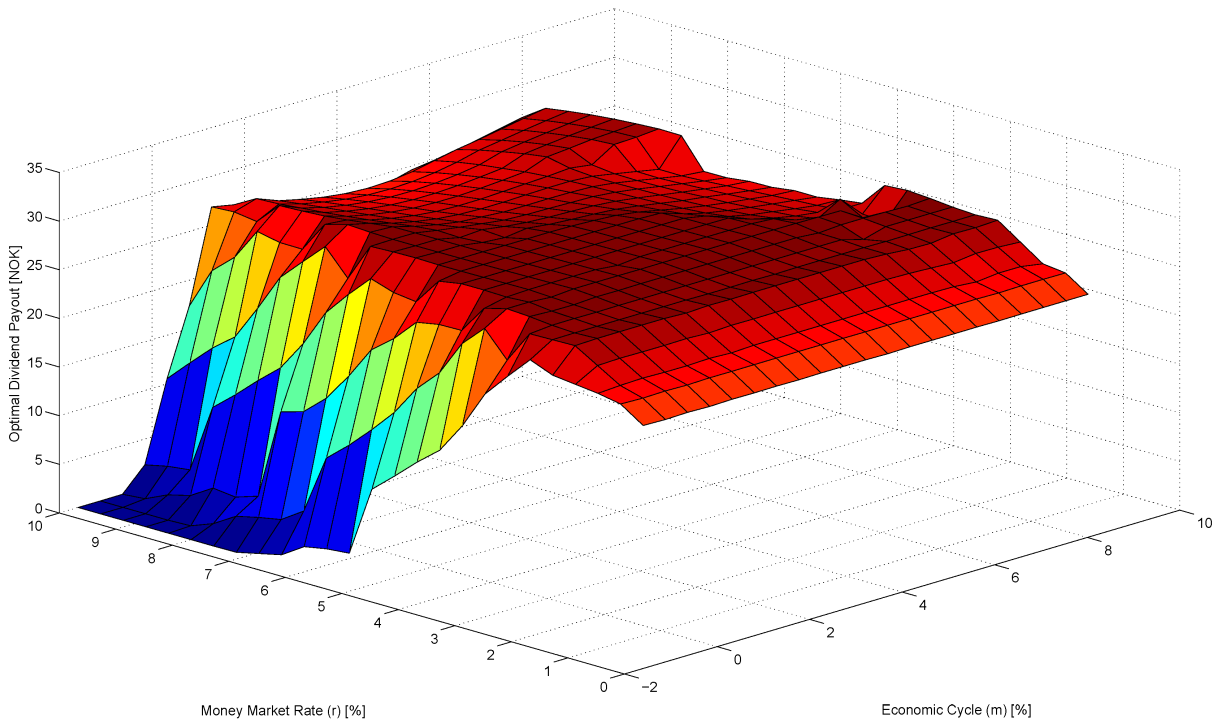

Figure 6.8: Optimal dividend policies for DNB ASA with $\Lambda_t = 6.0\%$

Figure 6.9 displays the optimal amount of bonds the model suggests DNB should issue at a CT1 requirement of 8.0%. We see that issuing debt is found to be optimal in most economic states, at an increasing rate in the GDP growth for almost all money market rates. In the liquidation region where dividends are paid, no debt is issued. The overall shape of the surface is coherent with the results of the HJB approach. Please refer to the discussion in subsection 6.1.2 for supporting economic intuition.

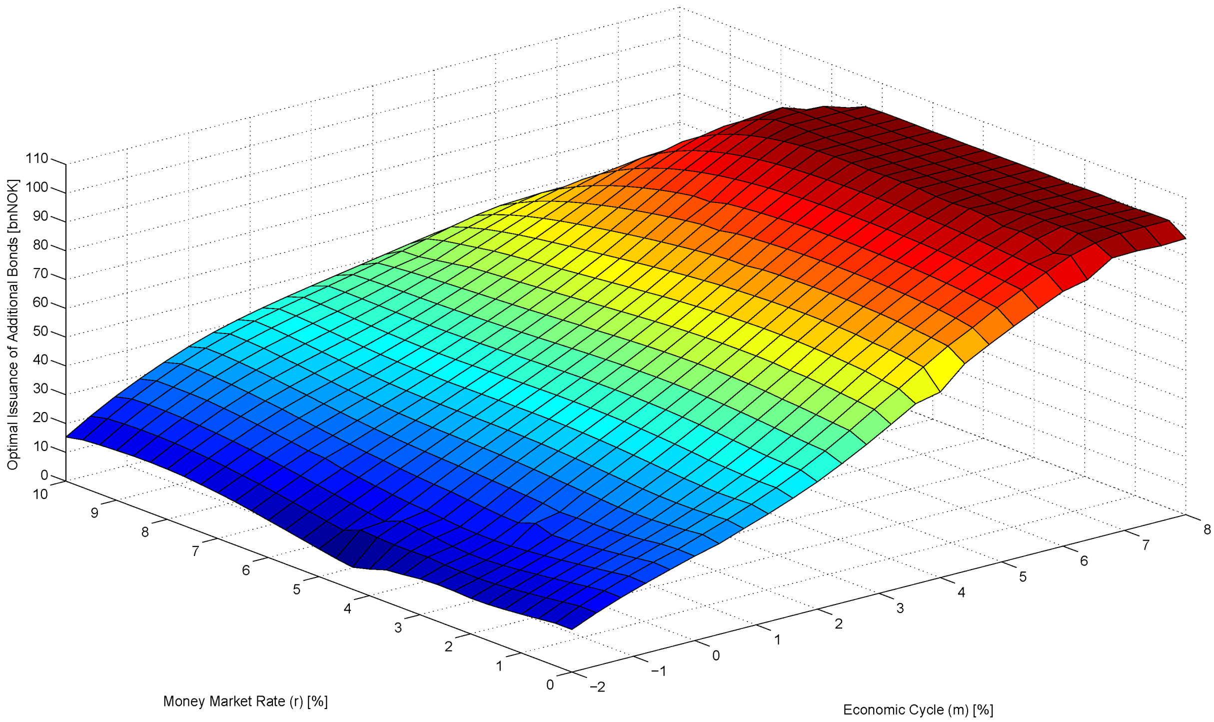

Figure 6.9: Optimal bond issuance for DNB ASA

Figure 6.10 displays the optimal amount of equity the model suggests DNB should issue at a CT1 requirement of 8.0%. Issuing equity impacts current shareholders negatively if new shareholders can acquire shares at a discount to the current market price due to the dilution effect. It is usually perceived as a negative signal by the market and should be used as a last resort when other sources of funding are not available. Our results comply with this; the only situation in which the model suggests DNB should issue equity, is when the probability of violating the CT1 capital requirement is significant. This is the case for scenarios of negative GDP growth combined with high money market rates. In this region all earnings are retained, but this is not sufficient to withstand the possibility of breaching the requirement if large losses occur.

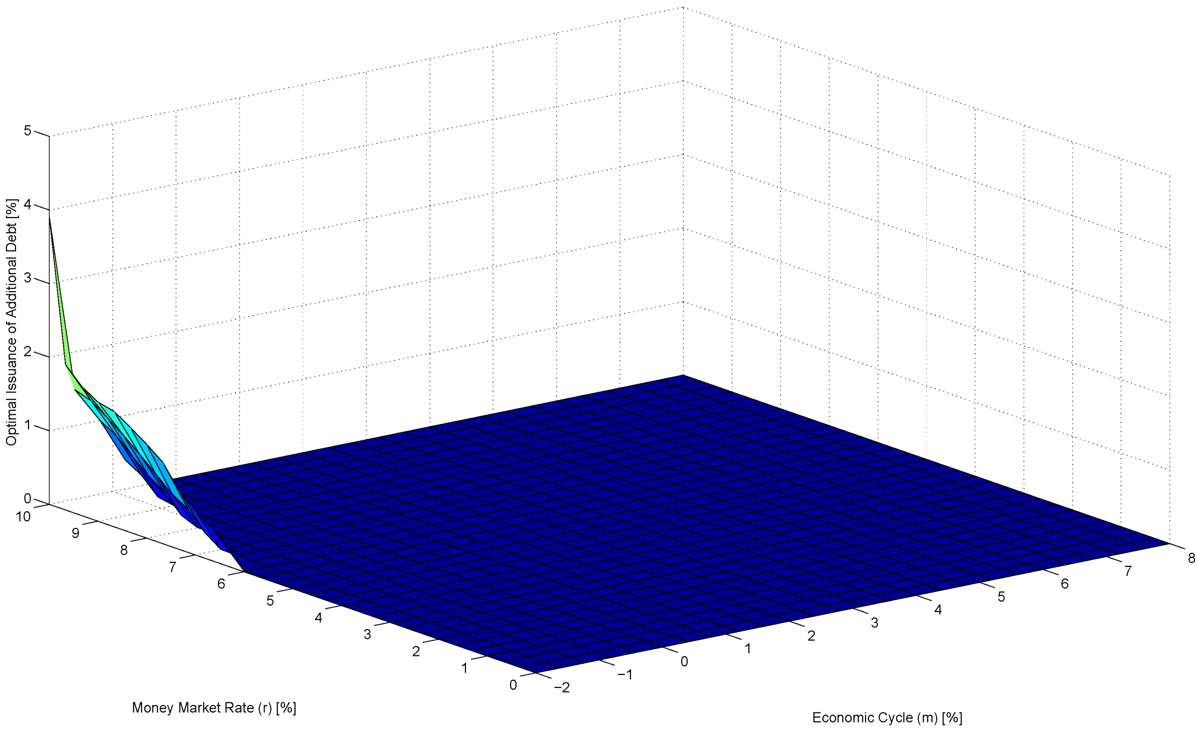

Figure 6.10: Optimal equity issuance for DNB ASA

6.2.2 Optimal Capital Structure

Figure 6.11 illustrates optimal DFI capital structure described by the CT1 ratio when the DFI is constrained by a capital requirement of 8.0%. As a reference one should also keep in mind that the branches of DNB we model have an initial CT1 ratio of 11.03%, well above the current regulatory requirement.

From the figure it is clear that the optimal CT1 ratio is well above the minimum requirement in all states, which is consistent with the resulting optimal CT1 ratio in section 6.1.2. The main trend we observe is that DNB should ideally hold more

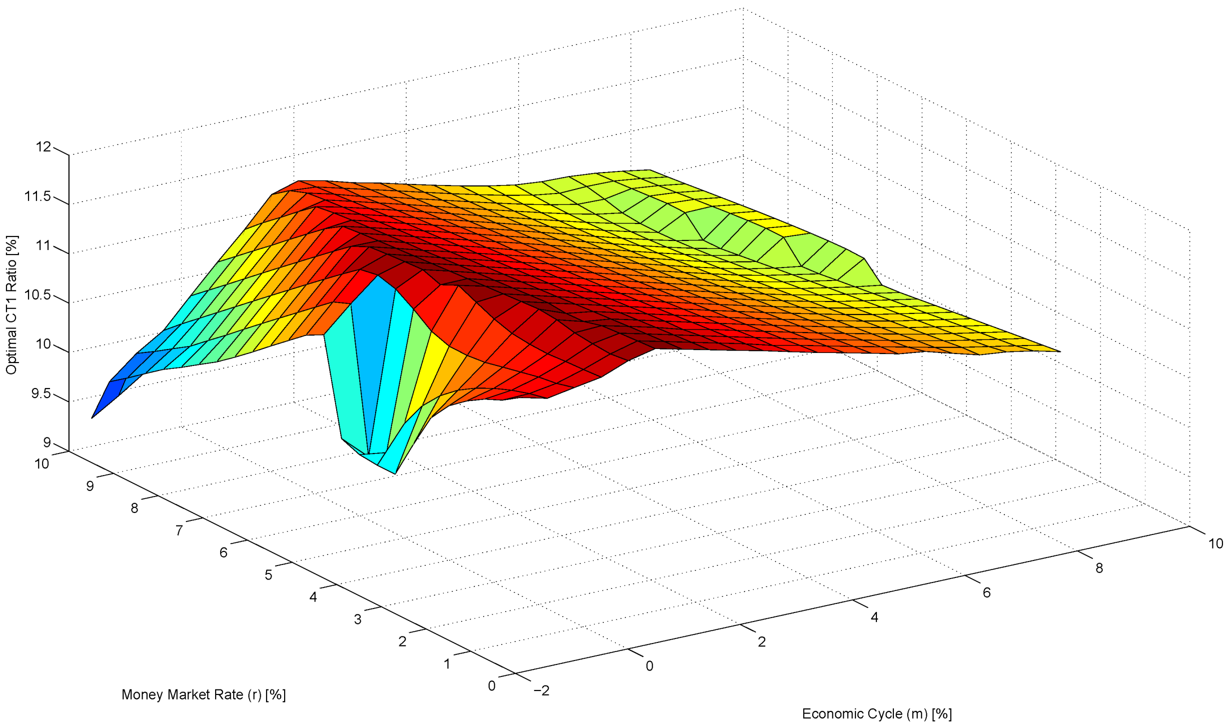

Figure 6.11: Optimal CT1 capital ratio for DNB ASA with $\Lambda_t = 8.0\%$

equity when the economy moves towards a recession and when the interbank rate grows. The largest CT1 ratio is 11.90%, 390 basis points above the requirement. The smallest CT1 ratio is 9.67%, 167 basis points above the requirement. On a risk-weighted asset base of NOK 1,439 billion this difference corresponds to NOK 32.1 billion, about one and a half times the size of DNB's net profit in a normal year. The average CT1 level is found to be 11.16%, 1.45 percentage points above the average CT1 level obtained with the HJB approach.

The overall shape of the CT1 ratio plots are similar. However, one region requires attention due to its special form. This region corresponds to the dividend paying region observed in figure 6.7, and is characterized by negative real GDP growth per capita and low money market rates. In this region the CT1 ratio is significantly lower than in other regions with the exception of the region characterized by negative real GDP growth and high money market rates. However, the reasons for the decrease in CT1 ratio in the two regions are different. In the region with the special form the ratio decreases due to an active choice from the manager's side, as positive earnings prompt higher dividend payouts. The economic intuition behind this choice was discussed in section 6.2.1. In the other region the decrease is due to negative earnings causing the book value of equity to decrease faster than risk-weighted assets on a relative basis, a decrease beyond the manager's control.

In the previous section we observed that the optimal dividend policy differed between the HJB solution and the NFQ solution when considering a capital requirement of 8.0%, and we presented the optimal dividend policy when the capital requirement was decreased to 6.0%. In figure 6.12 the corresponding optimal CT1 ratio under the 6.0% regime is presented. It bears greater resemblance to the HJB solution and is in line with the economic intuition drawn from the latter, as described in section 6.1.2.

When regarding the optimal CT1 ratio under the two different capital requirement regimes, it becomes clear that the capital requirement has a great impact on the optimal capital ratio. We observe that under a relaxed capital requirement, the optimal capital ratio is substantially lower, and on average 0.83 percentage points below the levels observed when the capital requirement was set at 8.0%. However, we also observe that there is a region characterized by a high capital ratio coinciding with states where the bank faces large unexpected losses. In other regions the bank can lower the capital ratio and move closer to the regulatory requirement.

Even though it is optimal to hold more capital in times of financial turmoil, it may not be possible to build capital from retained earnings in such periods, considering the bank incurs large losses. It must therefore build capital in periods where net profit is positive and the economy is stable. By considering figure 6.12, it is clear that if the capital requirement is low, the bank has an incentive to lower the capital ratio when the economy is expanding and earnings are high. Thus, instead of retaining earnings when it is possible, the bank pays higher dividends, causing the capital ratio to decrease. A too low capital requirement therefore increases

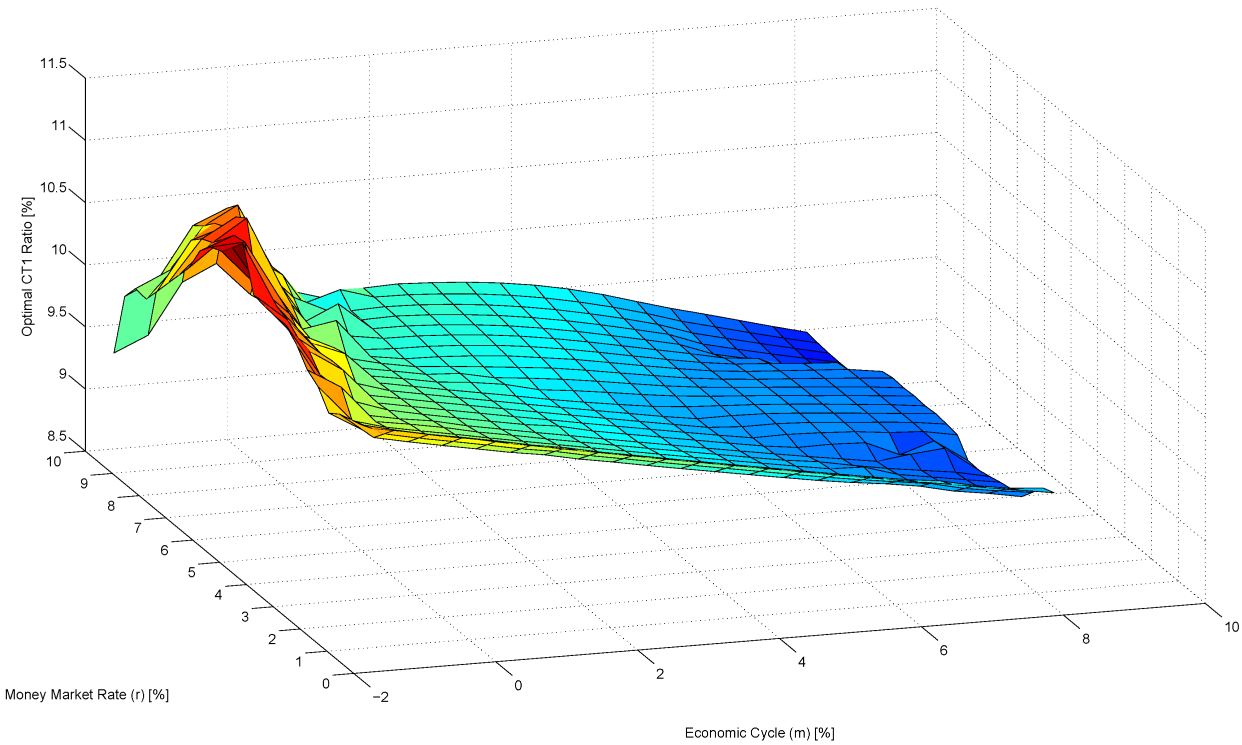

Figure 6.12: Optimal CT1 capital ratio for DNB ASA with $\Lambda_t = 6.0\%$

the likelihood of not withstanding large unexpected losses, as the bank would not have built the necessary capital when it had the chance.

From a regulatory perspective the capital requirement must therefore be set in such a way that the banks build capital in stable states so as to withstand losses if the economy should enter a recession. Hence a capital requirement of 6.0% may be too low. However, when considering the capital requirement of 8.0%, we observe that the bank retains earnings in most states, and hence builds capital, witnessed by the increased CT1 ratio in figure 6.11. Therefore, comparing these two requirements, we deem 8.0% a better choice from a regulatory perspective as it leads to desirable bank behavior for the society as a whole.

A capital requirement of 8.0% raises the optimal capital ratio in states where the economy is in growth, and the optimal capital ratio across all states becomes more aligned. This means that the standard deviation in the optimal capital ratio decreases, making the surface flatter. This can be seen in table 6.4, which describes key statistical properties for the optimal capital structure for selected capital requirements. The upward shift in the CT1 ratio when the requirement increases, is larger for states where the economy is in expansion than in a contraction. This is mathematically explained by the increasing mean, the decreasing standard deviation and the fact that the increase in maximum value is less than the increase in minimum value. Such an observation suggests that the optimal capital structure in a contraction is less affected by the capital requirements than the optimal capital structure in an expansion.

In the case where the capital requirement is set to 10%, we observe that in some cases the bank violates the requirement, as the minimum optimal CT1 ratio is below the requirement. This may indicate that in an economic downturn it could be optimal to violate, or that it might not be possible to comply with the regulatory requirement. The capital requirement must therefore be set such that it incentivizes the bank to build capital in stable and expansionary periods and that it minimizes unexpected behavior in periods of turmoil. Based on the table, such a level could be around 8-9%, and we note that this is the current level in the Norwegian legislation excluding the counter cyclical buffer and the buffer for systemically important banks like DNB.

	CT1 Requirement [%]			
Optimal CT1 [%]	0.0	4.0	**8.0**	10.0
Mean	9.32	10.33	**11.16**	11.17
Median	9.31	10.51	**11.19**	11.23
Standard Deviation	0.15	0.13	**0.10**	0.24
Maximum	11.45	11.71	**11.90**	12.17
Minimum	8.26	8.78	**9.67**	9.56

Table 6.4: Key statistical properties for optimal capital structure under different regulatory regimes

6.2.3 Value Function Representation based on ADP

Figure 6.13 shows the expected share price for DNB as a function of the economic cycle and the money market interest rate level. These prices assume the corresponding optimal dividends, bonds and equity policies presented in subsection 6.2.1 are followed.

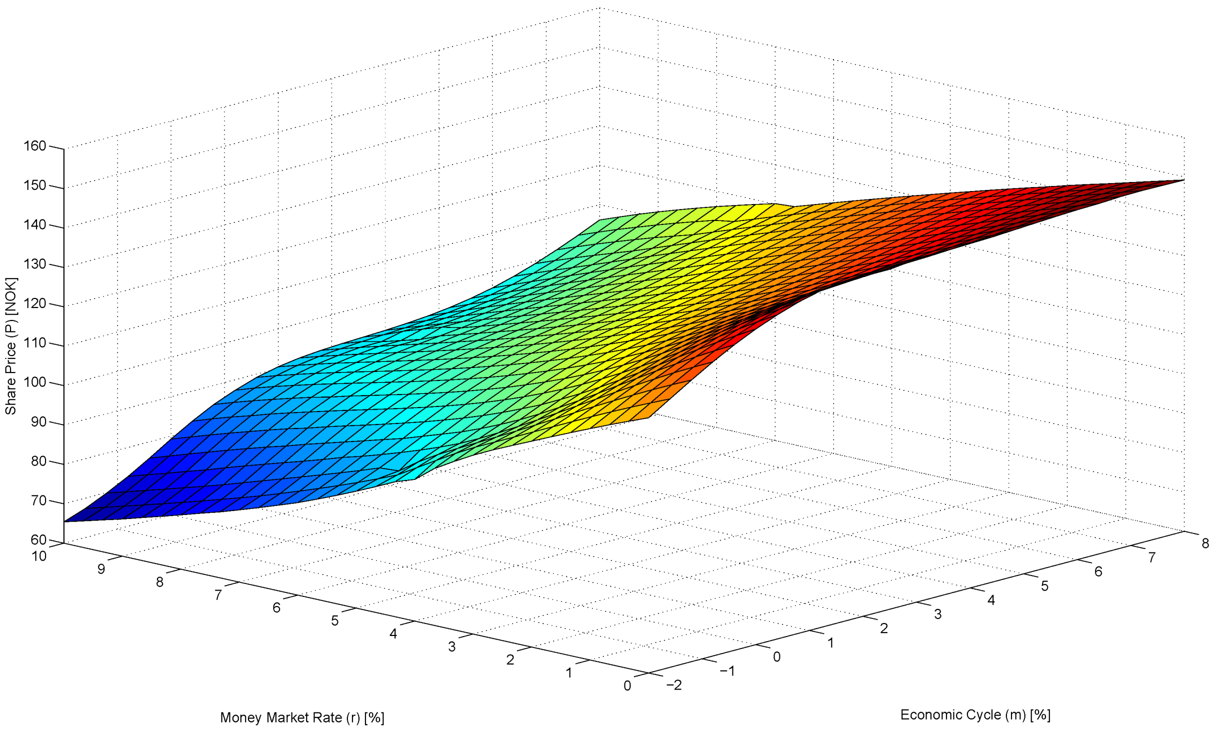

Figure 6.13: Expected share price of DNB ASA given optimal financing policies

The overall level and trend of the surface are coherent with the results from the HJB approach presented in section 6.1.3. The expected share price is monotonically increasing in the real GDP growth rate. In a growing economy, the demand for loans increases and the expected impairment of loans decreases. The net effect is that expected earnings increase, and earnings are the underlying driver of the share price.

Despite being equal in the overall shape, there are some differences in the dynamics of the two solutions. The HJB surface is a smooth, concave shape, while the NFQ surface has both concave and convex regions in the center of the plot. This implies that the second order derivative with respect to the economic state variables are positive in some regions and negative in others. We will elaborate on the reasons for these differences in section 6.3.

Table 6.5 shows some statistical properties for the expected share price under different regulatory regimes. Besides the 8.0% base case, the model was tested for CT1 capital ratio requirements of 0.0%, 4.0% and 10.0%, respectively. There are evident trends in the numbers, and these clearly show how capital regulations impact the share price; the stricter the capital requirement, the lower the share price.

	CT1 Requirement [%]			
Share Price [NOK]	0.0	4.0	**8.0**	10.0
Mean	173.04	156.11	**113.65**	80.77
Median	173.12	154.95	**114.20**	79.02
Standard Deviation	5.88	5.55	**1.79**	1.00
Maximum	203.53	187.27	**147.83**	118.78
Minimum	128.27	110.88	**68.37**	45.55

Table 6.5: Key statistical properties for share price under different regulatory regimes

6.3 Comparison of HJB and NFQ Solutions

In this section we compare the results given by the two solution approaches for the DNB test case. The comparison is performed on two levels. In the first part differences in the specific results for the optimal financing policies, capital structure and value function representation are discussed. In the second part the two solution methods are evaluated against each other and the quality of the solutions is discussed.

6.3.1 Differences in Results

The governing trends of the results for the two methods are in accordance with each other, but there are some interesting discrepancies. The most crucial differences are found in the dividend policies and optimal capital structure as the capital requirement increases past 7.0%. While the HJB-PDE gives a solution in which dividends are paid in all possible states, the NFQ solution only pays dividends in the small region of the surface where both the real GDP growth and the money market rate are at their lowest.

Economic intuition has been provided for both cases, and builds on the notion that management always has to asses whether shareholders are better off getting paid dividends now or if they believe they can create excess returns in the future by retaining earnings. The HJB approach seems in general to lean towards the first alternative, while the NFQ approach definitely leans towards the latter. From the plots of the optimal capital structures in figures 6.5 and 6.11, we can see an implication of this, as the NFQ approach consistently suggests a higher CT1 ratio than the HJB approach. We could interpret this as the NFQ algorithm offering a more risk-averse evaluation of the state space than the HJB model.

6.3.2 Comparison of HJB and NFQ as Solution Methods

The HJB-PDE and the NFQ algorithm solve the same stochastic optimal control problem, but they take quite different approaches in doing so. These fundamental differences are mainly due to the underlying assumptions governing the solution methods, how the stochastic nature of the problem is incorporated, and how optimal actions are found. It is hard for us to say whether one approach is better than the other, but we can point out the differences and their implications in light of our test case.

In subsection 4.3.3 we argued that the NFQ algorithm is a model-free approach, in the sense that we do not really need to assume anything about the underlying system model if we have sufficient training data available. It is a data-driven model, in which the system finds the optimal action by learning the consequences of taking specific actions in a given state. By assuming a sufficiently large training sample and enough iterations, the algorithm should converge to an optimal solution. The HJB method, however, is definitely a model-driven approach. Here all solutions are calculated from an analytically derived equation, and no simulations or data samplings are needed.

The stochastic nature of the optimal control problem is incorporated in different ways in the two solution methods. In the NFQ method Monte Carlo sampling is used to describe the stochastic processes in the underlying model. The assumption is that a sufficiently large sample will capture the underlying distribution of the stochastic variables, and thus the stochastic dynamics. In the HJB method the stochasticity is incorporated by taking the conditional expectation of a Taylor expansion describing the incremental change in the value function. This is then substituted into Bellman's equation in order to derive the HJB-PDE.

In general, one should be able to expect "richer" solutions from the NFQ approach if the algorithm is run long enough and the neural network is fed enough relevant training data. With the possibility of issuing new equity, the NFQ algorithm also has an additional decision variable at its disposal compared to the HJB-PDE. According to basic optimization theory, NFQ should hence be able to find better solutions than HJB (or at least as good). This comes at a price in terms of computational complexity, however, and there are many elements in the setup of the algorithm that need to be set appropriately for it to give stable results.

7

Concluding Remarks

We have developed a conceptual decision model for the capital structure of depository financial institutions, where the market value of equity is maximized in a dividend discount framework. The key objective of the model is to study the behavior of DFIs under uncertain market conditions, in particular how financing policies and capital structure decisions are affected when governmental capital regulations are imposed. We formulate a stochastic optimal control problem and solve this using two distinct approaches based on dynamic programming.

A case study of DNB ASA was carried out to test the model under different regulatory regimes. The results obtained suggest that there is a significant cost for equity holders associated with governmental capital regulations, seen from the reduction in market value of equity when requirements are strengthened. We find the impact of regulations to be stronger in times of high economic growth than during downturns. Moreover, given the current regulatory requirements, our model suggests that DNB ASA could increase shareholder value substantially by lowering its capital adequacy ratio and as such adopt a riskier capital structure. This is however not necessarily in the best interest of society as a whole, as the consequences of systemically important banks failing can be dramatic.

Despite showing promising results, we recognize that our model has its limitations.

The remainder of the paper is devoted to a discussion of these and our propositions for further research.

7.1 Modelling DFIs

The conceptual DFI we assume in our model, represents a simplified and limited version of such institutions in real life. We only assume one possible class of risky assets (loans), when in reality a DFI would hold a portfolio of assets from multiple risk classes, all with different risk and reward properties. We also do not allow the DFI to alter its market position, i.e. take or lose market shares; we force it to satisfy an exogenously given demand for loans. In reality, the DFI would be able to choose whether or not to satisfy this demand, and whether or not to position itself so as to capture new market shares. This could for instance be done by adjusting the interest rate margin on loans. Our model does not allow this, however; not do we only assume the spread not to be subject to the DFI's decision, we also assume it constant - independent of the business cycle and the money market rate. Another simplifying assumption we have made, is that the amount of deposits in the DFI at any time is given as a fixed fraction of total loans. A real DFI may try to keep such a fixed ratio, but is not guaranteed to do so; the deposits should ideally follow a stochastic process of their own. This could for instance be a jump-diffusion process to incorporate the possibility of extreme movements like bank-runs. The ADP model can easily be scaled, but an additional stochastic variable would complicate the HJB-PDE solution considerably. Moreover, the only income modelled is the interest income from interest generating activities. A commercial bank like DNB typically earns a substantial part of its income from investment banking activities.

All of these points add to the fact that our results are not directly comparable to real DFI numbers; the share prices we present for DNB ASA should not be directly compared to available market prices. Any extensions of our model should consider implementing these elements if a more realistic model is desirable.

7.2 Solution by Hamilton-Jacobi-Bellman PDE

The analytic form of our two stochastic processes makes finding an analytical solution to the HJB-PDE intractable. This means we have to rely on numerical methods. The finite difference method we have chosen, forces us to discretize our problem domain on a grid - both the state space and the action space. The finer the grid, the more accurate the solution, but no matter how fine discretization we use, we will never get the accuracy of a continuous solution. Also, finer discretization means longer running time for the algorithm.

The HJB-PDE approach does not incorporate equity issuance as a decision variable. This is out of considerations of the computational complexity; because the finite difference method relies on traversing the whole grid for each possible decision, the running time is exponential in the number of decision variables. If enough computational power were available, a natural extension of the model would be to include this decision variable.

7.3 Solution by Approximate Dynamic Programming

The most fundamental limitation to our ADP solution approach is the lack of available research on the topic. While the HJB-PDE approach is a well-established tool for solving stochastic problems in finance, the theory of ADP is in itself very young in an academic perspective. To the best of our knowledge, there are no one who has applied ADP techniques for DFI valuation in the way we do before. Consequently, we have had to do a bit of experimentation ourselves when it comes to the choices of learning procedure, approximation architecture, algorithm parameters, neural network settings, training algorithms etc. When starting the work, we did not know whether or not this approach would work at all. We have made the road by walking, trying and failing, adding elements along the way, adjusting the model as experienced was gathered. We still do not know if we have made the right choices for our particular problem and see a lot of opportunities for future investigations.

When it comes to the final ADP model, the most obvious limitation is the computational complexity. Even though we manage to handle the curses of dimensionality in exact dynamic programming, the running time is still a pressing issue. We solve a multidimensional all-continuous problem with strong nonlinearities, and as mentioned in subsection 4.3.2, any ADP algorithm must visit each state in the state space infinitely often to formally guarantee mathematical convergence of the value function approximation. In theory, the model should continuously improve as new training data is learned and could hence be run enduringly. MATLAB is not the fastest available programming language, however. Porting our code to some faster and more low-level programming language, like C or FORTRAN, to improve processing time, could be an interesting next step of research. The NFQ algorithm is also generally well suited for parallel processing, which could potentially give a huge speedup of the code if one had access to a cluster of computers with a lot of processors and built some multi-threading framework with message passing to handle the parallel execution.

Besides being computationally intensive, the NFQ model needs large amounts of relevant training data to be able to learn anything interesting. Commercial banks have access to a lot of real-time data which could potentially be utilized efficiently by an NFQ implementation. The data could be fed directly to the model, which could be kept running continually, incrementally learning from the data and consistently proposing better financing policies. How intelligent would such an agent be after months or years of learning? In theory it should eventually be able to dynamically control the capital structure of the bank by itself. We believe there are great opportunities for further work on such an application.

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Appendices

Appendix A

Capital Requirements for Norwegian DFIs

The following is taken directly from DNBs 2013 annual report, and explains the Norwegian interpretation and implementation of the Basel III Directive:

> *"On 22 June 2013, the Norwegian parliament decided to introduce new capital requirements as the first step in the adaptation to the CRD IV regulations. The new capital requirements in Norway entered into force on 1 July 2013 and imply a gradual increase in capital adequacy requirements over the coming years. Other requirements in the CRD IV regulations have not yet been introduced in Norway, though the Norwegian authorities are in the process of working out national rules that will apply until the CRD IV regulations are included in the EEA agreement. The new capital adequacy requirements for Norwegian banks imply that the minimum common equity Tier 1 capital requirement has been increased to 4.5 per cent. The minimum Tier 1 capital requirement has been increased to 6 per cent, of which up to 1.5 per cent may consist of hybrid capital. The minimum capital adequacy requirement has been retained at 8 per cent, of which Tier 2 capital can represent maximum 2 per cent. The new system entails that the banks will be required to hold significantly more capital than the minimum requirement in the form of various buffers. The international regulations require that all banks maintain a 2.5 per cent capital conservation buffer. In addition, Norway has introduced a 2 per cent systemic risk buffer, which will be increased to 3 per cent as of 1 July 2014. In addition, a special buffer of up to 1 per cent will be introduced for systemically important institutions with effect from 1 July 2015 and be increased to maximum 2 per cent as of 1 July 2016. Furthermore, a counter-cyclical capital element will be introduced, ranging between 0 and 2.5 per cent, determined by the national supervisory authorities. On 12*

December 2013, the Ministry of Finance concluded that the initial level of the counter-cyclical buffer should be 1 per cent. This requirement will enter into force on 30 June 2015. The level of the counter-cyclical capital buffer will be determined each quarter. A decision to increase the level will normally enter into force no earlier than 12 months after the decision was made. If the maximum countercyclical buffer requirement is applied, the total capital requirement will represent 18 per cent of risk-weighted assets. Of this, 8 percentage points represents the minimum primary capital requirement, while the buffer requirements that must be met exclusively by common equity Tier 1 capital constitute 10 percentage points. The Norwegian authorities have chosen to retain the Basel II transitional rules, which set a floor for how low a bank's risk-weighted volume can be relative to the Basel I rules, the so-called Basel I floor. This distinctively Norwegian supervisory practice will be of no consequence to domestic banks' actual capital adequacy, but will make Norwegian banks appear more weakly capitalised in international comparisons."

– DNB, Annual Report 2013

Appendix B

Derivation of the Hamilton-Jacobi-Bellman Equation

We will derive the Hamilton-Jacobi-Bellman Partial Differential Equation (HJB-PDE) from the continuous time version of the Bellman equation given by:

$$\rho P(m,r) = \max_{u \in \mathcal{U}} \left\{ u_1(m,r) + \frac{1}{dt}\mathbb{E}[dP] \right\}$$

We have that (m,r) is the known starting position at current time and $(m',r') = (m+dm, r+dr)$ is the random position at the end of an infinitesimal timestep dt. We can apply Itô's Lemma to or Taylor expand the incremental change in the value function dP around the known starting position (m,r) which yields:

$$\begin{aligned} dP = P_m(m,r)dm + P_r(m,r)dr + \\ \frac{1}{2}[P_{mm}(m,r)(dm)^2 + 2P_{mr}(m,r)dmdr + P_{rr}(m,r)(dr)^2] + o(dt) \end{aligned}$$

We reiterate that since we have an infinite horizon problem the calendar date has no effect since the value function is stationary and any partial derivative taken with respect to time is zero. Therefore we also omit the time label in our expressions. We can substitute for the stochastic increments dm and dr from equations 3.20 and 3.21 respectively and use the following multiplication rules:

	dt	dZ^m	dZ^r
dt	0	0	0
dZ^m	0	dt	γdt
dZ^r	0	γdt	dt

Using these and meanwhile acknowledging that $o(dt)$ captures terms that go to zero faster than dt we get:

$$dP = P_m(m,r)[\delta(\mu - m)dt + \eta dZ^m] + P_r(m,r)[\alpha(\beta - r)dt + \sigma\sqrt{r}dZ^r] + \frac{1}{2}[P_{mm}(m,r)\eta^2 dt + 2P_{mr}(m,r)\eta\sigma\sqrt{r}\gamma dt + P_{rr}(m,r)\sigma^2 r dt]$$

We can now evaluate the conditional expectation $\mathbb{E}[dP]$ on the current (m,r) and u, and remember that $dZ^m, dZ^r \sim \mathcal{N}(0, dt)$:

$$\mathbb{E}[dP] = \{\delta(\mu - m)P_m(m,r) + \alpha(\beta - r)P_r(m,r) + \frac{1}{2}[\eta^2 P_{mm}(m,r) + 2\gamma\eta\sigma\sqrt{r}P_{mr}(m,r) + \sigma^2 r P_{rr}(m,r)]\}dt$$

Substitution of this conditional expectation into the Bellman equation yields the HJB-PDE given by:

$$\max_{u \in \mathcal{U}} \left\{ \frac{1}{2}\eta^2 P_{mm}(m,r) + \gamma\eta\sigma\sqrt{r}P_{mr}(m,r) + \frac{1}{2}\sigma^2 r P_{rr}(m,r) + \delta(\mu - m)P_m(m,r) + \alpha(\beta - r)P_r(m,r) - \rho P(m,r) + u_1(m,r) \right\} = 0 \quad \text{(B.1)}$$

Appendix C

Derivation of Price-To-Book Ratio of Firm in Stabile Growth

The price-to-book valuation multiple for a firm in stabile growth is given by:

$$\left.\frac{P}{B}\right|_t = \frac{ROE - g}{COE - g}$$

Proof. The Dividend Discount Model (DDM), sometimes referred to as the Gordon Growth Model (GGM), can be used to derive an expression for the price-to-book ratio under the same set of assumptions. The DDM for the price of a stock is given by:

$$P_t = \frac{D_{t+1}}{COE - g}$$

If x represents the constant payout ratio of the firm's net income Π_{t+1}, we have that:

$$P_t = \frac{\Pi_{t+1} x}{COE - g}$$

The net income can be expressed as:

$$\Pi_{t+1} = ROE \cdot BVE_t$$

The sustainable growth rate a firm can achieve using only its own sources of funding is given by:

$$g = ROE(1 - x)$$

Since the model assumes a constant growth rate we will also assume that the ROE for a given terminal node captures the expected earnings in the next time period. Then we can substitute these expressions in the DDM, use $B_t = BVE_t$ and obtain the desired result:

$$\left.\frac{P}{B}\right|_t = \frac{ROE \cdot x}{COE - g}$$

$$\left.\frac{P}{B}\right|_t = \frac{ROE - g}{COE - g}$$

□

Appendix D

Derivation of Statistical Properties of the Stochastic Processes

D.1 Ornstein-Uhlenbeck Model

In this section we first derive the integral representation of the Ornstein-Uhlenbeck Process. This allow us to derive the conditional probability density function that variables with such dynamics satisfy.

The stochastic differential equation for the macroeconomic variable, m_t is given by equation (3.20). We can derive the integral representation by expanding $x = m_t e^{\delta t}$ using Itô's Lemma. We get:

$$\mathrm{d}x = \frac{\partial x}{\partial m_t}\mathrm{d}m_t + \frac{\partial x}{\partial t}\mathrm{d}t$$

where both higher order partials with respect to m_t and the crosspartial goes to zero due to the form of the expression under Itô expansion. The higher order terms with respect to t goes to zero due to the multiplication rule $\mathrm{d}t^2 = 0$. Then we get:

$$\mathrm{d}x = e^{\delta t}[\delta(\mu - m_t)\mathrm{d}t + \eta \mathrm{d}Z_t] + \delta m_t e^{\delta t}\mathrm{d}t$$

$$\mathrm{d}x = \mu\delta e^{\delta t}\mathrm{d}t + \eta e^{\delta t}\mathrm{d}Z_t$$

$$\mathrm{d}(m_t e^{\delta t}) = \mu\delta e^{\delta t}\mathrm{d}t + \eta e^{\delta t}\mathrm{d}Z_t$$

If we let the previous index t be a dummy for time, we can now integrate over a defined timeinterval t to $t + \Delta t$ on both sides to obtain:

$$m_{t+\Delta t}e^{\delta(t+\Delta t)} - m_t e^{\delta t} = \mu[e^{\delta(t+\Delta t)} - e^{\delta t}] + \int_t^{t+\Delta t} \eta e^{\delta s}\mathrm{d}Z_s$$

$$m_{t+\Delta t} = m_t e^{-\delta\Delta t} + \mu[1 - e^{-\delta\Delta t}] + e^{-\delta(t+\Delta t)}\int_t^{t+\Delta t} \eta e^{\delta s}\mathrm{d}Z_s$$

which is the representation of the economic cycle $m_{t+\Delta t}$ at time $t+\Delta t$, where Δt is the time step, conditioned on the previous economic cycle m_t. Consequently, we see that $m_{t+\Delta t}$ is the sum of two constant terms and a Riemann sum of independent and identically distributed stochastic terms due to $\mathrm{d}Z_s \sim \mathcal{N}(0, \mathrm{d}s)$. Then, by evaluating the moment generating function (mgf) of the normal distribution, or using a transformation with the appropriate Jacobian, it can be shown that $m_{t+\Delta t}$ also follows a normal distribution. If we take the expectation and variance of $m_{t+\Delta t}$, we get the formal result that:

$$m_{t+\Delta t} \sim \mathcal{N}(\mu + [m_t - \mu]e^{-\delta\Delta t}, \frac{\eta^2}{2\delta}[1 - e^{-2\delta\Delta t}]) \tag{D.1}$$

Thus, the closed form expression for $m_{t+\Delta t}$ at time $t + \Delta t$, conditioned on m_t, is given by:

$$m_{t+\Delta t} = \mu + [m_t - \mu]e^{-\delta\Delta t} + \eta\sqrt{\frac{1 - e^{-2\delta\Delta t}}{2\delta}}\mathcal{N}_{0,1} \tag{D.2}$$

where $\mathcal{N}_{0,1}$ is the standard normal distribution. We can then estimate the parameters δ, μ and η by the method of maximum likelihood. Let $U_1, ..., U_n$ be a sample of n observations from this distribution, where the data on real GDP growth per capita is taken from Statistics Norway. The time step in the sample is one year ($\Delta t = 1$), and we index the observations such that $i \in \{1, ..., n\}$ represent a set of n years. The conditional probability distribution of the variable U_i depends only on the realization of the variable in the previous time step, U_{i-1}. Hence, this stochastic process is a Markov process. The conditional probability density function is given by:

$$f_{U_i|U_{i-1}}(u_i|u_{i-1}; \delta, \mu, \eta) = \frac{1}{\sqrt{2\pi\xi^2}}\exp[-\frac{(u_i - u_{i-1}e^{-\delta\Delta t} - \mu(1 - e^{-\delta\Delta t}))^2}{2\xi^2}] \tag{D.3}$$

where ξ^2 is the variance of the conditional distribution given by $\xi^2 = \frac{\eta^2}{2\delta}[1-e^{-2\delta\Delta t}]$.

The log-likelihood function, given the vector $\mathbf{u}$ of sample data, is given by:

$$\ln\mathcal{L}(\delta,\mu,\eta;\mathbf{u}) = \sum_{i=1}^{n} \ln f_{U_i|U_{i-1}}(u_i|u_{i-1};\delta,\mu,\eta) =$$

$$-\frac{n}{2}\ln(2\pi) - n\ln(\xi) - \frac{1}{2\xi^2}\sum_{i=1}^{n}[u_i - u_{i-1}e^{-\delta\Delta t} - \mu(1-e^{-\delta\Delta t})]^2 \quad \text{(D.4)}$$

We can maximize the likelihood function with respect to δ, μ and η by looking at the first-order conditions. This yields the following maximum likelihood estimators (MLEs):

$$\hat{\mu} = \frac{u_y u_{xx} - u_x u_{xy}}{n(u_{xx} - u_{xy}) - (u_x^2 - u_x u_y)} \quad \text{(D.5)}$$

$$\hat{\delta} = -\frac{1}{\Delta t}\ln\frac{u_{xy} - \hat{\mu}u_x - \hat{\mu}u_y + n\hat{\mu}^2}{u_{xx} - 2\hat{\mu}u_x + n\hat{\mu}^2} \quad \text{(D.6)}$$

$$\hat{\xi}^2 = \frac{1}{n}[u_{yy} - 2\beta u_{xy} + \beta^2 u_{xx} - 2\hat{\mu}(1-\beta)(u_y - \beta u_x) + n\hat{\mu}^2(1-\beta)^2] \quad \text{(D.7)}$$

where $\beta = e^{-\delta\Delta t}$. Then by the invariance property of maximum likelihood estimators, it follows that the MLE for η is given by:

$$\hat{\eta} = \hat{\xi}\sqrt{\frac{2\hat{\delta}}{1-e^{-2\hat{\delta}\Delta t}}} \quad \text{(D.8)}$$

The sample statistics u_x, u_y, u_{xx}, u_{xy} and u_{yy} are given by the following relationships:

$$u_x = \sum_{i=1}^{n} u_{i-1} \quad \text{(D.9)}$$

$$u_y = \sum_{i=1}^{n} u_i \quad \text{(D.10)}$$

$$u_{xx} = \sum_{i=1}^{n} u_{i-1}^2 \quad \text{(D.11)}$$

$$u_{xy} = \sum_{i=1}^{n} u_{i-1}u_i \quad \text{(D.12)}$$

$$u_{yy} = \sum_{i=1}^{n} u_i^2 \quad \text{(D.13)}$$

D.2 Cox-Ingersoll-Ross Model

For the Cox-Ingersoll-Ross model we will not derive the conditional probability distribution, as it requires a very rigorous proof and is substantially more tedious than for the Ornstein-Uhlenbeck model. Hence we will merely state the conditional density function since the objective here is rather to outline the estimation procedure than to prove density functions. We can then estimate the parameters α, β and σ by the method of maximum likelihood. Let $R_1, ..., R_n$ be a sample of n observations from the conditional distribution of the CIR model, where the data on 3 month NIBOR is taken from Statistics Norway. The time step in the sample is one year ($\Delta t = 1$), where the data points correspond to annual averages. We index the observations such that $i \in \{1, ..., n\}$ represent a set of n years. The conditional probability distribution of the variable R_i depends only on the realization of the variable in the previous time step, R_{i-1}. Hence, this stochastic process is a Markov process. The conditional probability density function is given by:

$$f_{R_i|R_{i-1}}(r_i|r_{i-1}; \alpha, \beta, \sigma) = ce^{-u_{i-1}-v_i}\left(\frac{v_i}{u_{i-1}}\right)^{\frac{q}{2}} I_q(2\sqrt{u_{i-1}v_i})$$

where $I_q(\cdot)$ is the modified Bessel function of the first kind of order q and:

$$c = \frac{2\alpha}{(1 - e^{-\alpha\Delta t})\sigma^2} \tag{D.15}$$

$$u_{i-1} = cr_{i-1}e^{-\alpha\Delta t} \tag{D.16}$$

$$v_i = cr_i \tag{D.17}$$

$$q = \frac{2\alpha\beta}{\sigma^2} - 1 \tag{D.18}$$

The log-likelihood function, given the vector $\mathbf{r}$ of sample data, is given by:

$$\ln\mathcal{L}(\alpha, \beta, \sigma; \mathbf{r}) = \sum_{i=1}^{n} \ln f_{R_i|R_{i-1}}(r_i|r_{i-1}; \alpha, \beta, \sigma) =$$

$$n\ln(c) + \sum_{i=1}^{n}[-u_{i-1} - v_i + \frac{1}{2}q\ln\left(\frac{v_i}{u_{i-1}}\right) + \ln\{I_q(2\sqrt{u_{i-1}v_i}\}] \tag{D.19}$$

Then we need to maximize the likelihood function with respect to α, β and σ. This can be done analytically by looking at the first-order conditions or by maximizing the log-likelihood function with respect to the parameters in Matlab for instance. We chose the latter, being more convenient and efficient.

Appendix E

Financial Highlights in DNB's Annual Report 2014

Key Financial Items, DNB 2014		
Net Interest Income	32.487	[bn NOK]
Net Profit	20.617	[bn NOK]
Earnings per Share	12.67	[NOK]
Dividend per Share	3.80	[NOK]
Stock Price	134.20	[NOK]

Table E.1: Key financial items and metrics from DNB's annual report 2014